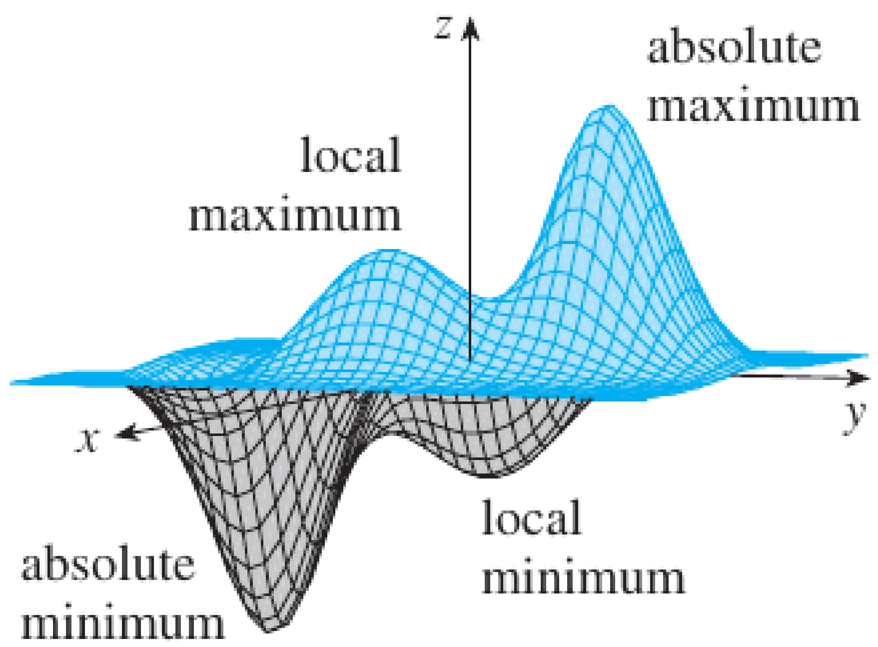
11.7 MAXIMUM AND MINIMUM VALUES



So far:

11.1 → visualization, contours, graphs, plots

11.2 → skipped | limits & continuity

11.3 → partial derivatives

11.4 → Tangent planes, differentials

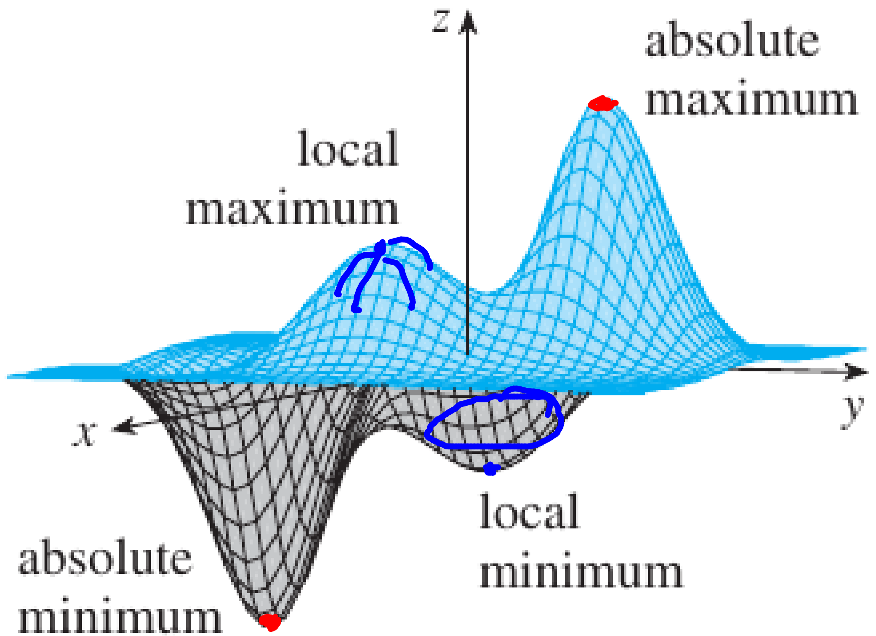
11.5 → chain rule

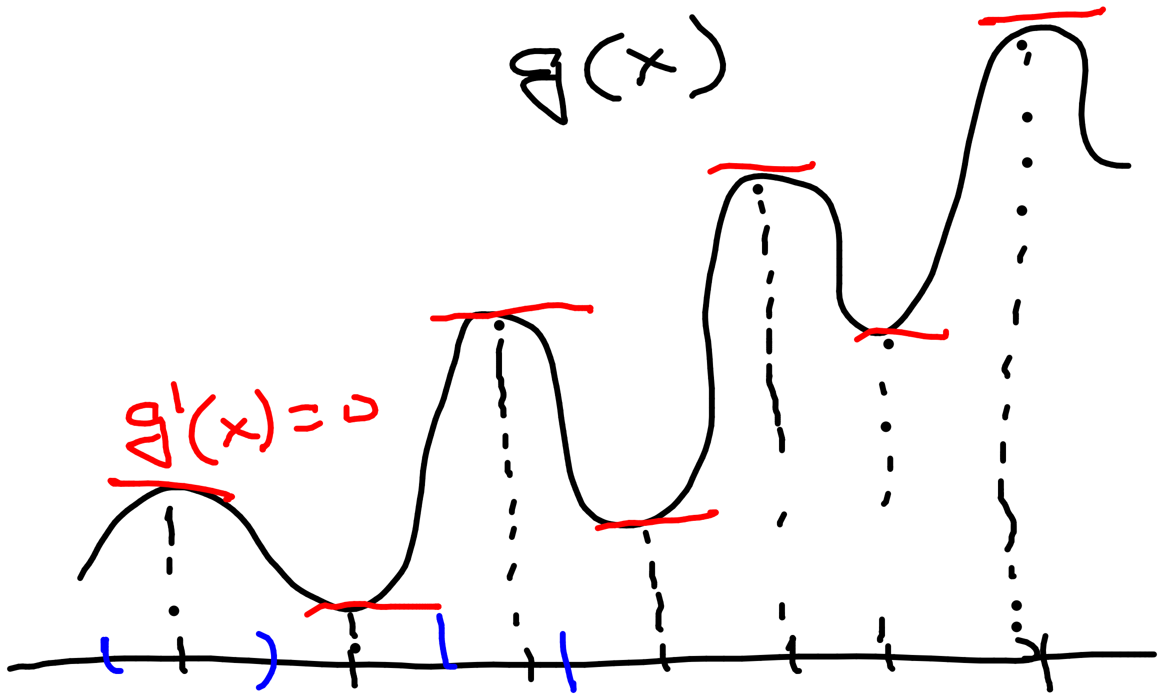
11.6 → directional derivatives & gradients

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

11.7 MAXIMUM AND MINIMUM VALUES

$f(x,y)$





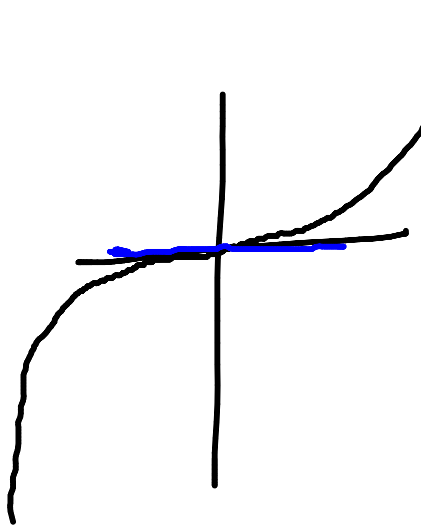
Q: find points of local max & min.

sol^n: solve $g'(x) = 0$

why??

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

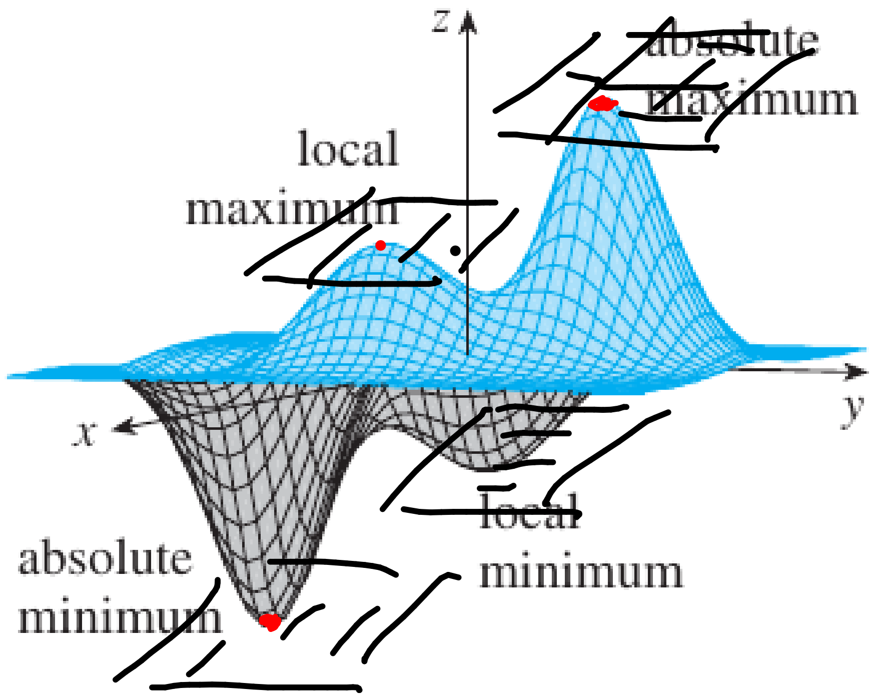
11.7 MAXIMUM AND MINIMUM VALUES



x^3

$3x^2$

$f(x,y)$



Q: tangent plane at local max should be horizontal or not ??

Q: recall eqⁿ of tangent plane

$$z - z_0 = \underbrace{\frac{\partial f}{\partial x}(x_0,y_0)}_{A}[x - x_0] + \underbrace{\frac{\partial f}{\partial y}(x_0,y_0)}_{B}[y - y_0]$$

$$z - z_0 = A(x - x_0) + B(y - y_0)$$

$A = 0$ & $B = 0$

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

$$\frac{\partial f}{\partial x}(x_0,y_0) = 0$$

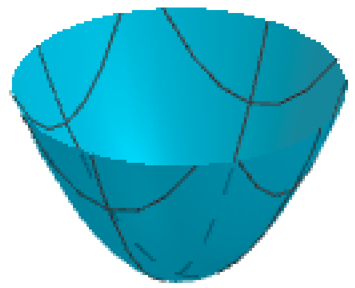
$$\frac{\partial f}{\partial y}(x_0,y_0) = 0$$

Eg.

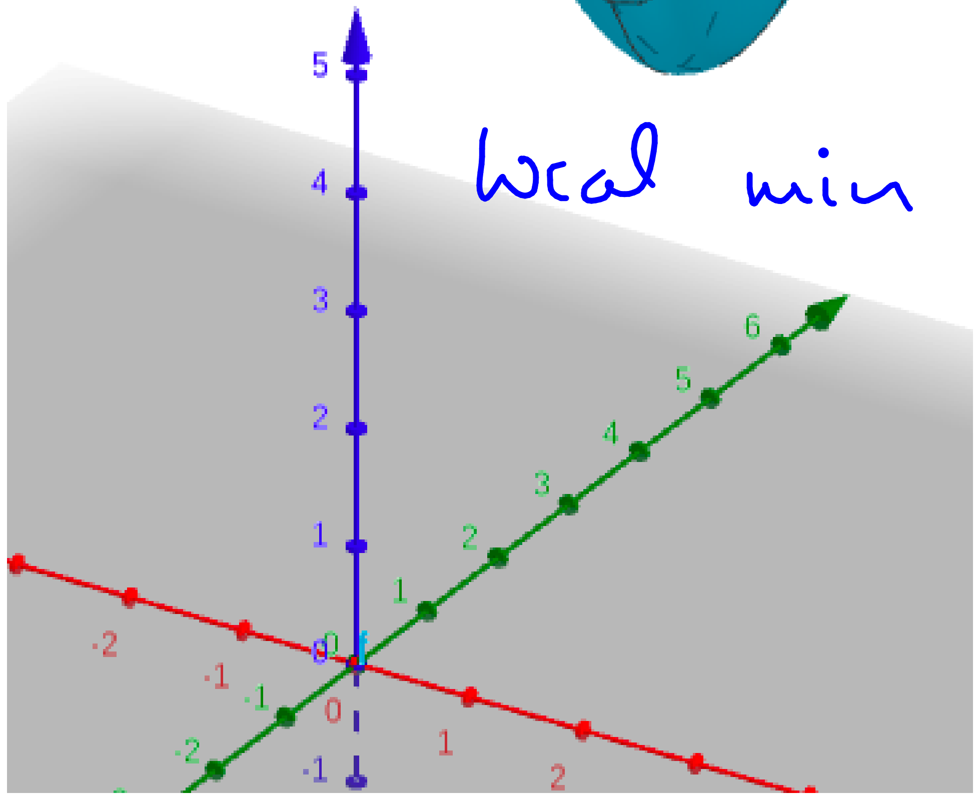
$T(x,y) = \dfrac{e^{xy}}{e^{-x^2-y^2}}$ temperature at point (x,y)

find hottest & coldest point

EXAMPLE 1 Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.



find critical points for $f(x,y)$:



local min

solve:

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$2x - 2 = 0 \qquad 2y - 6 = 0$$

$$x = 1 \qquad y = 3$$

Q: confirm that tangent plane should be horizontal at $(1,3)$ by plotting the graph of $f(x,y)$

