Numerical Methods for PDEs

Boundary Element Methods, Lecture 1: Introduction to the Discretization of Boundary Integral Equations

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1 Module Outline - 8 Lectures

SLIDE 1

Overview Integral Equation Methods
Applications, Collocation and Galerkin Basics.
Quadrature and Cubature for computing integrals
1^{st} **and** 2^{nd} **Kind Theory**
Fredholm Alternative,Nystrom Methods
Formulating Integral Equations
Potential Theory and Principle Values
1^{st} **and** 2^{nd} **Kind Revisited**
Radiation, Green's Thm, 1^{st} Kind Convergence
Helmholtz (wave) problems OR
Fast Multipole and FFT-based methods

2 Outline for Today

SLIDE 2

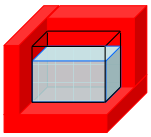
Background
Exterior versus interior problems
Point source approach
Test Function Selection
Collocation Method
Galerkin Method
Some issues in 3D
Singular integrals

3 Background

3.1 Interior vs Exterior Problems

SLIDE 3

Interior Exterior



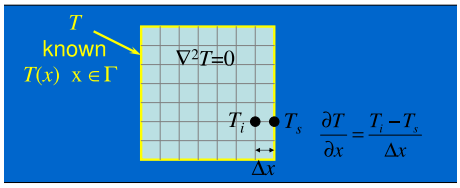
Temperature in a tank *Ice cube in a bath*

What is the heat distribution?
Heat flow = Thermal conductivity $\int_{surface} \frac{\partial T}{\partial n}$

3.1.1 The Interior Problem

SLIDE 4

Example: Heat Distribution in a Tank



How does one determine the heat distribution using the finite difference method?

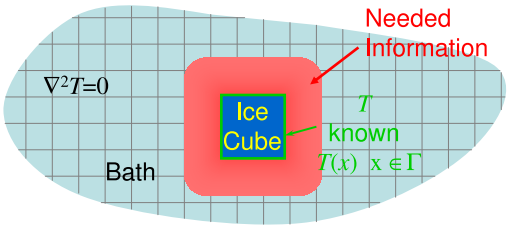
Use the above example of heat distribution in a tank, to see what is meant by an interior problem. This is a Dirichlet problem because $T(\vec{x})$, the temperature, is defined on the surface $\vec{x} \in \Gamma$. The steady state two dimensional heat flow equation is defined using the Laplace Equation, $\nabla^2 T(\vec{x}) = 0$. The domain of this problem is clearly the interior of the tank, $\vec{x} \in \Omega$.

3.1.2 The Exterior Problem

Up until this time, all problems that have been studied using finite-element and finite-difference methods have been *interior* problems, but now, we begin to wonder how to form an exterior problem. This poses some problems for these other methods such as generating the grid as well as mesh truncation.

SLIDE 5

Example: Ice Cube in a Bath

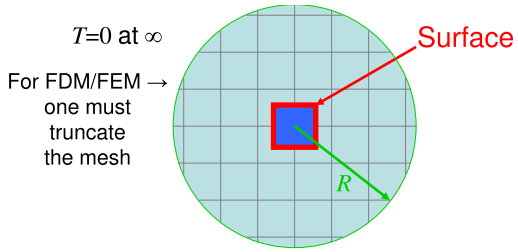


Above is a problem showing an *ice cube in a bath* which is used to indicate an exterior problem. Again, this is a Dirichlet problem because $T(\vec{x})$ is defined on the surface $\vec{x} \in \Gamma$ and described by the steady state two dimensional heat flow equation defined using Laplace's Equation, $\nabla^2 T(\vec{x}) = 0$. The problem domain is the infinitely extending region exterior to the ice cube. A point that we will expand in a later lecture is that with exterior problems, an additional boundary condition is needed to specify what happens at a large distance away from our point source. Assuming there are no heat sources exterior to the cube will impose the following *radiation* boundary condition

$$\lim_{\|\vec{x}\| \to \infty} T(\vec{x}) \to 0.$$

Suppose that for this specific problem, we are only interested in what occurs at the surface of the cube. It seems inefficient to use the finite-difference or finite-element methods where one needs to compute the temperature everywhere in Ω.

SLIDE 6

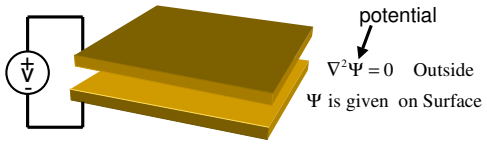


3.2 Examples

3.2.1 Computation of Capacitance

SLIDE 7

What is the capacitance?



$$\text{Capacitance} = \text{Dielectric Permittivity} \int \frac{\partial \Psi}{\partial n}$$

In the example in the slide, the yellow plates form a parallel-plate capacitor with an applied voltage V. In this 3-D electrostatics problem, the electrostatic potential Ψ satisfies Laplace's equation $\nabla^2 \Psi(x) = 0$ in the region exterior to the plates, and the potential is known on the surface of the plates (Dirichlet boundary condition). Furthermore, far from the plates,

$$\lim_{\|\vec{x}\| \to \infty} \Psi(\vec{x}) \to 0.$$

(Exterior Radiation Boundary Condition to be studied further in a future lecture). The value of interest is the capacitance, C, which satisfies

$$q = CV$$

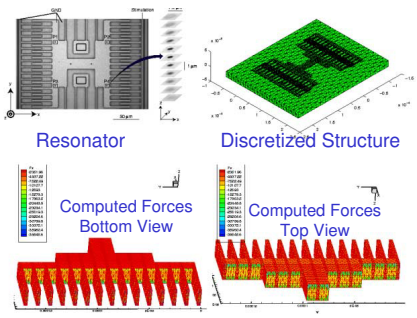
where q, the net charge on one of the plates, is given by the surface normal of the potential integrated over one plate and scaled by a dielectric permittivity.

Note 1 ***Example 1: Capacitance problem***

This is a typical application example, determining the charge density on the surface of conducting plates given an applied voltage. In this particular example, the top plate potential is $\Psi = 0.5V$ and the bottom plate potential is $\Psi = -0.5V$, where V is the voltage noted in the figure.
For this exterior Dirichlet problem, one can write an integral equation that relates the surface charge density on the plates σ to the potential on the plates. This integral equation, $\Psi(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')dS'$, is often referred to by physicists as the superposition integral. In the integral equation, x is any point on the plate surfaces and the surface being integrated over is the union of the top and bottom plate surfaces. Note that the integration surface is not a connected domain, but this presents no difficulties.

