Numerical Methods for PDEs

Integral Equation Methods, Lecture 4
First and Second Kind Equations

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1 Outline

SLIDE 1

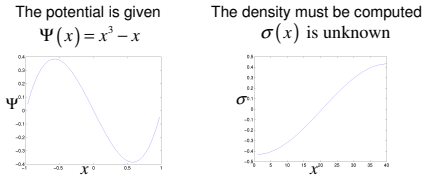
Convergence issues in 1D
First and second kind integral equations
Develop some intuition about the difficulties
Convergence for second kind equations
Consistency and stability issues
Nystrom Methods
High order convergence

2 Example Problems in 1D

2.1 First Kind Equation

SLIDE 2

$$\Psi(x) = \int_{-1}^{1} |x - x'|\sigma(x')dS' \qquad x \in [-1, 1]$$



In the next several slides we will investigate the convergence properties of integral equation discretization methods. How these methods converge depends on what kind of integral equation is being solving. Examining this issue will introduce one of the subtle points about integral equations.
To begin, consider the example one-dimensional first-kind integral equation on the above slide. For this equation, we assume that the potential, $\Phi(x)$, is known and that the charge density $\sigma(x)$ is unknown. Here, x is in the interval $[-1, 1]$, and the integration is over that same interval. Note that for this example, the Green's function is given by $G(x, x') = |x - x'|$.
In the left plot below the equation, an example potential, $x^3 - x$, is given and plotted as a function of x. On the right is a plot of charge density as a function of x which is a possible solution to the integral equation. As we will see shortly, the solution for the charge density is not so easy to find.

2.1.1 Collocation Discretization

SLIDE 3

$$\Psi(x) = \int_{-1}^{1} |x - x'|\sigma(x')dS' \qquad x \in [-1, 1] \tag{1}$$

Centroid Collocated Piecewise Constant Scheme

$$\Psi(x_{c_i}) = \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'| dS' \tag{2}$$

To compute the numerical solution to this one-dimensional problem, consider solving integral equation (1) using a piecewise-constant collocation scheme. In such a scheme, we first select $n+1$ points on the interval, in this case $[-1, 1]$. We denote those points as $\{x_0, x_1, ..., x_n\}$, as shown in the above figure. For this example, $x_0 = -1$ and $x_n = 1$. Over the subintervals define a set of basis functions, $\{\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)\}$, where

$$\varphi_i(x) = \left\{ \begin{array}{ll} 1 & x \in [x_{i-1}, x_i] \\ 0 & \text{otherwise} \end{array} \right\}. \tag{3}$$

The charge density σ can then be represented approximately as

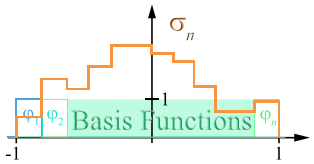
$$\sigma(x) \approx \sigma_n(x) \equiv \sum_{i=1}^{n} \sigma_{ni} \varphi_i(x), \tag{4}$$

where σ_{ni} is the weight associated with the i^{th} basis function. It may seem odd that we used the same letter to represent the density and the basis function weights, but there is a reason. The above basis set is such that only one basis function is nonzero for a given x, and basis functions only take on the value zero or one. Therefore, σ_{ni} will be equal to the approximate charge density when $x \in [x_{i-1}, x_i]$.

Charge Density Representation

SLIDE 4

$$\sigma_n(x) \equiv \sum_{i=1}^{n} \sigma_{ni} \varphi_i(x)$$



Plugging the basis function representation of the charge density, equation (4), into equation (1) yields

$$\Psi(x) = \int_{-1}^{1} |x - x'| \sum_{i=1}^{n} \sigma_{ni} \varphi_i(x') dS',$$

which can be simplified by exploiting equation (3). Now, let's introduce the residual, $R(x)$, and from this concept get:

$$R(x) = \Psi(x) - \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x - x'| dS'.$$

If collocation is used to solve this equation, then $R(x_{c_i}) = 0$ for all i, where x_{c_i} is the i^{th} collocation point. The collocation points shown in the figure are the subinterval center points, $x_{c_i} = \frac{1}{2}(x_{i-1} + x_i)$. Note that there are other choices for collocation points, such as $x_{c_i} = x_i$.
Using the fact that $R(x_{c_i}) = 0$ leads to

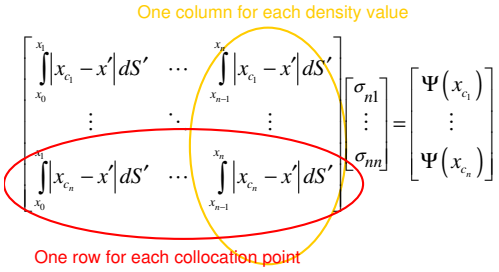
$$R(x_{c_i}) = \Psi(x_{c_i}) - \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'| dS' = 0$$

which can be reorganized into equation (2).

