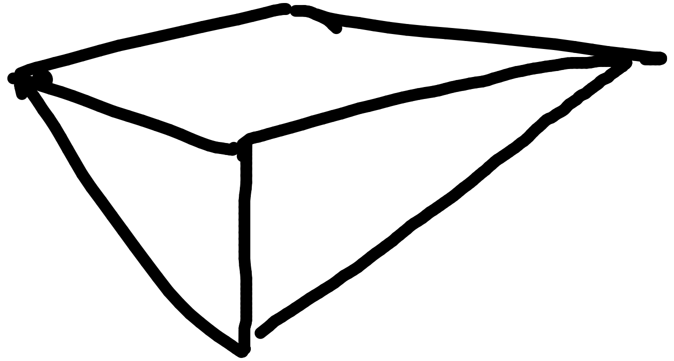
So far:

→ differentiation on multivariable functions

→ integration of multivariable functions

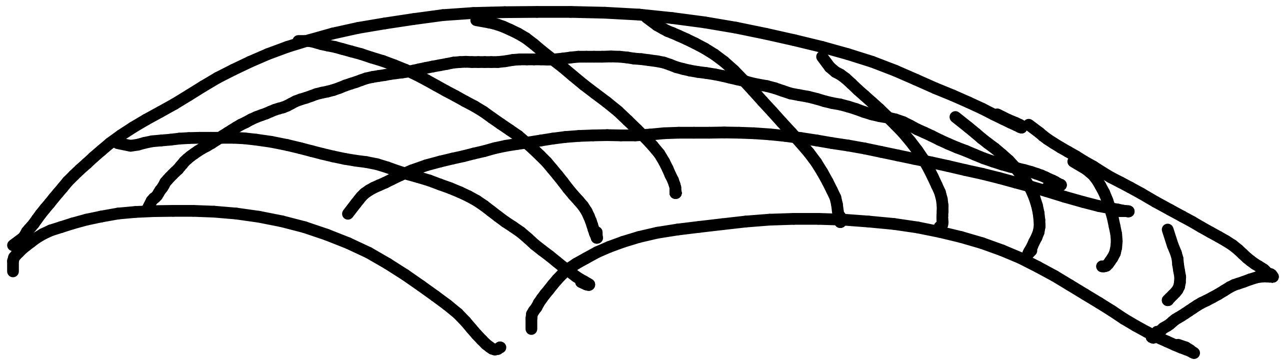
Sample applications of multivariable integration