3.2.2 Drag Force in a Microresonator

SLIDE 8



Note 2 ***Example 2: Drag force in a MEMS device***

The example in the slide is a microresonator, it is a structure that can be made to vibrate using electrostatic forces. The changing character of those vibrations can be used to sense rotation. The particulars of how the microresonator operates is not directly relevant to our discussion of integral equations, except for one point. In order to determine how much energy is needed to keep the microresonator vibrating, it is necessary to determine the fluid drag force on comb structures shown in the bottom part of the slide. The fluid is the air surrounding the structure, and at the micron-scale of these devices, air satisfies the incompressible Stokes equation,

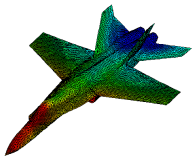
$$\begin{aligned} \nabla^2 u(x) &= \nabla p(x) \\ \nabla \cdot u(x) &= 0 \end{aligned} \tag{1}$$

where u is the fluid velocity and p is the pressure. By specifying the comb velocity, and then computing the surface pressure and the normal derivative of velocities tangent to the surface, one can determine the net drag force on the comb. Once again, this is a problem in which the known quantities (the

comb velocity) and the quantities of interest (the derivative of the tangential components of fluid velocity) are on the surface.

3.2.3 Aircraft Drag

SLIDE 9



Discretization for F-18 pressure simulation (no lift)

Inviscid, Irrotational, Steady Flow

Potential flow: $\nabla^2 u(x) = 0 \qquad \nabla u =$ velocity

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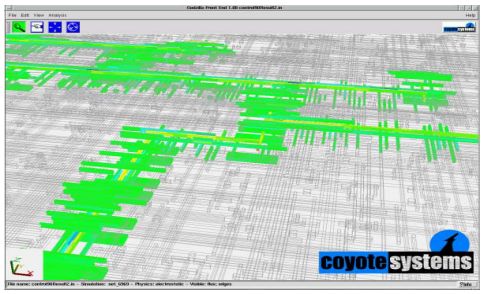
Note 3 ***Example 3: Aircraft Drag***

The potential flow model for aircraft drag computation will be discussed in more detail in subsequent lectures, so we only give a brief description here. In order to compute the drag on the wing of an aircraft, one must determine the difference between the wing velocity and the velocity of the air very close to the wing. If the air can be assumed inviscid, irrotational, and incompressible, the velocity is given by the gradient of a scalar potential which satisfies Laplace's equation. The boundary conditions for the Laplace's equation are given as a velocity boundary condition on the aircraft surface, equivalently a Neumann condition on the potential, and it is usually assumed that the potential approaches zero at infinity. The boundary condition at infinity is more subtle than it may seem, as we shall see in later lectures. Finally, it is common to introduce an artificial boundary in the domain, and specify a condition on that boundary to introduce rotational effects.

3.2.4 Capacitance of Microprocessor Signal Lines

SLIDE 10



Note 4 ***Example 4: Capacitance of microprocessor signal lines***

This last example in the above slide is a picture of the wiring on a microprocessor integrated circuit. A typical microprocessor has millions of wires, so we are only looking at a small piece of a processor. The critical problem in this example is determining how long signals take to get from the output of a logical gate to the input of the next gate. To compute that delay, one must determine the capacitance on each of the wires given in the slide picture. To do so requires computing charges given electrostatic potentials as noted above.

3.3 Advantages of Integral Equation Method

3.3.1 What is common about these examples?

SLIDE 11

Exterior Problems

MEMS device - fluid (air) creates drag

Aircraft Design - exterior air flow

Signal Line - Exterior fields.

Quantities of interest are on surface

MEMS device - Just want surface traction force

Aircraft Design - Just want surface tangent velocities

Signal Line - Just want surface charge.

Exterior problem is linear and space-invariant

MEMS device - Exterior Stoke's flow equation (linear)

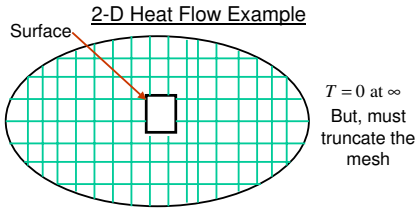
Aircraft Design - Laplace's equation, plus wakes.

Signal line - Laplace's equation in free spce (linear)

But problems are geometrically very complex

3.3.2 Why not use FDM / FEM?

SLIDE 12



Only need $\frac{\partial T}{\partial n}$ on the surface, but T is computed everywhere.
Must truncate the mesh, $\Rightarrow T(\infty) = 0$ becomes $T(R) = 0$.

Consider the two dimensional exterior heat conduction problem in the above figure in which the temperature is known on the surface of the square. Suppose the quantity of interest is the total heat flow out of the square.
The temperature T satisfies

$$\nabla^2 T(x) = 0 \qquad x \in \Omega$$

$$T(x) \; given \; x \in \Gamma \tag{2}$$
$$\lim_{\|x\| \to \infty} T(x) = 0$$

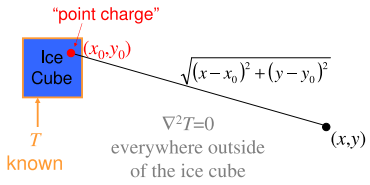
where Ω is the infinite domain outside the square and Γ is the region formed by the edges of the square.
Using finite-element or finite-difference methods to solve this problem requires introducing an additional approximation beyond discretization error. It is not possible to discretize all of Ω, as it is infinite, and therefore the domain must be truncated with an artificial finite boundary. In the slide,the artificial boundary is a large ellipse on which we assume the temperature is zero. Clearly, as the radius of the ellipse increases, the truncated problem more accurately represents the domain problem, but the number of unknowns in the discretization increases.

3.4 Point Source Approach

3.4.1 Green's Function

SLIDE 13

Heat Distribution in 2-D



Green's Function: $T = log\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$

From basic electrostatics, one knows that in 3-D, the potential field produced by a point charge decays inversely with the distance to the point charge. Since, roughly, one can represent any charge distribution using a sum of point charges, one can express the potential due to a charge density as a sum of point charge potentials. Therefore, point charge potentials play a special role, and are often referred to as Greens' functions for the problem.

SLIDE 14

In 2-D The potential due to a point charge is:

$$u = log\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right) \tag{3}$$
$$\forall (x,y) \neq (x_0, y_0)$$

In 3D The potential due to a point charge is:

$$u = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \tag{4}$$
$$\forall (x,y,z) \neq (x_0, y_0, z_0)$$

SLIDE 15

In 2D If $u = log\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)$
then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \forall \quad (x,y) \neq (x_0,y_0)$

In 3D If $u = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$
then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \forall \quad (x,y,z) \neq (x_0,y_0,z_0)$

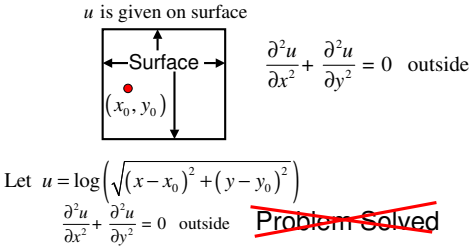
Proof: Just differentiate and see!

In the next few slides, we will use an informal semi-numerical approach to derive the integral form of Laplace's equation. We do this in part because such a derivation lends insight to the subsequent numerical procedures.
To start, recall from basic physics that the potential due to a point charge is related only to the distance between the point charge and the evaluation point. In 2-D the potential is given by the log of the distance, equation (3), and in 3-D the potential is inversely proportional to the distance, equation (4). These functions are sometimes referred to as Green's functions for Laplace's equation, but have the physical interpretation as the potential due to a point charge. We will be studying Green's functions in more depth later on.

