Finite element -- linear function space and basis

X is a linear function space

if ① for any number a and $f \in X$, $a \cdot f \in X$

② for any $f, g \in X$. $f + g \in X$

$B \subset X$ is a basis if

① Linear independence:

for any $f_1, f_2, \cdots, f_n \in B$,
there cannot be $a_1, \cdots, a_{n-1}$

$$f_n = a_1 f_1 + a_2 f_2 + \cdots a_{n-1} f_{n-1}$$

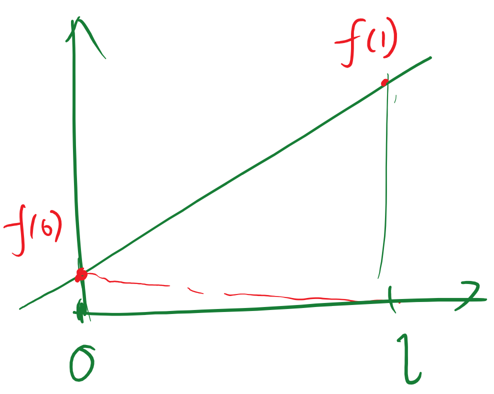
② B spans X

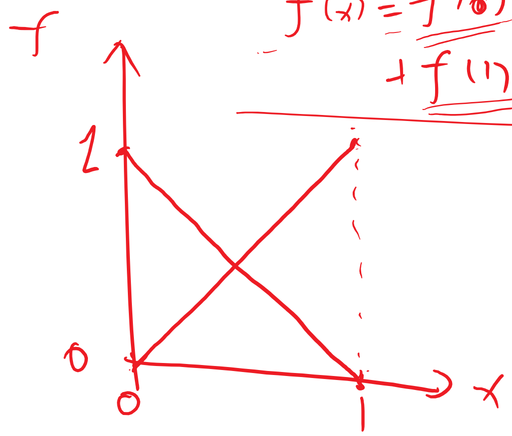
for any $f \in X$
there exist $f_1, \cdots, f_n \in B$

$$f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$$

If there exist a basis B with N "basis functions" in it,
then X is called N-dimension

$$f(x) = f(0) \cdot (1 - x) + f(1) \cdot x$$





Inner product in linear function space and Projection

$$\langle f, g\rangle = \langle g, f\rangle$$

$$\langle af, g\rangle = a\langle f, g\rangle$$

$$\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$$

$$\Rightarrow \langle f, f\rangle > 0 \text{ unless } f = 0$$

$$\boxed{\langle f, g\rangle_{L_2} := \int_\Omega f\cdot g \, dx}$$

$$\min_{f\in X} \langle f_0 - f, f_0 - f\rangle \Longrightarrow f \text{ is as close to } f_0 \text{ as possible}$$

$$\min_{a_1, a_2} \langle f_0 - a_1 h_1 - a_2 h_2, f_0 - a_1 h_1 - a_2 h_2\rangle := \min_{a_1, a_2} \langle f_0, f_0\rangle - a_1\langle h_1, f_0\rangle + \ldots$$

$$B = \{h_1, h_2\}$$

$$f = a_1 h_1 + a_2 h_2$$

$$\langle -a_2 h_2, -a_2 h_2\rangle$$

$$= -a_2\langle +h_2, -a_2 h_2\rangle$$

$$= (-a_2)^2 \langle h_2, h_2\rangle$$

Inner product in linear function space and Projection

$$\min_{f \in X} \langle f - f_0, f - f_0 \rangle$$

for any $df \in X$, $\langle (f+df) - f_0, (f+df) - f_0 \rangle \geq \langle f - f_0, f - f_0 \rangle$

$$\langle f - f_0 + df, f - f_0 + df \rangle \geq \langle f - f_0, f - f_0 \rangle$$

$\Rightarrow$ $\langle f - f_0, f - f_0 \rangle + \langle df, f - f_0 \rangle + \langle f - f_0, df \rangle + \langle df, df \rangle$

$\geq \langle f - f_0, f - f_0 \rangle$

$\Rightarrow$ $2 \langle df, f - f_0 \rangle + \langle df, df \rangle \geq 0 \quad \forall\, df$

$\Rightarrow$ $\langle df, f - f_0 \rangle = 0$

if $\langle df, f - f_0 \rangle < 0$,

then $\exists\, \varepsilon$ s.t. $\langle \varepsilon df, f - f_0 \rangle + \langle \varepsilon df, \varepsilon df \rangle < 0$

We have proved that

$$\min_{f \in X} \langle f - f_0, f - f_0 \rangle \Longrightarrow \forall\, df \in X, \ \langle df, f - f_0 \rangle = 0$$

