Convection Diffusion Equation: Limiting Cases

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = K \frac{\partial^2 u}{\partial x^2} + f$$

advection　　diffusion　　source

convection

1　$U = 0$, $f = 0$

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

heat equation

Parabolic equation

2. $U = 0$, $\frac{\partial u}{\partial t} = 0$

$$0 = K \cdot \frac{\partial^2 u}{\partial x^2} + f$$

Poisson's equation

Elliptic equation

3　$K = 0$. $f = 0$

$$\frac{\partial u}{\partial t} + U \cdot \frac{\partial u}{\partial x} = 0$$

linear advection equation

Hyperbolic equation

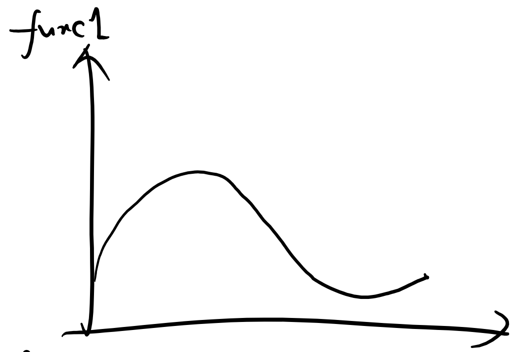
Limiting Case 1: Parabolic Equation and its Fourier Analysis

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Periodic B.C.

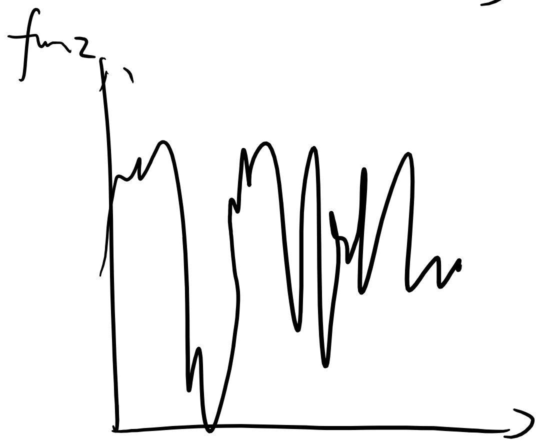
$$u(2\pi) = u(0)$$

$$u(x, t=0) = u_0(x)$$



$$u_0(x) = \sum_{k=-\infty}^{\infty} \hat{u}_{k0}\, e^{jkx}$$

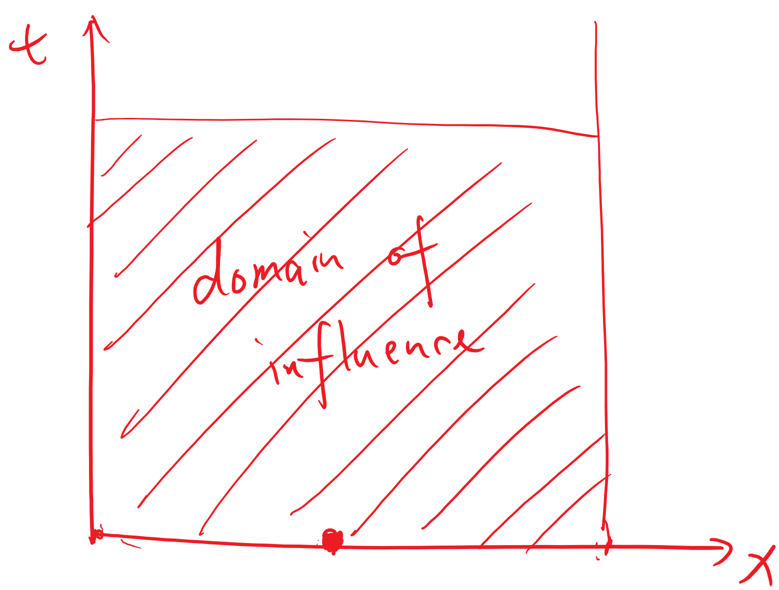
$$u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t)\, e^{jkx}$$