▷ ***Exercise 1*** Show by direct differentiation that the functions in equations (3) and (4) satisfy $\nabla^2 u = 0$, in the appropriate dimension almost everywhere. ■

3.4.2 Scaling Green's Function

SLIDE 16



Boundary conditions are not satisfied!

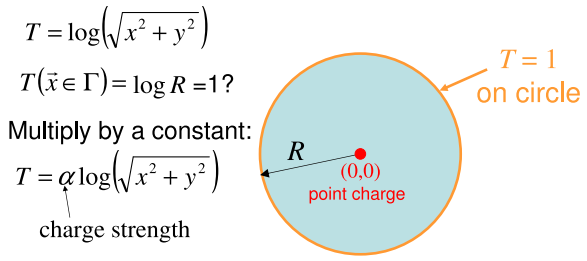
A simple idea for computing the solution of Laplace's equation outside the square is to let

$$u(x,y) = \alpha \log \sqrt{(x-x_0)^2+(y-y_0)^2}$$

where (x_0, y_0) is a point inside the square. Clearly u will always satisfy $\nabla^2 u = 0$ outside the square, but u may not match the boundary conditions. By adjusting

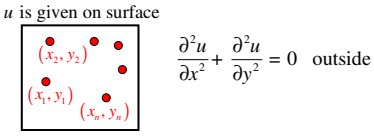
α, it is possible to make sure to match the boundary conditions at at least one point.
This concept is applied to a circle as a simple example of how to match the boundary conditions.

SLIDE 17



▷ ***Exercise 2*** Suppose the potential on the surface of the square is a constant. Can you match that constant potential everywhere on the perimeter of the square by judiciously selecting α ? ■

SLIDE 18

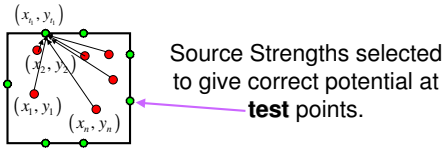


$$u = \sum_{i=1}^{n} \alpha_i \log\left(\sqrt{(x - x_i)^2 + (y - y_i)^2}\right) = \sum_{i=1}^{n} \alpha_i G(x - x_i, y - y_i)$$

Pick the α_i's to match the boundary conditions!

To construct a potential that satisfies Laplace's equation and matches the boundary conditions at more points, let u be represented by the potential due to a sum of n weighted point charges in the square's interior. As shown in the slide, we can think of the potential due to a sum of charges as a sum of Green's functions. Of course, we have to determine the weights on the n point charges, and the weight on the i^{th} charge is denoted hereby α_i.

SLIDE 19



$$\begin{bmatrix} G(x_{t_1} - x_1, y_{t_1} - y_1) & \cdots & G(x_{t_1} - x_n, y_{t_1} - y_n) \\ \vdots & \ddots & \vdots \\ G(x_{t_n} - x_1, y_{t_n} - y_1) & \cdots & G(x_{t_n} - x_n, y_{t_n} - y_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}, y_{t_1}) \\ \vdots \\ \Psi(x_{t_n}, y_{t_n}) \end{bmatrix}$$

To determine a system of n equations for the n α_i's,consider selecting a set of n test points, as shown in the slide above. Then, by superposition, for each test point (x_{t_i}, y_{t_i}),

$$u(x_{t_i}, y_{t_i}) = \sum_{i=1}^{n} \alpha_i \log \sqrt{(x_{t_i} - x_0)^2 + (y_{t_i} - y_0)^2} = \sum_{i=1}^{n} \alpha_i G(x_{t_i} - x_0, y_{t_i} - y_0). \tag{5}$$

Writing an equation like (5) for each test point yields the matrix equation

$$\begin{bmatrix} G(x_{t_1} - x_1, y_{t_1} - y_1) & \cdots & G(x_{t_1} - x_n, y_{t_1} - y_n) \\ \vdots & \ddots & \vdots \\ G(x_{t_n} - x_1, y_{t_n} - y_1) & \cdots & G(x_{t_n} - x_n, y_{t_n} - y_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{t_1}, y_{t_1}) \\ \vdots \\ \Psi(x_{t_n}, y_{t_n}) \end{bmatrix} \tag{6}$$

The matrix A in equation (6) has some properties worth noting:

- A is dense, that is $A_{i,j}$ never equals zero. This is because every charge contributes to every potential.
- If the test points and the charge points are ordered so that the i^{th} test point is nearest the i^{th} charge, then $A_{i,i}$ will be larger than $A_{i,j}$ for all j.

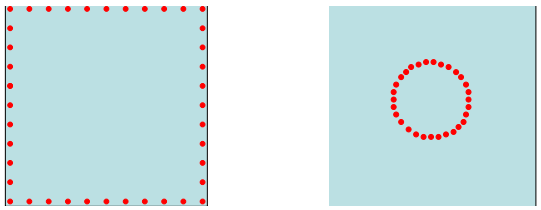
The 2^{nd} item above seems to suggest that A is diagonally dominant, but this is *not* the case. Diagonal dominance requires that the absolute *sum* of the off-diagonal entries is smaller than the magnitude of the diagonal. The matrix above easily violates that condition.

▷ ***Exercise 3*** Determine a set of test points and charge locations for the 2-D square problem that generates an A matrix where the magnitude of the diagonals are bigger than the absolute value of the off-diagonals, but the magnitude of the diagonal is smaller than the absolute sum of the off-diagonals. ■

3.4.3 Source Point Locations

SLIDE 20

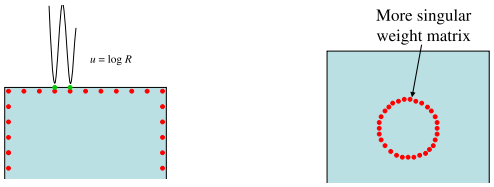
Where should the sources be located?



Close to the boundary Clustered in the center

SLIDE 21

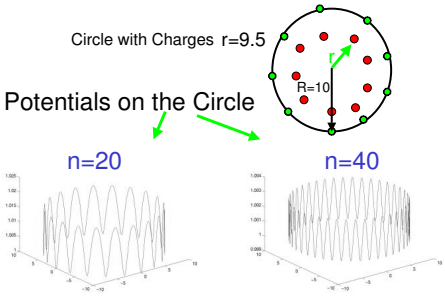
Problems with these placements:



Close to the boundary Clustered in the center

3.4.4 Computational Results

SLIDE 22



It is possible to construct a numerical scheme for solving exterior Laplace problems by adding progressively more point charges so as to match more boundary conditions. In the above graph, we show an example of using such a method to compute the potential exterior to a circle of radius 10, where the potential on the circle is given to be unity. In the example, charges are placed uniformly on a circle of radius 9.5, and test points are placed uniformly on the radius 10 circle. If 20 point charges are placed in a circle of radius 9.5, then the potential produced will be exactly one only at the 20 test points on the radius 10 circle. The potential produced by the twenty point charges on the radius 10 circle is plotted in the lower left corner of the slide above. As might be expected, the potential produced on the radius 10 circle is exactly one at the 20 test points, but then oscillates between 1 and 1.2 on the radius 10 circle. If 40 charges and test points are used, the situation improves. The potential on the circle still oscillates, as shown in the lower right hand corner, but now the amplitude is only between 1 and 1.004.

