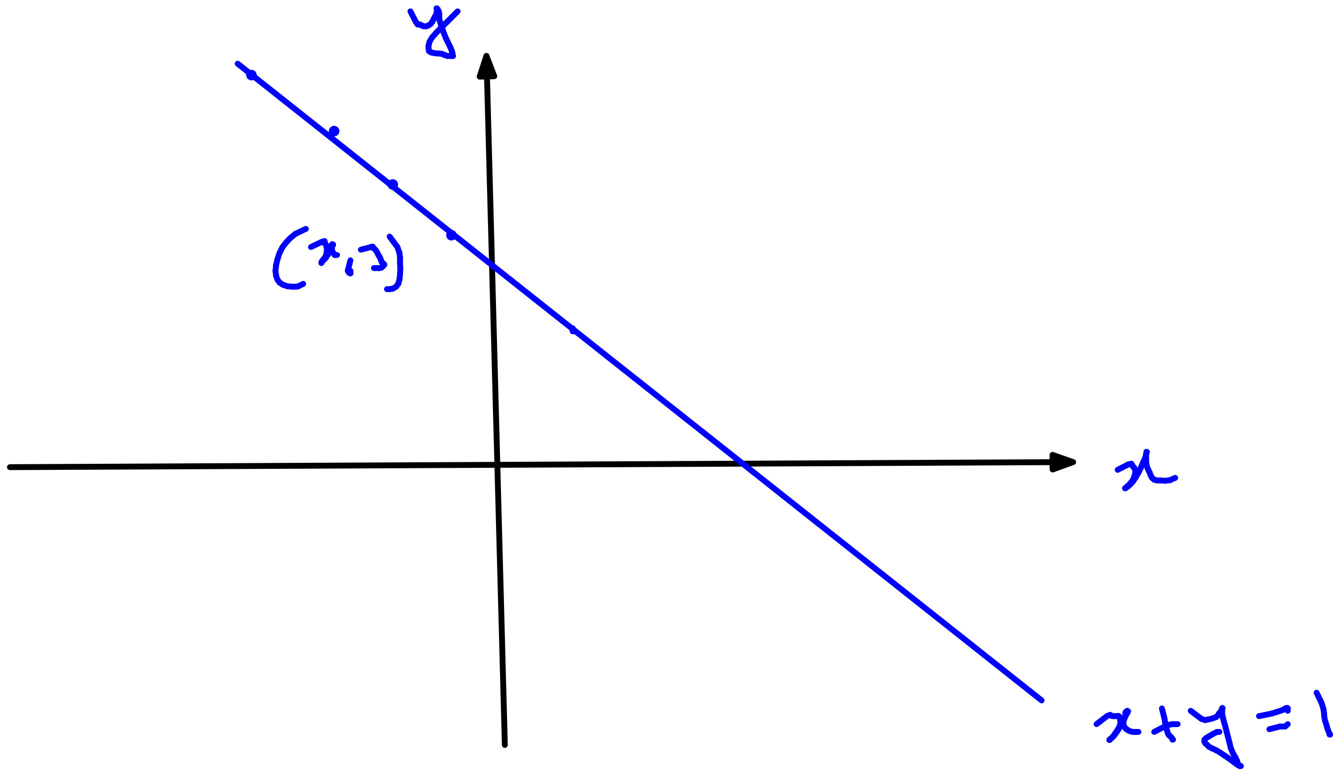
14.8

LAGRANGE MULTIPLIERS

Objective function: $f(x,y) = x^2+y^2$ $(0,0)$

s.t. $[\underbrace{x+y}_{g} = 1]$



$x+y=1$

$\nabla f = \lambda \nabla g$

$$\boxed{\begin{aligned} x+y &= 1 \\ 2x &= \lambda \\ 2y &= \lambda \end{aligned}}$$

