CFL Condition for advection equation

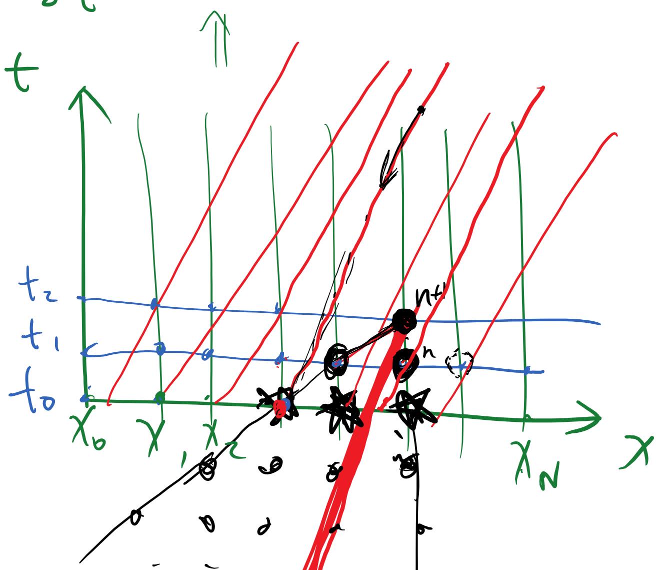
$$\frac{\partial u}{\partial t} + U\cdot\frac{\partial u}{\partial x} = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_i \simeq \frac{u_i - u_{i-1}}{\Delta x}$$

$$\frac{\partial u}{\partial t} \simeq \frac{u^{(n+1)} - u^{(n)}}{\Delta t}$$

$$\boxed{\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} + U\,\frac{u_i^{(n)} - u_{i-1}^{(n)}}{\Delta x} = 0}$$

1 2 2 3



CFL condition: Numerical D.o.D. must cover the analytical D.o.D.
physical

$\frac{\Delta t}{\Delta x}$: slope of numerical D.o.D.

$\frac{1}{U}$: slope of physical D.o.D.

$$\left|\frac{\Delta t}{\Delta x}\right| \leq \left|\frac{1}{U}\right| \qquad \Delta t \leq \frac{\Delta x}{|U|}$$

$$CFL = \frac{\Delta t}{\Delta x}|U| = \Delta t \max\left(\frac{|U|}{\Delta x}\right)$$

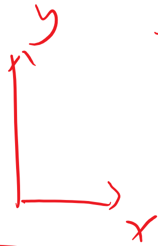
Finite difference for Poisson's equation (elliptic)

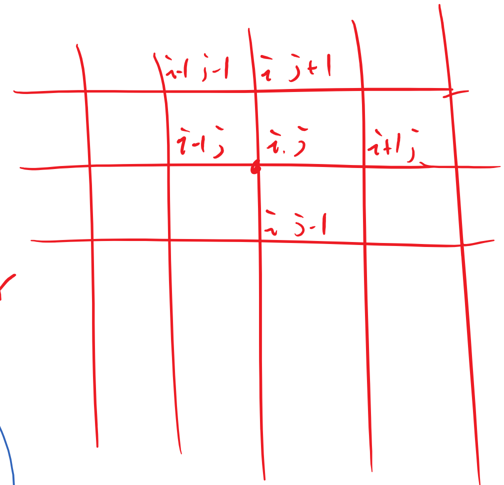
1. **How to approximate a derivative with arithmetic?**

