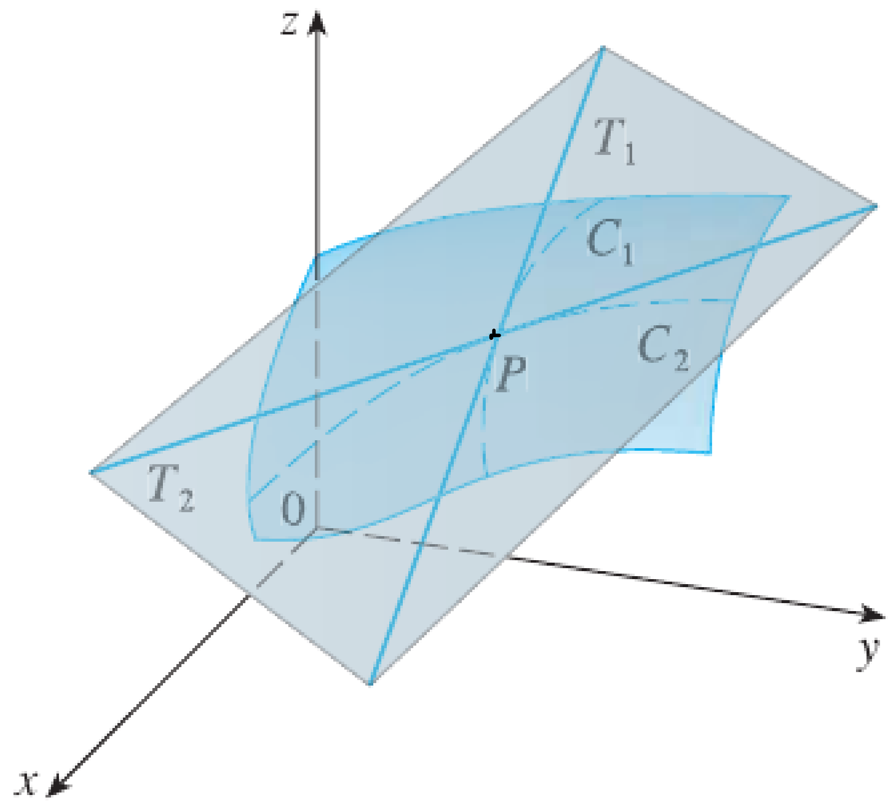
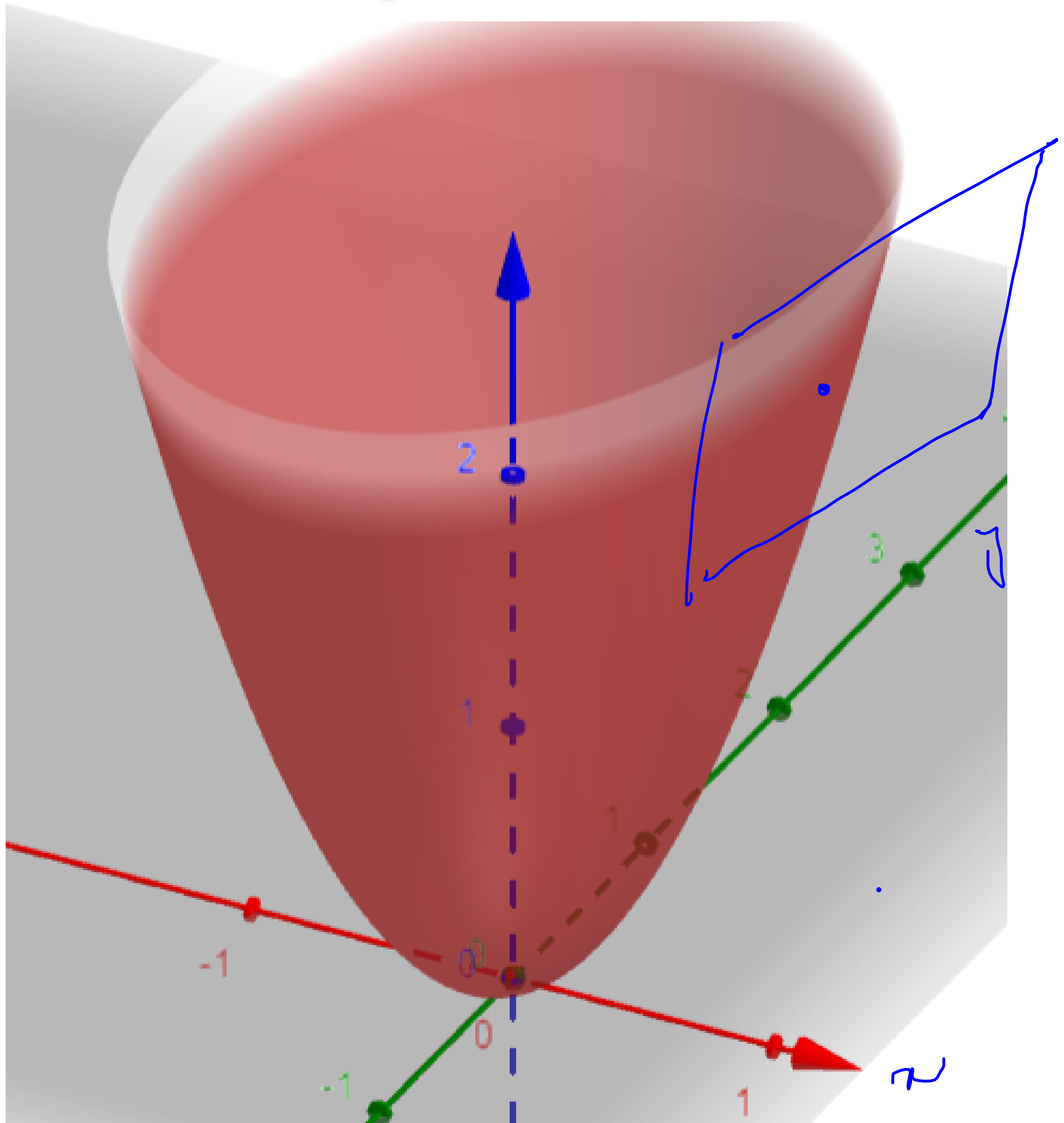
11.4 TANGENT PLANES AND LINEAR APPROXIMATIONS

