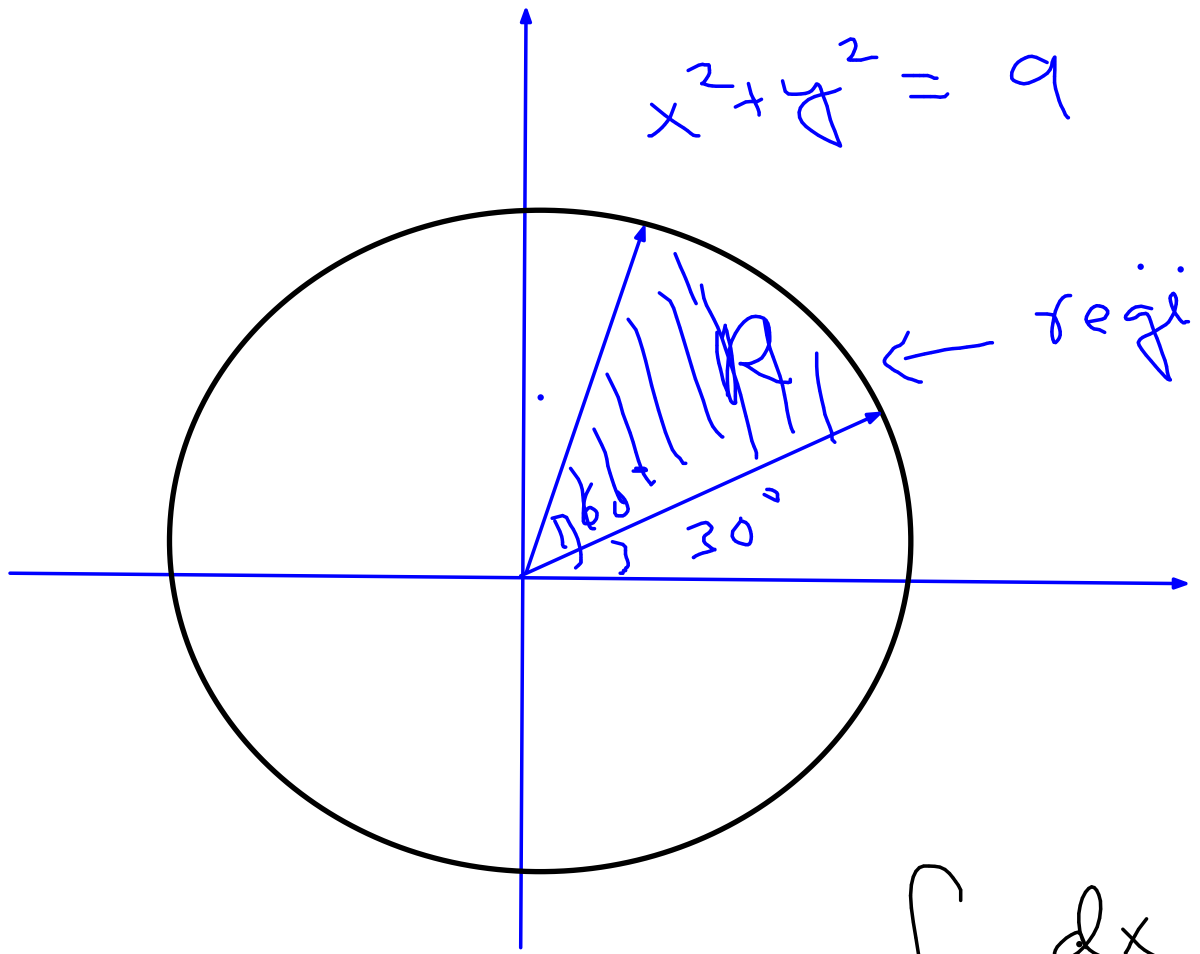
15.9

CHANGE OF VARIABLES IN MULTIPLE INTEGRALS



$$\iint_{R} \_\_\_ \, dx\,dy = \int_{0}^{3}\int_{30^\circ}^{60^\circ} d\theta\, dr$$

$$\int \_\_ \, dx$$

$x = f(u)$

$dx = f'(u)\, du$

Jacobian:

$$x = x(u,v)$$
$$y = y(u,v)$$

$$dx\,dy = (\text{Jacobian})\,du\,dv$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \text{abs}\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right)$$

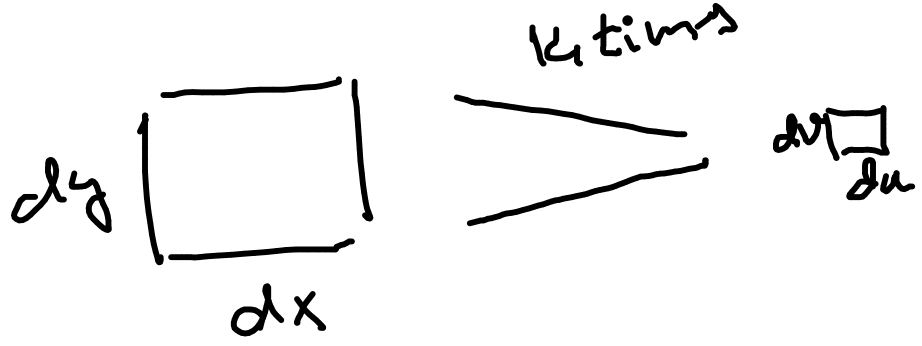
Find the Jacobian of the transformation.

$$x = u + 4v, \quad y = 3u - 2v$$

$$J = \text{abs}\left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right)$$

$$= \text{abs}\left(\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} \right) = 14$$

$$dx\, dy = 14\, du\, dv$$



Find the image of the set S under the given mapping

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ y = u - v$$

Jacobian:

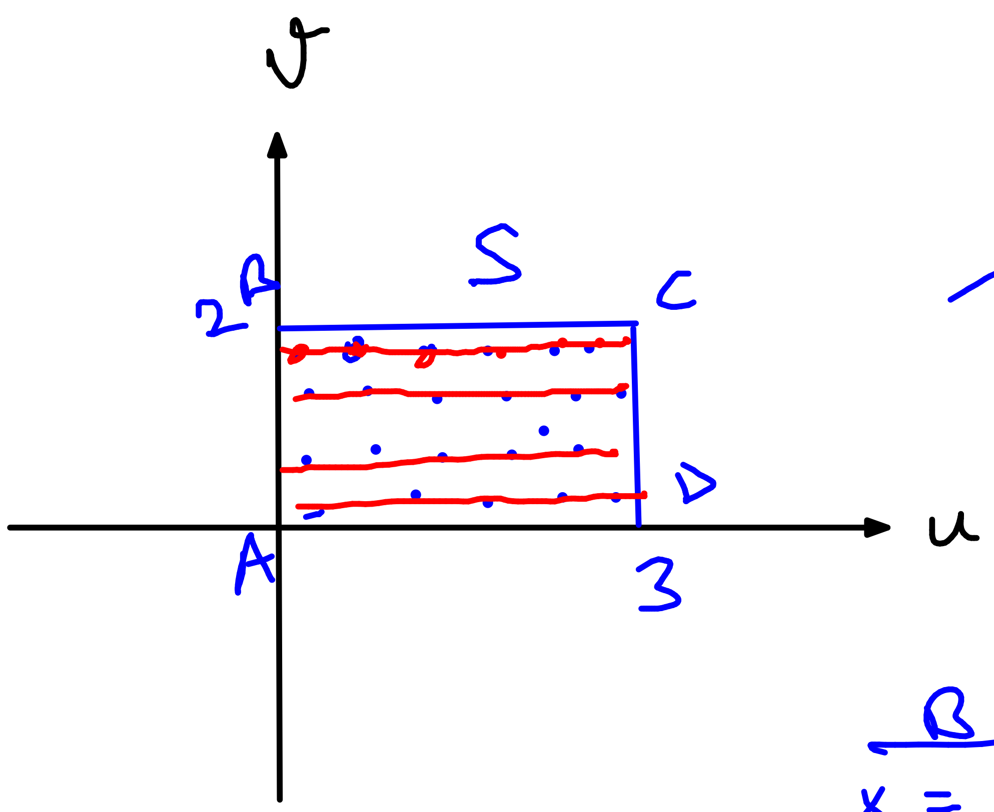
$x = 2u + 3v$

$y = u - v$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\right|$$

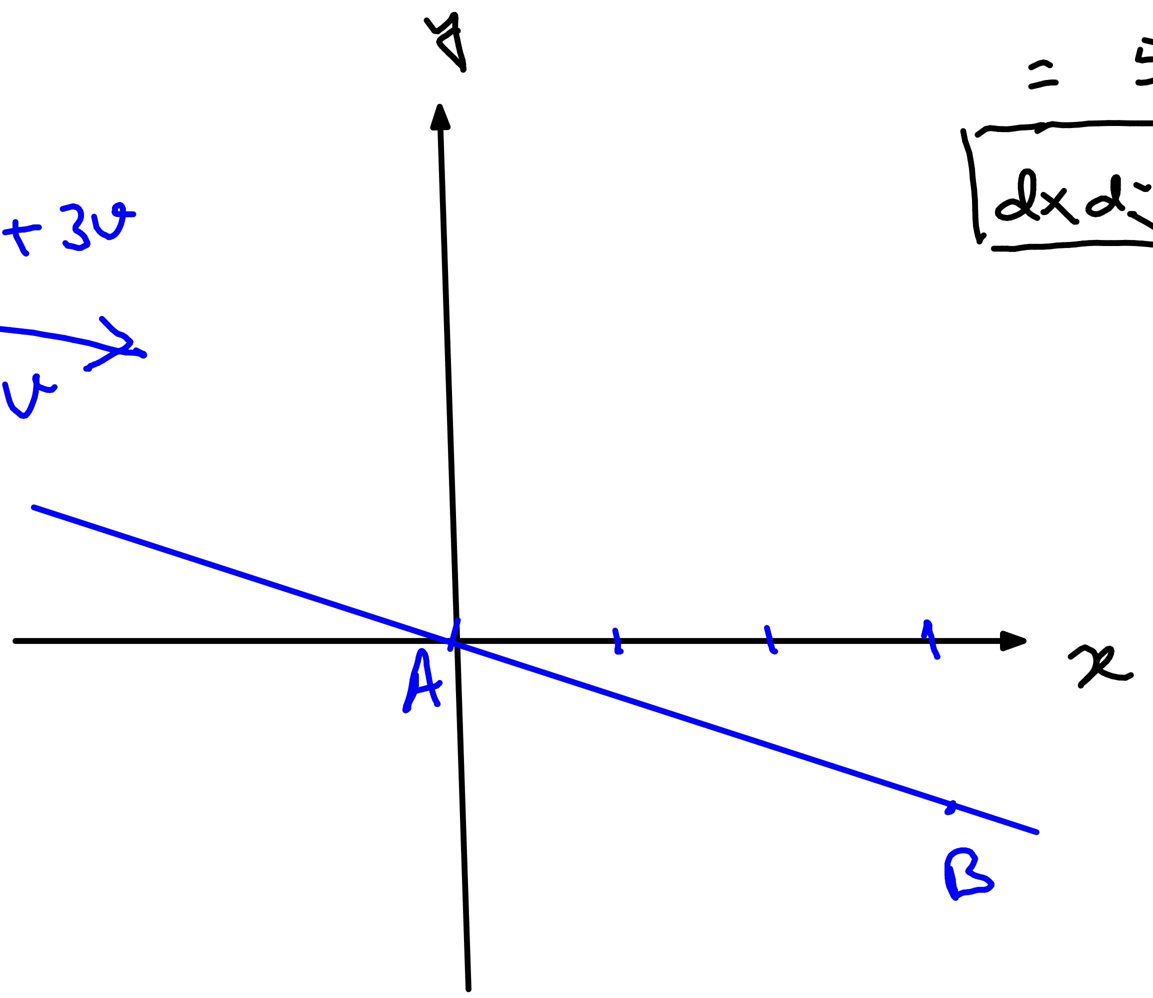
$= 5$

$$\boxed{dx\,dy = 5\,du\,dv}$$



$x = 2u + 3v$

$y = u - v$



$\underline{B}$

$x = 6$

$y = -2$

Image of AB in the xy plane

$u = 0,\quad 0 \leq v \leq 2$

$$\underbrace{x = 3v,\ y = -v}$$

$$\boxed{x = -3y}$$

$y = -x/3$

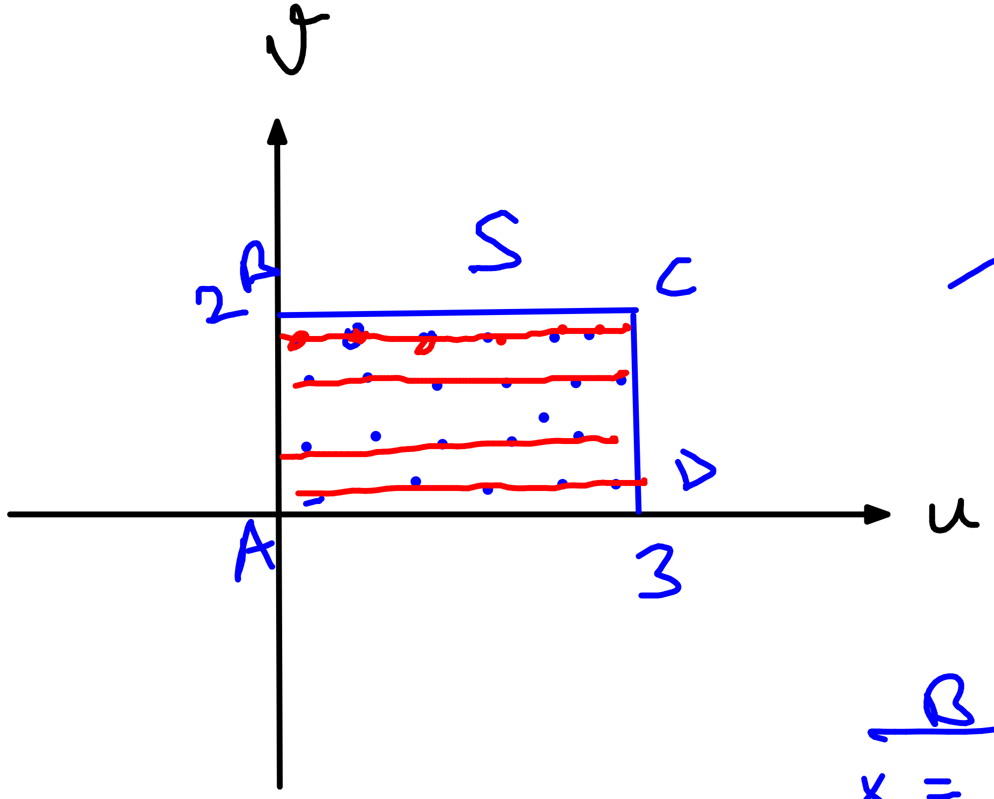
Find the image of the set S under the given mapping

