Taylor series and linearization

$$F(x) = 0$$

Let's guess $x \simeq x_0$

Let $x' = x - x_0$

$$\underbrace{F(x)}_{0} - \underbrace{F(x_0)}_{\text{computable}} = \left(\frac{\partial F}{\partial x}\right) x' + \overbrace{\left(\frac{1}{2} \left(\frac{\partial^2 F}{\partial x^2}\right)(x', x') + \cdots \right)}^{\text{ignored}}$$

Newton's method:

$$x_1 = x_0 + x'$$

$$F(x) - F(x_1) = \left(\frac{\partial F}{\partial x}\right) x_1' + \cdots$$

$$x_2 = x_1 + x_1'$$

$$\cdots$$

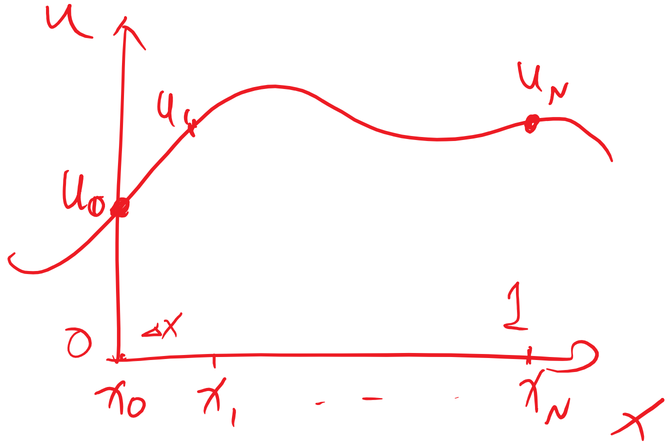
Finite difference for heat equation (parabolic)

1. **How to approximate a derivative with arithmetic?**
2. How much error does the approximation cause?
3. How does the approximation error affect the solution?

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

$$\Downarrow \; \text{FD} \; \Downarrow$$

$$\frac{du_i}{dt} = K \frac{u_{i-1} + u_{i+1} - 2u_i}{\Delta x^2}$$



$$u_0 = u_N = 0$$

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \end{pmatrix} = K \begin{pmatrix} -\frac{2}{\Delta x^2} & \frac{1}{\Delta x^2} & & \\ \frac{1}{\Delta x^2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{\Delta x^2} \\ & & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{N-1} \end{pmatrix} + K \begin{pmatrix} \frac{u_0}{\Delta x^2} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \frac{u_N}{\Delta x^2} \end{pmatrix}$$

Finite difference for heat equation (parabolic)

1. How to approximate a derivative with arithmetic?
2. **How much error does the approximation cause?**
3. How does the approximation error affect the solution?

$$\frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2} - \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} \approx \left(\frac{\partial^4 u}{\partial x^4}\right)\frac{\Delta x^2}{12}$$

Second order Scheme

$$= \frac{1}{\Delta x^2}\left(u_i + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \cdots\right)$$

$$+ \frac{1}{\Delta x^2}\left(u_i - \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} + \cdots\right)$$

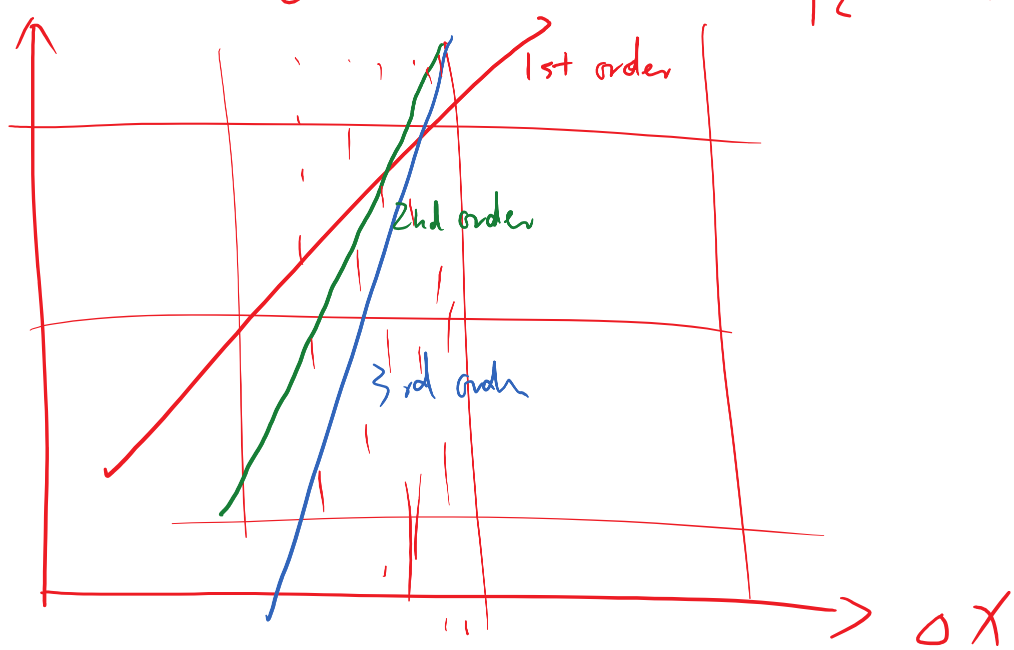
$$- \frac{2}{\Delta x^2} u_i$$

$$- \frac{\partial^2 u}{\partial x^2}$$

$$= 0 + 0 + 0 + 0 + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + O(\Delta x^3)$$