The Matrix

We can now generate a system of equations that can be used to solve for the σ_{ni}'s, the piecewise constant charge densities for each of the subintervals.

SLIDE 5



The right-hand side of this system of equations is a vector of known potentials at the interval centers (the collocation points). The i^{th} row of the matrix corresponds to unfolding the sum in the collocation equation

$$\Psi(x_{c_i}) = \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'| dS',$$

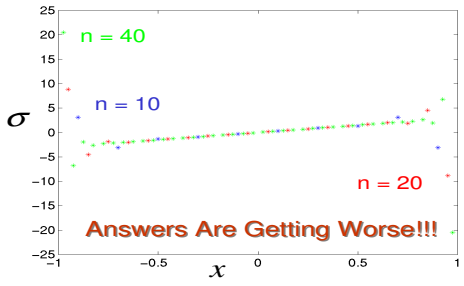
and the entries in the j^{th} column correspond to how much the charge on the j^{th} interval contributes to the i^{th} potential.
Note that the matrix is square and dense.

▷ ***Exercise 1*** Is the above matrix symmetric? If we used $x_{c_i} = x_i$, would the matrix still be symmetric? ■

2.1.2 Numerical Results with Increasing n

One usually believes that a discretization scheme should produce progressively more accurate answers as the discretization is refined. In this case, as we divide the interval into progressively finer subintervals, one might expect that the piecewise constant representation of the charge density given by $\sigma_n(x) \approx \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x)$ would become more accurate as n increases.

SLIDE 6



Unfortunately, the plot above indicates that the method is not converging. In the plot, which is hard to decipher without looking at a color version, the σ_{ni}'s produced using $n = 10$, $n = 20$ and $n = 40$ subintervals are shown. For each discretization, a point is plotted at σ_{ni}, x_i for $i = 1, .., n$, so there are ten points plotted for the coarsest discretization and forty points plotted for the finest discretization, but all sets of points span the interval $x \in [-1, 1]$.
What is clear from comparing the blue points ($n = 10$) to the red points ($n = 20$) and to the green points ($n = 40$), is that the charge density seems to be approaching infinity as the discretization is refined. The results are certainly not converging.
Why is this happening? Is the numerical technique at fault, or is the integral equation a problem?

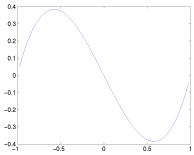
2.2 Second Kind Equation

We are going to postpone examining what went wrong in the First-Kind example, and now look at a Second Kind equation. Recall that we are assuming the potential, $\Psi(x)$, is known and that the charge density $\sigma(x)$ is unknown. Here, x is in the interval $[-1, 1]$, and the integration is over that same interval. Once again, the Green's function is given by $G(x, x') = |x - x'|$.

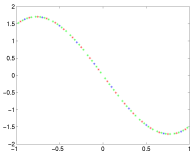
SLIDE 7

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} |x - x'|\sigma(x')dS' \qquad x \in [-1, 1]$$

The potential is given $\Psi(x) = x^3 - x$



The density must be computed $\sigma(x)$ is unknown



What makes this equation Second-Kind instead of First is because the unknown charge density appears outside of the integral (in addition to inside the integral). In the first-kind equation, the density appeared only inside the integral. This seemingly small difference has enormous numerical ramifications.
In the left plot above, an example given potential, $\Psi(x) = x^3 - x$ is plotted as a function of x. On the right is a plot of a charge density as a function of x which satisfies this second kind integral equation. As we will see below, this equation is easily solved numerically.

2.2.1 Collocation Discretization

SLIDE 8

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} |x - x'|\sigma(x')dS' \quad x \in [-1, 1] \tag{5}$$

Centroid Collocated Piecewise Constant Scheme

$$\Psi(x_{c_i}) = \sigma_{ni} + \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'|dS' \tag{6}$$

To compute the numerical solution to the one-dimensional second-kind equation (5), once again consider using a piecewise-constant collocation scheme. Again, we select $n+1$ points on the interval and denote those points as $\{x_0, x_1, ..., x_n\}$, as shown in the figure above. For this example, $x_0 = -1$ and $x_n = 1$. The corresponding basis functions, $\{\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)\}$, are the same as in equation (3):

$$\varphi_i(x) = \left\{ \begin{array}{ll} 1 & x \in [x_{i-1}, x_i] \\ 0 & \text{otherwise} \end{array} \right\}. \tag{7}$$

