Today:

11.4] Review problem] Tangent plane

Linear Approximation

Differentials

$f(x,y)$

11.6] Chain rule

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$f = f(x,y)$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

EXAMPLE 4 The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

$V(r,h) = \frac{1}{3}\pi r^2 h$

$dV = ??$

$$\boxed{dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh}$$

$dr = dh = 0.1$

$$dV = \underbrace{\frac{2}{3}\pi r h\, dr + \frac{1}{3}\pi r^2 dh}_{\substack{r=10,\ h=25 \\ dr = dh = 0.1}}$$

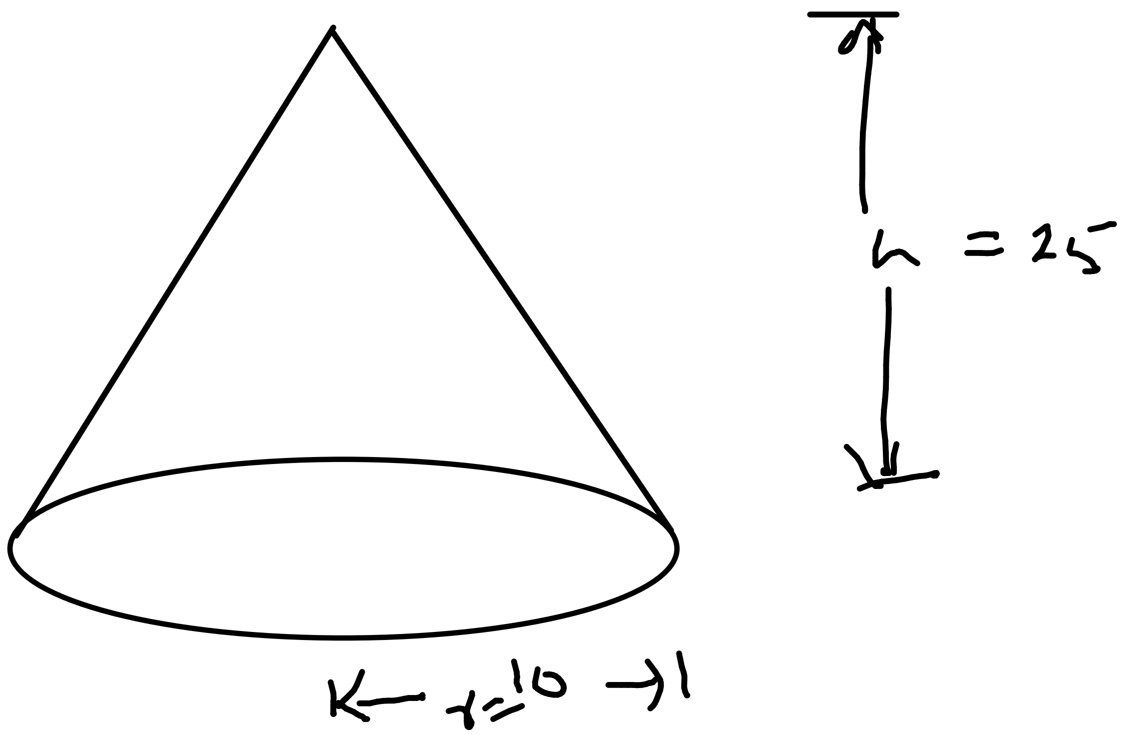
$= 20\pi$

$\frac{dV}{V} = ??\ \%$

$= \frac{3}{125}$

≈ 0.024

$= 2.4\ \%$



1–6 ▪ Find an equation of the tangent plane to the given surface at the specified point.

$$z = y\cos(x-y), \quad (2, 2, 2)$$

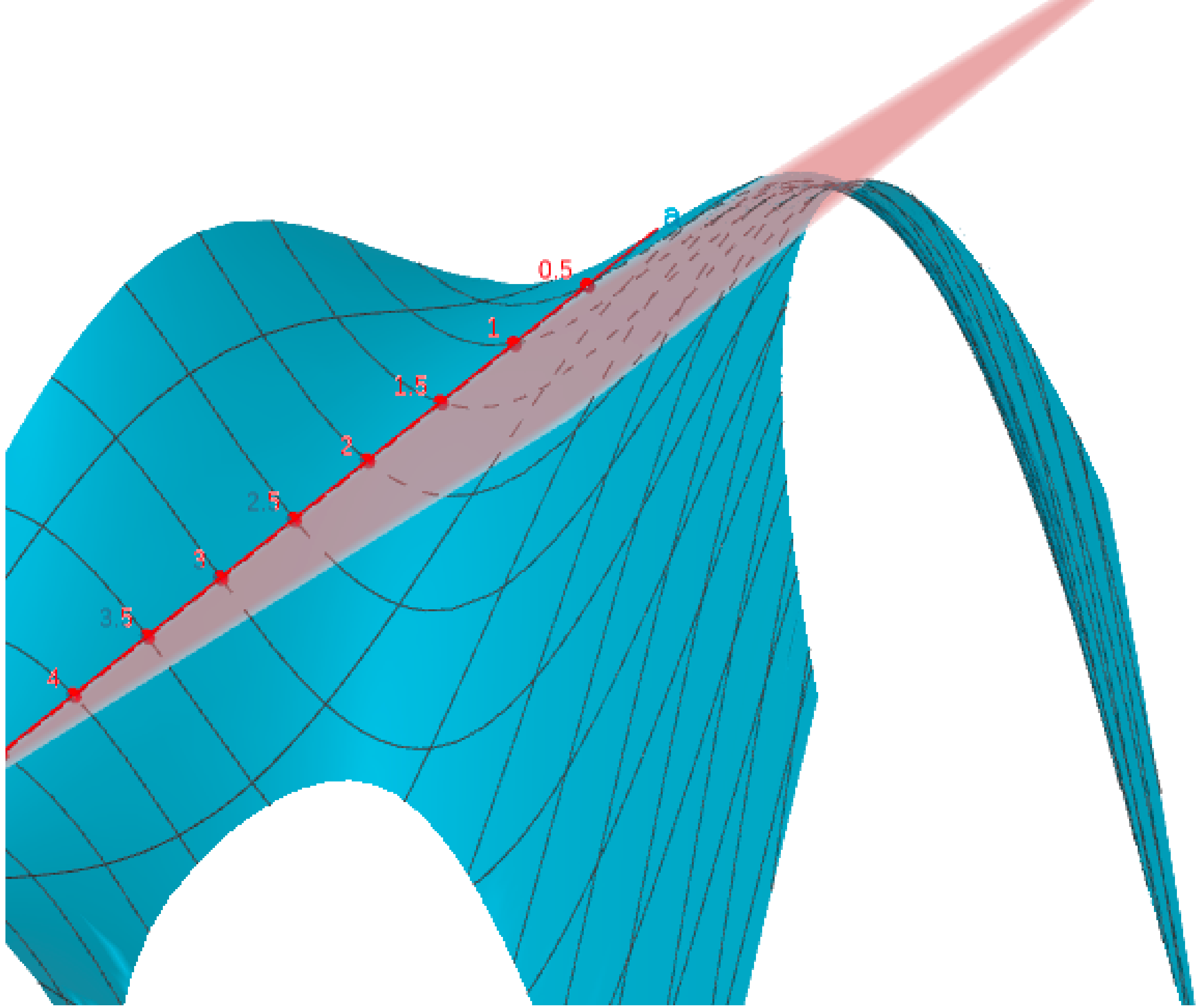
$$\boxed{z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)}$$

$$\frac{\partial z}{\partial x} = -y\sin(x-y)\Big|_{\substack{x=2\\y=2}} = 0$$

$$\frac{\partial z}{\partial y} = \Big[y\sin(x-y) + \cos(x-y)\Big]\Big|_{\substack{x=2\\y=2}} = 1$$

