Numerical Methods for PDEs

Integral Equation Methods, Lecture 4

Representing Potentials -> Jumps -> 2nd Kind -> Nystrom -> Fredholm

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1 Outline

SLIDE 1

Potential Representations

Monopole and Dipole Densities

Principle Value

Interior Neumann

Use Nystrom to Solve

Look at 2-D Problems

Fredholm Alternative

Connection to Linear Algebra

2 3D Problems

2.1 3D Laplace Equation

SLIDE 2

Laplace's equation in 3-D

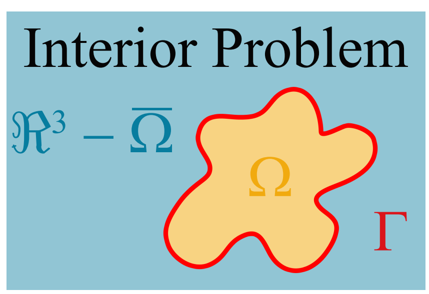
$$\nabla^2 u(\vec{x}) = \frac{\partial^2 u(\vec{x})}{\partial x^2} + \frac{\partial^2 u(\vec{x})}{\partial y^2} + \frac{\partial^2 u(\vec{x})}{\partial z^2} = 0$$

where

$$\vec{x} = (x, y, z) \in \Omega$$

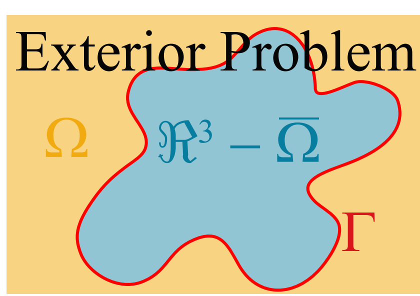
and Ω is bounded by Γ.

SLIDE 3



The exterior problem is simply the region $\Re - \bar{\Omega}$ of the interior problem.

SLIDE 4



One feature of using integral equation methods is that exterior problems can be solved using the same surface discretization required to solve an interior problem (assuming a linear space-invariant problem like Laplace's equation). This is true even though the exterior domain is infinite and the interior domain is finite. Exterior problems do introduce an additional complication, one must consider the boundary condition "at infinity" (later).

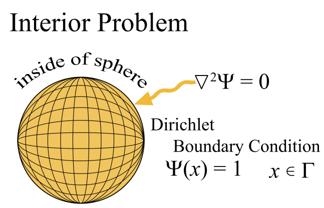
2.2 Boundary Conditions

2.2.1 Dirichlet

SLIDE 5

Dirichlet Condition

$$u(\vec{x}) = u_\Gamma(\vec{x}) \qquad \vec{x} \in \Gamma$$

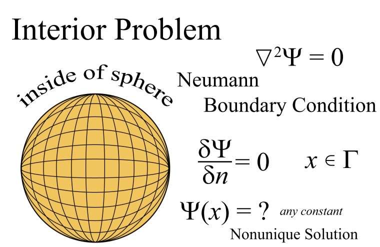


Can you determine the solution to Laplace's equation inside the sphere?

The solution of the interior Dirichlet problem is unique.

2.2.2 Neumann

SLIDE 6



The solution of the interior Neumann problem is not unique.
For the solution of the exterior Neumann problem to be unique, it is sufficient to impose a *radiation condition.* In this case, a radiation condition would be a specification of how $u(\vec{x})$ approaches zero as $\vec{x} \to \infty$.

2.2.3 Exterior

SLIDE 7

Dirichlet Boundary Condition

$$u(\vec{x}) = u_\Gamma(\vec{x}) \qquad \vec{x} \in \Gamma$$

Neumann Boundary Condition

$$\frac{\partial u(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} \qquad \vec{x} \in \Gamma$$

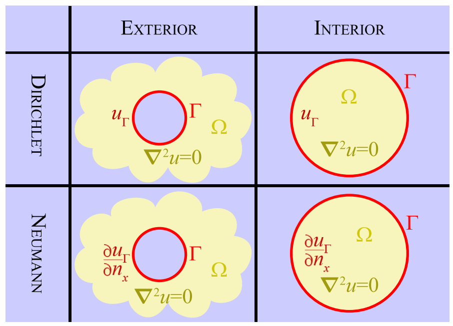
PLUS

A Radiation Condition (Next Week!)

