Finite elements for Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + f = 0$$

$$X_h = \{ f : \Omega \to \mathbb{R}, \text{ s.t. } f \text{ is continuous, piecewise linear} \}$$

$$u = \sum_{i=1}^{N} u_i g_i$$

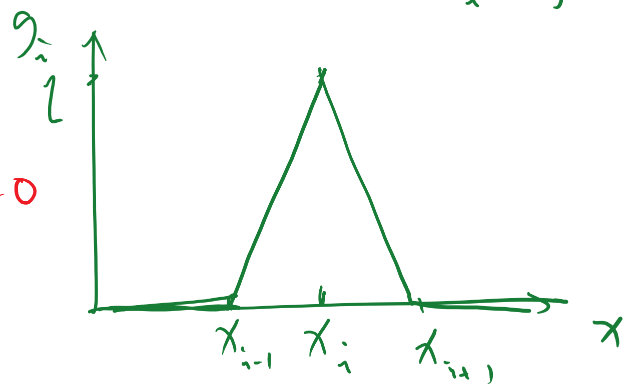
$$\left\langle g_j, \frac{\partial^2 u}{\partial x^2} + f \right\rangle = 0 \qquad j = 1, \dots, N$$

Basis:

$$g_i \in X_h, \text{ s.t. } g_i(x_i) = 1$$

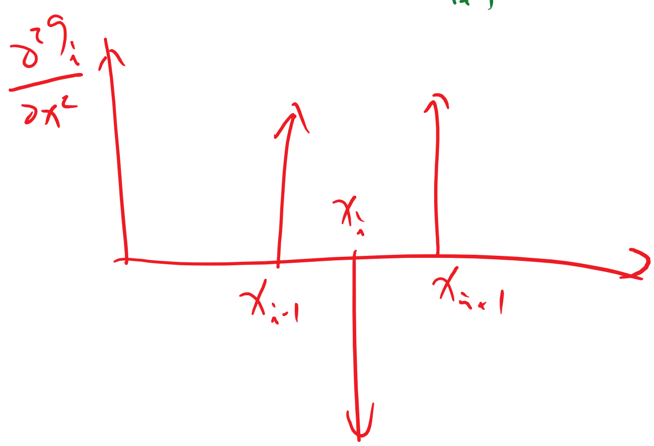
$$g_i(x_j) = 0 \quad j \neq i$$

$$\left\langle g_j, \frac{\partial^2 \sum u_i g_i}{\partial x^2} + f \right\rangle = 0$$



$$\sum u_i \left\langle g_j, \frac{\partial^2 g_i}{\partial x^2} \right\rangle + \langle g_j, f \rangle = 0$$

$$A \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = b$$



Finite elements for Poisson's equation: a weaker form

$$\frac{\partial^2 u}{\partial x^2} + f = 0 \; ; \quad u|_a = u|_b = 0$$

$$\int_\Omega g\left(\frac{\partial^2 u}{\partial x^2} + f\right) dx = 0 \qquad \int_a^b \frac{d(h \cdot g)}{dx} = h \cdot g\Big|_a^b$$

$$0 = \int_\Omega g \cdot f \, dx + g \cdot \frac{\partial u}{\partial x}\Big|_a^b - \int_\Omega \frac{\partial g}{\partial x}\frac{\partial u}{\partial x} dx \qquad \int_a^b h \frac{dg}{dx} + \int g \frac{dh}{dx} = hg\Big|_a^b$$

$$\Omega = [a, b] \qquad l \qquad a \qquad a \qquad h = \frac{\partial u}{\partial x}$$

weak form of the PDE/ODE

$$\boxed{a(u, g) + l(g) = 0} \quad \forall g \in X$$

$$a = -\int \frac{\partial g}{\partial x}\frac{\partial u}{\partial x} dx + g \cdot \frac{\partial u}{\partial x}\Big|_a^b \qquad a(u, g) \longrightarrow Au$$

$$l(g) \longrightarrow b$$

$$l = \int g f \, dx \qquad \left(\begin{array}{l} a(u_1 + u_2, g) = a(u_1, g) + a(u_2, g) \\ a(\lambda u_1, g) = \lambda \cdot a(u_1, g) \end{array} \right.$$