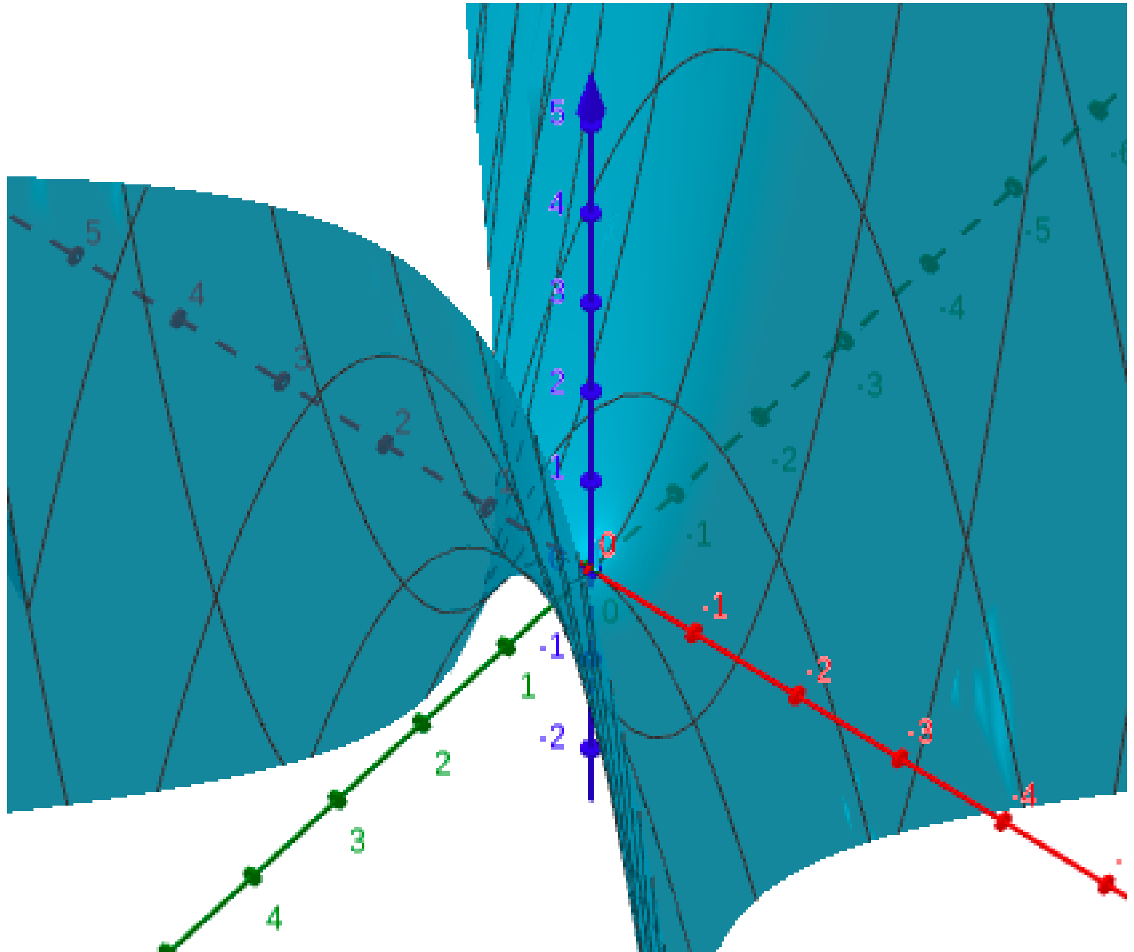
EXAMPLE 2 Find the extreme values of $f(x, y) = y^2 - x^2$.

find critical points :

$\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$

$-2x = 0$ $2y = 0$

critical point : $(0,0)$

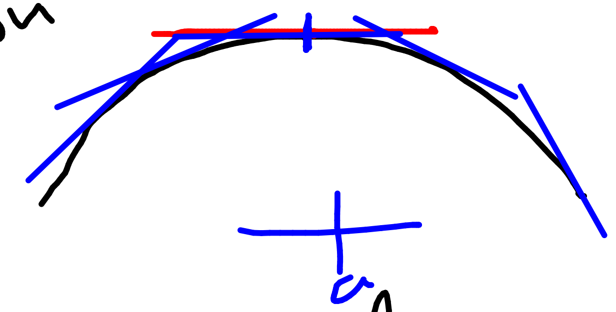


saddle points

Classification of critical points

$f(x)$, $f''(a)$

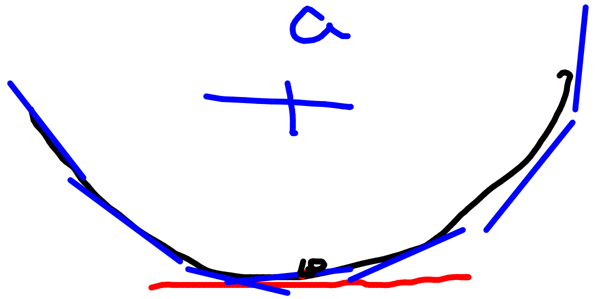
concave down $\begin{cases} f''(a) < 0 \\ \end{cases}$ $\Leftrightarrow$ f' is decreasing



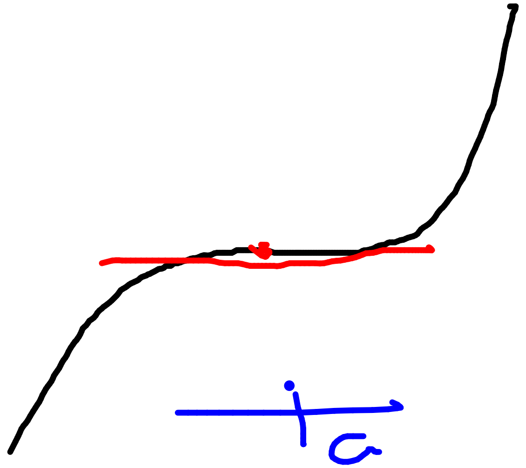
local max

concave up

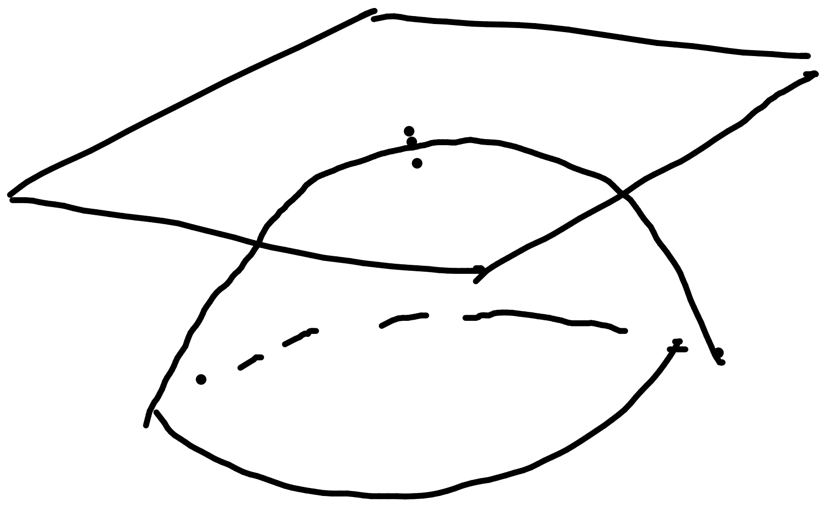
$f'' > 0$



local min

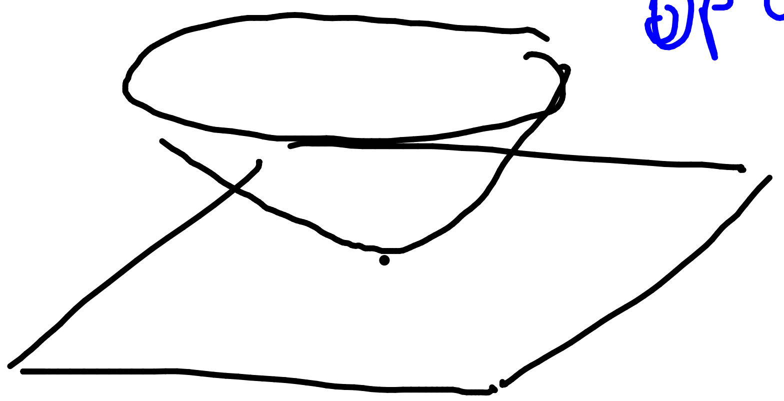


neither

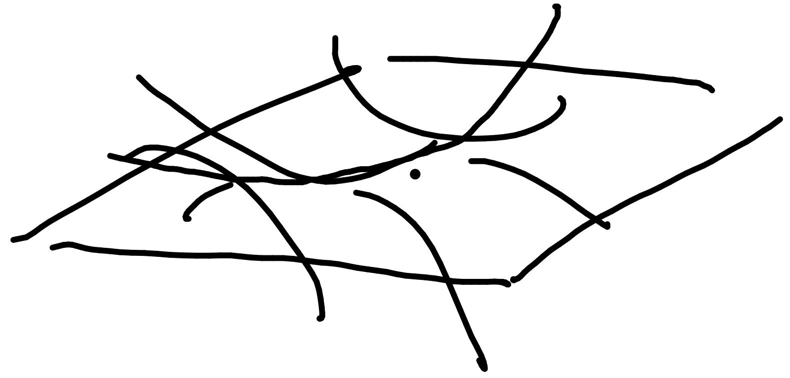


is f open
downward ??

if f open
upword



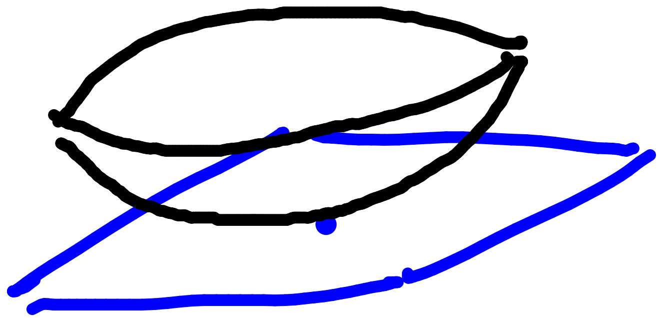
neither



Classify classify critical points as local max / min / neither

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

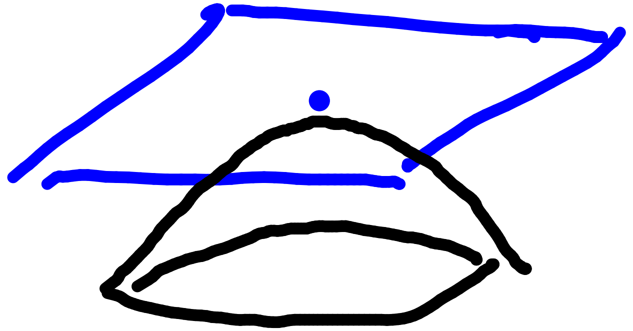
$f(x,y)$, (a,b) is a critical pt.



local min

$f_{xx} > 0$

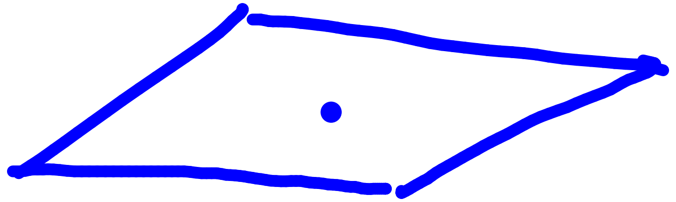
$D > 0$



local max

$f_{xx} < 0$

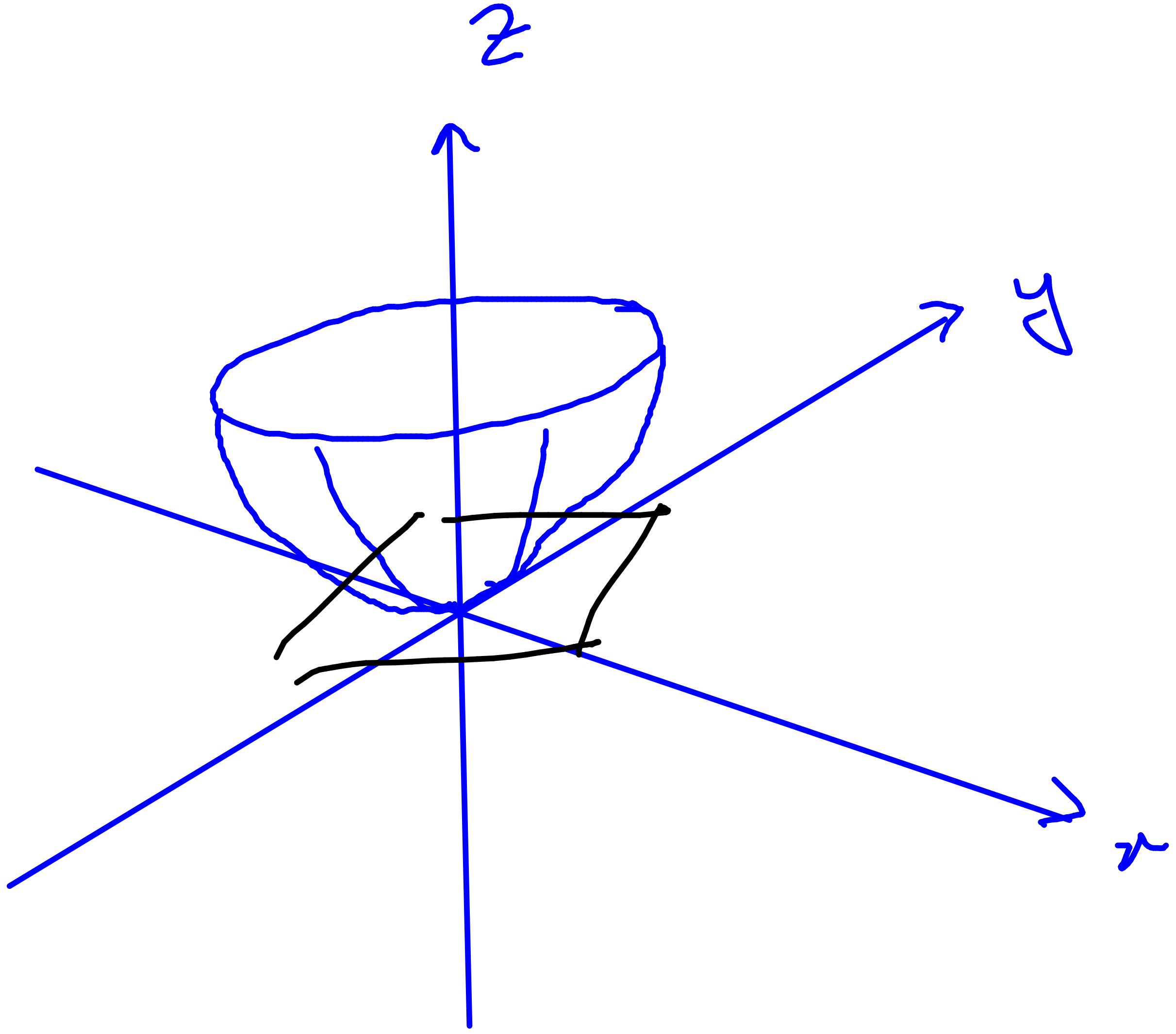
$D > 0$



neither

$D \leq 0$

Q: $f(x,y) = x^2+y^2$ | find & classify critical points



critical points:

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$2x = 0 \qquad 2y = 0$$

$$(x,y) = (0,0)$$

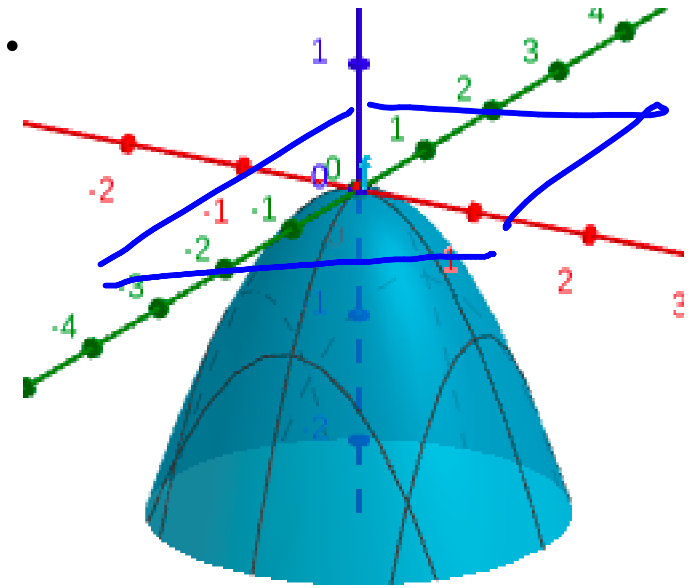
↳ classification of the critical point

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$D > 0$ & $f_{xx} = 2 > 0$

$\Rightarrow$ local min

Q: $f(x,y) = -x^2 - y^2$ | find & classify critical points



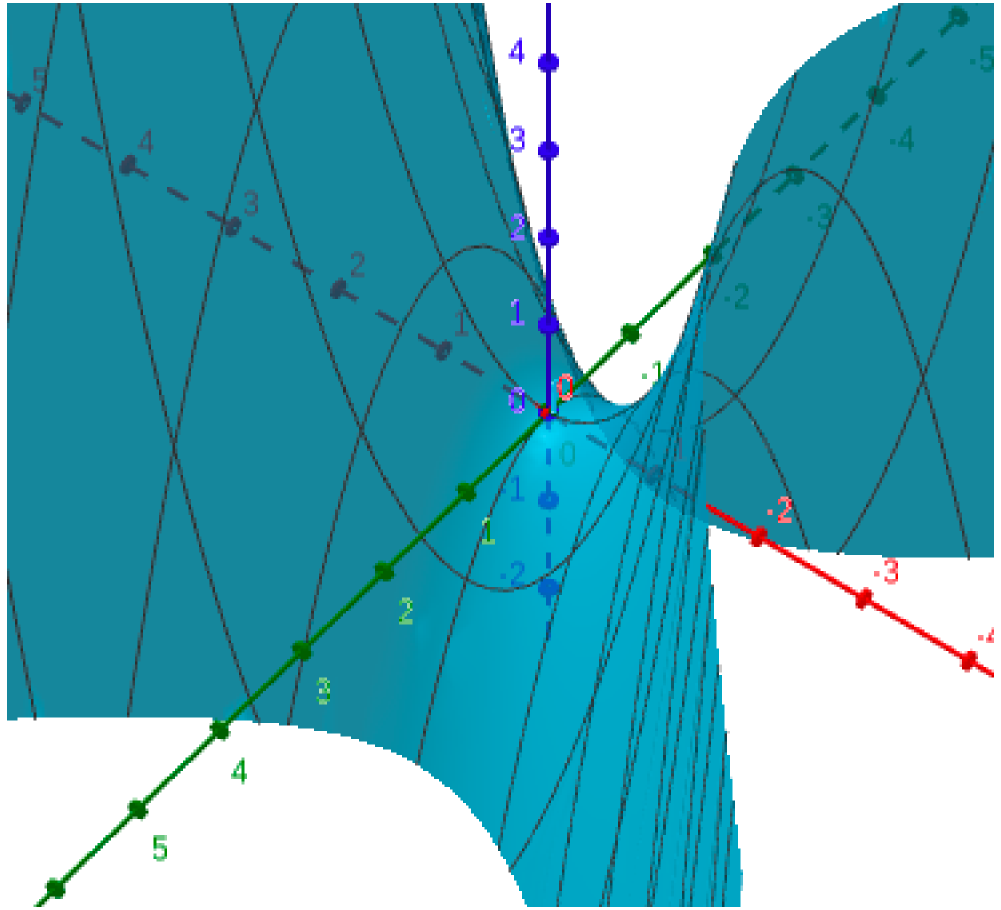
critical point $(0,0)$

classification:

$$D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$$

$D > 0$ & $f_{xx} < 0$ $\Rightarrow$ $(0,0)$ is a point of local max

Q_{11} $f(x,y) = x^2 - y^2$ | find & classify critical points



$f_x = 2x$ $f_y = -2y$

$D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = \underbrace{-4 < 0}_{\Downarrow}$

neither

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

midterm

Wed -20th

class time

5:30 - 7

3 SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

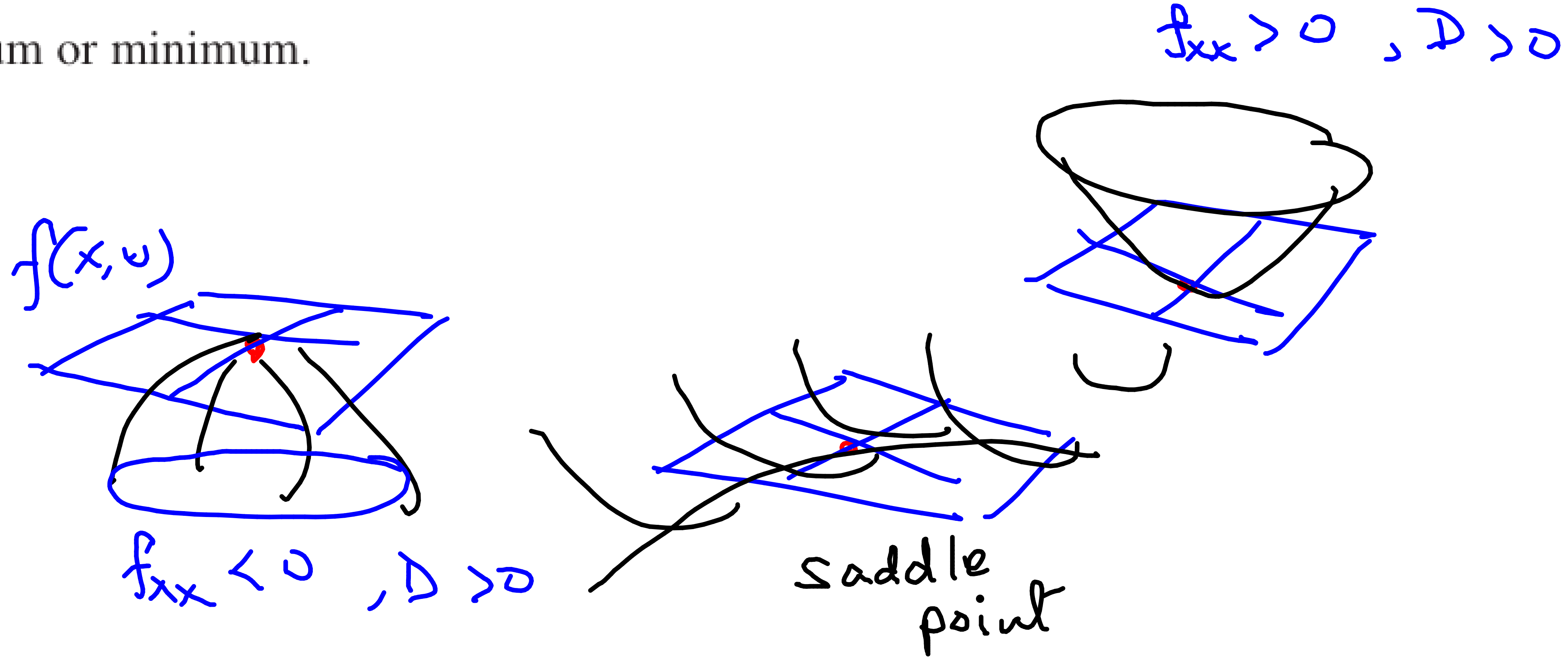
$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

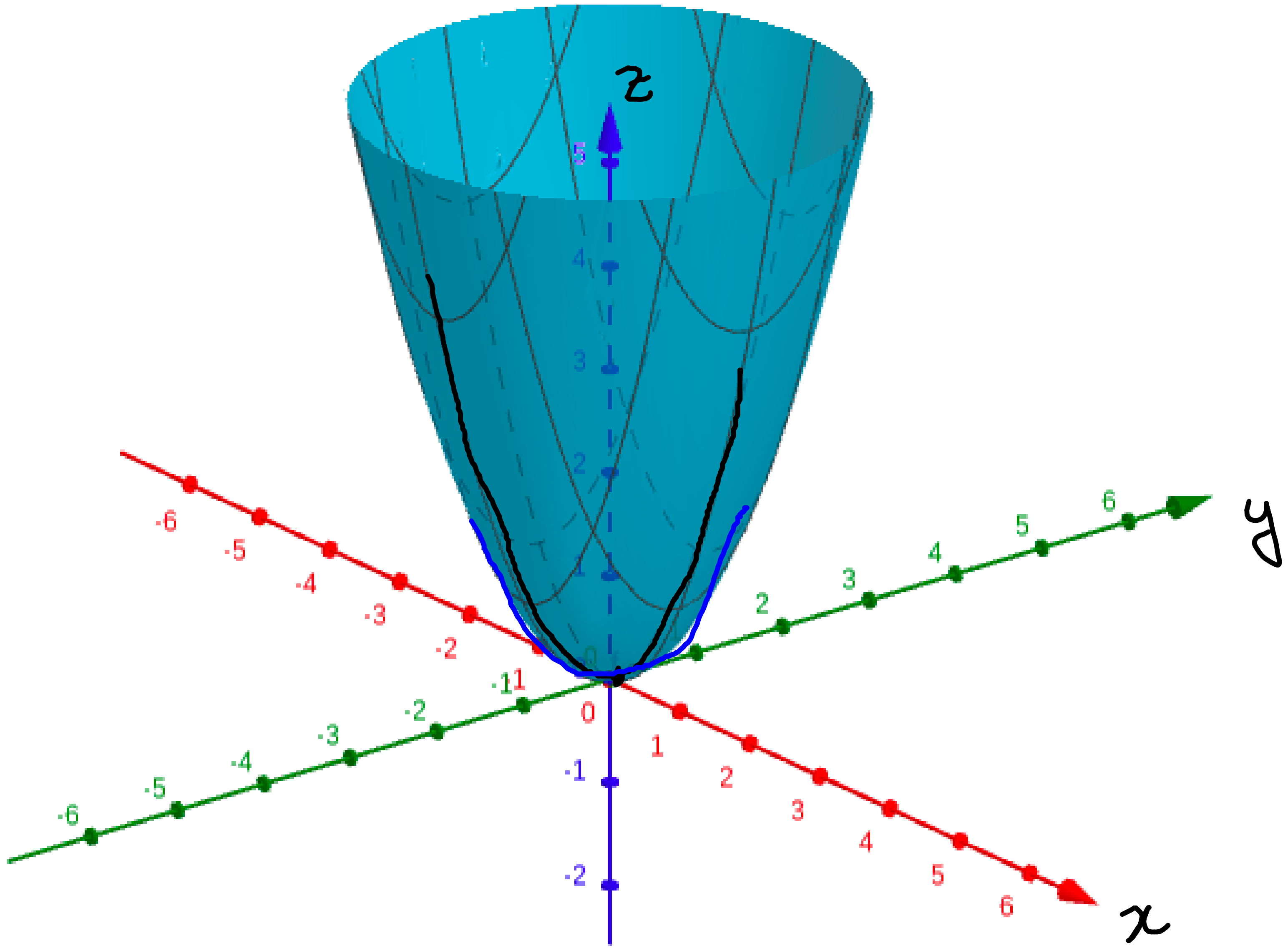
(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