3 Formulations – Problem Types

3.1 Single Domain

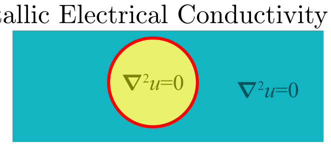
SLIDE 8



3.2 Coupled Domain

SLIDE 9

Example: Bimetallic Electrical Conductivity



Potential and Electric Current Continuity:

$$u(\vec{x}^+) = u(\vec{x}^-) \qquad \vec{x} \in \Gamma$$

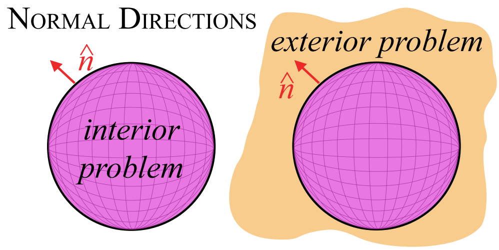
$$\alpha^+ \frac{\partial u(\vec{x}^+)}{\partial n_{\vec{x}}} = \alpha^- \frac{\partial u(\vec{x}^-)}{\partial n_{\vec{x}}} \qquad \vec{x} \in \Gamma$$

An example of a coupled domain problem would be a conductivity problem involving multiple materials. To determine the electrical conductivity between two terminals of an object made of multiple materials, one would determine the ratio of the voltage across the object's terminals and the current flowing through the object. Electrical current density in an ideal linear conductor is a vector quantity given by the gradient of the potential, ∇u, scaled by a factor known as the conductivity of the material. In an ideal linear conductor there is no accumulation of charge at any interior point, implying that the current density has zero divergence. Therefore, the potential in an ideal linear conductor satifies Laplace's equation, $\nabla^2 u = 0$. If an object is made of multiple materials with different electrical conductivities, then the boundary between materials satisfies interface conditions. At the boundary between materials, both the potential and the current density in the surface-normal direction are continuous. Since the conductivities of the two materials are different, continuity of the current density implies a jump in the gradient of the potential across the material boundary.

3.3 Normals

SLIDE 10

Normals usually point from Interior to Exterior.



Typically, the surface normal is assumed to point in the direction from the interior domain to the exterior domain. There are many situations where this typical practice is confusing or ambigious, so it is often necessary to be explicit about the direction of the normal.

4 Surface Density Integrals

4.1 Monopole & Dipole

SLIDE 11

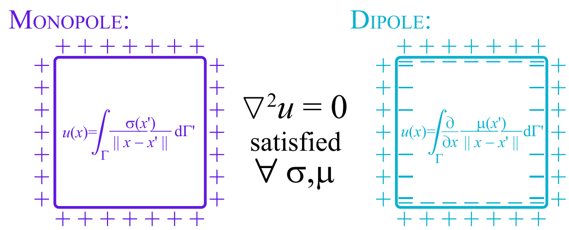
Potential due to a monopole density (σ):

$$u(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

Potential due to a dipole density (μ):

$$u(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \mu(\vec{x}') d\Gamma'$$

where the normal points out of the domain Ω bounded by Γ. SLIDE 12



Monopole or dipole densities can be used to generate potentials that satisfy $\nabla^2 u(\vec{x}) = 0$ for all $\vec{x} \in \Omega$. The monopole and dipole potentials differ in the radiation condition they satisfy. If the surface, γ, is finite in extent, then in the limit as $\|\vec{x}\| \to \infty$, the monopole potential decays like $\|\vec{x}\|^{-1}$, and the dipole

potential decays like $\|\vec{x}\|^{-1}$.
Either representation can be used to derive surface integral equations, but care must be used when evaluating the associated potentials when $\vec{x} \in \Gamma$.

