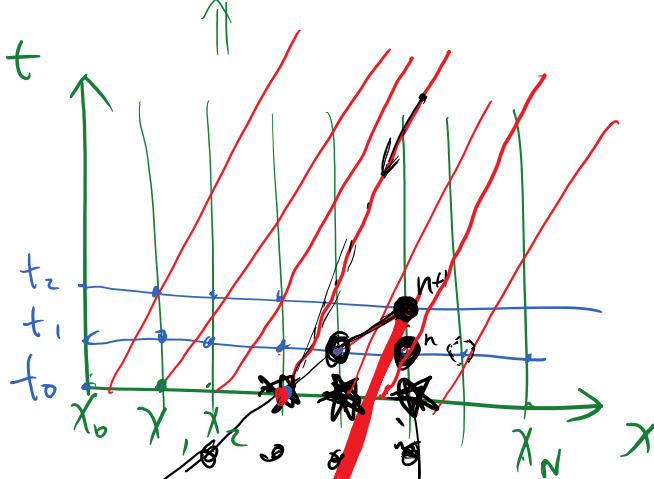


CFL Condition for advection equation

$$\frac{\partial u}{\partial t} + U \cdot \frac{\partial u}{\partial x} = 0$$



CFL condition: Numerical D.o.D.

must cover the analytical D.o.D.
physical

$\frac{\Delta t}{\Delta x}$: slope of numerical D.o.D.

$\frac{1}{U}$: slope of physical D.o.D.

$$\left| \frac{\Delta t}{\Delta x} \right| \leq \left| \frac{1}{U} \right| \quad \Delta t \leq \frac{\Delta x}{|U|}$$

$$\begin{aligned} CFL &= \frac{\Delta t}{\Delta x} |U| \\ &= \Delta t \max \left(\frac{|U|}{\Delta x} \right) \end{aligned}$$

$$\left. \frac{\partial u}{\partial x} \right|_i \approx \frac{u_i - u_{i-1}}{\Delta x}$$

$$\frac{\partial u}{\partial t} \approx \frac{u^{(n+1)} - u^{(n)}}{\Delta t}$$

$$\frac{u_i^{(n+1)} - u_i^{(n)}}{\Delta t} + U \frac{u_i^{(n)} - u_{i-1}^{(n)}}{\Delta x} = 0$$

\uparrow \uparrow \uparrow \uparrow
 (2) 2 2 3

Finite difference for Poisson's equation (elliptic)

1. How to approximate a derivative with arithmetic?
2. How much error does the approximation cause?
3. How does the approximation error affect the solution?

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} + f = 0$$

$\downarrow \quad \downarrow \quad \downarrow$

$$\frac{\partial^2 u}{\partial x^2} \Big|_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{\Delta y^2}$$

$\Delta x = \Delta y$

$$\begin{bmatrix} -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 & \dots & \frac{1}{\Delta x^2} & 0 & \dots & 0 \\ \frac{1}{\Delta x^2} & -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 & 0 & \frac{1}{\Delta x^2} & 0 & \dots \\ 0 & \frac{1}{\Delta x^2} & -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 & 0 & \frac{1}{\Delta x^2} & 0 \\ \vdots & 0 & \frac{1}{\Delta x^2} & -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 & 0 & \frac{1}{\Delta x^2} \\ 0 & 0 & 0 & \frac{1}{\Delta x^2} & -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 & 0 \\ \vdots & 0 & 0 & \frac{1}{\Delta x^2} & \frac{1}{\Delta x^2} & -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\Delta x^2} & \frac{1}{\Delta x^2} & -\frac{4}{\Delta x^2} & \frac{1}{\Delta x^2} \end{bmatrix}$$

$$\begin{pmatrix} u_{0,1} \\ u_{0,2} \\ \vdots \\ u_{0,N-1} \\ u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{1,N-1} \\ u_{2,1} \\ u_{2,2} \\ \vdots \\ u_{2,N-1} \\ \vdots \\ u_{N-1,1} \\ u_{N-1,2} \\ \vdots \\ u_{N-1,N-1} \end{pmatrix}$$

