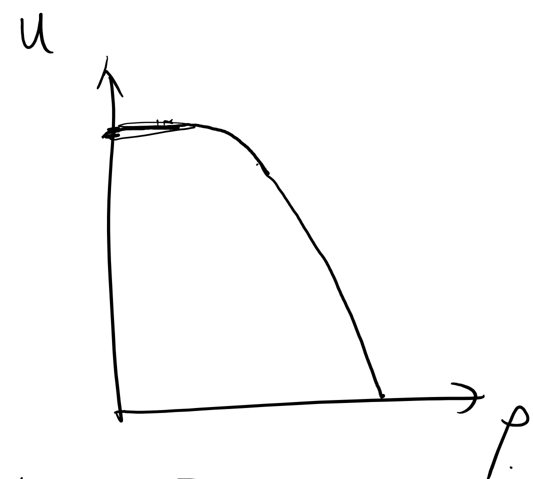
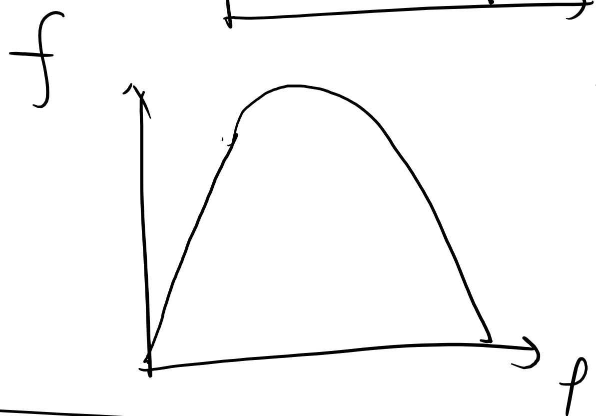
Scalar conservation laws -- Traffic flow equation and Buckley-Leverret equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0$$