: calculate volume/mass of any 3d shape

$\frac{4}{3}\pi r^3$

$\frac{1}{3}\pi r^2 h$



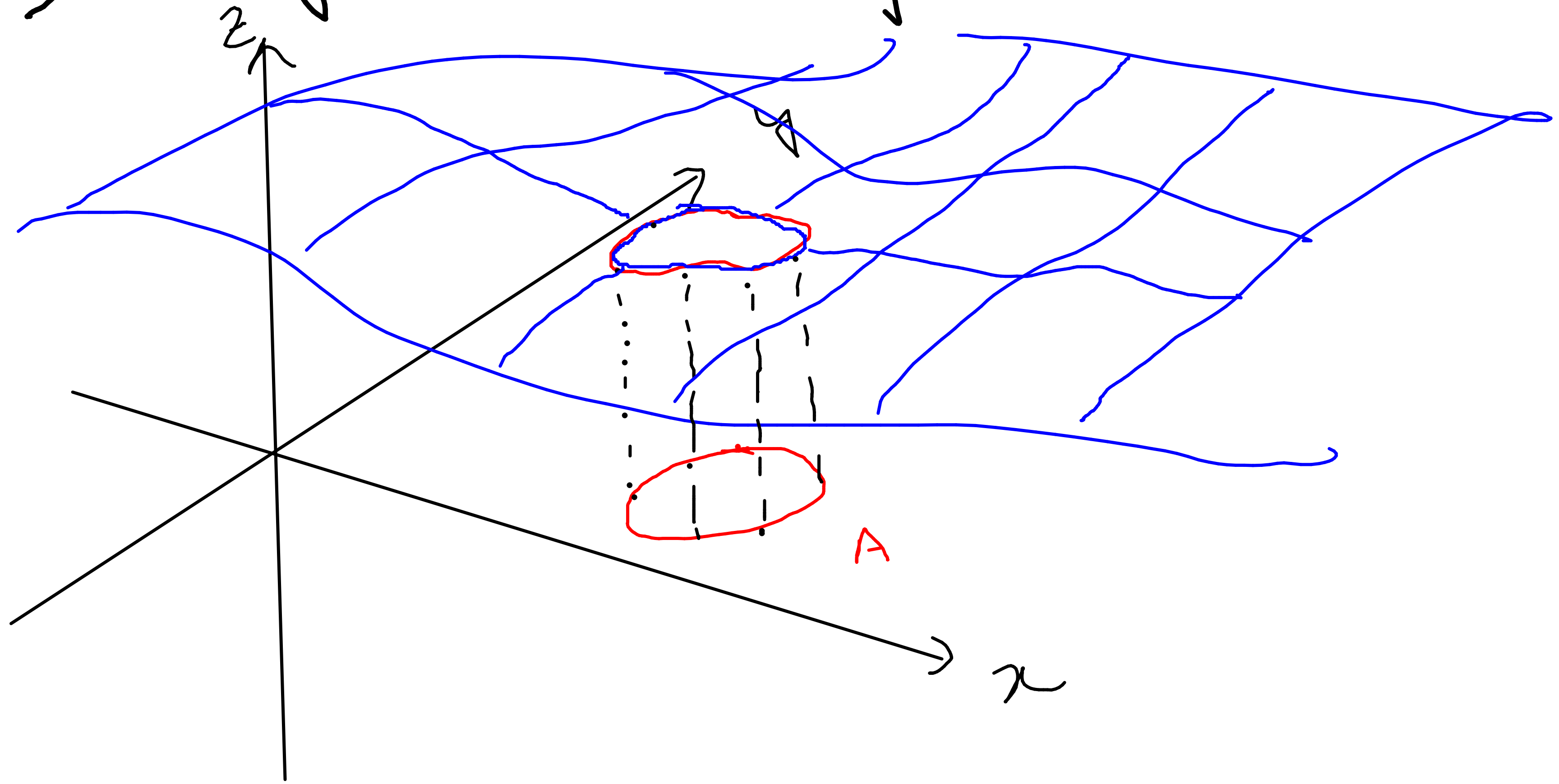
calculus on surfaces

Chapter 15

Calculate $\iint_A f(x,y)\, ds = ??$

= what would this might be

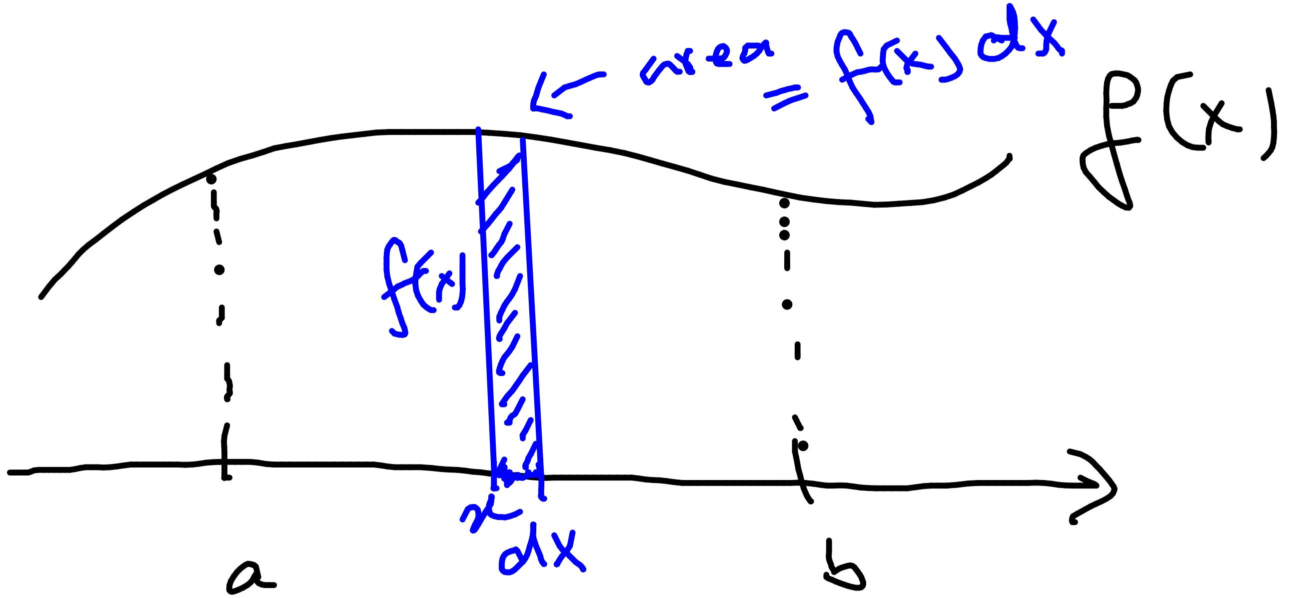
→ integration of $f(x,y)$ over a region A



