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Integral Equations

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6 lectures
1 project: posted Nov 1, due Nov 17
1 problem session: after Nov 1, TBD
perhaps Monday Nov 6, 7-8pm?

6 lectures

1. Intro to IEs, numerical sol^n via projection methods
2. Numerical Integration
3. Non-singular 1^{st} & 2^{nd} kind IEs, Nyström method
4. Fredholm theory
5. Green's thm & different IE formulations for Laplace
6. Fast algorithms for IEs: FMM, pFFT

maybe switch

[NB: notes online are more complete & serve to complement material of my lectures]

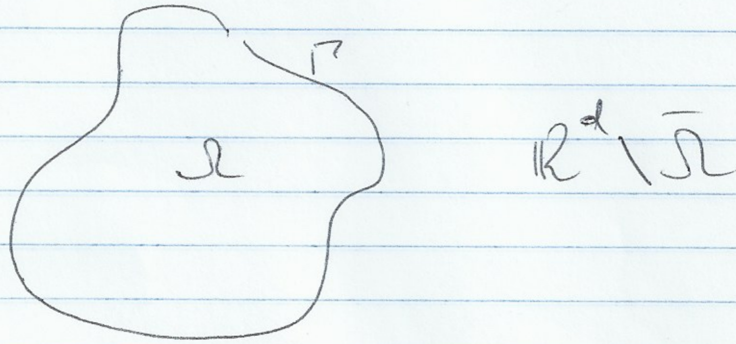
Lecture 1: Intro to integral equations (IEs)

- Many linear DEs can be reformulated as IEs (e.g., Maxwell, Helmholtz, Laplace, Lin. Elasticity)
- Restrict computation domain to boundary, hence reduce dimension by 1.
- Particularly useful for exterior problems where comp. dom. is ∞-te.
- Automatically satisfy radiation conditions at ∞
- Useful when only bdy. sol^n required, e.g. drag on aircraft wing.

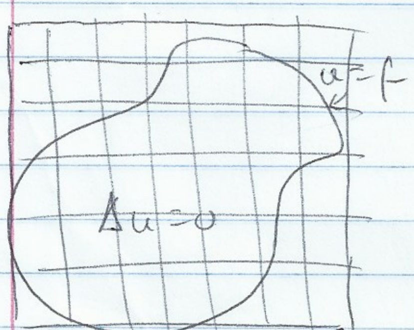
Consider the following BVPs for Laplace's eqn (steady heat flow; electrostatics; inviscid, irrotational flow).

Interior Dirichlet: $\Delta u(x) = 0, \quad x \in \Omega$
$u(x) = f(x), \quad x \in \Gamma := \partial\Omega$

Exterior Dirichlet: $\Delta u(x) = 0, \quad x \in \mathbb{R}^d \setminus \bar{\Omega}, \quad d = 2,3$
$u(x) = f(x), \quad x \in \Gamma$

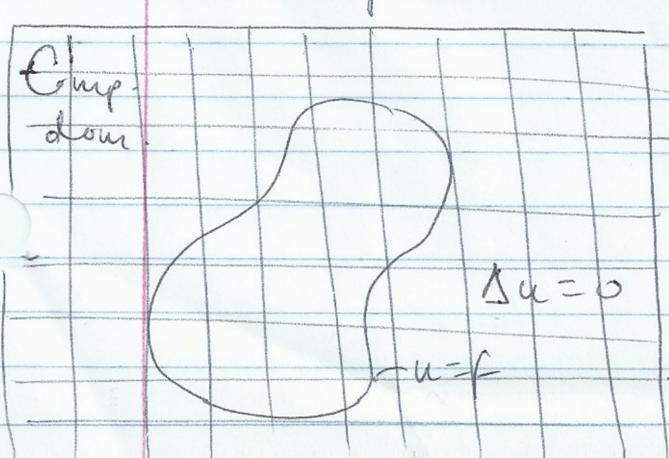


Interior problem



Straightforward to solve via FDM or FEM. Impose BC on edge elements.

Exterior problem.