why care for tangent planes?



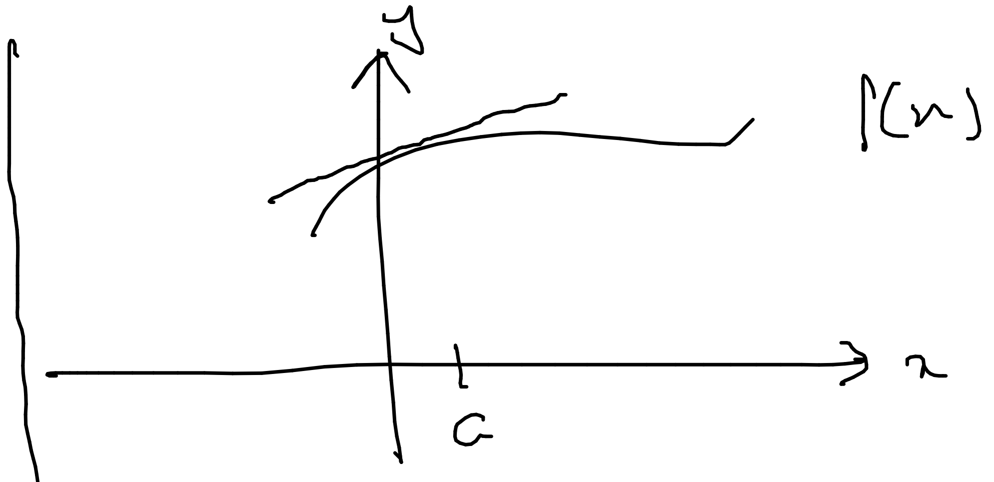
EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.



in 3d:

graph of $z = f(x,y)$

at point (a,b)