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ \ y = u - v$$

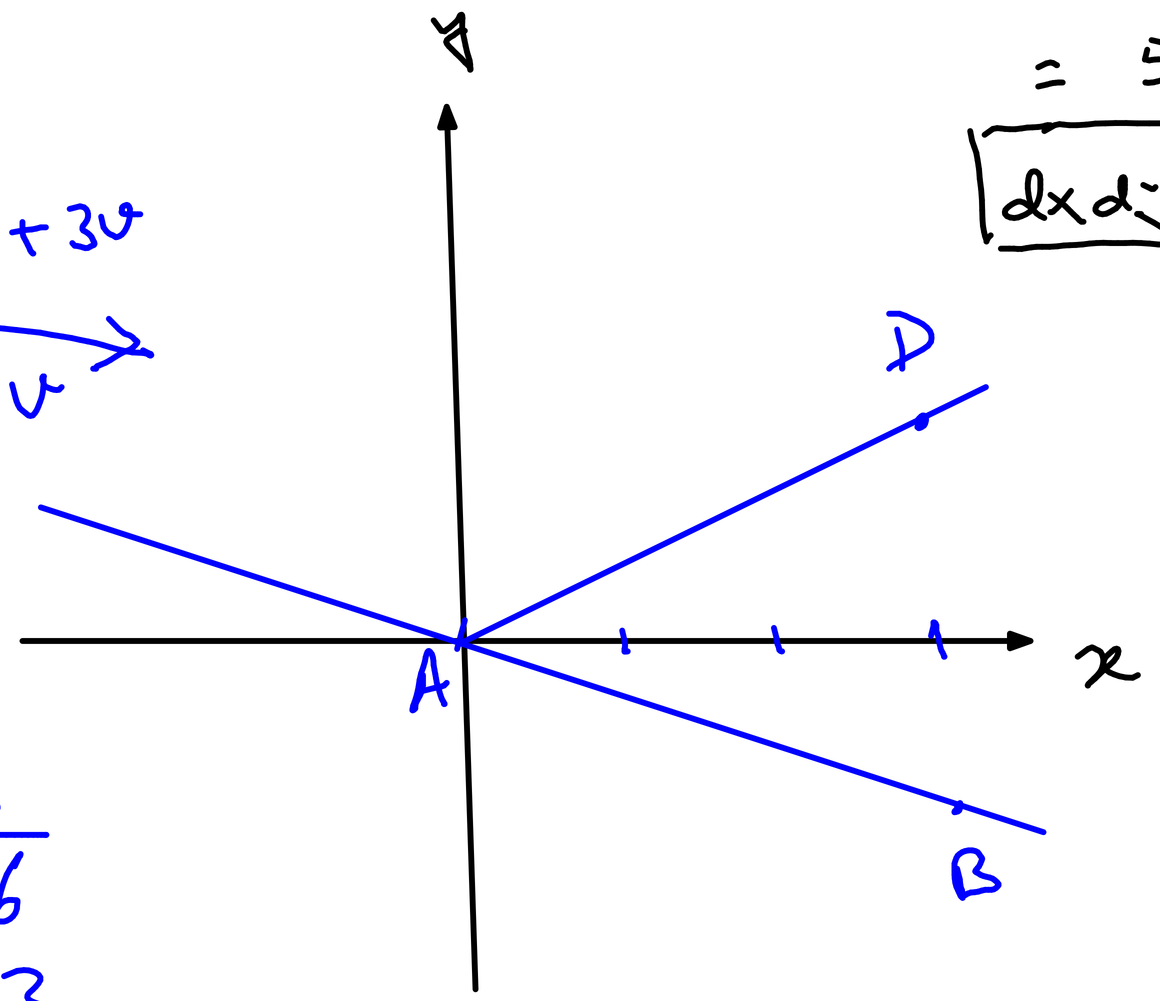
Jacobian:

$$x = 2u + 3v$$
$$y = u - v$$
$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\right|$$
$$= 5$$

$$\boxed{dx\,dy = 5\,du\,dv}$$



$$\xrightarrow[y = u - v]{x = 2u + 3v}$$



B	D
$x = 6$	$x = 6$
$y = -2$	$y = 3$

Image of AD

$0 \leq u \leq 3$ $v = 0$

$x = 2u$ $y = u$

$$x = 2y$$
$$y = x/2$$

Find the image of the set S under the given mapping

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ \ y = u - v$$

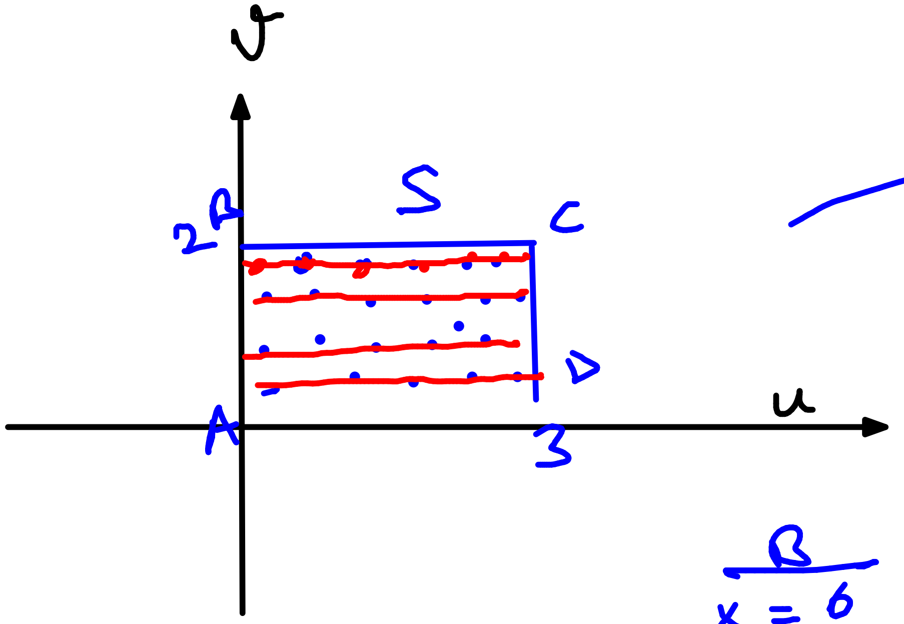
Jacobian:

$x = 2u + 3v$

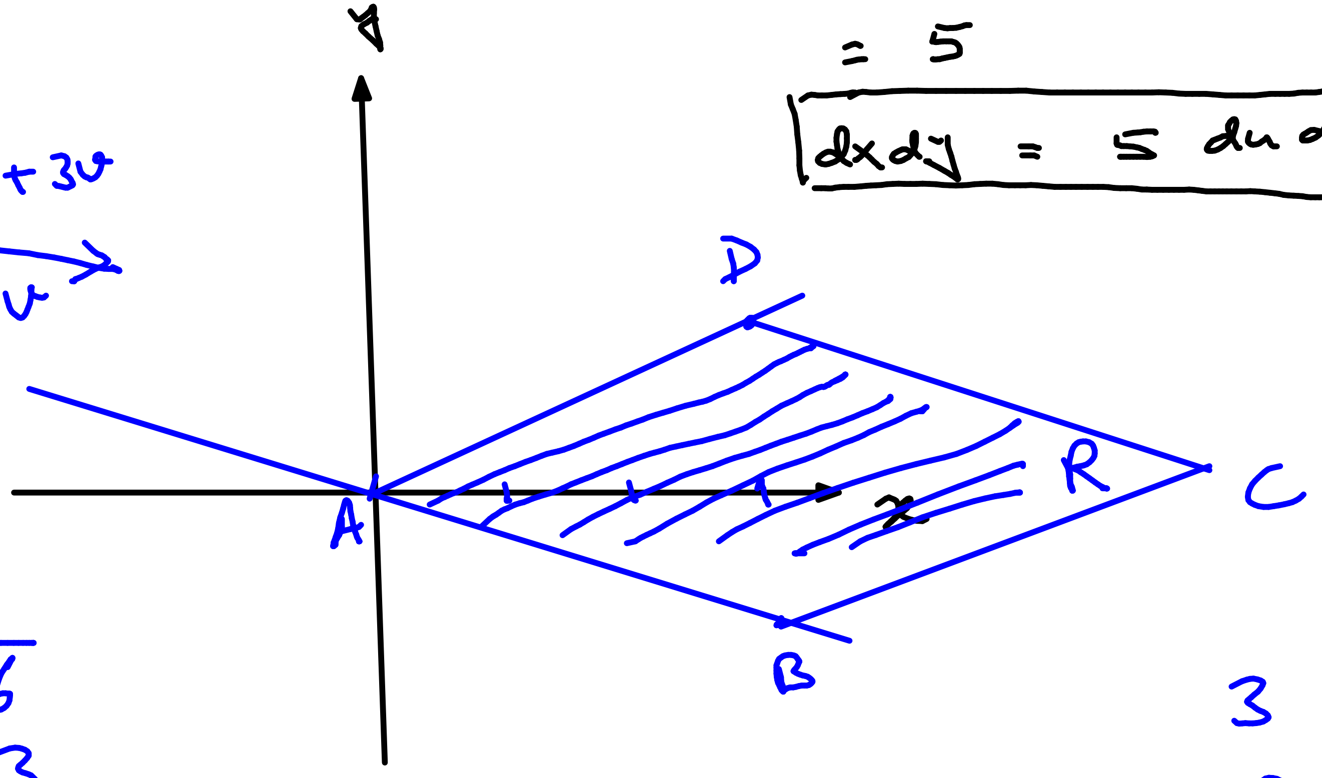
$y = u - v$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}\right|$$
$$= 5$$

$$\boxed{dx\,dy = 5\,du\,dv}$$



$$\xrightarrow[y = u - v]{x = 2u + 3v}$$



B	D
$x = 6$	$x = 6$
$y = -2$	$y = 3$

$$\iint_R \text{—}\ dx\,dy = \int_0^3 \int_0^2 \text{—}\ 5\,dv\,du$$

Find the image of the set S under the given

$$S = \{(u, v) \mid 0 \leqslant u \leqslant 3,\ 0 \leqslant v \leqslant 2\};$$
$$x = 2u + 3v,\ y = u - v$$

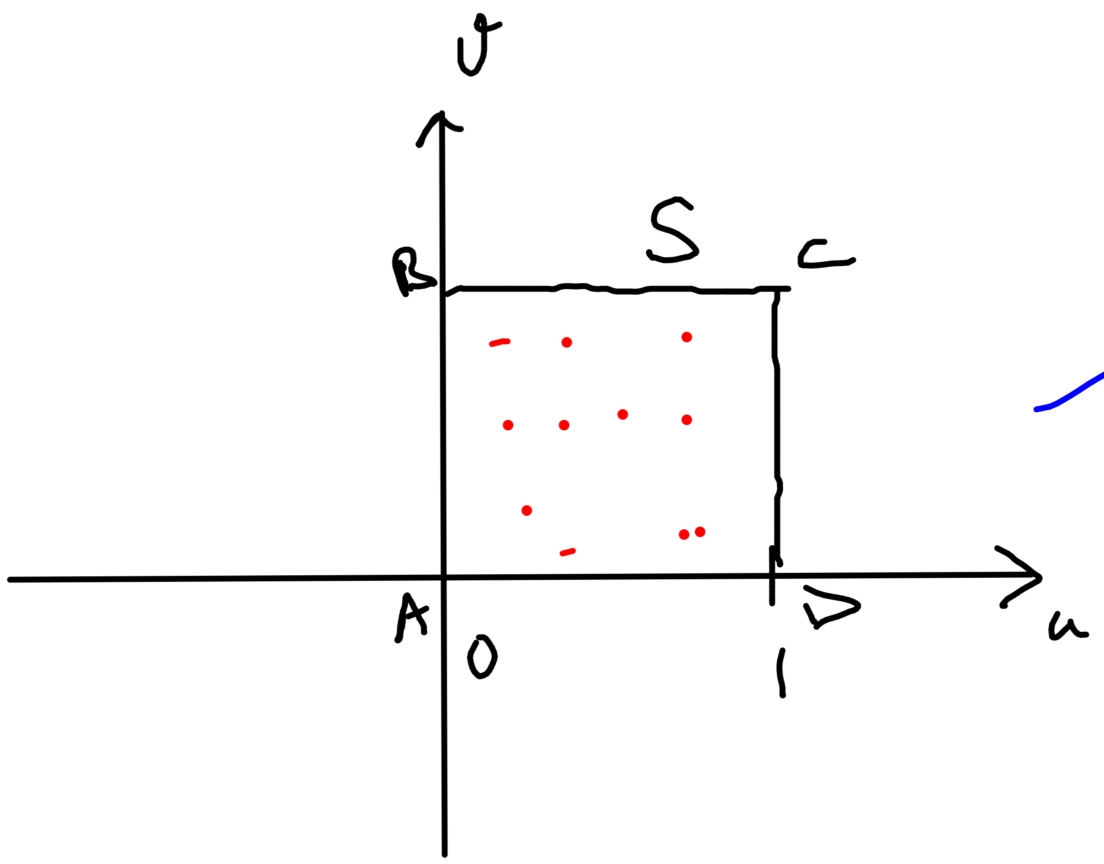
Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $\quad x = v,\ y = u(1 + v^2)$

AD

$v = 0,\ 0 \le u \le 1$

$x = 0,\ y = u$



$x = v$

$y = u(1+v^2) \rightarrow$

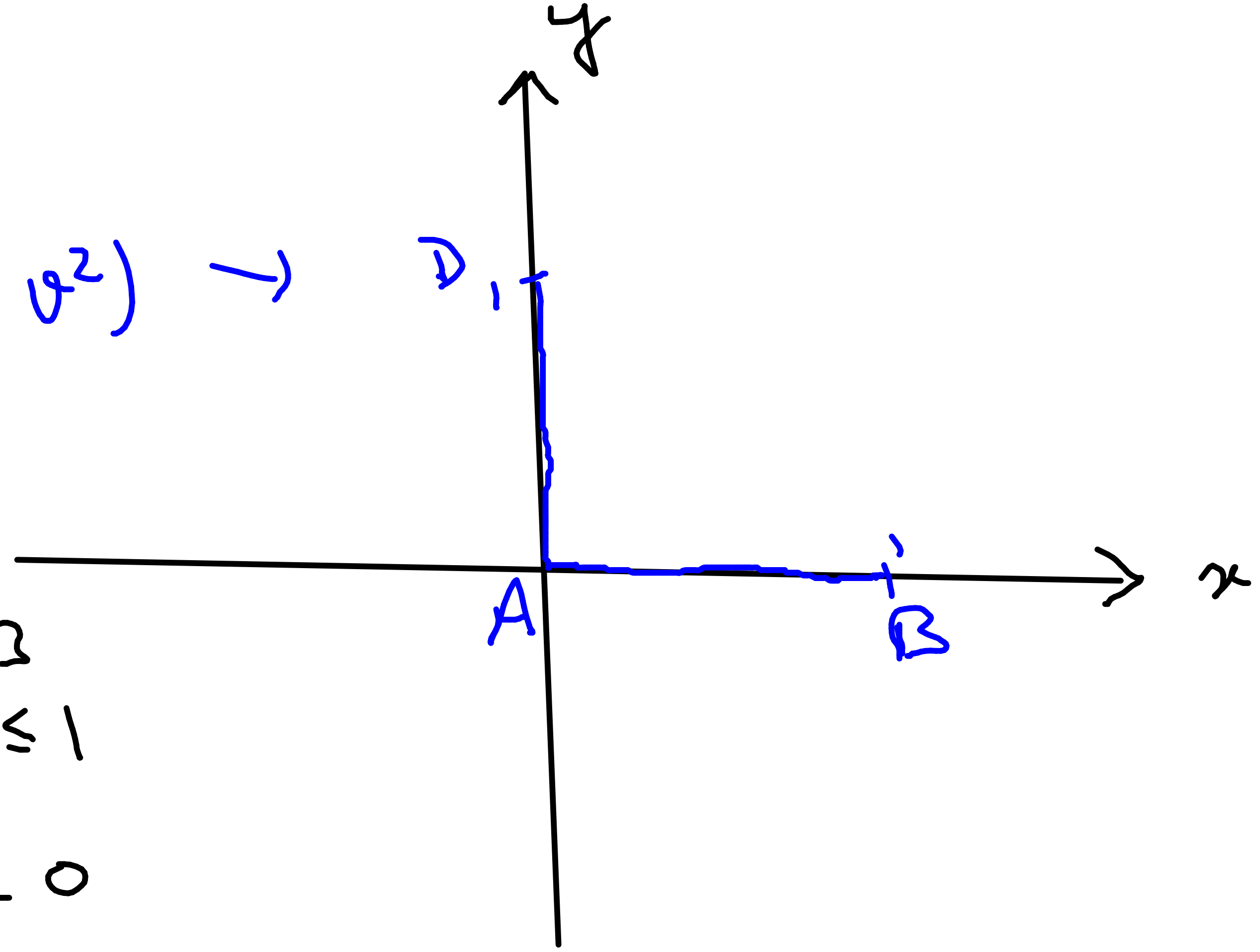


Image of AB

$u = 0,\ 0 \le v \le 1$

$x = v,\ y = 0$

$0 \le x \le 1$

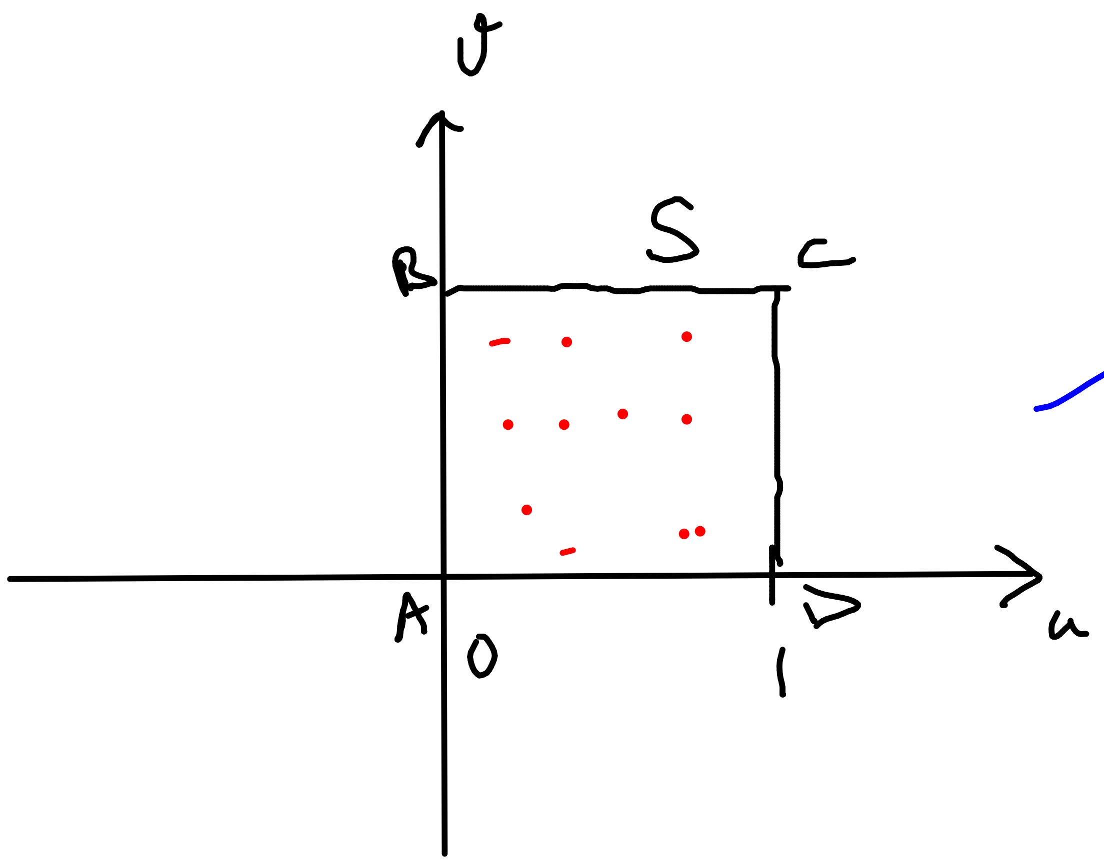
Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$