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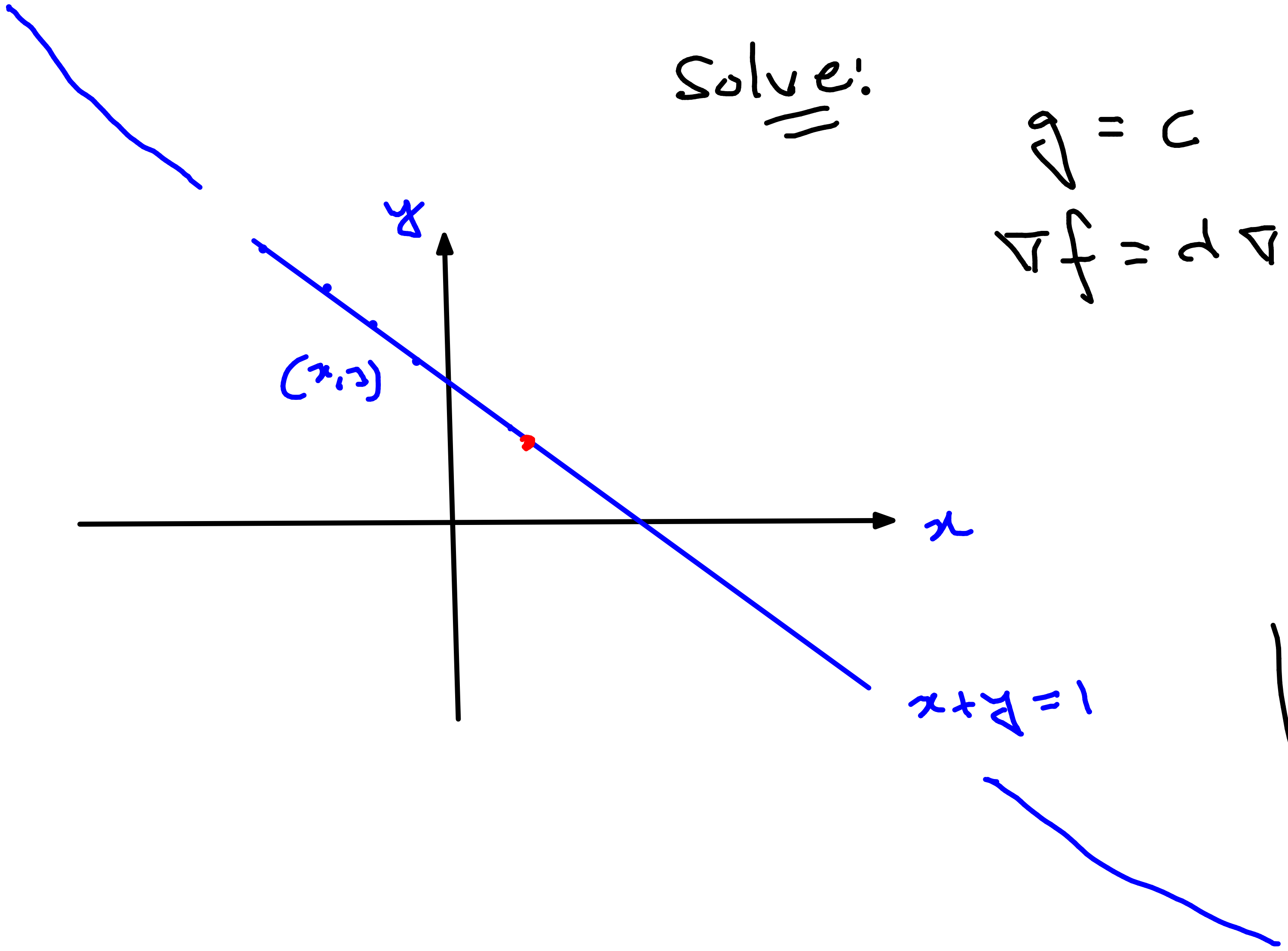


Solve:

$g = c$

$\nabla f = \lambda \nabla g$

$x+y=1$

$2x = \lambda(1)$

$2y = \lambda(1)$

$x+y=1$

$2x=\lambda$

$2y=\lambda$

$\lambda=1,\ x=0.5,\ y=0.5$

λ: f takes max or min at $(0.5, 0.5)$ s.t. $x+y=1$

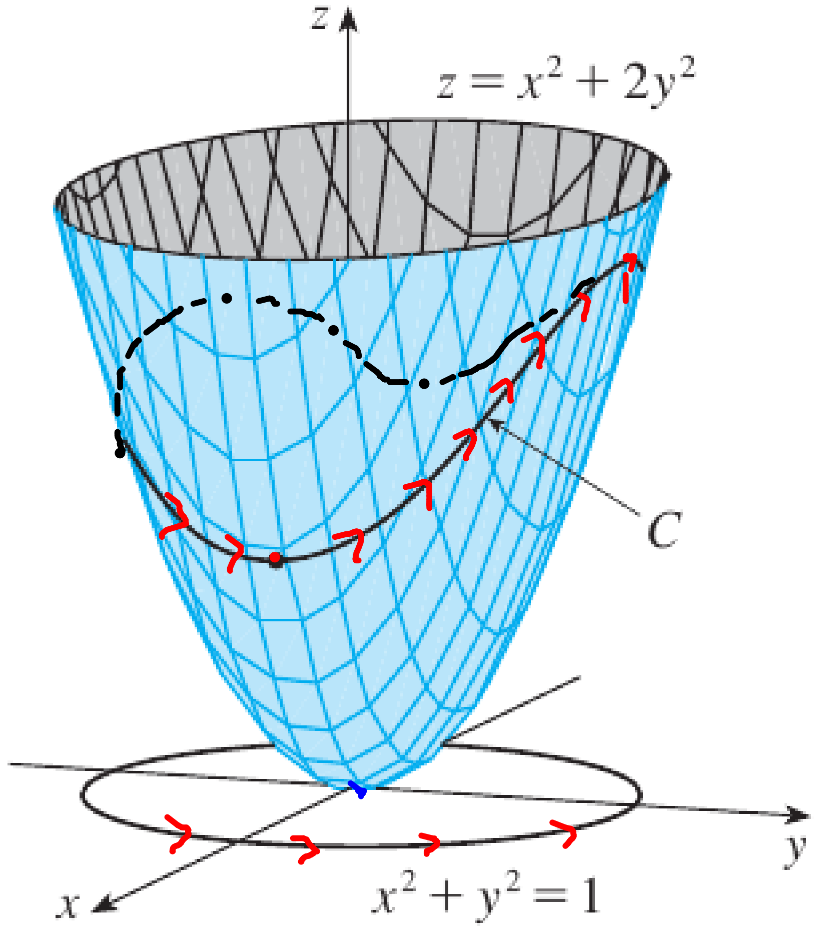
$\to$ min at $(0.5, 0.5)$

& no max

V EXAMPLE 2 Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

maximize & minimize $f = x^2 + 2y^2$

s.t. $x^2 + y^2 = 1$



(x,y)	λ	f	
$(1,0)$	1	1	– min
$(-1,0)$	1	1	– min
$(0,1)$	2	2	– max
$(0,-1)$	2	2	– max

$g = c$

$\nabla f = \lambda \nabla g$

$$\left.\begin{array}{l} x^2+y^2=1 \\ 2x = \lambda 2x \\ 4y = \lambda 2y \end{array}\right\} \Rightarrow \begin{array}{l} x^2+y^2=1 \\ x(1-\lambda)=0 \\ y(2-\lambda)=0 \end{array}$$

lets start with: $x(1-\lambda) = 0$

if $x \neq 0$:

$\lambda = 1$

$y = 0$

$x^2 = 1$

$x = \pm 1$

if $x = 0$:

??