A : is an area in xy plane

= volume under the graph of $f(x,y)$ & above the region A

Recall one variable integration

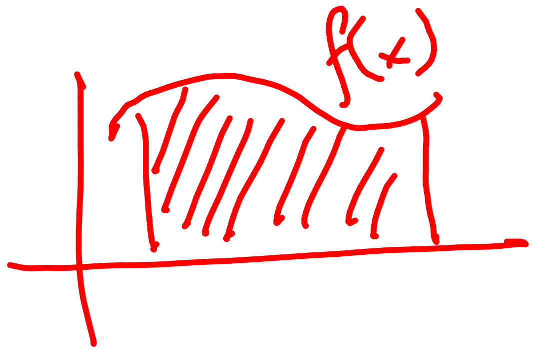


$$\int_a^b f(x)\,dx = \text{area under the graph } f(x)$$

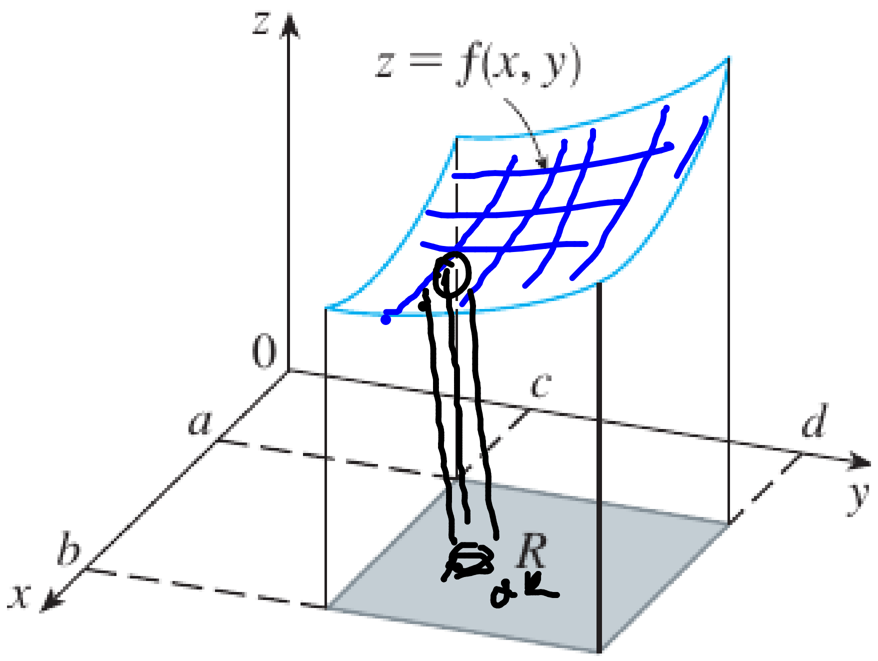
$$= \int_a^b f(x)\,dx$$

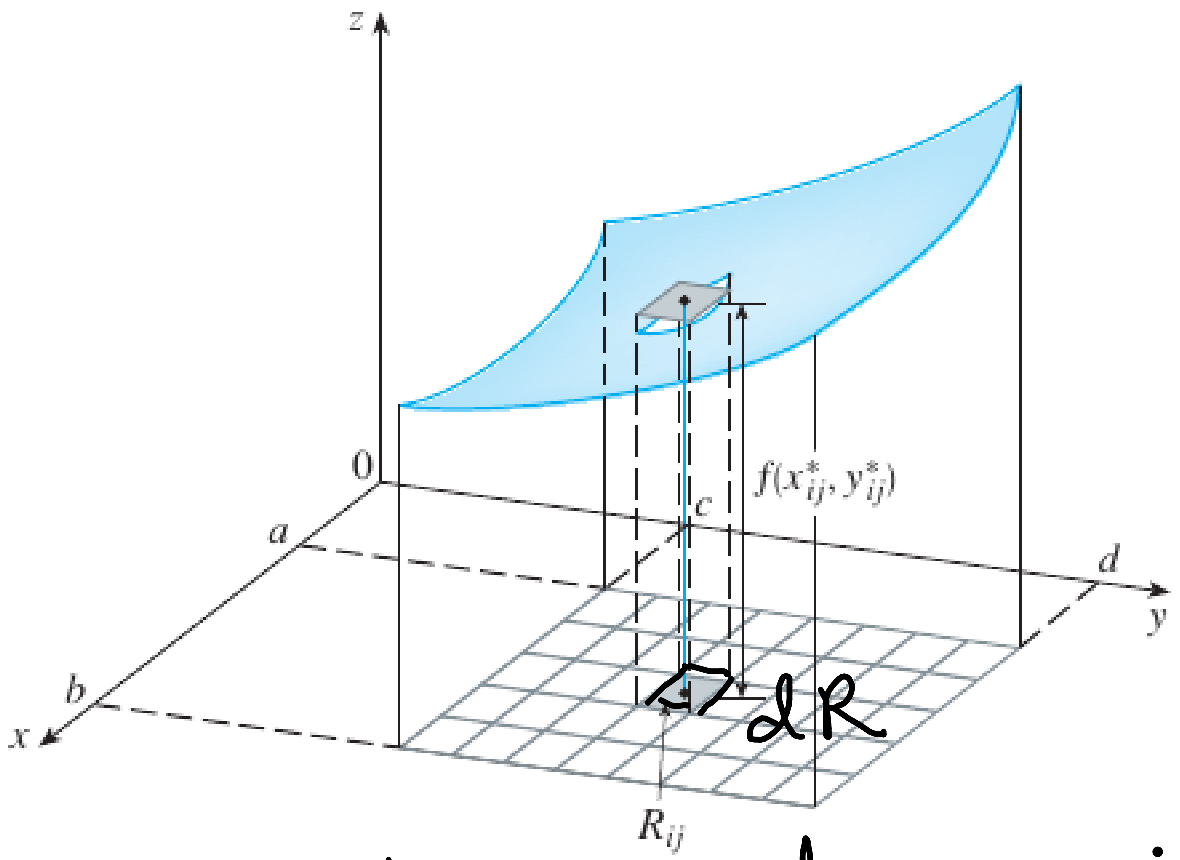
15

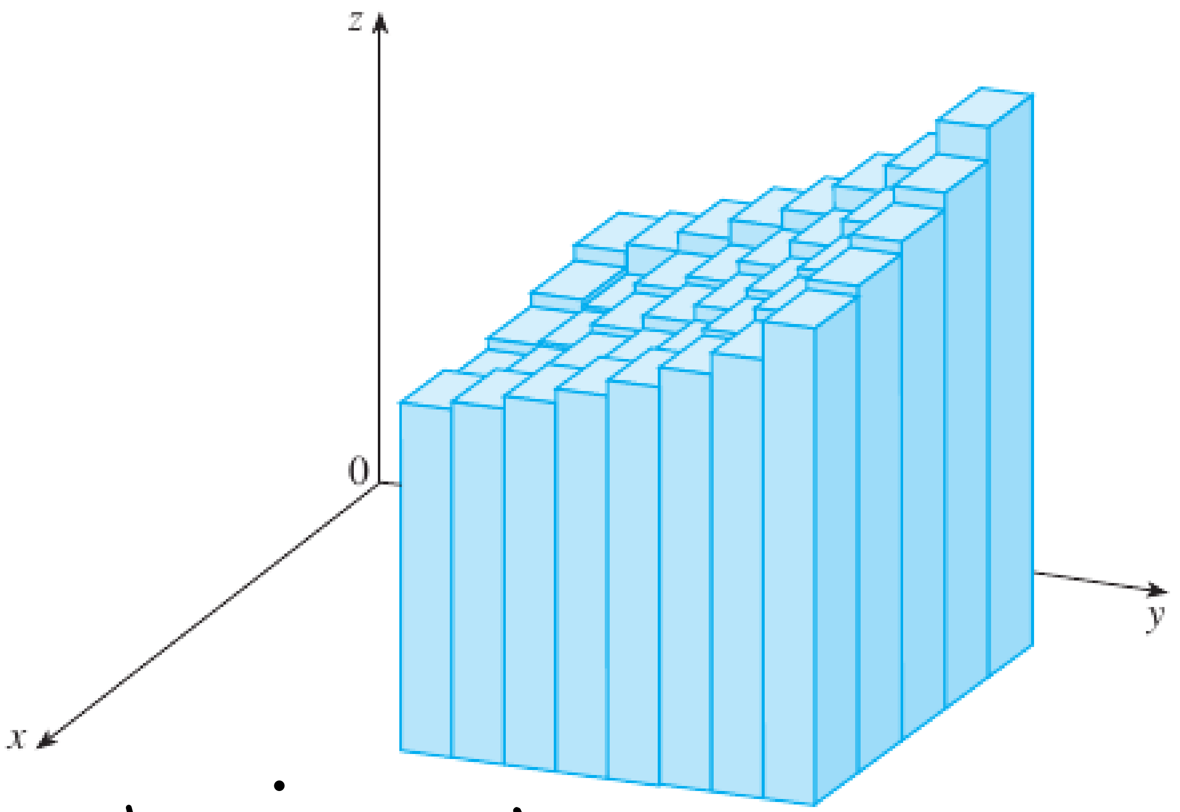
MULTIPLE INTEGRALS



$$\iint_R f(x,y)\, dR$$