4.2 Surface Potentials

SLIDE 13

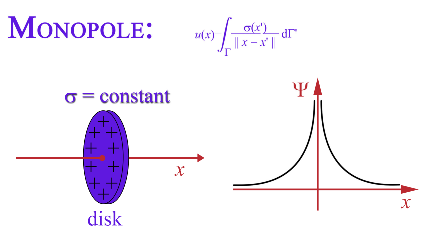
The monopole potential is continuous as x passes through Γ, so

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

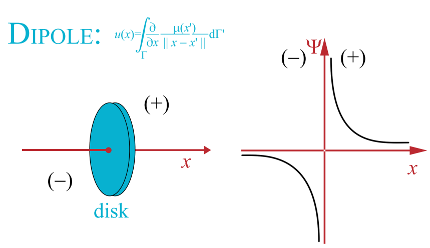
The dipole potential "jumps" as x passes through Γ, so the limit as $\vec{x} \to \Gamma$ of

$$u(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \mu(\vec{x}') d\Gamma'$$

depends on how Γ is approached. SLIDE 14



Don't be put-off by the graph above. The monopole potential is continuous (it does not go off to infinity, as it may seem to in the above figure), but it is not continuously differentiable, there will be a discontinuity in the derivative at x_0. SLIDE 15



4.2.1 Principle Value Integral

SLIDE 16

If $f(y)$ is singular for some $y = x_0$, where $x_0 \in \Gamma$, then the principle value integral is

$$\int_\Gamma^{PV} f(\vec{y}) d\Gamma \equiv \lim_{\epsilon \to 0} \int_{\Gamma - B(x_0,\epsilon) \cap \Gamma} f(\vec{y}) d\Gamma$$

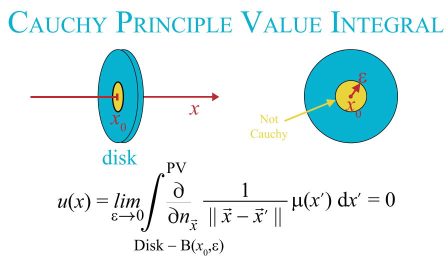
when $B(x_0, \epsilon)$ is the ϵ radius ball about x_0.

The P.V. is a special kind of limit

Limit of deleting and ever shrinking portion of the integration domain.

NOT EQUIVALENT TO limiting processes on f!

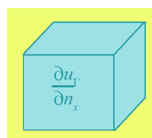
SLIDE 17



4.2.2 Monopole Derivative (MD)

SLIDE 18

Consider a cube geometry:



$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \lim_{\vec{x}\to\Gamma^+} \frac{\partial}{\partial n_{\vec{x}}} \int_\Gamma \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma' \qquad \vec{x}\in\Gamma$$

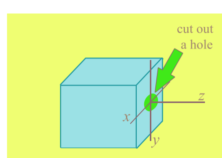
The plus $(+)$ in Γ^+ indicates exterior approach.

In the above slide, we consider computing the normal derivative of the monopole potential just outside the boundary γ. As will be shown in the next few slides, the derivative can be represented as the sum of a principle value integral and an extra term.

4.2.3 MD Disk Removal

SLIDE 19

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \lim_{\vec{x}\to\Gamma^+,\epsilon\to 0}\left[\frac{\partial}{\partial n_{\vec{x}}}\int_{\Gamma-B(x,\epsilon)}\frac{\sigma(\vec{x}')}{\|\vec{x}-\vec{x}'\|}d\Gamma' + \frac{\partial}{\partial n_{\vec{x}}}\int_{B(x,\epsilon)}\frac{\sigma(\vec{x}')}{\|\vec{x}-\vec{x}'\|}d\Gamma'\right]$$

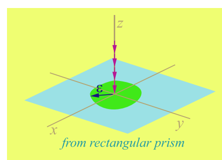


Consider the entire side panel on the right of the cube in the above slide, and consider evaluating the normal derivative of the potential generated by a monopole distribution on the suface of the cube's right side. Specifically, assume that we wish to evaluate the derivative at a point $\vec{x}$ in the center of the green disk on the cube's right side. The matter is complicated by the fact that the integrand goes to infinity when $\vec{x} = \vec{x}'$. Thus, we need to break up the integral into two pieces. One piece is the entire panel minus the green disk, and the other piece is just the green disk.