The charge density σ is approximately represented by

$$\sigma(x) \simeq \sigma_n(x) \equiv \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x), \tag{8}$$

where σ_i is the weight associated with the i^{th} basis function.
Plugging the basis function representation of the charge density, equation (8) into the second kind integral equation (5) gives:

$$\Psi(x) = \sum_{j=1}^{n} \sigma_{nj}\varphi_j(x) + \int_{-1}^{1} |x - x'| \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x')dS',$$

which can be simplified by exploiting the specific basis functions, equation (7) to

$$\Psi(x) = \sum_{j=1}^{n} \sigma_{nj}\varphi_j(x) + \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x - x'|dS'. \tag{9}$$

As shown in the above figure, the collocation points are the subinterval center points, $x_{c_i} = \frac{1}{2}(x_{i-1} + x_i)$. When collocation is used, equation (9) must be satisfied exactly at the collocation points and therefore

$$\Psi(x_{c_i}) = \sum_{i=1}^{n} \sigma_{ni}\varphi_j(x_{c_i}) + \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'|dS'. \tag{10}$$

Note that $\varphi_j(x_{c_i}) = 0$ when $i \neq j$, and $\varphi_i(x_{c_i}) = 1$. Using this fact yields equation (6).

The Matrix

Just as in the discretized first-kind equation, we generate a system of equations that can be used to solve for the σ_{ni}'s, the piecewise constant charge densities for each of the subintervals.

SLIDE 9

$$\begin{bmatrix} 1+\int_{x_0}^{x_1}\left|x_{c_1}-x'\right|dS' & \cdots & \int_{x_{n-1}}^{x_n}\left|x_{c_1}-x'\right|dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1}\left|x_{c_n}-x'\right|dS' & \cdots & 1+\int_{x_{n-1}}^{x_n}\left|x_{c_n}-x'\right|dS' \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi\left(x_{c_1}\right) \\ \vdots \\ \Psi\left(x_{c_n}\right) \end{bmatrix}$$

The right-hand side of this system of equations is a vector of known potentials at interval centers (the collocation points). The i^{th} row of the matrix corresponds to unfolding the sum in the collocation equation

$$\Psi(x_{c_i}) = \sigma_{ni} + \sum_{j=1}^{n} \sigma_{nj} \int_{x_{j-1}}^{x_j} |x_{c_i} - x'|dS'$$

and the entries in the j^{th} column corresponds to how much the charge on the j^{th} interval contributes to the i^{th} potential.

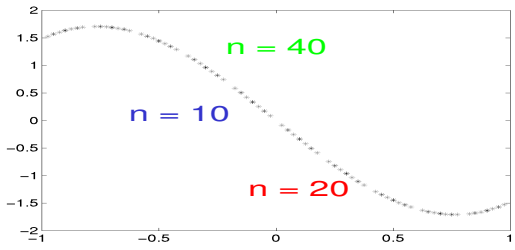
The major difference between the matrix in this discretized second-kind example and the first-kind example is circled on the above slide. There is an additional one on the diagonal of the discretized second-kind equation that did not appear in the first-kind equation. In other words,

$$A_{second\ \ kind} = I + A_{first\ \ kind}.$$

2.2.2 Numerical Results with Increasing n

Unlike the results from discretizing the first kind equation, progressively refining the discretization of the second kind equation produces more accurate answers.

SLIDE 10



Answers Are Improving!!!

Once again, the plot is a little hard to decipher without looking at the color version. It shows the σ_{ni}'s produced using $n = 10$, $n = 20$ and $n = 40$ subintervals. For each discretization, a point is plotted at σ_{ni}, x_i for $i = 1, .., n$, so there are ten points plotted for the coarsest discretization and forty points plotted for the finest discretization, but all sets of points span the interval $x \in [-1, 1]$.
What is clear from comparing the blue points ($n = 10$) to the red points ($n = 20$) and to the green points ($n = 40$), is that the charge density seems to be approaching a smooth solution.
What is the essential difference between first and second kind equations?
Is it some aspect of the numerical technique or are these two equations really that different? In the next slides, we will try to answer this question.

2.3 Difficulty with the First Kind Equation

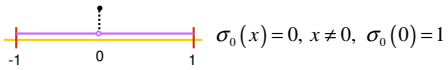
2.3.1 Singular Integral Operator

SLIDE 11

Denote the integral operator as K

$$K\sigma \equiv \int_{-1}^{1} |x - x'|\sigma(x')dS' \Rightarrow K\sigma = \Psi \tag{11}$$

The integral operator is **singular** : K **has a null space**



$$K\sigma_0 = \int_{-1}^{1} |x - x'|\sigma_0(x')dS' = 0 \tag{12}$$

$$\text{If} \quad K\sigma^a = \Psi \quad \text{then} \quad K(\sigma^a + \sigma_0) = \Psi$$

In equation (11), we introduce the abstract notion that

$$\int_{-1}^{1} |x - x'|\sigma(x')dS'$$

is an operator on the function σ, which we denote with the symbol K. As shown on the top of the slide, this notation makes writing the integral equation look just like writing a matrix equation.

