Today:

11.4] Review problem] Tangent plane
Linear Approximation
Differentials
$f(x,y)$

11.6] Chain rule

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$f = f(x,y)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

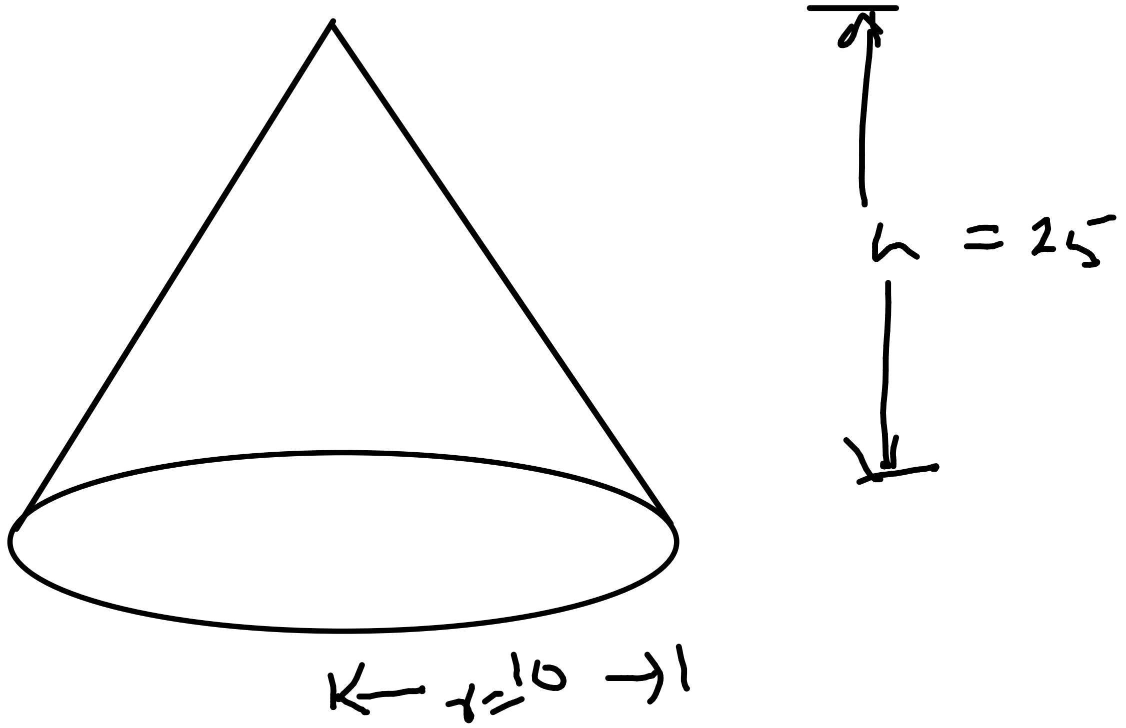
EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

$$V(r,h) = \frac{1}{3}\pi r^2 h$$

$$dV = ??$$

$$\boxed{dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh}$$

$$dr = dh = 0.1$$



$$dV = \underbrace{\frac{2}{3}\pi r h\, dr + \frac{1}{3}\pi r^2 dh}$$

$$r = 10, \quad h = 25$$

$$dr = dh = 0.1$$

$$= 20\pi$$

$$\frac{dV}{V} = ??\ \%$$

$$= \frac{3}{125}$$

$$\approx 0.024$$

$$= 2.4\ \%$$

1–6 ▪ Find an equation of the tangent plane to the given surface at the specified point.

$$z = y\cos(x-y), \quad (2, 2, 2)$$

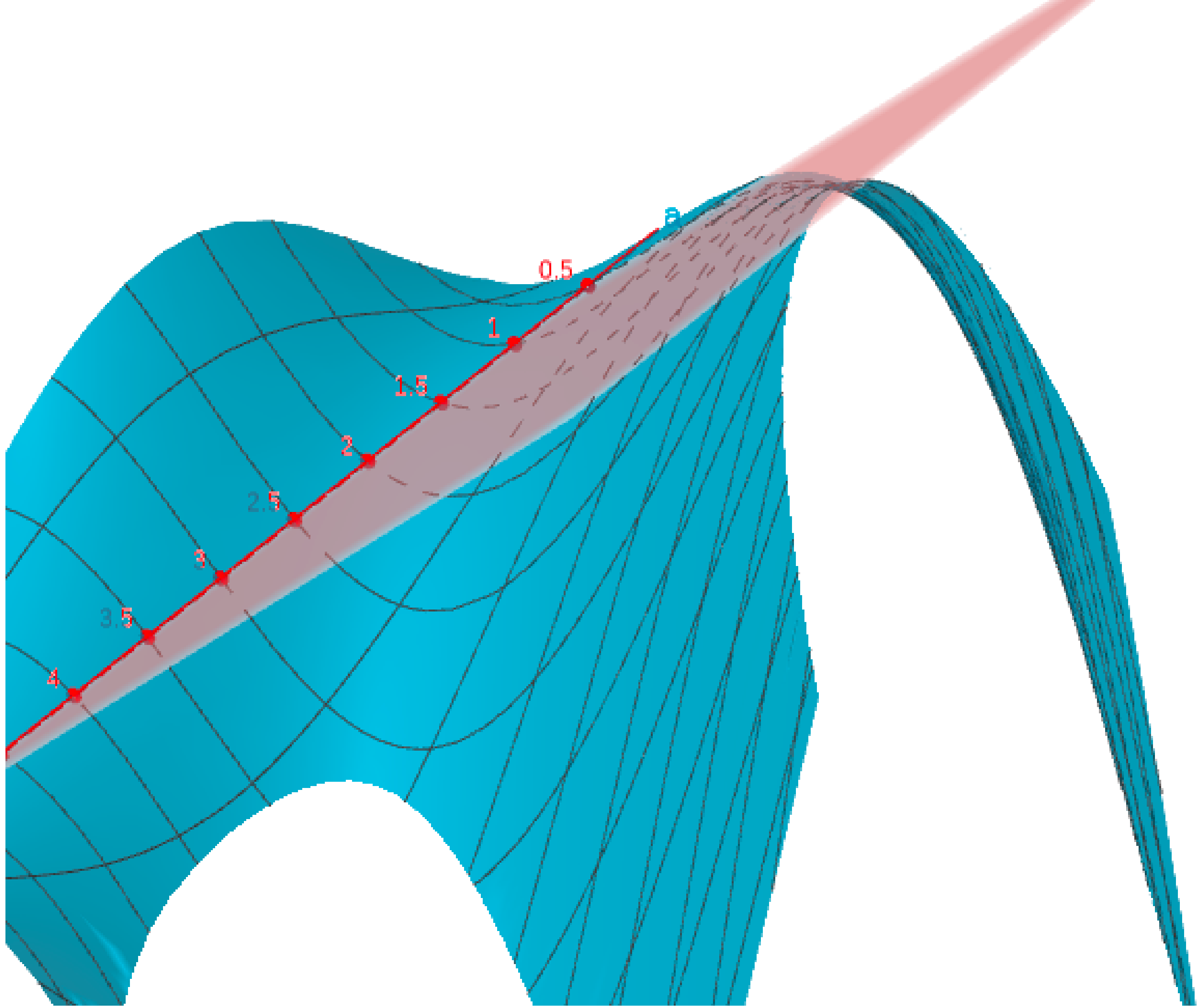
$$\boxed{z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)}$$