$$z - 2 = 0(x-2) + 1(y-2)$$

$$\boxed{z = y}$$



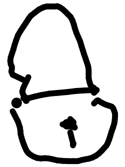
a
0.5
1
1.5
2
2.5
3
3.5
4

30. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

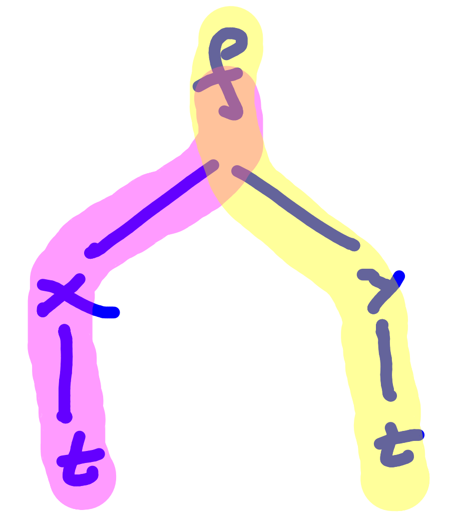
$$P = 8.31\,T/V$$

$dP = ??$ given $V = 12$, $T = 310$, $dV = 0.3$, $dT = -5$

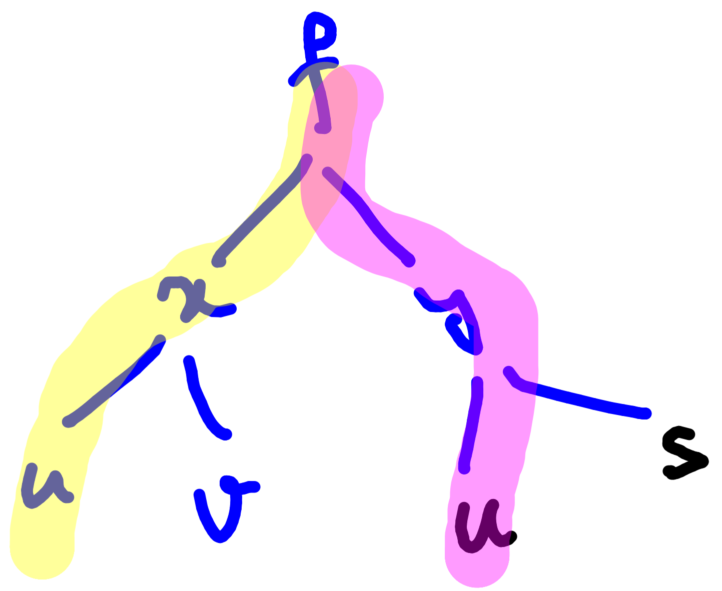
$$dP = \frac{\partial P}{\partial V}\,dV + \frac{\partial P}{\partial T}\,dT$$

$$= -8.8$$ 

11.5 THE CHAIN RULE



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$



$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Q: $T(x,y) = x+y$

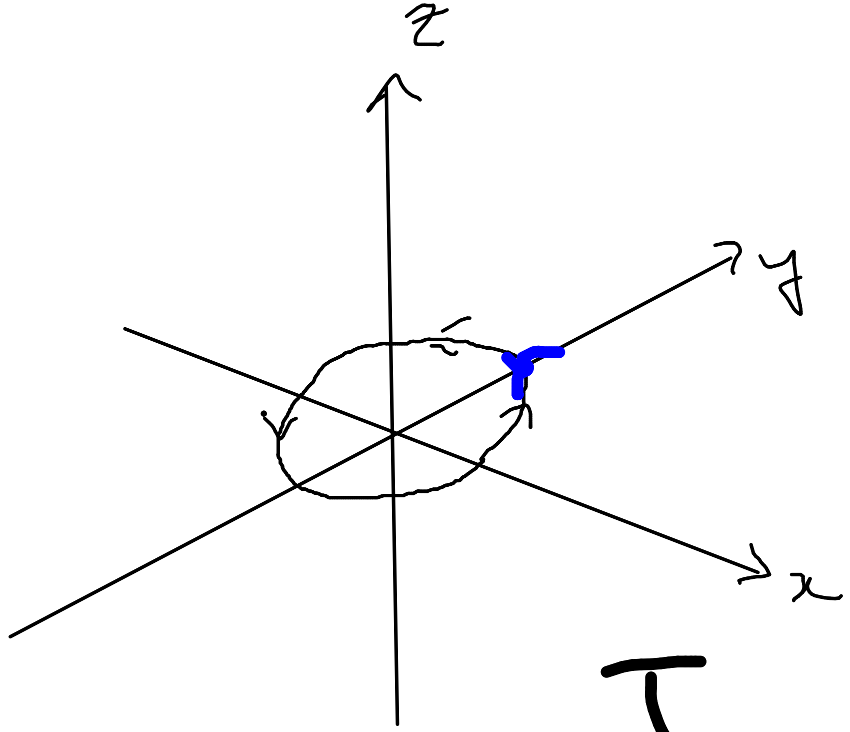
Temperature at point (x,y)

$x = \cos(t)$

$\underbrace{y = \sin(t)}$, $0 < t < \infty$

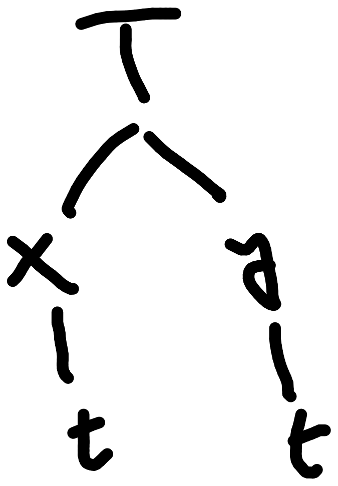
what path is this ??

$\vec{r}(t) = \cos t\,\hat{i} + \sin(t)\,\hat{j}$



Q: rate of change of temperature T w.r.t. time t

$\frac{dT}{dt} = ??$ at $t = \pi/2$

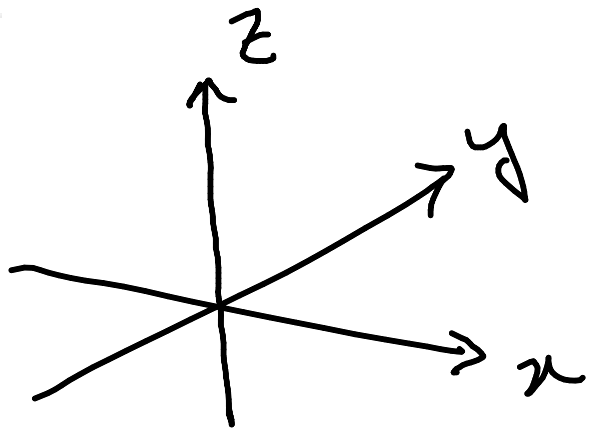


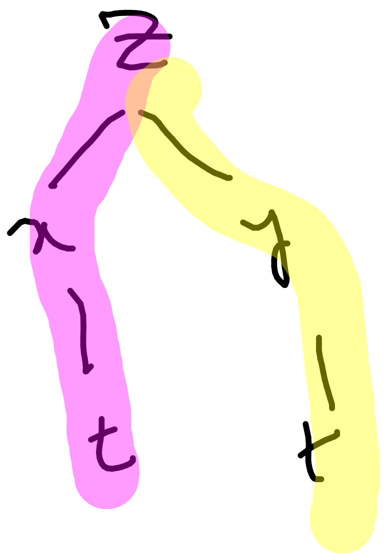
$$= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= \Big[1(-\sin t) + 1 \cdot \cos(t)\Big]_{t=\pi/2}$$

$$= -1$$

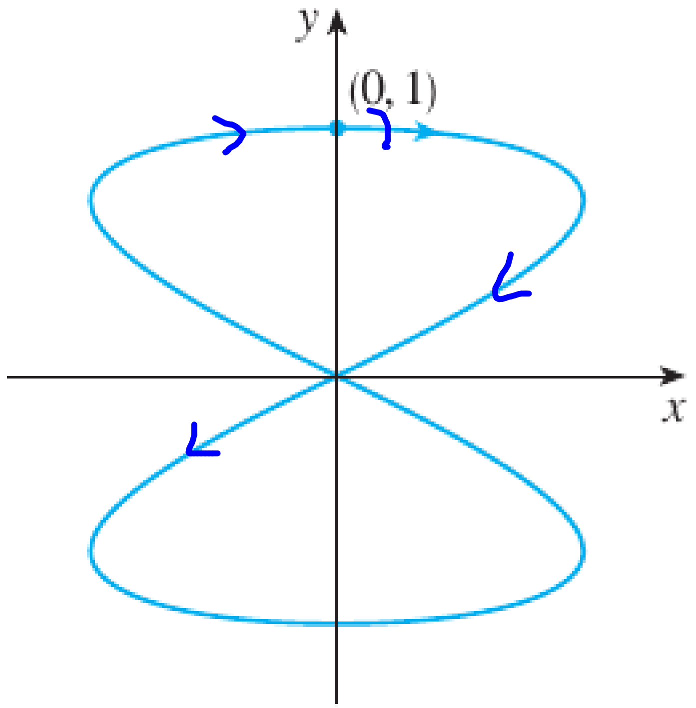
EXAMPLE 1 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.





