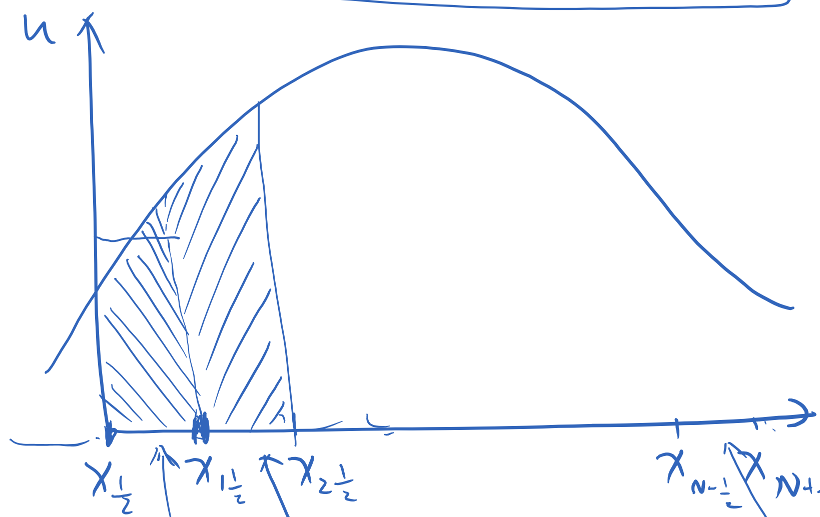
Finite volume -- a discretely conservative method

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$$\boxed{\frac{d}{dt}\int_a^b u\,dx + f(u(b)) - f(u(a)) = 0}$$ integral form



$u_1 \cdot \Delta x_1$ $\quad u_2 \Delta x_2$ $\quad u_N \Delta x_N$

$$= \int_{x_{\frac12}}^{x_{1+\frac12}} u\,dx \qquad\qquad u_i := \frac{1}{\Delta x_i}\int_{x_{i-\frac12}}^{x_{i+\frac12}} u\,dx$$

$$\frac{d}{dt}u_i = \frac{1}{\Delta x_i}\frac{d}{dt}\int_{x_{i-\frac12}}^{x_{i+\frac12}} u\,dx = \frac{\underbrace{f(u(x_{i-\frac12}))}_{f_{i-\frac12}} - \underbrace{f(u(x_{i+\frac12}))}_{f_{i+\frac12}}}{\Delta x_i} \qquad ①$$

Finite volume scheme

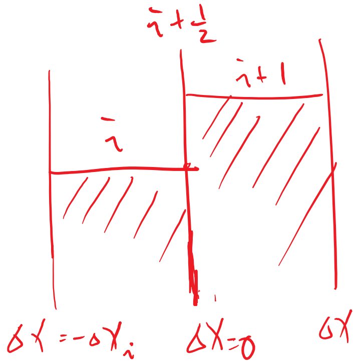
1: ① ✓

2: Flux reconstruction: $f(u(x_{i-\frac12})) \simeq F(u_{i-1}, u_i)$

$$\frac{d}{dt}\sum u_i \Delta x_i = \sum_i f_{i-\frac12} - f_{i+\frac12} = f_{\frac12} - f_{N+\frac12}$$ FV is discretely conservative

$$\frac{d}{dt}\sum u_i \Delta x_i = \sum_i f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}} = f_{\frac{1}{2}} - f_{N+\frac{1}{2}}$$ FV is discretely conservative

Flux reconstruction -- central flux scheme



$$\delta x = x - x_{i+\frac{1}{2}} \qquad x = x_{i+\frac{1}{2}} + \delta x$$

$$u(x_{i+\frac{1}{2}}+\delta x) = u_{i+\frac{1}{2}} + \delta x \frac{\partial u}{\partial x} + \frac{\delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \cdots$$

$$u_i = \frac{1}{\delta x_i}\int_{-\delta x_i}^{0} u(x_{i+\frac{1}{2}}+\delta x)\, d\delta x = u_{i+\frac{1}{2}} - \frac{\delta x_i^2}{2\delta x_i}\frac{\partial u}{\partial x} + \frac{\delta x_i^3}{6\delta x_i}\frac{\partial^2 u}{\partial x^2} + \cdots$$