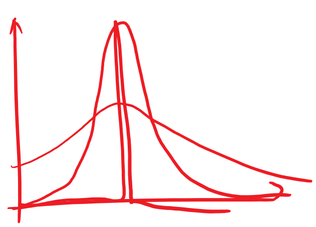


$$\left(\frac{\partial u}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{d\hat{u}_k}{dt} e^{jkx}\right)$$

$$= k\left(\frac{\partial^2 u}{\partial x^2} = \sum_{k=-\infty}^{\infty} \hat{u}_k(-k^2)\, e^{jkx}\right)$$

$$\frac{d\hat{u}_k}{dt} = -k\, k^2\, \hat{u}_k \quad \text{for all } k$$





Limiting Case 2: Elliptic Equation and its Fourier Analysis

$$\frac{\partial^2 u}{\partial x^2} + f = 0 \quad \left(\frac{\partial u}{\partial \tau}\right)$$

$$\left(f = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} \right)$$

$$u = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikx}$$

$$\left(\frac{\partial^2 u}{\partial x^2} = \sum_{k=-\infty}^{\infty} \hat{u}_k (-k^2) e^{ikx} \right)$$

$$\hat{f}_k - k^2 \hat{u}_k = 0 \quad \text{for all } k$$

$$\hat{u}_k = \frac{1}{k^2} \hat{f}_k$$

Elliptic

Limiting Case 3: Hyperbolic Equation and its Fourier Analysis

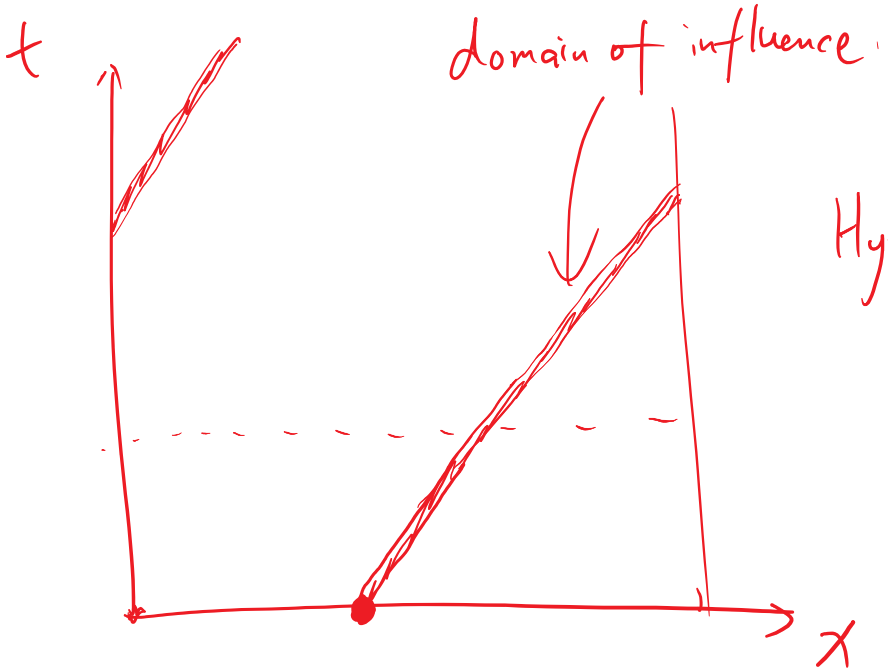
$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0 \qquad u(x, t=0) = u_0(x)$$

$$u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) \cdot e^{jkx}$$

$$\left(\frac{\partial u}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{d\hat{u}_k}{dt} e^{jkx}\right)$$

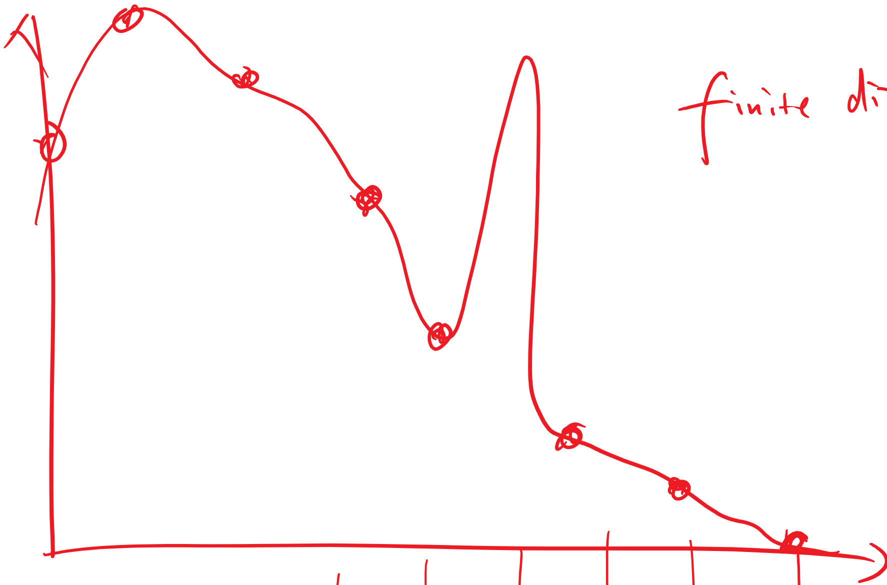
$$+U\left(\frac{\partial u}{\partial x} = \sum_{k=-\infty}^{\infty} \hat{u}_k \cdot jk\, e^{jkx}\right) = 0$$

