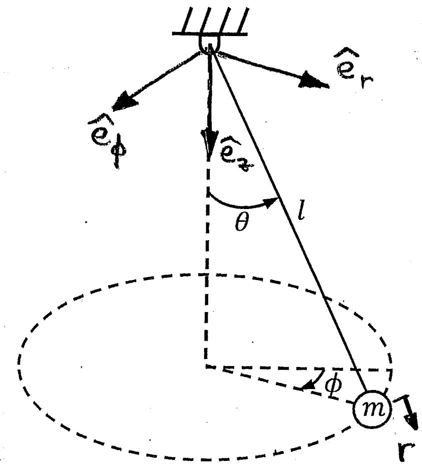
1) $n = 2$

$m = 3 \Rightarrow q_1 = \theta,\ q_2 = \phi,\ q_3 = r$




$$\vec{\omega} = \dot{\phi}\,\hat{e}_z$$

$$\vec{r} = (l+r)\sin\theta\,\hat{e}_r + (l+r)\cos\theta\,\hat{e}_z$$

$$(\dot{\vec{r}})_{rel} = \left[(l+r)\dot{\theta}\cos\theta + \dot{r}\sin\theta\right]\hat{e}_r$$
$$+ \left[-(l+r)\dot{\theta}\sin\theta + \dot{r}\cos\theta\right]\hat{e}_z$$

$$\vec{\omega}\times\vec{r} = (l+r)\dot{\phi}\sin\theta\,\hat{e}_\phi$$

$$\dot{\vec{r}} = (\dot{\vec{r}})_{rel} + \vec{\omega}\times r$$

$$\dot{\vec{r}} = \left[(l+r)\dot{\theta}\cos\theta + \dot{r}\sin\theta\right]\hat{e}_r + (l+r)\dot{\phi}\sin\theta\,\hat{e}_\phi$$
$$+ \left[-(l+r)\dot{\theta}\sin\theta + \dot{r}\cos\theta\right]\hat{e}_z$$

$$V = -mg(l+r)\cos\theta$$

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}} + mg(l+r)\cos\theta$$

$$L = \frac{m}{2}\Big[(l+r)^2\dot{\theta}^2\cos^2\theta + 2(l+r)\dot{r}\dot{\theta}\sin\theta\cos\theta + \dot{r}^2\sin^2\theta$$
$$+ (l+r)^2\dot{\phi}^2\sin^2\theta + (l+r)^2\dot{\theta}^2\sin^2\theta$$
$$- 2(l+r)\dot{r}\dot{\theta}\sin\theta\cos\theta + \dot{r}^2\cos^2\theta\Big]$$
$$+ mg(l+r)\cos\theta$$

$$L = \frac{m}{2}\left[(l+r)^2\dot{\theta}^2 + (l+r)^2\dot{\phi}^2\sin^2\theta + \dot{r}^2\right] + mg(l+r)\cos\theta$$

There is one kinematic constraint $(p=1)$:

$$j=1: \quad r=0 \Rightarrow \dot{r}=0 \Rightarrow \ddot{r}=0$$
$$\Downarrow$$
$$a_{11}=0, \quad a_{12}=0, \quad a_{13}=1$$

<u>$k=1 \ (q_1=\theta)$</u>

$$\frac{\partial L}{\partial \dot{\theta}} = m(\ell+r)^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m(\ell+r)^2\ddot{\theta} + 2m(\ell+r)\dot{r}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = m(\ell+r)^2\dot{\phi}^2 \sin\theta\cos\theta - mg(\ell+r)\sin\theta$$

$$m(\ell+r)^2\ddot{\theta} + 2m(\ell+r)\dot{r}\dot{\theta} - m(\ell+r)^2\dot{\phi}^2\sin\theta\cos\theta + mg(\ell+r)\sin\theta = \lambda_1 \cancelto{0}{a_{11}}$$

Let $r=\dot{r}=\ddot{r}=0$

$$\boxed{\begin{gathered} m\ell^2\ddot{\theta} - m\ell^2\dot{\phi}^2\sin\theta\cos\theta + mg\ell\sin\theta = 0 \\ \text{or} \\ \ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta + \frac{g}{\ell}\sin\theta = 0 \end{gathered}}$$