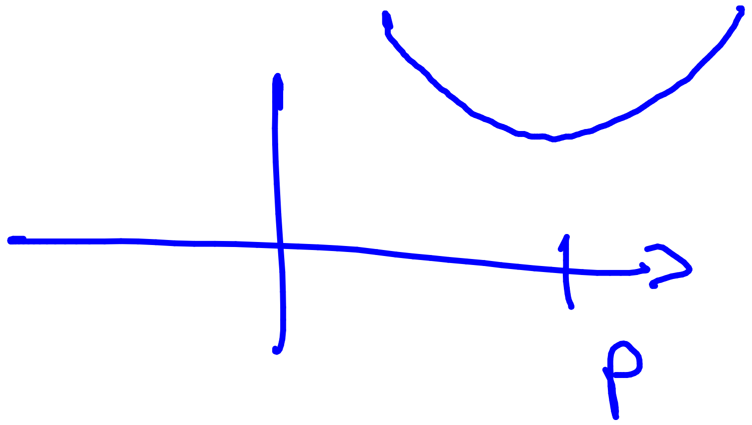


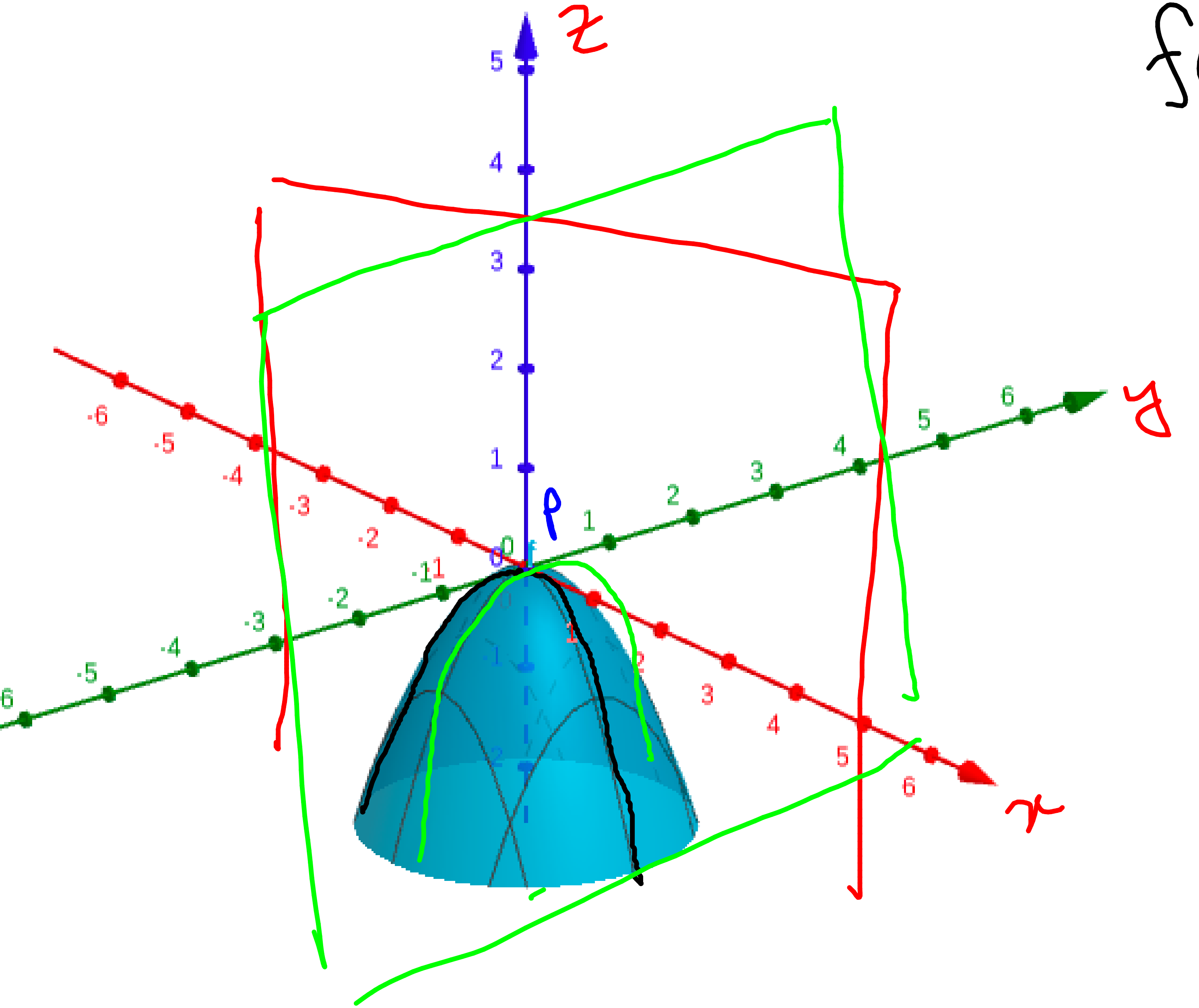
$f(x,y)$



$f_{xx} > 0$ at P & $D > 0$

$\Rightarrow f_{yy} > 0$





$f(x,y)$ p is a critical point

$f_{xx} < 0$ & $f_{yy} < 0$

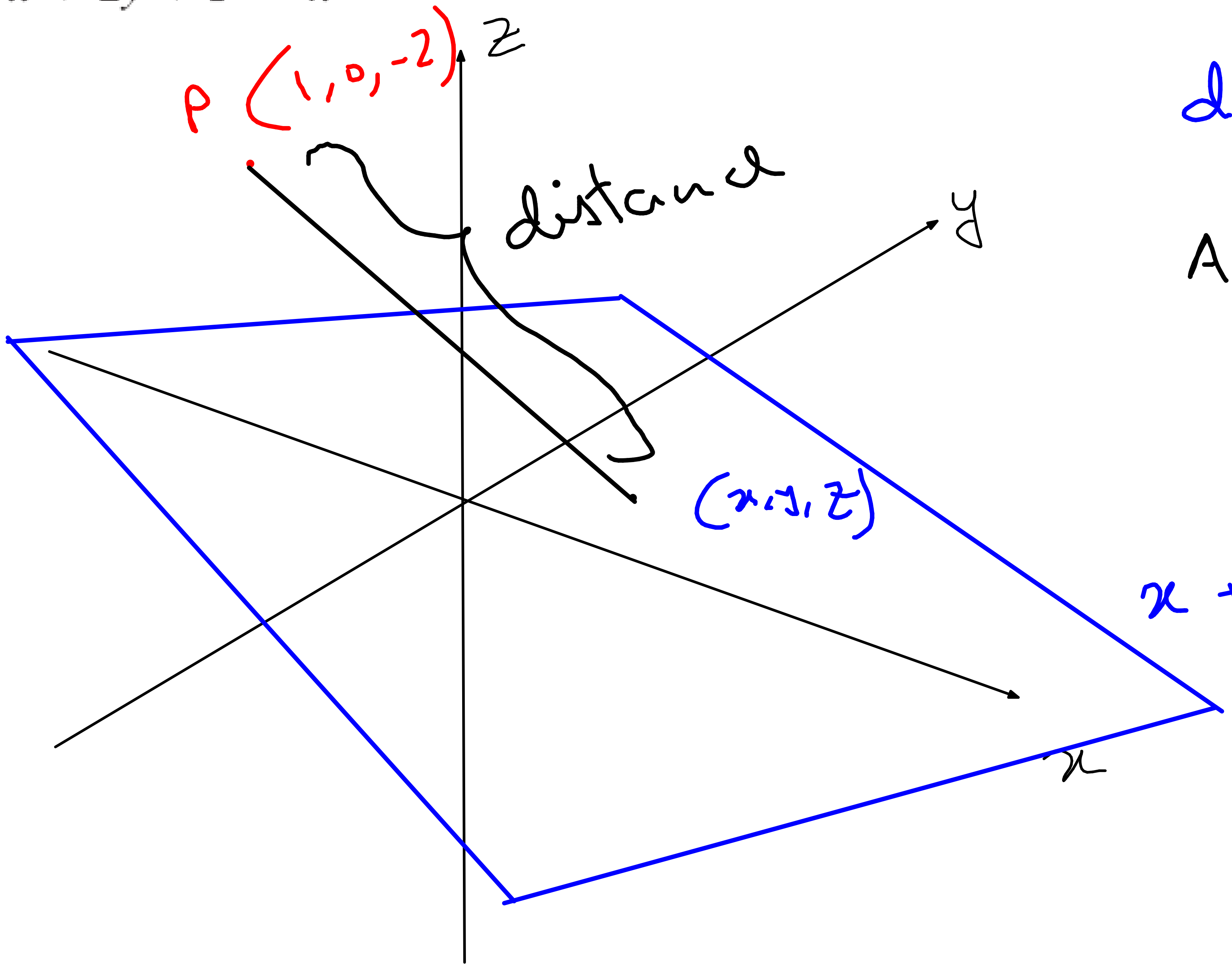
$\rightarrow$ p is a point of local max

$f_{xx} < 0$ mean ??

f_{xx} = usual second derivative of the black curve < 0

$\rightarrow$ $f_{yy} < 0$

EXAMPLE 4 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.



$x + 2y + z = 4$

$$d = \sqrt{(1-x)^2 + (0-y)^2 + (-2-z)^2}$$

Aim: find point (x,y,z) on the plane where d is minimum

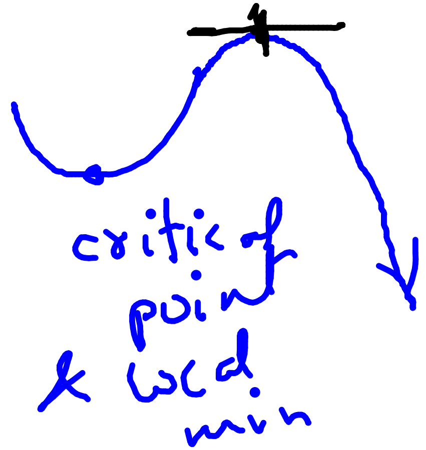
$\Leftrightarrow$ minimize d^2

One variable calculus

$\rightarrow$ $f(x)$ a function

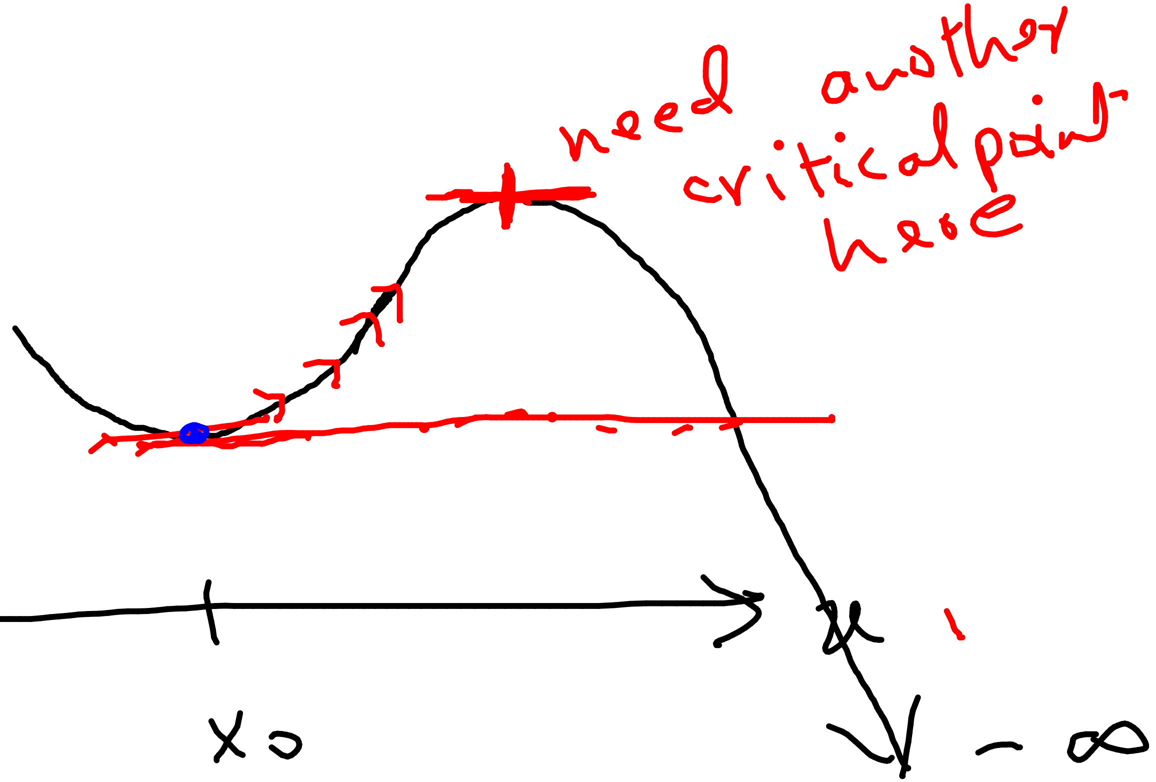
$\rightarrow$ x_0 : only one critical point

$\rightarrow$ $f''(x_0) > 0$



x_0 : local min

x_0 is also an absolute minimum



why??

EXAMPLE 4 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

minimize distontance square $= (1-x)^2 + (0-y)^2 + (-2-z)^2$

s.t. $x + 2y + z = 4$] use the plane eqn to eliminate z

minimize $f(x,y) = (1-x)^2 + y^2 + (2+4-x-2y)^2$

→ start with finding & classifying the critical points

→ critical points: $\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$

$x = \frac{11}{6}$; $y = 5/3$

classification:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 4 & 10 \end{vmatrix}$$

$$= 24 > 0$$

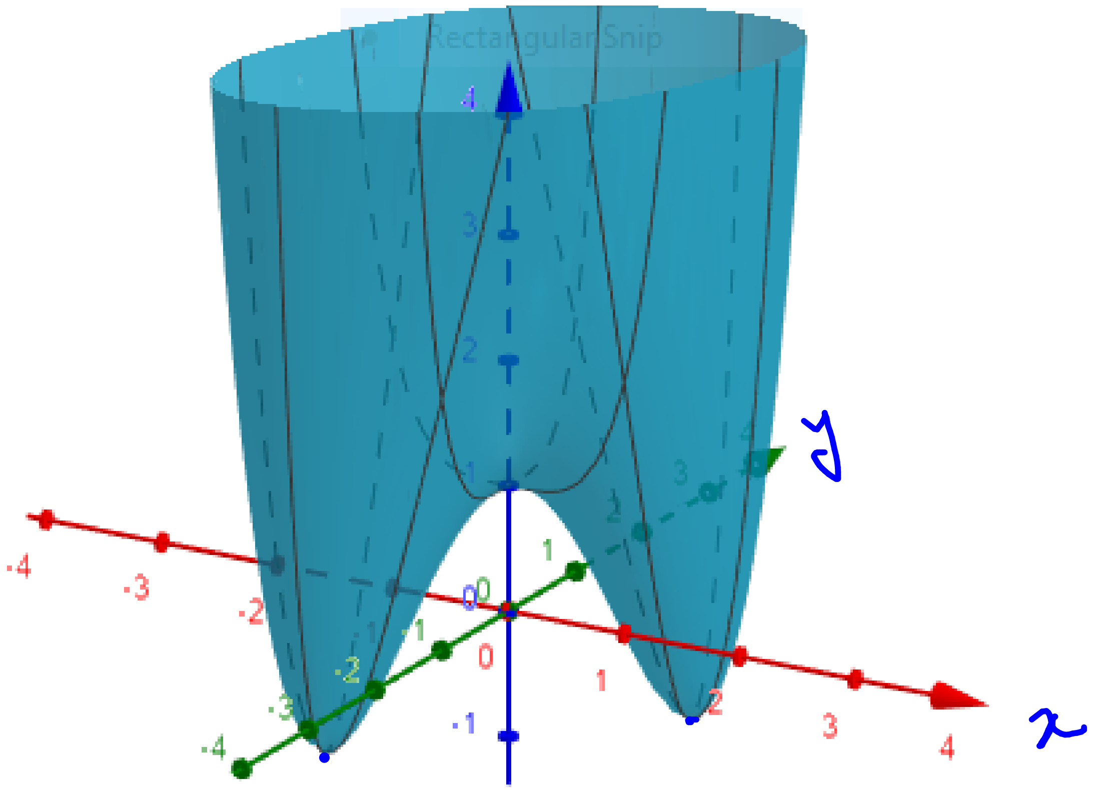
$D > 0$ & $f_{xx} > 0 \Rightarrow$
critical point is local min
& only critical point
$\Rightarrow$ absolute minimum

$$x = \frac{11}{6}, \quad y = \frac{5}{3}, \quad z = 5/6$$

$\left(\frac{11}{6}, \frac{5}{3}, \frac{5}{6}\right)$ point on the plane closest to $P(1, 0, -2)$