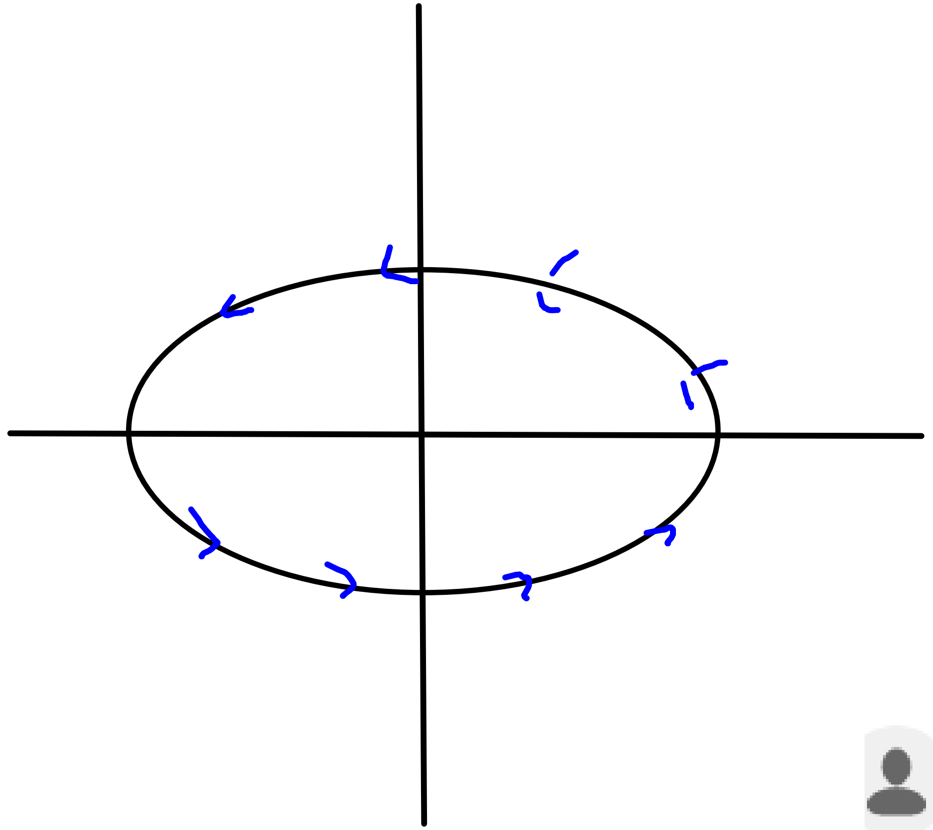
$y^2 = 1$, $y = \pm 1$

$\lambda = 2$

Q:

maximize & minimize $f(x,y) = x^2 y$

s.t. $x^2 + 2y^2 = 6$



$g = c$

$\nabla f = \lambda \nabla g$

$$\left.\begin{aligned} x^2 + 2y^2 &= 6 \\ 2xy &= \lambda 2x \\ x^2 &= \lambda 4y \end{aligned}\right\}$$

H.W. solve & check if

CHANDRAMOULI BHATTACHA... 6:53 PM

sir i am getting (0,0)(2,1)(-2,-1) as points

SHREEYA MITTAL 6:58 PM

sir fmax and fmin are 16 and -16?

yes sir

Lagrange Multipliers:

maximize/minimize $f(x,y)$

s.t. $g(x,y) = c$ (constant)

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Solve: 3x3 eq^n

$$g(x,y) = c$$

2 eqns.: $\left[\nabla f = \lambda \nabla g\right.$

$$\boxed{\begin{aligned} g &= c \\ \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \end{aligned}}$$

3 eqns.

& 3 unknowns: (x, y, λ)

Recall: 14.8 Lagrange Multipliers:

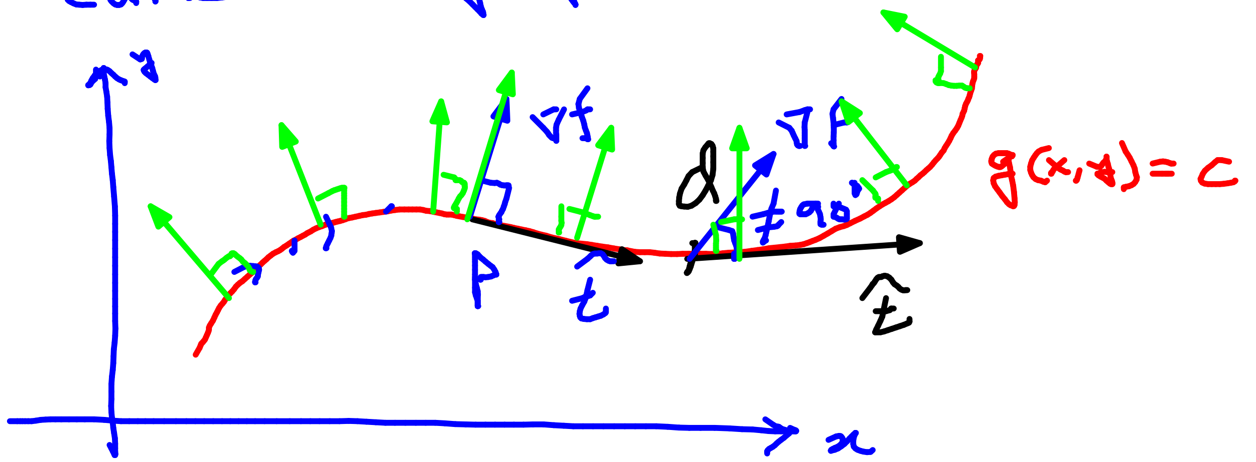
$$\boxed{\begin{array}{l}\text{maximize/minimize } f(x,y) \\ \text{s.t. } \quad g(x,y) = c\end{array}}$$

$$\left.\begin{array}{l} g(x,y) = c \\ \nabla f = \lambda \nabla g \end{array}\right] \quad \begin{array}{l}\text{3 eq}^{\text{n}}\text{s} \\ \text{\& 3 unknowns}\end{array}$$