3.5 Charge Density

SLIDE 23

Want to smear point charges to the surface

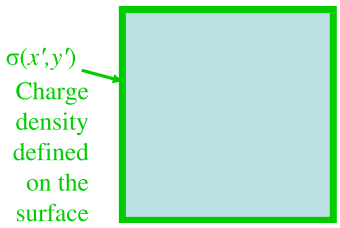
Results in an integral equation

$$\Psi(\vec{x}) = \int_\Gamma G(\vec{x}, \vec{x}')\sigma(\vec{x}')dS' \tag{7}$$

How do we solve the integral equation?

In equation (7) for which variable are we trying to solve?

SLIDE 24



$$\Psi(x, y) = \int_\Gamma \sigma(x', y') \log \sqrt{(x - x')^2 + (y - y')^2} dx' dy'$$

Single Layer Potential

The oscillating potential produced by the point charge method is due to the rapid change in potential as the separation between evaluation point and point charge shrinks. If the point charges could be smeared out, so that the produced potential did not rise to infinity with decreasing separation, then the resulting computed potential would not have the oscillation noted on the previous slide. In addition, it makes the most sense to smear the point charges onto the surface, as then the charge density and the known potential have the same associated geometry. The result is the integral equation (7), where now the unknown is a charge *density* on the surface and the potential due to that charge density is given by the well-known superposition integral. In the case of two or three dimensional Laplace problems, $G(\vec{x}, \vec{x}')$ can be written as $\hat{G}(\vec{x} - \vec{x}')$, as the potential is only a function of distance to the charge density and not a function of absolute position. For such a Green's function, this equation is,

$$\Psi(\vec{x}) = \int_\Gamma \hat{G}(\vec{x} - \vec{x}')\sigma(\vec{x}')dS', \tag{8}$$

which one may recognize from system theory as a convolution integral. This connection is quite precise. A space-invariant system has an impulse response, which is usually referred to as a Green's function. The output, in this case, the potential, is a convolution of the impulse response with the input, in this case, the charge density. Such an integral form of the potential is referred to as a *single layer potential.*

The single layer potential is an example of a class of integral equations known as "Fredholm integral equation of the First Kind". A Fredholm integral equation of the Second Kind results when the unknown charge density exists not only under the integral sign but also outside it. An example of such an equation is

$$\Psi(\vec{x}) = \sigma(\vec{x}) + \int_{\Gamma} K(\vec{x} - \vec{x}')\sigma(\vec{x}')dS'. \tag{9}$$

Fredholm integral equations, in which the domain of integration is fixed, usually arise out of boundary value problems. Initial value problems typically give rise to the so-called Volterra integral equations, where the domain of integration depends on the output of interest. For example, consider the initial value problem

$$\frac{dx(t)}{dt} = tx(t); \quad t \in [0, T],\ T > 0.$$
$$x(t = 0) = x_0$$

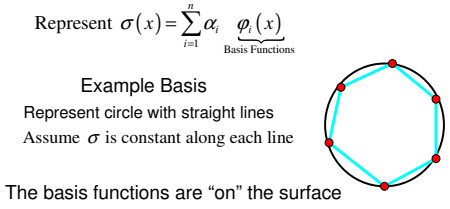
The "solution" of this equation is the following Volterra integral equation:

$$x(t) = x_0 + \int_0^t \xi x(\xi)d\xi.$$

4 Basis Functions

4.1 Basic Idea

SLIDE 25



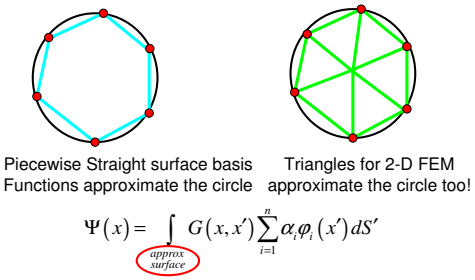
Basis Functions can be used to approximate the surface charge density in a similar way in which they approximate geometry for finite elements.

Numerical solution of the single layer potential
As we have studied extensively in the finite-element section of the course, one approach to numerically computing solutions to partial differential equations is to represent the solution approximately as a weighted sum of basis functions. Then, the original problem is replaced with the problem of determining the basis function weights. In finite-element methods, the basis functions exist in a volume, for integral equations they typically exist on a surface. For 2-D problems that means the basis functions are restricted to curves and in 3-D the

basic functions are on physical surfaces.
As an example, consider the circle in the above figure. One could try to represent the charge density on the circle by breaking the circle into n sub-arcs, and then assume the charge density is a constant on each sub-arc. Such an approach is not commonly used. Instead the geometry is usually approximated along with the charge density. In this example case, shown in the center right of the slide, the sub-arcs of the circle are replaced with straight sections, thus forming a polygon. The charge density is assumed constant on each edge of the polygon. The result is a piecewise constant representation of the charge density on a polygon.

4.2 Geometric Approximation

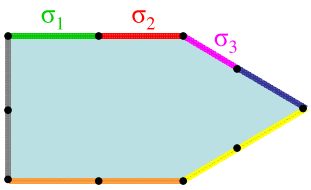
SLIDE 26



The idea that both the geometry and the unknown charge density has been approximated is not actually a new issue. As shown in the figure in the above slide, if FEM methods are used to solve an interior problem, and triangular elements are used, then the circle is approximated to exactly the same degree as when straight sections replace the sub-arcs for the surface integral equation. As shown at the bottom of the above slide, we can substitute the basis function representation into the integral equation, but then we should also note that the integral is now over the approximated geometry. It is common, but not mathematically justified, to ignore the errors generated by the geometry approximation. We will also ignore the error in the geometric approximation in our analyses, just for simplicity. In the case of polygonal geometries, there is no geometric approximation, so there is at least one case where the assumption is precise. It should be noted, however, that there are often analytic results only for smooth geometries, and then before making comparisons to such analytic results, it is necessary to examine the effect of the approximated geometry.

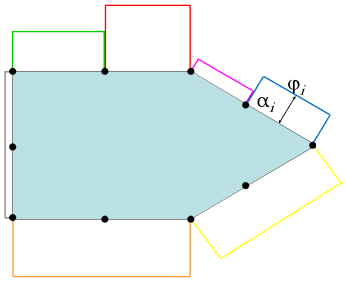
SLIDE 27

If the original problem is a polygon



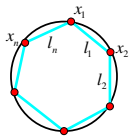
there is no geometric approximation

SLIDE 28



4.2.1 Piecewise Constant Straight Sections

SLIDE 29



1) Pick a set of n Points on the surface
2) Define a new surface by connecting points with n lines.
3) Define $\varphi_i(x) = 1$ if x is on line l_i otherwise, $\varphi_i(x) = 0$

$$\Psi(\vec{x}) = \int_{\substack{\text{approx} \\ \text{surface}}} G(\vec{x},\vec{x}') \sum_{i=1}^{n} \alpha_i \varphi_i(\vec{x}') dS' = \sum_{i=1}^{n} \alpha_i \int_{line\ l_i} G(\vec{x},\vec{x}') dS'$$

How do we determine the α_i's?