$$\frac{d\hat{u}_k}{dt} + U \cdot jk\, \hat{u}_k = 0$$

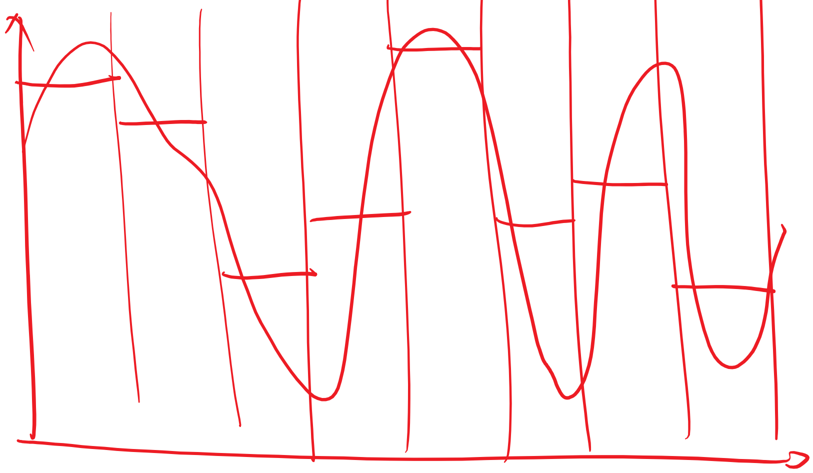


Hyperbolic equation

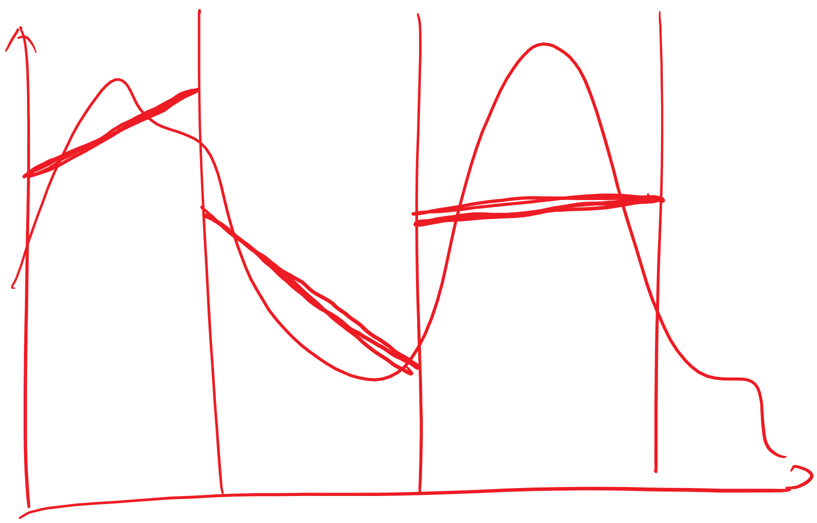
Finite difference, Finite Volume, and Finite Element



finite diffence



finite volume

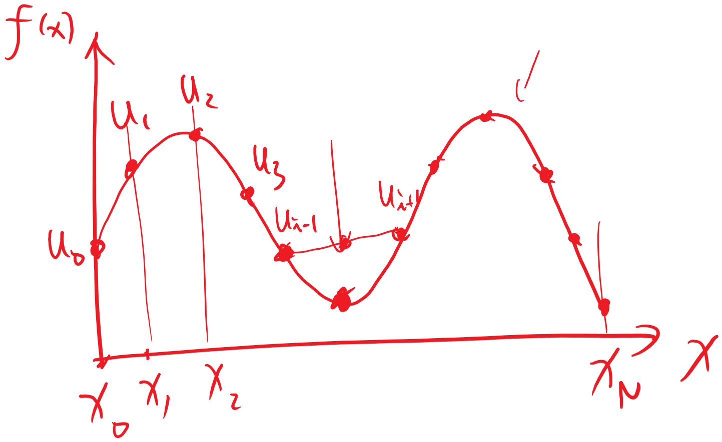


finite element

Finite difference for heat equation (parabolic)

