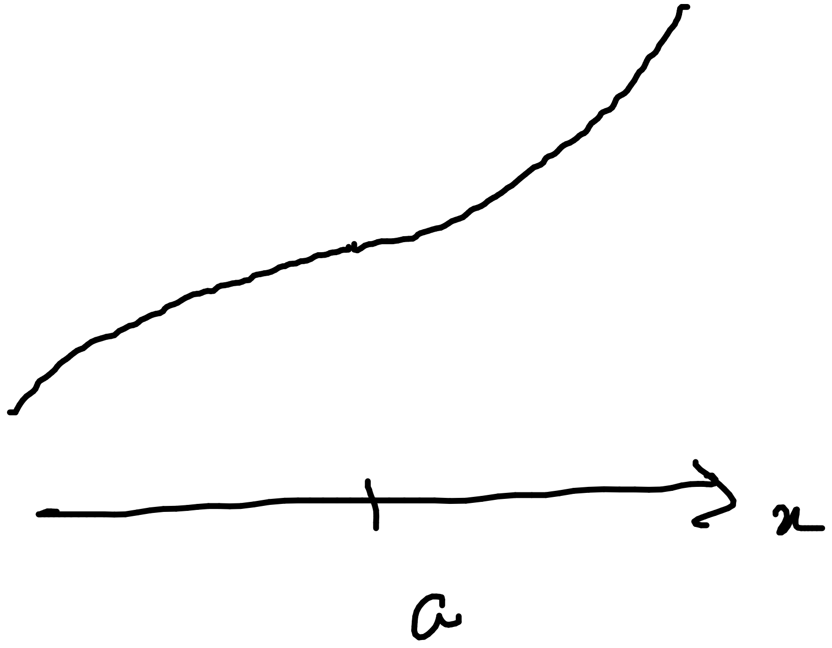
11.2 LIMITS AND CONTINUITY

for future sections we will assume functions are continuous

skip

11.3 PARTIAL DERIVATIVES

Today



$$\lim_{x \to a} f(x) = ??$$

Today's agenda:

Section 11.3

Partial Derivatives

Q. $f(x,y) = x^2y^3 + x$

partial derivative w.r.t. x

$\frac{\partial f}{\partial x}$ = [differentiate f w.r.t. x, assuming all variables other than x are constants]

$$= \frac{\partial}{\partial x}\left(x^2y^3 + x\right) = \frac{\partial}{\partial x}\left(x^2y^3\right) + \frac{\partial}{\partial x}(x)$$

$$= y^3 \frac{\partial}{\partial x}(x^2) + 1 = y^3(2x) + 1$$

$$f(x,y) = x^2 y^3 + x$$

$$\frac{\partial f}{\partial y} = ?? = \frac{\partial}{\partial y}(x^2 y^3) + \frac{\partial}{\partial y}(x)$$