We complete the description of using constant charge densities on straight sections as the basis. If we substitute this example basis function into the integral equation, as is done above, the result is to replace the original integration of the product of the Green's function and the density with a weighted sum of integrals over straight lines of just the Green's function. The next step is then to develop an approach for determining the weights, denoted here by α_i's.

5 Test Points

5.1 Residuals

5.1.1 Definition and Minimization

SLIDE 30

$$R(\vec{x}) = \Psi(\vec{x}) - \int_{\substack{\text{approx}\\ \text{surface}}} G(\vec{x},\vec{x}') \sum_{i=1}^{n} \alpha_i \varphi_i(\vec{x}') dS' \tag{10}$$

Pick the α_i's to minimize $R(\vec{x})$
General Approach: Pick a set of test functions $\phi_1, \ldots, \phi_n$ and force $R(\vec{x})$ to be orthogonal to the set:

$$\int \phi_i(\vec{x}) R(\vec{x}) dS = 0 \quad \forall i \tag{11}$$

One way of assessing the accuracy of the basis function based approximation of the charge density is to examine how well the approximation satisfies the integral equation. To be more precise, we define the residual associated with the integral equation and an approximate solution, equation (10). Note that $R(\vec{x})$ is just the difference between the given potential on the surface and the potential produced by the approximated charge density. Note also that the equation is now over the approximate geometry and therefore $\vec{x}$ and $\vec{x}'$ are both on the approximated surface.
If the representation satisfies the integral equation exactly, then the residual $R(\vec{x})$ will be zero for all $\vec{x}$ and the approximate solution is equal to the exact solution (provided the integral equation has a unique exact solution ... more on this later). In general, though, this is not possible, and instead we will try to pick the basis function weights, the α_i's, to somehow minimize $R(\vec{x})$. One approach to minimizing $R(\vec{x})$ is to make it orthogonal to a collection of test functions, which may or may not be related to the basis functions, equation (11). Enforcing orthogonality in this case means ensuring that the integral of the product of $R(\vec{x})$ and $\phi(\vec{x})$ over the surface is zero.

5.1.2 Residual Minimization Using Test Functions

SLIDE 31

$\boxed{\int \phi_i(\vec{x}) R(\vec{x}) dS = 0} \Rightarrow$

$$\int \phi_i(\vec{x}) \Psi(\vec{x}) dS - \iint_{\substack{\text{approx}\\ \text{surface}}} \phi_i(\vec{x}) G(\vec{x},\vec{x}') \sum_{j=1}^{n} \alpha_j \varphi_j(\vec{x}') dS' dS = 0 \tag{12}$$

We will generate different methods by choosing the $\phi_1, \ldots, \phi_n$
Collocation : $\phi_i(\vec{x}) = \delta(\vec{x} - \vec{x}_{t_i})$ (point matching)
Galerkin Method : $\phi_i(\vec{x}) = \varphi_i(\vec{x})$ (basis = test)
Weighted Residual Method : $\phi_i(\vec{x}) = 1$ if $\varphi_i(\vec{x}) \neq 0$ (averages)

As noted in the equation (12), by substituting the definition of the residual into the equation (11), it is possible to generate n equations, one for each test function. The generated equation has two integrals. The first is a surface integral of the product of the given potential with the test function. The second integral is a double integral over the surface. The integrand of the double integral is a product of the test function, the Green's function, and the charge density representation.
Three different numerical techniques can be derived by altering the test functions.

5.2 Collocation

SLIDE 32

Collocation: $\phi_i(\vec{x}) = \delta(\vec{x} - \vec{x}_{t_i})$ **(point matching)**

$\boxed{\int \delta(\vec{x}-\vec{x}_{t_i})R(\vec{x})dS=R(\vec{x}_{t_i})=0} \Rightarrow$

$$\sum_{j=1}^{n} \alpha_j \overbrace{\int_{\substack{\text{approx} \\ \text{surface}}} G(\vec{x}_{t_i}, \vec{x}')\varphi_j(\vec{x}')dS'}^{A_{i,j}} = \Psi(\vec{x}_{t_i}) \Rightarrow$$

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(\vec{x}_{t_1}) \\ \vdots \\ \Psi(\vec{x}_{t_n}) \end{bmatrix}$$

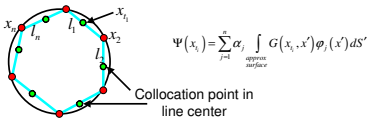
The collocation method, described in the above slide, uses shifted impulse functions as test functions, $\phi_i(\vec{x}) = \delta(\vec{x} - \vec{x}_i)$. Impulse functions, also called "delta" functions, have a *sifting* property when integrated with a smooth function $f(\vec{x})$,

$$\int f(\vec{x})\delta(\vec{x} - \vec{x}_i)dx = f(\vec{x}_i).$$

Impulse functions are also referred to as generalized functions, and they are specified only by their behavior when integrated with a smooth function. In the case of the impulse function, one can think of the function as being zero except for a very narrow interval around $\vec{x}_i$, and then being so large in that narrow interval that $\int \delta(\vec{x} - \vec{x}_i)dx = 1$.
As the summation equation in the middle of the above slide indicates, testing with impulse functions is equivalent to insisting that $R(\vec{x}_i) = 0$, or in words, that the potential produced by the approximated charge density should match the given potential at n test points. That the potentials match at the test points gives rise to the method's name, the point where the potential is exactly matched is "co-located" with a set of test points.
The $n \times n$ matrix equation at the bottom of the above slide has as its right-hand side the potentials at the test points. The unknowns are the basis function weights. The j^{th} matrix element for the i^{th} row is the potential produced at test point x_i by a charge density equal to basis function φ_j.

5.2.1 Centroid Collocation for Piecewise Constant Bases

SLIDE 33



$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(\vec{x}_{t_1}) \\ \vdots \\ \Psi(\vec{x}_{t_1}) \end{bmatrix} \Rightarrow \Psi(\vec{x}_{t_i}) = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{linej} G(\vec{x}_{t_i}, \vec{x}')dS'}_{A_{i,j}}$$

In the above slide, a specific collocation algorithm is described. First, the basis being used is the constant charge density on n straight sections or lines, as described above. Note that therefore the geometry is being approximated. Second, the collocation points being selected are the centroids of the basis functions, in this case just the center of each straight line. Note that the collocation point is on the approximated geometry, not the original geometry. So, one can think of the problem as having been restated to be on a polygon instead of the original circle. One could also have selected the collocation points on the original circle, but then the replacement interpretation does not hold.
In collocation, or point-matching, the charge densities on each of the straight lines are selected so that the resulting potential at the line centers matches the given potential. As the equations on this slide make clear, the matrix element $A_{i,j}$ is the potential at the center of line i due to a unit charge density along line j.
It should be noted that the matrix A is dense, the charge on line j contributes to the potential everywhere. Also note that if line j is far away from line i, then