$\underline{AD}$

$v = 0$, $0 \le u \le 1$

$x = 0$, $y = u$



$x = v$

$y = u(1+v^2) \rightarrow$

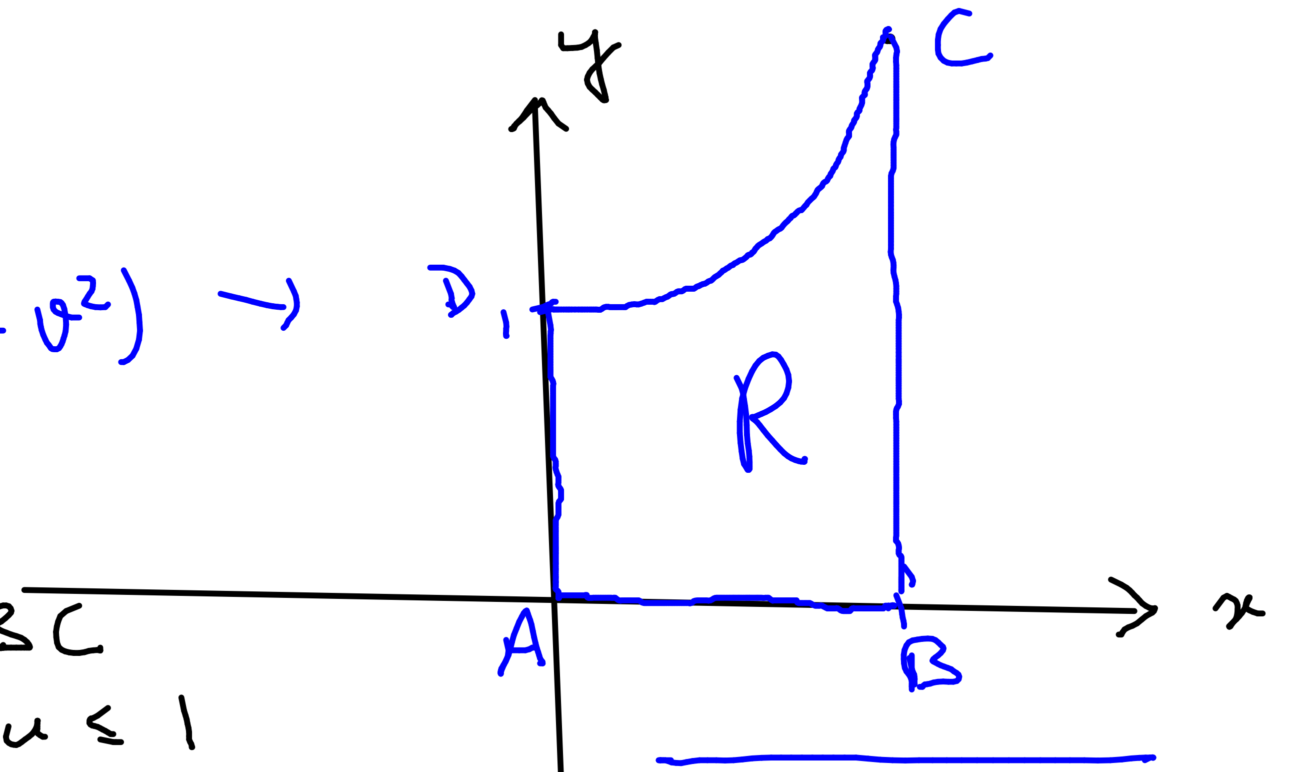


Image of BC

$v = 1$, $0 \le u \le 1$

$x = 1$, $y = 2u$

$0 \le y \le 2$

Image of DC

$u = 1$, $0 \le v \le 1$

$x = v$, $y = 1 + v^2$

$\boxed{y = 1 + x^2}$

Find the image of the set S under the given transformation.

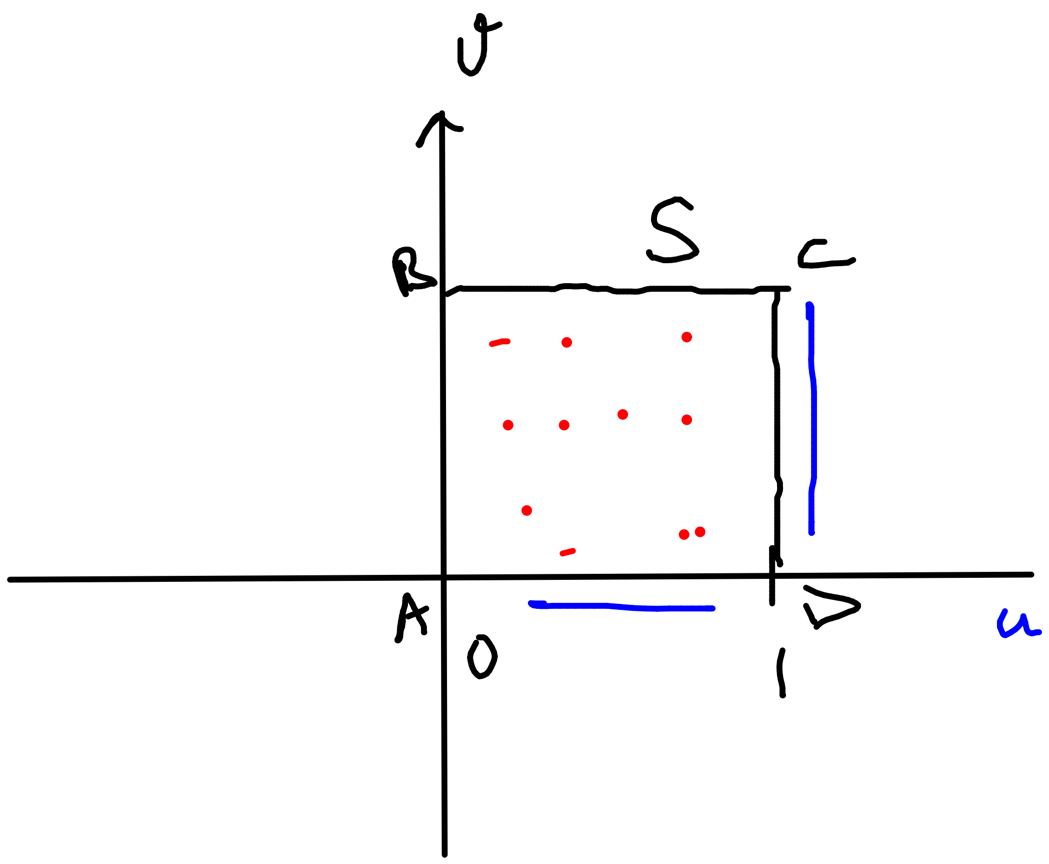
S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{vmatrix} = (1+v^2)$$

AD

$v = 0$, $0 \le u \le 1$

$x = 0$, $y = u$



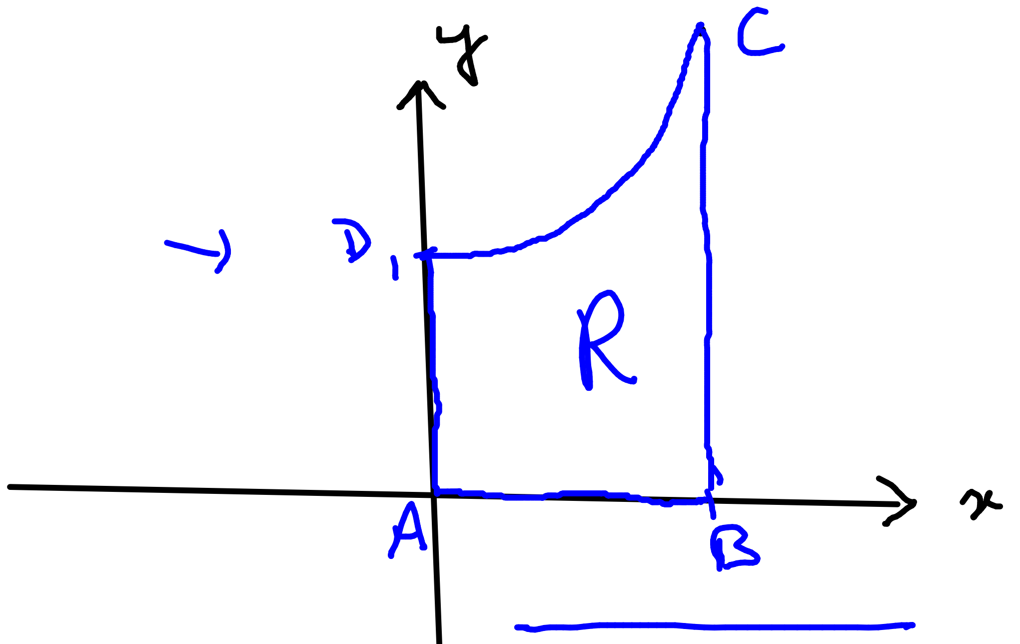


Image of DC

$u = 1$, $0 \le v \le 1$

$x = v$, $y = 1 + v^2$

$\boxed{y = 1 + x^2}$

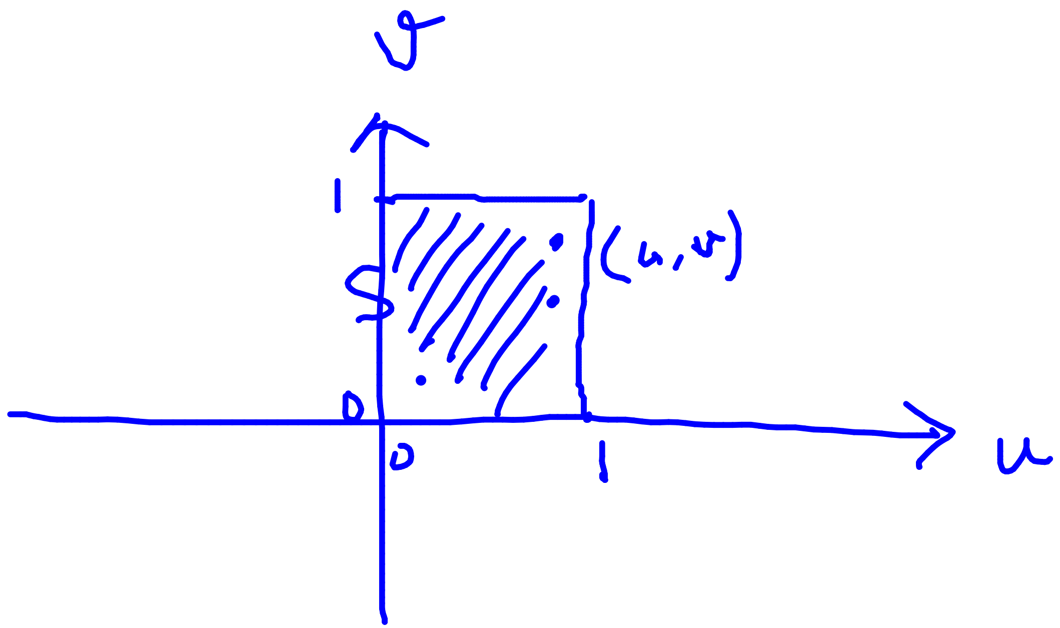
$$\iint_R \_\_\_ \, dx\,dy$$

$$= \int_?^? \int_?^? \_\_\_ \;??\; du\,dv$$

$$= \int_0^1 \int_0^1 (\qquad)(1+v^2)\,du\,dv$$

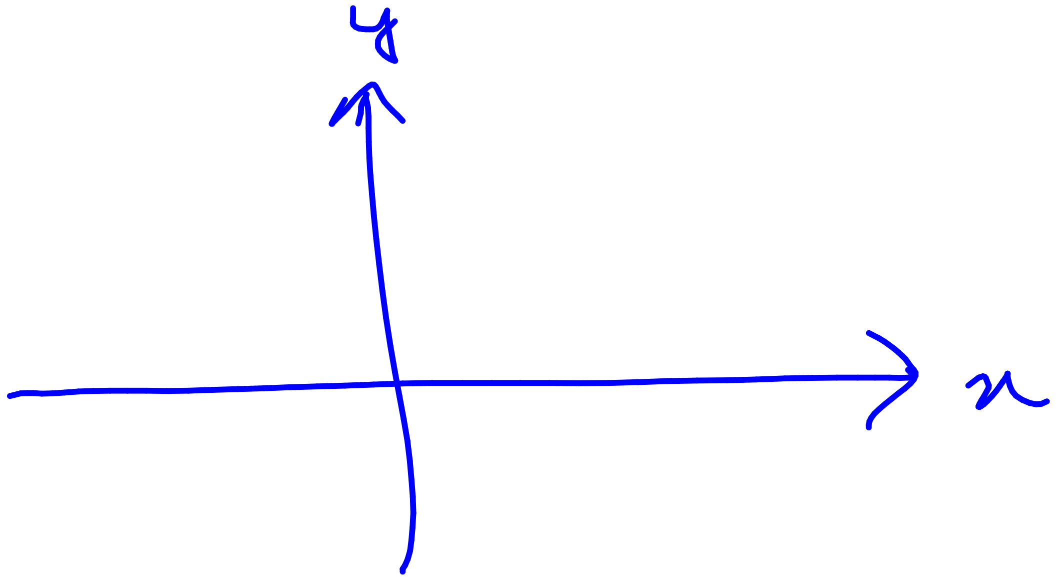
Find the image of the set S under the given transformation.