$$\begin{pmatrix} f_{1,1} \\ f_{1,2} \\ \vdots \\ f_{1,N-1} \\ f_{2,1} \\ f_{2,2} \\ \vdots \\ f_{2,N-1} \\ \vdots \\ f_{N-1,1} \\ f_{N-1,2} \\ \vdots \\ f_{N-1,N-1} \end{pmatrix} + \begin{pmatrix} \frac{u_{0,1}}{\Delta x^2} \\ \frac{u_{0,2}}{\Delta y^2} \\ \vdots \\ \frac{u_{0,N-1}}{\Delta y^2} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} u_{0,j} = 1$$

Dirichlet B.C.

Neuman B.C.

Robin B.C.

$u|_{\partial C} = \text{known value}$

$\nabla u \cdot n|_{\partial C} = \text{known value}$

$a u + b \nabla u \cdot n|_{\partial C} = \dots$

Finite difference for Poisson's equation (elliptic)

$$x=0, \quad \frac{\partial u}{\partial x} = 0 \quad \downarrow$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} \Big|_{i=0,j} &\approx a u_{0,j} + b u_{1,j} + c \left(\frac{\partial u}{\partial x} \Big|_{0,j} \right) + d u_{2,j} \\ &= a u_{0,j} + b \left(u_{0,j} + \frac{\partial u}{\partial x} \Delta x + \frac{\partial^2 u}{\partial x^2} \frac{\Delta x^2}{2} + \frac{\partial^3 u}{\partial x^3} \frac{\Delta x^3}{6} \right) \\ &\quad + c \left(\frac{\partial u}{\partial x} \right) + d \left(u_{0,j} + \frac{\partial u}{\partial x} 2\Delta x + \frac{\partial^2 u}{\partial x^2} 2\Delta x^2 + \frac{\partial^3 u}{\partial x^3} \frac{4\Delta x^3}{3} \right) \\ a+b &= 0 \end{aligned}$$

$$\rightarrow b \cdot \Delta x + c = 0$$

$$b \cdot \frac{\Delta x^2}{2} = 1$$

$$\frac{\partial^2 u}{\partial x^2} \Big|_{0,j} = \frac{2 u_{1,j} - 2 u_{0,j}}{\Delta x^2}$$

$$b = \frac{2}{\Delta x^2}$$

$$a = -\frac{2}{\Delta x^2}$$

$$c = -\frac{2}{\Delta x}$$

Finite difference for Poisson's equation (elliptic)

1. How to approximate a derivative with arithmetic?
2. How much error does the approximation cause?
3. How does the approximation error affect the solution?

The image contains handwritten notes illustrating the finite difference approximation for Poisson's equation. On the left, a 3x3 stencil is shown with a central node and four surrounding nodes. The stencil is labeled with the following coefficients:

- Top-left: $-\frac{4}{6x^2}$
- Top: $\frac{1}{6x^2}$
- Top-right: $\frac{1}{6x^2}$
- Left: $\frac{1}{6x^2}$
- Center: $\frac{2}{6x^2}$
- Right: $\frac{1}{6x^2}$
- Bottom-left: $\frac{1}{6x^2}$
- Bottom: $\frac{1}{6x^2}$
- Bottom-right: $\frac{1}{6x^2}$

The word "Same" is written in the center of the page. On the right, a matrix equation is shown, representing the finite difference approximation of Poisson's equation:

$$\begin{pmatrix} u_{0,1} \\ \vdots \\ u_{0,N-1} \end{pmatrix} + \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix} - \frac{2}{6x^2} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

Eigenvalue problem -- creating the Matlab logo

$$\begin{pmatrix} A \end{pmatrix} u + \lambda u = 0$$