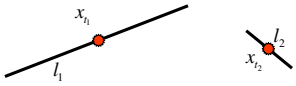
$$A_{i,j} \approx \text{length}(line_j) \times G(\vec{x}_{t_i}, \vec{x}_{t_j}) \tag{13}$$

▷ ***Exercise 4*** Suppose we are using piecewise constant centroid collocation to solve a 2-D Laplace problem, so $G(x, y, x', y') = \log\sqrt{(x-x')^2 + (y-y')^2}$. Roughly how far apart do line sections i and j have to be for equation (13) to be accurate to within one percent? Assume line j has length of one. Does your answer depend on the orientation of line j? Does your answer depend on the orientation of line i? (You should answer yes to one of these and no to the other, do you see why?) ■

5.2.2 Centroid Collocation Generates Nonsymmetric A

SLIDE 34

$$\Psi(\vec{x}_{t_i}) = \sum_{j=1}^{n} \alpha_j \overbrace{\int_{\text{line } j} G(\vec{x}_{t_i}, \vec{x}')dS'}^{A_{i,j}}$$



$$A_{1,2} = \int_{\text{line } 2} G(\vec{x}_{t_1}, \vec{x}')dS' \neq \int_{\text{line } 1} G(\vec{x}_{t_2}, \vec{x}')dS' = A_{2,1} \tag{14}$$

Consider the two line sections, l_1 and l_2 given in the above figure. For Laplace problems, $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$, which suggests a symmetry in the underlying integral equation that is not represented in the collocation discretization. This asymmetry is shown in equation (14) by noting that $A_{1,2} \neq A_{2,1}$. That is, the potential at the center of l_2 due to a unit charge density on l_1 is not equal to the potential at the center of l_1 due to a unit charge on l_2.
It is possible to scale the variables to improve the symmetry, consider a change of variables

$$\hat{\alpha}_i = \alpha_i \times \text{length}(line_j).$$

In this change of variables, the unknowns $\hat{\alpha}_i$ are now the net line charges rather than the line charge densities. In this new system, $\hat{A}\hat{\alpha} = \Psi$, where the elements of the matrix $\hat{A}$ are given by

$$\hat{A}_{i,j} = \frac{1}{\text{length}(line_j)} \int_{line_j} G(\vec{x}_{t_i}, \vec{x}')dS'.$$

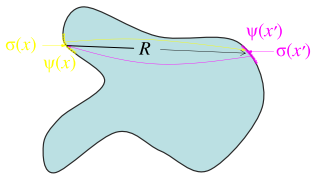
Under the change of variables, if line j is far away from line i, then

$$\hat{A}_{i,j} \approx G(\vec{x}_{t_i}, \vec{x}_{t_j}) \approx \hat{A}_{j,i}. \tag{15}$$

In other words, the elements of $\hat{A}$ corresponding to distant terms are approximately symmetric.

SLIDE 35

Is $\Psi(\vec{x})$ due to $\sigma(\vec{x}')$ the same as $\Psi(\vec{x}')$ due to $\sigma(\vec{x})$?



Green's Function is due to $\log R$

▷ ***Exercise 5*** Give an example which shows that the scaled entries of $\hat{A}$ can be far from symmetric. Assume we are using piecewise constant straight sections with centroid collocation and the 2-D Laplace's equation Green's function. ■

5.3 Galerkin

SLIDE 36

Galerkin: $\phi_i(x) = \varphi_i(x)$ (test=basis)

$$\int \varphi_i(x)\, R(x)\, dS = \int \varphi_i(x)\, \Psi(x)\, dS - \int_{\substack{\text{approx} \\ \text{surface}}} \varphi_i(x) G(x,x') \sum_{j=1}^{n} \alpha_j \varphi_j(x')\, dS'\, dS = 0$$

$$\underbrace{\int_{\substack{\text{approx} \\ \text{surface}}} \varphi_i(x)\, \Psi(x)\, dS}_{b_i} = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{\substack{\text{approx} \\ \text{surface}}} \int_{\substack{\text{approx} \\ \text{surface}}} G(x,x') \varphi_i(x) \varphi_j(x')\, dS'\, dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

If $G(\vec{x}, \vec{x}') = G(\vec{x}', \vec{x})$ then $A_{i,j} = A_{j,i} \Rightarrow$ **A is symmetric**

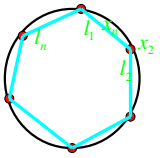
In the Galerkin method, the test functions are equal to the basis functions. In particular, one generates n equations for the basis function weights by insisting that $R(\vec{x})$ is orthogonal to each of the basis functions. Enforcing orthogonality corresponds to setting

$$\int \varphi(\vec{x}) R(\vec{x}) dS = 0$$

and substituting the definition of $R(\vec{x})$ into the orthogonality condition yields the equation in the center of the above slide.
Note that the Galerkin method yields a system of n equations, one for each orthogonality condition, and n unknowns, one for each basis function weight. Also, the system does not have the potential explicitly as the right hand side. Instead, the i^{th} right-hand side entry is the average of the product of the potential and the i^{th} basis function.

5.3.1 Galerkin for Piecewise Constant Bases

SLIDE 37



$$\underbrace{\int_{\text{line}_i} \Psi(x)\, dS}_{b_i} = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{\text{line}_i} \int_{\text{line}_j} G(x,x')\, dS'\, dS}_{A_{i,j}}$$

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

In the Galerkin method, the basis has constant charge density on n straight sections or lines. We will think of the problem as having been restated to be on a polygon instead of the original circle. The charge densities on each of the straight lines are selected so that the resulting line averaged potential matches the line averaged given potential. As the equations on the above slide make clear, the matrix element $A_{i,j}$ is the average potential over line i, scaled by the length of line i, due to a unit charge density along line j.

As with the collocation method, the matrix A is dense because the the charge on line j contributes to the averaged potentials everywhere. Also note that if line j is far away from line i, then

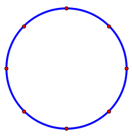
$$A_{i,j} \approx \text{length}(line_j) \times \text{length}(line_i) \times G(\vec{x}_{t_i}, \vec{x}_{t_j}) \tag{16}$$

▷ ***Exercise 6*** Suppose we are using piecewise constant centroid collocation to solve a 2-D Laplace problem, so $G(x, y, x', y') = \log \sqrt{(x - x')^2 + (y - y')^2}$. Roughly how far apart do line sections i and j have to be for equation (16) to be accurate to within one percent? Assume line j has length of one. Does your answer depend on the orientation of line j? Does your answer depend on the orientation of line i? (Your answer should be different than the answer you gave for the collocation method. Do you see why?) ■

5.4 Summary

SLIDE 38

Compare the Collocation and Galerkin methods on a two-dimensional circle.

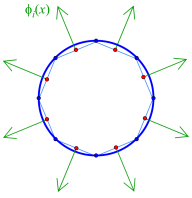


What do the test functions look like?
What do the Residuals look like?