$f(x,y)\, dS$ = volume of an infinitesimally thin rectangular pipe with base dS.

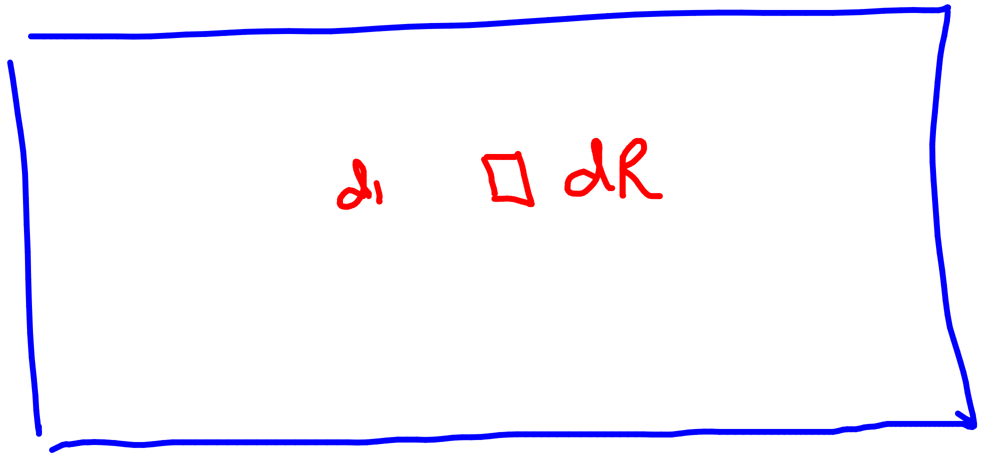
$$\iint_R f(x,y)\, dR$$

typically:

$f(x,y)$: some kind of density

e.g. mass per unit area

or charge per unit area



R

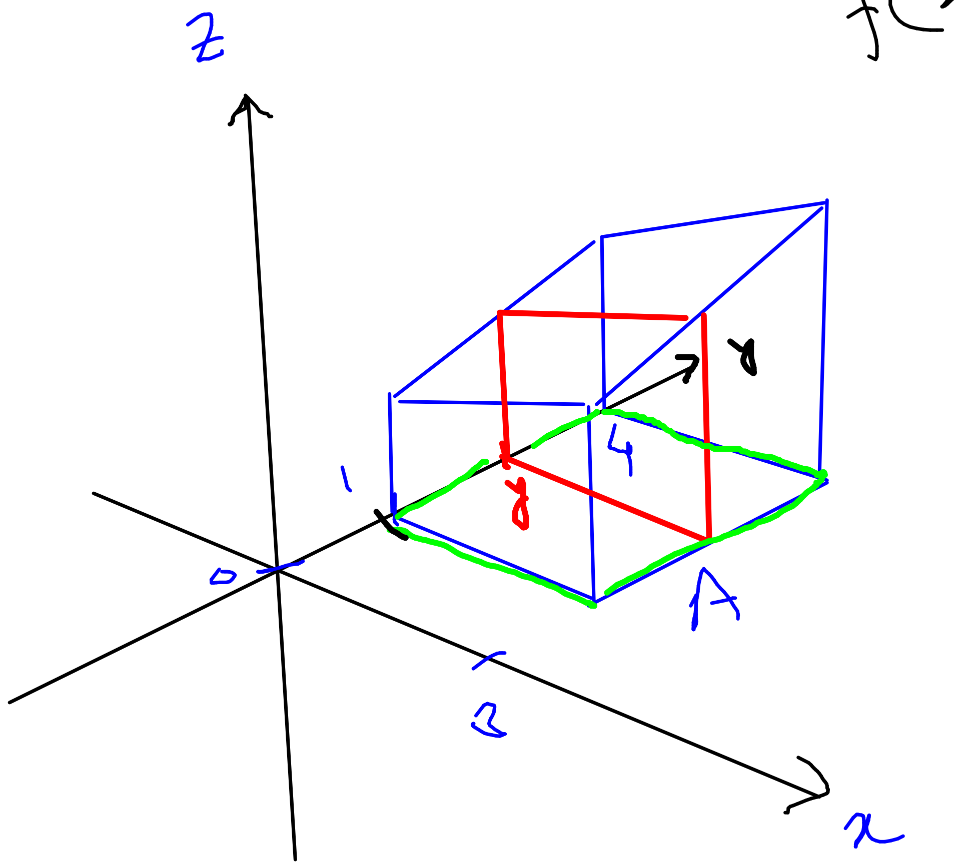
$dm = f(x,y)\, dR$ = mass of dR

$$\iint_R f(x,y)\, dR = \iint_R dm = \text{mass of } R$$

8.

$$\iint_A (x+y)\, ds \quad \text{where} \quad A = \{(x,y) \mid 0 \le x \le 3,\ 1 \le y \le 4\}$$