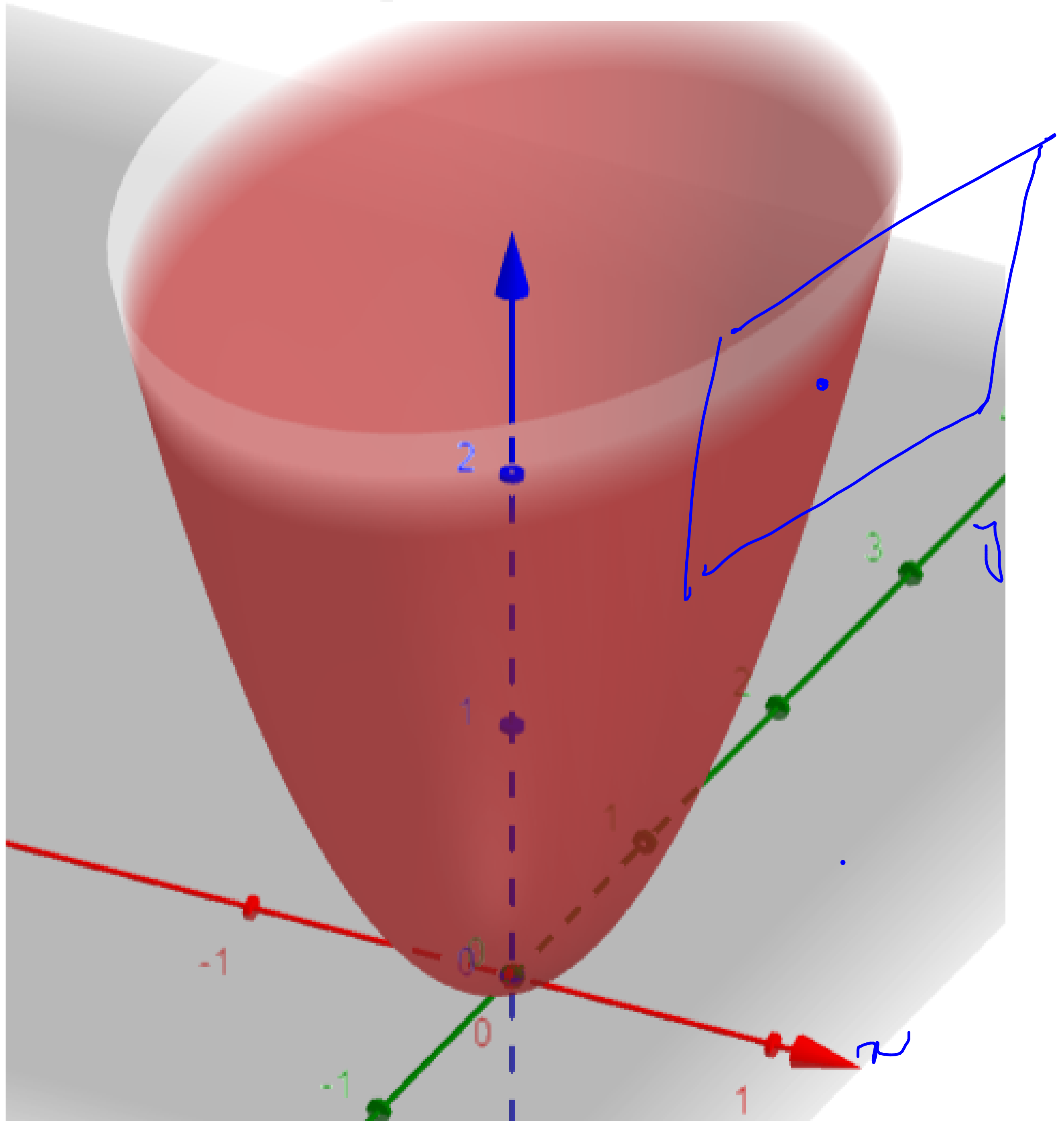


tangent line

$$y - f(a) = \frac{df}{dx}(a)\,(x-a)$$

$$\boxed{z - f(a,b) = \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b)}$$

EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.



in 3d:

graph of $z = f(x,y)$
at point (a,b)

$$\boxed{z - f(a,b) = \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b)}$$

$$z - \underbrace{f(1,1)}_{??} = \underbrace{\frac{\partial f}{\partial x}(1,1)}_{??}\,(x-1) + \underbrace{\frac{\partial f}{\partial y}(1,1)}\,(y-1)$$

$f(1,1) = 3$

$\frac{\partial f}{\partial x}(1,1) = 4x\Big|_{\substack{x=1\\y=1}} = 4$

$\frac{\partial f}{\partial y}(1,1) = 2y\Big|_{\substack{x=1\\y=1}} = 2$

the tangent plane :

