Nonzero Dirichlet boundary condition for finite elements

$$\frac{\partial^2 u}{\partial x^2} + f = 0 \qquad u|_0 = u|_1 = 0 \qquad \text{strong form}$$

$$a(u,g) + l(g) = 0 \quad \text{for all } g \in X \qquad \text{weak form}$$

$$a(u,g) = -\int_0^1 \frac{\partial u}{\partial x}\cdot\frac{\partial g}{\partial x}\,dx \qquad \text{where } u \in X$$

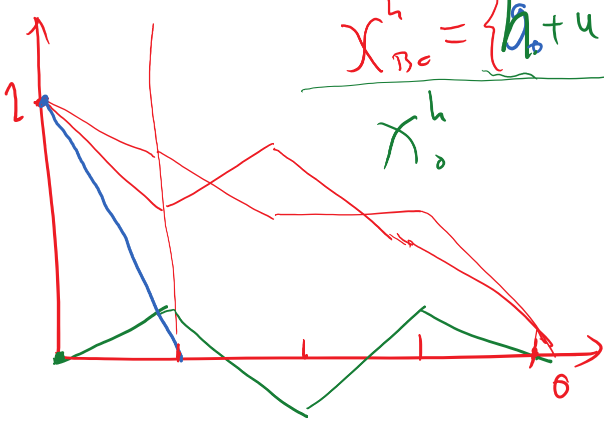
$$X = \{u,\ u \in H^1_{[0,1]},\ u|_0 = u|_1 = 0\}$$

$$u|_0 = 1,\quad u|_1 = 0$$

$$a(u,g) + l(g) = 0, \quad \text{for all } g \in X_0$$

$$X^h_{B.C} = \{b_0 + u,\ u \in X^h_0\} \quad \& \quad u \in X_{B.C}$$

X^h_0



$$u \in X_{B.C} = \{u,\ u \in H^1_{[0,1]}\ u|_0 = 1,\ u|_1 = 0\}$$

Affine space

$$g \in X_0 = \{u,\ u \in H^1_{[0,1]},\ u|_0 = 0,\ u|_1 = 0\}$$

$$a(u,g) + l(g) = 0, \quad \text{for all } g \in X^h_0 \subset X_0$$

$$\text{for some } u \in X^h_{B.C} \subset X_{B.C}$$

For Galerkin method X^h_0 need to be the homogeneous version of $X^h_{B.C}$

$$X_0^h = \{ u - v \ , \ u, v \in X^h_{B.C} \}$$

Nonzero Dirichlet boundary condition for finite elements

$$u = h_0 \cdot u_0 + \sum_{i=1}^{n-1} h_i u_i + h_n \cdot u_n$$

affine space

$$g = \sum_{j=1}^{n-1} h_j g_j$$

$$a(u,g) + l(g) = 0 \qquad \forall g$$

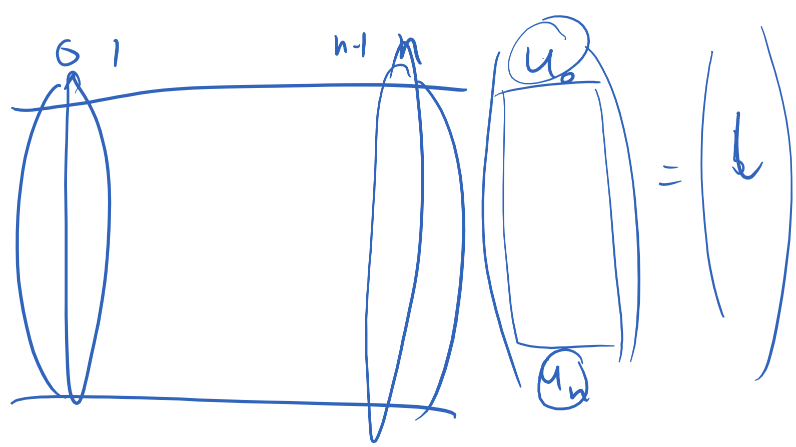
$$\Downarrow$$

$$a(u,h_j) + l(h_j) = 0 \qquad j = 1, \dots, n-1$$

$$u_0 a(h_0,h_j) + \sum_{i=1}^{n-1} u_i a(h_i,h_j) + u_n \cdot a(h_n,h_j) + l(h_j) = 0$$

$$A \cdot u + b = 0$$

$$A_{ij} = a(h_i,h_j)$$



Neumann boundary condition for finite elements

$$\frac{\partial^2 u}{\partial x^2} + f = 0 \qquad \frac{\partial u}{\partial x}\Big|_0 = 0 \qquad u\big|_1 = 0$$

$$\int_0^1 g\cdot\left(\frac{\partial^2 u}{\partial x^2} + f\right) dx = 0$$

$$= \boxed{g\cdot\frac{\partial u}{\partial x}\Big|_0^1} - \int_0^1 \frac{\partial g}{\partial x}\frac{\partial u}{\partial x}\, dx + \int_0^1 g\cdot f\, dx = 0$$

$$a = -\int_0^1 \frac{\partial g}{\partial x}\frac{\partial u}{\partial x}\, dx \qquad a(u,g) + l(g) = 0$$

$$l = \int_0^1 g\cdot f\, dx \qquad X = \{u \in H^1,\ u\big|_1 = 0\}$$

Zero Neumann B.C. Not enforce Anything

What if $\frac{\partial u}{\partial x} = 1$

$$-g\Big|_{x=0} - \int_0^1 \frac{\partial g}{\partial x}\frac{\partial u}{\partial x}\, dx + \int_0^1 g f\, dx = 0$$

