2.097J / 6.339J / 16.920J

Numerical Methods for Partial Differential Equations

Massachusetts Institute of Technology – Fall 2017

Handed Out: Sept. 27, 2017 | *Due: Oct. 11, 2017*

Problem 1 - Traffic Flow

Problem Statement

The occurrence of car pile-up crashes (article) on highways is a frequent occurrence. Such pile-up crashes are often likened to a "domino effect", wherein successive cars "fall off like dominoes". Such a domino effect is often caused when a car decelerates abruptly – this then induces a chain reaction that involves successive cars having to slam on their brakes and decelerate abruptly. This chain reaction propagates through the highway, and often leads to pile-up crashes. In this problem, we explore some of the origins of this "domino effect."

Mathematically, this domino effect can be likened to a shock wave occurring in the traffic. The notion of a shock wave has its origins in gas dynamics - wherein a compression wave passing through a gas can become stronger and stronger and ultimately develop into a shock wave. Across this shock wave, near discontinuous transitions occur in density and velocity of the gas.

In the traffic scenario, we define density as the number of vehicles per unit length of the highway. It can be visualized that if a car decelerates abruptly, it sets up a compression wave behind it. This compression wave then passes through the "vehicular fluid" and may develop into a shock wave.

We shall first examine the formation of the shock wave using the non-linear hyperbolic equation, for lanes $1 \leq l \leq n$:

$$\frac{\partial \rho^{(l)}}{\partial t} + \frac{\partial \rho^{(l)} v^{(l)}}{\partial x} = s^{(l)} \tag{1}$$

with $\rho^{(l)} = \rho^{(l)}(x,t)$ the density of cars (vehicles/km), and $v^{(l)} = v^{(l)}(x,t)$ the (average) velocity of the cars. Assume that the velocity v_i is given as a function of ρ_i:

$$v^{(l)}(\rho) = v_{\max}\left(1 - \frac{\rho^{(l)2}}{\rho_{\max}^2}\right) \tag{2}$$

With $v_{\max}$ the maximum speed and $\rho_{\max}$ is the maximum car density per lane, where $0 \leq \rho^{(l)} \leq \rho_{\max}$. The flux of cars is therefore given by:

$$f^{(l)}(\rho) = v_{\max}\left(\rho^{(l)} - \frac{\rho^{(l)3}}{\rho_{\max}^2}\right) \tag{3}$$

The source term $s^{(l)}$ models commuters who switch lanes, α is the fraction of drivers that change lanes, according to

$$s^{(l)} = \alpha \sum_{|k-l|=1, n\geq k, l\geq 1} (\rho^{(k)} - \rho^{(l)}) \tag{4}$$

Questions

1) (40 pts) Implement a first-order conservative finite volume scheme for a single lane

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right) \tag{5}$$

a. (20 pts) For the numerical flux function use Godunov's scheme in which the flux is the exact solution to the Riemann problem at the interface between two volumes, i.e.,

$$F^G_{i+\frac{1}{2}} = f\left(\rho\left(x_{i+\frac{1}{2}}, t^{n+}\right)\right) = \begin{cases} \min\limits_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max\limits_{\rho\in[\rho_i,\rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1} \end{cases} \tag{6}$$

Analyzing $f(\rho)$ determine the Godunov scheme flux in a manner that does not require a brute force search for the minimum/maximum of the flux between ρ_i and ρ_{i+1}.

b. (20 pts) Look at the problem of a traffic accident causing a lane ($x \in [0; 10]$) to be blocked at time $t = 0$ and solve the continuity equation (1). Use the following problem parameters:

$$\rho_{\max} = 1.0, \quad v_{\max} = 1.0 \tag{7}$$

An accident happened at $t = 0$ at $x = 5$ and is cleared at $t = 1$. Due to the accident, the lane is completely blocked or the velocity is zero at $x = 5$.
The initial condition is

$$\rho(x, t = 0) = \rho_0 \tag{8}$$

The boundary conditions, if applicable, are:

$$\rho(0, t) = \rho_0 \ , \rho(10, t) = \rho_0 \ ; \tag{9}$$

Consider two conditions:

i. Light traffic: $\rho_0 = 0.2\ \rho_{\max}$,

ii. Traffic jam: $\rho_0 = 0.8\ \rho_{\max}$

Solve the PDE from $t = 0$ to $t = 2$. When are the specified boundary conditions not applicable? Why? How do you modify the set of boundary conditions? In what conditions can we prescribe a density on the left side of the boundary? In what conditions can we prescribe a density on the right side of the boundary? Why? Describe what happens to ρ and v as time evolves due to the blockage. For the traffic jam conditions, how is this related to the domino effect?

2) (40 pts) Now develop a second order finite volume algorithm in which the interface values are found using the limited κ-reconstructions

$$\rho^-_{i+\frac{1}{2}} = \rho_i + \frac{1-\kappa}{4}\Psi(R_i)\Delta_{i-\frac{1}{2}}\rho + \frac{1+\kappa}{4}\Psi(1/R_i)\Delta_{i+\frac{1}{2}}\rho \tag{10}$$

$$\rho^+_{i-\frac{1}{2}} = \rho_i - \frac{1-\kappa}{4}\Psi(1/R_i)\Delta_{i+\frac{1}{2}}\rho - \frac{1+\kappa}{4}\Psi(R_i)\Delta_{i-\frac{1}{2}}\rho \tag{11}$$

where

$$R_i = \frac{\Delta_{i+\frac{1}{2}}\rho}{\Delta_{i-\frac{1}{2}}\rho} \quad \text{and} \quad \Delta_{i+\frac{1}{2}}\rho = \rho_{i+1} - \rho_i \tag{12}$$

and $\Psi(R_i)$ is determined by the choice of limiters. For the time integration, implement a four-stage Runge-Kutta algorithm.

a. (20 pts) Using the same mesh size and time step, solve the previous traffic flow problems with the second order algorithm without limiting (i.e. set $\Psi = 1$) for $\kappa = -1, 0$, and $1/3$. Describe the quality of these solutions. In particular, describe the presence of oscillations. You can for instance look at the solution at time $t = 2.0$.

b. (20 pts) Now employ the limited algorithm with the MinMod, Van Leer, and Superbee limiters. Again, utilize the same mesh size and time step as above, and all three values of $\kappa = -1$, 0, and 1/3. Describe the quality of these solutions, in particular how they compare with each other and with the solution obtained without limiters for the light traffic case. Again, it might be useful to look at the solutions at time $t = 2.0$ obtained with the different methods.

3) (40 pts) Implement a multi-lane algorithm to answer the following questions.

a. (20 pts) Ethan Hunter (Mission Impossible III - video) reasons that traffic behaves like a living organism and takes the example when a driver brakes abruptly on a highway. Effectively the abrupt breaking can be modeled as an impulsive source term S at $t = 0$ at $x = 5$ in some initial traffic ρ_{init} where $\alpha = 0.1$. Find a initial condition or relationship between S and ρ_{init} such that a shock wave forms later in time for the single, double and a triple lane case.

b. (20 Bonus pts)

Ethan drives his Tesla Model S over to Boston from New York City to visit Qiqi. With just miles to spare, the battery goes flat on the Massachusetts Turnpike (Mass Pike). Ethan is stranded in the middle of the interstate. To pass the time while he waits for a tow car he decides to model the traffic conundrum he has created. Ethan knows from his professional work that his model will be more representative if he accounts for a time lag, τ, in cars changing lanes. Now the source term $s^{(l)}$ is

$$s^{(l)} = \alpha \sum_{|k-l|=1, n \geq k, l \geq 1} \left(\rho^{(k)}(x, t-\tau) - \rho^{(l)}(x, t-\tau)\right) \tag{13}$$

Lets follow suit and model the situation. Mass Pike is a three lane interstate, Ethan comes to a halt at $t = 0$ at $x = 5$ on the middle lane. Ethan estimates τ to be 0.01. Find a condition or relationship for ρ_{init} and α such that all three lanes upstream of Ethan will form a shock wave.