$x = \sin 2t$

$y = \cos t$



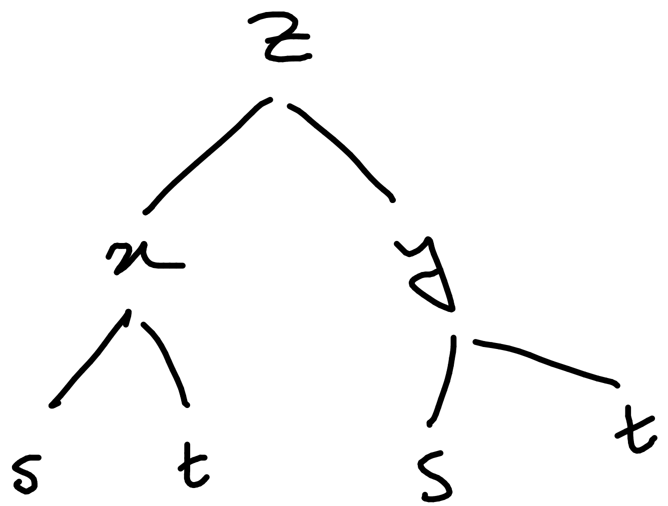
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$= \underbrace{(2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)}_{t = 0,\ x = 0,\ y = 1}$$

$$= 6$$

EXAMPLE 3 If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

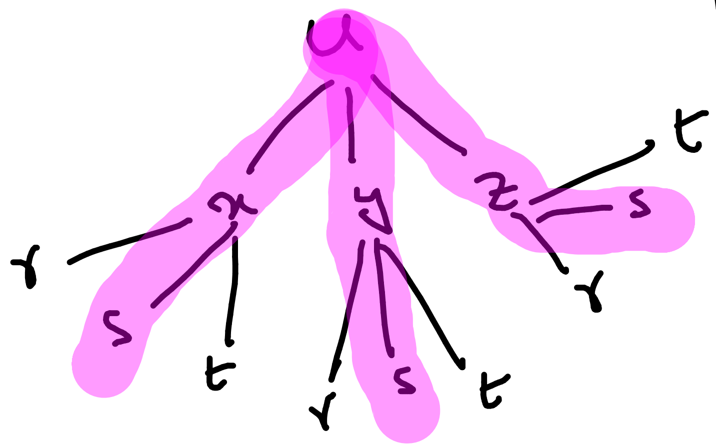
$$z = e^x \sin y$$



$$\frac{\partial z}{\partial s} = e^x \sin y \; t^2 + e^x \cos y \; 2st$$

$$\frac{\partial z}{\partial t} = e^x \sin y \; 2st + e^x \cos y \; s^2$$

EXAMPLE 5 If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= \underbrace{4x^3y\, re^t + (x^4 + 2yz^3)\, 2rse^{-t} + (3y^2z^2)\, r^2\sin t}_{\substack{r=2,\ s=1,\ t=0 \\ x=2,\ y=2,\ z=0}}$$

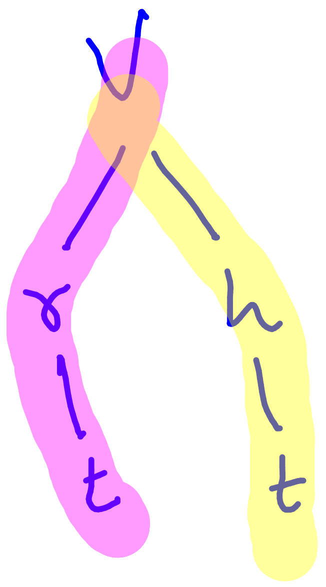
$$= 192$$

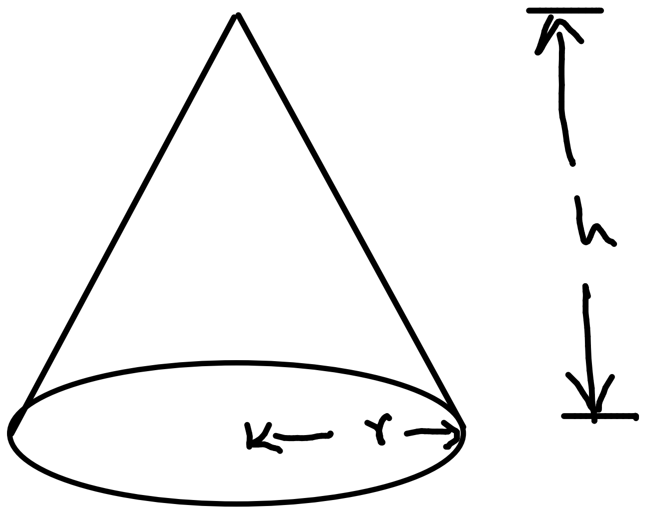
Today:

→ Review : chain rule

→ 11.6 : Directional Derivatives
& gradients

32. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?





$$\frac{dh}{dt} = -2.5 \qquad \frac{dr}{dt} = 1.8$$

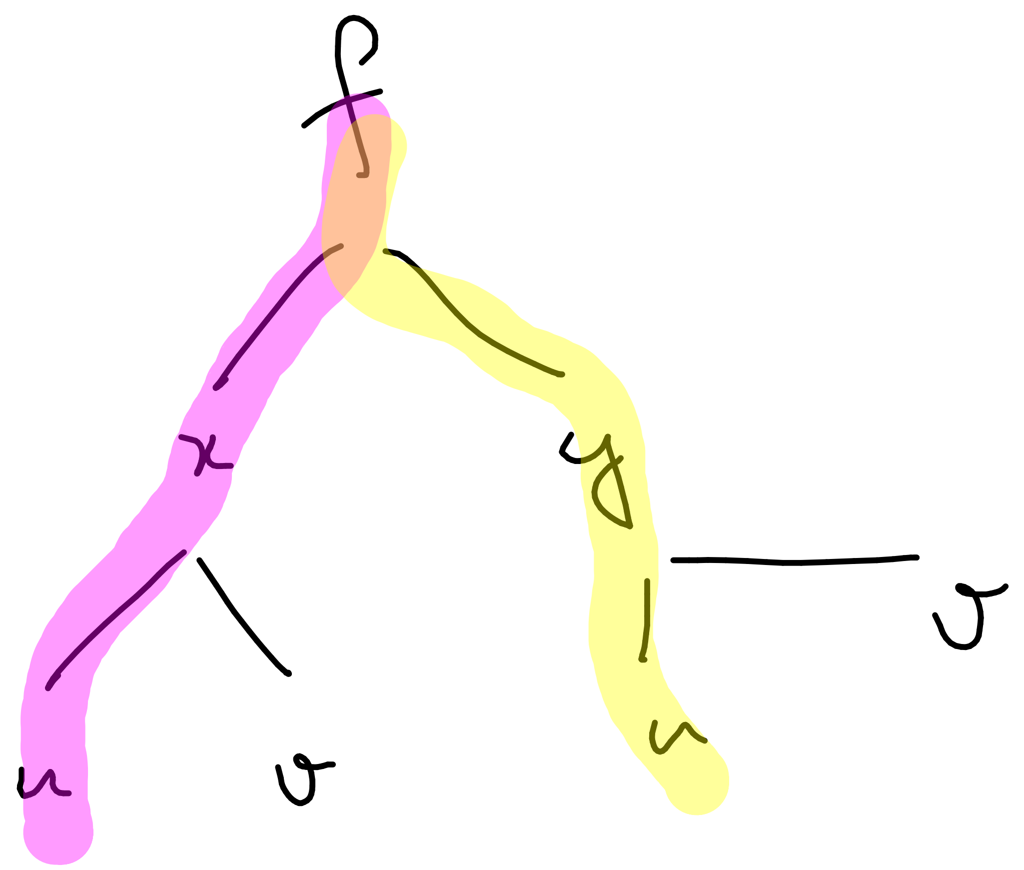
$$V = \frac{1}{3}\pi r^2 h$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=120\\h=140}} = ?? = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi(120)(140)(1.8) + \frac{\pi}{3}(120)^3(-2.5)$$