$$\boxed{z-3 = 4(x-1) + 2(y-1)}$$

Q. Exercise:

find eqn of tangent plane

$$f(x,y) = xe^{y} \quad , \quad (1,0)$$

$$z - z_0 = \frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)(y-0)$$

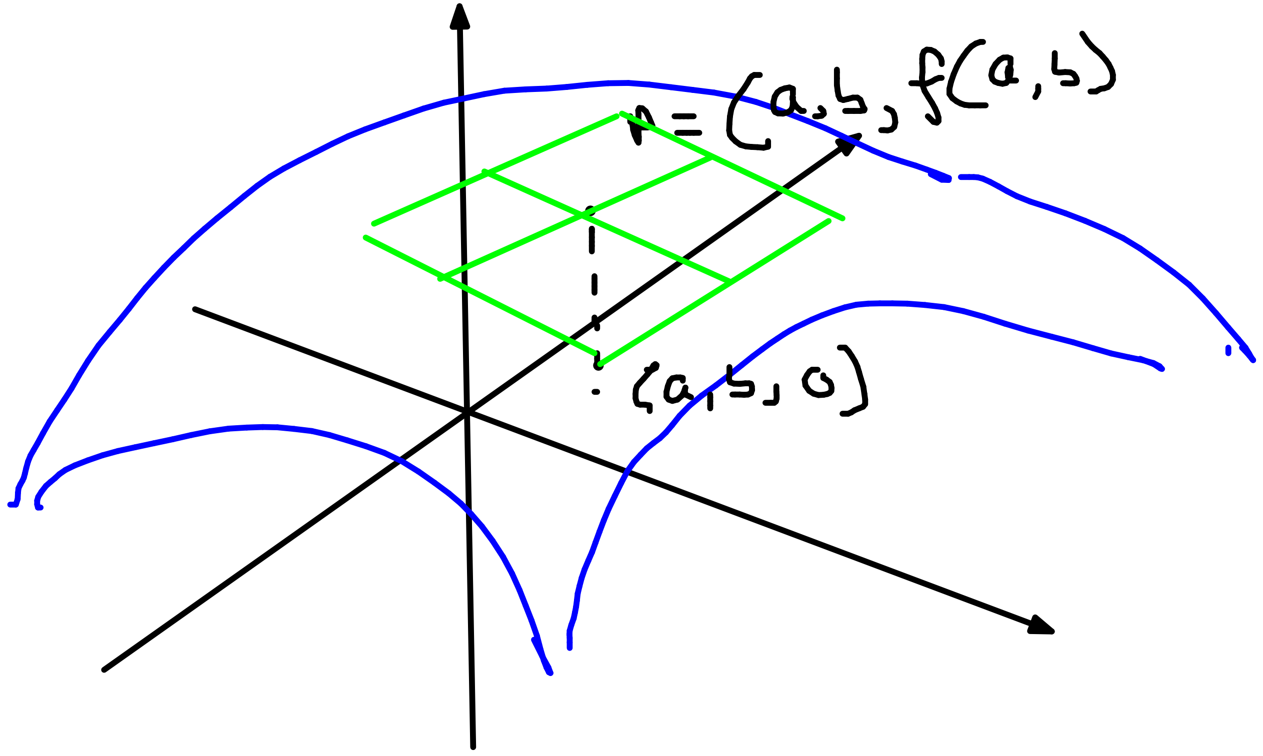
$$z_0 = f(1,0) = 1$$

$$\frac{\partial f}{\partial x} = e^{y}\Big|_{\substack{x=1 \\ y=0}} = 1$$

$$\frac{\partial f}{\partial y} = xe^{y}\Big|_{x=1, y=0} = 1$$

$$z - 1 = 1(x-1) + 1(y-0)$$

$$\boxed{z = x + y}$$



formula of tangent to graph of $f(x,y)$ at (a,b) is

$$z - z_0 = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$z_0 = f(a,b)$

plane contains the point P (a,b,z_0)