S is the square bounded by the lines $u = 0$, $u = 1$, $v = 0$, $v = 1$; $x = v$, $y = u(1 + v^2)$

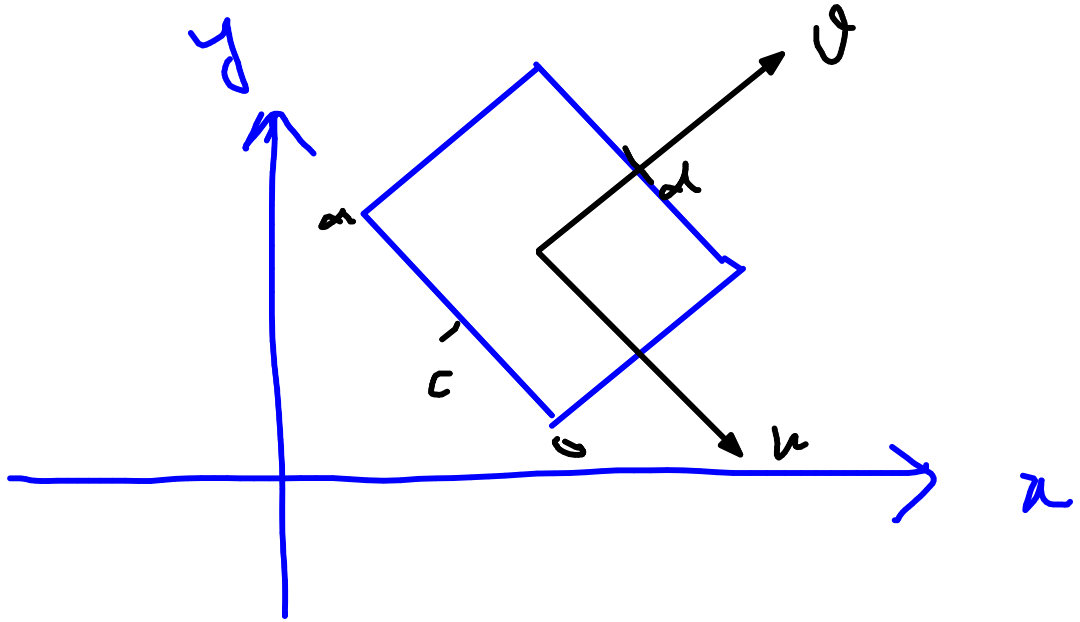


$x = v$

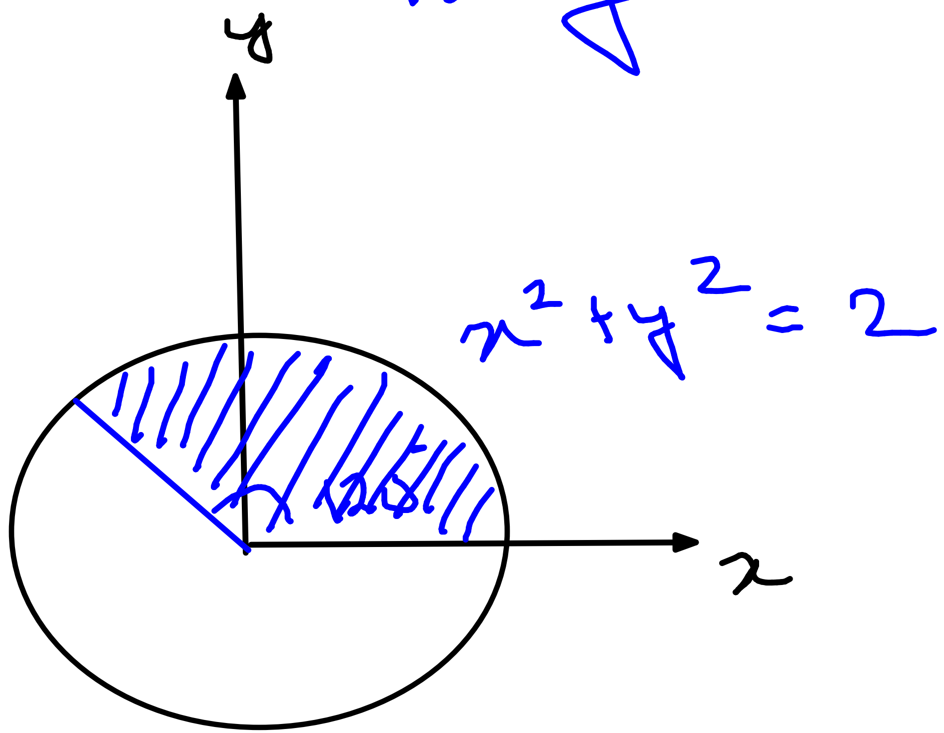
$y = u(1+v^2)$



15.8 Change of Variables in multivariable integration



$$\int_{\text{messy}}^{\text{messy}} \int f(x,y)\ dy\,dx$$



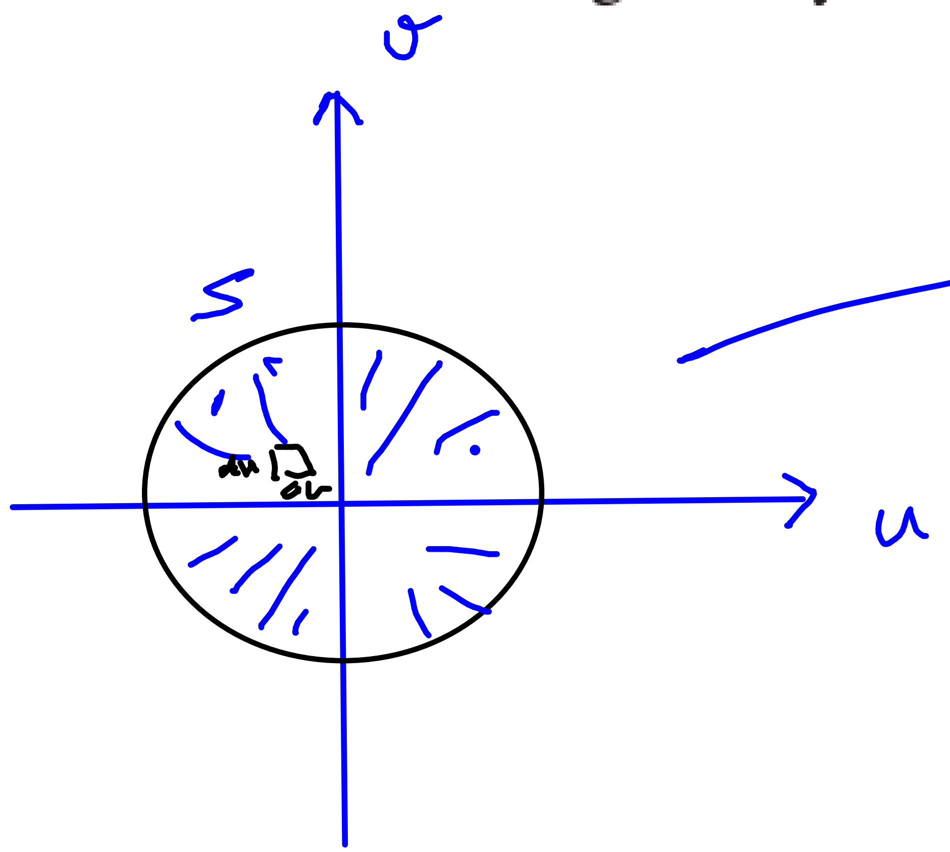
$0 \le r \le \sqrt{2}$

$0 \le \theta \le 2\pi/3$

Find the image of the set S under the given transformation.

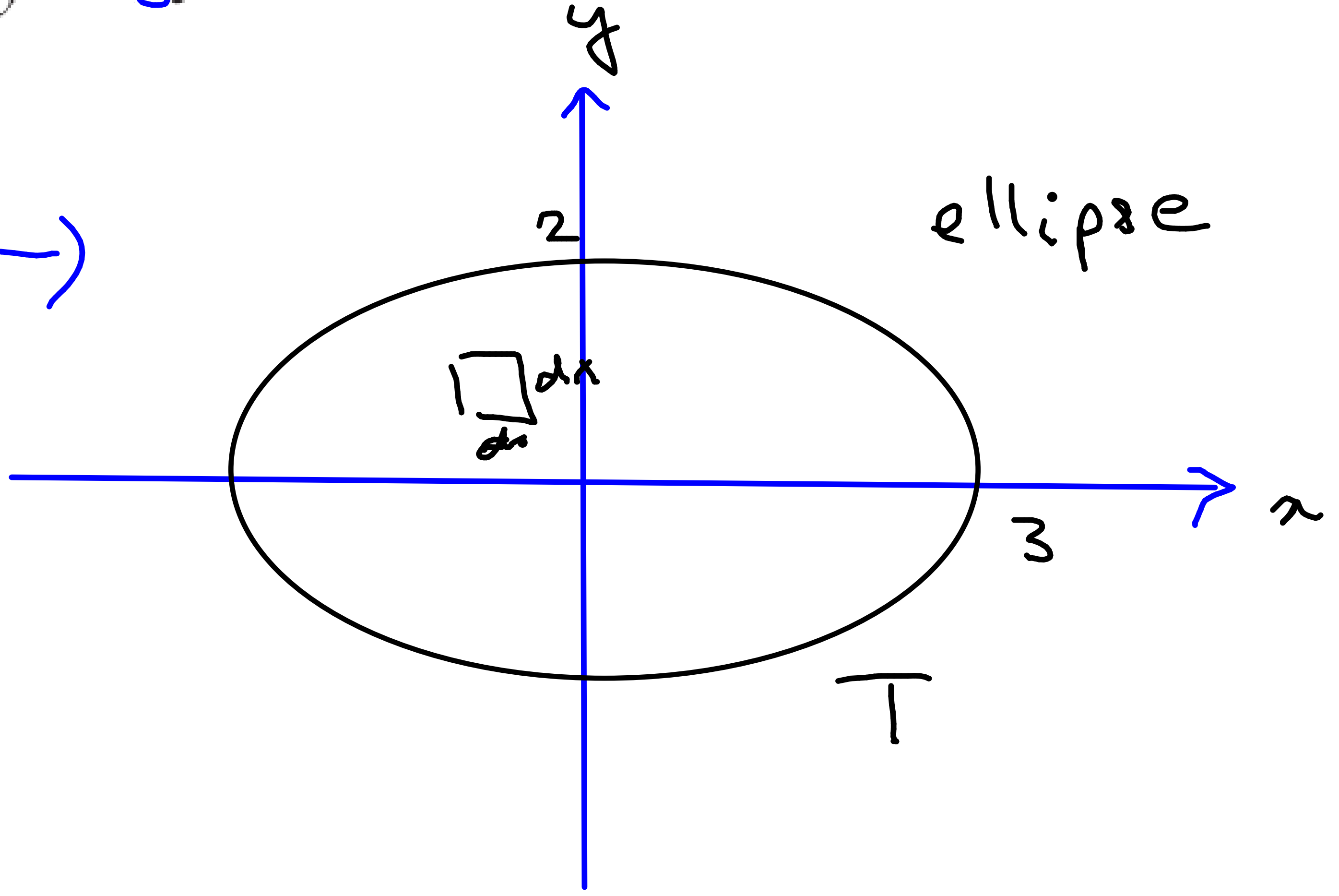
S is the disk given by $u^2 + v^2 \leqslant 1$; $x = 2u$, $y = 3v$

Jacobian $= \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 6$



$x = 2u$

$y = 3v$



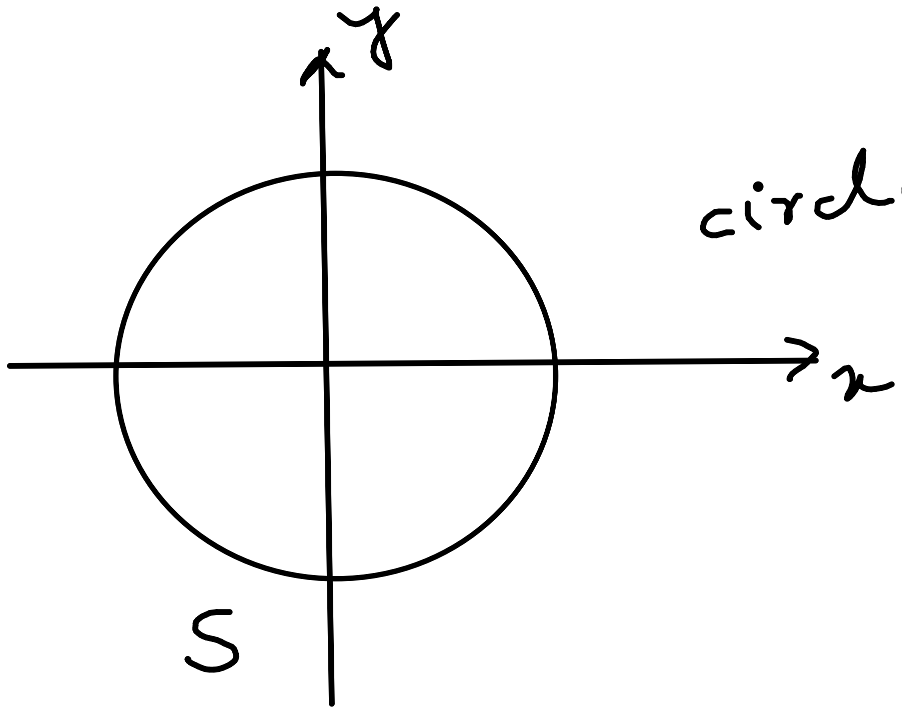
$u^2 + v^2 = 1$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$$\iint_T f(x,y)\,dx\,dy = \iint_S f(2u,3v)\;6\;du\,dv$$

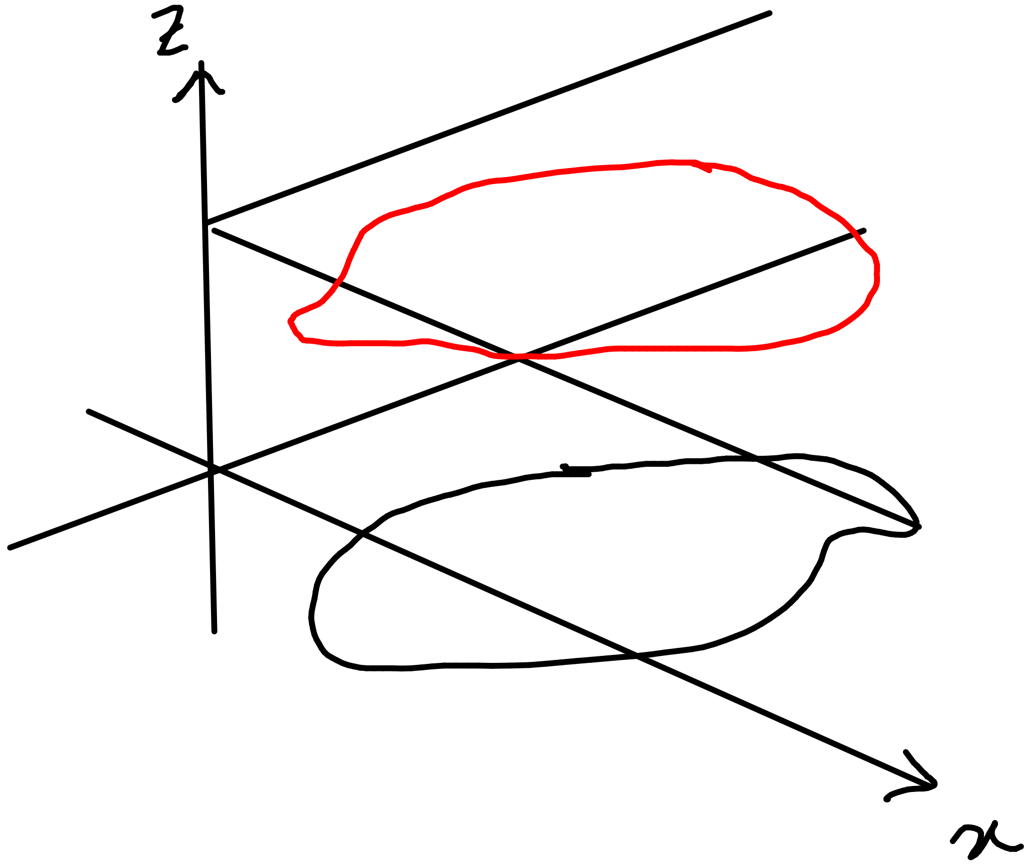
Jacobian

Q.



circle of radius 5

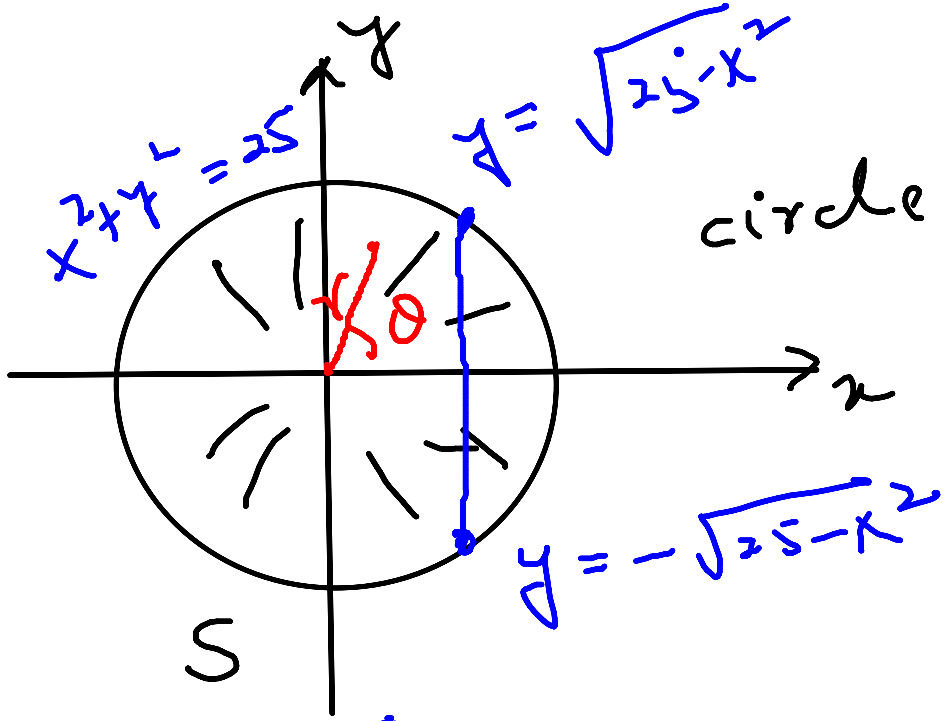
area $= \pi 5^2$



$$\iint_S \underline{1}\, dA \overset{??}{=} \text{area of } S$$

$$\iint_{\underline{S}} 1\, dA = (\text{base area}) \times \underline{1}$$

Q.