The key problem is that the operator K is singular. And if

$$K\sigma = \Psi$$

were a matrix equation with a singular K, one would not be surprised to discover the system of equations is hard, or impossible, to solve.

We will not try, in this lecture, to be formal about the concept of a singular operator. To do so, we would necessarily be examining details about certain types of function spaces. Instead, we will try to develop some intuition. In particular, we will draw an analogy to matrices and note that if an operator is singular, it must have a null space.

To see that K does have a null space, consider the spike function $\sigma_0(x)$ depicted in equation (12). This spike function is one at $x = 0$ and zero otherwise. Note, this function is *not* an impulse function. Unlike the impulse function, the spike's value at $x = 0$ is finite and the area under its curve is obviously zero.

As noted in equation (12), $K\sigma_0 = 0$. To see this consider that since σ_0 is nonzero only at $x = 0$, and therefore

$$\int_{-1}^{1} |x - x'|\sigma_0(x')dS' = |x| \int_{-1}^{1} \sigma_0(x')dS'.$$

Since $\int_{-1}^{1} \sigma_0(x')dS' = 0$, as the area under σ_0's curve is zero, then $K\sigma_0 = 0$.

The statement "If $K\sigma^a = \Psi$ then $K(\sigma^a + \sigma_0) = \Psi$" says that if K has a null space, and there exists a solution, then there exist infinitely many solutions.

One last comment should be made. The spike function we generated is not unique. Simply shifting the nonzero point would generate and infinite number of spike functions which would all be in the null space of K. That is, K has an incredibly rich null space.

2.3.2 Eigenvalues

Difficulty from the Matrix

SLIDE 12

Collocation generates a discrete form of K

$$K\sigma = \Psi \quad \rightarrow \quad K_n\sigma_n = \Psi_n$$

$$\underbrace{\begin{bmatrix} \int_{x_0}^{x_1} |x_{c_1} - x'| dS' & \cdots & \int_{x_{n-1}}^{x_n} |x_{c_1} - x'| dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1} |x_{c_n} - x'| dS' & \cdots & \int_{x_{n-1}}^{x_n} |x_{c_n} - x'| dS' \end{bmatrix}}_{\underline{K}n} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi(x_{c_1}) \\ \vdots \\ \Psi(x_{c_n}) \end{bmatrix}$$

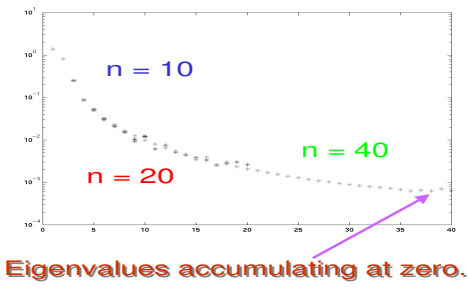
The matrix $\underline{K}_n$ is the not the operator K_n!

As shown above, discretizing the integral equation by combining a piecewise constant charge density representation with collocation at subinterval centers results in a system of equations which relates the subinterval σ_i's to the collocation point potentials. From this perspective, the matrix above can be thought of as a discrete representation of the operator K. We denote the matrix with $\underline{K}_n$ to indicate the matrix was generated using a discretization with n basis functions.
Next, we will have to be more precise about the discrete representation of the operator K, but for the moment, the matrix is sufficient.

Numerical Results with Increasing n

If the operator K is singular, one might expect to see that reflected in the eigenvalues of a matrix generated by discretizing K. In particular, one would expect the matrix to have eigenvalues that are near zero.

SLIDE 13

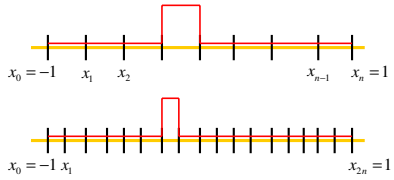


In the figure above, the eigenvalues of matrices generated by discretizing K for the 1-D problem are plotted. Discretizing using 10 subintervals generates a matrix with 10 eigenvalues plotted in blue. The blue eigenvalue closest to zero is ≈ 0.01. As the discretization is refined to 20 subintervals, the minimum eigenvalue (plotted in red) drops to ≈ 0.003, and with 40 subintervals the minimum eigenvalue (plotted in green) drops to ≈ 0.0009. Examining this

data suggests that as the discretization is refined, the generated matrix more accurately reflects the operator K, and therefore the matrix is becoming closer to being singular.
As the discretization is refined, the matrix is larger and has more eigenvalues. Notice that as the discretization is refined from $n = 10$ to $n = 20$ to $n = 40$, all the additional eigenvalues are closer to zero.