$$= x^2 \, 3y^2 + 0$$

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

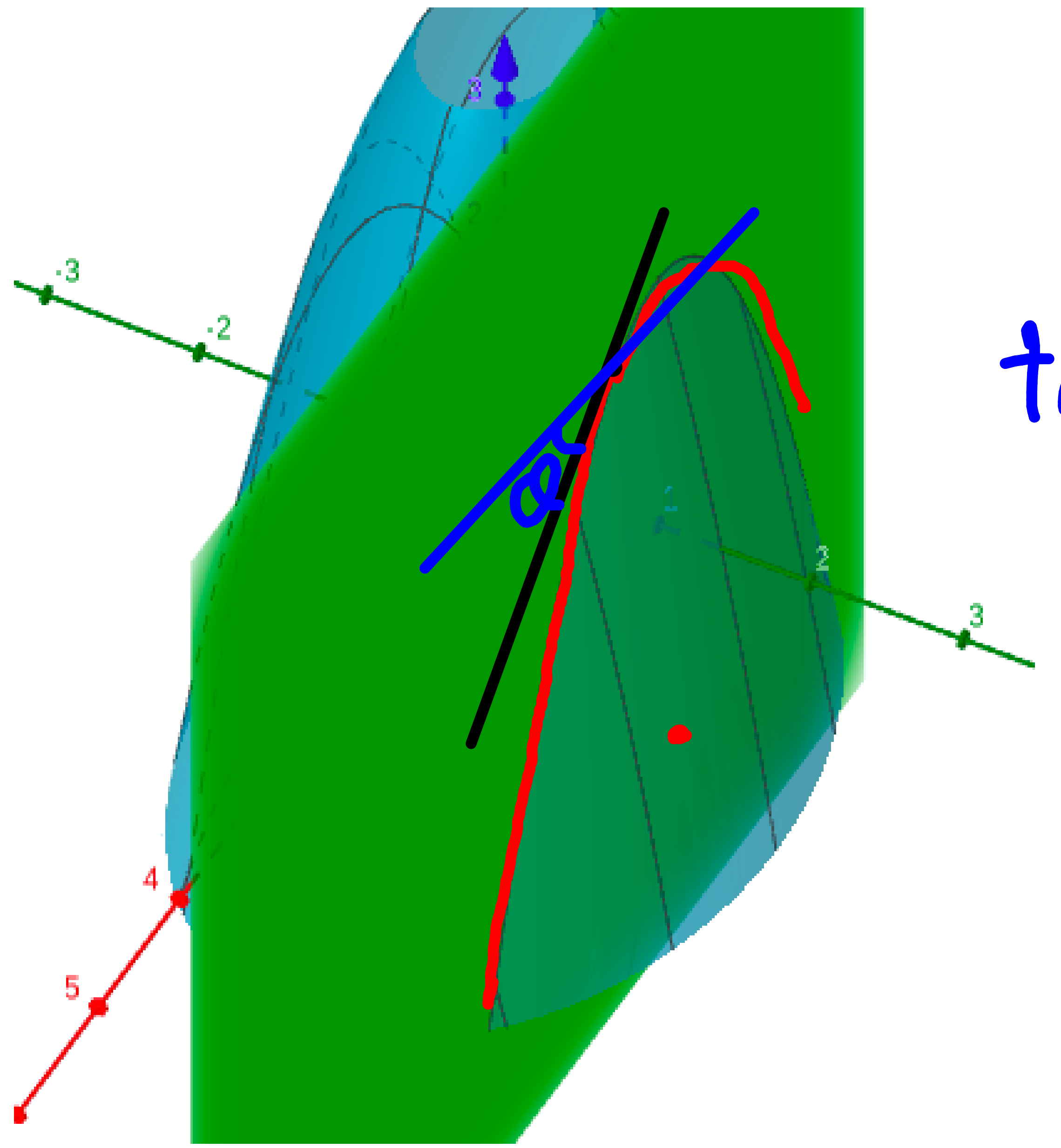
$$f_x = \frac{\partial f}{\partial x} = -2x$$

$$f_y = \frac{\partial f}{\partial y} = -4y$$

$$f_x(1,1) = -2$$

$$f_y(1,1) = -4$$

what are these ??