= Ans

V EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.



$$\frac{\partial f}{\partial x} = 0 \qquad\qquad \frac{\partial f}{\partial y} = 0$$

$$4x^3 - 4y = 0 \qquad\qquad 4y^3 - 4x = 0$$

$$\left.\begin{aligned} x^3 - y &= 0 \\ y^3 - x &= 0 \end{aligned}\right\} \text{ how to solve ??}$$

$$y = x^3$$

$$(x^3)^3 - x = 0$$
$$x^9 - x = 0$$
$$x(x^8 - 1) = 0$$
$$x = 0 \quad \& \quad \underbrace{x^8 = 1}_{\Rightarrow\, x = \pm 1}$$

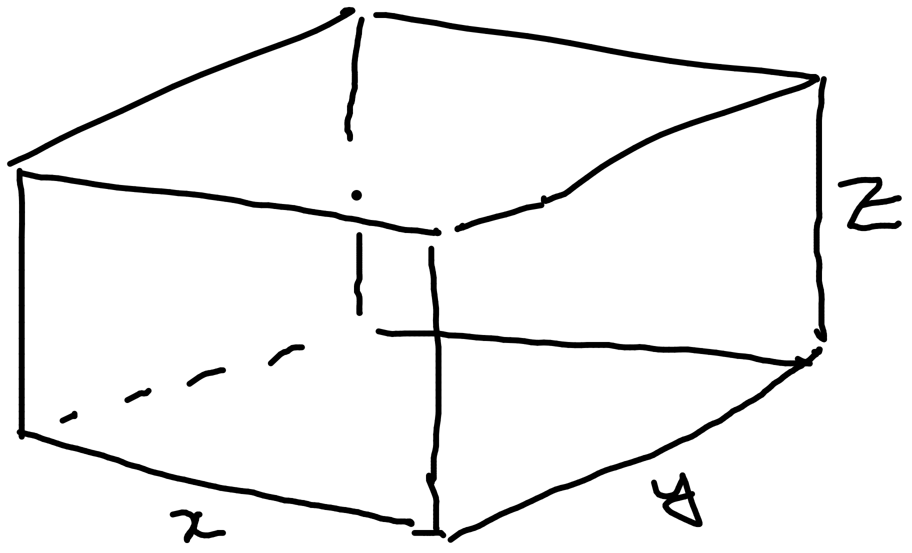
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$$

$$= 144x^2y^2 - 16$$

x	$y = x^3$		
0	0	$D < 0$	saddle point
1	1	$D > 0, f_{xx} > 0$	local min
-1	-1	"	"

V EXAMPLE 4 Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

EXAMPLE 5 A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.



maximize $V(x,y,z) = xyz$

$xy + 2xz + 2yz = 12$

$z = \dfrac{12 - xy}{2(x+y)}$

$V = xy\dfrac{(12 - xy)}{2(x+y)}$

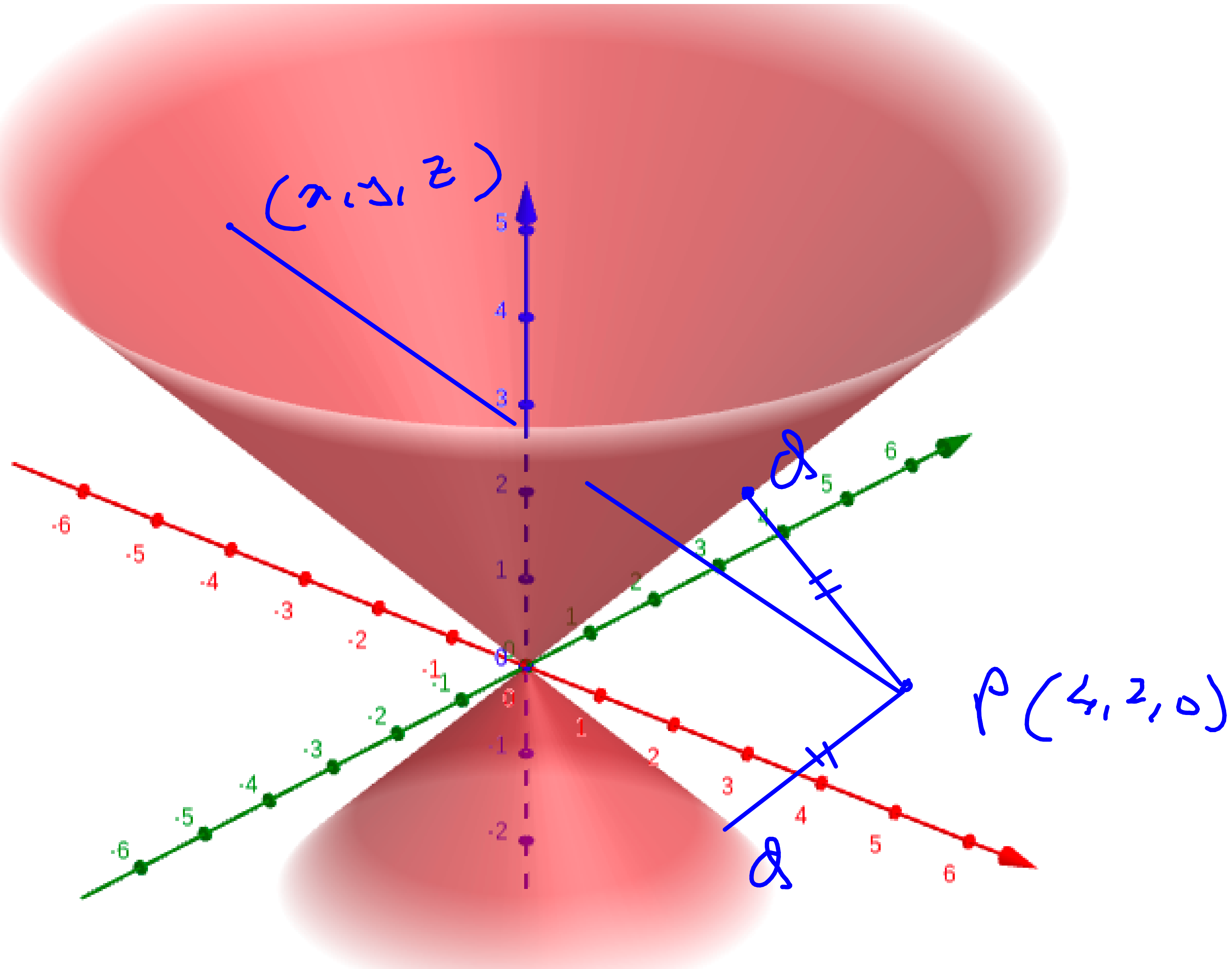
$\dfrac{\partial V}{\partial x} = 0$, $\dfrac{\partial V}{\partial y} = 0$

→ solve for critical point,

→ there should be only one critical point

→ classify it as local max & argue that local max is absolute max

33. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.



$d = ??$

$$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$$

$$\boxed{z^2 = x^2 + y^2}$$

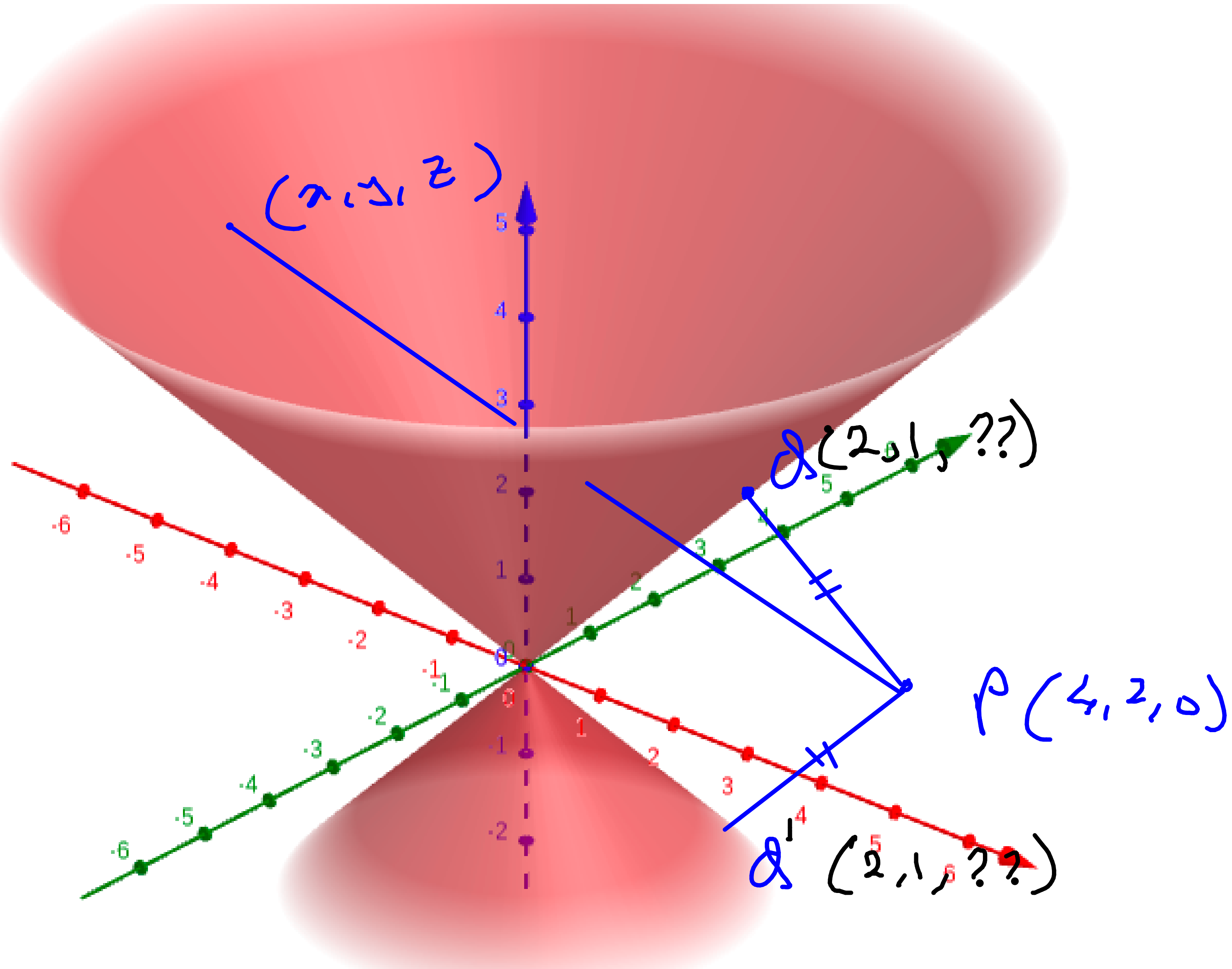
$$f(x,y) = (x-4)^2 + (y-2)^2 + (x^2 + y^2)$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

→ solve for critical point

→ & classify them

33. Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.