Today: Why Lagrange Multiplier Works.

$$\boxed{\begin{array}{ll} \text{maximize} & f(x,y) \\ \text{s.t.} & g(x,y) = c \end{array}}$$

Typically $g(x,y) = c$ is a curve in xy plane



∇g always $\perp$ to g

at max point P

∇f is also $\perp$ to g

$$\nabla f = \lambda \nabla g$$

say: max occurs at $\underline{\underline{P}}$

we will see that $\boxed{\nabla f(P) = \lambda \nabla g(P)}$

Q.① $\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j}$ = vector ??

Q.② Recall directional derivatives:

Q: a random point on the curve

$\hat{t}$: unit tangential vector at Q

what will be the rate of change of f at Q along the curve ??

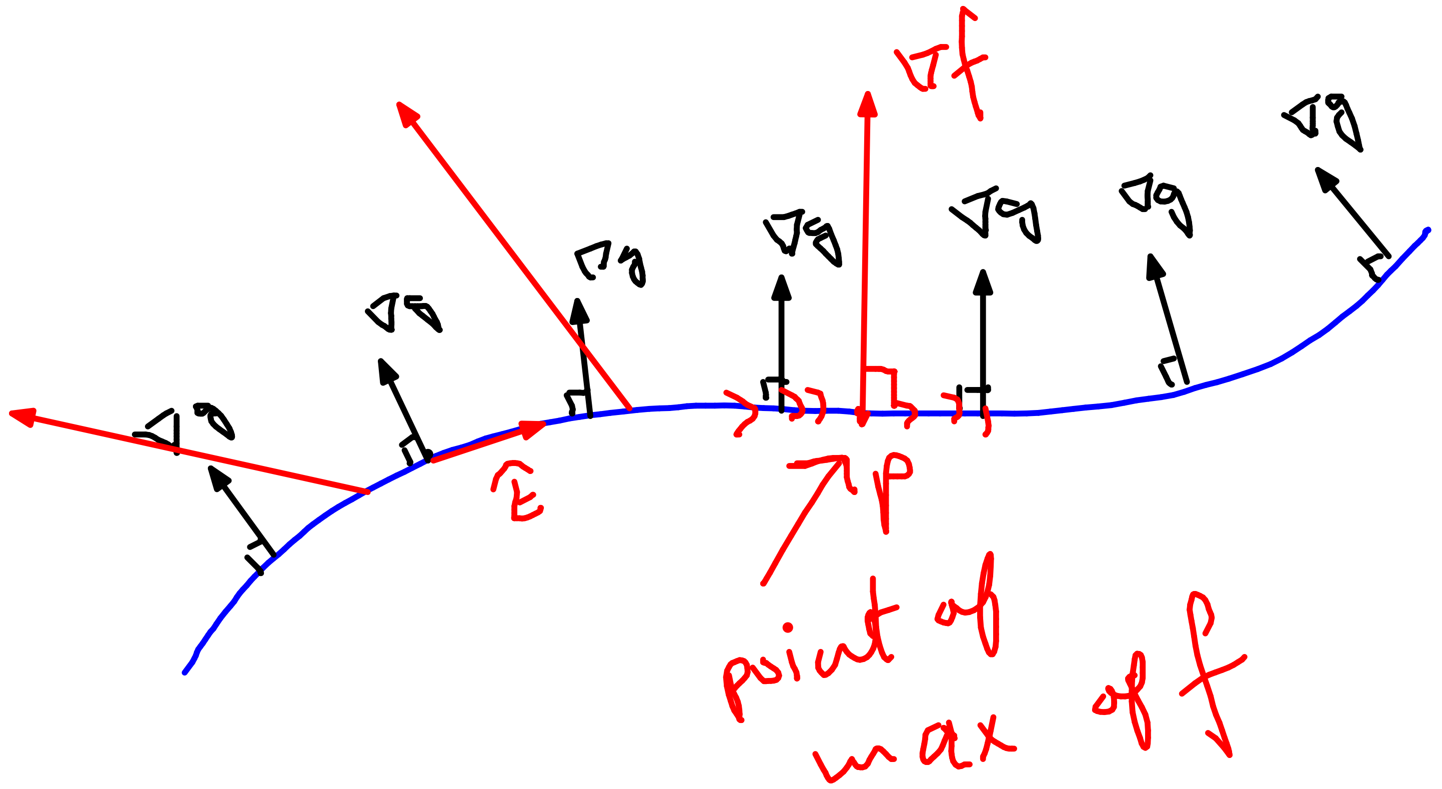
$= D_{\hat{t}} f(Q) = \nabla f \cdot \hat{t}$

component of ∇f along $\hat{t}$

Q③ at P, what will be the rate of change of f along the curve ?? 0 ?? $\nabla f(P) \cdot \hat{t} = 0$

Q③ Take any point on the curve. What will be direction of ∇g?

g is constant on the curve. $D_{\hat{t}}\, g(\text{any point}) = 0 = \nabla g \cdot \hat{t}$