4.2.4 MD Disk Picture

SLIDE 20

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \lim_{\vec{x}\to\Gamma^+,\epsilon\to 0}\left[\frac{\partial}{\partial n_{\vec{x}}}\int_{\Gamma - B(x,\epsilon)} \frac{\sigma(\vec{x}')}{\|\vec{x}-\vec{x}'\|}d\Gamma' + \frac{\partial}{\partial n_{\vec{x}}}\int_{B(x,\epsilon)} \frac{\sigma(\vec{x}')}{\|\vec{x}-\vec{x}'\|}d\Gamma'\right]$$

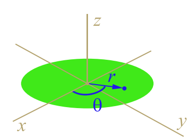


The first integral is the Principle Value Integral and the second integral is the integral of just the disk.
Given that the disk was extracted from right the surface of a cube, the disk is flat, and the normal is in the z-axis direction.

4.2.5 MD Disk Eval

SLIDE 21

$$\begin{aligned}
&\lim_{\vec{x}\to\Gamma^+}\frac{\partial}{\partial n_{\vec{x}}}\int_{B(x,\epsilon)}\frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma' \\
&\qquad\approx \lim_{z\to 0^+}\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\epsilon}\frac{\sigma(\vec{x})}{\sqrt{r^2+z^2}}r\,dr\,d\theta \\
&\qquad= \lim_{z\to 0^+}\frac{\partial}{\partial z}2\pi\sigma(\vec{x})\left[\sqrt{\epsilon^2+z^2}-|z|\right] \\
&\qquad= -2\pi\sigma(\vec{x}).
\end{aligned}$$



Note ?? **Disk Evaluation Math**

For this problem, it is quite straightforward to see how one changes from cartesian to cylindrical coordinates. But, the algebra and calculus involved in solving this integral may not be as straightforward, herein is presented one method, broken-down into bite-size pieces:

$$\lim_{z\to 0^+}\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\epsilon}\frac{\sigma(\vec{x})}{\sqrt{r^2+z^2}}r\,dr\,d\theta.$$

Use trigonometric substitution to solve the integral with respect to r. Substitute $r = z\tan\alpha$ and $dr = z\sec^2\alpha\, d\alpha$ and simplify to get the following integral:

$$= \lim_{z\to 0^+}\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\alpha(\epsilon)}\frac{\sigma(\vec{x})z\sin\alpha}{\cos^2\alpha}d\alpha\, d\theta.$$

This integral is easily solved using direct substitution of $u = \cos\alpha$:

$$= \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_0^{2\pi} \int_0^{u[\alpha(\epsilon)]} -z\frac{\sigma(\vec{x})}{u^2} du d\theta = \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_0^{2\pi} z\frac{\sigma(\vec{x})}{u}\bigg|_0^{u[\alpha(\epsilon)]} d\theta.$$

Plug back in for $u = \cos\alpha = \frac{z}{\sqrt{r^2+z^2}}$

$$= \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_0^{2\pi} \sigma(\vec{x})\sqrt{r^2+z^2}\bigg|_0^{r=\epsilon} d\theta$$

$$= \lim_{z\to 0^+} \frac{\partial}{\partial z} \int_0^{2\pi} \sigma(\vec{x}) \left[\sqrt{\epsilon^2+z^2} - \sqrt{z^2}\right] d\theta.$$

