2.097J / 6.339J / 16.920J

Numerical Methods for Partial Differential Equations

Massachusetts Institute of Technology – Fall 2017

Handed Out: Oct. 16, 2017 | *Due: Oct. 30, 2017*

Finite Element Method - Deflection of Beams

Problem Statement

Small elastic deformation of isotropic solid materials in two dimensions under a force $\vec{F}(\vec{x})$ is governed by the linear elasticity equation

$$\frac{\partial \vec{\sigma_x}(\vec{u})}{\partial x} + \frac{\partial \vec{\sigma_y}(\vec{u})}{\partial y} + \vec{F} = 0\ , \quad x, y \in \Omega \tag{1}$$

where the deformation, a vector field $\vec{u} = (u_x, u_y)$, is the unknown to be solved. Both $\vec{\sigma_x} = (\sigma_{xx}, \sigma_{xy})$ and $\vec{\sigma_y} = (\sigma_{yx}, \sigma_{yy})$ are vector fields in the two-dimensional space. Together they are known as the Cauchy stress tensor. The Cauchy stress tensor depends on the deformation $\vec{u} = (u_x, u_y)$ in the following way:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yx} \\ \sigma_{yy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & 0 & 0 & \nu \\ 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ 0 & \frac{1-\nu}{2} & \frac{1-\nu}{2} & 0 \\ \nu & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial y} \end{pmatrix} \tag{2}$$

where E is the Young's modulus and ν is the Poisson ratio of the material. The material of choice is a titanium alloy with the pliable name Ti-6Al-2Nb-1Ta-0.8Mo. You can assume $E = 1.18 \times 10^{11}\ N/m^2$ and $\nu = 0.31$ for the reminder of the project. We will consider two types of boundary conditions on the domain boundary:

1) Dirichlet boundary conditions on $\partial\Omega_D$:

$$u_x = u_y = 0. \tag{3}$$

2) Free boundary conditions on $\partial\Omega_N$:

$$n_x \vec{\sigma_x} + n_y \vec{\sigma_y} = \vec{0}, \tag{4}$$

where $\vec{n} = (n_x, n_y)$ is the normal vector of the boundary of the domain.

Questions

1) (50 pts) Mathematical foundation

a. (20 pts) Derive the weak form of the problem considering only the free boundary condition (4). Show that the weak form is symmetric and positive semi-definite.

b. (10 pts) Discretize the weak form derived in Part a. in a square domain $x \in [-1, 1], y \in [-1, 1]$. Reduce the system of equations to a matrix form using the following bilinear basis functions. (Represent each entry of the matrix and the right hand side analytically in terms of the basis functions and the force (F_x, F_y)).

$$
\begin{aligned}
g_{--} &:= \frac{(1+x)(1+y)}{4}, \qquad g_{-+} := \frac{(1+x)(1-y)}{4}, \\
g_{+-} &:= \frac{(1-x)(1+y)}{4}, \qquad g_{++} := \frac{(1-x)(1-y)}{4}
\end{aligned}
\tag{5}
$$

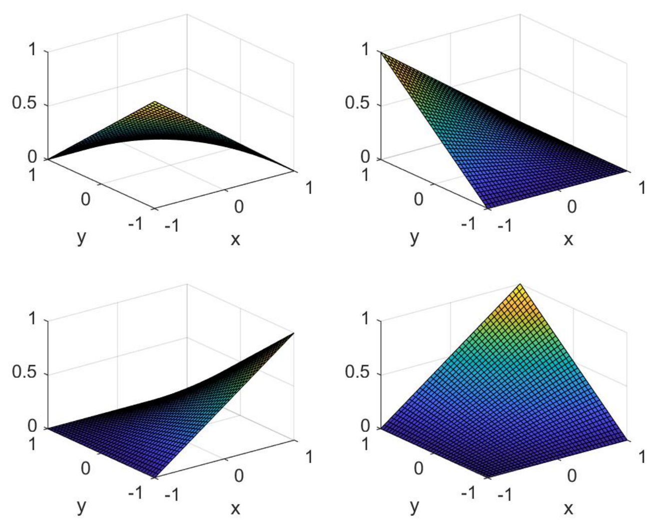


Figure 1: The basis functions in Equation (5).

c. (5 pts) If a two-dimensional Gauss quadrature is used to integrate each entry of the matrix in Part b., how many points will be necessary to compute an exact answer? Numerically compute the matrix for a single element, assuming the free boundary conditions for all boundaries.

