stability analysis for Poisson's equation (elliptic)

1. How to approximate a derivative with arithmetic?
2. How much error does the approximation cause?
3. **How does the approximation error affect the solution?**

$$\frac{\partial^2 u}{\partial x^2} + f = 0$$

$$A\hat{u} + f = 0$$

$$Au + f = \tau$$

$$A(\underbrace{u - \hat{u}}_{e}) = \tau$$

$$e = A^{-1}\tau$$

$$Au_i : \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2}$$

$$Au + f : \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2} + \cancel{f} - \left(\frac{\partial^2 u}{\partial x^2} + \cancel{f}\right)$$

$$\tau = \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2} - \frac{\partial^2 u}{\partial x^2}$$

How does error in solving $Au = -f$ affect the solution?

$$Au_r + f = r \qquad Au + f = 0 \qquad A(u_r - u) = r$$

$$\frac{\|r\|}{\|f\|} = \varepsilon \qquad \text{how does } \frac{\|u_r - u\|}{\|u\|} \text{ depend on } \varepsilon\text{?}$$

$$\frac{\frac{\|u_r - u\|}{\|u\|}}{\frac{\|r\|}{\|f\|}} = \frac{\|u_r - u\|}{\|u\|}\frac{\|f\|}{\|r\|} = \frac{\|A^{-1}r\|}{\|u\|}\frac{\|-Au\|}{\|r\|}$$

$$= \frac{\|A^{-1}r\|}{\|r\|}\frac{\|Au\|}{\|u\|} \le \|A^{-1}\|\|A\| =: \operatorname{cnd}(A)$$

Nonlinear conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g$$

$$\frac{\partial \vec{u}}{\partial t} + \nabla \cdot \vec{\vec{F}}(\vec{u}) = \vec{g}$$

$\vec{u}$: m-dimensional vector

$\nabla \cdot$: d-dimensional divergence

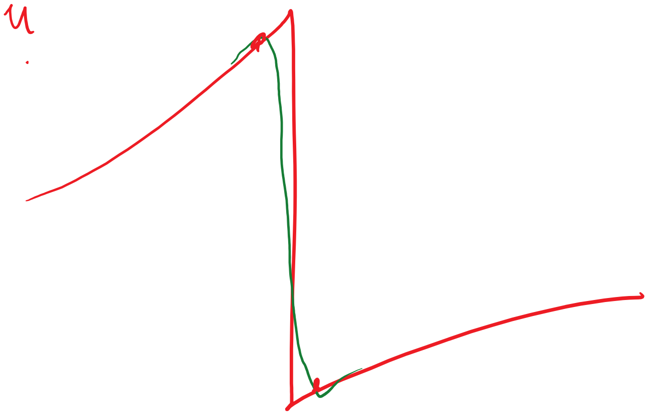
$\vec{\vec{F}}$: d x m-dimensional tensor

$f(u) = U \cdot u \qquad \frac{df}{du} = U.$

Linear advection

$f(u) = \frac{u^2}{2} \qquad \frac{df}{du} = u$

Burgers equation



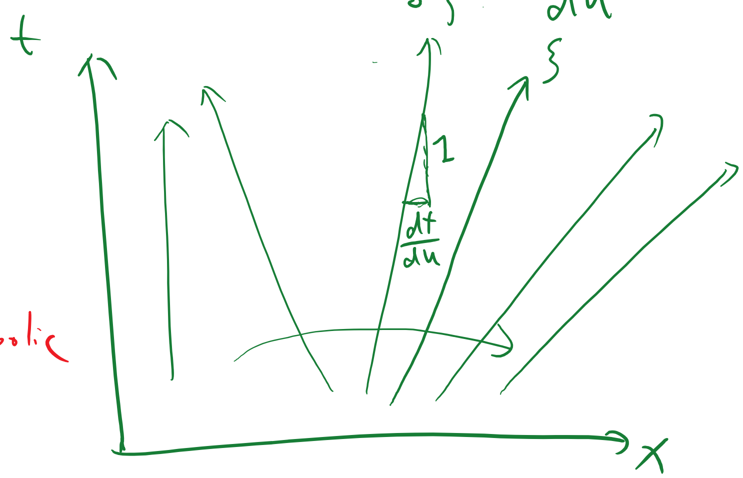
Scalar conservation laws -- smooth region

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad g.$$ conservative form.

$$\frac{\partial u}{\partial t} + \frac{df}{du} \cdot \frac{\partial u}{\partial x} = 0 \quad g.$$ primitive form.