$$u = \sum u_i \cdot h_i \qquad a(u, g)$$

$$g = h_j \qquad = \sum u_i \, a(h_i, h_j)$$

$$l(g) = l(h_j)$$

$$\sum_{i=1}^N a(h_i, h_j) \cdot u_i + l(h_j) = 0 \qquad j = 1, \dots, N$$

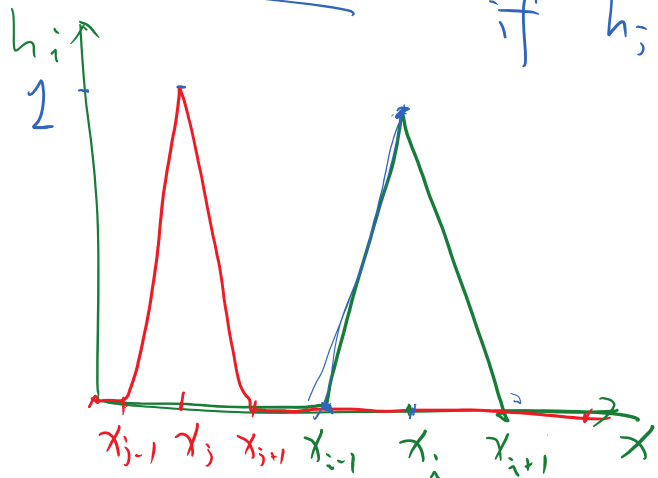
$$Au + b = 0 \qquad A_{ij} = a(h_i, h_j)$$

Finite elements for Poisson's equation: a weaker form

$$A_{ij} = a(h_i, h_j) = -\int_0^1 \frac{\partial h_i}{\partial x}\cdot\frac{\partial h_j}{\partial x} + h_j \frac{\partial h_i}{\partial x}\Big|_0^1$$

$$b_j = l(h_j) = \int_0^1 h_j \cdot f\, dx$$

$h_j \frac{\partial h_i}{\partial x}\Big|_0^1 = 0$ if $h_j = 0$ at Bndry.



$$\int_0^1 \frac{\partial h_i}{\partial x}\frac{\partial h_i}{\partial x}\, dx = \int_{x_{i-1}}^{x_i} \left(\frac{\partial h_i}{\partial x}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{\partial h_i}{\partial x}\right)^2 dx$$

$$= \left(\frac{1}{x_i - x_{i-1}}\right)^{\cancel{2}} \cancel{(x_i - x_{i-1})} + \left(\frac{1}{x_{i+1} - x_i}\right)^{\cancel{2}} \cancel{(x_{i+1} - x_i)}$$

$$\int_0^1 \frac{\partial h_i}{\partial x}\cdot\frac{\partial h_{i+1}}{\partial x}\, dx = \int_{x_i}^{x_{i+1}} -\frac{1}{x_{i+1} - x_i}\,\frac{1}{x_{i+1} - x_i}\, dx$$

$$= -\left(\frac{1}{x_{i+1} - x_i}\right)^{\cancel{2}} \cancel{(x_{i+1} - x_i)}$$

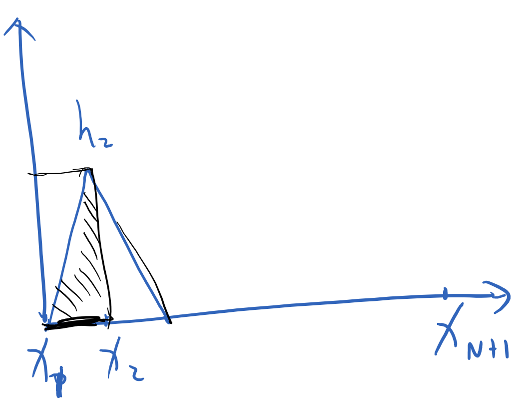
$f = 1$

$$b_j = \int_0^1 h_j\, dx$$

$$= \int_{x_{j-1}}^{x_j} h_j + \int_{x_j}^{x_{j+1}} h_j$$

$$= \frac{x_j - x_{j-1}}{2} + \frac{x_{j+1} - x_j}{2}$$

$$= \frac{x_{j+1} - x_{j-1}}{2}$$



Weak form in Sobolev spaces

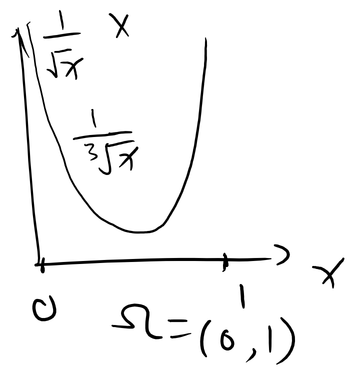
$$a(u,g) + l(g) = 0 \quad , \quad u \in X_u$$

$$g \in X_g$$

Galerkin method $\underline{X_u} = \underline{X_g}$

$$L^2_\Omega = \left\{ f : \int_\Omega f^2 dx < \infty \right\}$$

$$H^1_\Omega = \left\{ f : \int_\Omega f^2 dx < \infty , \int_\Omega \|\nabla \cdot f\|^2 dx < \infty \right\}$$