$f - f_0 \perp X$

Projection into finite dimensional linear function space

$$f = \sum_{i=1}^{N} a_i h_i \qquad \{h_1, \dots, h_N\} = B$$

if $\quad \langle h_j, f_0 - \sum_{i=1}^{N} a_i h_i \rangle = 0 \quad \forall j = 1, \dots, N$

then, $\quad \langle df, f_0 - f \rangle = 0 \quad \forall df$

finding f in an infinite set to satisfy ∞ equation

finding N numbers to satisfy N equation

$$\langle h_j, f_0 \rangle - \sum_{i=1}^{N} a_i \langle h_j, h_i \rangle = 0$$

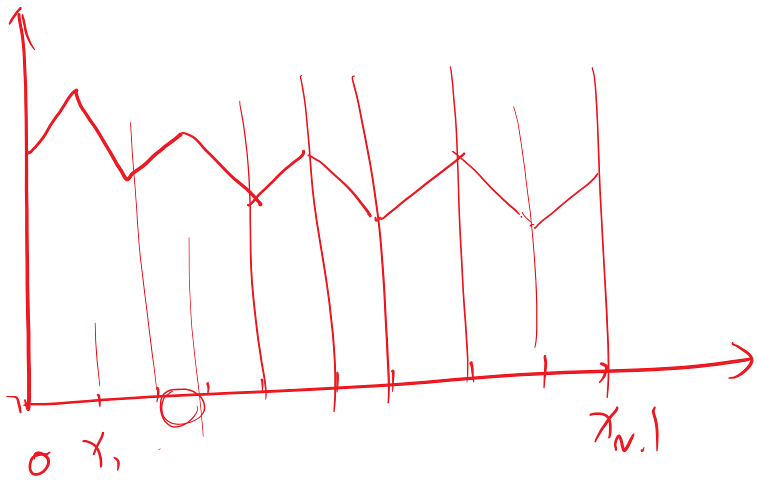
$Ax = b$. where $x = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$, $b = \begin{pmatrix} \langle h_1, f_0 \rangle \\ \langle h_2, f_0 \rangle \\ \vdots \\ \langle h_N, f_0 \rangle \end{pmatrix}$

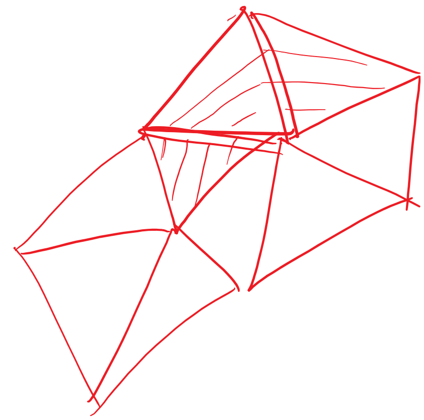
$$A = \begin{pmatrix} \langle h_1, h_1 \rangle & \cdots & \langle h_1, h_N \rangle \\ & & \\ \langle h_N, h_1 \rangle & \cdots & \langle h_N, h_N \rangle \end{pmatrix}$$

$$b_i = \int_\Omega f_0 \, h_i \, dx$$

$$A_{ij} = \int_\Omega h_i h_j \, dx$$

Example: Projection into the space of continuous, piecewise linear functions



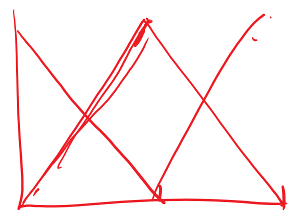


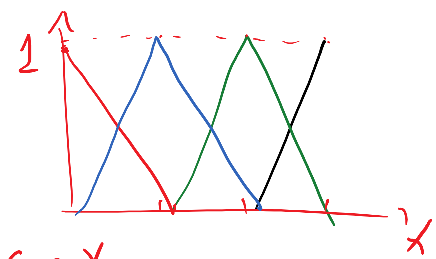
$N = 1$ $\quad D = 2$

$N = 2$ $\quad D = 3$

$N = 3$ $\quad D = 4$

$D = N + 1$





$f \in X$

$f = \sum a_i h_i$

where $a_i = f(x_i)$

$Ax = b.$

$A_{ij} = \int_\Omega h_i h_j \, dx$

$b_i = \int_\Omega f_0 h_i \, dx$

Using projection for solving Poisson's equation

with 0 Dirichlet B.C.

$$\frac{d^2u}{dx^2} + f = 0$$

$$u \in X$$

$$u = \sum_{i=1}^{N} a_i h_i$$

$$\frac{d^2u}{dx^2} + f = r, \qquad \underset{f \in X}{\operatorname{argmin}} \langle r - f, r - f \rangle = 0$$

$$\Updownarrow$$

$$\forall\, df, \quad \langle df, r - 0 \rangle = 0$$

$$\langle df, \frac{d^2u}{dx^2} + f \rangle = 0$$

$$\langle df, \sum_{i=1}^{N} a_i \frac{d^2h_i}{dx^2} + f \rangle = 0$$

$$\forall\, df \quad \sum_{i=1}^{N} a_i \langle df, \frac{d^2h_i}{dx^2} \rangle + \langle df, f \rangle = 0$$

$$\forall j = 1, \dots, N \quad \sum_{i=1}^{N} a_i \langle h_j, \frac{d^2h_i}{dx^2} \rangle + \langle h_j, f \rangle = 0$$

$$Ax = b. \qquad x = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \qquad b = \begin{pmatrix} \langle h_1, f \rangle \\ \vdots \\ \langle h_N, f \rangle \end{pmatrix}.$$

$$A_{ji} = \langle h_j, \frac{d^2h_i}{dx^2} \rangle.$$