SLIDE 39

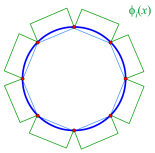
Test Functions

Collocation



$\phi_i(\vec{x}) = \delta(\vec{x} - \vec{x}_i)$

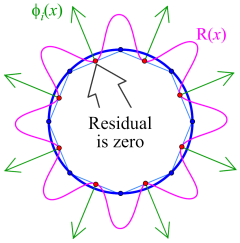
Galerkin



$\phi_i(\vec{x}) = \varphi_i(\vec{x})$

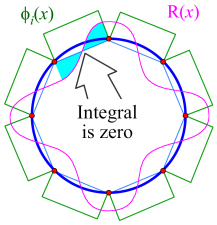
SLIDE 40

Collocation Method



SLIDE 41

Galerkin Method



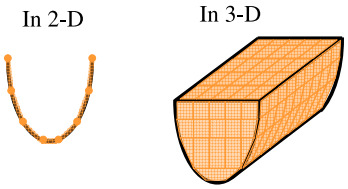
6 Issues in 3D

6.1 Geometric Representation

6.1.1 Introduction

SLIDE 42

Example: Ship's Hull
More errors are introduced with expansion of dimensions

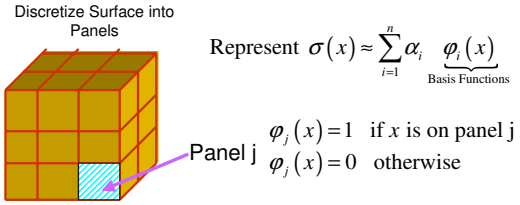


Note 6 ***"Leaky Panels"***

Many papers in the literature on solving integral equations refer to "panel methods". The name is derived from the idea of breaking a surface into flat panels. In the application area of analyzing ocean wave forces on ship hulls, panel methods are commonly used. However, it is not possible to represent a curved hull with quadrilateral flat panels. Researchers in the area often create a best fit panelled surface in which there are gaps between the edges of the panels. Such a discretization technique is often referred to as using "leaky panels", a very compelling image.

SLIDE 43

Integral Equation : $\Psi(x) = \int_{surface} \frac{1}{||x-x'||}\sigma(x')dS'$



Represent $\sigma(x) \approx \sum_{i=1}^{n} \alpha_i \underbrace{\varphi_i(x)}_{\text{Basis Functions}}$

$\varphi_j(x) = 1$ if x is on panel j
$\varphi_j(x) = 0$ otherwise

Consider solving the integral equation where the surface is the surface of the cube shown. The first step, as we have mentioned in previous lectures, is to develop a basis in which to represent the surface charge density σ.
The cube pictured in the slide has had its surface divided into panels, and a basis is derived from the panels. In particular, one can associate a basis function φ_j with each panel j by assigning $\varphi_j(\vec{x})$ the value one when $\vec{x}$ is a point on panel j, and setting $\varphi_j(\vec{x}) = 0$ otherwise. If σ is approximated by a weighted combination of these basis functions, then the approximation is a piecewise constant representation of the charge density on the surface of the cube.
A few aspects of this basis set should be noted.

- The basis functions are orthogonal, that is if $i \neq j$,

$$\int \varphi_j(\vec{x})\varphi_i(\vec{x})dx = 0.$$

- These basis functions are normalized with respect to l_∞, not l_2. That is, $\|\varphi\|_\infty = 1$ but

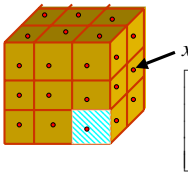
$$\|\varphi_j\|_2^2 = \int \varphi_i(\vec{x})\varphi_i(\vec{x})dx = \text{panel area}.$$

6.2 Centroid Collocation

SLIDE 44

Put collocation points at panel centroids

$$\Psi(x_{c_i}) = \sum_{j=1}^{n} \alpha_j \underbrace{\int_{panel\ j} \frac{1}{\|x_{c_i} - x'\|} dS'}_{A_{i,j}}$$



$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & A_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n,1} & \cdots & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \Psi(x_{c_1}) \\ \vdots \\ \vdots \\ \Psi(x_{c_n}) \end{bmatrix}$$

After one has decided on a basis with which to approximately represent the surface charge density, the next step is to develop a system of equations from which to determine the basis weights, denoted as the α_i's. The most commonly used approach to forming such a system is to use collocation, though Galerkin methods are also quite widely used. Recall that in collocation, the basis function weights are determined by ensuring the the integral equation is exactly satisfied at a collection of "collocation" points. For panel methods, the most common choice for the position of the collocation points are the panel centroids, as shown in the cube diagram above.

The equation in the top of the above slide relates the potential at collocation point $\vec{x}_{c_i}$ to the weights for the panel-based basis functions. To see how the equation was derived, consider evaluting the potential at the i^{th} collocation point using the original integral equation

$$\Phi(\vec{x}_{c_i}) = \int_{surface} \frac{1}{\|\vec{x}_{c_i} - \vec{x}'\|} \sigma(\vec{x}') dS', \tag{17}$$

where Φ is the know potential on the problem surface and σ is the unknown charge density. Substituting the approximate representation for σ,

$$\sigma(\vec{x}) \approx \sum_{j=1}^{n} \alpha_j \varphi_j(\vec{x})$$

into the integral equation results in

$$\Phi(\vec{x}_{c_i}) = \int_{surface} G(\vec{x}_{c_i}, \vec{x}') \sum_{j=1}^{n} \alpha_j \varphi_j(\vec{x}') dS', \tag{18}$$

where $G(\vec{x}, \vec{x}') \equiv \frac{1}{\|\vec{x} - \vec{x}'\|}$ is used to simplify the formula. Exploiting the fact that $\varphi_j(\vec{x}) = 1$ if $\vec{x}$ is on panel j, and zero otherwise, results in the formula at the top of the above slide.

The system of equations from which to determine the basis function weights is given in the lower corner of the slide. The right hand side of the system is the vector of known potentials at the collocation points. The i, j^{th} element of the matrix A is the potential produced at collocation point i due to a unit charge density on panel j. The vector of α's are the unknown panel charge densities.

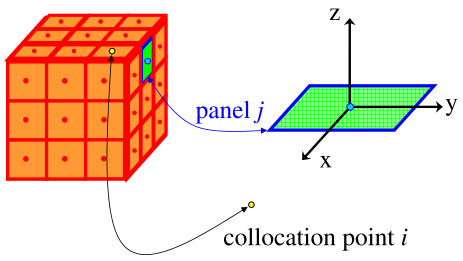
▷ ***Exercise 7*** Determine a scaling of the α's ($\hat{\alpha}_i = c_i \alpha_i$) such that the scaled matrix $\hat{A}$ has the property

$$A_{i,j} \approx \frac{1}{\|\vec{x}_{c_i} - \vec{x}_{c_j}\|}$$

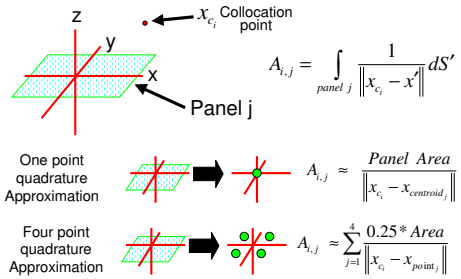
when $\|\vec{x}_{c_i} - \vec{x}_{c_j}\|$ is much larger than a panel diameter. ■

6.2.1 Calculating Matrix Elements

SLIDE 45



SLIDE 46



In order to calculate the matrix entries for the system of equations described in the previous slide, recall that $A_{i,j}$ is the potential produced at collocation point i due to a unit charge density on panel j. The formula for $A_{i,j}$ is given on the top right of the above slide.