$$\frac{\partial z}{\partial x} = -y\sin(x-y)\Big|_{\substack{x=2\\y=2}} = 0$$

$$\frac{\partial z}{\partial y} = \left[y\sin(x-y) + \cos(x-y)\right]\Big|_{\substack{x=2\\y=2}} = 1$$

$$z - 2 = 0(x-2) + 1(y-2)$$

$$\boxed{z = y}$$



0.5
1
1.5
2
2.5
3
3.5
4

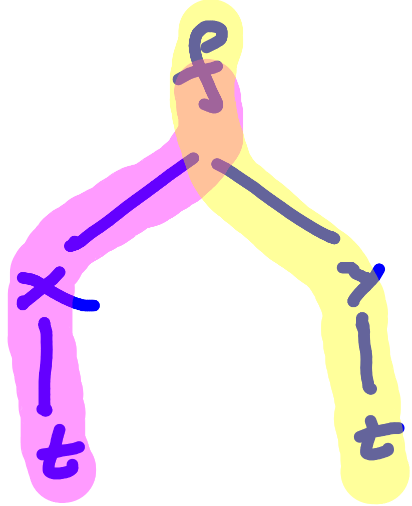
30. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

$$P = 8.31\,T/V$$

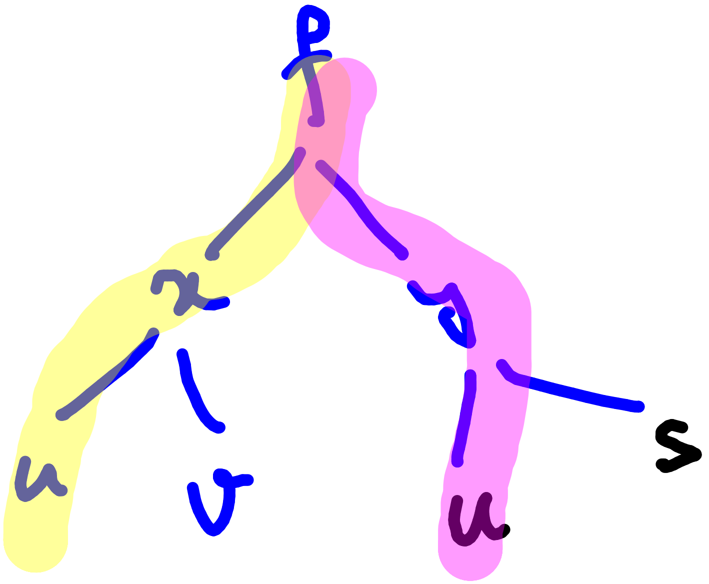
$dP = ??$ given $V = 12$, $T = 310$, $dV = 0.3$, $dT = -5$

