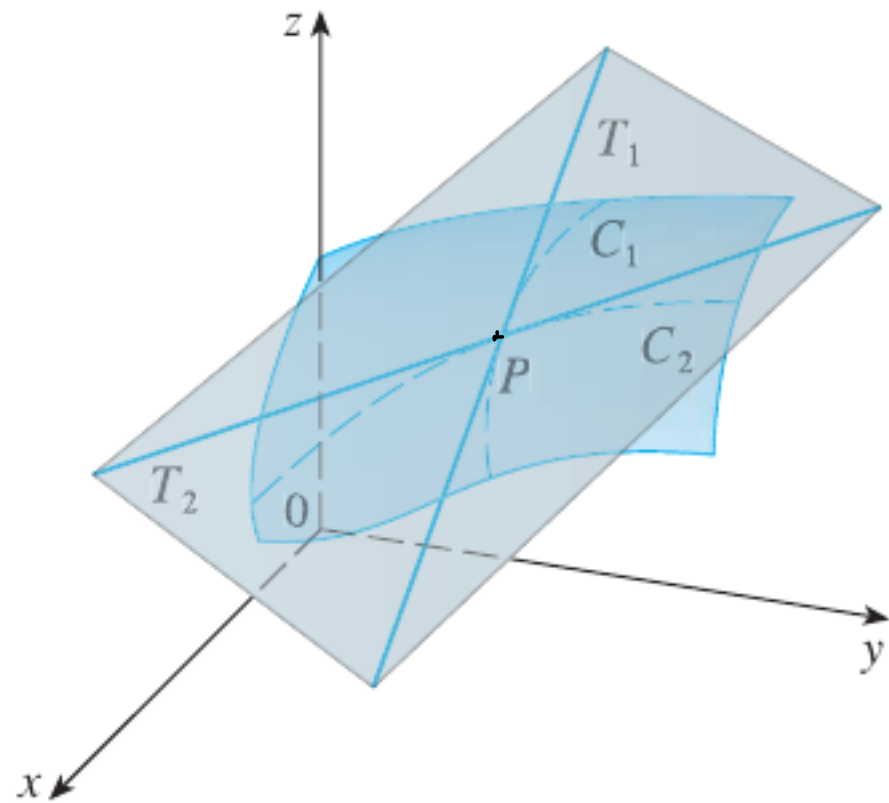


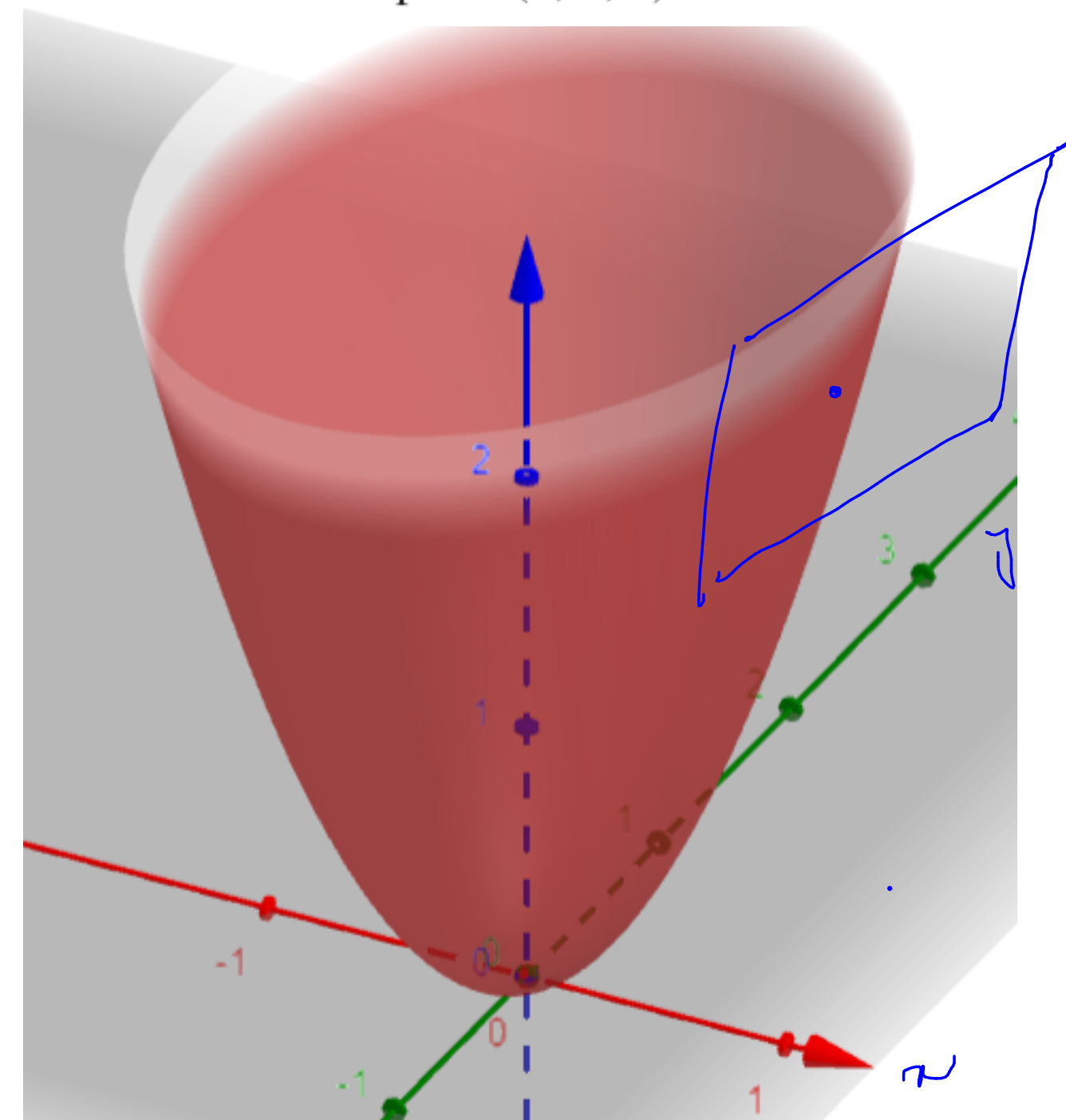
11.4