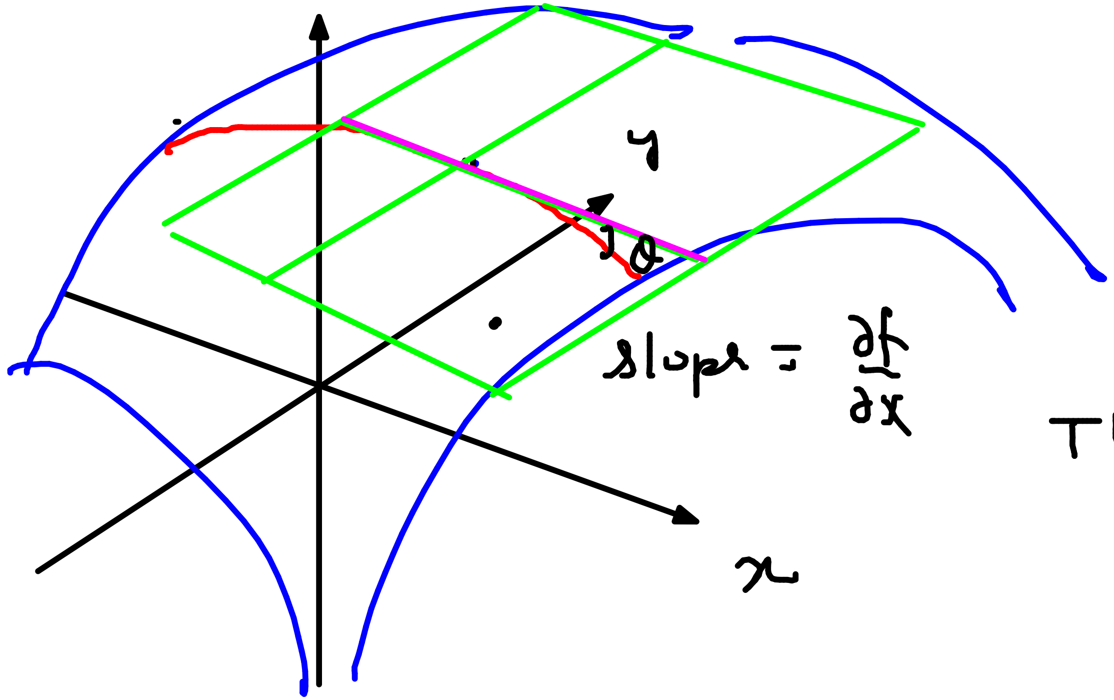
plane must look like

$$A(x-a) + B(y-b) + c(z-z_0) = 0$$

if $c = 0$, plane becomes vertical
not possible

→ simplified gen eqn passing trough (a,b,z_0)

$z = f(x,y)$



$[\,z - z_0 = A(x-a) + B(y-b)\,]$ green plane

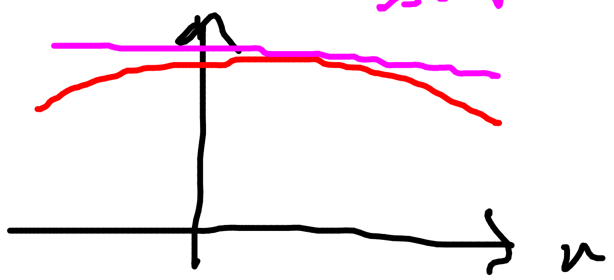
Q: $A = \frac{\partial f}{\partial x}(a,b)$, $B = \frac{\partial f}{\partial y}(a,b)$ } why??

Think: slice the graph $z = f(x,y)$ with plane $y = b$

& also slice the plane $z - z_0 = A(x-a) + B(y-b)$ with $y = b$

$z - z_0 = A(x-a)$

$p = (a, b, z_0)$



Q: what will be the slope of pink line??

A = slope of the pink line

$= \frac{\partial z}{\partial x}$

$$z - z_0 = A(x-a) + B(y-b)$$

we just discussed: $A = \frac{\partial f}{\partial x}(a,b)$:

H.W. repeat the argument to convince why

$$B = \frac{\partial f}{\partial y}(a,b)$$

Today:

11.4 remaining topics:

11.5: chain rule

$$f(x,y) \qquad \frac{\partial f}{\partial u} = ??$$

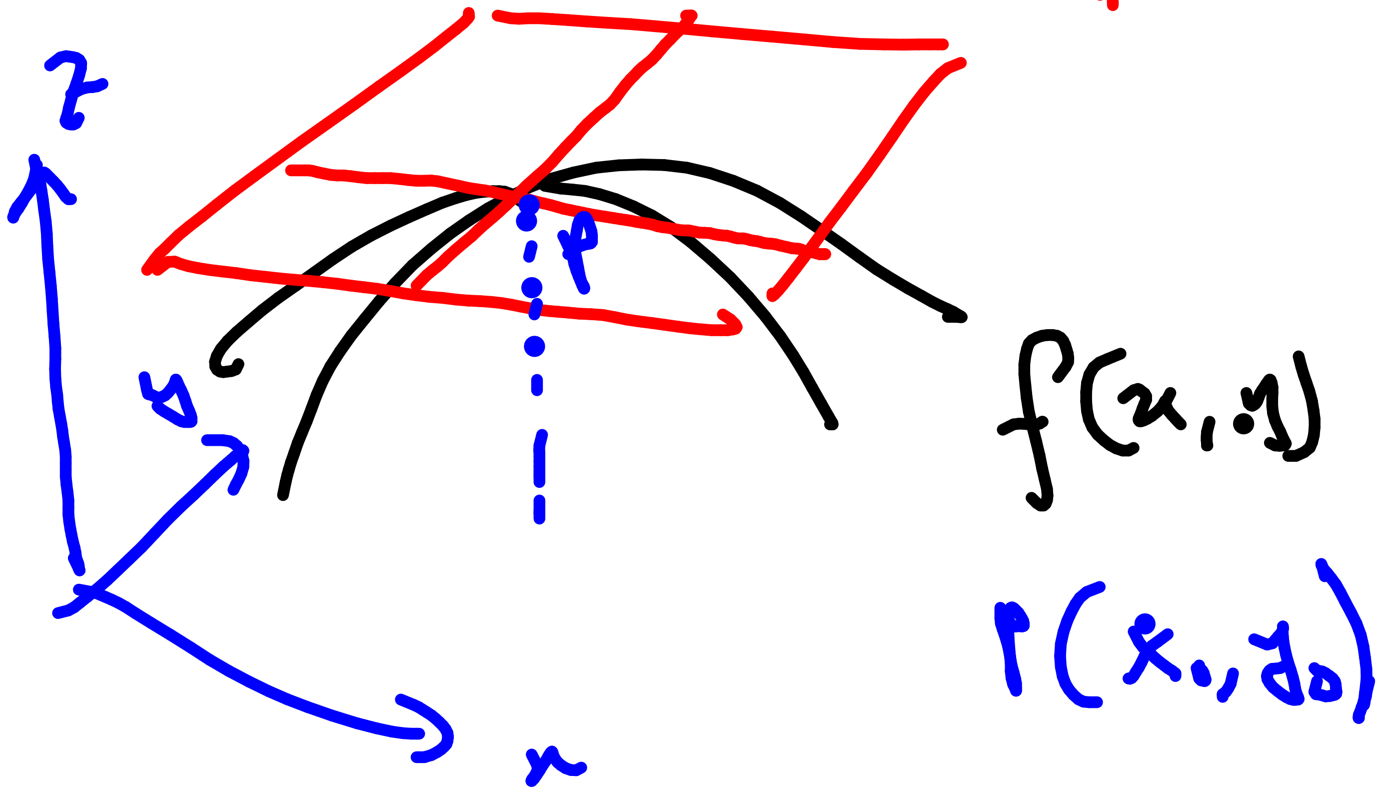
$$x = (u,v)$$

$$y = (u,v)$$

Recall: eqⁿ for the tangent plane

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

where $z_0 = f(x_0, y_0)$



8 THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

$f(x, y)$

$\frac{\partial f}{\partial x}$ ✓

$\frac{\partial f}{\partial y}$ ✓

$g(x)$

$g(x)$ is differentiable if $\frac{dg}{dx}$ exist

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Q. f is differentiable if $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ exist & are continuous

$$\frac{\partial f}{\partial x} = xy\,e^{xy} + e^{xy}$$

$$\frac{\partial f}{\partial y} = x^2 e^{xy}$$

are these continuous at $(1,0)$ or not ??

Yes: visual inspection of the graphs

V EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

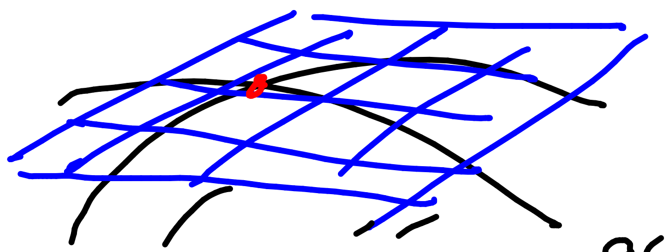
$L(1.1, -0.1) = ??$

→ find its linearization:

simply a reformulation of the tangent Pane

find a formula

$L(x,y) = ??$ whose graph is the tangent plane.