$$dP = \frac{\partial P}{\partial V}\,dV + \frac{\partial P}{\partial T}\,dT$$

$$= -8.8$$



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}$$

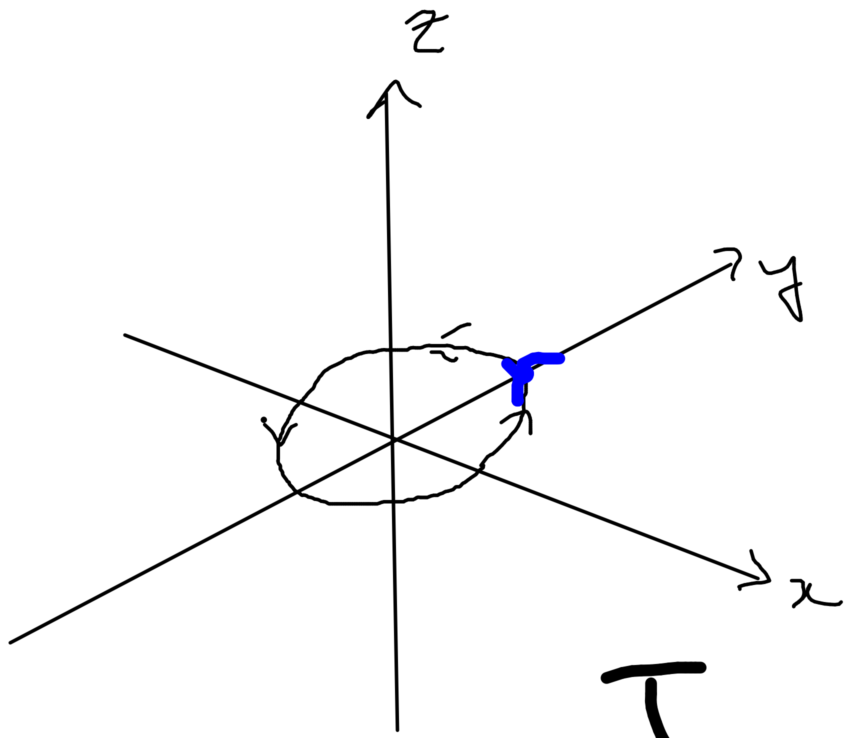
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Q: $T(x,y) = x+y$

Temperature at point (x,y)

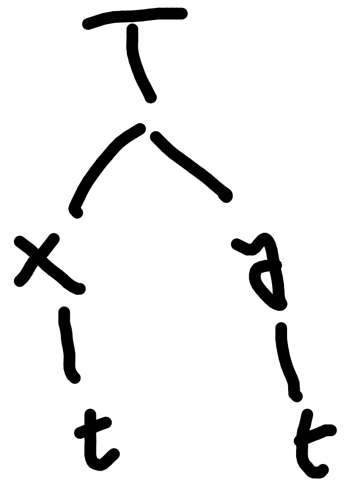
$$\underbrace{x = \cos(t), \quad y = \sin(t)}_{\text{what path is this ??}}, \quad 0 < t < \infty$$

$$\vec{r}(t) = \cos t\,\hat{i} + \sin(t)\,\hat{j}$$



Q: rate of change of temperature T w.r.t. time t

$$\frac{dT}{dt} = ?? \quad \text{at } t = \pi/2$$

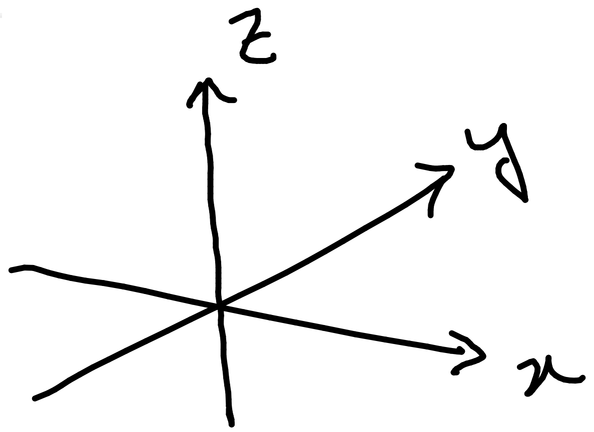


$$= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

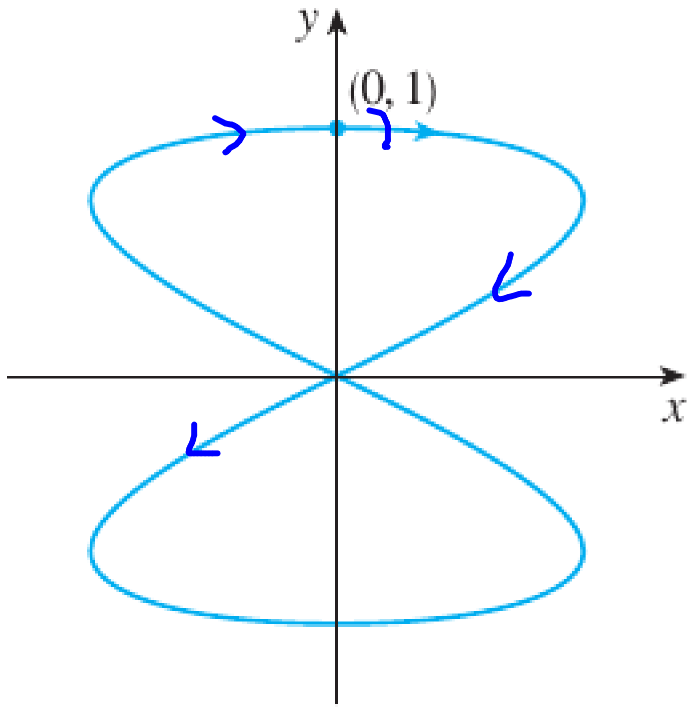
$$= \Big[1(-\sin t) + 1 \cdot \cos(t)\Big]_{t=\pi/2}$$

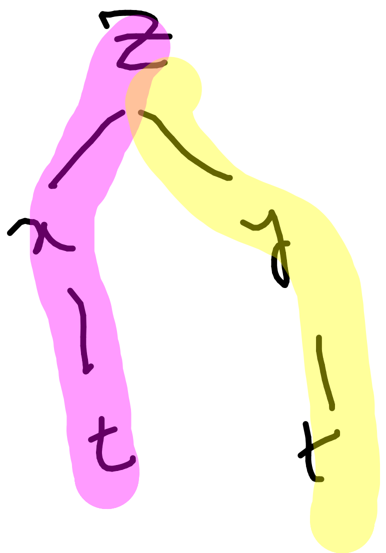
$$= -1$$

EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.