TANGENT PLANES AND LINEAR APPROXIMATIONS

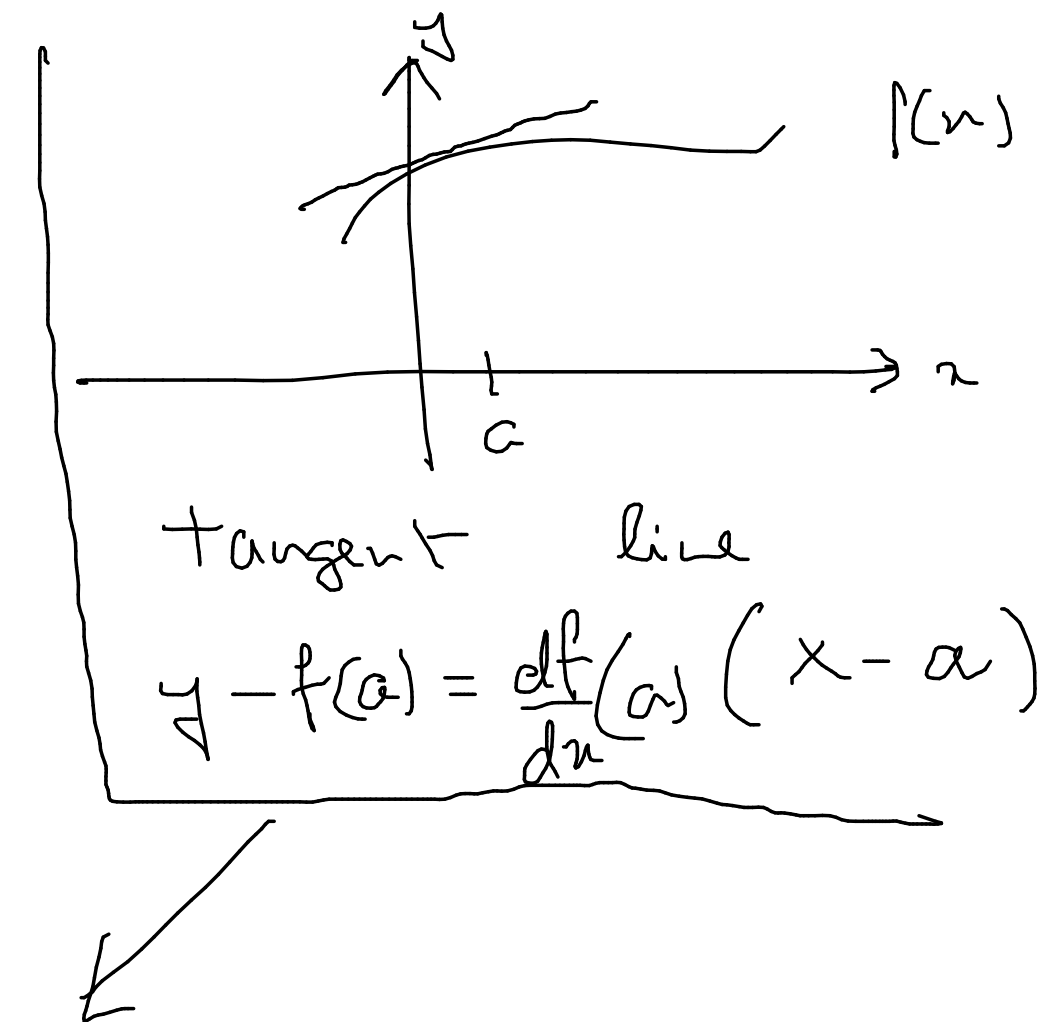
why care for tangent planes?