$$f(\rho) = \rho \cdot u(\rho)$$

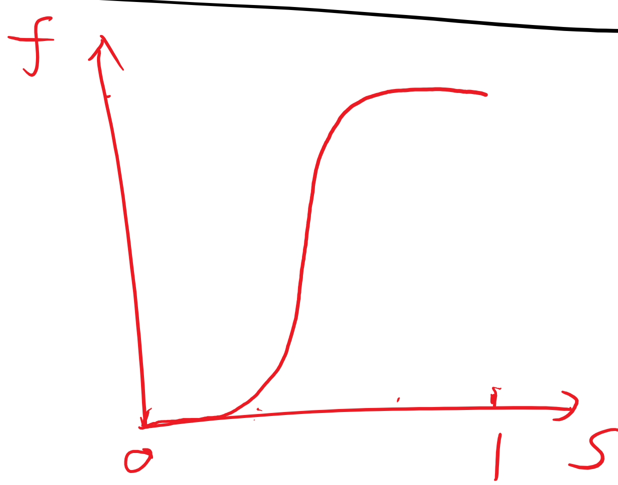


$$\frac{df}{d\rho} \searrow \text{ as } \rho \nearrow$$

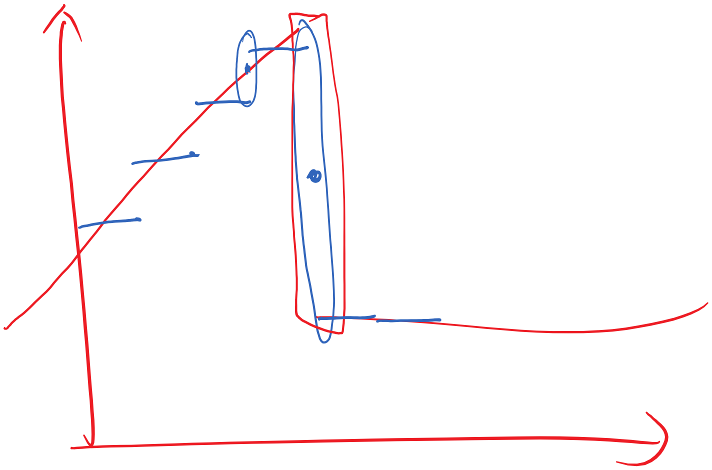


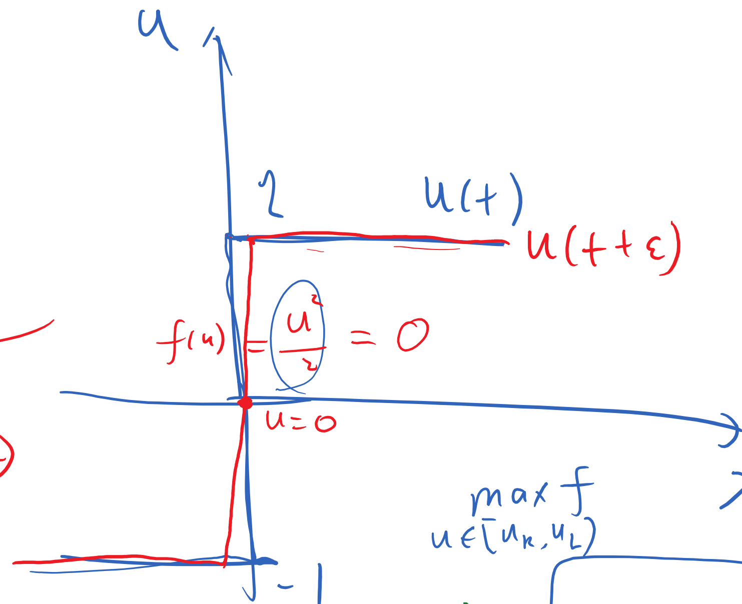
$$\frac{\partial s}{\partial t} + \frac{\partial f(s)}{\partial x} = 0$$

