Von Neumann stability analysis for heat equation (parabolic)

1. How to approximate a derivative with arithmetic?
2. How much error does the approximation cause?
3. **How does the approximation error affect the solution?**

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

$$\frac{d\hat{u}_i}{dt} = K \frac{\hat{u}_{i-1} + \hat{u}_{i+1} - 2\hat{u}_i}{\Delta x^2}$$

define $e_i = \hat{u}_i - u_i$

$$\frac{de_i}{dt} = K \frac{e_{i-1} + e_{i+1} - 2e_i}{\Delta x^2} + K\left(\frac{u_{i-1} + u_{i+1} - 2u_i}{\Delta x^2} - \left.\frac{\partial^2 u}{\partial x^2}\right|_i\right)$$

Idea: expand $\hat{u}_i$ or e_i with DFS and substitute into the FD operator

$$\hat{u}_i = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} u^{(k)} e^{jik\frac{2\pi}{N}} \quad (j = \sqrt{-1})$$

N is the # of grid points in a periodic domain of $[0, 2\pi)$

$$\frac{\hat{u}_{i+1} + \hat{u}_{i-1} - 2\hat{u}_i}{\Delta x^2}$$

$$= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{u^{(k)}\left(e^{j(i+1)k\frac{2\pi}{N}} + e^{j(i-1)k\frac{2\pi}{N}} - 2e^{jik\frac{2\pi}{N}}\right)}{\Delta x^2}$$

$$= \sum u^{(k)} e^{jik\frac{2\pi}{N}} \frac{e^{jk\frac{2\pi}{N}} + e^{-jk\frac{2\pi}{N}} - 2}{\Delta x^2}$$

$$= \sum \underbrace{u^{(k)} e^{jik\frac{2\pi}{N}}}_{\hat{u}} \frac{2\left(\cos\frac{2\pi}{N}k - 1\right)}{\Delta x^2}$$

If only $u^{(k)}$ is nonzero, then

$$K\frac{\hat{u}_{i+1} + \hat{u}_{i-1} - 2\hat{u}_i}{\Delta x^2} = \hat{u}_i \underbrace{\boxed{K\frac{2\left(\cos\frac{2\pi}{N}k - 1\right)}{\Delta x^2}}}_{\lambda_k}$$

$$\frac{d\hat{u}_i}{dt} = \lambda_k \hat{u}_i$$

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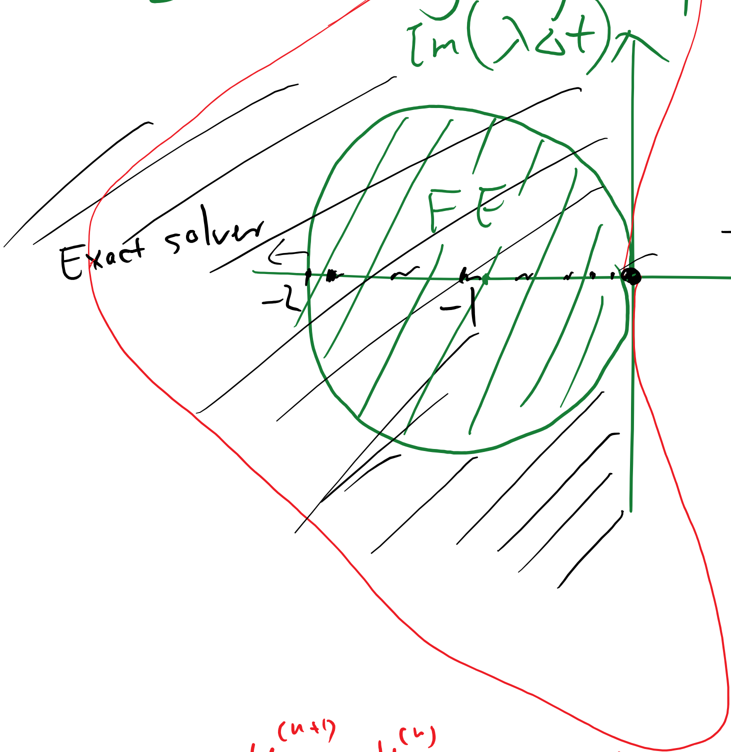
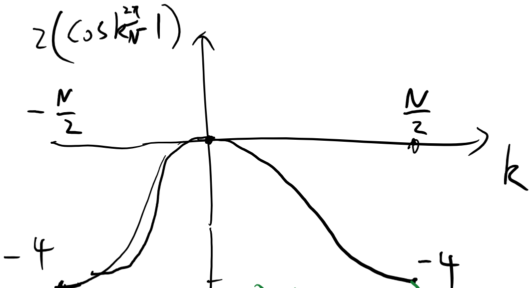
$$\frac{du}{dt} = \lambda u$$