$$\left[z - z_0 = \frac{\partial f}{\partial x}(1,0)\,(x-1) + \frac{\partial f}{\partial y}(1,0)\,(y-0) \right]$$ tangent plane

$$z - 1 = 1\,(x-1) + 1\,(y-0)$$

$$z = 1 + 1\,(x-1) + 1\,(y-0)$$

$$L(x,y) = 1 + 1\,(x-1) + 1\,(y-0) = x+y$$

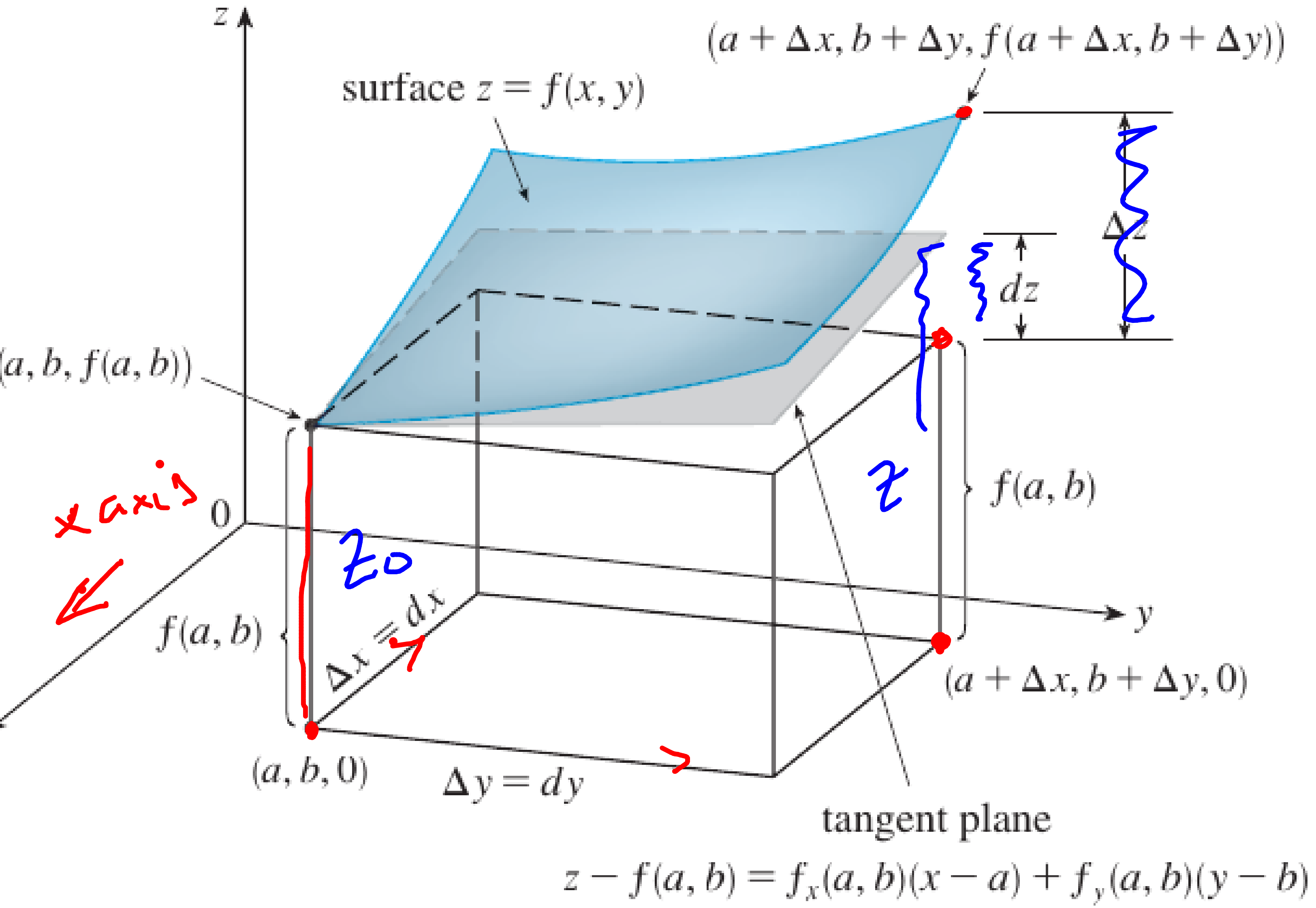
$$f(1.1,-0.1) = 1.1\,e^{(1.1)(-0.1)} = 0.986$$