Problem domain is ∞ te!
Must impose extra BC at ∞ to ensure sol^n decays.
Radiation condition:
$$\lim_{|x|\to\infty} |u(x)| < \infty, \quad 2D \quad (\text{i.e., bdd.})$$
$$|u(x)| = O\left(\frac{1}{|x|}\right), \quad 3D.$$
approximate radiation cond^n somehow

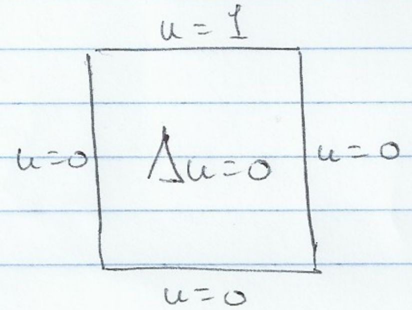
In FDM/FEM, must truncate mesh & approx. rad. cond. Not ideal.

IEs avoid this challenge. They satisfy the rad. cond. by <u>construction</u>.

In both int. & ext. problems, IEs reduce dim. by 1. So useful for interior as well!

"Rough" IE derivation via point-source approach

Consider int. Laplace problem for square.



Know from electrostatics, can represent (roughly) the potential due to a charge dist^n as sum of pt. charge potentials, i.e.,

$$u(x) \simeq \sum_{i=1}^{n} \alpha_i G(x, y_i), \quad x \in \Omega, \ y_i \in \mathbb{R}^2 \setminus \overline{\Omega} \ (\text{s.t. } y_i \neq x)$$

where

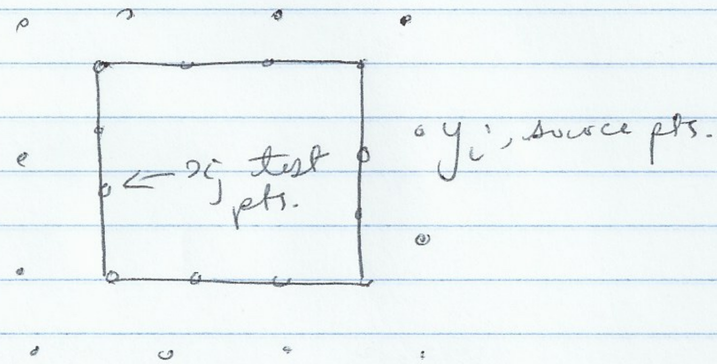
$$G(x, y_i) = \begin{cases} -\frac{1}{2\pi} \log|x - y_i|, & 2D \\ \frac{1}{4\pi} \frac{1}{|x - y_i|}, & 3D \end{cases}$$

is the potential at x due to a pt. charge at y_i.

[Ex.: check that these satisfy $\Delta G = 0$]

G is the "Green's func^n" for Laplace. [More later].

Locate pt. sources around square to avoid $x = y_i$.



To determine α_i, choose test pts. $x_j \in \Gamma$ where we enforce BC.

So for each j, have that

$$u(x_j) = \sum_{i=1}^{n} \alpha_i G(x_j, y_i), \quad j = 1, \dots, m$$

Choosing $m = n \longrightarrow$ square system of eq^{n}s.

$$\underbrace{\begin{pmatrix} G(x_1, y_1) & G(x_1, y_2) & \cdots & G(x_1, y_n) \\ \vdots & & & \\ G(x_n, y_1) & & \cdots & G(x_n, y_n) \end{pmatrix}}_{A:\ \text{Green's fn. evaluations}} \underbrace{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}}_{\text{unknowns}} = \underbrace{\begin{pmatrix} u(x_1) \\ \vdots \\ u(x_n) \end{pmatrix}}_{\text{~~known from BC~~ bdy data.}}$$

NB

- ~~Denses~~ Dense $n \times n$ matrix – all charges "talk" to each other.
- If order charges s.t. i^{th} test pt. is nearest i^{th} charge, then A_{ii} larger than A_{ij} $\forall i, j$, $i \neq j$.

Qs: how many pts. and where to locate them?
does sol^n converge as no. pts. $\to \infty$?