$f(x,y)$: where $f(x,y)$ is max on $g(x,y) = c$

$g(x,y) = c$ | $\rightarrow$ ∇g is always $\perp$ to curve $g = c$

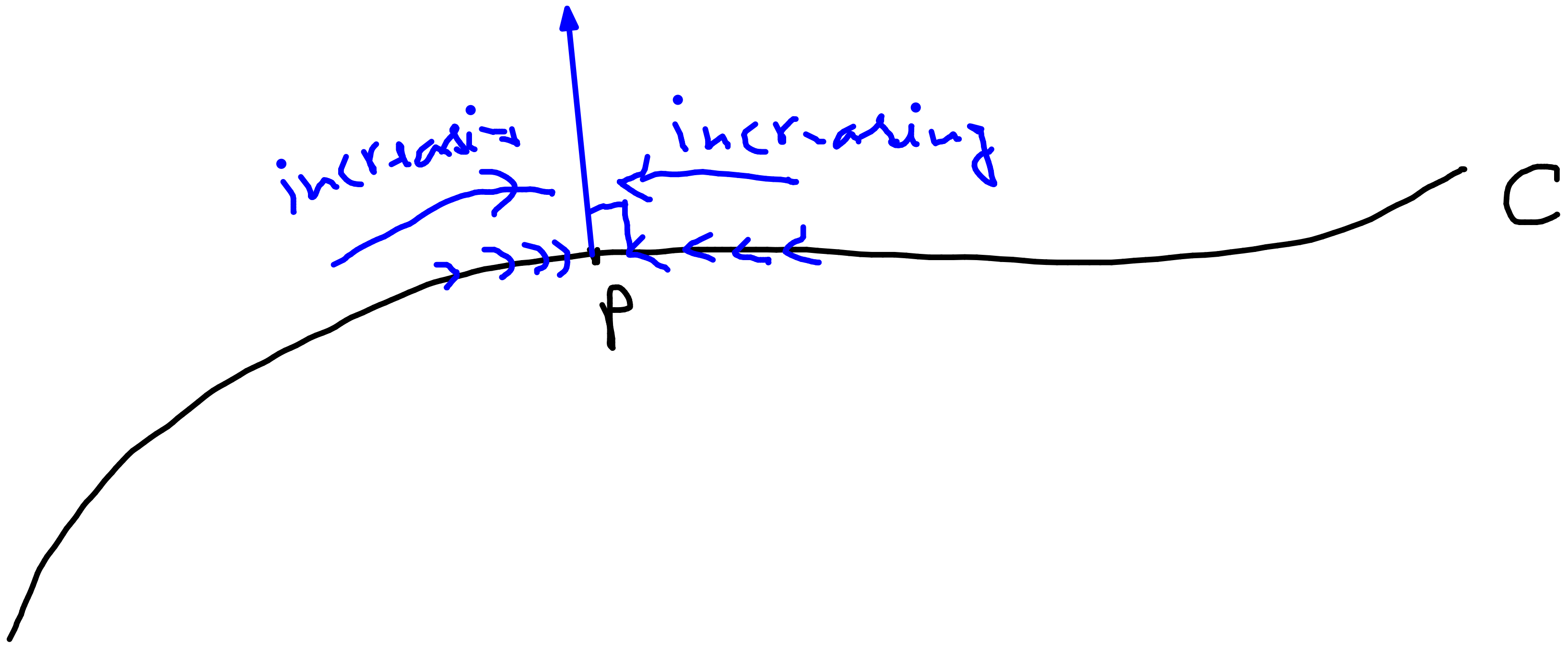
why: $\nabla g \cdot \hat{t} = 0$

$\rightarrow$ at point of max

$\nabla f \cdot \hat{t} = 0$

i.e. at p

$$\boxed{\nabla f = \lambda \nabla g}$$



→ $f(x,y)$ along C

→ f has a local max along the curve at P

→ $\nabla f(P) \perp$ curve

14.8 Continued: Lagrange Multipliers

maximize $f(x,y)$

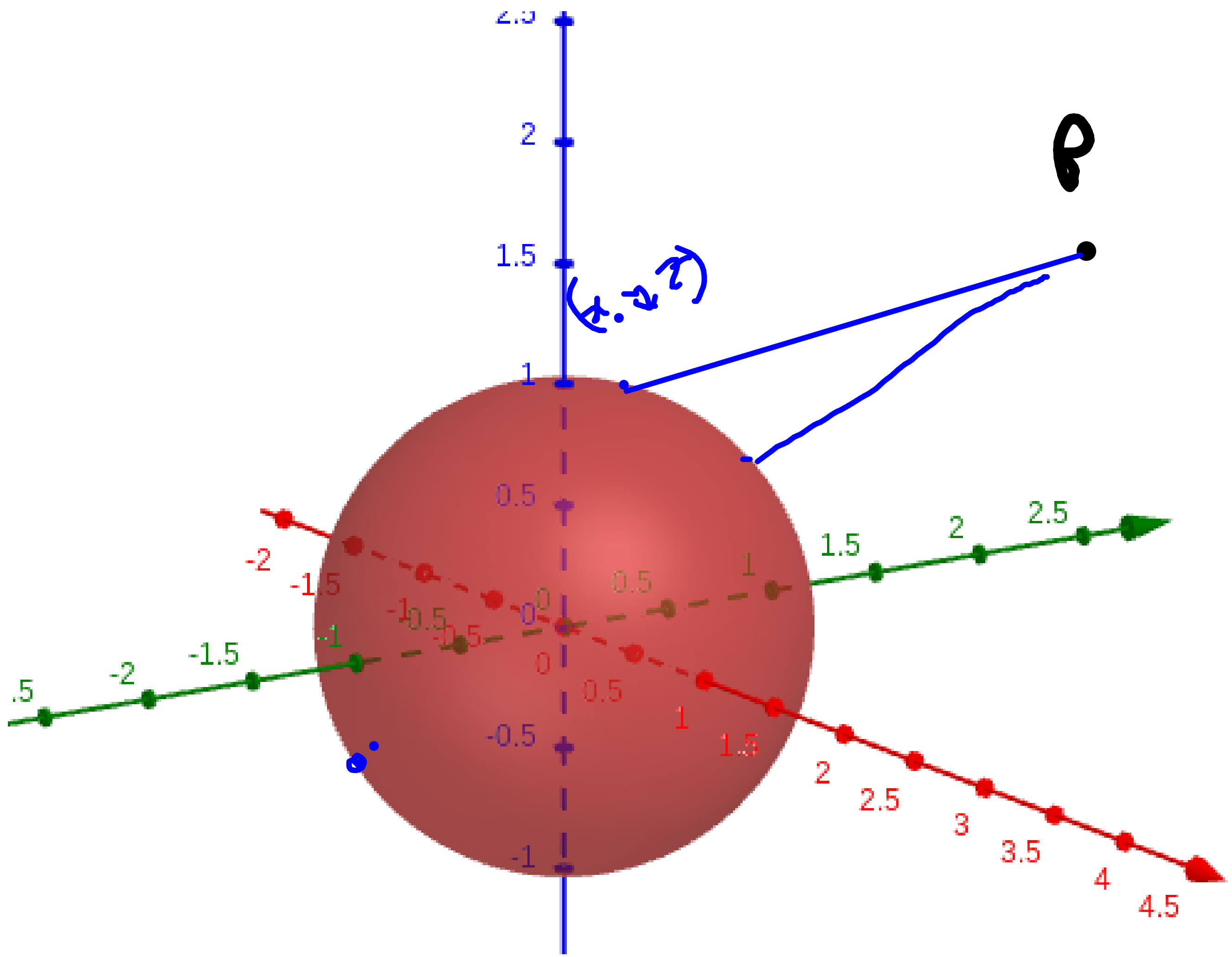
s.t. $g(x,y) = c$

$\Longleftrightarrow$

solve $g(x,y) = c$

$\nabla f = \lambda \nabla g$

Q: $x^2+y^2+z^2=1$ & Point $P(5,2,3)$



d: find nearest & farthest point on the sphere from P.