Integrating the last part is quite simple,

$$= \lim_{z\to 0^+} \frac{\partial}{\partial z}\sigma(\vec{x}) \left(\sqrt{\epsilon^2+z^2} - \sqrt{z^2}\right)\theta\bigg|_0^{2\pi} = 2\pi\sigma(\vec{x}) \lim_{z\to 0^+} \frac{\partial}{\partial z}\left(\sqrt{\epsilon^2+z^2} - \sqrt{z^2}\right).$$

Finally, take the derivatives with respect to z, the normal,

$$= 2\pi\sigma(\vec{x}) \lim_{z\to 0^+} \left(\frac{z}{\sqrt{\epsilon^2+z^2}} - \frac{z}{\sqrt{z^2}}\right).$$

It can now be seen that, since $\lim_{z\to 0^+} \frac{z}{\sqrt{\epsilon^2+z^2}} = 0$ and $\lim_{z\to 0^+} \frac{z}{\sqrt{z^2}} = \text{sign}\, z$ that

$$\lim_{z\to 0^+} \frac{\partial}{\partial z} \int_0^{2\pi} \int_0^{\epsilon} \frac{\sigma(\vec{x})}{\sqrt{r^2+z^2}} r dr d\theta = -2\pi\sigma(\vec{x}).$$

4.2.6 MD Final

Slide 22

$$\begin{aligned}
\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} &= \lim_{\vec{x}\to\Gamma^+} \frac{\partial}{\partial n_{\vec{x}}} \int_\Gamma \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma' \\
&= \lim_{\vec{x}\to\Gamma^+} \lim_{\epsilon\to 0} \left[\frac{\partial}{\partial n_{\vec{x}}} \int_{\Gamma - B(x,\epsilon)} \frac{\sigma(\vec{x}')}{\|\vec{x}-\vec{x}'\|} d\Gamma' + \frac{\partial}{\partial n_{\vec{x}}} \int_{B(x,\epsilon)} \frac{\sigma(\vec{x}')}{\|\vec{x}-\vec{x}'\|} d\Gamma'\right] \\
&= \int_\Gamma^{PV} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma' - 2\pi\sigma(\vec{x}')
\end{aligned}$$

5 Exterior Formulations

5.1 Dirichlet Problem

5.1.1 Monopole Potential

Slide 23

For an 3D exterior problem:

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma'$$

For an 2D exterior problem:

$$u_\Gamma(\vec{x}) = \int_\Gamma \log\|\vec{x}-\vec{x}'\|\sigma(\vec{x}')d\Gamma'$$

5.1.2 Dipole Potential

Slide 24

For a 3-D exterior problem:

$$u_\Gamma(\vec{x}) = 2\pi\mu(\vec{x}) + \int_\Gamma^{PV} \frac{(\vec{x}-\vec{x}')^T n_{\vec{x}'}}{(\|\vec{x}-\vec{x}'\|)^3}\mu(\vec{x}')d\Gamma'$$

For a 2-D exterior problem:

$$u_\Gamma(\vec{x}) = \pi\mu(\vec{x}) + \int_\Gamma^{PV} \frac{(\vec{x}-\vec{x}')^T n_{\vec{x}'}}{(\|\vec{x}-\vec{x}'\|)^2}\mu(\vec{x}')d\Gamma'$$

Normal points from interior to exterior.

5.2 Neumann Problem

5.2.1 Monopole

Slide 25

For an exterior 3D problem

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = -2\pi\sigma(\vec{x}) - \int_\Gamma^{PV} \frac{(\vec{x}-\vec{x}')^T n_{\vec{x}}}{(\|\vec{x}-\vec{x}'\|)^3}\sigma(\vec{x}')d\Gamma'$$

For an exterior 2D problem

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = -\pi\sigma(\vec{x}) - \int_\Gamma^{PV} \frac{(\vec{x}-\vec{x}')^T n_{\vec{x}}}{(\|\vec{x}-\vec{x}'\|)^2}\sigma(\vec{x}')d\Gamma'$$

Normal points from interior to exterior.