Intuition About Eigenvalues SLIDE 14

As the discretization is refined, $\sigma_0(x)$ becomes better approximated



As the discretization is refined, K's null space can be more accurately represented.

As an alternative view of why refining the discretization for the first kind equation produces a matrix with more and more smaller eigenvalues, consider the figures above. In the top plot, one of the basis functions is plotted for a coarse discretization. In the bottom plot, one of the basis functions is plotted for a finer discretization. Notice that as the discretization is refined, these basis functions look progressively more like the spike function mentioned previously. And since the spike function is in the null space of K, one would expect that finer discretizations would generate "spikier" basis functions whose associated eigenvalues would be near zero.

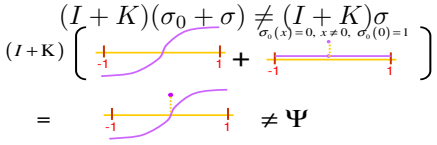
2.4 Second Kind Equation Has Fewer Problems

SLIDE 15

Second Kind equation

$$(I + K)\sigma \equiv \sigma(x) + \int_{-1}^{1} |x - x'|\sigma(x')dS'$$
$$\Rightarrow (I + K)^{-1}\sigma = \Psi \tag{13}$$

$$(I + K)(\sigma_0 + \sigma) \neq (I + K)\sigma$$



As shown in equation (13), the abstract operator for the second-kind equation is denoted by $I + K$, where I here is just the identity operator and K is the integral operator.
To see why the spike function, σ_0, is not in the null space of the operator $I + K$, or equivalently that

$$(I + K)(\sigma_0 + \sigma) \neq (I + K)(\sigma)$$

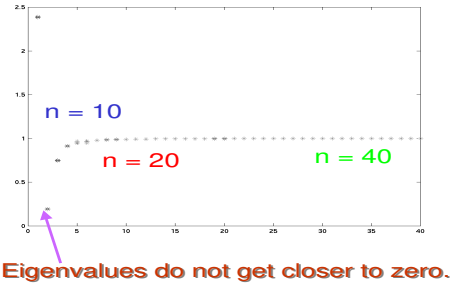
consider the figures above. If a spike is added to a smooth σ, the $(I + K)$ operator will preserve the spike. Another way to see this is to consider that since σ_0 is in the null space of K,

$$(I + K)\sigma_0 = (I)\sigma_0 + K\sigma_0 = \sigma_0 \neq 0.$$

2.4.1 Eigenvalues

Numerical Results with Increasing n

SLIDE 16



As we noted before, the matrix associated with discretizing the operator $I + K$ is identical to the sum of the identity matrix and the matrix associated with discretizing K alone. In the plot above, we once again present the eigenvalues generated by discretizing the 1-D example problem. Discretizing using 10 subintervals generates a matrix with 10 eigenvalues plotted in blue. The blue eigenvalue closest to zero is ≈ 0.2. As the discretization is refined to 20 subintervals, the minimum eigenvalue (plotted in red) is still ≈ 0.2, and with 40 subintervals the minimum eigenvalue (plotted in green) is still ≈ 0.2. Examining this data suggests that as the discretization is refined, and the generated matrix more accurately reflects the operator $I + K$, the matrix is not becoming more singular. In fact, the eigenvalues are accumulating near one, an unsurprising result given that the eigenvalues of the discretized K operator were accumulating at zero.

▷ ***Exercise 2*** Estimate how many iterations will be needed for a Krylov-subspace based algorithm to converge for the 1-D discretized second-kind example. Will the number of iterations increase as the discretization is refined? ■

▷ ***Exercise 3*** Suppose the integral equation were changed to

$$\Psi(x) = \sigma(x) + \frac{1}{\lambda}\int_{-1}^{1} |x - x'|\sigma(x')dS'.$$

For what value of λ would the solution no longer be unique. (you can answer this just by looking at the eigenplot above). ■

As the above exercise makes clear, a second-kind integral equation does not always have a unique solution. However, a first-kind equation almost never has a unique solution, the exception being when the Green's function is singular, as we will investigate next lecture.

3 Theory of 2^{nd} Kind Equations

3.1 General Framework

SLIDE 17

General Second kind integral equation

$$\Psi(x) = \sigma(x) + \int G(x,x')\sigma(x')dx' \Rightarrow \Psi = (I+K)\sigma \tag{14}$$

Discrete equivalent

$$\Psi_n = (I + K_n)\,\sigma_n \tag{15}$$

where Ψ_n and σ_n are functions of x.
What is Ψ_n ? K_n?