$$x = \sin 2t$$
$$y = \cos t$$





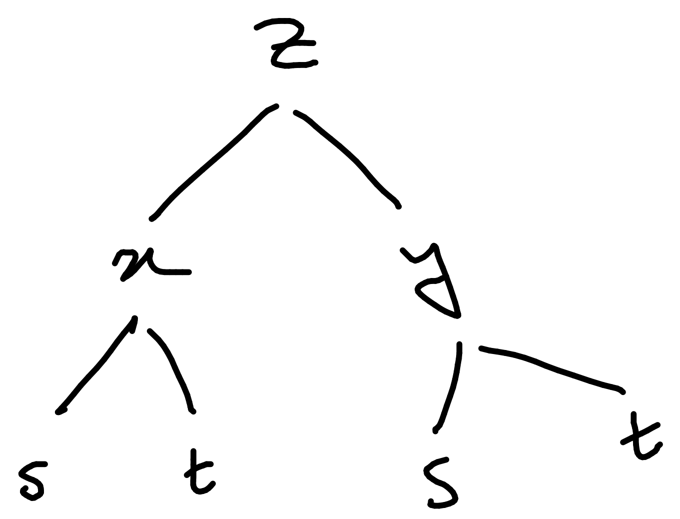
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= \underbrace{(2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)}_{t=0,\ x=0,\ y=1}$$

$$= 6$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

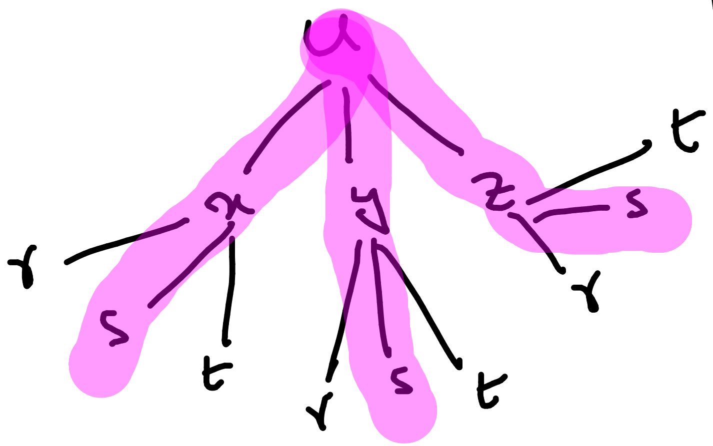
$$z = e^x \sin y$$



$$\frac{\partial z}{\partial s} = e^x \sin y \; t^2 + e^x \cos y \; 2st$$

$$\frac{\partial z}{\partial t} = e^x \sin y \; 2st + e^x \cos y \; s^2$$

V EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= \underbrace{4x^3y\, re^t + (x^4 + 2yz^3)\, 2rse^{-t} + (3y^2z^2)\, r^2\sin t}_{r=2,\ s=1,\ t=0}$$

$x = 2$, $y = 2$, $z = 0$

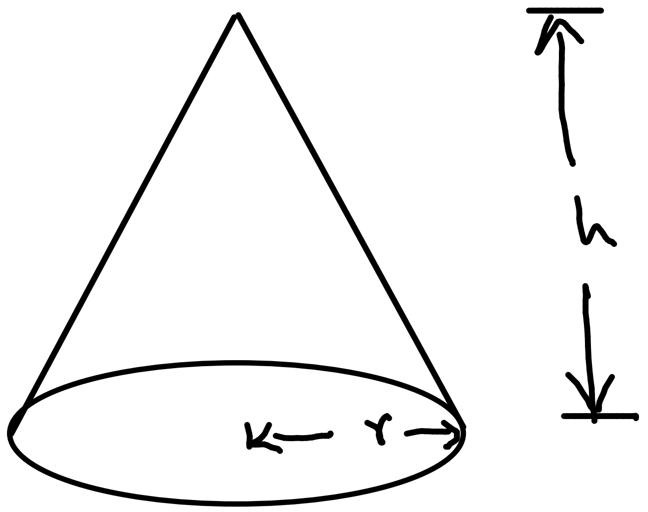
$$= 192$$

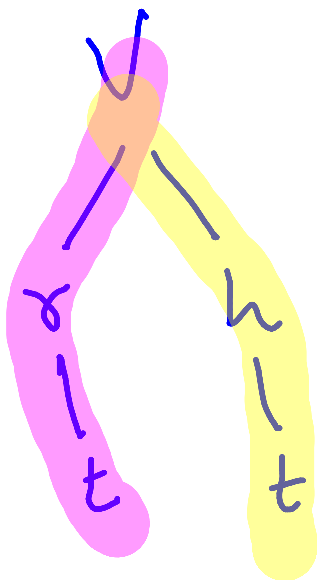
Today:

→ Review : chain rule

→ 11.6 : Directional Derivatives & gradients

32. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?