$$H^2_\Omega = \left\{ f : f \in H^1_\Omega , \int_\Omega \left\| \frac{\partial^2 f}{\partial x^2} \right\|^2 dx < \infty \right\}$$

$$\underline{a(u,g)} + \underline{l(g)} = 0 \qquad \boxed{\nabla \cdot \nabla u + f = 0}$$

$$\boxed{\begin{matrix} \underline{u \in H^1_\Omega} , & u|_{\partial\Omega} = 0 \\ \underline{g \in H^1_\Omega} & g|_{\partial\Omega} = 0 \end{matrix}}$$

$$\int_\Omega g \left(\nabla \cdot \nabla u + f \right) = 0$$

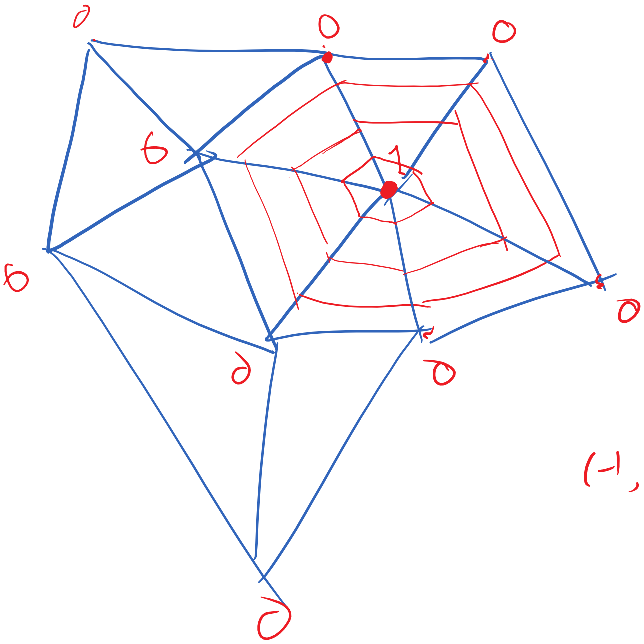
$$\underline{\int_\Omega \nabla \cdot (g \vec{h}) = \int_{\partial\Omega} \vec{n} \cdot (g \vec{h})}$$

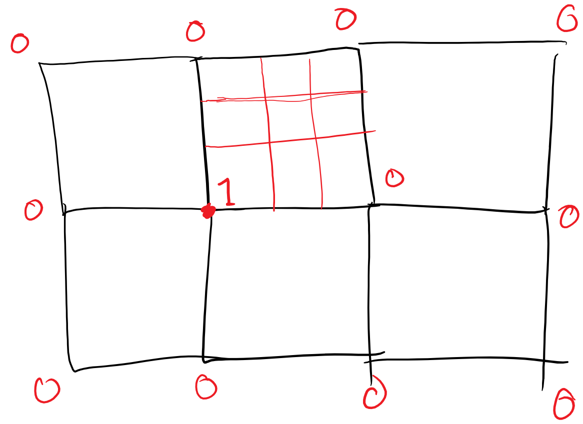
$$= \int_{\partial\Omega} g \, \vec{n} \cdot \nabla u \underbrace{- \int_\Omega \nabla g \cdot \nabla u}_{a(u,g)} + \underbrace{\int g \cdot f}_{l(g)} = 0$$

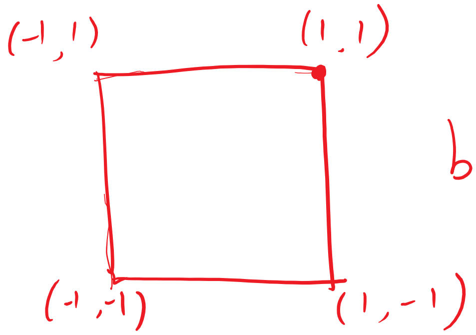
$$\int_\Omega g \, \nabla \cdot \vec{h} + \int \nabla g \cdot \vec{h} = \int_{\partial\Omega} g \, \vec{n} \cdot \vec{h}$$

$$\vec{h} = \nabla u$$

Finite elements in 2D and Gauss Quadrature







bilinear

$(x+1)(y+1)/4$

$(-x+1)(y+1)/4$

$(x+1)(-y+1)/4$

$(-x+1)(-y+1)/4$

$A_{ij} = a(h_i, h_j)$ e.g. $a(h_i, h_j) = \int \nabla h_i \cdot \nabla h_j \, dx$

$\boxed{\text{Gauss Quadrature}}$

$$\int_a^b f(x)\,dx \simeq \int_a^b f(x)\,\delta_i(x)\,dx$$

δ_i has weight w_i and location x_i

$$= \sum_{i=1}^{n} f(x_i)\cdot w_i$$