$\tan\theta = -2$

EXAMPLE 2 If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

$$f_y = \frac{\partial f}{\partial y} = -4y$$

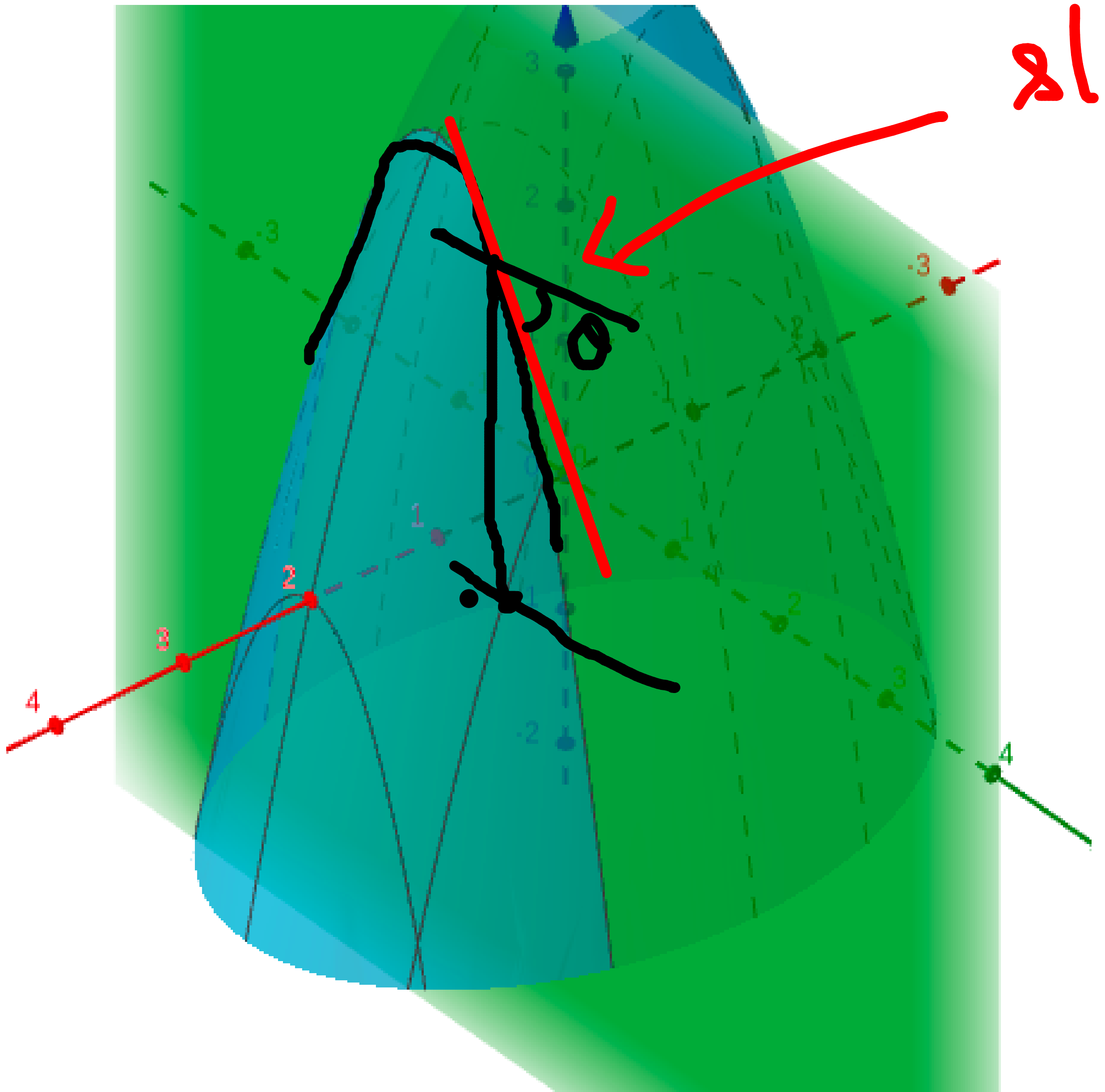
$$f_y(1,1) = -4$$

→ think of graph of $f(x,y)$

→ imagine standing at point $(1,1)$ on (x,y) plane

→ in $\frac{\partial f}{\partial y}$ calculation: x is fixed

→ $\frac{\partial f}{\partial y}$: slope of the tangent line of the curve obtained by intersection of graph & plane $x=1$



slope of this

$= -4$

at $(1,1)$

EXAMPLE 3 If $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot x \cdot \frac{-1}{(1+y)^2}$$

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

EXAMPLE 6 Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

$$f_{xx} = 6x + 2y^3$$

$$f_{yy} = 6x^2y - 4$$

$$f_{xy} = 6xy^2$$

$$f_{yx} = 6xy^2$$

it was observed that in general

$$f_{xy} = f_{yx}$$

Q: Is it true always??