$f(x,y) = x+y$



$$\iint_A (x+y)\, ds = \int_1^4 \int_0^3 (x+y)\, dx\, dy$$

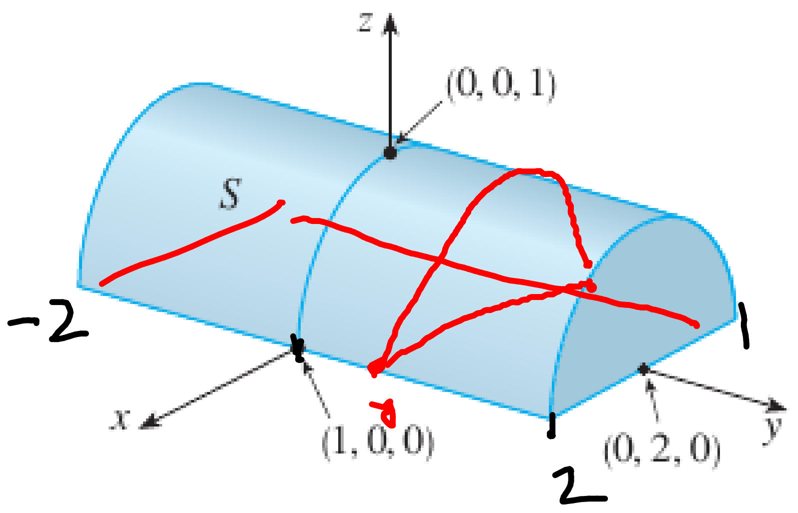
= integrate inside out.

$$= \int_1^4 \left[\frac{x^2}{2} + xy\right]_{x=0}^{x=3} dy = \int_1^4 \left(\frac{9}{2} + 3y\right) dy$$

$$= 36$$

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$



$$z = \sqrt{1-x^2}$$

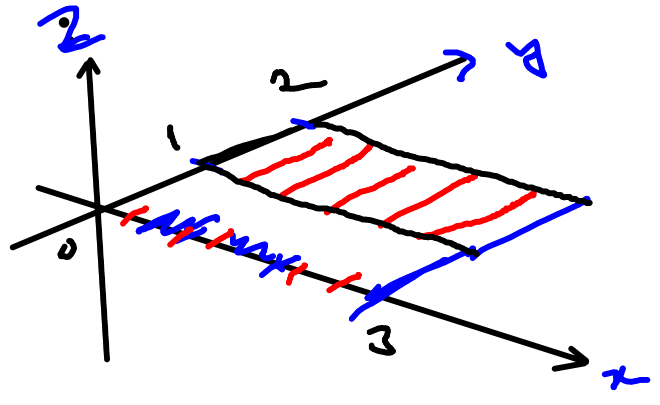
$$\iint_R \sqrt{1-x^2}\, dA = \int_{-2}^{2} \int_{-1}^{1} \sqrt{1-x^2}\, dx\, dy$$

EXAMPLE 4 Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

sketch the region of integration

a)

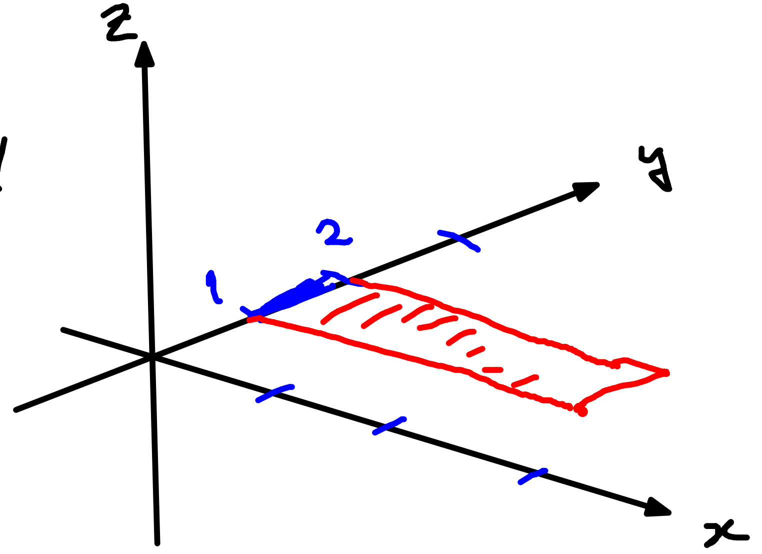


$$\int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx$$

$$= \int_0^3 \left| \frac{x^2 y^2}{2} \right|_{y=1}^{y=2} dx$$

$$= \int_0^3 \frac{3}{2} x^2 \, dx = \frac{27}{2}$$

b) DIY



$$\int_1^2 \left(\int_0^3 x^2 y \, dx \right) dy = \int_1^2 \left| \frac{x^3}{3} y \right|_{x=0}^{x=3} dy$$

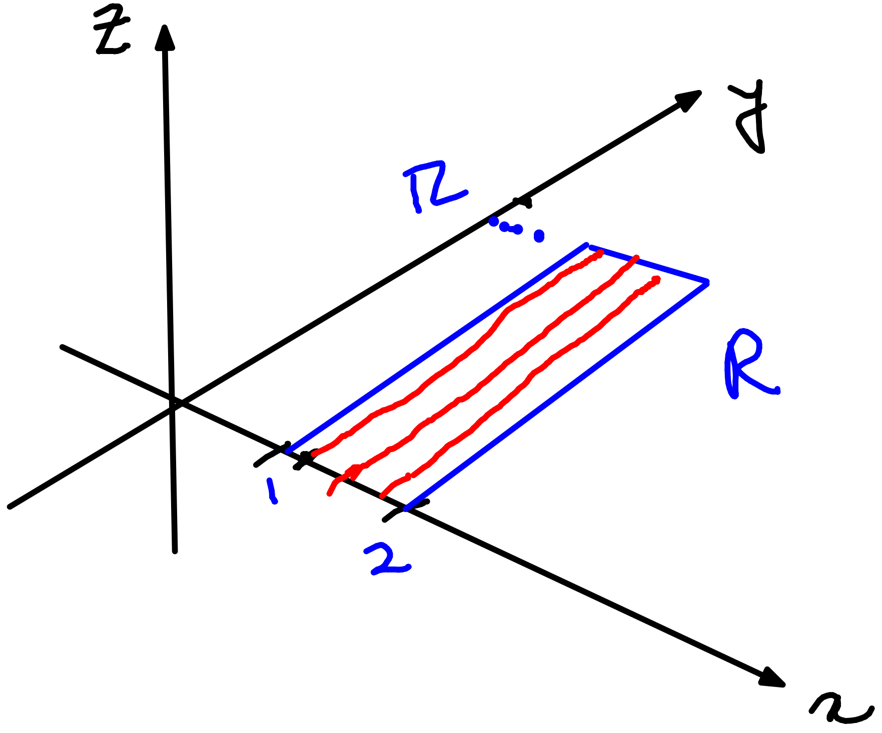
$$= \int_1^2 \frac{27}{3} y \, dy = \frac{27}{2}$$