1. **How to approximate a derivative with arithmetic?**
2. How much error does the approximation cause?
3. How does the approximation error affect the solution?

$$\frac{\partial u}{\partial t} = k \cdot \underbrace{\frac{\partial^2 u}{\partial x^2}}$$



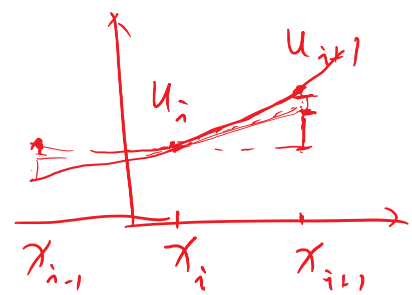
$$\left.\frac{\partial u}{\partial t}\right|_{x=x_i} = ?\left(u\big|_{x=x_0},\ u\big|_{x=x_1},\ \dots\ u\big|_{x=x_N} \right)$$

Idea from curvature:

$$\frac{\partial^2 u}{\partial x^2} > 0 \quad \overset{\sim}{\Longrightarrow} \quad \frac{u_{i+1}+u_{i-1}}{2} > u_i$$

$$< 0 \qquad \frac{2}{\Delta x^2} \qquad <$$

$$(*) \quad \boxed{\frac{\partial^2 u}{\partial x^2} \simeq ? \left(\frac{u_{i+1}+u_{i-1}}{2} - u_i \right)}$$



$$\rightarrow u_{i+1} = u(x_{i+1}) = u(x_i + \Delta x)$$

$$(\text{Taylor}) \qquad = \underbrace{u(x_i)}_{u_i} + \Delta x \left.\frac{\partial u}{\partial x}\right|_{x_i} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \dots$$

$$\rightarrow u_{i-1} = u(x_{i-1}) = u(x_i - \Delta x)$$

$$= u_i - \Delta x \left.\frac{\partial u}{\partial x}\right|_{x_i} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \dots$$

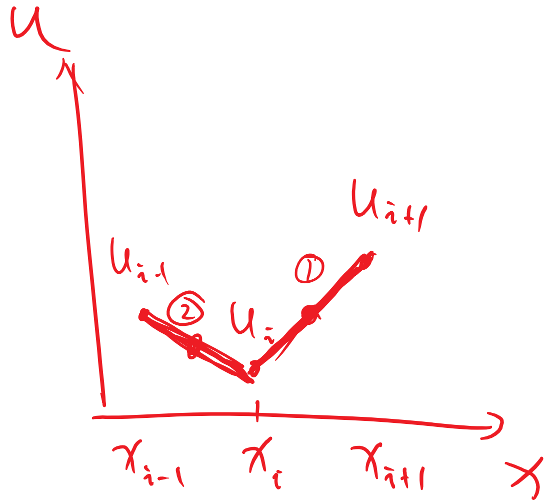
$$\left(\frac{u_{i+1}+u_{i-1}}{2} - u_i \right) = u_i - u_i + 0 + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \dots$$

Finite difference for heat equation (parabolic)

1. **How to approximate a derivative with arithmetic?**
2. How much error does the approximation cause?
3. How does the approximation error affect the solution?

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2}$$

$$= \frac{\overset{\textcircled{1}}{\dfrac{u_{i+1} - u_i}{\Delta x}} - \overset{\textcircled{2}}{\dfrac{u_i - u_{i-1}}{\Delta x}}}{\Delta x}$$



$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

with approximation :

$$\frac{du_i}{dt} = k \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2}$$

$i = 1, 2, 3, \ldots, N-1$

need u_0 & u_N

as B.C.

need $u_1, \ldots, u_{N-1}$

at $t = 0$

as I.C.