Two factors decide the stability of ODE solver:

1 $\lambda \cdot \Delta t = K \dfrac{2(\cos k\frac{2\pi}{N} - 1)}{\Delta x^2} \Delta t$ $\qquad k \sim -\frac{N}{2}, \dots, \frac{N}{2} - 1$

2. Stability Region of the ODE solver





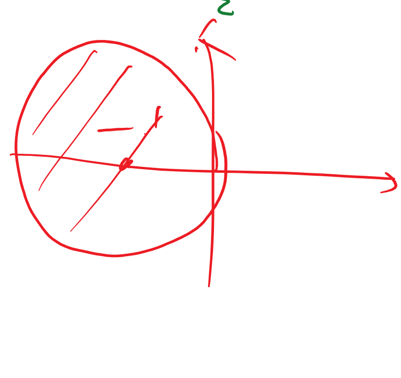
$$-4K\frac{\Delta t}{\Delta x^2} \overset{?}{<} -2 \quad \text{unstable}$$

$$K\frac{\Delta t}{\Delta x^2} > \frac{1}{2} \quad \text{or}$$

$$K\Delta t \le \frac{\Delta x^2}{2} \quad \text{stable}$$

FE: $\dfrac{u^{(n+1)} - u^{(n)}}{\Delta t} = \lambda u^{(n)}$

$$u^{(n+1)} = \underline{(1 + \lambda\Delta t)}\, u^{(n)}$$



Finite difference for advection equation (hyperbolic)

1. **How to approximate a derivative with arithmetic?**
2. **How much error does the approximation cause?**
3. How does the approximation error affect the solution?

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_i = a u_{i-1} + b u_i + c u_{i+1}$$

$$= a\left(u_i - \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$+ b\, u_i$$

$$+ c\left(u_i + \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}\right)$$

$$\begin{cases} a+b+c=0 \\ -a\Delta x + c\Delta x = 1 \\ a\frac{\cancel{\Delta x^2}}{\cancel{2}} + c\frac{\cancel{\Delta x^2}}{\cancel{2}} = 0 \end{cases}$$

$$= \left.\frac{\partial u}{\partial x}\right|_i + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}$$

$$\begin{cases} a = -\frac{1}{2\Delta x} \\ c = \frac{1}{2\Delta x} \end{cases} \Longrightarrow \frac{\partial u}{\partial x} = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

central difference

$$b = 0$$

$$\frac{du_i}{dt} + U\frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0$$

Von Neumann stability analysis for advection equation (hyperbolic)