$$= 8160\pi \quad in^3/s$$

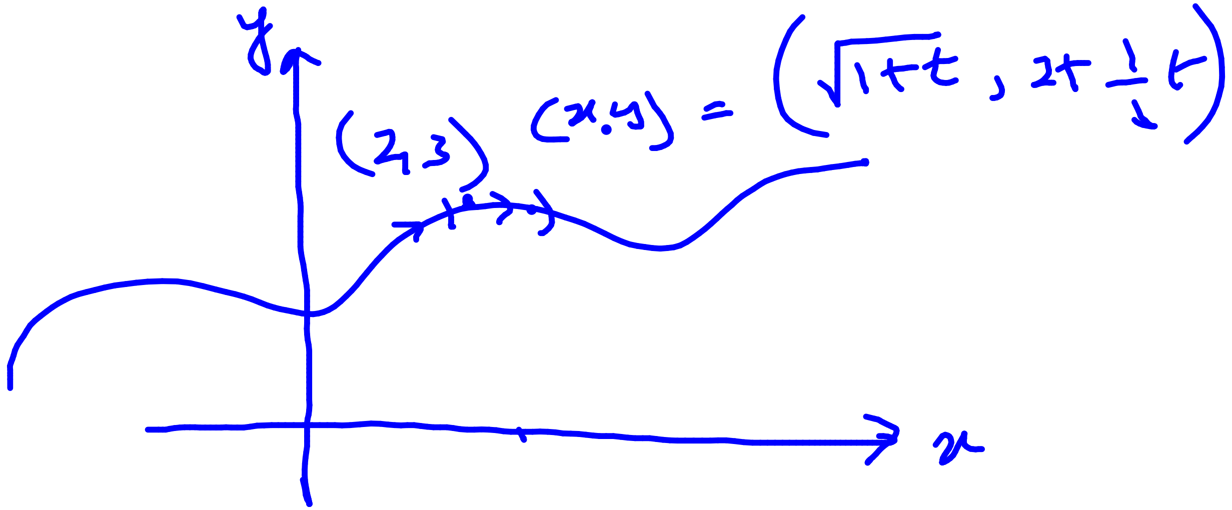


$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

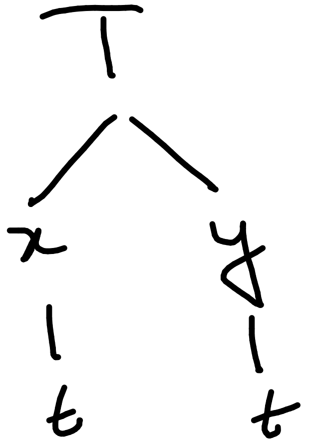
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$T_x = \frac{\partial T}{\partial x}$$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?



$t = 3 \qquad x = 2 \qquad y = 3$



$$\left.\frac{dT}{dt}\right|_{t=3} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= \left.\left(4 \cdot \frac{1}{2\sqrt{1+t}} + 3 \cdot \frac{1}{3}\right)\right|_{t=3}$$

$$= 1 + 1 = 2$$

29. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

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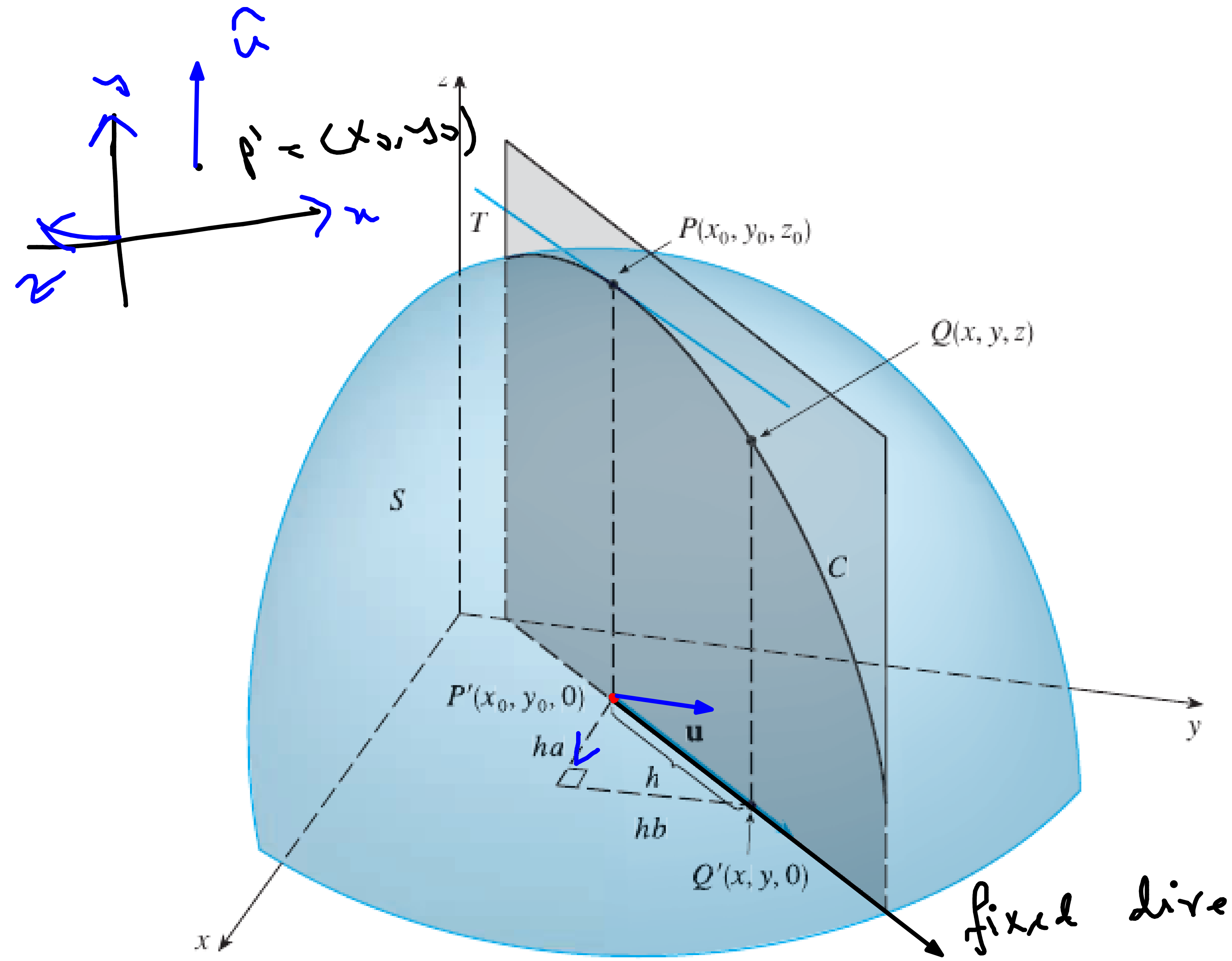


fixed direction

2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

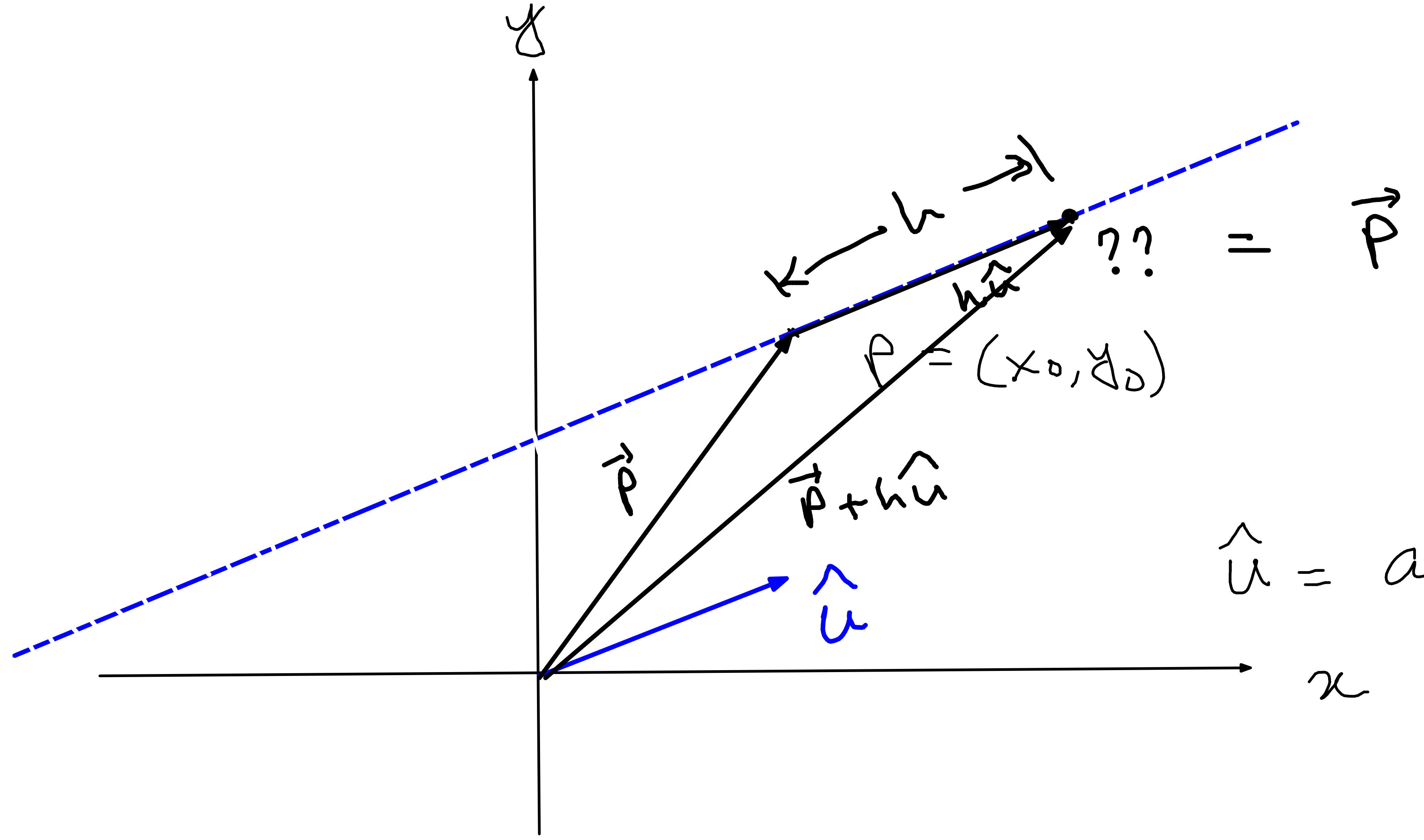
$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