6 Interior Example

6.1 3D Case

6.1.1 Monopole Potential

Slide 26

Surface Potential

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma' \qquad \vec{x}\in\Gamma$$

Surface Normal Derivative

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial}{\partial n_{\vec{x}}}\int_\Gamma \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma' \qquad \vec{x}\in\Gamma$$

Normal points to exterior.

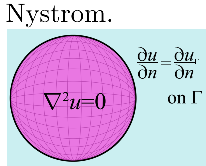
6.1.2 Interior Neumann

Slide 27

Monopole Potential Using the P.V. Integral

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}) + \int_\Gamma^{PV} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}-\vec{x}'\|}\sigma(\vec{x}')d\Gamma'$$

looks 2^{nd} Kind Equation, Try Nystrom.



6.1.3 Nystrom Method

Slide 28

Set **quadrature points** $=$ **collocation points**

$$\frac{\partial u_\Gamma(\vec{x}_1)}{\partial n_{\vec{x}}} = 2\pi\sigma_{n1} + \sum_{j=1}^{n} w_{1,j} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}_1-\vec{x}_j\|}\sigma_{nj}$$

$$\vdots$$

$$\frac{\partial u_\Gamma(\vec{x}_n)}{\partial n_{\vec{x}}} = 2\pi\sigma_{nn} + \sum_{j=1}^{n} w_{n,j} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}_n-\vec{x}_j\|}\sigma_{nj}$$

n equations in n unknowns

$j = i$ **case (self-term)?**

6.1.4 $i = j$

Slide 29

For the monopole 3-D Neumann Formulation,

$$G(\vec{x},\vec{x}') = \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}-\vec{x}'\|} = -\frac{(\vec{x}-\vec{x}')^T n_{\vec{x}}}{(\|\vec{x}-\vec{x}'\|)^3}$$

PROBLEM: $G(\vec{x},\vec{x}')$ **blows up as** $\vec{x} \to \vec{x}'$**.**

6.2 2-D Case

Slide 30

Monopole Neumann Formulation

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \pi\sigma(\vec{x}) + \int_\Gamma^{PV} \frac{\partial}{\partial n_{\vec{x}}} \log\|\vec{x}-\vec{x}'\|\sigma(\vec{x}')d\Gamma'$$

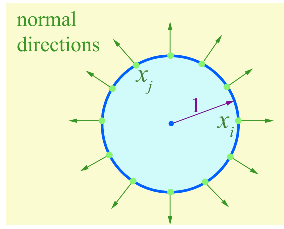
Simplifying the Green's function,

$$G(\vec{x},\vec{x}') = \frac{\partial}{\partial n_{\vec{x}}} \log\|\vec{x}-\vec{x}'\| = -\frac{(\vec{x}-\vec{x}')^T n_{\vec{x}}}{(\|\vec{x}-\vec{x}'\|)^2}$$

6.2.1 Smooth Γ

SLIDE 31

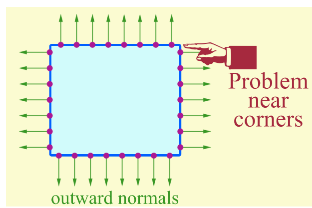
$G(\vec{x}, \vec{x}')$ finite as $\vec{x} \to \vec{x}'$ if Γ smooth.



6.2.2 Nonsmooth Γ

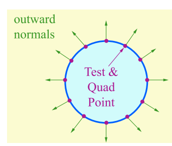
SLIDE 32

$G(\vec{x}, \vec{x}')$ not finite as $\vec{x} \to \vec{x}'$ if x' is on a corner.



6.2.3 Disk Example

SLIDE 33



$$\frac{\partial u_\Gamma(x_i)}{\partial n_{x_i}} = \pi \sigma(x_i) - \frac{2\pi}{N} \sum \frac{\vec{n}_{x_i} \cdot (x_i - x_j)}{\|x_j - x_i\|^2} \sigma(x_j)$$

Note uniform quadrature weights on the circle.
Resulting matrix is singular! Why?