1. How to approximate a derivative with arithmetic?

2. How much error does the approximation cause?

3. **How does the approximation error affect the solution?**

$$\frac{du_i}{dt} + U\,\frac{u_{i+1}-u_{i-1}}{2\Delta x} = 0$$

$$u_i = u^{(k)} e^{j i k \frac{2\pi}{N}} = u^{(k)} e^{j i k \Delta x}$$

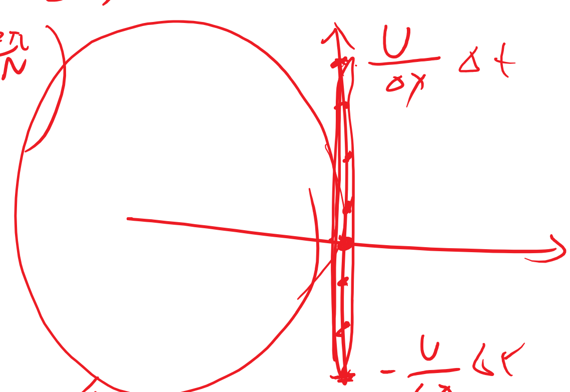
in a domain of $[0, 2\pi]$

$$\Delta x = \frac{2\pi}{N}$$

$$U\,\frac{u_{i+1}-u_{i-1}}{2\Delta x} = U u^{(k)}\,\frac{e^{j(i+1)k\frac{2\pi}{N}} - e^{j(i-1)k\frac{2\pi}{N}}}{2\Delta x}$$

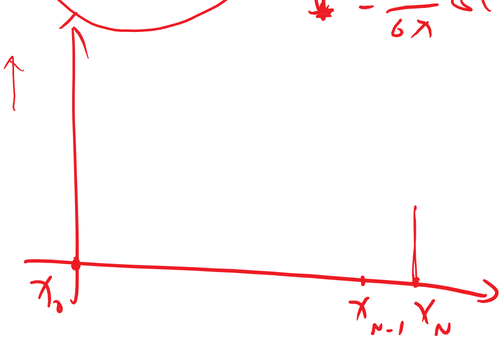
$$= U u^{(k)} e^{jik\frac{2\pi}{N}}\,\frac{e^{jk\frac{2\pi}{N}} - e^{-jk\frac{2\pi}{N}}}{2\Delta x}$$

$$= u_i \Big(\underbrace{U\,\frac{j\cdot \sin k\frac{2\pi}{N}}{\Delta x}}_{\lambda_k}\Big)$$



$$\frac{du_i}{dt} = \lambda_k \cdot u_i$$

$$\frac{d}{dt}\begin{pmatrix} u_0 \\ \vdots \\ \vdots \\ u_{N-1} \end{pmatrix} = -U \cdot \begin{pmatrix} 0 & \frac{1}{2\Delta x} & & & -\frac{1}{2\Delta x} \\ -\frac{1}{2\Delta x} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{1}{2\Delta x} \\ \frac{1}{2\Delta x} & & & -\frac{1}{2\Delta x} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ \vdots \\ u_{N-1} \end{pmatrix}$$



Upwinding for advection equation (hyperbolic)

1. How to approximate a derivative with arithmetic?
2. How much error does the approximation cause?
3. **How does the approximation error affect the solution?**

$$\left.\frac{\partial u}{\partial x}\right|_i = a\,u_{i-1} + b\,u_i + c\,u_{i+1}$$

$$= a\left(u_i - \Delta x \frac{\partial u}{\partial x} + \underbrace{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \cdots\right)$$

$$+\, b\, u_i$$

$$+\, c\left(u_i + \Delta x \frac{\partial u}{\partial x} + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}} + \cdots\right)$$

$$a + b + c = 0$$

$$-a\Delta x + c\Delta x = 1$$

if $a = 0$: $c = \frac{1}{\Delta x}$, $b = -\frac{1}{\Delta x}$

if $c = 0$: $a = -\frac{1}{\Delta x}$, $b = \frac{1}{\Delta x}$

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1} - u_i}{\Delta x} \quad \text{or} \quad \frac{u_i - u_{i-1}}{\Delta x}$$

$$\frac{du_i}{dt} = -U\,\frac{u_{i+1} - u_i}{\Delta x} \qquad u_i = \cancel{u(t)}\; e^{jik\Delta x}$$

$$= -U \cdot e^{jik\Delta x} \cdot \left(\frac{e^{jk\Delta x} - 1}{\Delta x}\right) \qquad \left(\frac{1 - e^{-jk\Delta x}}{\Delta x}\right)$$

$$= u_i \cdot \left(U\,\frac{1 - e^{jk\Delta x}}{\Delta x}\right) \qquad \left(U\,\frac{e^{-jk\Delta x} - 1}{\Delta x}\right)$$

λ_k