$\Leftrightarrow$

maximize & minimize $f(x,y,z) = (x-5)^2 + (y-2)^2 + (z-3)^2$

s.t. $\underbrace{x^2+y^2+z^2}_{=g(x,y,z)} = 1$

$\Leftrightarrow$ Lagrange multipliers

$x^2+y^2+z^2=1$

$\nabla f = \lambda \nabla g$

$x^2+y^2+z^2=1$

$2(x-5) = \lambda 2x$

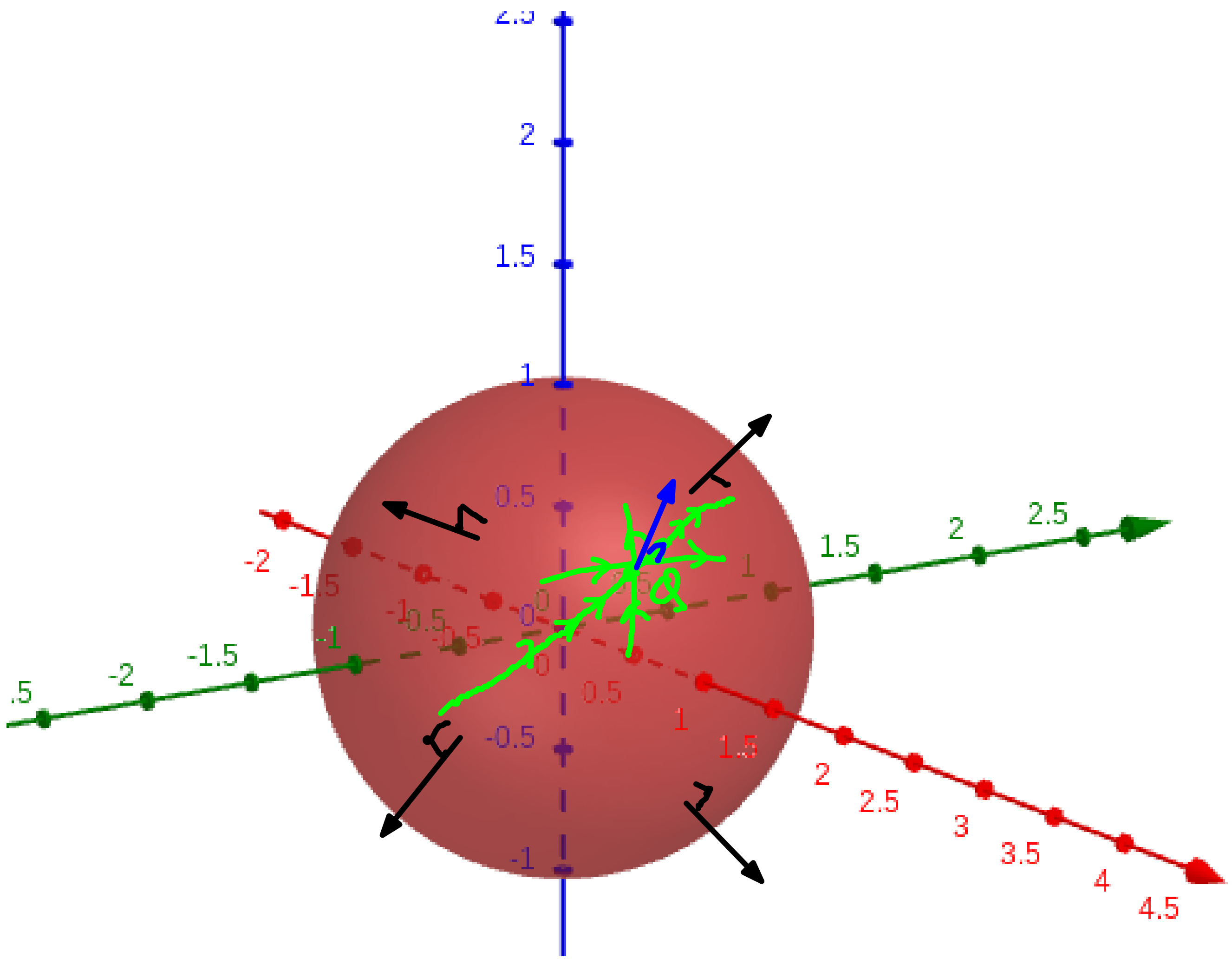
$2(y-5) = \lambda 2y$

$2(z-3) = \lambda 2z$

Q. $x^2+y^2+z^2=1$ & Point $P(5,2,3)$

maximize $f(x,y,z) = (x-5)^2+(y-2)^2+(z-3)^2$

s.t. $\underbrace{x^2+y^2+z^2}_{g} = 1$



Q: If f maxes at a point Q on the sphere

$$\nabla f = \lambda \nabla g$$

Q. The direction of ∇g is always $\perp$ to the surface