V EXAMPLE I Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.



in 3d:
graph of $z = f(x, y)$
at point (a, b)



$$z - f(a, b) = \frac{\partial f}{\partial x}(a, b) (x - a) + \frac{\partial f}{\partial y}(a, b) (y - b)$$

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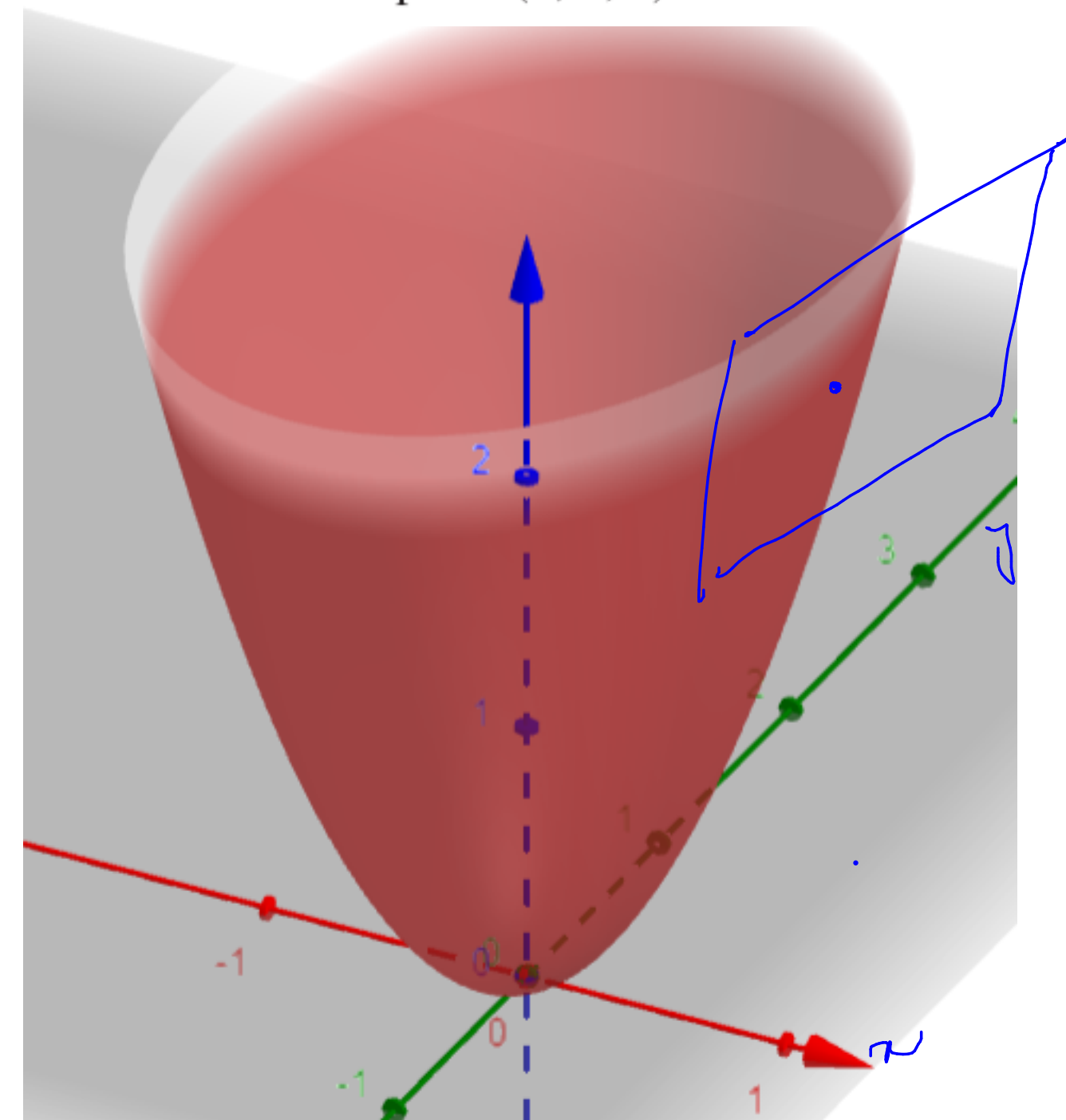
$$z - f(a, b) = \frac{\partial f}{\partial x}(a, b) (x - a) + \frac{\partial f}{\partial y}(a, b) (y - b)$$

$$z - \underbrace{f(1, 1)}_{??} = \underbrace{\frac{\partial f}{\partial x}(1, 1)}_{??} (x - 1) + \underbrace{\frac{\partial f}{\partial y}(1, 1)}_{??} (y - 1)$$

$$f(1, 1) = 3$$

$$\frac{\partial f}{\partial x}(1, 1) = 4x \bigg|_{\substack{x=1 \\ y=1}} = 4$$

$$\frac{\partial f}{\partial y}(1, 1) = 2y \bigg|_{\substack{x=1 \\ y=1}} = 2$$



the tangent plane:

$$z - 3 = 4(x - 1) + 2(y - 1)$$

Q. Exercise:
find eqⁿ of tangent plane

$$f(x, y) = x e^y, \quad (1, 0)$$

$$z - z_0 = \frac{\partial f}{\partial x}(1, 0)(x - 1) + \frac{\partial f}{\partial y}(1, 0)(y - 0)$$