2. **How much error does the approximation cause?**

3. How does the approximation error affect the solution?

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} + f = 0$$

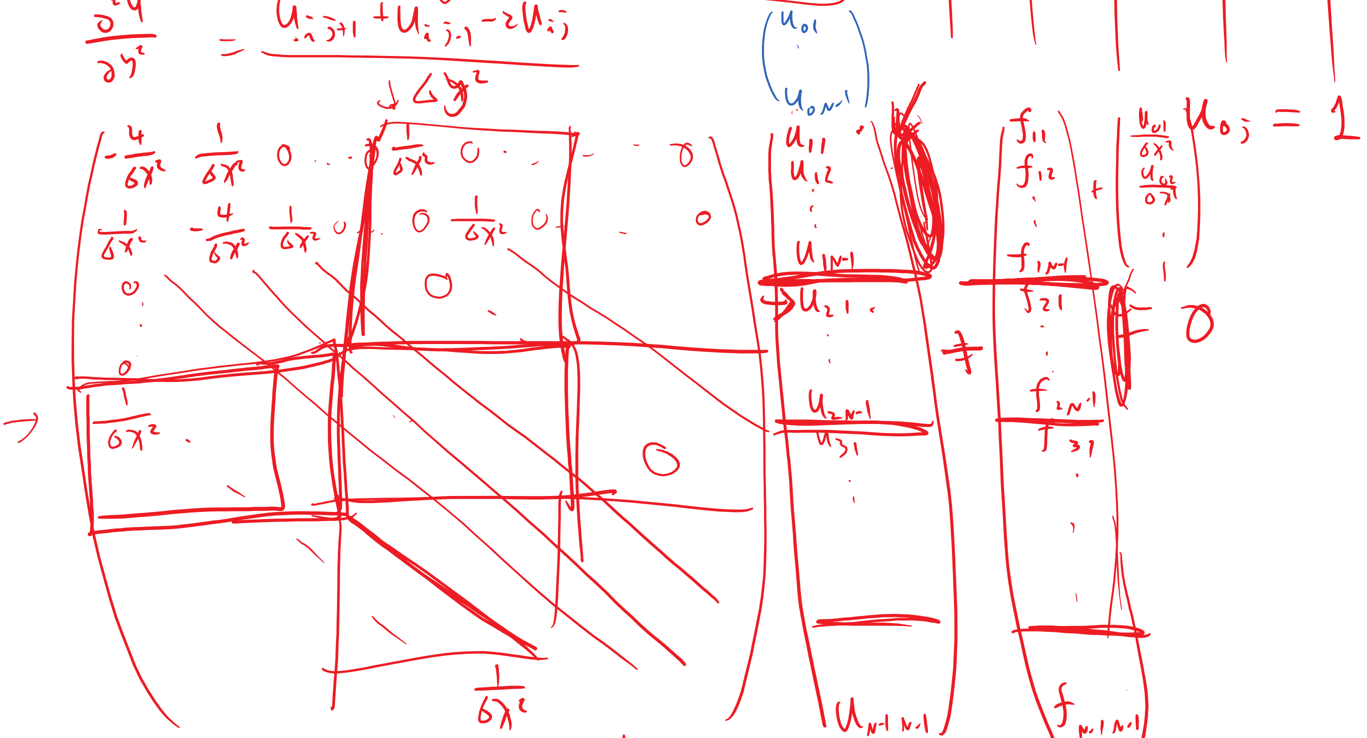




$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{ij} = \frac{u_{i+1\,j} + u_{i-1\,j} - 2u_{ij}}{\Delta x^2}$$

$$\boxed{\Delta x = \Delta y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i\,j+1} + u_{i\,j-1} - 2u_{ij}}{\Delta y^2}$$



Dirichlet B.C. $\qquad u|_{BC}$ = known value

Neuman B.C. $\qquad \nabla u \cdot n|_{BC}$ = known value

Robin B.C. $\qquad a u + b \nabla u \cdot n|_{BC} = \ldots$

Finite difference for Poisson's equation (elliptic)

$x = 0, \quad \underline{\dfrac{\partial u}{\partial x} = \cancel{0}\ 1}$

$$\underline{\left.\frac{\partial^2 u}{\partial x^2}\right|_{i=0,j}} \cong a\,u_{0,j} + b\,u_{1,j} + c\left(\left.\frac{\partial u}{\partial x}\right|_{0,j}\right) + d\,u_{2,j}$$

$$= a\,u_{0,j}$$

$$+ b\left(u_{0,j} + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + \frac{\partial^3 u}{\partial x^3}\cdot\frac{\Delta x^3}{6}\right)$$

$$+ c\,\frac{\partial u}{\partial x}$$

$$+ d\left(u_{0,j} + \frac{\partial u}{\partial x}2\Delta x + \frac{\partial^2 u}{\partial x^2}2\Delta x^2 + \frac{\partial^3 u}{\partial x^3}\cdot\frac{4\Delta x^3}{3}\cdots\right)$$