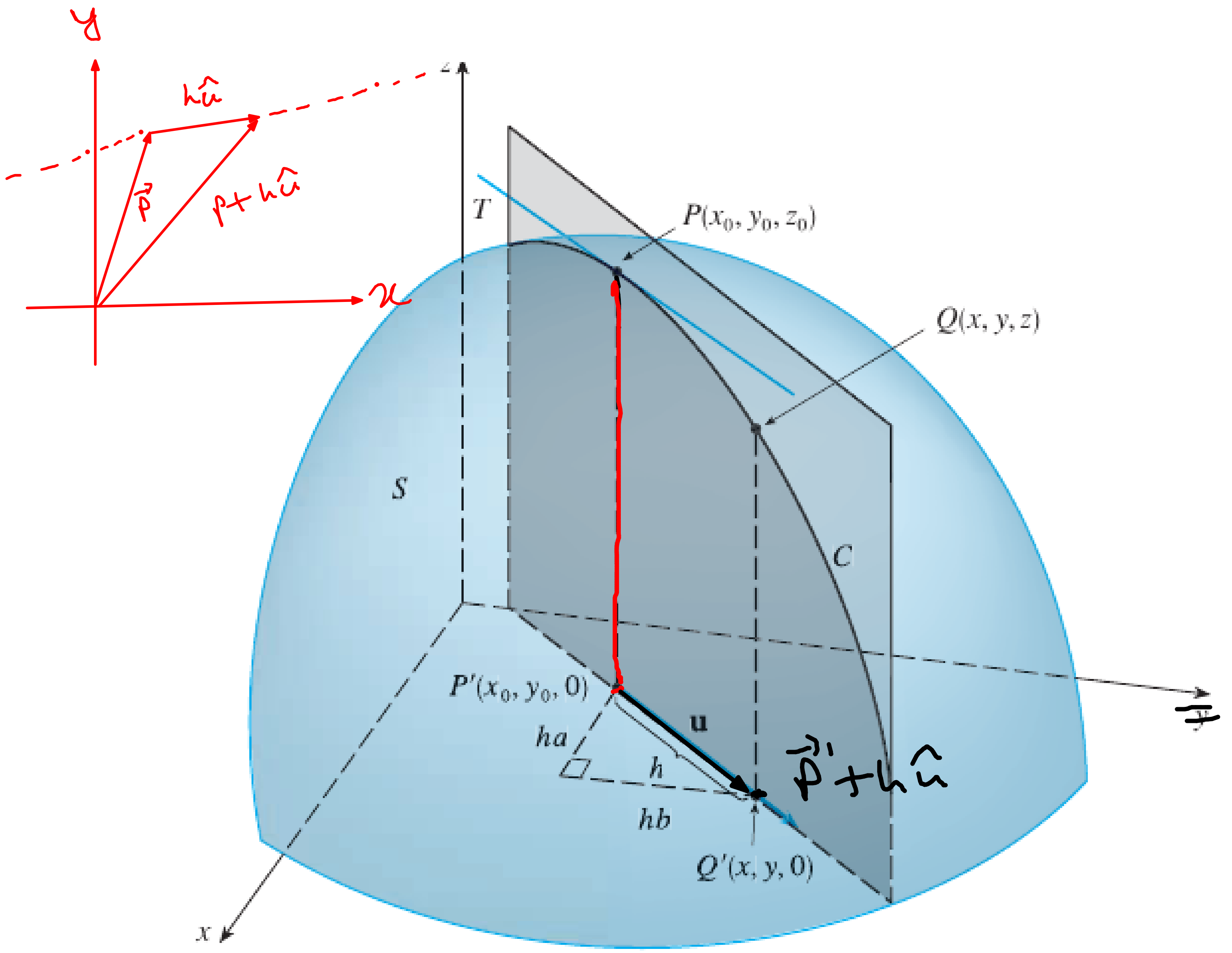
- blue surface is the graph of $f(x,y)$
- base point $(x_0, y_0) = P'$
- Q: directional derivative at point P', in the direction $\hat{u}$

  $=$ rate of change of f if we start moving from point P' along the direction $\hat{u}$

Q: sketch the line passing through P & parallel to $\hat{u}$



$\hat{u} = a\hat{i} + b\hat{j}$



2 DEFINITION The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

$\hat{u} = a\hat{i} + b\hat{j}$

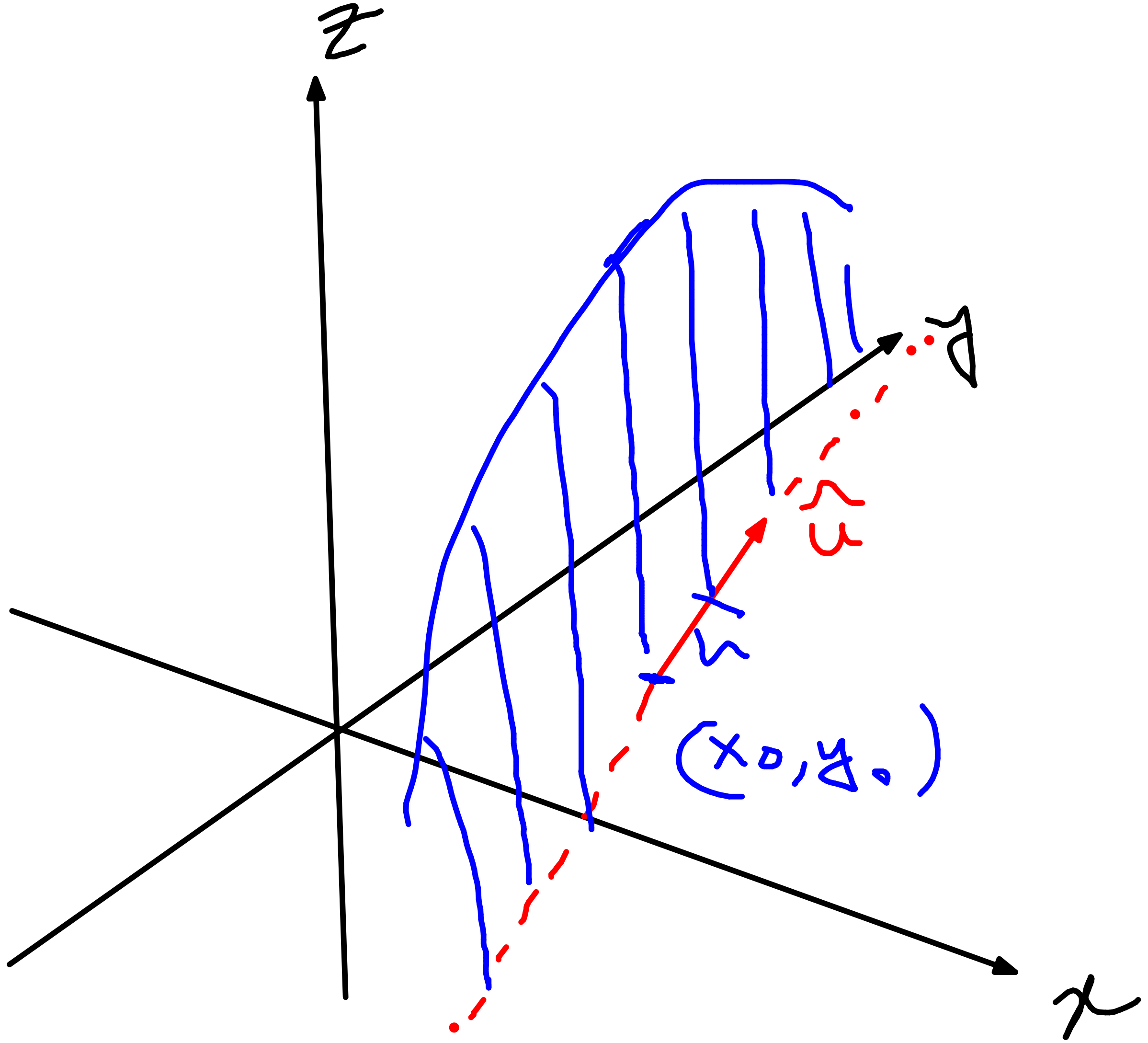
Directional derivative at point p' in the direction of $\hat{u}$

$$\lim_{h \to 0} \frac{f(\vec{p}' + h\hat{u}) - f(\vec{p}')}{h}$$

Today:

→ Finish 11.6

→ 11.7



$f(x,y)$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$D_{\hat{u}} f(x_0,y_0) = ??$ = rate of change of f at (x_0,y_0) in the direction of $\hat{u}$

$$= \nabla f \cdot \hat{u}$$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

3 THEOREM If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

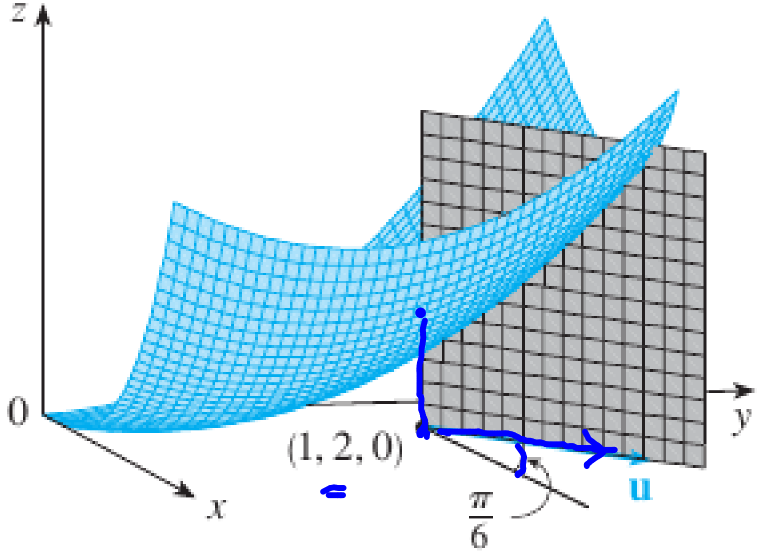
$$D_u f(x,y) = \lim_{h \to 0} \frac{f[(x,y) + h(x,y)] - f(x,y)}{h} = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

ok??

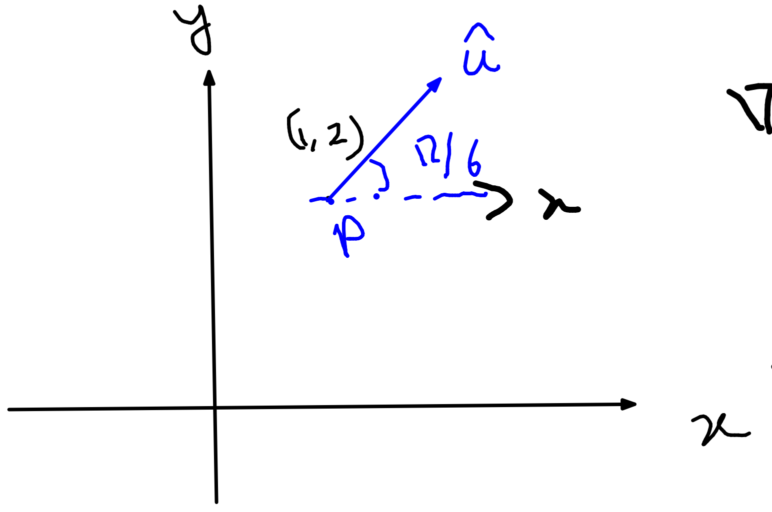
$$= \underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)}_{\text{gradient } \nabla f} \cdot \hat{u}$$

$$\boxed{D_u f(x,y) = \nabla f \cdot \hat{u}}$$

EXAMPLE 1 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?



$$D_u f(1,2) = \nabla f \cdot \hat{u}$$



$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= (3x^2 - 3y)\hat{i} + (-3x + 8y)\hat{j}$$

$$\nabla f(1,2) = -3\hat{i} + 13\hat{j}$$

$$\hat{u} = ?? = \cos\frac{\pi}{6}\hat{i} + \sin\frac{\pi}{6}\hat{j}$$

$$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$D_u f(1,2) = (-3\hat{i} + 13\hat{j}) \cdot \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$= (13 - 3\sqrt{3})/2$$

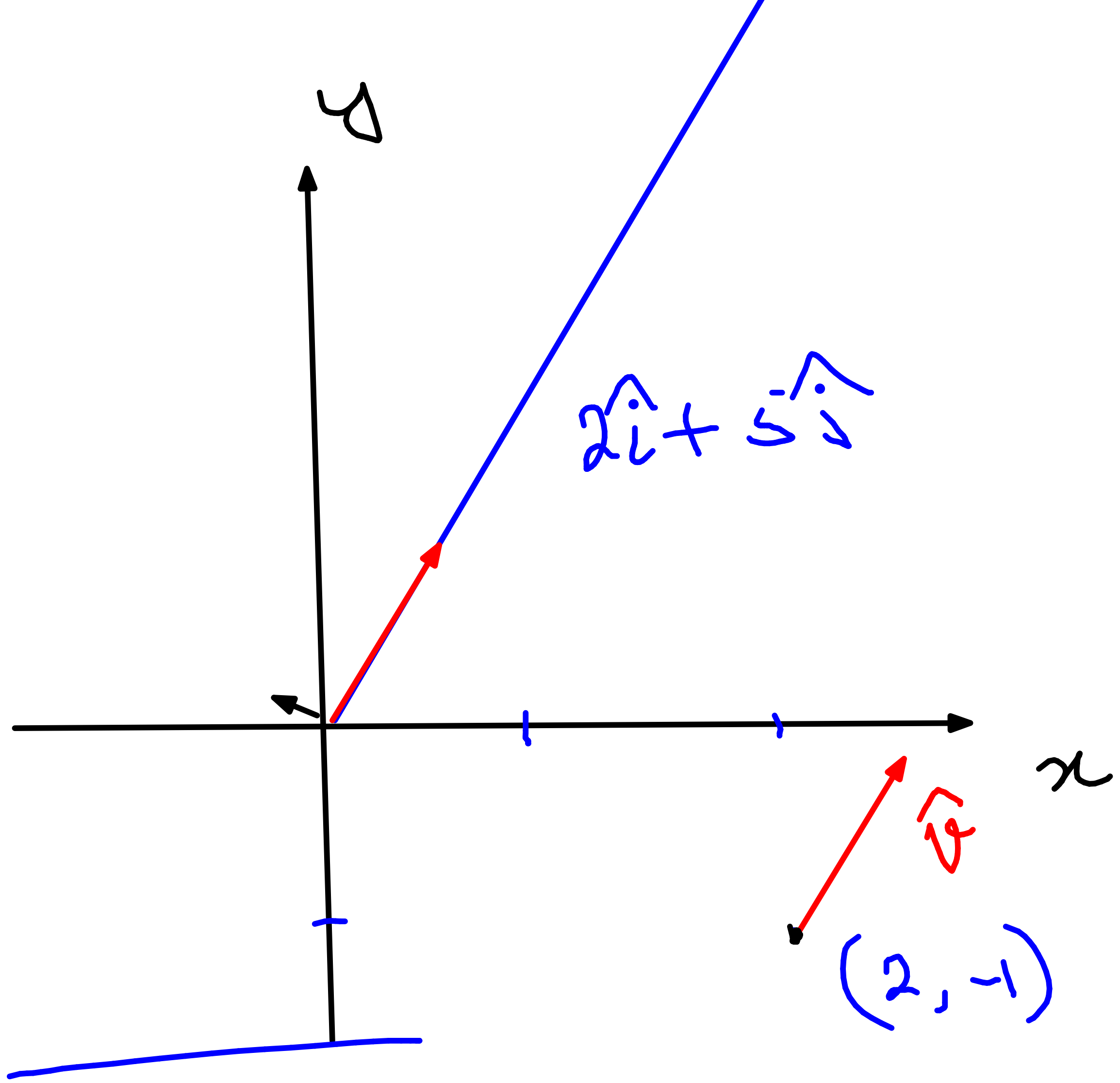
8 DEFINITION If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