$$u = e^x - \sin x + x^2$$

$$\left\| \frac{\partial^2 u}{\partial x^2} - \frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2} \right\|$$



Finite difference for heat equation (parabolic)

1. How to approximate a derivative with arithmetic?
2. How much error does the approximation cause?
3. **How does the approximation error affect the solution?**

$$\frac{du}{dt} = K(Au+b)$$

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = K(Au^{(n)}+b)$$

$$u^{(n+1)} = u^{(n)} + K\Delta t\left(Au^{(n)}+b\right)$$

Denote numerical solution as $\hat{u}_i$

exact - - - - u_i

Define $e_i = \hat{u}_i - u_i$

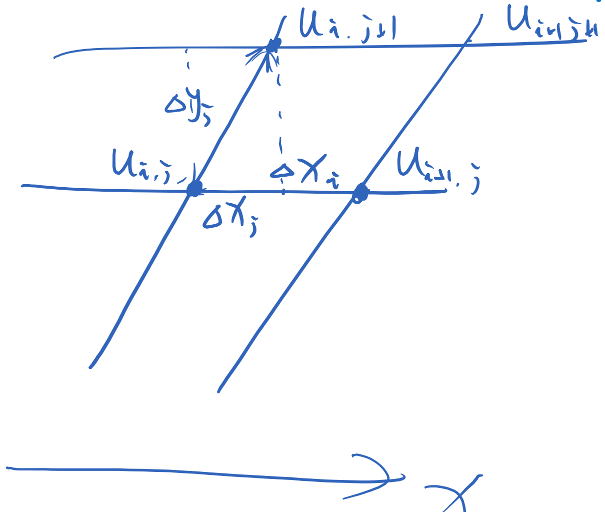
? $$\frac{de_i}{dt} = \frac{d\hat{u}_i}{dt} - \frac{du_i}{dt}$$

$$= K\left(\frac{\hat{u}_{i+1}+\hat{u}_{i-1}-2\hat{u}_i}{\Delta x^2} - \frac{\partial^2 u}{\partial x^2}\right)$$

$$= K\left(\underbrace{\frac{\hat{u}_{i+1}+\hat{u}_{i-1}-2\hat{u}_i}{\Delta x^2} - \frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2}} + \underbrace{\frac{u_{i+1}+u_{i-1}-2u_i}{\Delta x^2} - \frac{\partial^2 u}{\partial x^2}}\right)$$

$$\frac{de_i}{dt} = K\,\frac{e_{i+1}+e_{i-1}-2e_i}{\Delta x^2} + K\left(\frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right) + O(\Delta x^3)\right)$$

General finite difference operators



$$u_{i+1,j} = u_{i,j} + \Delta x_i \frac{\partial u}{\partial x} + \frac{\Delta x_i^2}{2} \frac{\partial^2 u}{\partial x^2} + \cdots$$

$$u_{i,j+1} = u_{i,j} + \Delta x_j \frac{\partial u}{\partial x} + \Delta y_j \boxed{\frac{\partial u}{\partial y}}$$

$$+ \frac{\Delta x_j^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Delta y_j^2}{2} \frac{\partial^2 u}{\partial y^2}$$

$$+ \Delta x_j \Delta y_j \frac{\partial^2 u}{\partial x \partial y} + \cdots$$

$$a u_{i+1,j} + b u_{i,j+1} + c u_{i,j} - \frac{\partial u}{\partial y}$$ as small as possible

$$a\left(u_{i,j} + \Delta x_i \frac{\partial u}{\partial x} + \frac{\Delta x_i^2}{2} \frac{\partial^2 u}{\partial x^2}\right)$$

$$+ b\left(u_{i,j} + \Delta x_j \frac{\partial u}{\partial x} + \Delta y_j \frac{\partial u}{\partial y} + \frac{\Delta x_j^2}{2} \frac{\partial^2 u}{\partial x^2} + \cdots\right.$$

$$+ c\left(u_{i,j}\right)$$

$$- \frac{\partial u}{\partial y}$$

$$\begin{cases} a + b + c = 0 \\ a \Delta x_i + b \Delta x_j = 0 \\ b \Delta y_j = 1 \end{cases}$$