10 FUBINI'S THEOREM If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

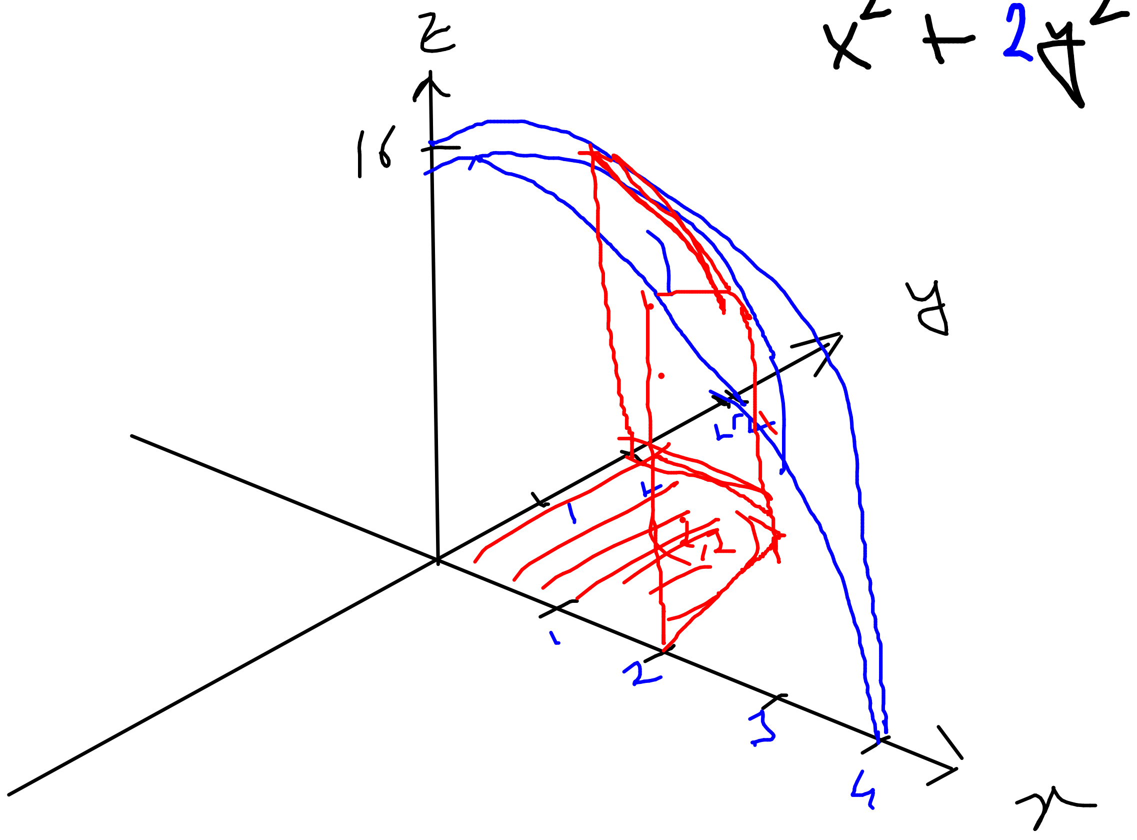
$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$



DIY

EXAMPLE 7 Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

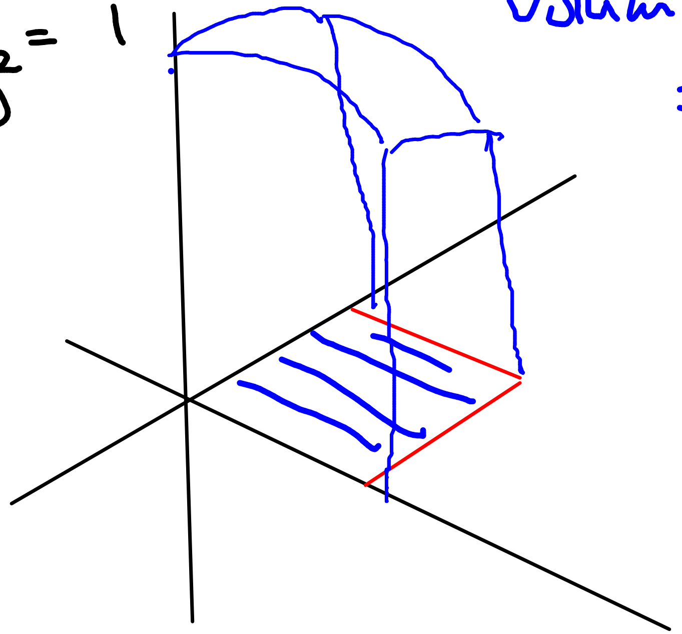
Try to sketch the solid S



$x^2 + 2y^2 + z = 16$ | related to $z = -x^2 - 2y^2 + 16$

$x^2 + 2y^2 = 16$

$\frac{x^2}{4^2} + \frac{y^2}{(2\sqrt{2})^2} = 1$



volume $= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$

symbolab.com/solver/step-by-step/%5Cint_%7B0%7D%5E%7B2%7D%20%5Cint_%7B0%7D%5E%7B2%7D%2016%20-%20x%5E%7B