3.2 Discrete Equivalent for Galerkin

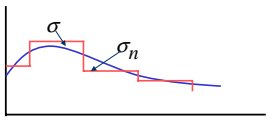
SLIDE 18

Representation $\sigma_n(x) = \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x)$

Projection $\sigma_n = P\sigma$

$$P\sigma \equiv \sum_{i=1}^{n} \overbrace{\left(\int \sigma(x)\varphi_i(x)dx\right)}^{\sigma_{ni}} \varphi_i(x)$$

$$P\sigma = \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x)$$



Note: $K\sigma_n(x) = KP\sigma(x) = \sum_{i=1}^{n} \sigma_{ni} \int G(x,x')\varphi_i(x')dx'$

SLIDE 19

$$P(KP\sigma) = \sum_{j=1}^{n} \left(\int \varphi_j(x) KP\sigma(x)dx\right) \varphi_j(x)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \sigma_{ni} \int\int \varphi_j(x)G(x,x')\varphi_i(x')dxdx'\right) \varphi_j(x)$$

$$(I + PKP)\sigma_n = P\Psi$$

$$(I + K_n)\sigma_n = \Psi_n$$

For second-kind integral equations, one can prove a convergence theory for almost any reasonable discretization scheme, assuming that the integral equation has a unique solution. As noted above, this is not necessarily the case, but we will assume it to analyze convergence. In particular, as noted on the above slide, we will assume that the second-kind integral equation operator has a bounded inverse.

Before beginning the derivation given in the slide, let's readdress the notation definitions.

Let K denote the integral operator, and therefore the general form is

$$K\sigma = \int G(x, x')\sigma(x')dx'$$

Let σ_n denote a numerical approximation to σ on x based on using n basis functions. Note here that σ_n is a function of x and would typically be given by

$$\sigma_n(x) = \sum_{i=1}^{n} \sigma_{ni}\varphi_i(x).$$

Let K_n be the discrete representation of the integral operator. Note that K_n is not a matrix ($\underline{K}_n$ is a matrix), but an operator which maps a function of x into another function of x. For example, if the discretization scheme uses a basis to approximate σ, and the coefficients of the discretization were combined with a collocation scheme, a not necessarily unique associated K_n could be given by

$$K_n\sigma = V\left(\int G(x, x')P\sigma(x')dx'\right)$$

where

$$P\sigma(x) = \sum_{i=1}^{n}\left(\int \sigma(x')\varphi_i(x')dx'\right)\varphi_i(x), \tag{16}$$

and

$$Vu(x) = \sum_{i=1}^{n} u(x_{c_i})\varphi_i(x). \tag{17}$$

Equations (16) and (17) deserve some explanation. The piecewise constant basis is orthonormal, so the formula in equation (16) is a simple projection of $\sigma(x)$ onto the basis. If centroid collocation is used, then the discrete potentials computed by evaluating the integral operator at the collocation points must be converted to a function of x by interpolation. In equation (17), the $\varphi_i(x)$'s act as interpolation functions.

The main theorem is given in the next slide. The theorem states that if the discretization scheme generates progressively more accurate representations of the integral operator, then the discretization method converges. That is,

$$\lim_{n\to\infty} \|\sigma - \sigma_n\| \to 0.$$

▷ ***Exercise 4*** Suppose a nonorthogonal basis is used to represent σ. How would the projection operator in equation (16) change? ■

3.3 Main Theorem

SLIDE 20

$$\text{Given} \quad (I+K)\sigma = \Psi \quad \& \quad \|(I+K)^{-1}\| < C$$

Equation uniquely solvable

$$(I+K_n)\sigma_n = \Psi_n$$

Discrete Equivalent

Consistency:

If $\lim_{n\to\infty} \max_{\|\sigma_{smooth}\|=1} \|(K-K_n)\sigma\| \to 0$ and $\lim_{n\to\infty} \|\Psi - \Psi_n\| \to 0$

Then $\lim_{n\to\infty} \|\sigma - \sigma_n\| \to 0$

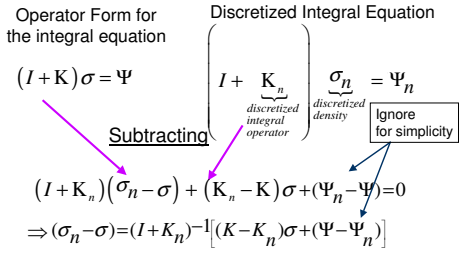
3.4 Rough Proof

To derive a relationship between the errors in the computed solution and the errors in the operator representation, we write the exact equation alongside the discrete equation.