Change in the weak form

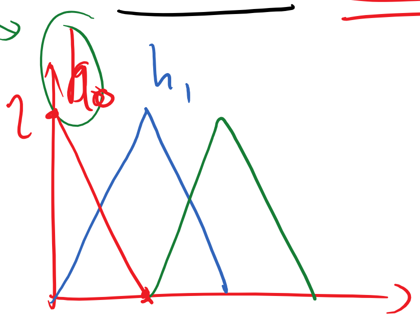
$$a(u,g) + l(g) = 0$$

1 for $g = h_0$

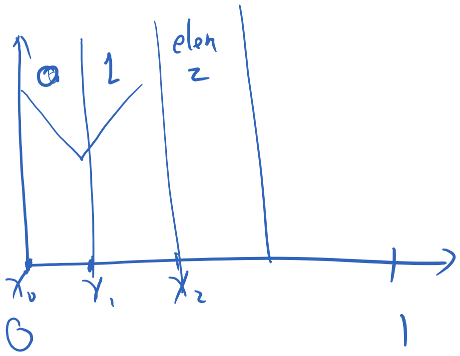
$$l(g) = -g\Big|_{x=0} + \int_0^1 g\cdot f\, dx$$

$$b = \begin{pmatrix} l(h_0) \\ l(h_1) \\ \vdots \\ \vdots \end{pmatrix}$$

$\neq 0$ at $x=0$



Element-by-element construction of finite element matrix and right hand side

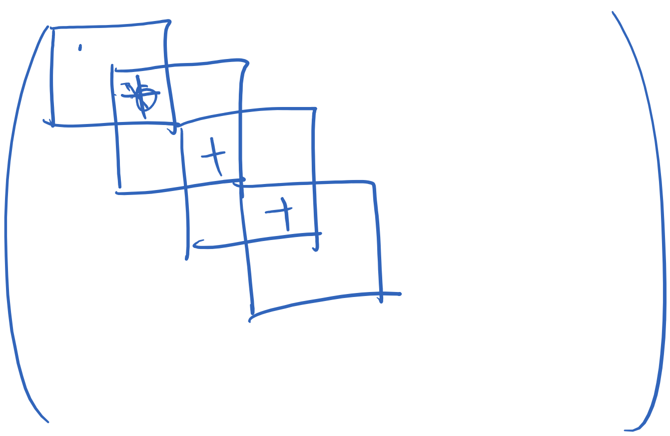


$$\underline{A_{ij}} = a(g_i, g_j) = -\int_0^1 \frac{\partial g_i}{\partial x}\cdot\frac{\partial g_j}{\partial x}$$

$$\underline{A^{elem_k}_{ij}} = \cdots - \int_{x_{k-1}}^{x_k} \frac{\partial g_i}{\partial x}\cdot\frac{\partial g_j}{\partial x}$$

2x2

$$\Delta x_k := x_{k+1} - x_k \qquad A^k_{k:k+1,\,k:k+1} = \begin{pmatrix} -\frac{1}{\Delta x_k} & \frac{1}{\Delta x_k} \\ \frac{1}{\Delta x_k} & -\frac{1}{\Delta x_k} \end{pmatrix}$$



Mixed boundary condition for finite elements

$$\frac{\partial u}{\partial x} + au = b \quad \text{at } x=0, \qquad u=0, \text{ at } x=1$$

$$\int_0^1 g\left(\frac{\partial^2 u}{\partial x^2} + f\right) dx = 0$$

$$= g\frac{\partial u}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial g}{\partial x}\cdot\frac{\partial u}{\partial x}\, dx + \int_0^1 g\cdot f\, dx = 0$$

$$\parallel$$

$$-g\cdot\frac{\partial u}{\partial x}\Big|_0$$

$$= \boxed{-g(b-au)\Big|_{x=0} - \int_0^1 \frac{\partial g}{\partial x}\cdot\frac{\partial u}{\partial x}\, dx + \int_0^1 g\cdot f\, dx = 0}$$

$$a(u,g) + l(g) = 0$$

Essential and natural boundary conditions

$\underbrace{\text{Essential}}$ → work on X

$\underbrace{\text{natural}}$ → work on a, ℓ l

Solving time-dependent problems in finite element

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 = \int_0^1 g\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) dx \qquad g(x) \text{ space only}$$

$$= \underbrace{\frac{d}{dt}\int_0^1 g \cdot u \, dx}_{m(g,u)} + \underbrace{\int_0^1 \frac{\partial g}{\partial x} \cdot \frac{\partial u}{\partial x} \, dx}_{a(g,u)}$$

$$\frac{d}{dt} m(g,u) + a(g,u) = 0 \qquad u(t) \in X_0, \quad g \in X_0$$

FEM: $X_0^h \subset X_0$

$u(t) \in X_0^h$

$g \in X_0^h$

$$\frac{d}{dt} m\left(h_j, \sum_{i=1}^n u_i(t) \cdot h_i\right) + a\left(h_j, \sum u_i h_i\right) = 0$$

$$\sum_{i=1}^n \frac{d}{dt}\left(u_i(t) \cdot m(h_j, h_i)\right) + \sum_{i=1}^n u_i a(h_j, h_i) = 0$$

$$\sum_{i=1}^n m(h_j, h_i) \frac{du_i}{dt} + \sum_{i=1}^n a(h_j, h_i) \cdot u_i = 0$$

$$M \frac{du}{dt} + Au = 0 \qquad \frac{du}{dt} = M^{-1}Au$$

$$M \frac{u^{(m+1)} - u^{(m)}}{\Delta t} + A \frac{u^{(m+1)} + u^{(m)}}{2} = 0$$

$$\left(\frac{M}{\Delta t} + \frac{A}{2}\right) u^{(m+1)} = \left(\frac{M}{\Delta t} - \frac{A}{2}\right) u^{(m)}$$