$$\frac{dh}{dt} = -2.5 \qquad \frac{dr}{dt} = 1.8$$

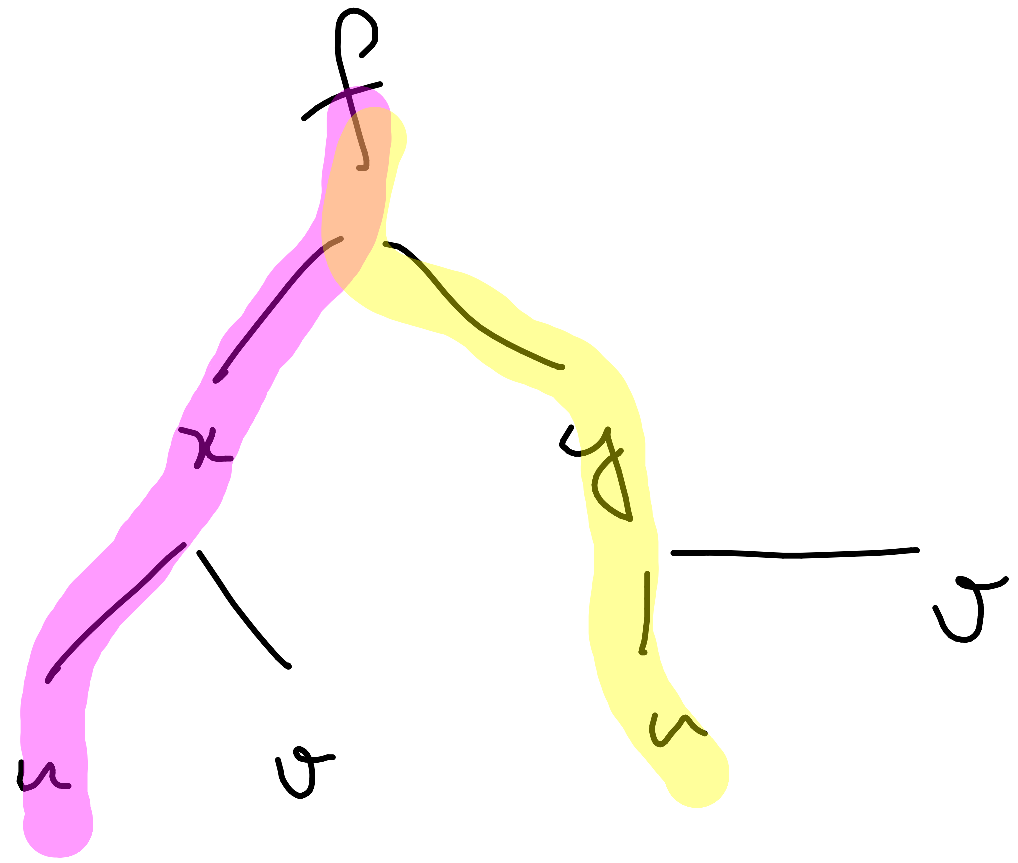
$$V = \frac{1}{3}\pi r^2 h$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=120\\ h=140}} = ?? = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi(120)(140)(1.8) + \frac{\pi}{3}(120)^3(-2.5)$$

$$= 8160\pi \text{ in}^3/\text{s}$$

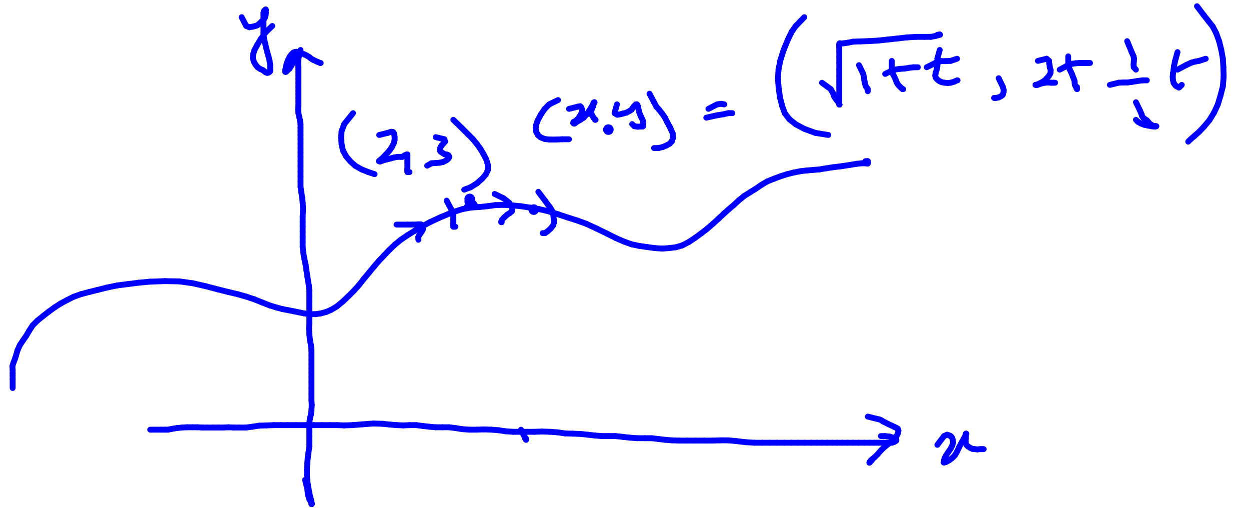


$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

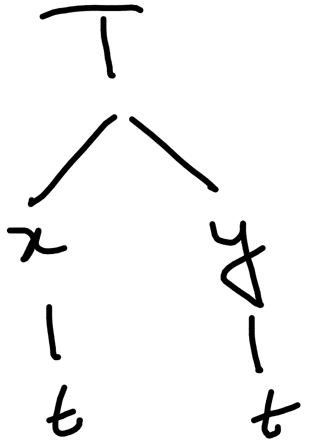
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$T_x = \frac{\partial T}{\partial x}$$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?