Exact Equation	Discretized Equation
$\Psi = (I+K)\sigma$	$\Psi_n = (I+K_n)\sigma_n$

$$\begin{aligned}
&\Rightarrow \Psi - \Psi_n = (I+K)\sigma - (I+K_n)\sigma_n \\
&\Rightarrow (\Psi - \Psi_n) = (\sigma - \sigma_n) + K\sigma - K_n\sigma_n \\
&\Rightarrow (\Psi - \Psi_n) = (\sigma - \sigma_n) + K\sigma - K_n\sigma + K_n\sigma - K_n\sigma_n \\
&\Rightarrow (\Psi - \Psi_n) = (\sigma - \sigma_n)(I+K_n) + (K-K_n)\sigma \\
&\Rightarrow (\Psi - \Psi_n) - (K-K_n)\sigma = (\sigma - \sigma_n)(I+K_n) \\
&\Rightarrow [(\Psi - \Psi_n) - (K-K_n)\sigma]\,(I+K_n)^{-1} = (\sigma - \sigma_n)
\end{aligned}$$

SLIDE 21



Then, subtraction combined with operator inversion yields the above equation.

SLIDE 22

The equation for the solution error (previous slide)

$$\underbrace{(\sigma_n - \sigma)}_{\text{solution error}} = (I + K_n)^{-1}(K - K_n)\sigma$$

Taking norms

$$\underbrace{||\sigma_n - \sigma||}_{\substack{\text{Error which} \\ \text{should go to} \\ \text{zero as } n \\ \text{increases}}} \leq \underbrace{||(I + K_n)^{-1}||}_{\substack{\text{Needs a} \\ \text{bound, that is} \\ \textbf{\textit{stability}}}} \underbrace{||(K - K_n)\sigma||}_{\substack{\text{Goes to} \\ \text{zero} \\ \text{by } \textbf{\textit{consistency}}}}$$

We complete deriving a relationship between the errors in the computed solution and the errors in the operator representation. In order to establish that consistency implies convergence, the inverse of the discretized operator must be bounded.

3.4.1 Stability Bound

SLIDE 23

Norm of solution error
$||(\sigma_n - \sigma)|| \leq ||(I + K_n)^{-1}||\ ||(K - K_n)\sigma||$

Deriving the stability bound
$(I + K_n)^{-1} = [I + K - (K - K_n)]^{-1} = [(I - (I + K)^{-1}(K - K_n)]^{-1}(I + K)^{-1}$

Taking norms

$$||(I + K_n)^{-1}|| \leq \underbrace{||(I + K)^{-1}||}_{\substack{\text{Bounded by C} \\ \text{by Assumption}}} \ || \left(I - (I + K)^{-1}(K - K_n)\right)^{-1} ||$$

SLIDE 24

Repeating from last slide

$$||(I + K_n)^{-1}|| \leq \underbrace{||(I + K)^{-1}||}_{\substack{\text{Bounded by C} \\ \text{by Assumption}}} \ || \left(I - (I + K)^{-1}(K - K_n)\right)^{-1} ||$$

Bounding terms

$$||(I + K_n)^{-1}|| \leq \frac{C}{1 - \underbrace{||(I + K)^{-1}(K - K_n)||}_{\substack{\text{Less than } \epsilon \text{ for n larger} \\ \text{than } n_0 \text{ by consistency}}}} \leq \frac{C}{1-\epsilon} < C \quad \text{for} \quad n \geq n_0$$

Bounding the inverse of the discretized operator requires several steps of algebra,

as described above. Notice that

$$\left\| \left[I - (I+K)^{-1}(K-K_n) \right]^{-1} \right\| = \frac{1}{\inf_{||x||=1} \left\| I - (I+K)^{-1}(K-K_n) \right\|}, \text{ by definition}$$

$$\leq \frac{1}{\inf_{||x||=1} ||x|| \, (1-\epsilon)}$$

where $\epsilon = \left\| (I+K)^{-1}(K-K_n) \right\| < 1$

Slide 25

Final result

$$\lim_{n\to\infty} ||(\sigma_n - \sigma)|| \leq C \qquad \lim_{n\to\infty} ||(K-K_n)\sigma|| = 0$$

What does this mean?

The discretization convergence of a second kind integral equation solver depends on how well the integral is approximated.

The final result, noted on the above slide, is that the solution error is bounded by a constant multiplying the error in the integral operator representation. This suggests that any method which can accurately represent the integral operator can be used to discretize a second-kind integral equation.