$a + b = 0$

$\rightarrow b\cdot\Delta x + c = 0$

$b\cdot\dfrac{\Delta x^2}{2} = 1$

$b = \dfrac{2}{\Delta x^2}$

$a = -\dfrac{2}{\Delta x^2}$

$c = -\dfrac{2}{\Delta x}$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{0,j} = \frac{2u_{1,j} - 2u_{0,j}}{\Delta x^2}$$

Finite difference for Poisson's equation (elliptic)

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2. **How much error does the approximation cause?**

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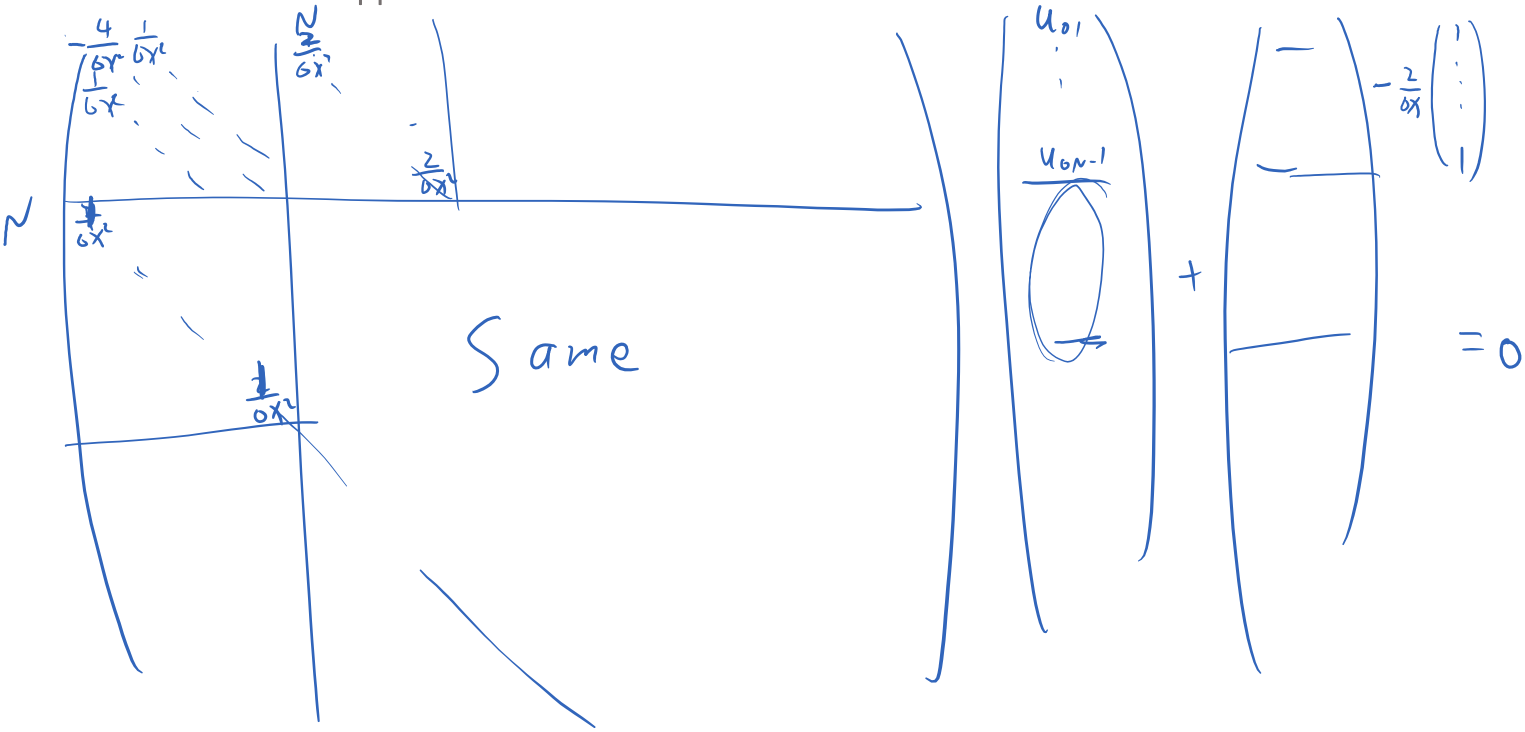

Eigenvalue problem -- creating the Matlab logo

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} u + \lambda u = 0$$