circle of radius 5

area = $\pi 5^2$

change of variables:

$x = r\cos\theta$

$y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} 1 \, dy\, dx = \int_0^{2\pi}\int_0^{5} 1 \;(\text{Jacobian})\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{5} r \, dr\, d\theta = ??$$

= area of circ

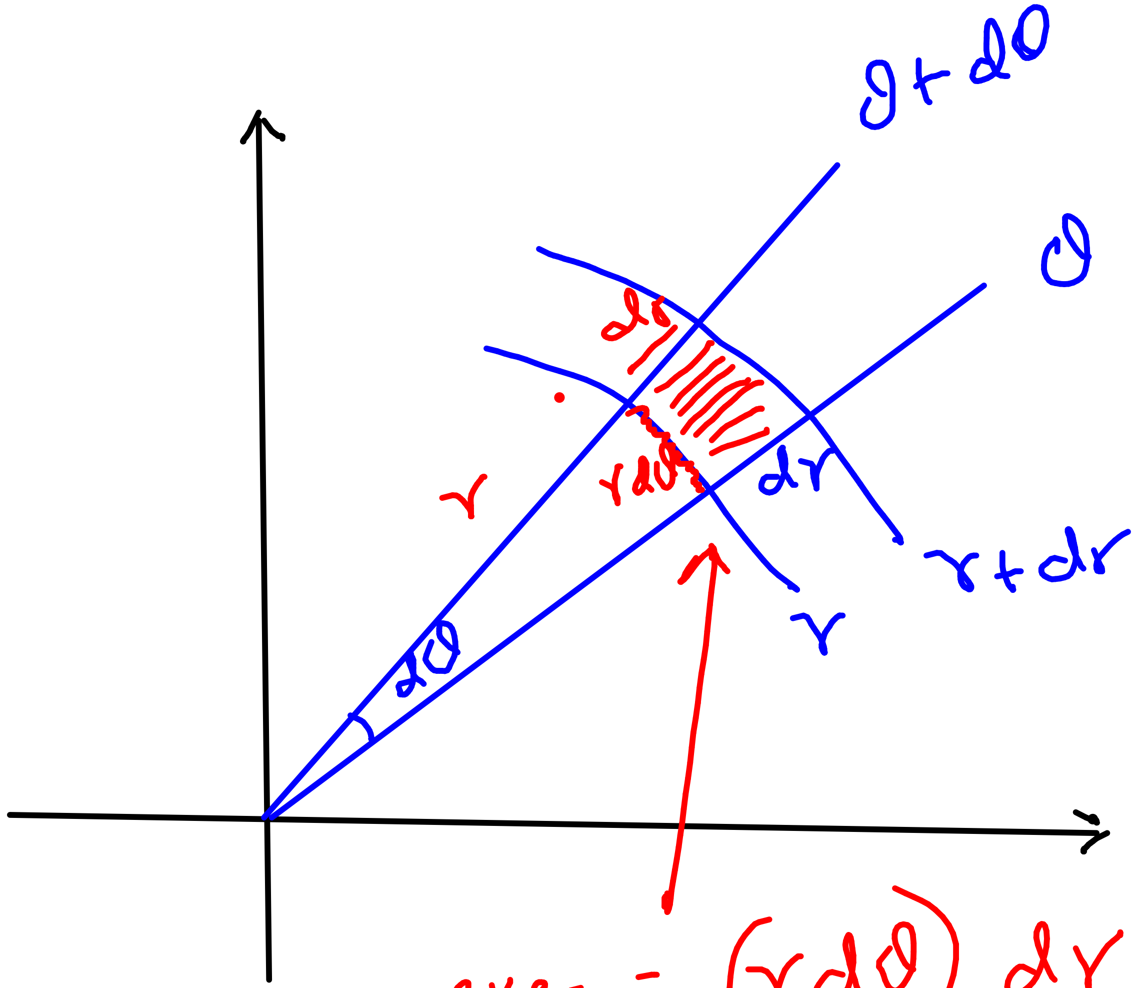
$= 25\pi$

To remember:

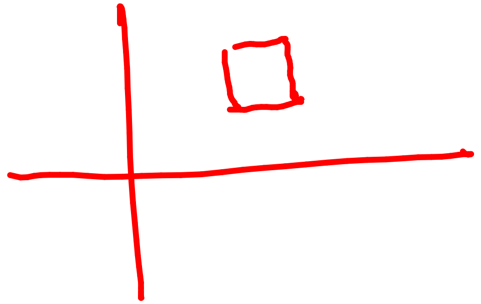
Switching to polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$



$$\text{area} = (r\,d\theta)\,dr = r\,dr\,d\theta$$



Q. Find the image of S under the given transformation:

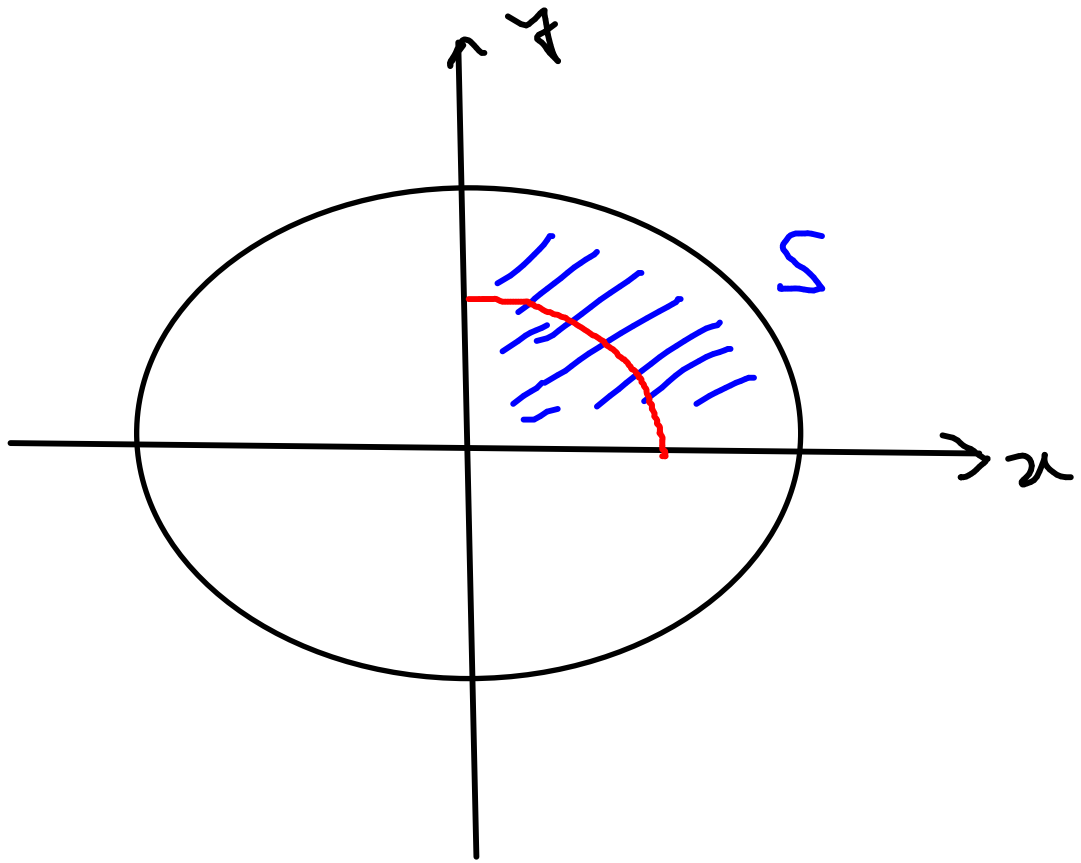
$$S = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

Q: $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy$

$x = \sqrt{1-y^2}$

$x^2 + y^2 = 1$



→ sketch the region of integration

→ set up the same integration in polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

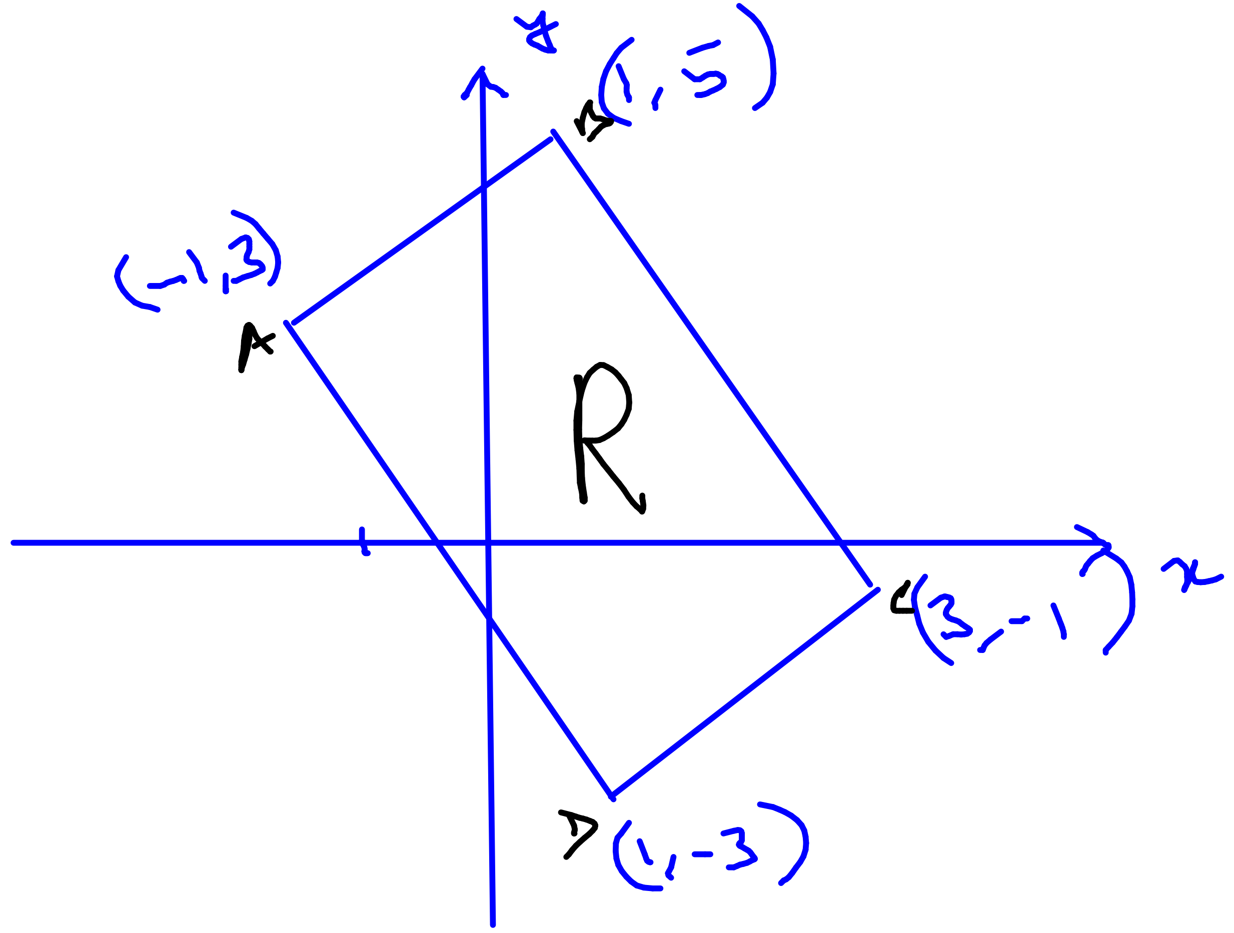
$$\int_0^1 \int_0^{\pi/2} (r\cos\theta)(r\sin\theta)\, r \, d\theta \, dr = \int_0^1 \int_0^{\pi/2} r^3 \cos\theta \sin\theta \, d\theta \, dr$$

Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;
$x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

$$\iint_R (4x+8y)\,dA = \int_{?}^{?}\int_{?}^{?} (??)\,du\,dv$$

Jacobian



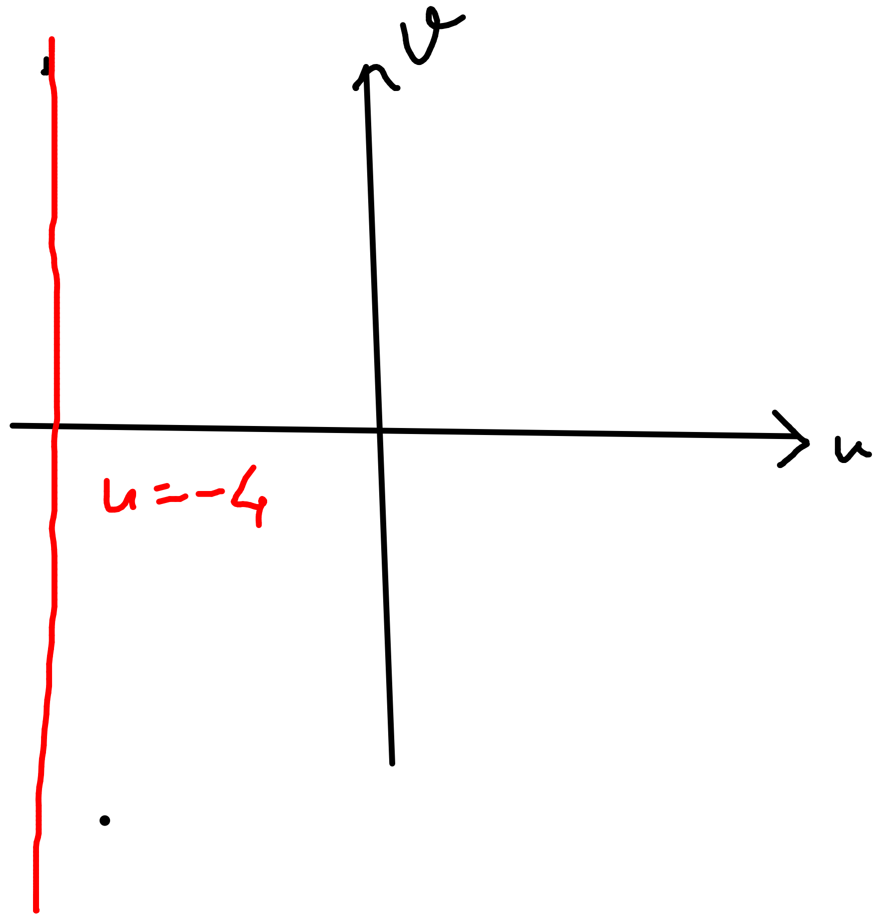
AB : eqn ??