$t = 3 \qquad x = 2 \qquad y = 3$



$$\left.\frac{dT}{dt}\right|_{t=3} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= \left.\left(4 \cdot \frac{1}{2\sqrt{1+t}} + 3 \cdot \frac{1}{3}\right)\right|_{t=3}$$

$$= 1 + 1 = 2$$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

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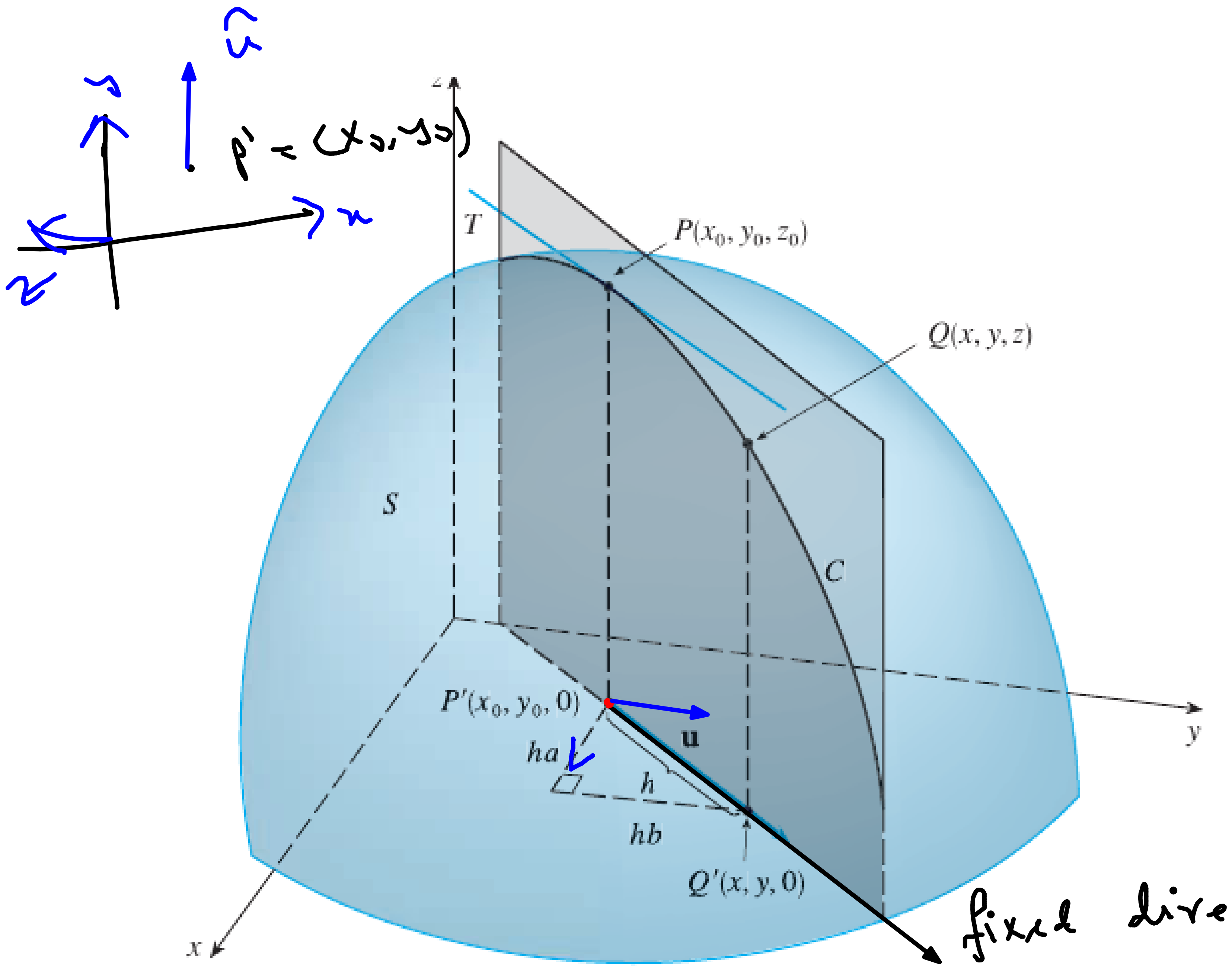