Q. At max point Q, ∇f is ALSO $\perp$ to surface

$\Rightarrow$ at max/min points $[\nabla f = \lambda \nabla g]$

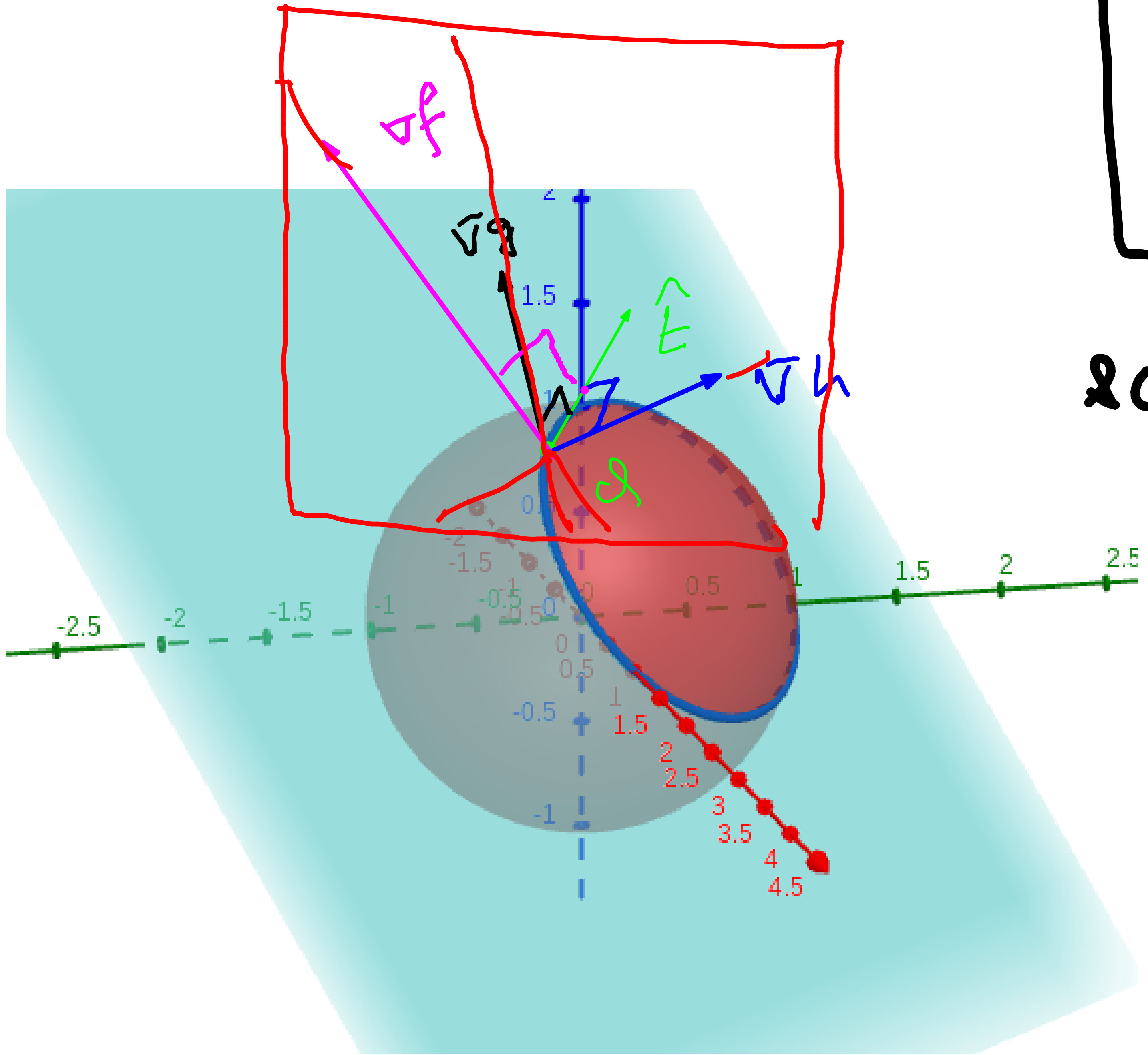
Q. imagine $x^2+y^2+z^2=1$ & Point $P(5,2,3)$

intersects with $x+y+z=1$

$$\boxed{\begin{array}{ll} \text{maximize} & f(x,y,z) = (x-5)^2+(y-2)^2+(z-3)^2 \\ \text{\& minimize} & \\ \text{s.t.} & g\ [x^2+y^2+z^2=1 \\ & h\ [x+y+z=1 \end{array}}$$

IIIly: at max/min points

∇f will be $\perp$ to the circle



max $f(x,y,z) =$ ______

s.t. $g(x,y,z) = c_1$] sphere

& $h(x,y,z) = c_2$] plane

soy: this max occurs at $\mathcal{Q}$.