$$u_{i+1} = \frac{1}{\delta x_{i+1}}\int_0^{\delta x_{i+1}} u(x_{i+\frac{1}{2}}+\delta x)\, d\delta x = u_{i+\frac{1}{2}} + \frac{\delta x_{i+1}^2}{2\delta x_{i+1}}\frac{\partial u}{\partial x} + \cdots$$

$$a u_i + b\cdot u_{i+1} = u_{i+\frac{1}{2}}$$

$$\begin{cases} a+b=1 \\ -a\cdot\delta x_i + b\cdot\delta x_{i+1} = 0 \end{cases}$$

if $\delta x_i = \delta x_{i+1}$

$$a = b = \frac{1}{2}$$

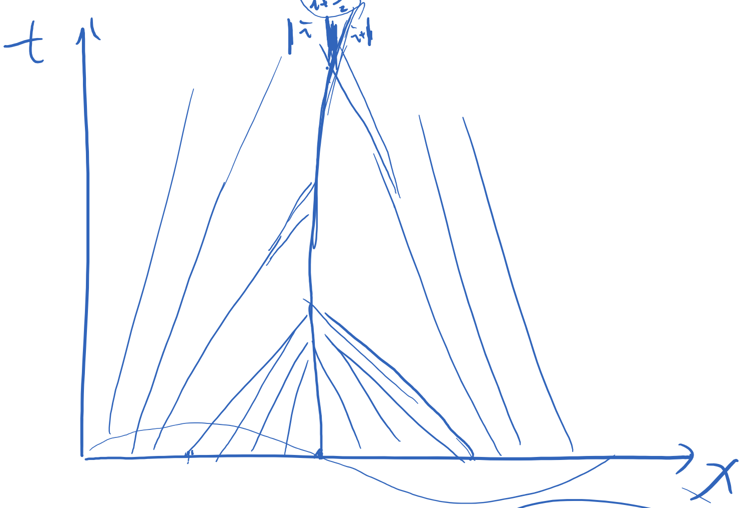
$$a = \frac{\delta x_{i+1}}{\delta x_i + \delta x_{i+1}} \qquad b = \frac{\delta x_i}{\delta x_i + \delta x_{i+1}}$$

$$f_{i+\frac{1}{2}} = \begin{cases} f(a u_i + b u_{i+1}) \\ a\cdot f(u_i) + b\cdot f(u_{i+1}) \end{cases}$$

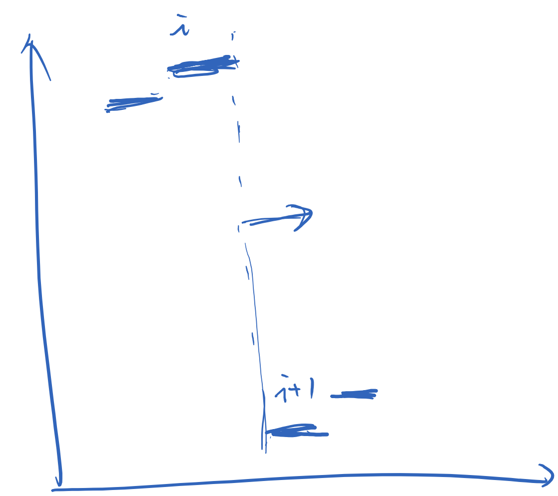
$$u = \sin(2\pi x)$$

$$u_i = \frac{1}{\delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sin(2\pi x)\, dx = \frac{1}{2\pi\,\delta x_i}\left(\cos^{2\pi} x_{i-\frac{1}{2}} - \cos^{2\pi} x_{i+\frac{1}{2}}\right)$$