$$\approx L(1.1,-0.1) = 1$$

DIFFERENTIALS

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$f(x,y)$



$x \to x + \Delta x$ & $y \to y + \Delta y$

differentials estimates change in f

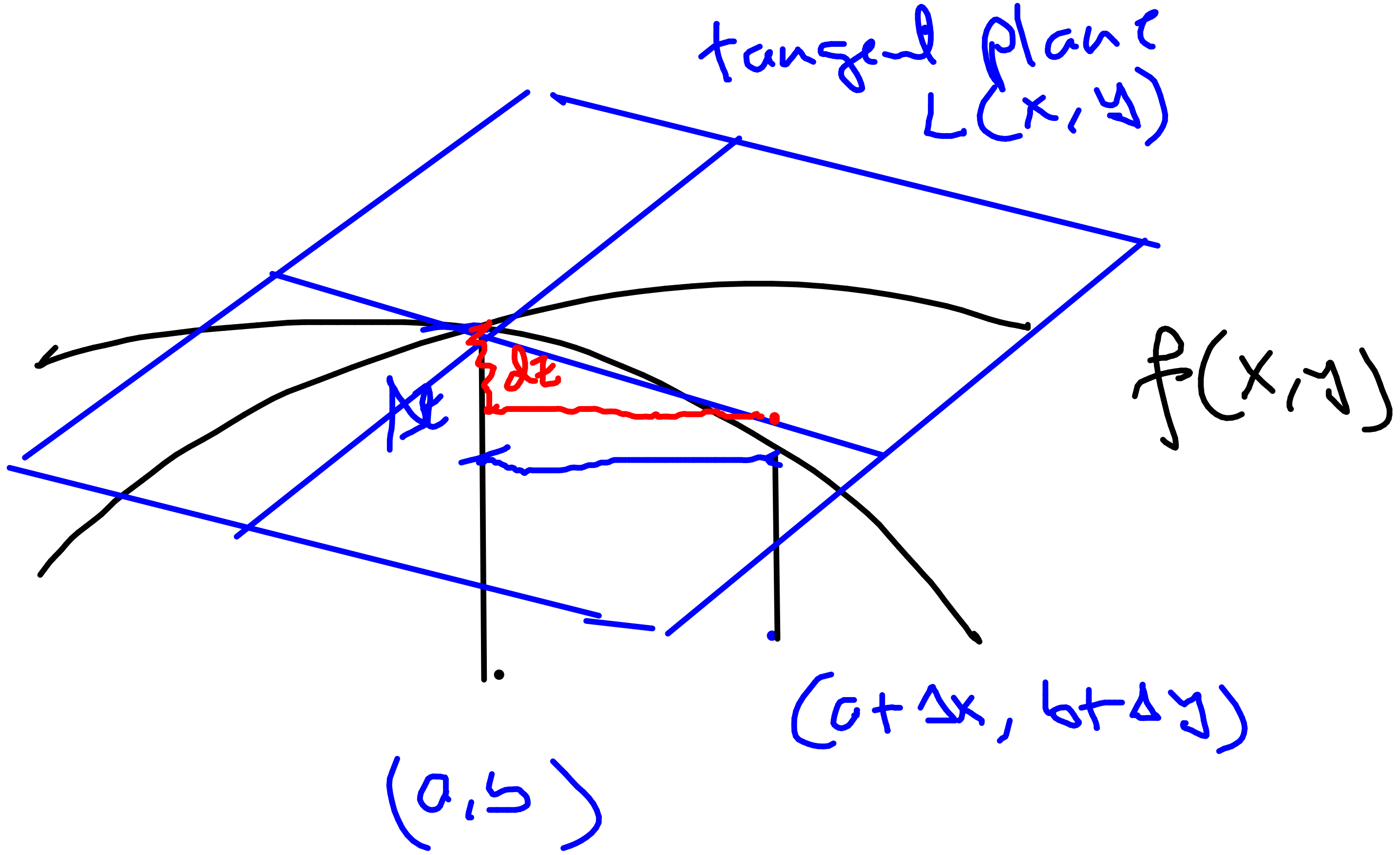
$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$

$\approx$

$dz =$ actual change in the tangent plane

$$z - z_0 = \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)$$

$$\boxed{dz = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}$$



$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

$\Delta z \approx dz =$ change in z in the tangent plane

$$\boxed{z - z_0 = \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)}$$

$$\boxed{dz = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}$$

V EXAMPLE 3

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

a)

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$dz = (2x + 3y) dx + (3x - 2y) dy$$

b)

dz , $x = 2$, $y = 3$, $dx = 0.05$, $dy = -0.04$

$$dz = 13(0.05) + 0 \cdot dy = 0.65$$

$$\Delta z = f(2.05, 2.96) - f(2, 3)$$

EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.