fixed direction

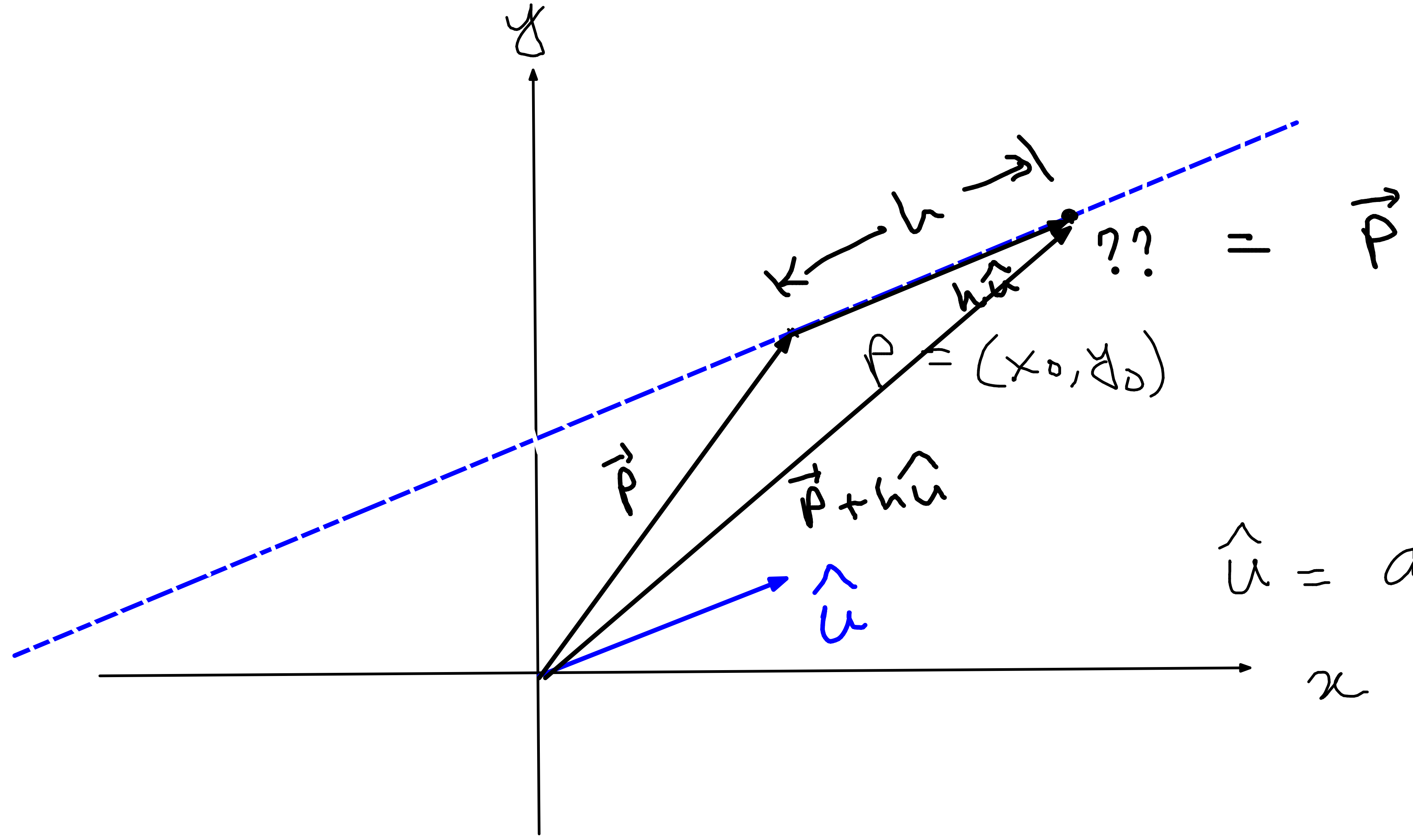
2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

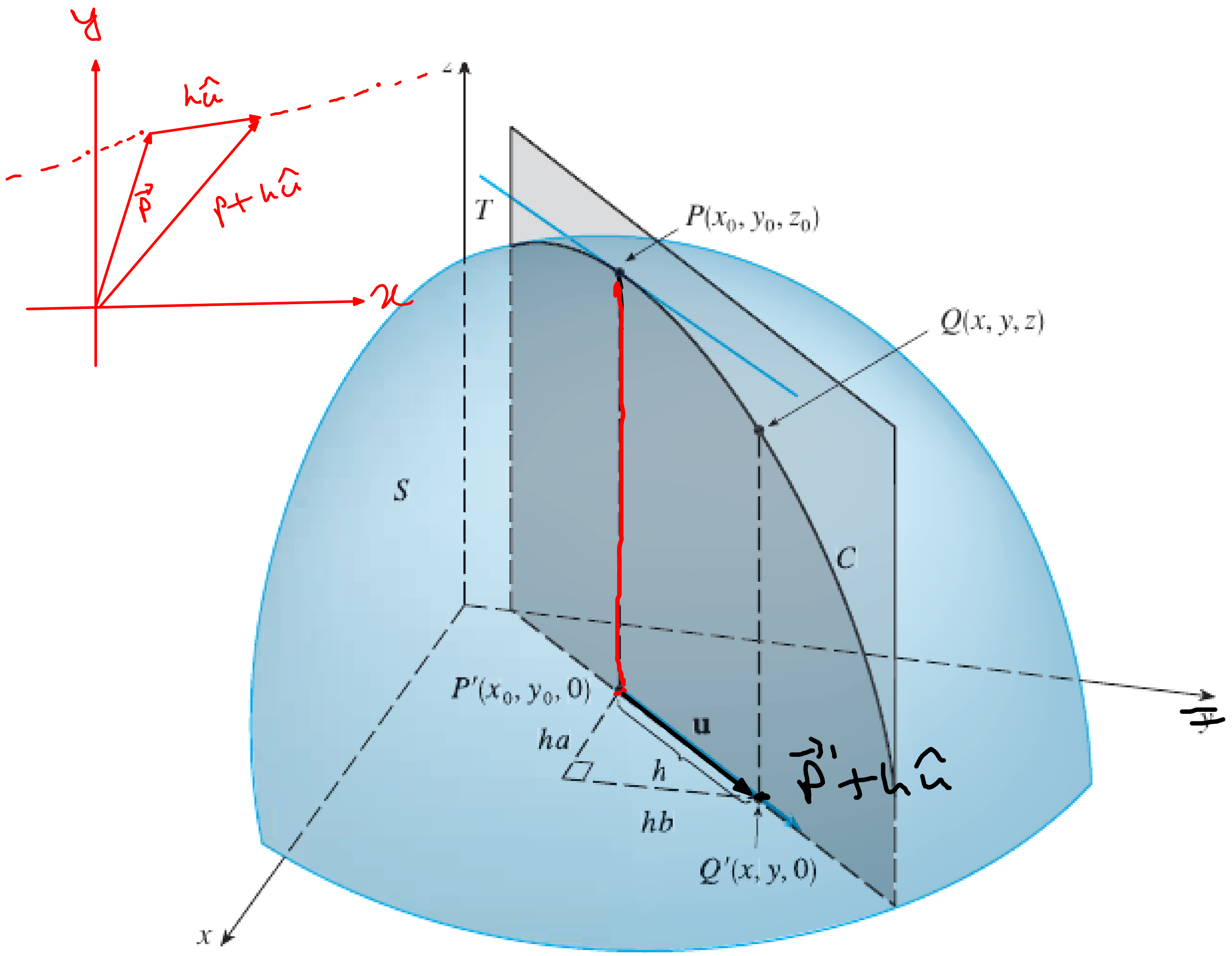
- blue surface is the graph of $f(x,y)$
- base point $(x_0, y_0) = P'$
- Q: directional derivative at point P', in the direction $\hat{u}$ = rate of change of f if we start moving from point P' along the direction $\hat{u}$