$$\frac{df}{ds} \nearrow \searrow \text{ as } s \nearrow$$

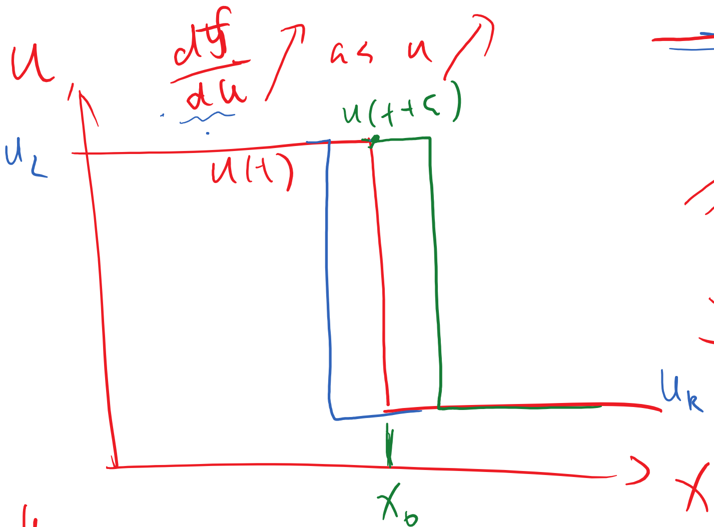


Behavior of a discontinuity -- the Riemann problem





$\max_{u\in[u_R,u_L)} f$

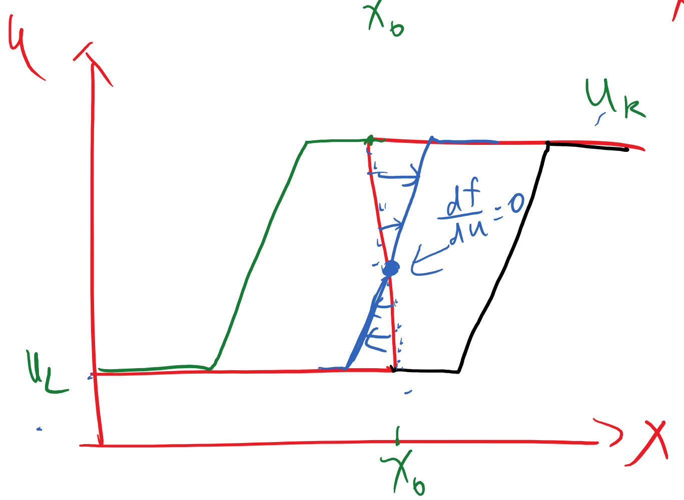


$\boxed{\frac{\Delta f}{\Delta u} > 0}$

$\frac{\Delta f}{\Delta u} < 0$

$f|_{x_0} = f(u_L)$

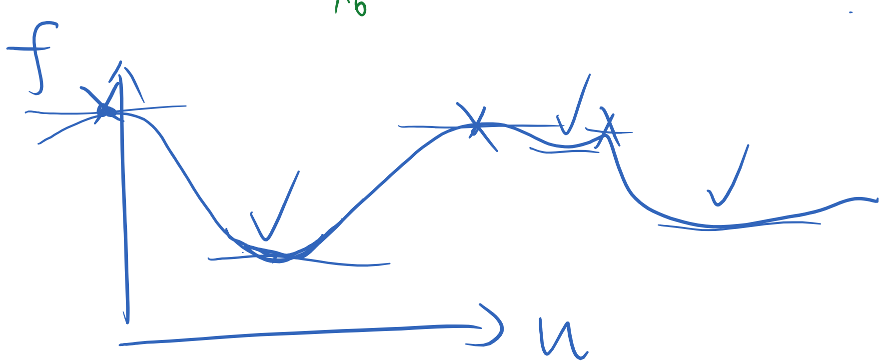
$f|_{x_0} = f(u_R)$



① $f|_{x_0} = f(u_L)$

② $f|_{x_0} = f\left(u \,\middle|\, \frac{df}{du}=0\right) = \min_{u\in[u_L,u_R]} f(u)$

③ $f|_{x_0} = f(u_R)$



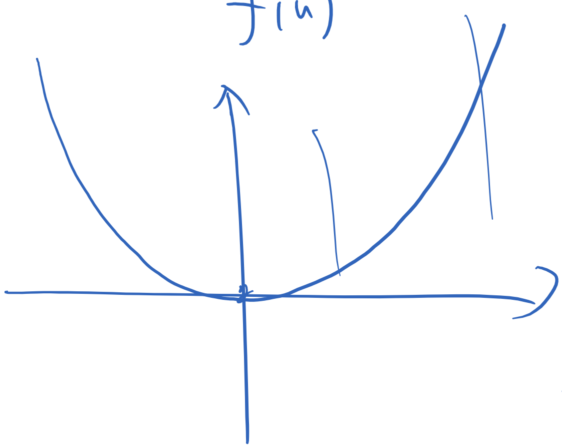
$f = \min_{u\in[u_L,u_R]} f(u)$ if $u_L < u_R$

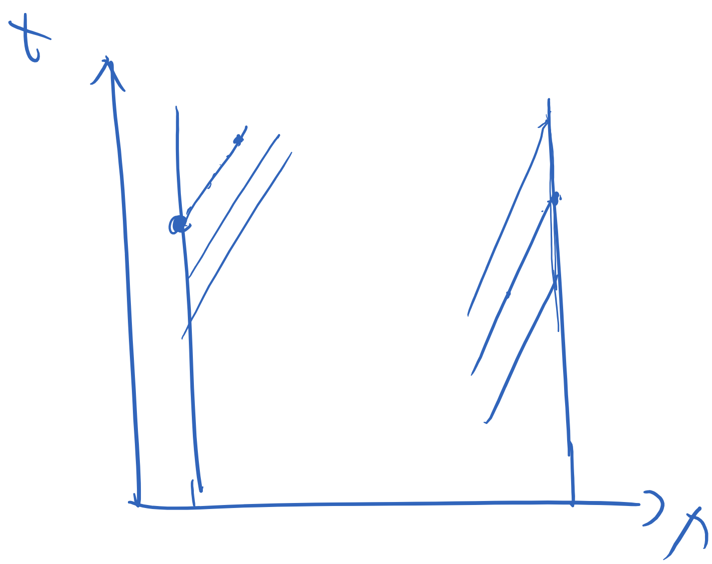
$= \max_{u\in[u_R,u_L]} f(u)$ $\quad u_L > u_R$