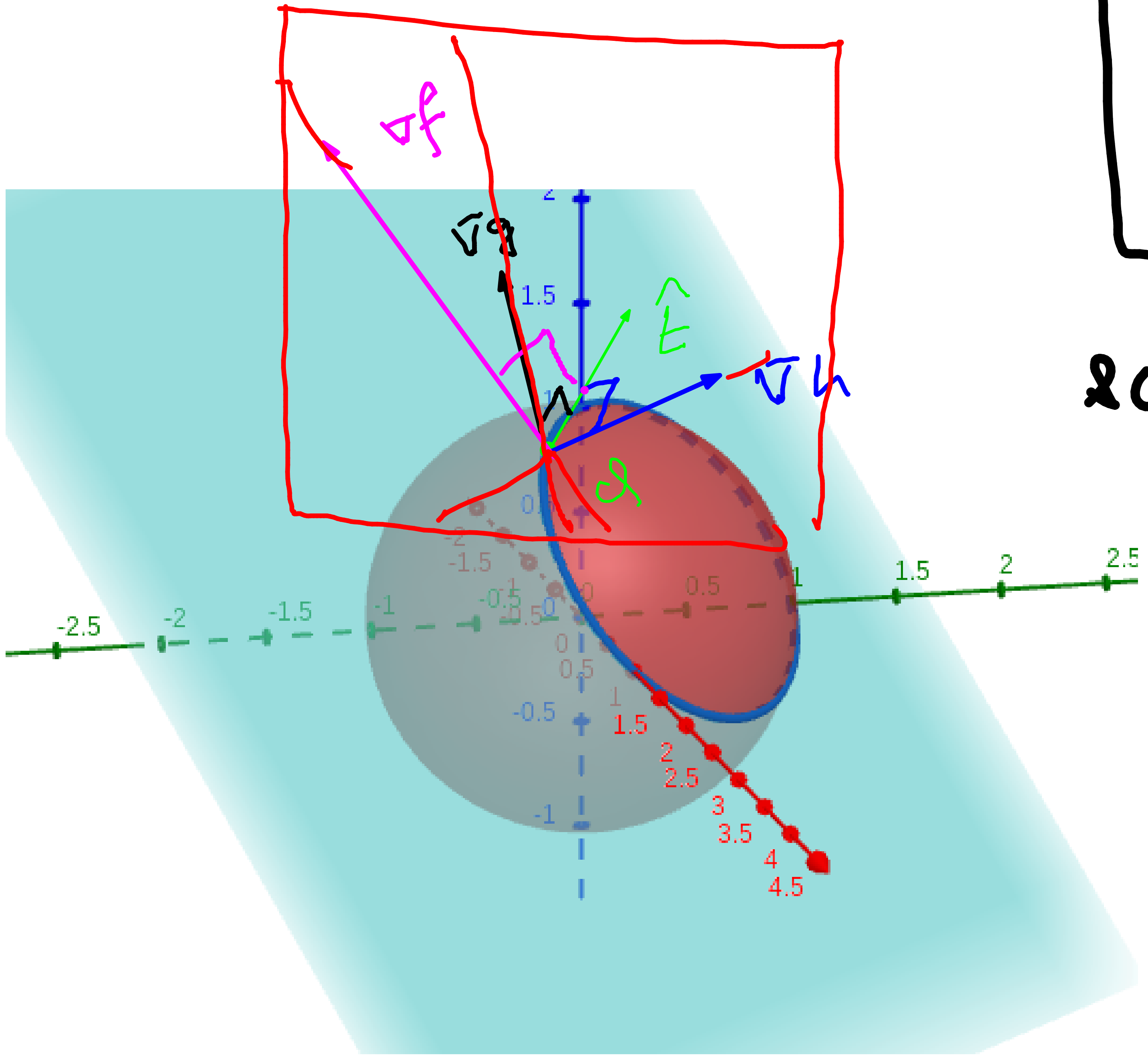
note ① $\nabla f \perp \hat{t}$ at $\mathcal{Q}$

note ② $\nabla g \perp \hat{t}$ at $\mathcal{Q}$. why?

note ③ $\nabla h \perp \hat{t}$ at $\mathcal{Q}$. why?!

$\Rightarrow$ $\nabla f, \nabla g, \nabla h$ are in same plane

$\therefore$ $\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$



$\max \quad f(x,y,z) = \underline{\qquad}$

s.t. $\quad g(x,y,z) = c_1$] sphere

& $\quad h(x,y,z) = c_2$] plane

say: this max occurs at Q.

$$g(x,y,z) = c_1$$

$$h(x,y,z) = c_2$$

$$\therefore \boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

Q. imagine $x^2+y^2+z^2=1$ & Point $P(5,2,3)$

intersects with

$x+y+z=1$

maximize & minimize $f(x,y,z) = (x-5)^2+(y-2)^2+(z-3)^2$

s.t. $g\left[x^2+y^2+z^2=1\right.$

$h\left[x+y+z=1\right.$

Lagrange multiplier

$$\boxed{\begin{aligned} &x^2+y^2+z^2=1 \\ &x+y+z=1 \\ &\nabla f = \lambda \nabla g + \mu \nabla h \end{aligned}}$$