$$y = x + 4$$

$$\frac{1}{4}(v-3u) = \frac{1}{4}(u+v) + 4$$

$$\cancel{v} - 3u = u + \cancel{v} + 16$$

$$u = -4$$

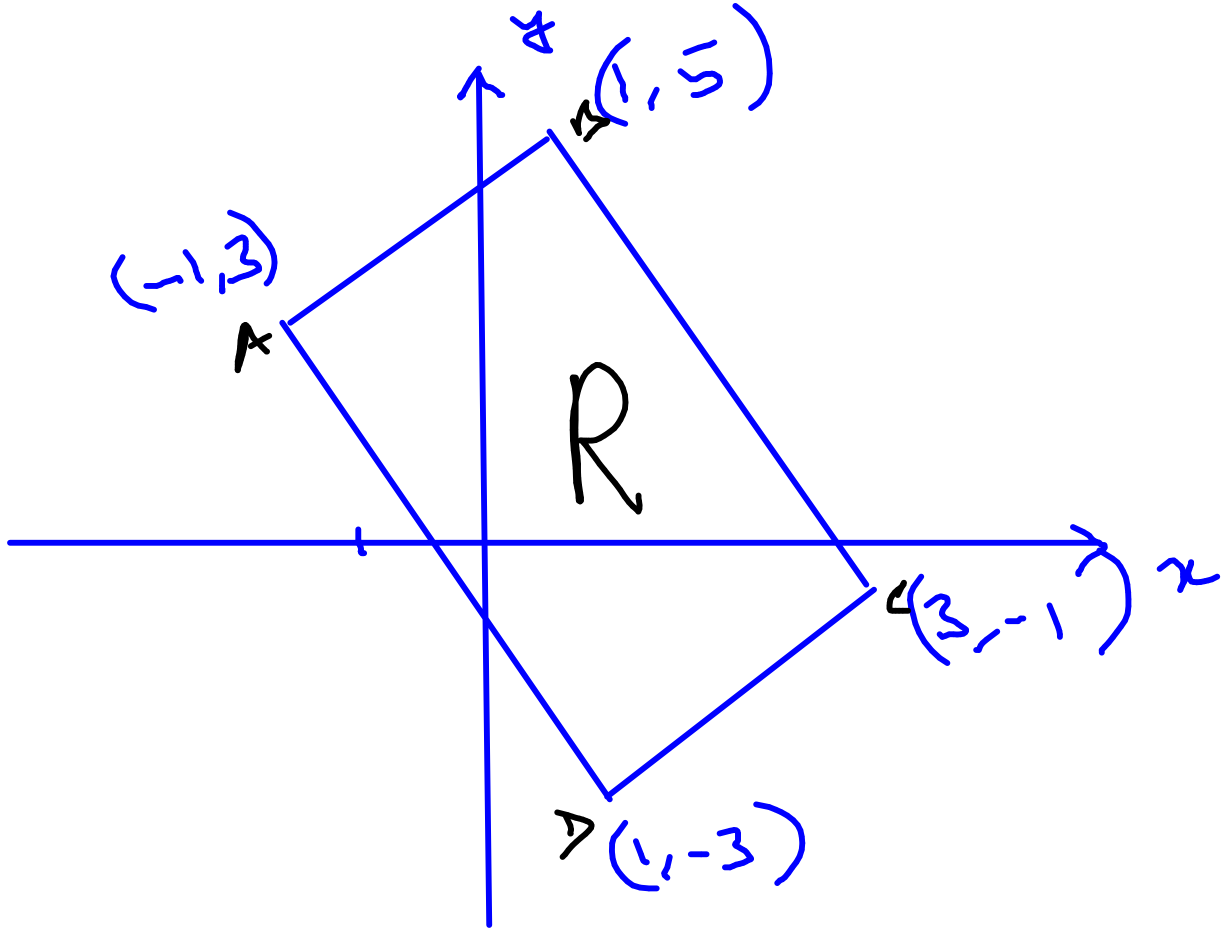


Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\, dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$;
$x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

$$\iint_R (4x+8y)\,dA = \int_{?}^{?}\int_{?}^{?} (??)\,du\,dv$$

Jacobian



DC

$y = x - 4$

$\downarrow$

$u = 4$

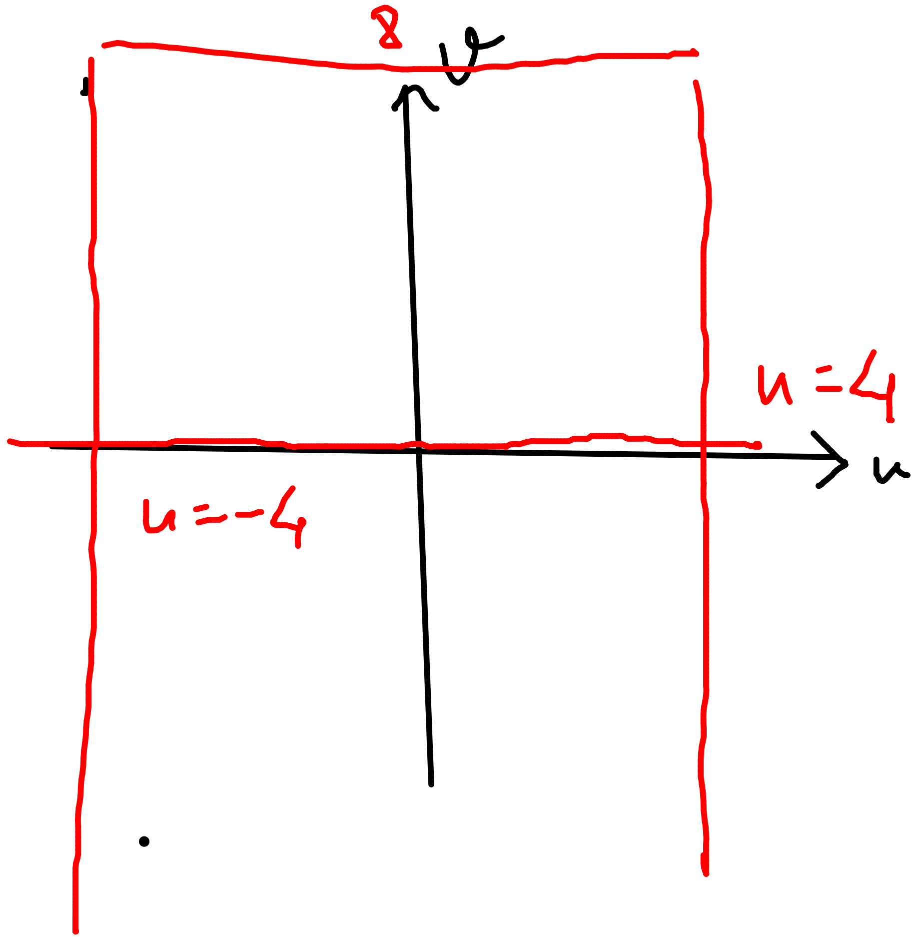
AD $\quad y = -3x$

$v = 0$

BC

$y = -3x + 8$

$v = 8$

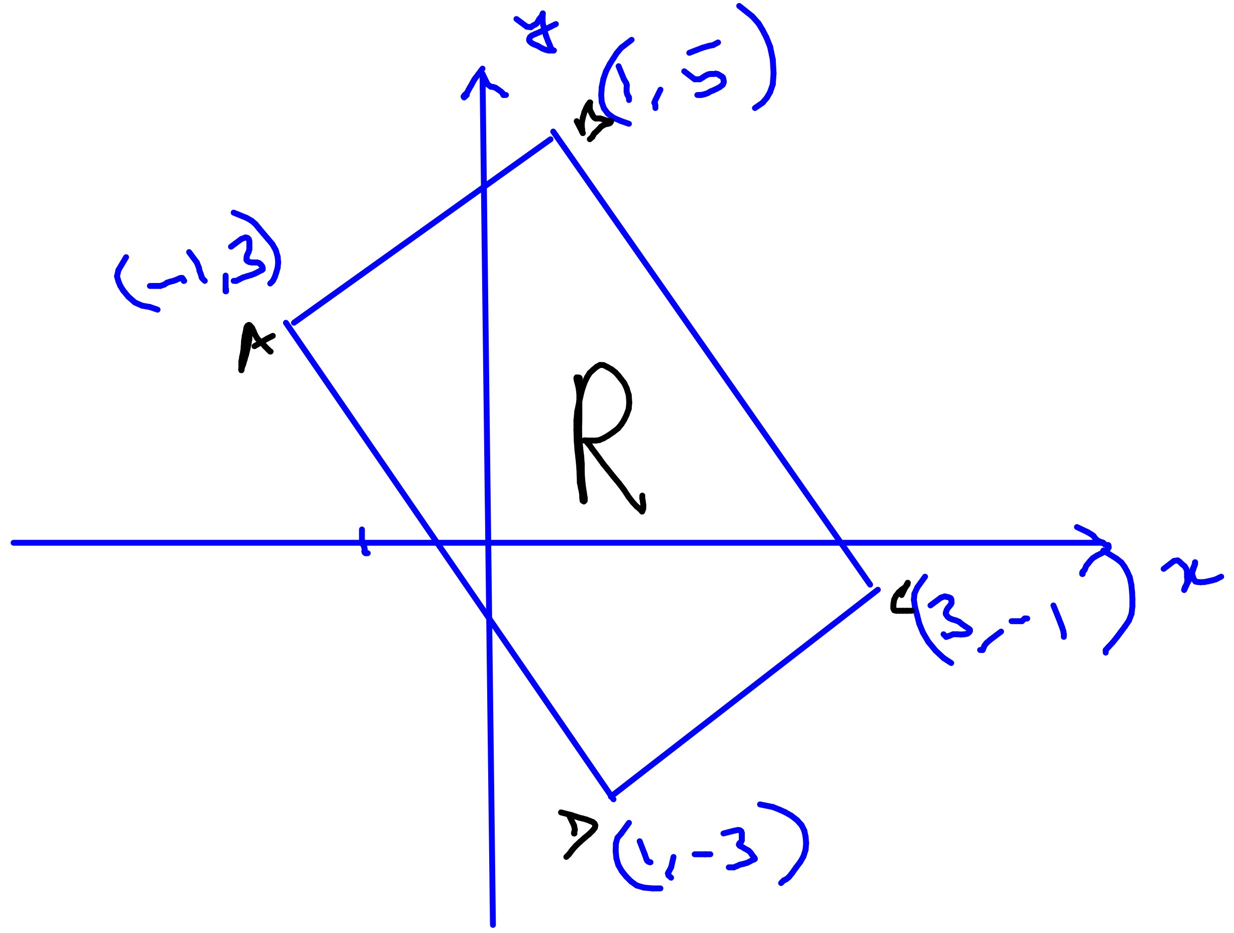


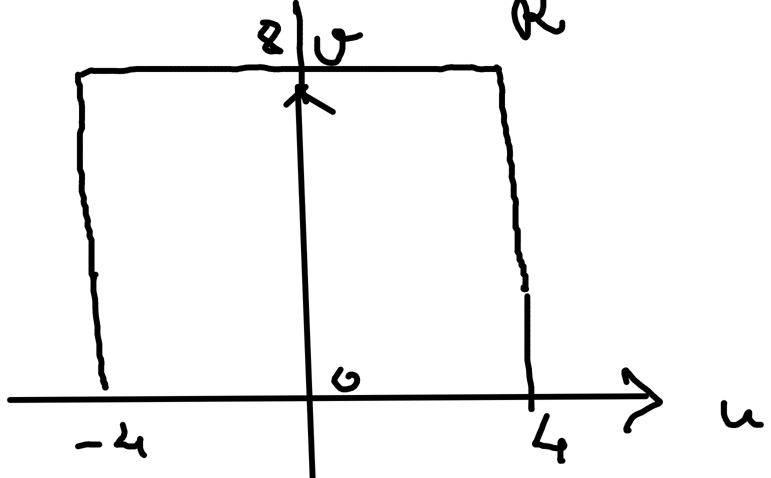
Use the given transformation to evaluate the integral.

$\iint_R (4x + 8y)\,dA$, where R is the parallelogram with vertices $(-1, 3)$, $(1, -3)$, $(3, -1)$, and $(1, 5)$; $x = \frac{1}{4}(u + v)$, $y = \frac{1}{4}(v - 3u)$

$$\iint_R (4x+8y)\,dA = \int_?^?\int_?^? (??)\,du\,dv$$

Jacobian





$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \left| \begin{vmatrix} 1/4 & 1/4 \\ 1/4 & \frac{-3}{4} \end{vmatrix} \right| = \frac{1}{4}$$

$$\int_{-4}^{4}\int_{0}^{8} \left[(u+v) + 2(v-3u)\right](\text{Jacobian})\,dv\,du$$

$$= \int_{-4}^{4}\int_{0}^{8} \left[(u+v) + 2(v-3u)\right]\frac{1}{4}\,dv\,du$$

specific
change of var.
polar coordinates

14.1 , 14.8

graphs

Lagrange multipliers

15.1 , 15.2 , 15.9 , 15.3 , 15.6

general
change
of variables

Use the given transformation to evaluate the integral.

$\iint_R x^2\,dA$, where R is the region bounded by the ellipse $9x^2+4y^2=36$; $\quad x=2u,\ y=3v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}2&0\\0&3\end{vmatrix}$$

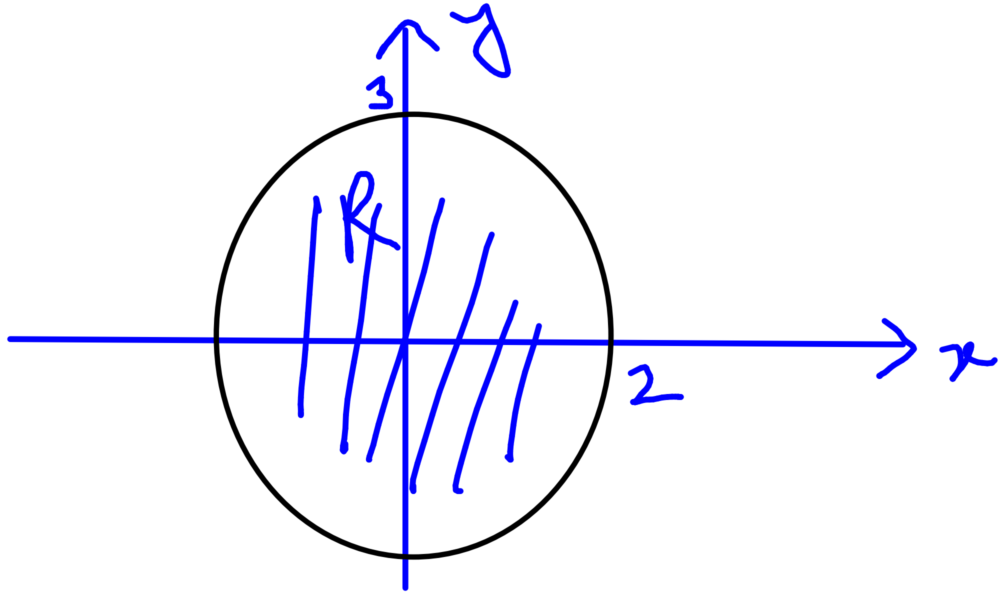
$$9x^2+4y^2=36$$

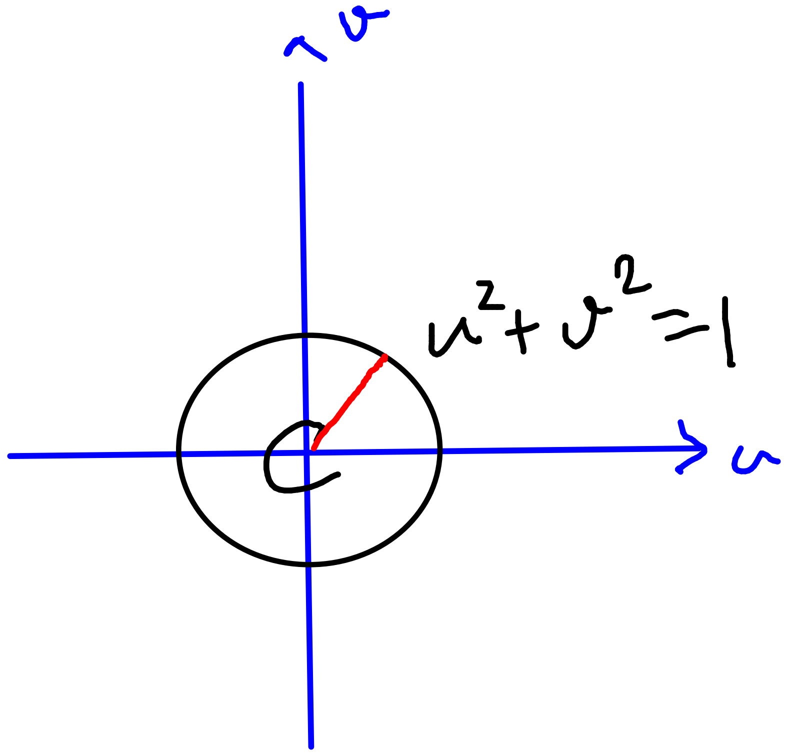
$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\frac{(2u)^2}{4}+\frac{(3v)^2}{9}=1$$

$$u^2+v^2=1$$

$$\iint_R x^2\,dA = \int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} (2u)^2\,6\,du\,dv = \int_0^{2\pi}\int_0^{1} 24\,(r\cos\theta)^2\,r\,dr\,d\theta$$





$u = r\cos\theta$

$v = r\sin\theta$

Use the given transformation to evaluate the integral.