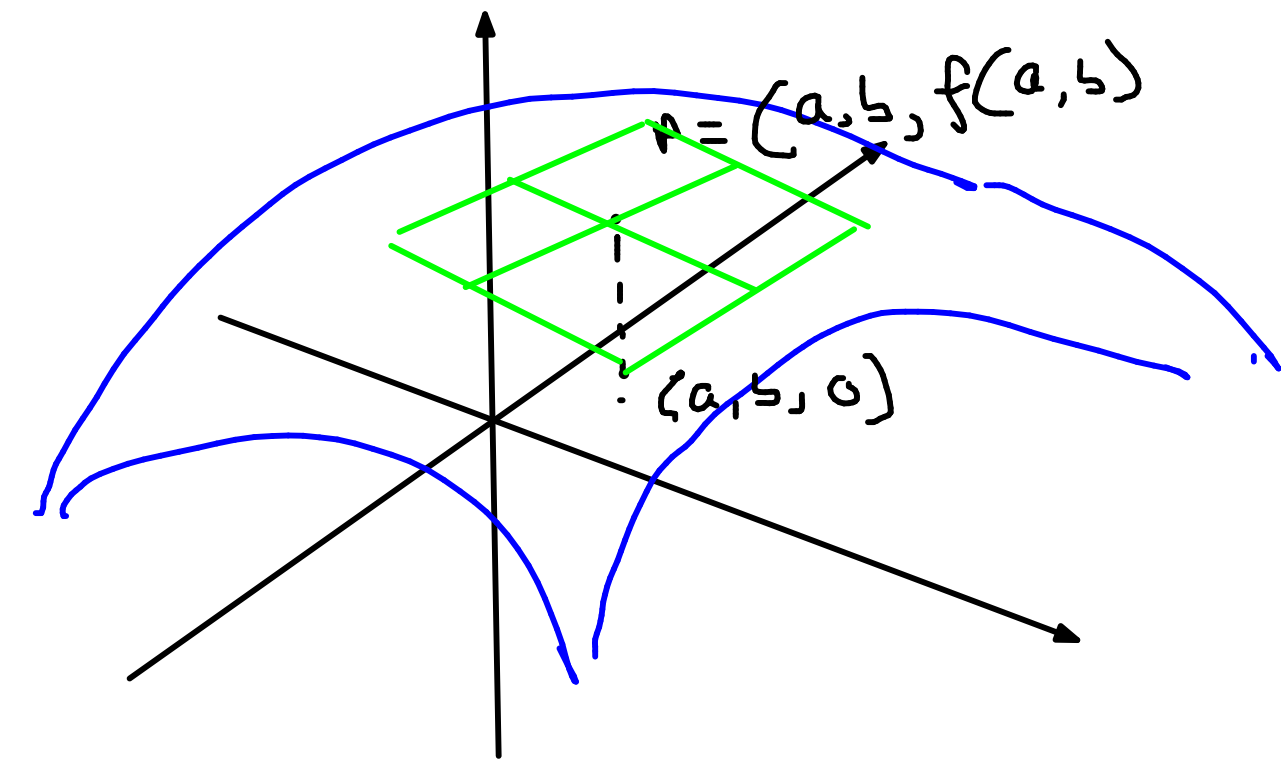
$$z_0 = f(1, 0) = 1$$

$$\frac{\partial f}{\partial x} = e^y \Big|_{\substack{x=1 \\ y=0}} = 1$$

$$\frac{\partial f}{\partial y} = x e^y \Big|_{x=1, y=0} = 1$$

$$z - 1 = 1(x - 1) + 1(y - 0)$$

$$\boxed{z = x + y}$$



formula of tangent to graph of
 $f(x, y)$ at (a, b) is

$$z - z_0 = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

plane must look like

$$A(x - a) + B(y - b) + C(z - z_0) = 0$$

$$z_0 = f(a, b)$$

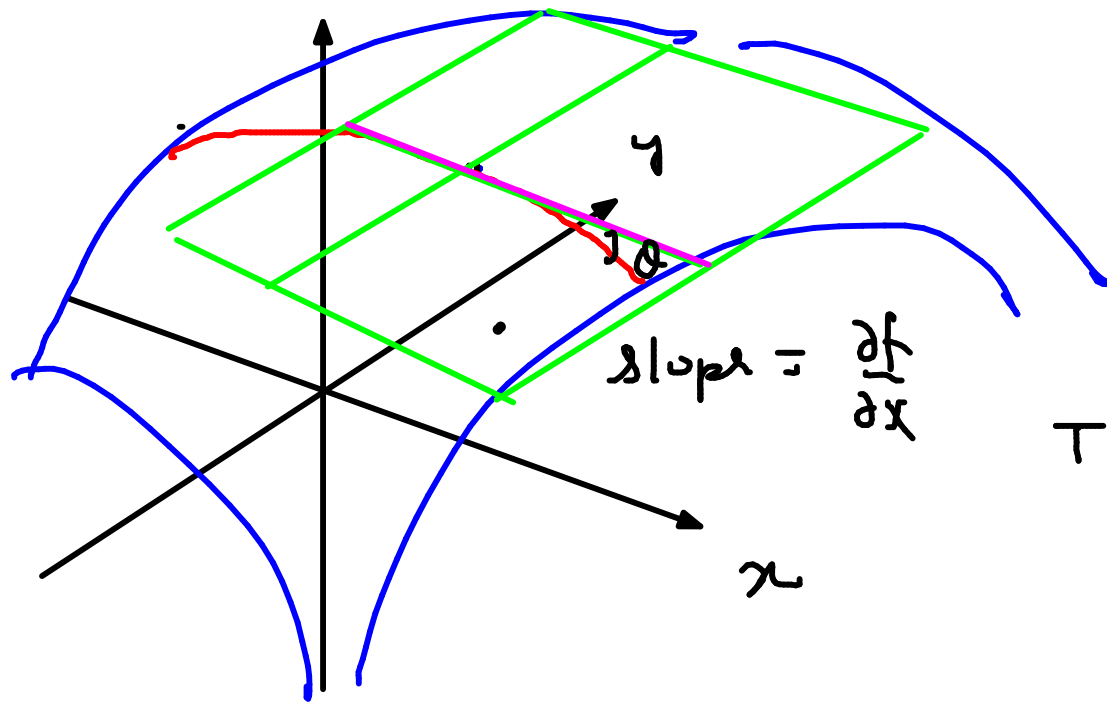
plane contains the point P
 (a, b, z_0)

if $c = 0$,
 plane becomes
 vertical
 not possible

→ simplified gen eqn passing through
 (a, b, z_0)

$$z = f(x, y)$$

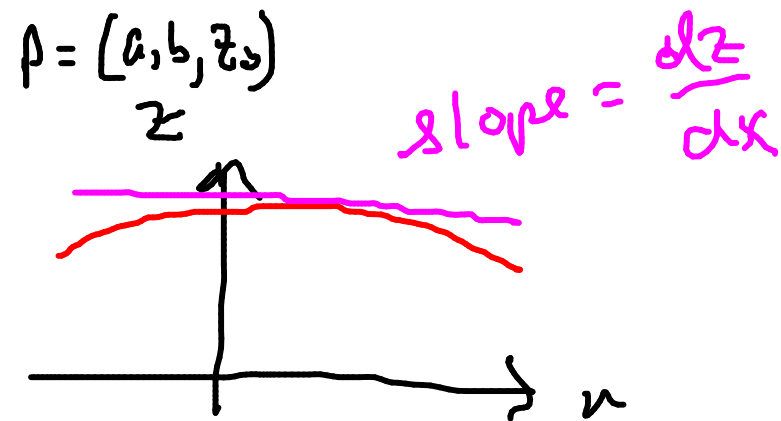
$$[z - z_0 = A(x - a) + B(y - b)] \text{ green plane}$$



$$\underline{Q:} \quad \left. \begin{aligned} A &= \frac{\partial f}{\partial x}(a, b) \\ B &= \frac{\partial f}{\partial y}(a, b) \end{aligned} \right\} \text{ why ??}$$

Think: slice the graph $z = f(x, y)$ with plane $y = b$
 I also slice the plane $z - z_0 = A(x - a) + B(y - b)$ with $y = b$

$$z - z_0 = A(x - a)$$



Q: what will be the slope of pink line??

$A = \text{slope of the pink line}$

$$= \frac{dz}{dx}$$

$$z - z_0 = A(x - a) + B(y - b)$$

we just discussed: $A = \frac{\partial f}{\partial x}(a, b)$:

it.w. repeat the argument to convince why

$$B = \frac{\partial f}{\partial y}(a, b)$$

Today:

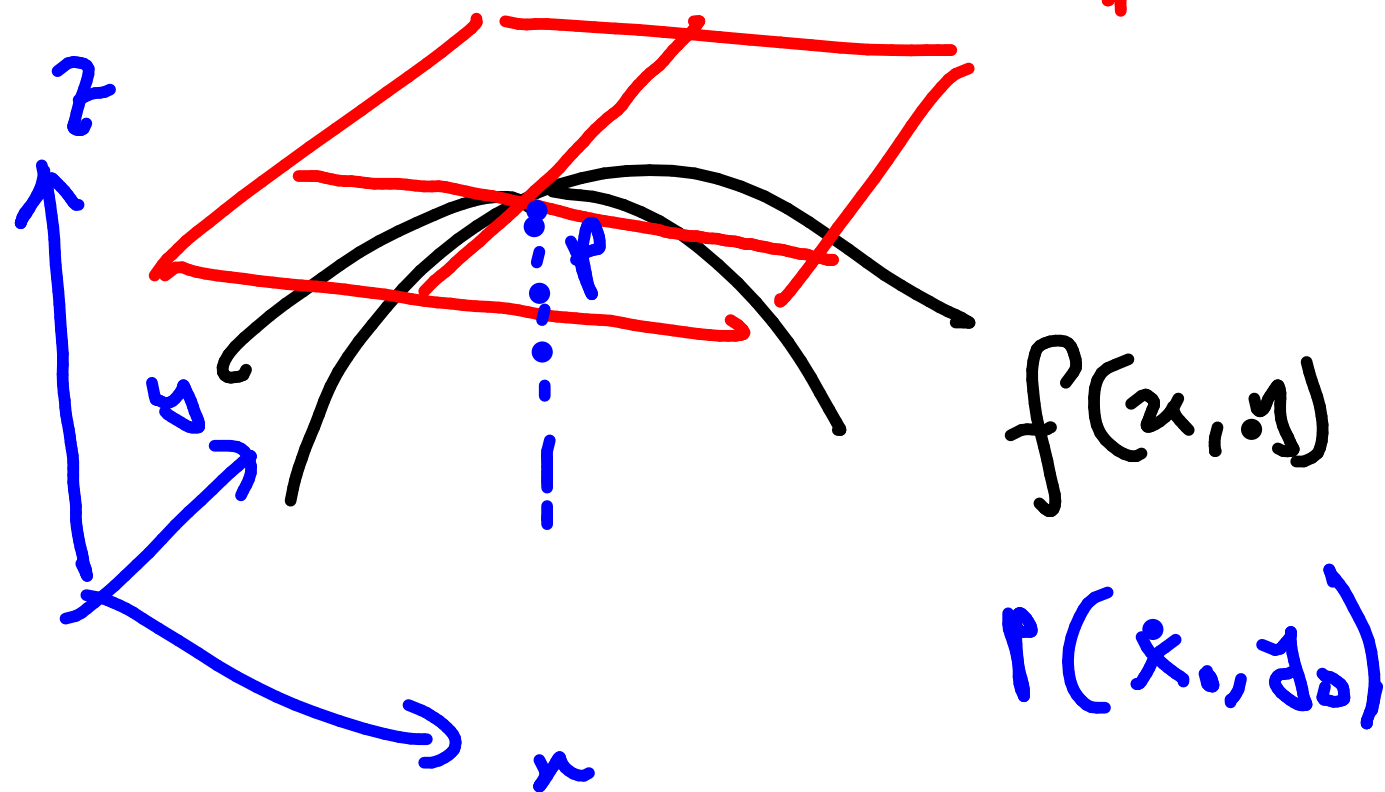
11.4 remaining topics:

11.5: chain rule

$$\left. \begin{array}{l} f(x, y) \\ x = (u, v) \\ y = (u, v) \end{array} \right| \frac{\partial f}{\partial u} = ??$$

Recall: eqⁿ for the tangent plane

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$



where

$$z_0 = f(x_0, y_0)$$

8 THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are con-
tinuous at (a, b) , then f is differentiable at (a, b) .

$$f(x, y)$$

$$\frac{\partial f}{\partial x}$$

✓

$$\frac{\partial f}{\partial y}$$

✓

$g(x)$
 $g(x)$ is differentiable
if $\frac{dg}{dx}$ exist

V EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Q. f is differentiable if $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ exist & are continuous

$$\begin{aligned}\frac{\partial f}{\partial x} &= xy e^{xy} + e^{xy} \\ \frac{\partial f}{\partial y} &= x^2 e^{xy}\end{aligned}$$

are these continuous
at $(1, 0)$ or not??

Yes:

visual inspection of the
graphs

V EXAMPLE 2 Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

→ find its linearization:

simply a reformulation of the tangent plane

find a formula

$$L(x, y) = ??$$

whose graph is the tangent plane.

$$\left[z - z_0 = \frac{\partial f}{\partial x}(1, 0)(x - 1) + \frac{\partial f}{\partial y}(1, 0)(y - 0) \right] \text{ tangent plane}$$