4 Nystrom Method

4.1 1-D Second Kind Example

4.1.1 Collocation Discretization

Slide 26

Integral Equation

$$\Psi(x) = \sigma(x) + \int_{-1}^{1} G(x,x')\sigma(x')dS' \qquad x \in [-1,1]$$

Apply **quadrature** to **Collocation equation**

$$\Psi(x_i) = \sigma(x_i) + \int_{-1}^{1} G(x_i,x')\sigma(x')dS'$$

$$\Rightarrow \Psi(x_i) = \sigma(x_i) + \sum_{j=1}^{n} w_j G(x_i,x_j)\sigma(x_j)$$

After applying quadrature to Collocation:

$$\Psi(x_i) = \sigma(x_i) + \sum_{j=1}^{n} w_j G(x_i,x_j)\sigma(x_j)$$

x_i is a collocation point
x_j's are quadrature points
Now set **quadrature points = collocation points**

In Gaussian quadrature, described in the previous lecture, an integral is approximated using weighted combinations of the integrand. As a reminder, the Gaussian quadrature formula for integrating a function on the unit interval is

$$\int_0^1 f(x)dx \simeq \sum_{i=1}^{n} w_i f(x_i)$$

where the x_i's are the evaluation points given by the zeros of an n^{th}-order orthogonal polynomial on the unit interval, and the w_i's are the weights determined by solving exactness equations.
The key idea behind a Nystrom method for discretizing an integral equation is to use the Gaussian quadrature evaluation points as the test points in a collocation method for solving an integral equation. Then, the collocation method integrals can be approximated using the Gaussian quadrature scheme, resulting in a system of equations which only require evaluations of the integrand at the test=quadrature points. The second kind theory predicts that the error in such a scheme is proportional to the error in the quadrature scheme for computing the collocation integrals.

SLIDE 27

Set **quadrature points = collocation points**

$$\Psi(x_1) = \sigma_{n1} + \sum_{j=1}^{n} w_j G(x_1, x_j)\sigma_{nj}$$

$$\vdots$$

$$\Psi(x_n) = \sigma_{nn} + \sum_{j=1}^{n} w_j G(x_n, x_j)\sigma_{nj}$$

System of n equations in n unknowns
Collocation equation per quad/colloc point
Unknown density per quad/colloc point

4.1.2 Discretization-Matrix Comparison

SLIDE 28

Nystrom Matrix

$$\begin{bmatrix} 1+w_1G(x_1,x_1) & \cdots & w_nG(x_1,x_n) \\ \vdots & \ddots & \vdots \\ w_1G(x_n,x_1) & \cdots & 1+w_nG(x_n,x_n) \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi(x_1) \\ \vdots \\ \Psi(x_n) \end{bmatrix}$$

Piecewise Constant Collocation Matrix

$$\begin{bmatrix} 1+\int_{x_0}^{x_1} G(x_{c_1},x')dS' & \cdots & \int_{x_{n-1}}^{x_n} G(x_{c_1},x')dS' \\ \vdots & \ddots & \vdots \\ \int_{x_0}^{x_1} G(x_{c_n},x')dS' & \cdots & 1+\int_{x_{n-1}}^{x_n} G(x_{c_n},x')dS' \end{bmatrix} \begin{bmatrix} \sigma_{n1} \\ \vdots \\ \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \Psi(x_{c_1}) \\ \vdots \\ \Psi(x_{c_n}) \end{bmatrix}$$

Nystrom Matrix

SLIDE 29

Just Green's function evaluations – No integrals
Entries each have a quadrature weight
Collocation points are quadrature points
High order quadrature=faster convergence?

Piecewise Constant Collocation Matrix

Integrals of Green's function over line sections
Distant terms equal Green's function
Collocation points are basis function centroids
Low order method always

4.2 K_n and Ψ_n for Nystrom Method

SLIDE 30

$$K_n\sigma = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} w_j G(x_i, x_j)\sigma(x_j) \right) \varphi_i(x)$$

$$\Psi_n = \sum_{i=1}^{n} \Psi(x_i)\varphi_i(x)$$

4.3 Convergence

4.3.1 Theorem

SLIDE 31

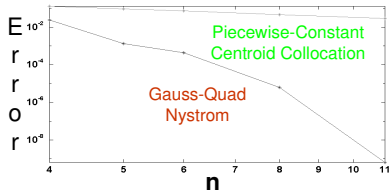
In the limit as $n \to \infty$ (number of quad points $\to \infty$)
The discretization error $= \max_{||\sigma||=1} ||(K - K_n)\sigma|| \to 0$
AT THE SAME RATE as the underlying quadrature!!

Gauss Quadrature $\Rightarrow$ Exponential Convergence!

4.3.2 Comparison

SLIDE 32

$$\cos 2\pi x = \sigma(x) + \int_{-1}^{1} (x - x')^2 \sigma(x') dS'$$



4.3.3 Caveat

SLIDE 33

If Nystrom method can have exponential convergence, why use anything else?

Gaussian quadrature has exponential convergence for **nonsingular** kernels

Most physical problems of interest have **singular kernels** ($\frac{1}{r}$, $\frac{\exp ikr}{r}$, etc)

5 Summary

SLIDE 34

Convergence Issues in 1D
: 1st and 2nd kind integral equations, null spaces

Convergence for second kind equations
: Show consistency and stability issues

Nystrom methods
: High order convergence

Did not address singular integrands