$Q = ??$

$f(x,y,z) = (x-4)^2 + (y-2)^2 + z^2$

$\boxed{z^2 = x^2+y^2}$

$f(x,y) = (x-4)^2 + (y-2)^2 + (x^2+y^2)$

$\frac{\partial f}{\partial x} = 0$ | $\frac{\partial f}{\partial y} = 0$

$\rightarrow 2(x-4) + 2x = 0$ | $2(y-2) + 2y = 0$

$x = 2$ | $y = 1$

$\rightarrow$ only critical point : $(2,1)$

$D = \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} \Rightarrow \left.\begin{matrix} D > 0 \\ f_{xx} > 0 \end{matrix}\right\} \rightarrow$ local min

$$z^2 = x^2 + y^2 \Big|_{\substack{x=2 \\ y=1}} =$$

$$z^2 = 5$$

$$z = \pm\sqrt{5}$$

Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid

$$9x^2 + 36y^2 + 4z^2 = 36$$

$$\frac{x^2}{2^2} + y^2 + \frac{z^2}{3^2} = 1$$

maximize

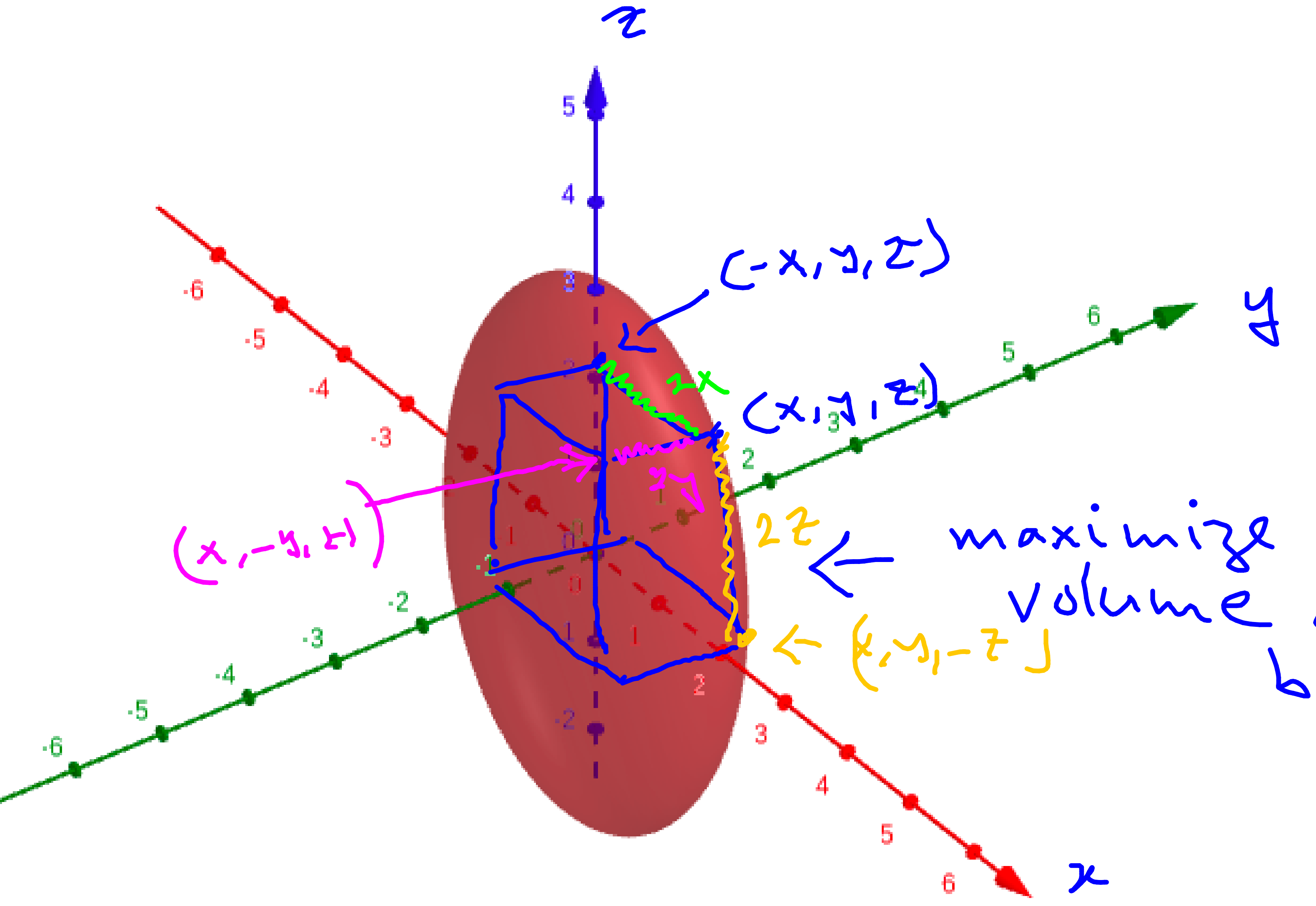
$$V = 8xyz$$

$$9x^2 + 36y^2 + 4z^2 = 36$$

$$y^2 = \frac{36 - 9x^2 - 4z^2}{36}$$

$$y^2 = 1 - \frac{x^2}{4} - \frac{z^2}{9}$$

$$y = \sqrt{1 - \frac{x^2}{4} - \frac{z^2}{9}}$$



maximize the volume of this box

$$V = 8xz\sqrt{1 - \frac{x^2}{4} - \frac{z^2}{9}}$$

$$\frac{\partial V}{\partial x} = 0 \qquad \frac{\partial V}{\partial z} = 0$$

$\rightarrow$ your work ....

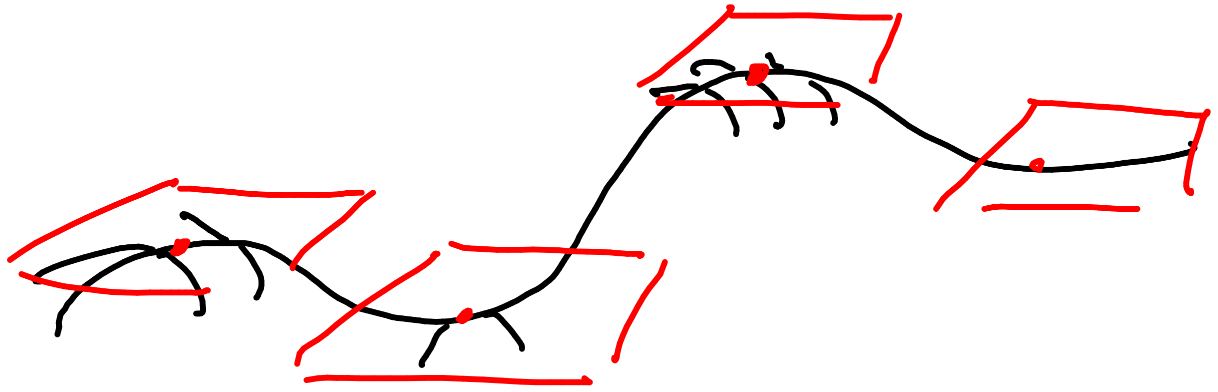
14.7 Solving max min problems in multivariable calculus

$f(x,y)$ $\longrightarrow$ find local/absolute max/min points

$\longrightarrow$ solve what equation ??

find critical points

$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$



$\longrightarrow$ Classification of critical points

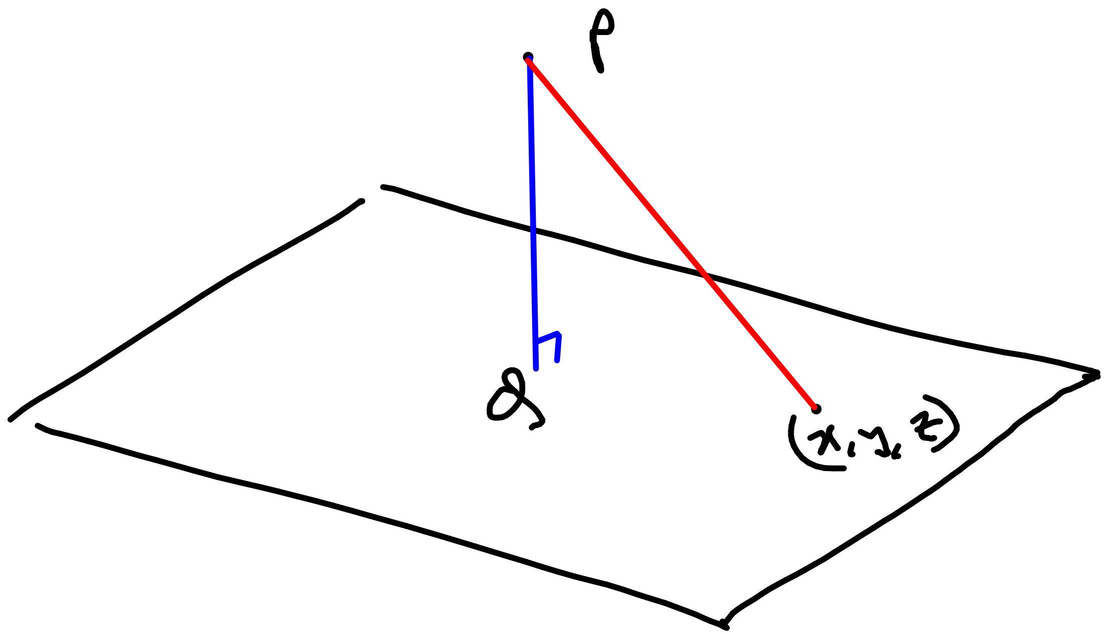
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

$D > 0$ & $f_{xx} > 0$ } $\Rightarrow$ local min

$D > 0$ & $f_{xx} < 0$ } $\Rightarrow$ local max

$D < 0$ } $\Rightarrow$ neither

Q:



Plane given

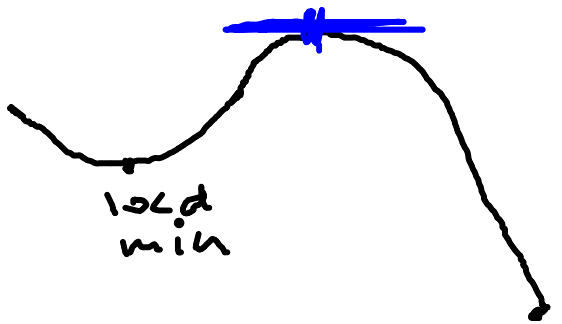
point P given

$Q = ??$

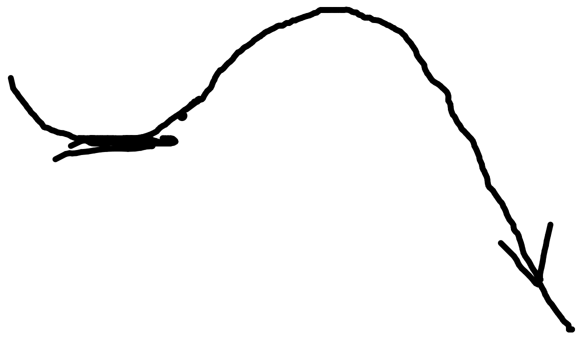
simply minimize

$$f(x,y,z) = \text{distance}^2\big(P, (x,y,z)\big)$$

Q: how local minimum becomes an absolute minimum

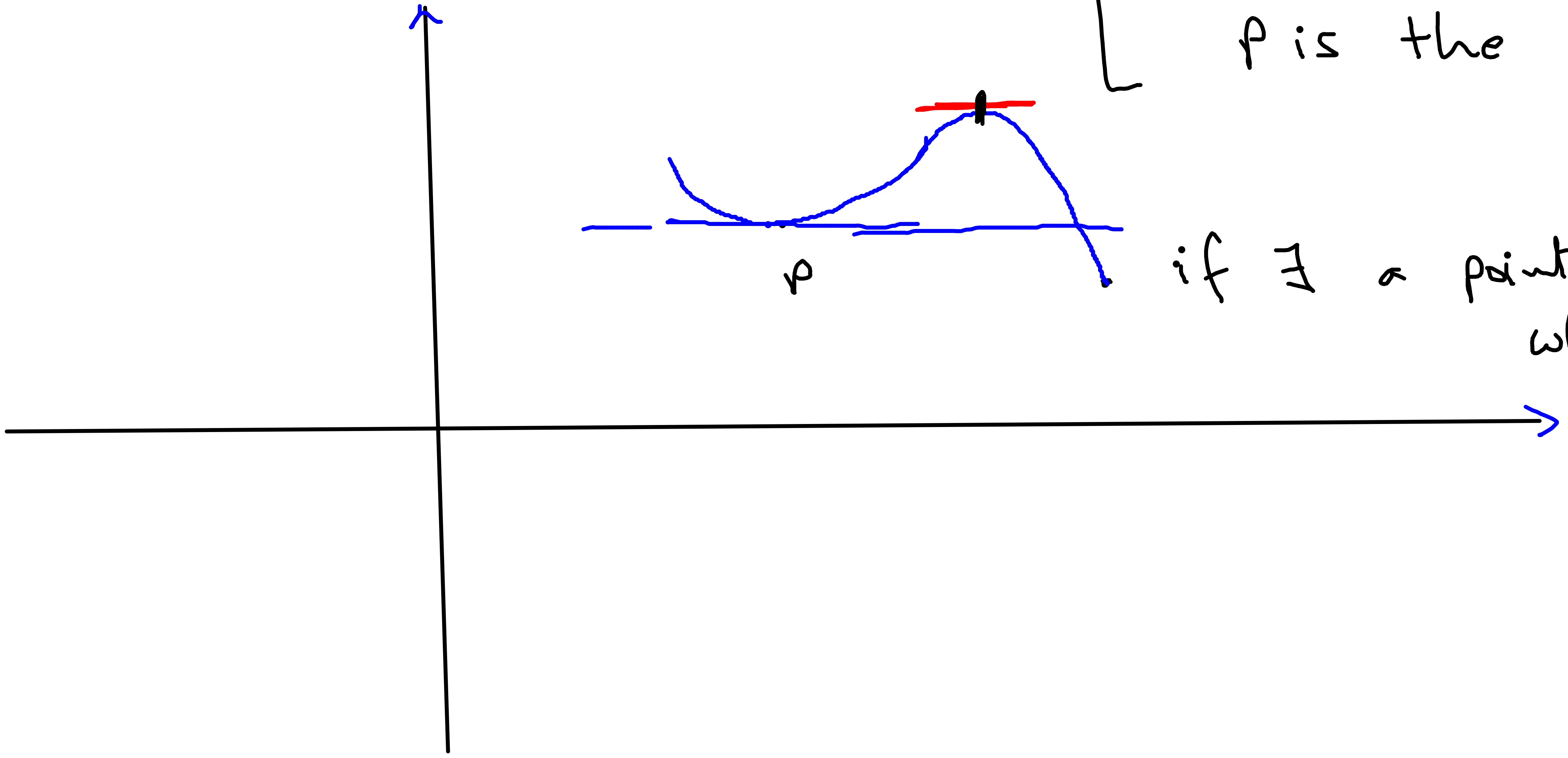


lower value



Suppose p: is point of local min

[& also given:
p is the only critical point]