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial t} + \frac{df}{du} \cdot \frac{\partial u}{\partial x} = 0 \; g \quad \text{if} \quad \frac{\partial t}{\partial \xi} = 1$$

$$\& \quad \frac{\partial x}{\partial \xi} = \frac{df}{du}$$



$$\frac{\partial u}{\partial \xi} = g$$

All conservation laws are hyperbolic

Scalar conservation laws -- shock wave

For any Ω ← control volume.

$$\frac{d}{dt}\left(\int_\Omega u\, dV\right) = \text{Influx through } \partial\Omega$$

$$= \int_{\partial\Omega} \vec{f}(u)\cdot(-\vec{n})\, dA + \int_\Omega g\cdot dV$$

$$\Updownarrow$$

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{f}(u) = g$$

Divergence Thm:

$$\boxed{\int_{\partial\Omega} \vec{f}\cdot\vec{n}\, dA = \int_\Omega \nabla\cdot\vec{f}\, dV}$$

$$\frac{d}{dt}\int_\Omega u\, dV + \int_\Omega \nabla\cdot\vec{f}\, dV = \int_\Omega g\, dV \quad \forall\ \Omega$$

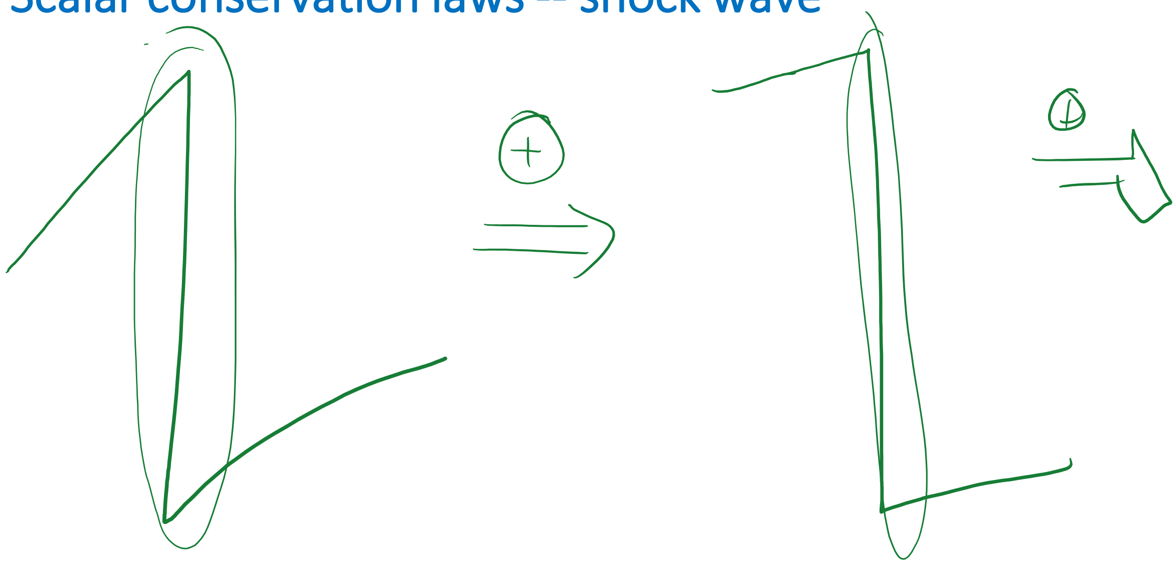
$\Omega = [a, b]$

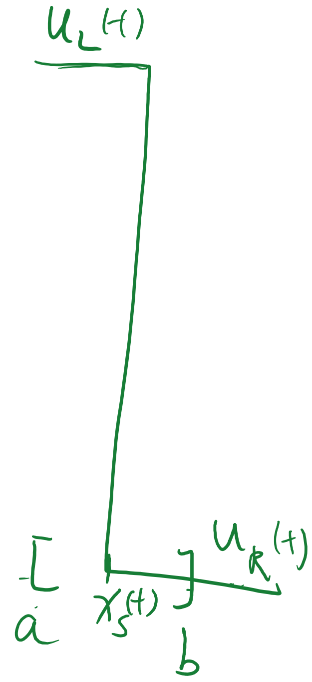
$n=-1 \qquad n=1$

$$\frac{d}{dt}\int_a^b u\, dx = f(u)\Big|_a - f(u)\Big|_b + \int_a^b g\, dx$$

integral form

Scalar conservation laws -- shock wave

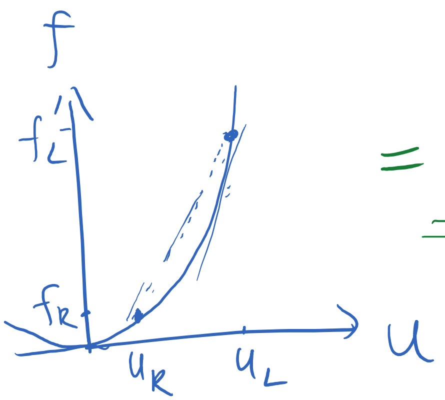




$$\int_a^b u\,dx = \int_a^{x_s} + \int_{x_s}^b$$

$$= u_L(x_s - a) + u_R(b - x_s)$$

$$\frac{d}{dt}\int_a^b u\,dx = \cancel{\underset{=0}{\frac{du_L}{dt}}(x_s - a) + \underset{=0}{\frac{du_R}{dt}}(b - x_s)} + \frac{dx_s}{dt}(u_L - u_R)$$



$$= f(u_L) - f(u_R)$$

$$\frac{dx_s}{dt} = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

Burgers : $f = \frac{u^2}{2}$

$$= \frac{u_L^2 - u_R^2}{2(u_L - u_R)}$$

$$= \frac{u_L + u_R}{2}$$

Characteristic speed : $\frac{df}{du}$