The figure on the left of the above slide is a diagram of how one typically computes the panel integral given on the top right. First, consider a shift and rotation of the coordinate system so that the panel lies in the x-y plane at $z = 0$, with the panel's center at $x = 0$, $y = 0$. The figure in the top left shows the panel in the shifted and rotated coordinate system. Note that the collocation point must also be placed in the new coordinate system.

If panel j is reasonably well separated from collocation point i, it is possible to approximate the integral given in the top right by a single point quadrature. More specifically, one could approximates the integral of $\frac{1}{\|\vec{x}_{c_i} - \vec{x}'\|}$ by a product of $\frac{1}{\|\vec{x}_{c_i} - \vec{x}_{centroid_j}\|}$ and the panel area. As show in the middle figure, a single point quadrature is like treating the panel as if it were a point charge at the panel's centroid, where the point charge's strength is equal to the panel area.

If the collocation point is close to the panel, then a single point quadrature will be insufficiently accurate. Instead, a more accurate four point quadrature scheme would be to break the panel into four subpanels, and then treat each of the subpanels as point charges at their respective centers. This simple idea is shown in the figure at the bottom of the above slide. This four point scheme is

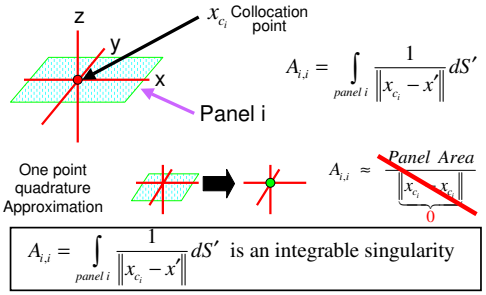
equivalent to

$$\int_{panel_j} \frac{1}{\|\vec{x}_{c_i} - \vec{x}'\|} dS' \approx \sum_{j=1}^{4} \frac{0.25 * Area}{\|\vec{x}_{c_i} - \vec{x}_{point_j}\|}.$$

If the panel is a unit square in the x-y plane whose center is at the coordinate system origin, then the four $\vec{x}_{point_j}$'s are $(x, y, z) = (0.25, 0.25, 0)$, $(x, y, z) = (-0.25, 0.25, 0)$, $(x, y, z) = (-0.25, -0.25, 0)$, and $(x, y, z) = (0.25, -0.25, 0)$.

6.2.2 Calculating "Self Term"

SLIDE 47

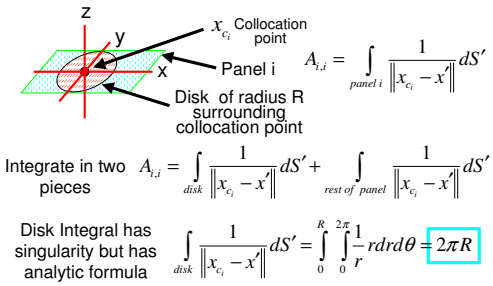


The diagonal terms $A_{i,i}$ can not be computed using the quadrature approximation given on the previous slide. To see this, consider the figure at the top left of the above slide, where a panel has been shifted and rotated into the x-y plane, and the collocation point is the center of the panel. The integral that must be computed is given on the right side of the top of the above slide.
As shown in the middle of the slide, using a single point quadrature scheme will fail, because the distance between the point charge approximation to the panel and the collocation point will be zero. Therefore, the single point formula will require computing the reciprocal of zero, which is infinite. The problem is that the integrand in

$$\int_{panel_i} \frac{1}{\|x_{c_i} - x'\|} dS' \tag{19}$$

is singular. That is, the integrand approaches infinity at a point x' which is in the domain of integration. What is not so obvious is that (19) is an integrable singularity. Therefore, even though the integrand approaches infinity at some point, the "area under the curve" is finite.

SLIDE 48



In the above slide, we both show that

$$\int_{panel_i} \frac{1}{\|\vec{x}_{c_i} - \vec{x}'\|} dS'$$

is integrable, and also give an idea about how to compute the integral.
As shown in the slide, first rotate and shift the coordinate system so that the panel is in the x-y plane at $z = 0$, and so that the collocation point (or equivalently the panel centroid) is at the origin. In this new coordinate system, the integral can be written as

$$A_{i,i} = \int_{panel(rs)_i} \frac{1}{\|\vec{x}'\|} dS'$$

where the notation $panel(rs)$ is used to indicate that the integral is over the rotated and shifted panel.
On the top left of the above slide, a circular disk of radius R and center at the collocation point is inscribed in the rotated panel. In the equations that follow the figure, it is noted that the panel integral can be recast as the sum of an integral over the disk plus an integral over the rest of the panel. The integrand in the integral over the rest of the panel is no longer singular, but the integrand in the integral over the disk is still singular.
The integral over the disk can be computed analytically by using a change of variables. After rotating and shifting the panel, the disk is in the x-y plane and its center, equal to the collocation point, is at zero. Therefore,

$$\int_{disk} \frac{1}{\|\vec{x}_{c_i} - \vec{x}'\|} dS' = \int_{disk} \frac{1}{\|\vec{x}'\|} dS'$$

Apply a change of variables as transformations of two-dimensional regions. Recall this mapping from an earlier lecture in the finite difference method. Suppose that a region $\hat{\Omega}$ in the r-θ plane is transformed one-to-one into the region Ω by differentiable equations of the form

$$x = rcos\theta, \qquad y = rsin\theta.$$

Any function $f(x, y)$ defined on Ω can be thought of as a function $f(x(r, \theta), y(r, \theta))$ on $\hat{\Omega}$. The integrals of these functions are related by

$$\iint_{\Omega} f(x, y) dx dy = \iint_{\hat{\Omega}} f(x(r, \theta), y(r, \theta)) \, |J(r, \theta)| \, dr d\theta$$

where $J(r,\theta)$ is the Jacobian determinant of the coordinate transformation where

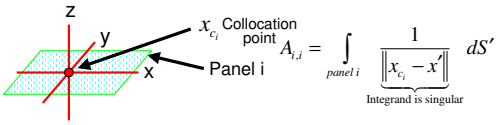
$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r(cos^2\theta + sin^2\theta) = r$$

So, now the integral may be put into the transformed coordinates using this transformation

$$\int_{disk} \frac{1}{\|\vec{x}_{c_i} - \vec{x}'\|} dS' = \int_0^R \int_0^{2\pi} \frac{1}{r} r d\theta dr$$

The integral over the disk is easily seen to be $2\pi R$.

SLIDE 49



1. If panel is a flat polygon, analytical formulas exist.

2. Curved panels can be handled with projection.

7 Summary

SLIDE 50

Integral Equation Methods
Exterior versus interior problems
Start with using point sources
Standard Solution Methods
Collocation Method
Galerkin Method
Integrals for 3D Problems
Singular Integrals