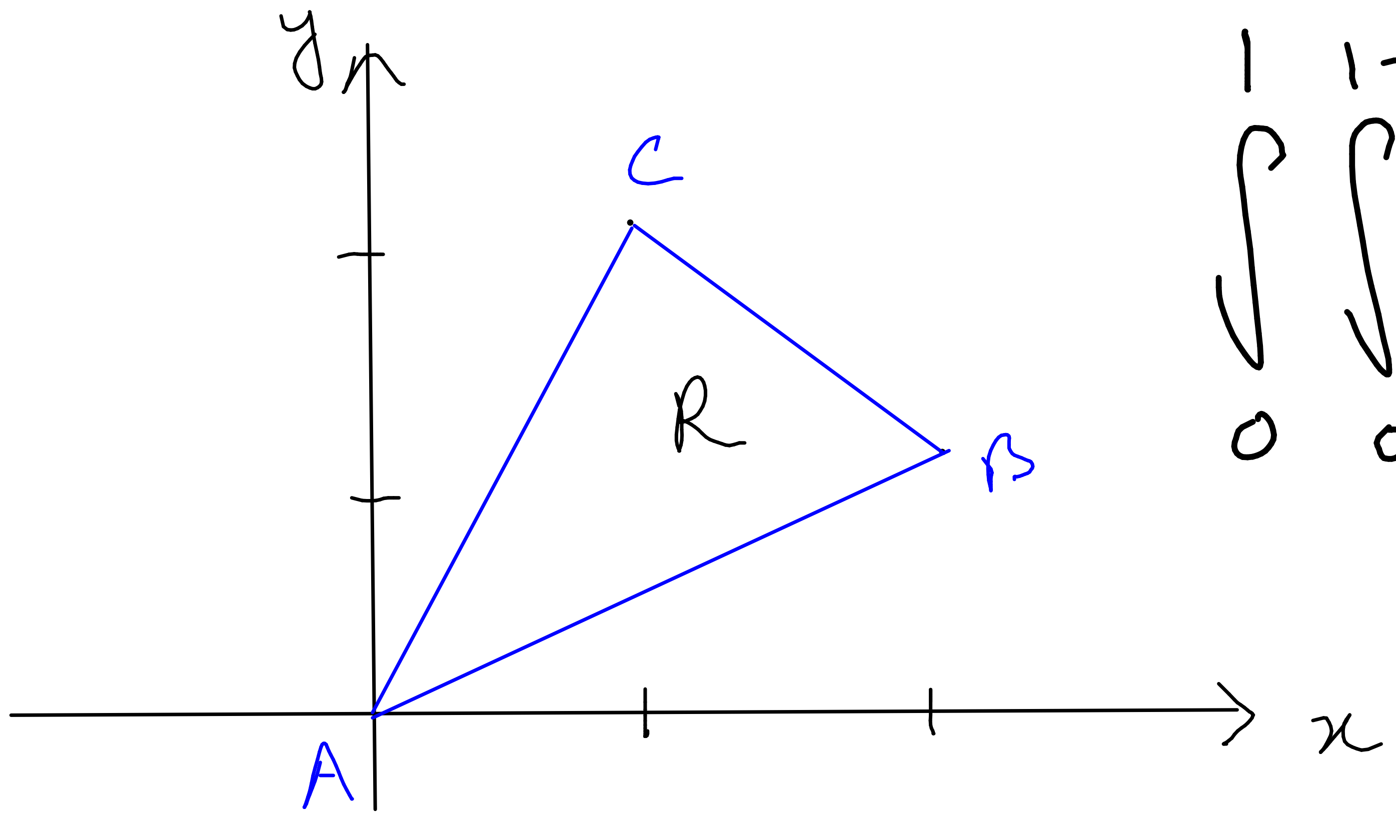
$\iint_R (x - 3y)\, dA$, where R is the triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$; $\quad x = 2u + v,\ y = u + 2v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

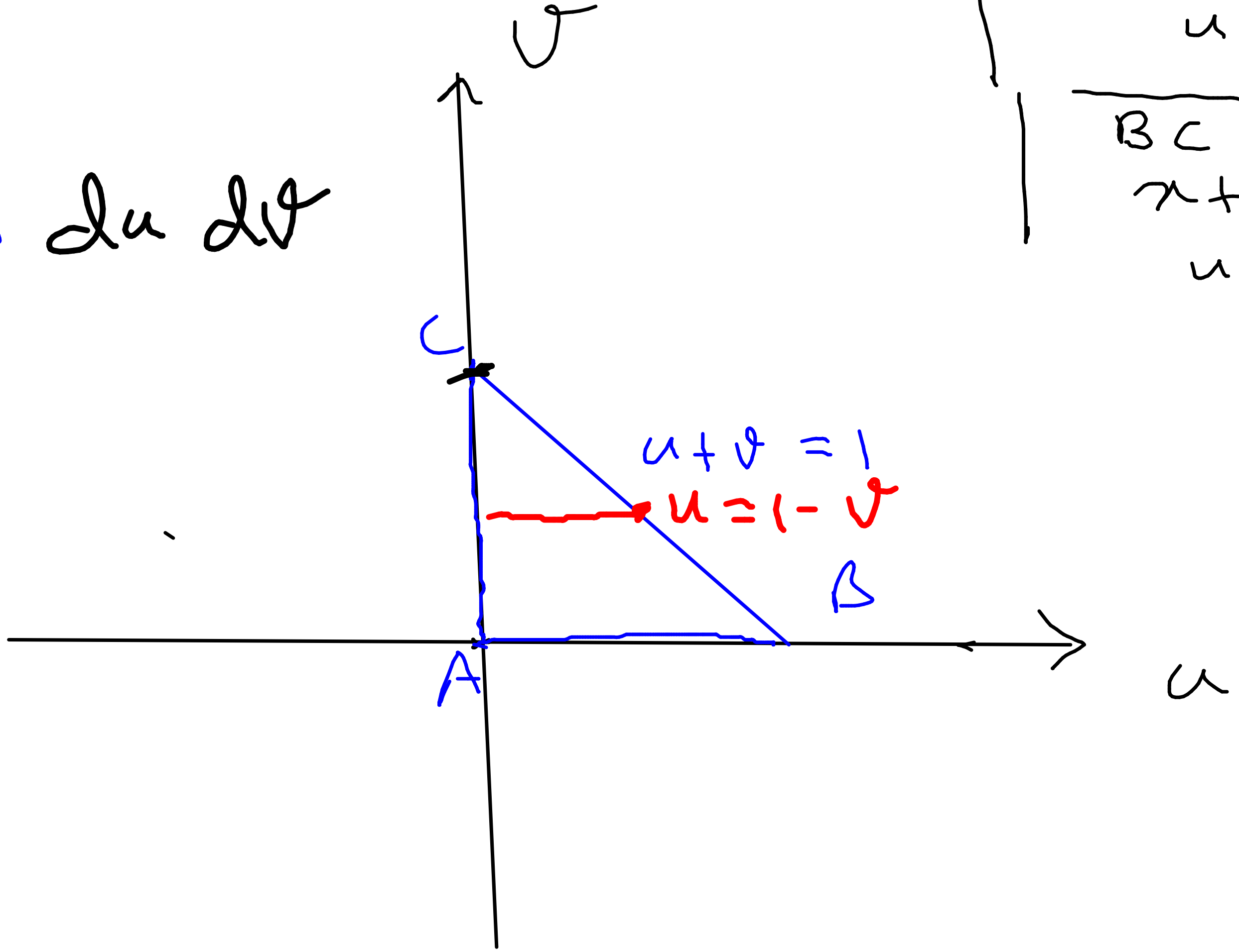
AB
$x = 2y$
$2u + v = 2(u + 2v)$
$v = 0$

AC
$y = 2x$
$u = 0$

BC
$x + y = 3$
$u + v = 1$



$$\int_0^1 \int_0^{1-v} (-u - 5v)\, 3\, du\, dv$$

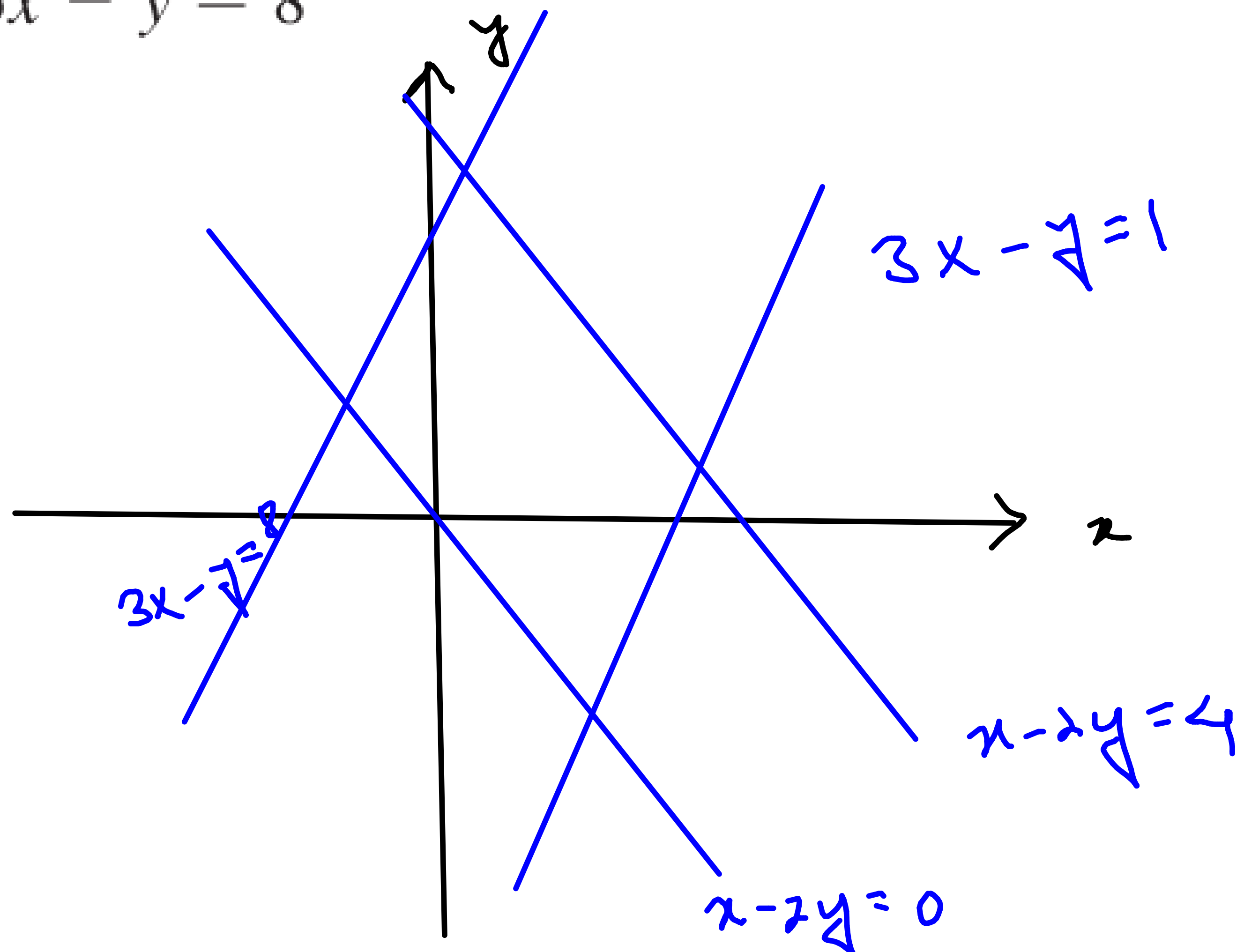


Evaluate the integral by making an appropriate change of variables

$$\iint_R \frac{x-2y}{3x-y}\, dA,$$ where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$

$u = x - 2y$ | $v = 3x - y$

$0 \le u \le 4$ | $1 \le v \le 8$



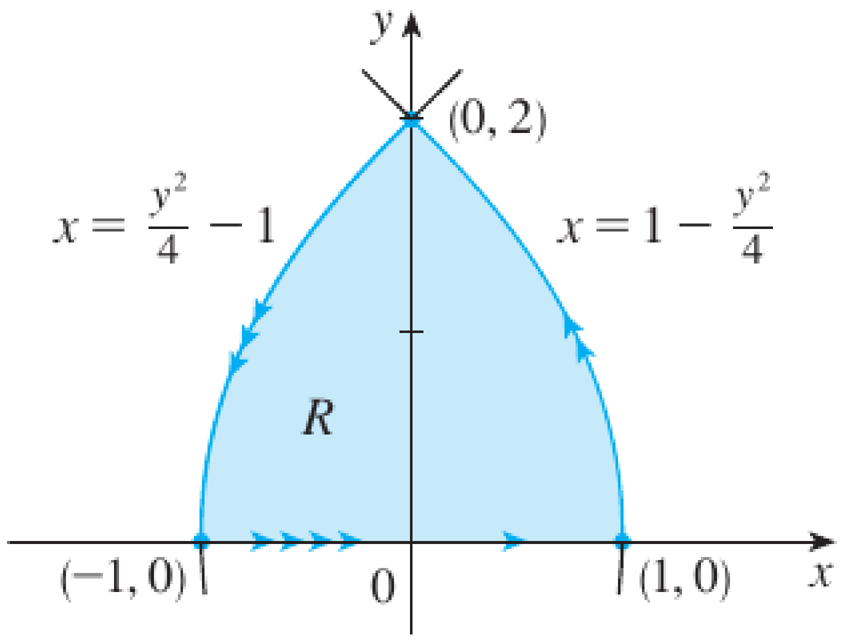
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}$$

$$\left| \boxed{\int_1^8 \int_0^4 \frac{u}{v}\,\frac{1}{5}\,du\,dv} \right| = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}} = \frac{1}{5}$$

Evaluate the integral by making an appropriate change

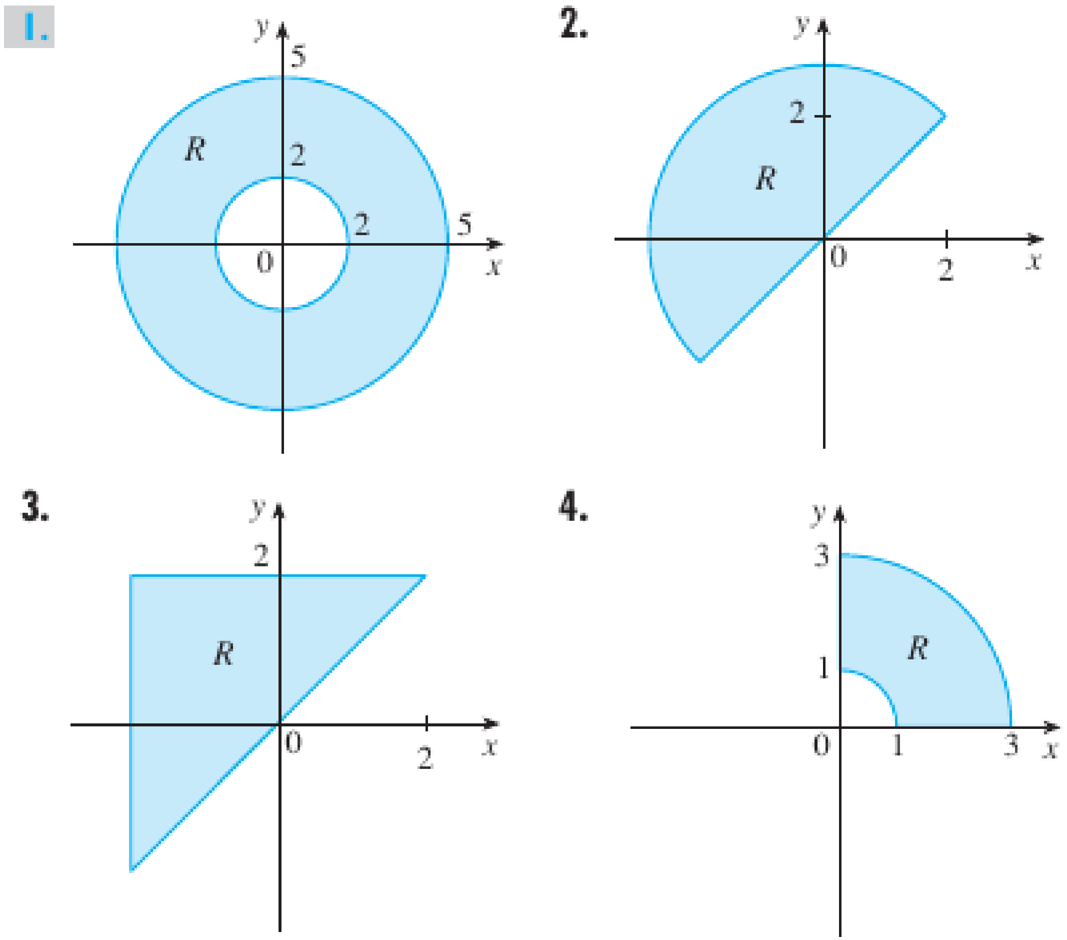
$\iint_R e^{x+y}\, dA$, where R is given by the inequality
$|x| + |y| \leqslant 1$

V EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.