Q: sketch the line passing through p & parallel to $\hat{u}$



$\hat{u} = a\hat{i} + b\hat{j}$

11.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR



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if this limit exists.

$\hat{u} = a\hat{i} + b\hat{j}$

Directional derivative at point p' in the direction of $\hat{u}$

$$\lim_{h \to 0} \frac{f(\vec{p}' + h\hat{u}) - f(\vec{p}')}{h}$$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

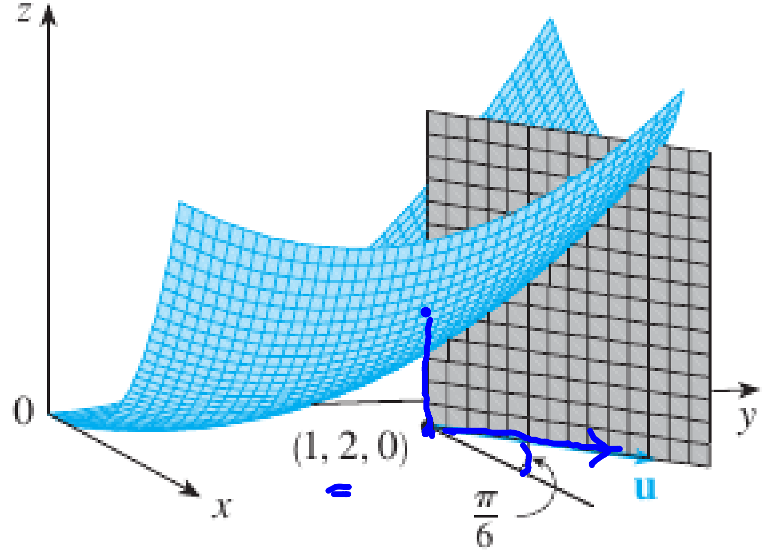
$$D_u f(x,y) = \lim_{h \to 0} \frac{f[(x,y) + h(x,y)] - f(x,y)}{h} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

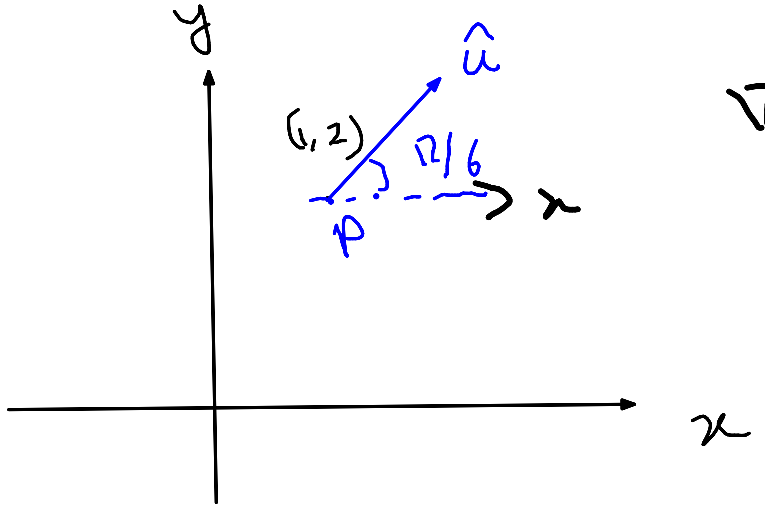
$$= \underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)}_{\text{gradient}\ \nabla f} \cdot \hat{u}$$

ok??

$$\boxed{D_u f(x,y) = \nabla f \cdot \hat{u}}$$

EXAMPLE 1 Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}}f(1, 2)$?





$$D_u f(1,2) = \nabla f \cdot \hat{u}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= (3x^2 - 3y)\hat{i} + (-3x + 8y)\hat{j}$$

$$\nabla f(1,2) = -3\hat{i} + 13\hat{j}$$

$$\hat{u} = ?? = \cos\frac{\pi}{6}\hat{i} + \sin\frac{\pi}{6}\hat{j}$$

$$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$D_u f(1,2) = (-3\hat{i} + 13\hat{j}) \cdot \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$= (13 - 3\sqrt{3})/2$$

8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

next time

V EXAMPLE 3 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

V EXAMPLE 4 If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

MAXIMIZING THE DIRECTIONAL DERIVATIVE

EXAMPLE 5

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

EXAMPLE 6 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?