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

almost always $f_{xy} = f_{yx}$

when: each of them should exist

& each of them are continuous

good news: these conditions mostly holds

■ Alexis Clairaut was a child prodigy in mathematics, having read l'Hospital's textbook on calculus when he was ten and presented a paper on geometry to the French Academy of Sciences when he was 13. At the age of 18, Clairaut published *Recherches sur les courbes à double courbure*, which was the first systematic treatise on three-dimensional analytic geometry and included the calculus of space curves.

8.

The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.

ellipsoid ??

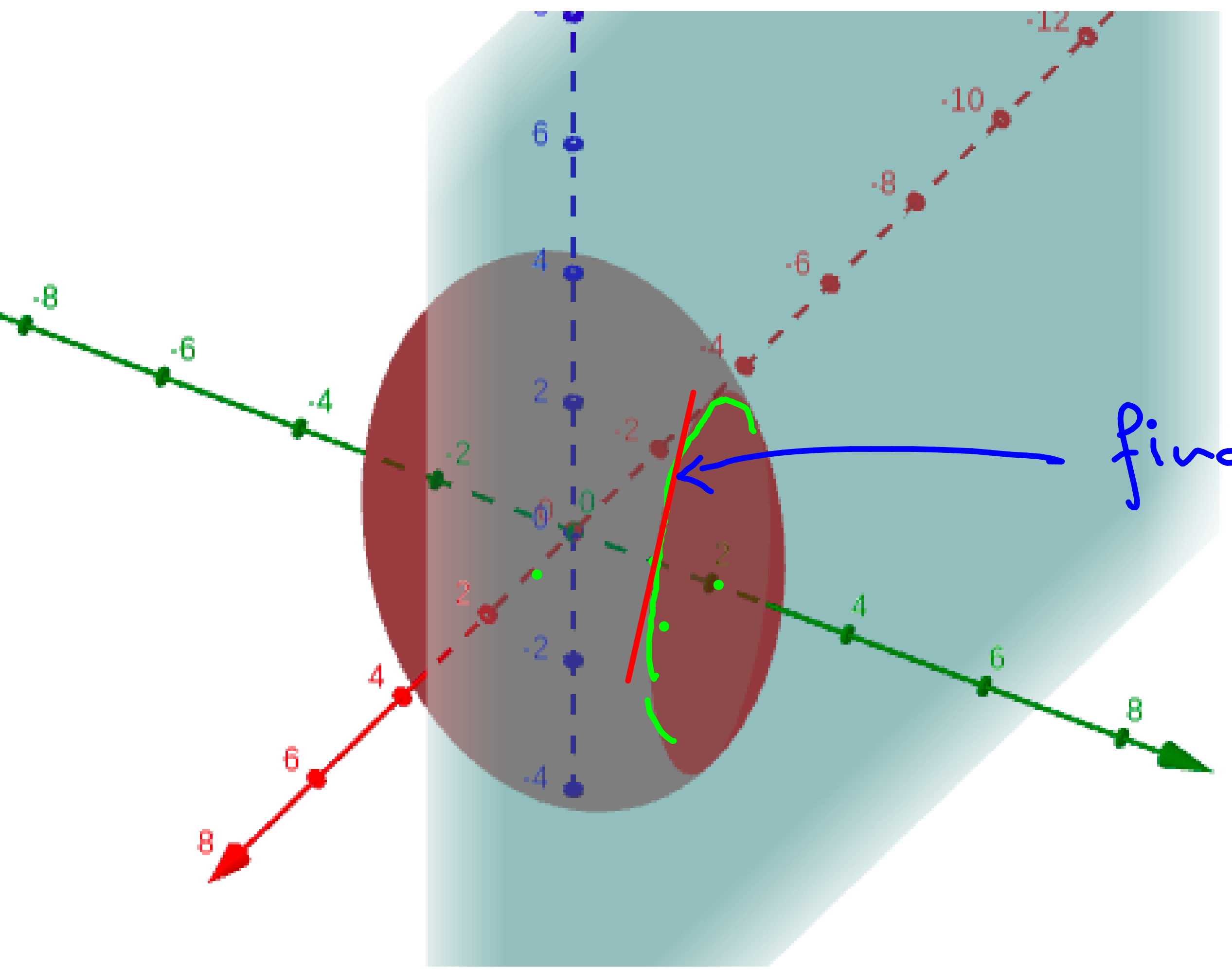
$$x^2+y^2+z^2 = 3^2 \quad ??$$

$$\left(\frac{x}{3}\right)^2+\left(\frac{y}{3}\right)^2+\left(\frac{z}{3}\right)^2 = 1$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2 = 1$$