Flux reconstruction -- the Godunov scheme

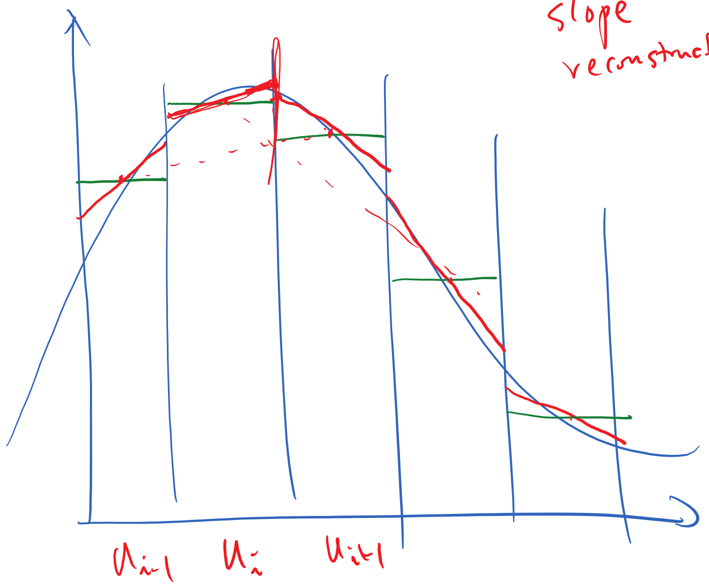
$$f_{i+\frac{1}{2}} = \text{Godunov}(u_i, u_{i+1})$$

$$\text{Godunov}(u_L, u_R) = \begin{cases} \max\limits_{u \in [u_L, u_R]} f & u_L > u_R \\ \min\limits_{u \in [u_R, u_L]} f & u_R > u_L \\ f(u) & u_R = u_L = u \end{cases}$$





Second order Godunov scheme



slope reconstruction

$$S_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

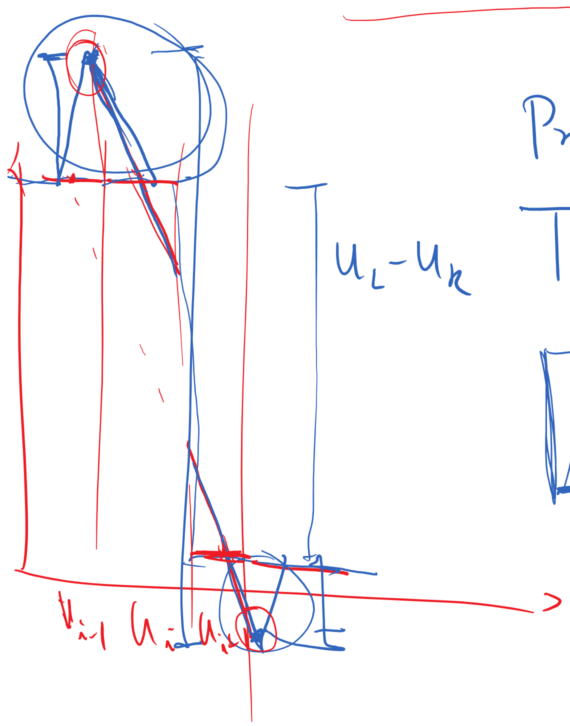
$$u^-_{i+\frac{1}{2}} = u_i + \frac{\Delta x}{2} \cdot S_i$$