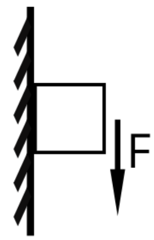
d. (5 pts) Now consider a square domain $x \in [x_0 - a, x_0 + a], y \in [y_0 - a, y_0 + a]$. How would the result of Part c. change?

e. (5 pts) How does the weak form derived in Part a. change if part of the domain boundary $\partial\Omega_D$ satisfies the Dirichlet boundary condition (3), while the rest of the domain boundary $\partial\Omega_N$ satisfies the free boundary condition (4)?

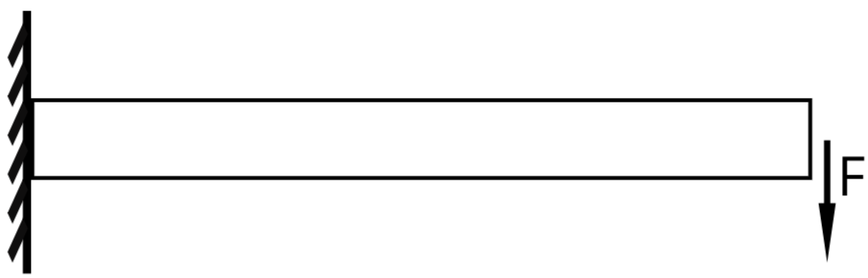
f. (5 pts) How does the discretization derived in Part b. and matrix computed in Part c. change if the left boundary, $x = -1, y \in [-1, 1]$, has the Dirichlet boundary condition while the rest of the domain boundary satisfies the free boundary condition?

2) (50 pts) Stepwise, build up the capability of your numerical solver.

a. (15 pts) Program a finite element solver for a single element, assuming Dirichlet boundary condition on the left boundary, $x = -1, y \in [-1, 1]$, and free boundary condition on all other three edges. The force exerted in the domain is a force in the negative y direction located at $x = 1, y = 0$. That is, $F_x \equiv 0$, and F_y is a Dirac delta function with magnitude -8×10^4 N at $(1, 0)$. Plot the computed deflec ion u_x and u_y of the domain.



b. (20 pts) Build on top of your solver to accommodate beam with a single row of N elements. The domain is now $\Omega = (x, y) : x \in [0, 2N], y \in [-1, 1]$. Your solution and test functions are in the space of continuous functions that are bilinear in each element $\{x \in [2i - 2, 2i], y \in [-1, 1], i = 1, \ldots, N\}$. Apply the same boundary conditions as in Part a., i.e., Dirichlet boundary condition (Equation 3) at $x = 0, y \in [-1, 1]$, and free boundary condition on other parts of the boundary. Apply the same concentrated external load $\vec{F}$ as in Part a. on the right end of the beam, at $x = 2N, y = 0$. Plot the deformation field u_x and u_y for $N = 2, 4$, and 10.



c. (15 pts) Make your solver capable of simulating a beam with multiple elements in the y direction. Consider the same geometry, boundary condition, and force as in Part b., but discretized using $M \times N$ by M smaller square elements for $M > 1$. Plot the deformation u_x and u_y for $(N = 10, M = 2), (N = 10, M = 4)$, and $(N = 10, M = 8)$. Also plot the deflection at $x = N, y = 0$ as a function of M.

3) (20 Bonus pts) Quantify the performance of your solver.

a. (5 pts) Estimate the order of accuracy of your solver, i.e. the convergence rate of the error. You may assume a high fidelity solution (using a large M) being the analytical solution to reference of your error.

b. (15 pts) Discuss and explore how the accuracy of the solver can be improved.