Most Used Actions

simplify | solve for | inverse | tangent | line | See All

$\int_0^2 \int_0^2 16 - x^2 - 2 \cdot y^2 \, dx \, dy$

Go

Examples »

Solution

Keep Practicing >

Show Steps

$\int_0^2 \int_0^2 16 - x^2 - 2y^2 dxdy = 48$

Steps

$\int_0^2 \int_0^2 16 - x^2 - 2y^2 dxdy$

$\int_0^2 \left(16 - x^2 - 2y^2\right) dx = 32 - \frac{8}{3} - 4y^2$ Show Steps

$= \int_0^2 \left(32 - \frac{8}{3} - 4y^2\right) dy$ $= 48$

Relat

PROPERTIES OF DOUBLE INTEGRALS

12 $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

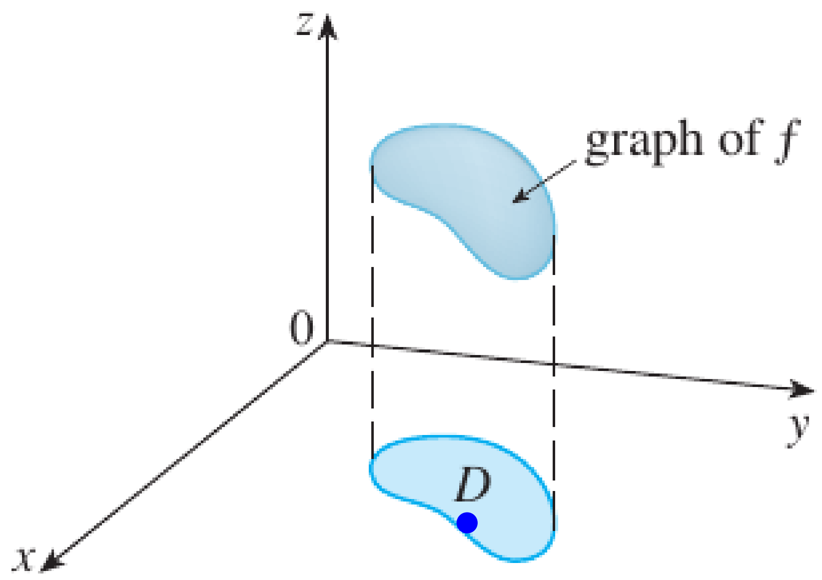
13 $$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in R, then

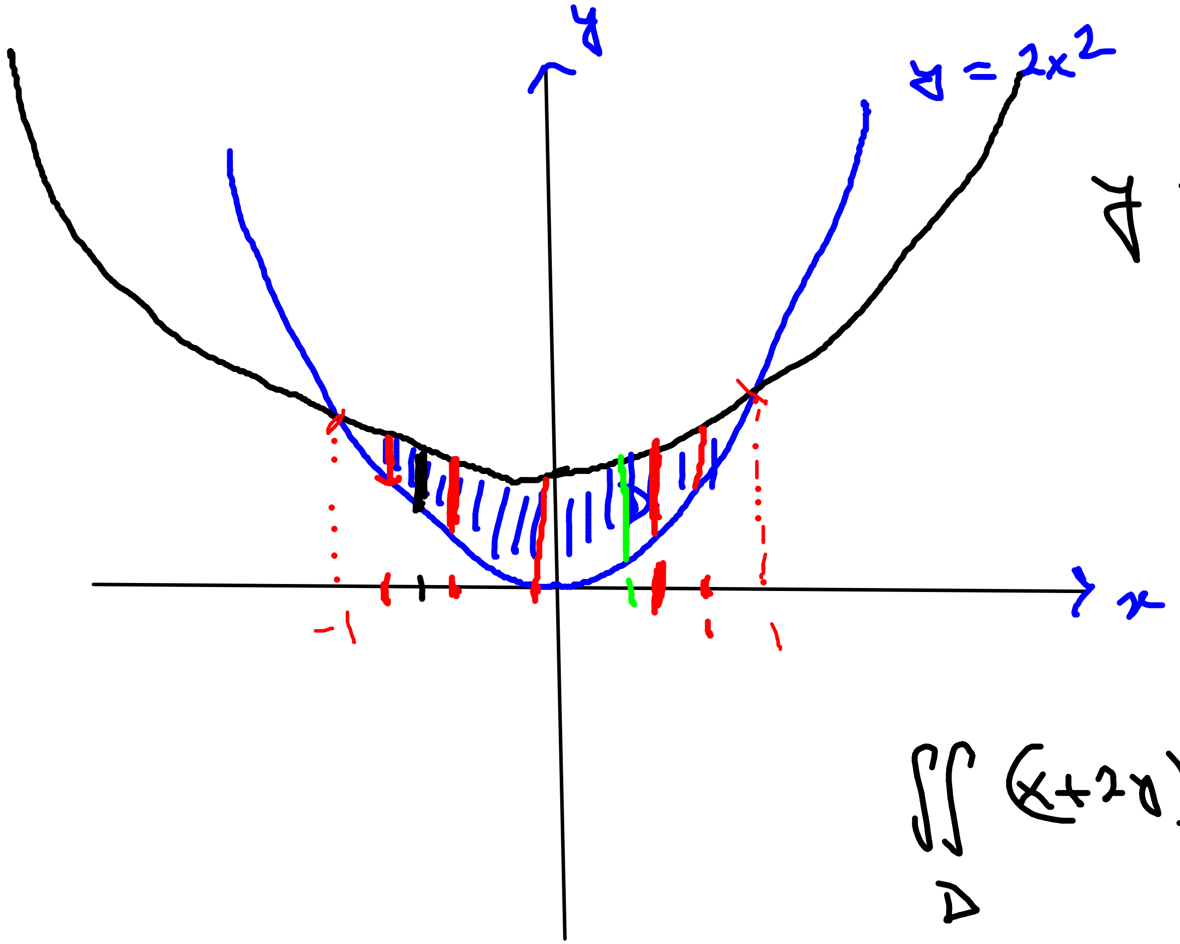
14 $$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