<u>MATLAB demo</u>

sideSrc = 10; nPts = 50, 100, 200 ← gets bad

sideSrc = 6; Still high cond.
sideSrc = 5.1; low cond – still oscillations ~~near corn~~
[Explain poor conditioning]

- Not necessarily the case that more pts. $\to$ more accuracy
- Trade off between conditioning & oscillations at bdy.
- Suggests smearing pts. onto bdy the best thing.

$$\text{I.E.} \quad u(x) = \int_\Gamma G(x,y)\,\alpha(x)\,ds(y), \quad x \in \Gamma \qquad (\dagger)$$

"Single-layer potential" (SLP) – simplest type of I.E.

<u>Numerical sol^n of SLP</u>: collocation & Galerkin (projection methods)

Suppose $u \in X$ – infinite dim^l space (e.g., $L^2(\Gamma)$, $C^2(\Gamma)$, $H^1(\Gamma)$)

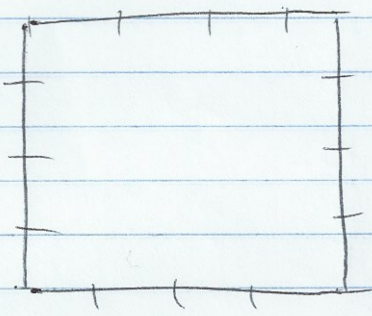
Choose a finite dim^l subspace $X_n \subset X$, with dim. n
Let X_n have basis

$$\{\phi_1, \ldots, \phi_n\},$$

e.g., space of piecewise const. fns. over elts.

Then seek $\alpha_n \in X_n$, i.e.,

$$\alpha_n(x) = \sum_{i=1}^{n} c_i \phi_i(x), \quad x \in \Gamma \qquad (\dagger\dagger).$$



Sub $(\dagger\dagger)$ into $(\dagger)$, then coeffs c_i are determined by forcing the eq^n to be exact in some sense.

Define <u>residual</u>

$$r_n(x) = \int_\Gamma G(x,y)\, \alpha_n(x)\, ds(y) - u(x), \quad x \in \Gamma$$

$$= \sum_{i=1}^{n} c_i \int_\Gamma G(x,y)\, \phi_i(y)\, ds(y) - u(x).$$

<u>Collocation.</u>
Pick pts. $x_1, \dots, x_n \in \Gamma$ and require

$$r_n(x_j) = 0, \quad j = 1, \dots, n.$$

This leads to the system of n eq^{n}s:

$$\sum_{i=1}^{n} c_i \int_\Gamma G(x_j, y)\, \phi_i(y)\, ds(y) = u(x_j), \quad j = 1, \dots, n \quad (*)$$

Similar to pt. source approach but with integrals over Γ.

If each basis fn. lives only on i^{th} elt. Γ_i, then (*) becomes

$$\sum_{i=1}^{n} c_i \int_{\Gamma_i} G(x_j, y)\, \phi_i(y)\, ds(y) = u(x_j), \quad j=1,\dots,n.$$

NB: when $x_j \in \Gamma_i$, integrals are singular.

Typically collocate at elt. centers.

Galerkin.

Require r_n to satisfy

$$\underbrace{\langle r_n, \phi_j \rangle}_{L^2 \text{ inner prod.}} = 0, \quad j=1,\dots,n, \quad \phi_j \in X_n.$$

(ϕ_j: test fn.)

[Test & trial spaces do not have to be equal but often are. When different, called "Petrov-Galerkin"].

Yields system

$$\sum_{i=1}^{n} c_i \left\langle \int_{\Gamma} G(x,y)\, \phi_i(y)\, ds(y),\ \phi_j \right\rangle = \langle u, \phi_j \rangle$$

i.e.,

$$\sum_{i=1}^{n} c_i \int_{\Gamma_j}\int_{\Gamma_i} G(x,y)\, \phi_i(y)\, \phi_j(x)\, ds(y)\, ds(x) = \int_{\Gamma_j} u(x)\, \phi_j(x)\, ds(x)$$

Similar to collocation but now with <u>double integrals</u>

<u>NB</u>: when $\phi_i = \phi_j$, integral singular, also when ϕ_i, ϕ_j are neighbors.

MATLAB demo: Galerkin method, piecewise const. fns.