V EXAMPLE 3 Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

$$\vec{v} = 2\hat{i} + 5\hat{j}$$

$$\hat{v} = \frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}$$



$$\boxed{D_{\hat{v}} f = \nabla f \cdot \hat{v}}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

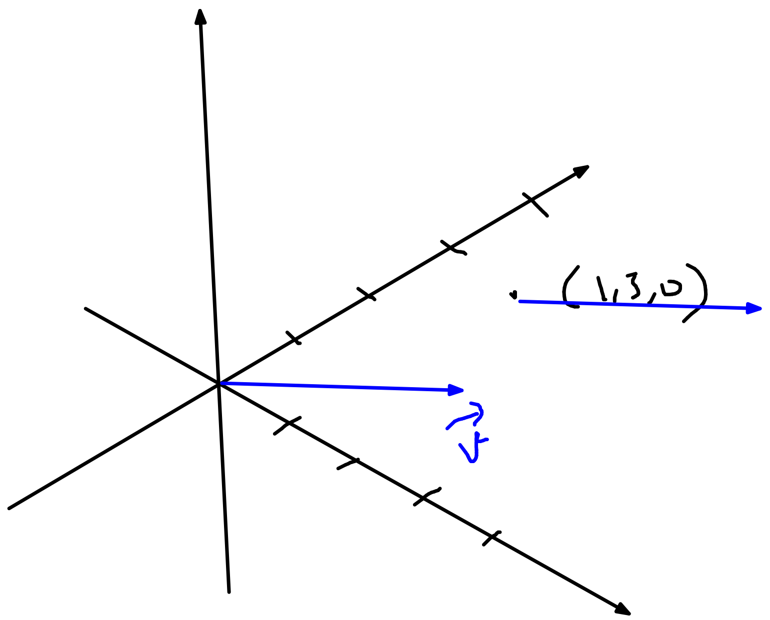
$$= \Big[2xy^3\,\hat{i} + (3x^2y^2 - 4)\,\hat{j}\Big]_{\substack{x=2\\ y=1}}$$

$$= ?? = -4\hat{i} + 8\hat{j}$$

$$D_{\hat{v}} f = \left(-4\hat{i} + 8\hat{j}\right) \cdot \left(\frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}\right) = \frac{32}{\sqrt{29}}$$

$$\text{length}\left(2\hat{i} + 5\hat{j}\right) = \sqrt{2^2 + 5^2}$$

EXAMPLE 4 If $f(x, y, z) = x \sin(yz)$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.



$$\left[\nabla f(1,3,0)\right] \cdot \hat{v}$$

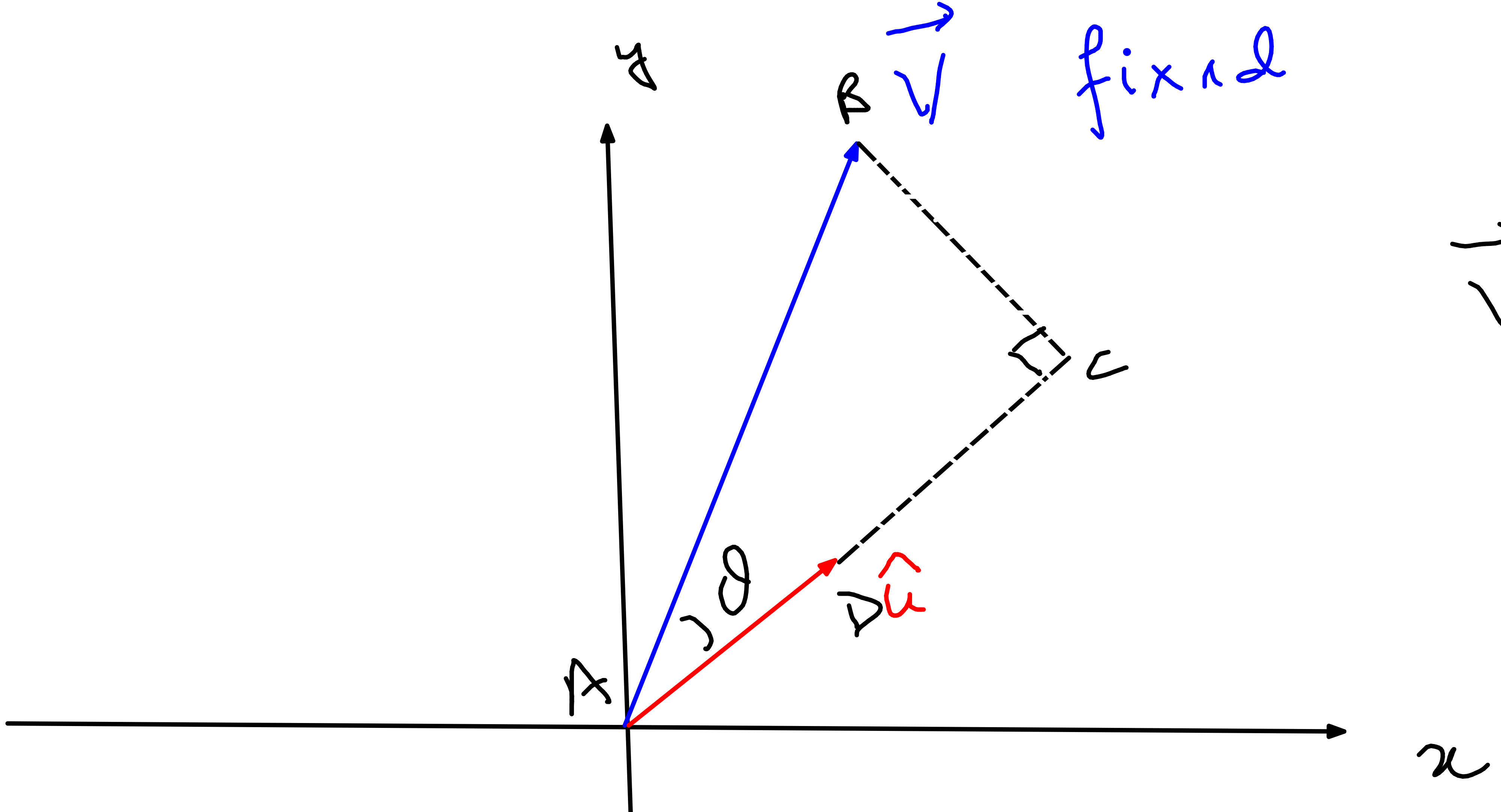
$$\hat{v} = \frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$$

$$\nabla f = \underbrace{\sin(y z)\,\hat{i} + xz\cos(yz)\,\hat{j} + xy\cos(yz)\,\hat{k}}_{x=1,\ y=3,\ z=0}$$

$$= 3\hat{k}$$

$$D_{\hat{v}} f(1,3,0) = (3\hat{k}) \cdot \left(\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}\right)$$

$$= \frac{-3}{\sqrt{6}}$$



$$|\vec{v}| = AB$$

$$\vec{v} \cdot \hat{u} = |\vec{v}|\,|\hat{u}| \cos\theta$$
$$= |\vec{v}| \cos\theta$$
$$= |AB| \cos\theta$$
$$= AC$$