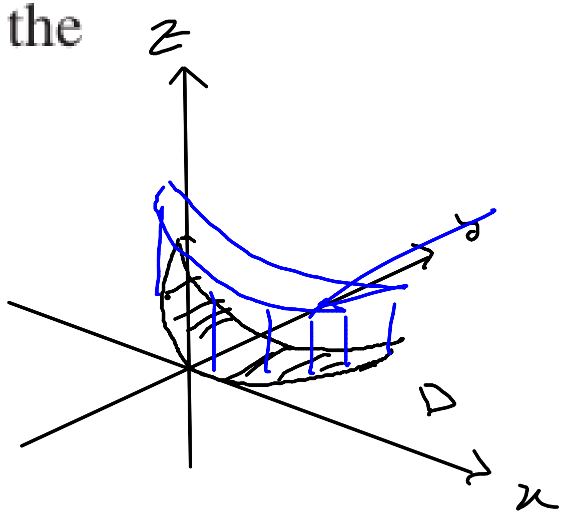
15.2

DOUBLE INTEGRALS OVER GENERAL REGIONS



V EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.





$y = 1 + x^2$

$1 + x^2 = 2x^2$

$1 = x^2$

$x = \pm 1$

$$\iint_D (x+2y)\, dA = \int_{-1}^{1} \cancel{\int_{0}^{2}} x+2y \, dy\, dx = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy \; dx$$

symbolab.com/solver/step-by-step/%5Cint_%7B-1%7D%5E%7B1%7D%20%5Cint_...

$$\int_{-1}^{1} \int_{2\cdot x^2}^{1+x^2} x + 2\cdot y\, dy\, dx$$

Examples »

Solution

Keep

Show St

$$\int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y dy dx = \frac{32}{15}$$ (Decimal: $2.13333\ldots$)

Steps

$$\int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y dy dx$$

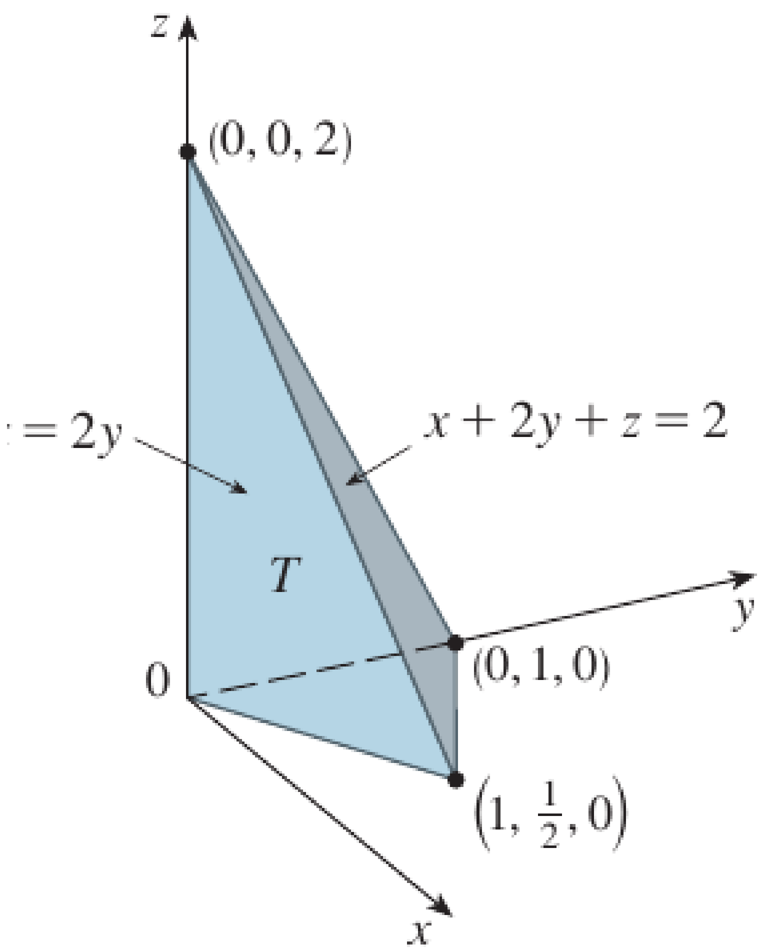
$$\int_{2x^2}^{1+x^2} (x + 2y) dy = x - x^3 + 1 + 2x^2 - 3x^4$$

$$= \int_{-1}^{1} \left(x - x^3 + 1 + 2x^2 - 3x^4\right) dx$$

$$\int_{-1}^{1} \left(x - x^3 + 1 + 2x^2 - 3x^4\right) dx = \frac{32}{15}$$

EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.



EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

V EXAMPLE 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

V EXAMPLE 3 Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

1–6 ▪ Evaluate the iterated integral.

1. $\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx$

2. $\int_1^2 \int_y^2 xy\, dx\, dy$

1–6 ▪ Evaluate the iterated integral.

1. $\int_0^1 \int_0^{x^2} (x + 2y)\, dy\, dx$

2. $\int_1^2 \int_y^2 xy\, dx\, dy$

1–6 ▪ Evaluate the iterated integral.

2. $\int_1^2 \int_y^2 xy \, dx \, dy$

31–36 ■ Sketch the region of integration and change the order of integration.

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x, y)\, dx\, dy$$

31–36 ▪ Sketch the region of integration and change the order of integration.

$$\int_0^4 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$$

37–42 ▪ Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$$