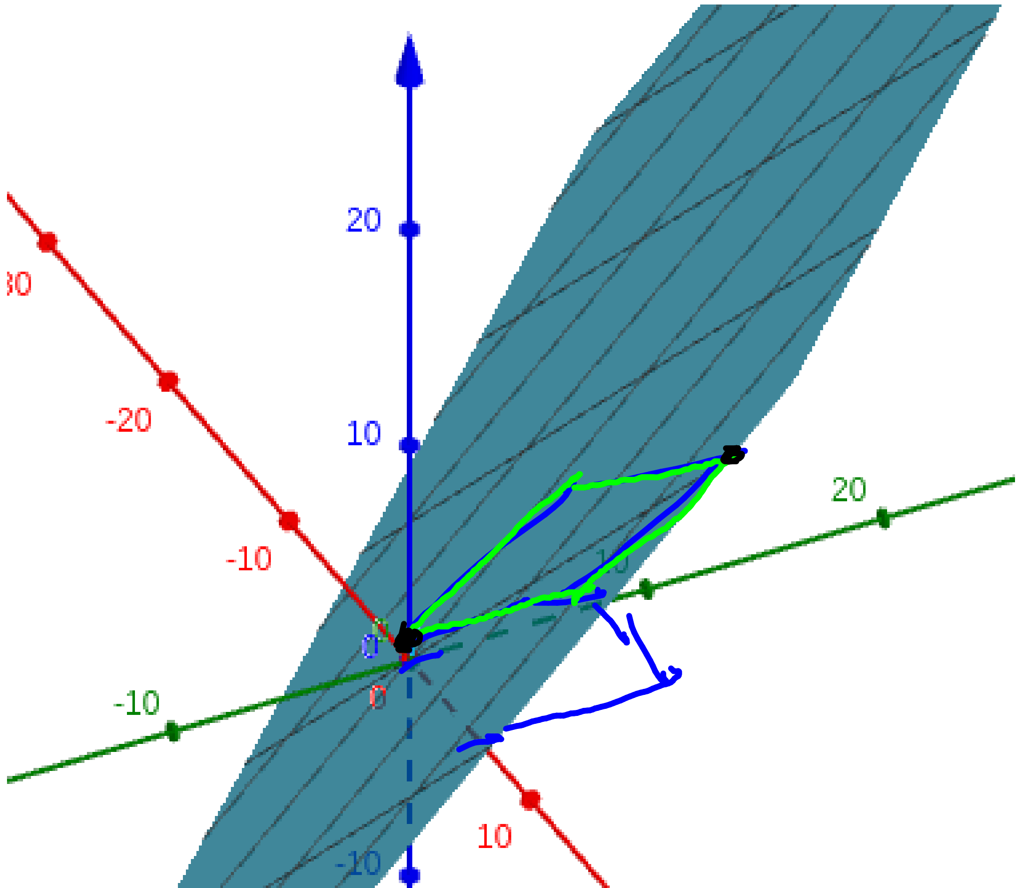
if $\exists$ a point on graph which is lower than p

Solving max/min problems on a bounded domain

$f(x,y) = x+y$

$0 \le x \le 1$

$0 \le y \le 1$



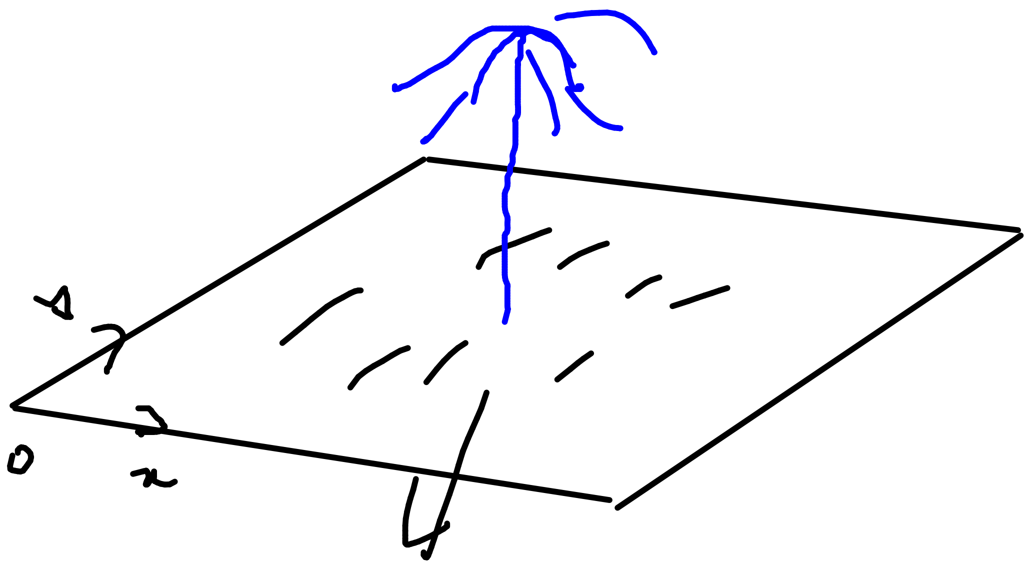
message ① on finite domain a "continuous" function always has a max & min.

message ② max & min points are generally not critical points

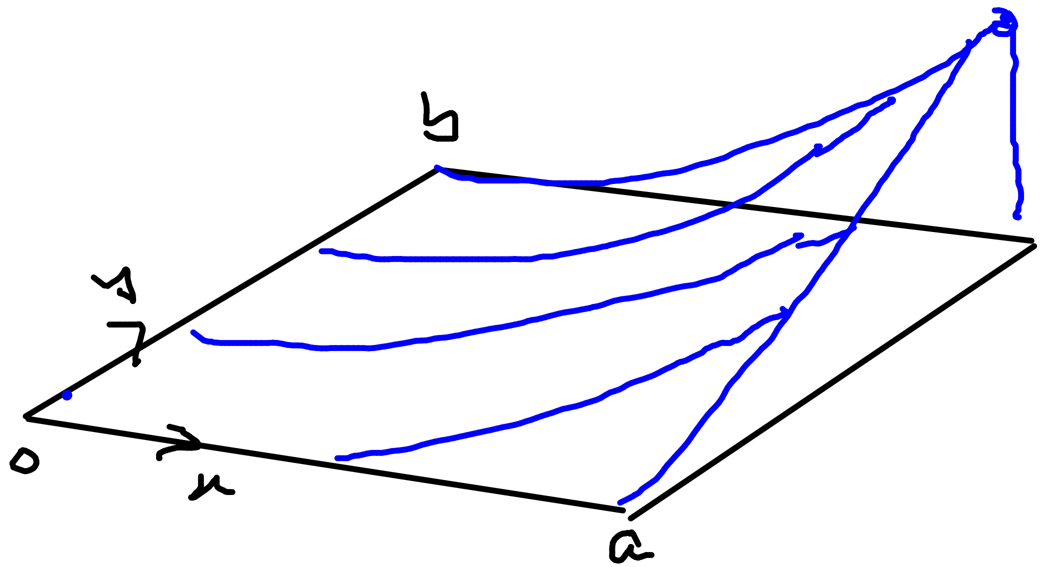
$f(x,y)$

interior
boundary

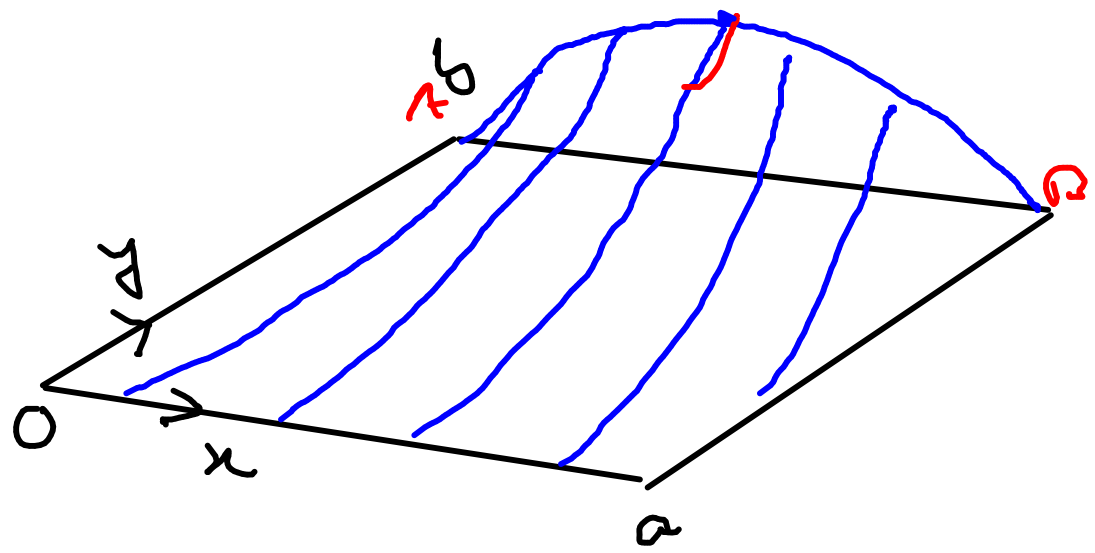
Q: if $\boxed{f(x,y)}$ has a max in the interior, then that point is a critical point.



$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$



Q: if max occur at corners, does it have to be a critical point / No

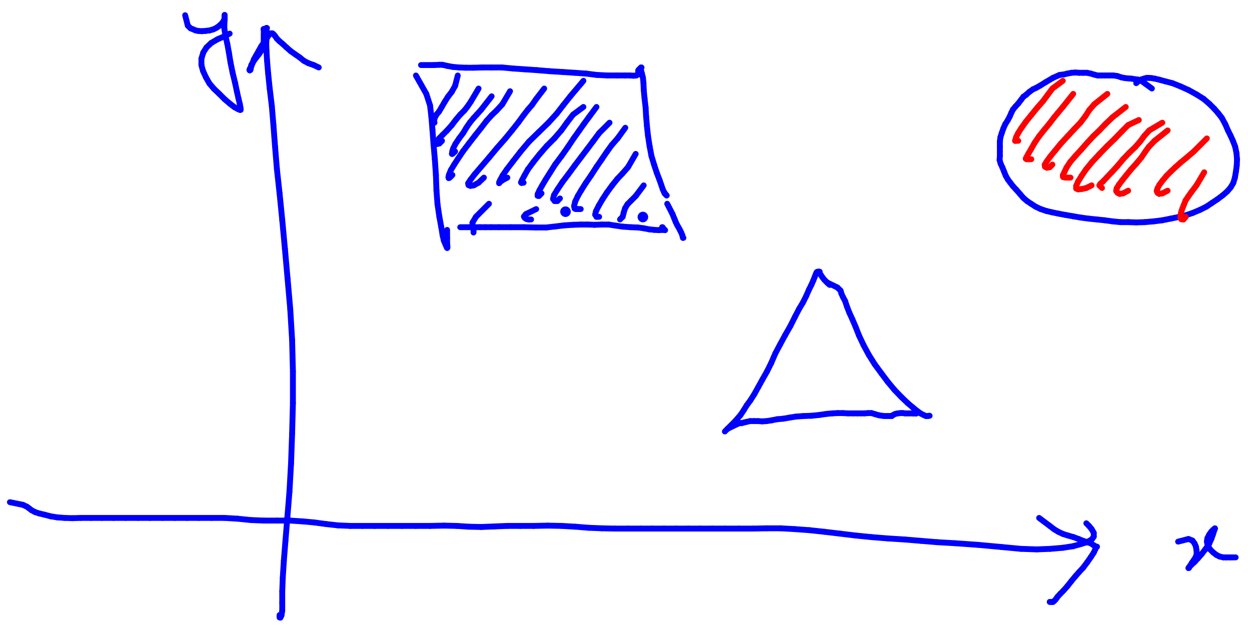


Q: if max does not occur in the interior, does max have to be at the corner ??

Recall : 11.7 (old edition) / 14.7 (new edition)

we want to find max/min $f(x,y)$

$(x,y) \in$ bounded domain

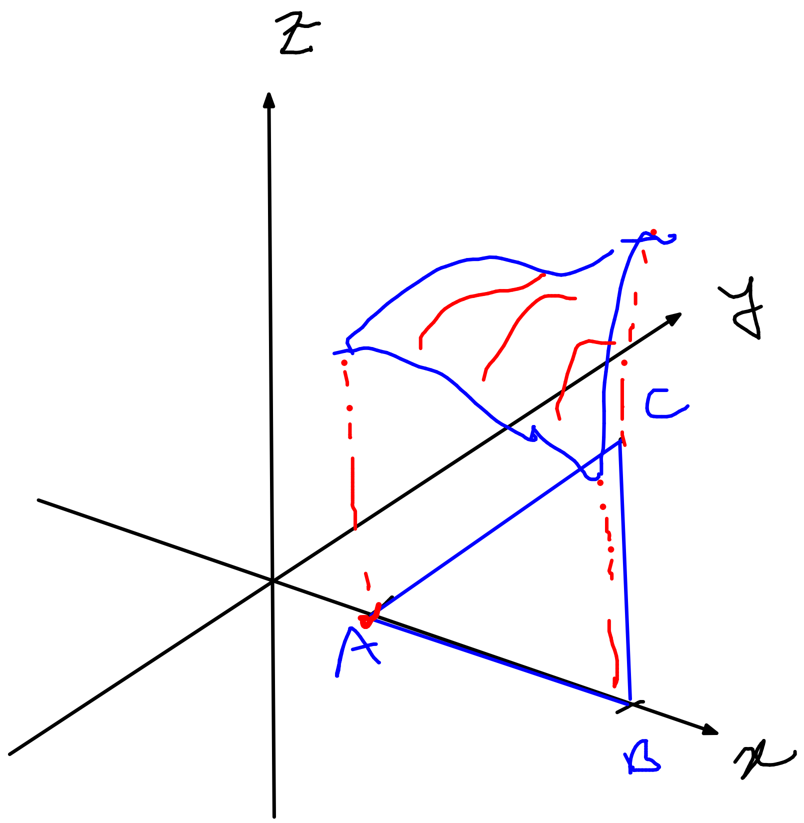


Q.