Q.

The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects the plane $y = 2$ in an ellipse. Find parametric equations for the tangent line to this ellipse at the point $(1, 2, 2)$.



$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{8}}\right)^2 + \left(\frac{z}{4}\right)^2 = 1$$

find eqn of this tangent line.

. line in 3d

→ find the parametric eq

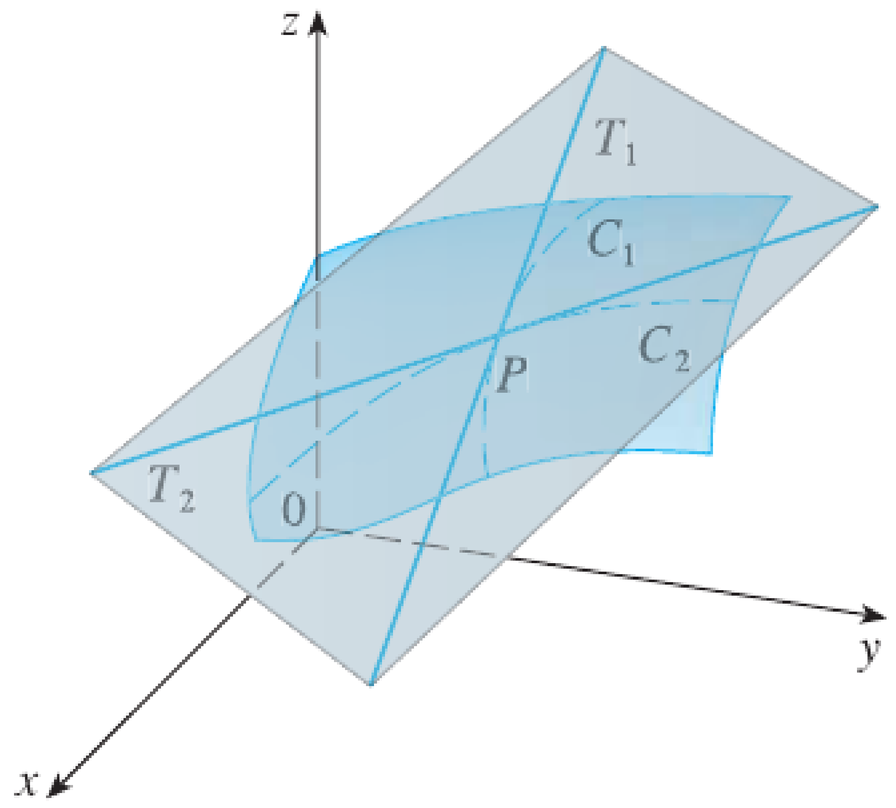
$$\frac{x-1}{??} = \frac{y-2}{??} = \frac{z-2}{??}$$

Let

$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) Use a computer to graph f.
(b) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.
(c) Find $f_x(0, 0)$ and $f_y(0, 0)$ using Equations 2 and 3.
(d) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.
(e) Does the result of part (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