Flux reconstruction -- upwind scheme



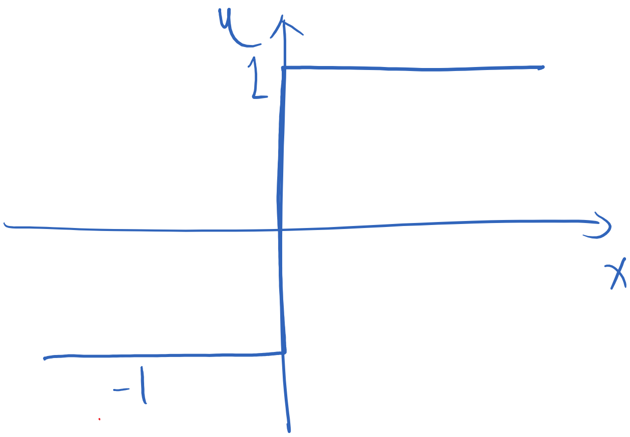
$$\frac{\partial f}{\partial u} = u$$

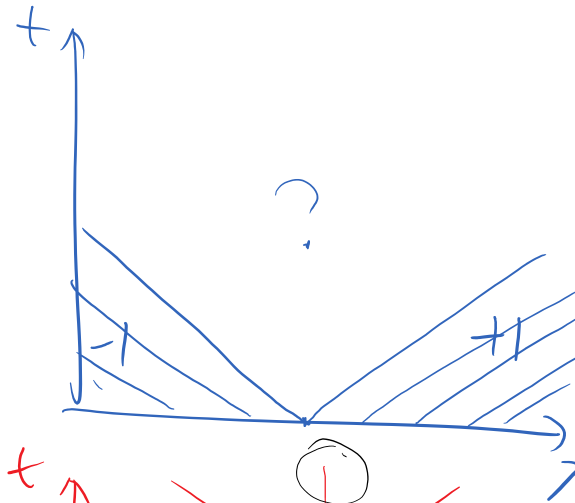


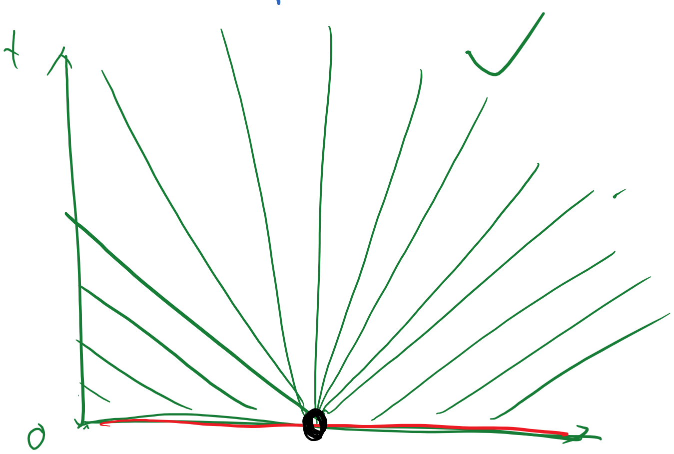
$$\frac{\Delta f}{\Delta u} = \frac{f(u_i) - f(u_{i+1})}{u_i - u_{i+1}}$$

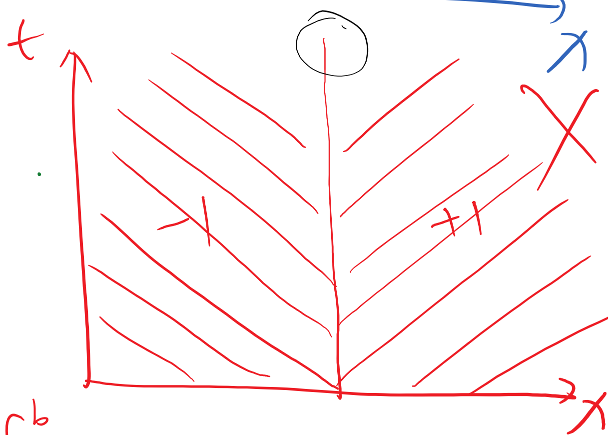
or $\left.\frac{df}{du}\right|_{u_i}$ if $u_i = u_{i+1}$

Non-unique solution and the entropy condition









$$\frac{d}{dt}\int_a^b u\,dx = f(a) - f(b) \quad ?$$

$$f(x) = \frac{u^2(x)}{2} \equiv \frac{1}{2}$$