$f(x, y) = 3 + xy - x - 2y$, D is the closed triangular region with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$

A B C

find max/min on $\triangle ABC$



BC ~ $x + y = 5$

Q. Does $f(x,y)$ has a critical point in $\triangle ABC$??

$\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial y} = 0$

$y - 1 = 0$ | $x - 2 = 0$

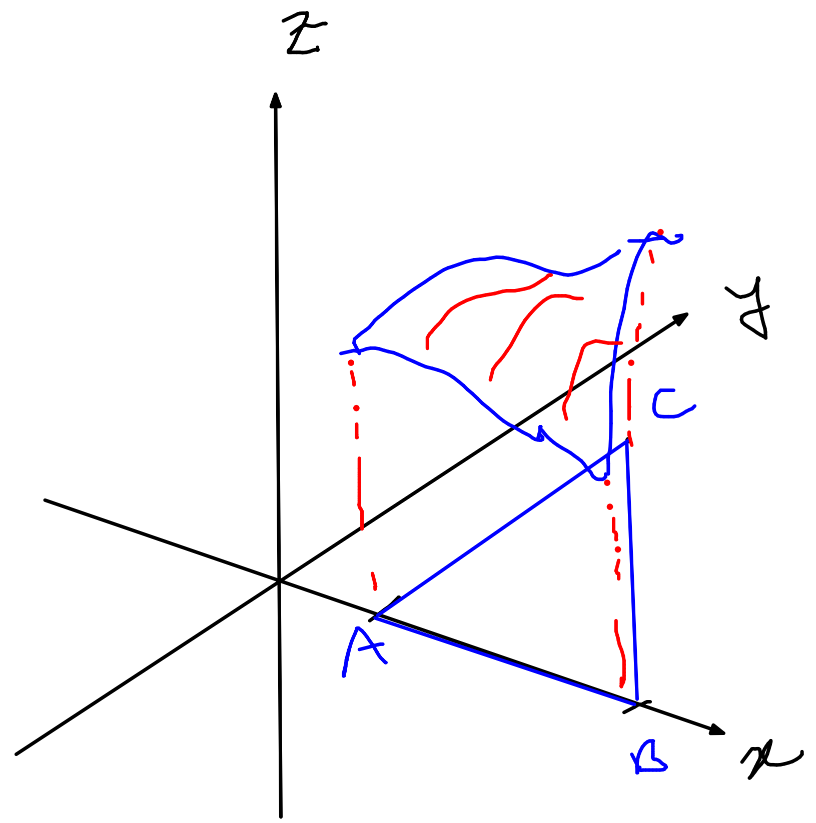
$(x, y) = (2, 1) \overset{??}{\in} \triangle ABC$

(2,1) is a possible candidate for max/min point.

8. $f(x, y) = 3 + xy - x - 2y$, D is the closed triangular region with vertices (1, 0), (5, 0), and (1, 4)

A B C

find max/min on $\triangle ABC$



BC ~ $x+y=5$

Restriction of $f(x,y)$ on line AB

$f = 3 - x$, $1 \le x \le 5$

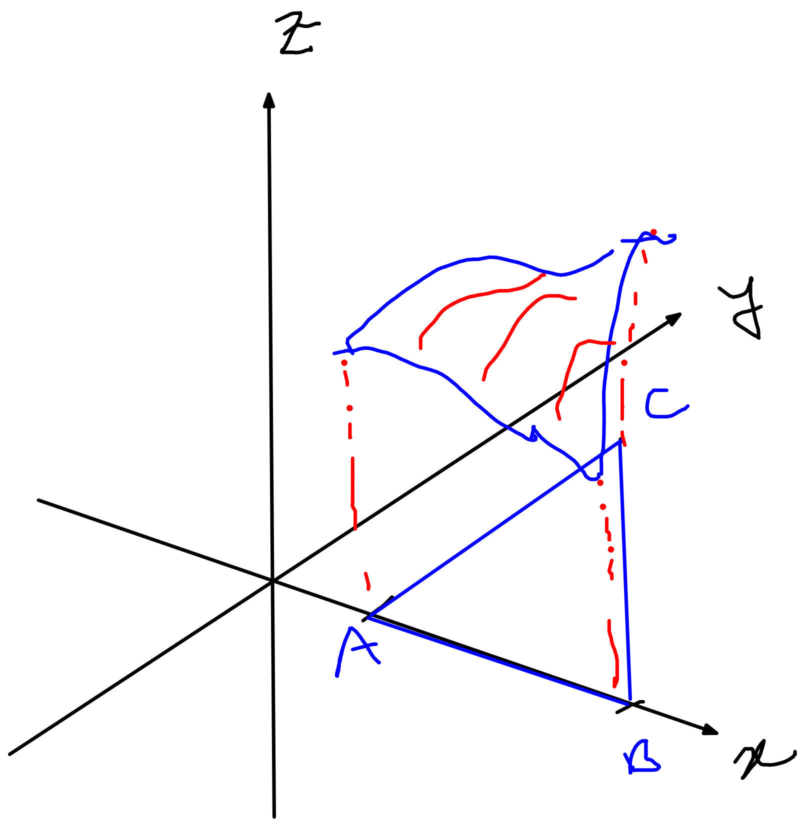
		y	f	
max at	$x=1$	0	2	A
min at	$x=5$	0	-2	B

8.

$f(x, y) = 3 + xy - x - 2y$, D is the closed triangular region with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$

A B C

find max/min on ΔABC



BC ~ $x+y=5$

Restriction on line AC

use: $x=1$

$f = 3+y-1-2y = 2-y$, $0 \le y \le 4$

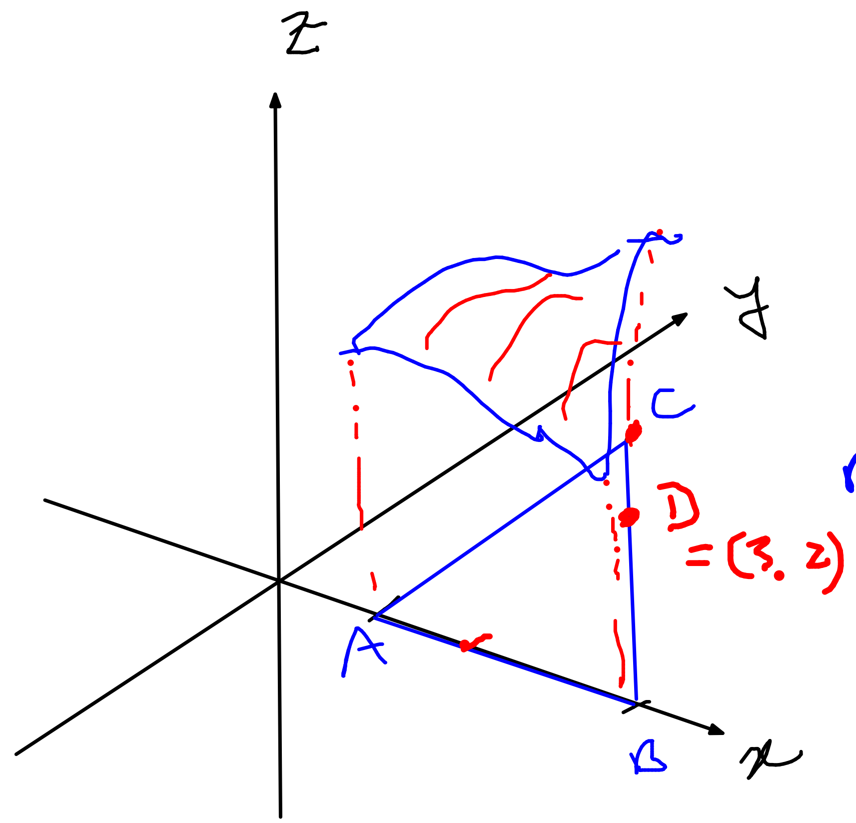
	x	y	f	
max	1	0	2	A
	1	4	-2	C

8.

$f(x, y) = 3 + xy - x - 2y$, D is the closed triangular region with vertices $(1, 0)$, $(5, 0)$, and $(1, 4)$

A B C

find max/min on ΔABC



$BC \sim x + y = 5$

Restriction on line BC

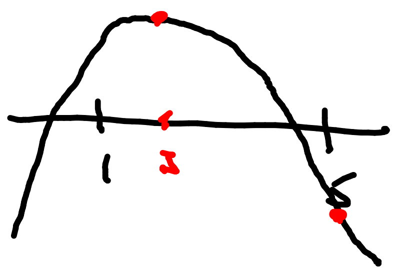
$x + y = 5$

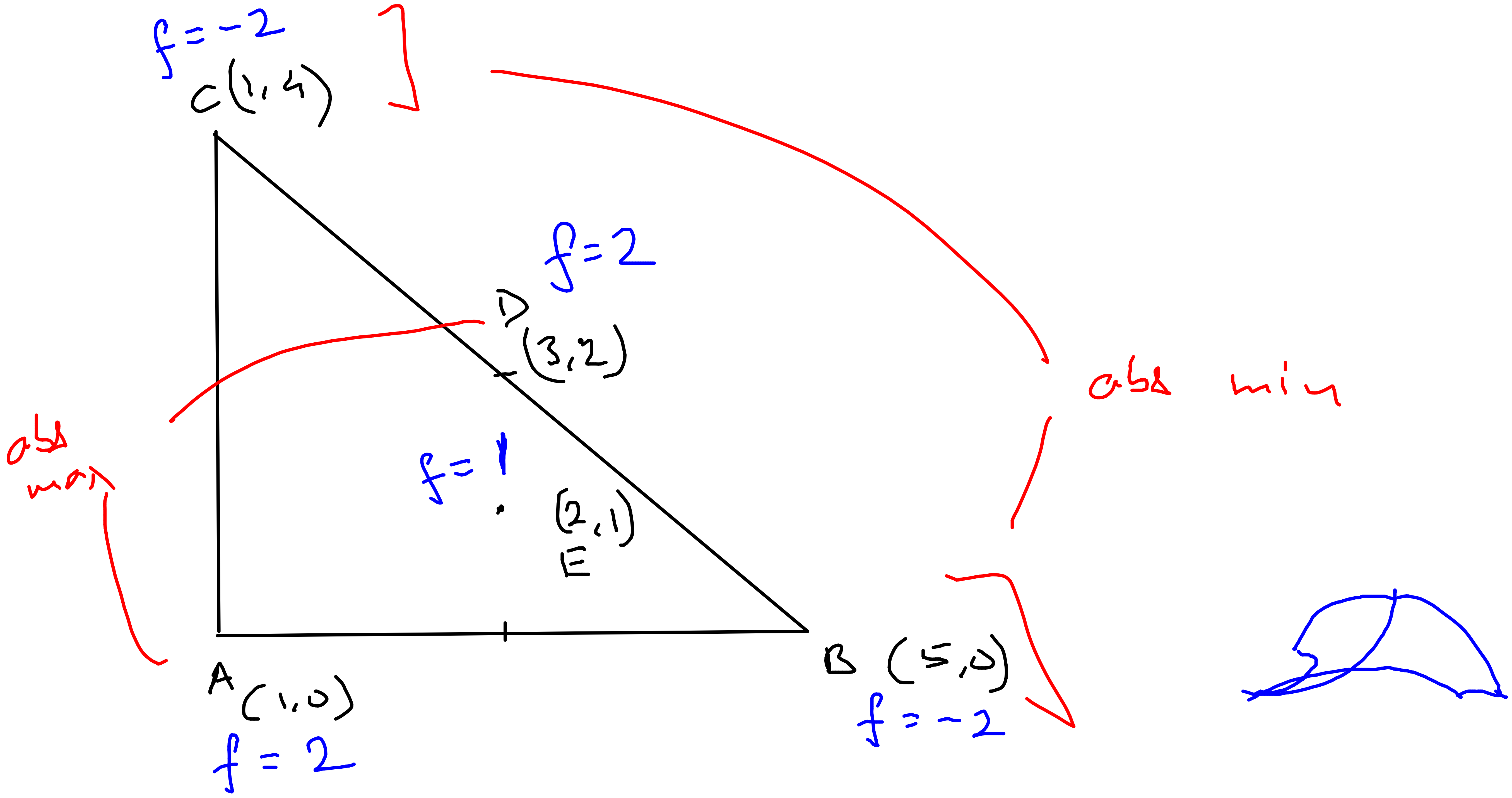
$$f = 3 + x(5-x) - x - 2(5-x)$$

$$= -7 + 6x - x^2, \quad 1 \le x \le 5$$

Q: max/min of $-7 + 6x - x^2$, if $1 \le x \le 5$

x	y	f
1	4	-2
5	0	-2
$6 - 2x = 0$ $\quad x = 3$	2	2



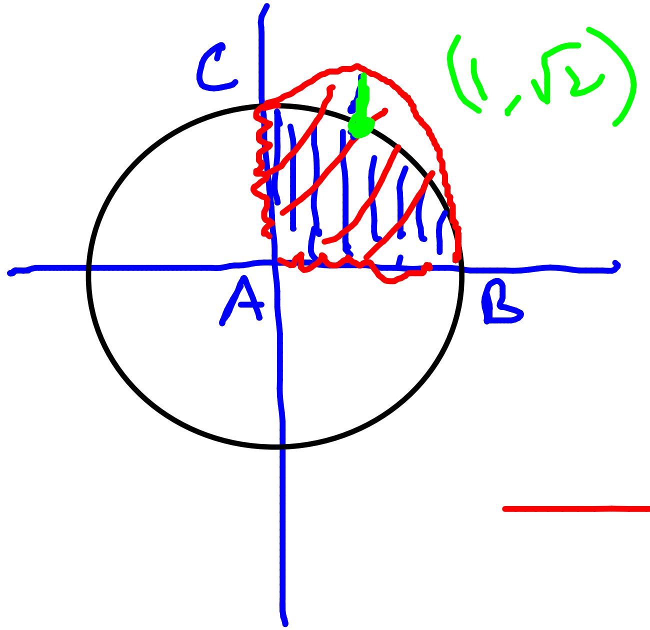


f = -2
C(1,4)
f = 2
D
(3,2)
abs max
f = 1
(2,1)
E
abs min
A (1,0)
f = 2
B (5,0)
f = -2

$$f(x, y) = xy^2, \quad D = \{(x, y) \mid x \geqslant 0,\ y \geqslant 0,\ x^2 + y^2 \leqslant 3\}$$

d. Sketch the D

$x^2 + y^2 \leq 3$



→ check for max/min in the interior

$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$

$y^2 = 0 \qquad 2xy = 0$

$(x, 0)$: for any x

→ not in the interior

→ AB: $f = 0$

AC: $f = 0$

BC: $x^2 + y^2 = 3$

$y^2 = 3 - x^2$

$f = x(3 - x^2), \quad 0 \leq x \leq \sqrt{3}$

$f' = 3 - 3x^2 = 0$

$x = 1$

	x	y	f
	0	$\sqrt{3}$	0
	$\sqrt{3}$	0	0
max	1	$\sqrt{2}$	2