11.4 TANGENT PLANES AND LINEAR APPROXIMATIONS



V EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

8 THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

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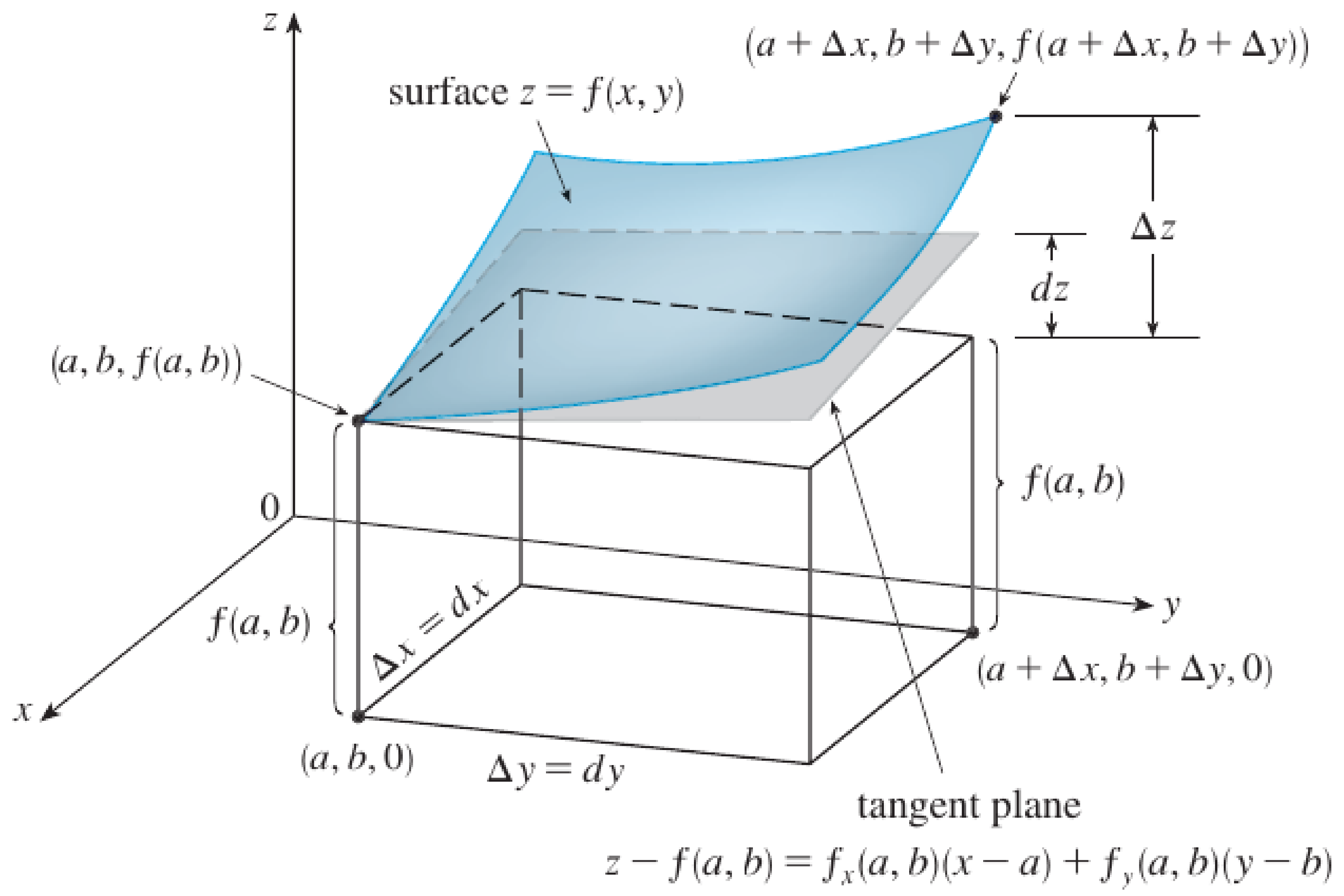
DIFFERENTIALS

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$



EXAMPLE 3

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.