Q: pick $\hat{u}$ in reference to $\vec{v}$

s.t. $\vec{v} \cdot \hat{u}$ is max

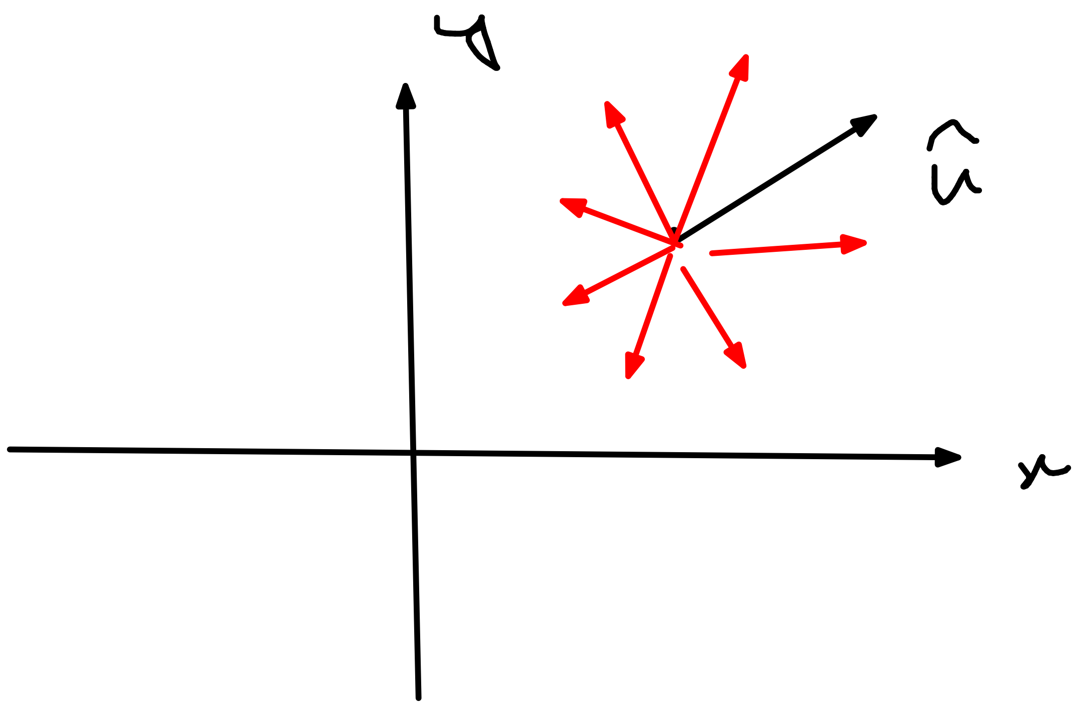
Ans: $\hat{u}$ is parallel to $\vec{v}$

i.e. $\hat{u} = \vec{v} / |\vec{v}|$

MAXIMIZING THE DIRECTIONAL DERIVATIVE

| Why Gradients are so famous ??

$f(x,y)$



Directional derivatives

: given a direction we can find out rate of change in that direction

: $\nabla f \cdot \hat{u}$

i.e find $\hat{u}_0$ s.t.

$$D_{\hat{u}_0} f \geq D_{\hat{u}} f \quad \text{for all } \hat{u}$$

Q. Imagine : $\vec{V}$ fixed vector , $\hat{u}$ we can choose

s.t $\vec{V} \cdot \hat{u}$ is as max as possible

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

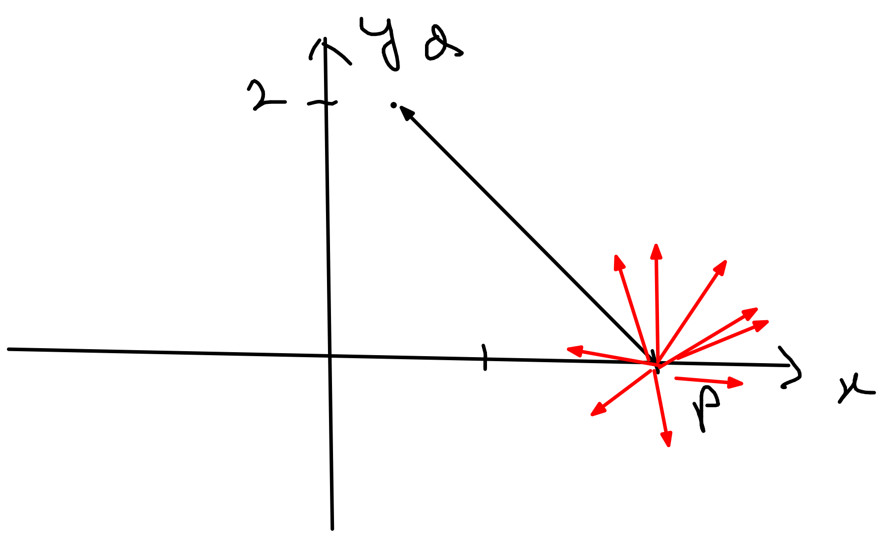
$\Rightarrow$ if we want to move in the direction of fastest increment, then what should be our $\hat{u}$??

Ans: ?? go along the gradient

EXAMPLE 5

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$. $\hat{u}$: given

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?



a)

direction $\vec{u} = \vec{Q} - \vec{P}$

$= -\frac{3}{2}\hat{i} + 2\hat{j}$

$\hat{u} = ??$

$\nabla f(2,0) = \left[e^y\hat{i} + (xe^y)\hat{j}\right]_{\substack{x=2 \\ y=0}}$

$= \hat{i} + 2\hat{j}$

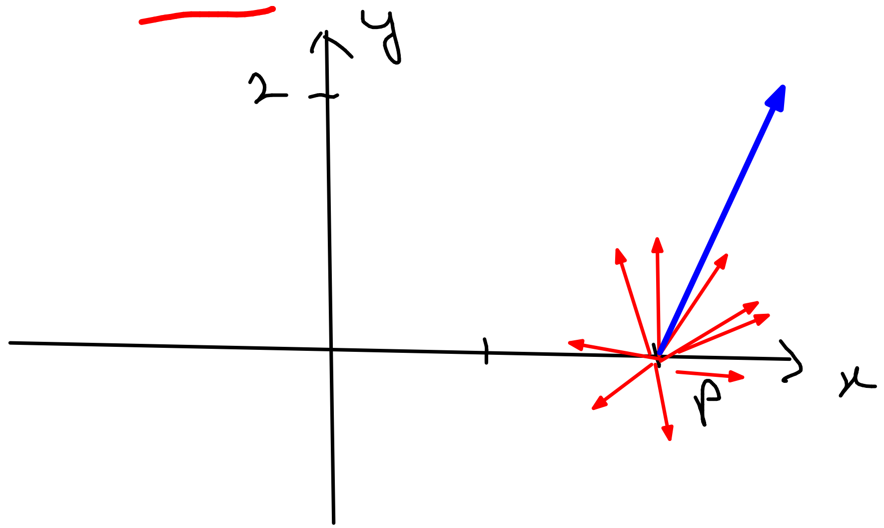
$(\hat{i} + 2\hat{j}) \cdot \left(-\frac{3}{2}\hat{i} + 2\hat{j}\right) \Big/ \sqrt{\frac{9}{4} + 4} = 1$

EXAMPLE 5

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$. $\hat{u}$: given

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

$\vec{v} \cdot \dfrac{\vec{v}}{|\vec{v}|} =$



(b)

$$\nabla f(2,0) = \left[e^y \hat{i} + (xe^y)\hat{j}\right]_{\substack{x=2\\y=0}}$$

$$= \hat{i} + 2\hat{j}$$

Direction of max rate of change is along $\nabla f = \dfrac{\nabla f}{|\nabla f|}$

$$\rightarrow \nabla f \cdot \left(\frac{\nabla f}{|\nabla f|}\right) = |\nabla f| = |\hat{i} + 2\hat{j}| = \sqrt{5}$$

$$\vec{v} \cdot \underbrace{\left(\frac{\vec{v}}{|\vec{v}|} \right)}_{\text{unit vector}} = ?? = |\vec{v}|$$

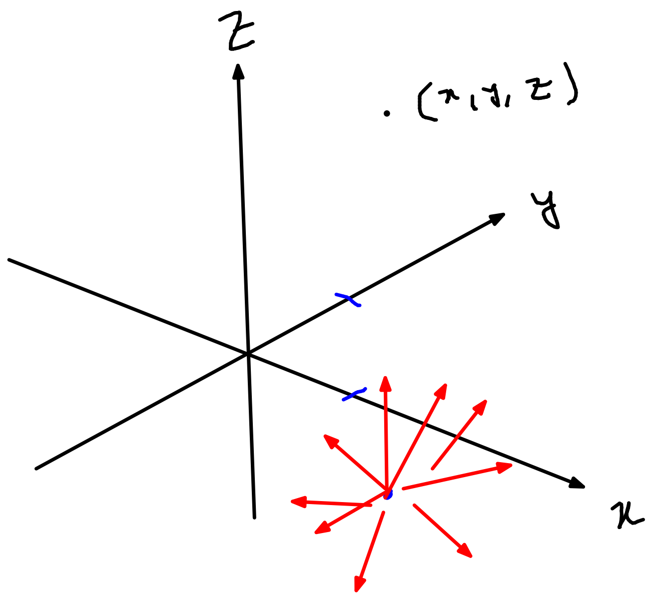
$$\longrightarrow \quad \vec{v} = a\hat{i} + b\hat{j}$$

$$\vec{v} \cdot \left(\frac{\vec{v}}{|\vec{v}|} \right) = (a\hat{i} + b\hat{j}) \cdot \left(\frac{a\hat{i} + b\hat{j}}{\sqrt{a^2 + b^2}} \right)$$

$$= \frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$$

$$= |\vec{v}|$$

EXAMPLE 6 Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?



direction of max increase rate

$$= \nabla T(1,1,-2) = \frac{5}{8}(-\hat{i} - 2\hat{j} + 6\hat{k})$$

max rate of change

$$= |\nabla T(1,1,-2)|$$

$$= \frac{5}{8}\sqrt{41}$$