$$S_{i+1} = \frac{u_{i+2} - u_i}{2\Delta x}$$

$$u^+_{i+\frac{1}{2}} = u_{i+1} - \frac{\Delta x}{2} S_{i+1}$$

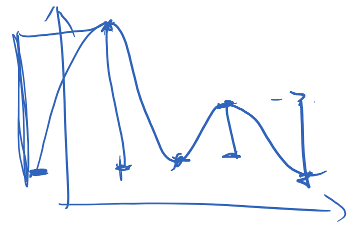
$$f_{i+\frac{1}{2}} = \text{Godunov}\left(u^-_{i+\frac{1}{2}},\ u^+_{i+\frac{1}{2}}\right)$$

$$\text{Godunov}(u_L, u_R) = \begin{cases} \max f \ \ldots \\ \min f \ \ldots \end{cases}$$



Property of scalar conservation law

TVD : (Total variation diminishing)

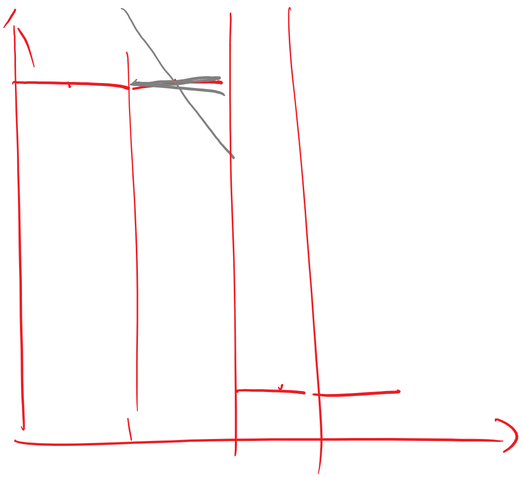


$$\frac{dTV}{dt} \leq 0$$

To numerically preserve TVD : no new extrema in reconstruction

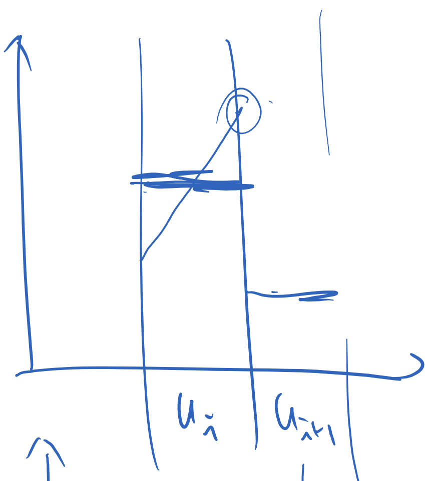
Total variation diminishing property and flux limiter

Thm: no TVD scheme can be linear and second order unless the CFL number is an integer $\left(\frac{df}{du}\cdot\frac{\Delta t}{\Delta x}\right)$



$$u_{i+\frac{1}{2}}^{-} = u_i + S_i\,\phi\left(\frac{(u_{i+1}-u_i)/\Delta x}{S_i}\right)\frac{\Delta x}{2}$$

Property 1: if S_i has different sign as $u_{i+1}-u_i$

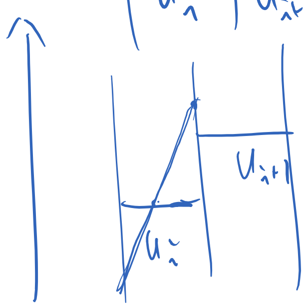


$$\phi = 0$$

$$\phi(r) = 0 \text{ if } r \le 0$$

$$\phi(r) < 2r \text{ if } r > 0$$

$$S_i \cdot \phi\frac{\Delta x}{2} < 2\,\frac{u_{i+1}-u_i}{\Delta x}\,\frac{\Delta x}{2} = u_{i+1}-u_i$$



$$\phi(1) = 1$$

Total variation diminishing property and flux limiter