7 Second Kind Theorem

7.1 Theorem

SLIDE 34

Given

$$(I + K)\sigma = \Psi \quad \text{(Integral Eqn.)}$$
$$(I + K_n)\sigma_n = \Psi_n \quad \text{(Discretized Eqn.)}$$

AND

$$\|(I + K)^{-1}\| < C \quad \text{Unique solvability}$$

If

$$lim_{n\to\infty} \|(K - K_n)\| \to 0 \; and \; \|\Psi - \Psi_n\| \to 0$$

Then

$$\lim_{n\to\infty} \|\sigma - \sigma_n\| \to 0$$

7.1.1 Scaled Example

SLIDE 35

Define Scaled Variables

$$\Psi \equiv \frac{1}{\pi} \frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}}$$

$$K \equiv \frac{1}{\pi} \int_\Gamma^{PV} \frac{\partial}{\partial n_{\vec{x}}} \log \|\vec{x} - \vec{x}'\| \sigma(\vec{x}') d\Gamma'$$

The 2-D Neumann problem becomes

$$(I + K)\sigma = \Psi$$

7.1.2 Key Property

SLIDE 36

Main assumption of second kind theory:

$(I + K)^{-1}$ is bounded.

Is $(I + K)^{-1}$ bounded for Interior Neumann Problem?

7.2 Linear Algebra

SLIDE 37

Given $Ax = b$, $\quad A \in \Re^{n \times n}$, $\quad x, b \in \Re^n$

A^{-1} exists and is bounded iff

$Ay = 0$ implies $y = 0$ (no null space)

If $Ay = 0$ for $y \neq 0$ then either

$Ax = b$ has an infinite # of solutions

$Ax = b$ then $A(x + \alpha y) = b$

OR

$Ax = b$ does not have a solution

b is not in the column space of A

7.3 3-D Null Space

SLIDE 38

Consider $\tilde{\sigma}$ defined by

$$u_\Gamma(\vec{x}) = 1 = \int_\Gamma \frac{1}{\|\vec{x} - \vec{x}'\|} \tilde{\sigma}(\vec{x}') d\Gamma' \qquad \vec{x} \in \Gamma$$

Then

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 0 = 2\pi\tilde{\sigma}(\vec{x}) + \int_\Gamma^{PV} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x}-\vec{x}'\|}\tilde{\sigma}(\vec{x}')d\Gamma'$$

$\tilde{\sigma}$ is in the Null space of $I+K$

$$(I+K)^{-1} \text{ is not bounded!!}$$

7.4 Fredholm Alternative

SLIDE 39

General Theorem
For $I+K$ either

$(I+K)\sigma = \Psi$ has an infinite # of solutions
OR
$(I+K)\sigma = \Psi$ has no solution

7.4.1 2D Example

SLIDE 40

Scaled Equations:

$$\frac{1}{\pi}\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \sigma(\vec{x}) + \frac{1}{\pi}\int_\Gamma^{PV} \frac{\partial}{\partial n_{\vec{x}}} \log\|\vec{x}-\vec{x}'\|\sigma(\vec{x}')d\Gamma'$$

For a solution to exist

$$\int_\Gamma \frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} d\Gamma = 0$$

2D Neumann Second Kind Integral equation
has a one-dimensional Null space.

7.4.2 Fixes

SLIDE 41

Add a point constraint
Fix u at some point
Force σ orthogonal to null space
Need the null space
May need to solve 1^{st} kind equation
Use SVD to solve singular system
Can be computationally expensive

8 Summary

SLIDE 42

Potential Representations
Monopole and Dipole Densities
Principle Value
Interior Neumann

Use Nystrom to Solve

Look at 2-D Problems

Fredholm Alternative

Connection to Linear Algebra