12.3 DOUBLE INTEGRALS IN POLAR COORDINATES

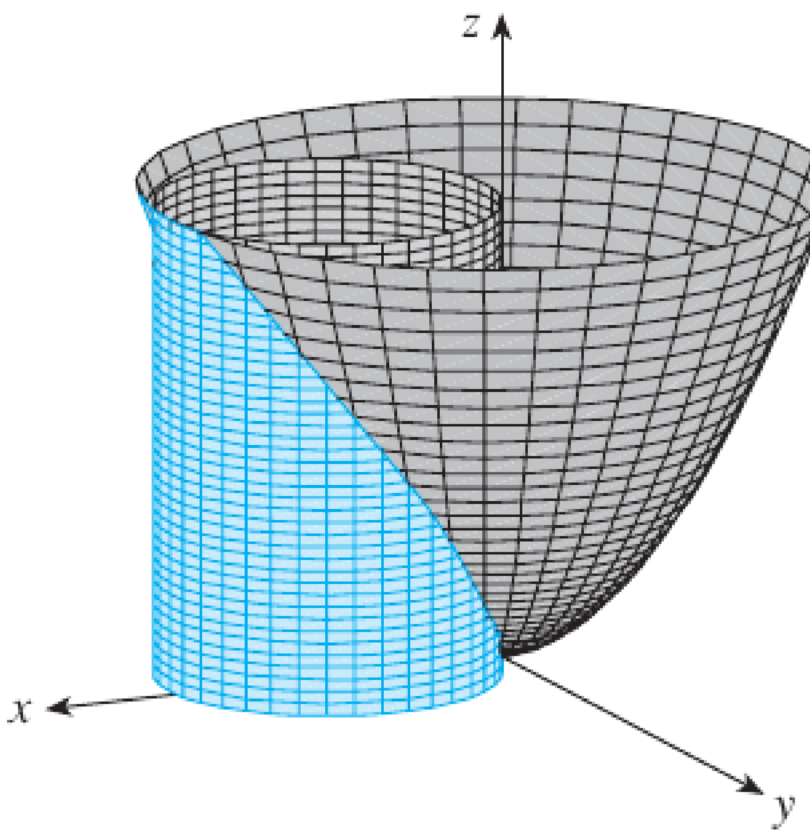
Q. for each region: choose whether it is more convenient to describe the region in xy- or r-θ



EXAMPLE 1 Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

V EXAMPLE 3 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.



Use polar coordinates to find the volume of the given solid.

Under the cone $z = \sqrt{x^2 + y^2}$ and above the disk $x^2 + y^2 \leqslant 4$

Use polar coordinates to find the volume of the given solid.

A sphere of radius a

29. Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy \, dy \, dx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy \, dy \, dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy \, dy \, dx$$

into one double integral. Then evaluate the double integral.

30. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

30. (a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \lim_{a \to \infty} \iint_{D_a} e^{-(x^2+y^2)}\, dA$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

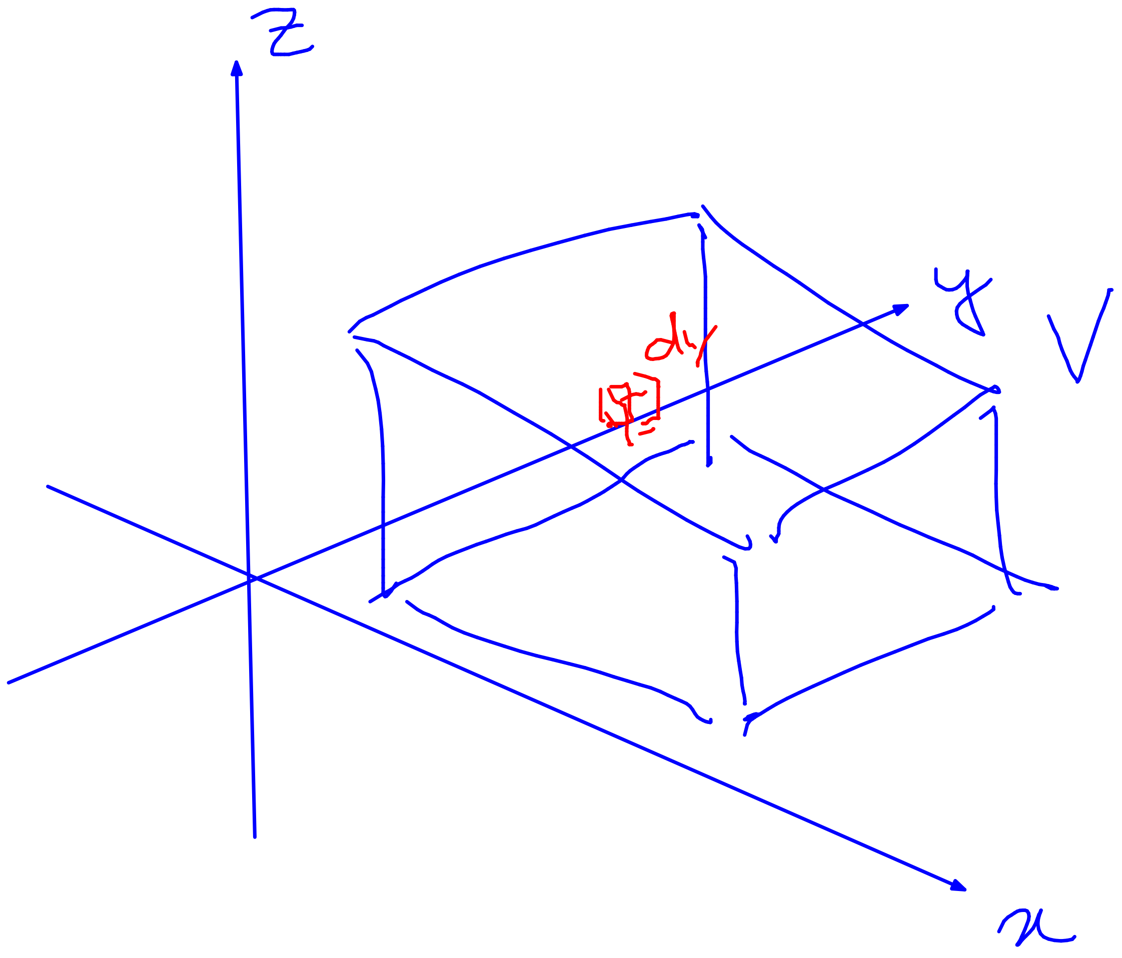
(d) By making the change of variable $t = \sqrt{2}\,x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

(This is a fundamental result for probability and statistics.)

15.6 Triple Integration :

$$\iiint_V f(x,y,z)\,dV$$

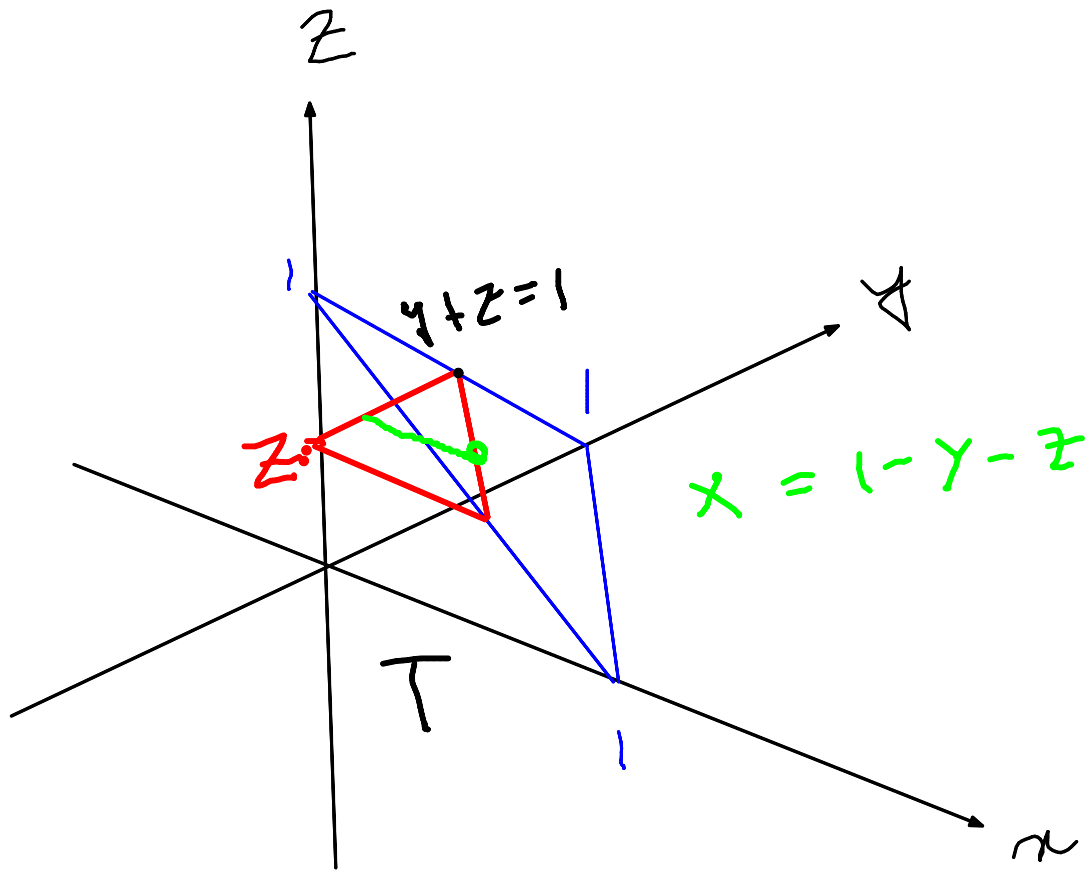


f : density

$f\,dV$ = mass of dV

$$\iiint_V f\,dV = \iiint_V (\text{small mass}) = \text{total mass}$$

Q.

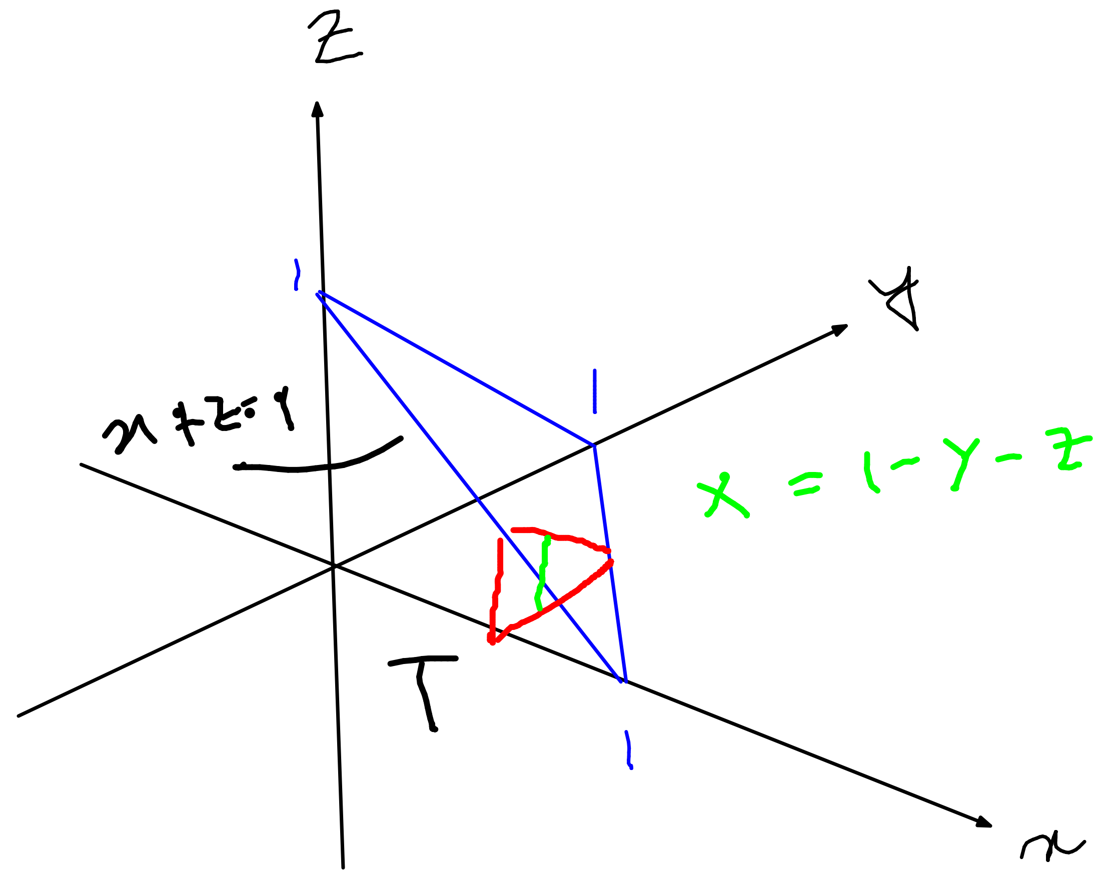


Imagine the tetrahedron bounded by the plane $x+y+z=1$ & xy plane, yz plane, xz plane

Aim: find the volume of this tetrahedron using triple integration

$$V = \iiint_T dV = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx\,dy\,dz$$

Q.

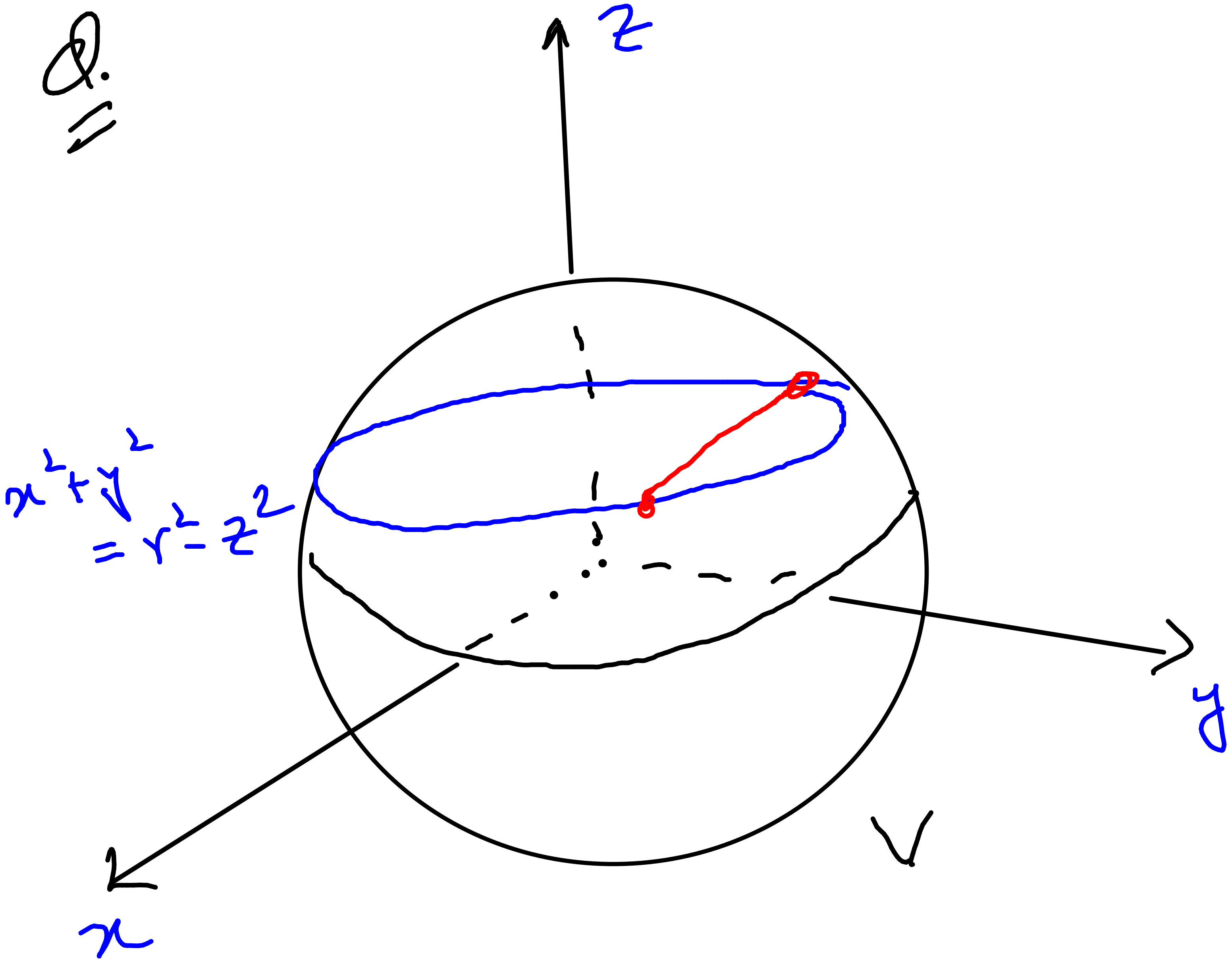


Imagine the tetrahedron bounded by the plane $x+y+z=1$ & xy plane, yz plane, xz plane

Aim: find the volume of this tetrahedron using triple integration

$$V = \iiint_T dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-z} dy\, dz\, dx =$$

Q.



$$x^2+y^2+z^2 = r^2$$

$$\text{Volume} = \frac{4}{3}\pi r^3$$

$$\iiint_V dV$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} \int_{-\sqrt{r^2-z^2-y^2}}^{\sqrt{r^2-z^2-y^2}} dx\,dy\,dz$$

$$= \frac{4}{3}\pi r^3$$