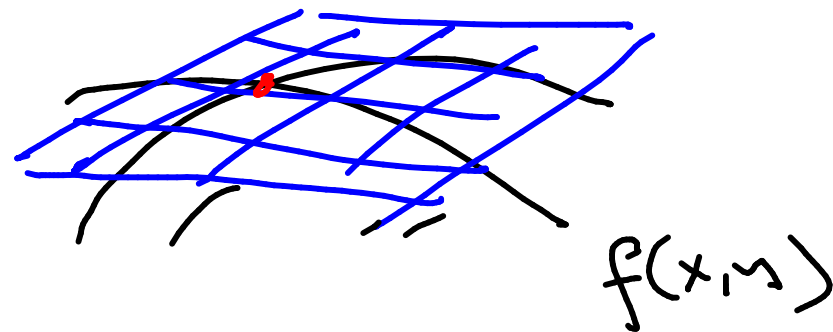
$$z - 1 = 1(x - 1) + 1(y - 0)$$

$$z = 1 + 1(x - 1) + 1(y - 0)$$

$$L(x, y) = 1 + 1(x - 1) + 1(y - 0) = x + y$$

$$f(1.1, -0.1) = 1.1 e^{(1.1)(-0.1)} = 0.986$$

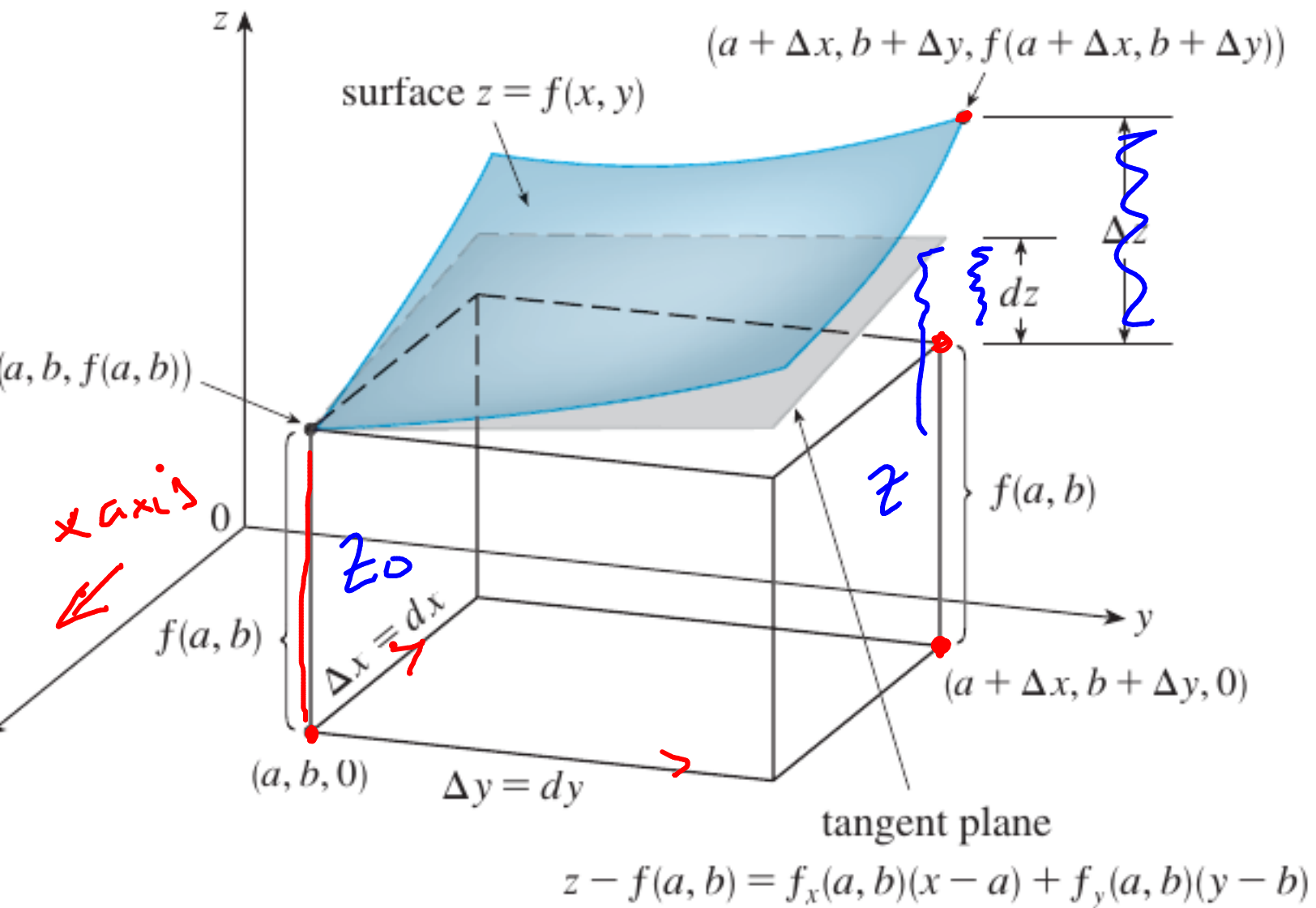
$$\approx L(1.1, -0.1) = 1$$



DIFFERENTIALS

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$f(x, y)$$



$$x \rightarrow x + \Delta x \quad \& \quad y \rightarrow y + \Delta y$$

differentials estimates change in f

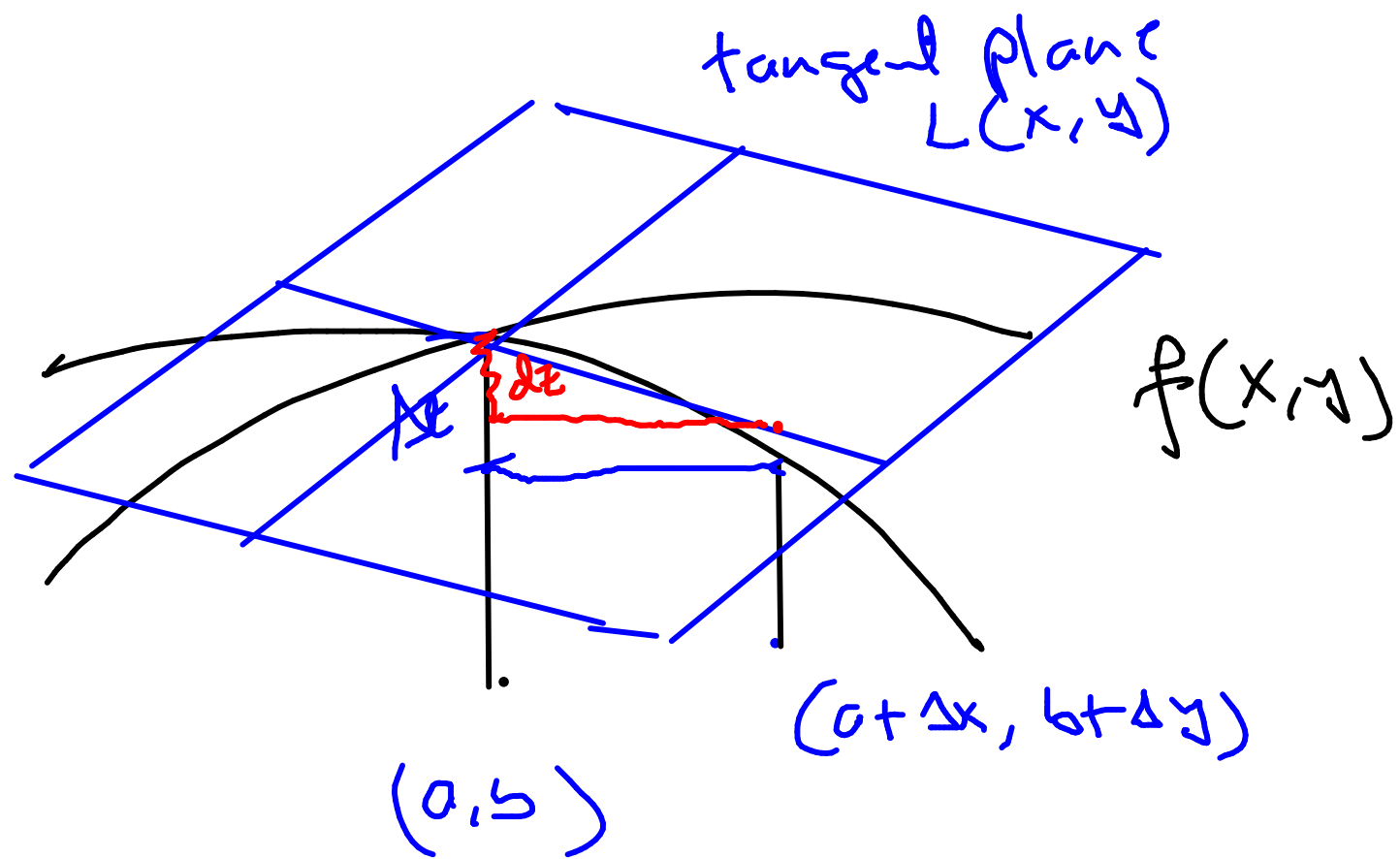
$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

SS

dz = actual change in the tangent plane

$$z - z_0 = \frac{\partial f}{\partial x} (x - a) + \frac{\partial f}{\partial y} (y - b)$$

$$dz = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$



$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

$\Delta z \approx dz =$ change in z in the tangent plane

$$z - z_0 = \frac{\partial f}{\partial x} (x - a) + \frac{\partial f}{\partial y} (y - b)$$

$$dz = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

V EXAMPLE 3

- (a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz .
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz .

a)

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$dz = (2x + 3y) dx + (3x - 2y) dy$$

b)

$$dz, \quad x = 2, \quad y = 3, \quad dx = 0.05, \quad dy = -0.04$$
$$dz = 13(0.05) + 0 \cdot dy = 0.65$$

$$\Delta z = f(2.05, 2.96) - f(2, 3)$$

EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.