<u>$k=2 \ (q_2=\phi)$</u>

$$\frac{\partial L}{\partial \dot{\phi}} = m(\ell+r)^2\dot{\phi}\sin^2\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = m(\ell+r)^2\ddot{\phi}\sin^2\theta + 2m(\ell+r)\dot{r}\dot{\phi}\sin^2\theta + 2m(\ell+r)^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$m\ell^2\ddot{\phi}\sin^2\theta + 2m\ell^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = \lambda_1 \cancelto{0}{a_{12}}$$

$$\boxed{\begin{gathered} m\ell^2\ddot{\phi}\sin^2\theta + 2m\ell^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0 \\ \text{or} \\ \ddot{\phi}\sin^2\theta + 2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0 \end{gathered}}$$

$\underline{k=3\ (q_3=r)}$

$$\frac{\partial L}{\partial \dot r} = m\dot r \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = m\ddot r$$

$$\frac{\partial L}{\partial r} = m(\ell+r)\dot\theta^2 + m(\ell+r)\dot\phi^2\sin^2\theta + mg\cos\theta$$

$$m\ddot r - m(\ell+r)\dot\theta^2 - m(\ell+r)\dot\phi^2\sin^2\theta - mg\cos\theta = \overbrace{\lambda_1 a_{13}}^{C_3} = \overbrace{\lambda_1}^{C_3}$$

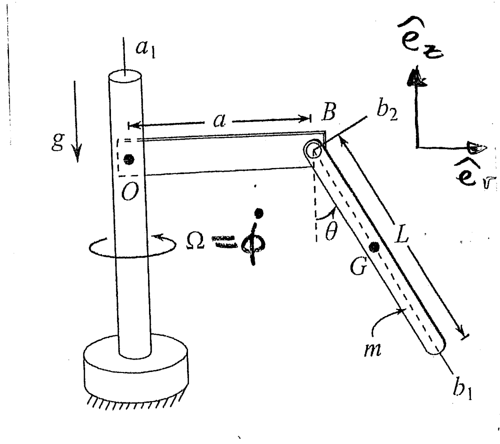
let $r = \dot r = \ddot r = 0$

$$\boxed{C_3 = -m\ell\left(\dot\theta^2 + \dot\phi^2\sin^2\theta\right) - mg\cos\theta}$$

2)

$N = 2$

$n = 2 \Rightarrow q_1 = \phi,\ q_2 = \theta$




$V = -\frac{mgL}{2}\cos\theta$

$\vec{r}_G = (a + \frac{L}{2}\sin\theta)\hat{e}_r - \frac{L}{2}\cos\theta\,\hat{e}_z$

$\dot{\vec{r}}_G = \frac{L}{2}\dot{\theta}\cos\theta\,\hat{e}_r + (a + \frac{L}{2}\sin\theta)\dot{\phi}\,\hat{e}_\phi$
$\quad + \frac{L}{2}\dot{\theta}\sin\theta\,\hat{e}_z$

$\dot{\vec{r}}_G \cdot \dot{\vec{r}}_G = \frac{L^2}{4}\dot{\theta}^2\cos^2\theta + (a + \frac{L}{2}\sin\theta)^2\dot{\phi}^2 + \frac{L^2}{4}\dot{\theta}^2\sin^2\theta$

$\dot{\vec{r}}_G \cdot \dot{\vec{r}}_G = \frac{L^2}{4}\dot{\theta}^2 + (a + \frac{L}{2}\sin\theta)^2\dot{\phi}^2$

For the swinging arm:

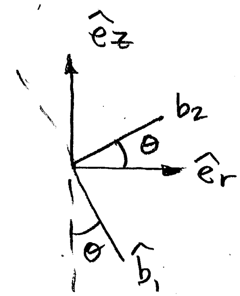
$T_{trans} = \frac{1}{2}m\dot{\vec{r}}_G \cdot \dot{\vec{r}}_G = \frac{mL^2}{8}\dot{\theta}^2 + \frac{m}{2}(a + \frac{L}{2}\sin\theta)^2\dot{\phi}^2$

$T_{rot} = \frac{1}{2}\{\omega\}^T[I_G]\{\omega\}$ (using the $b_1\, b_2\, b_3$ frame)

Where $\vec{\omega} = \dot{\phi}\,\hat{e}_z + \dot{\theta}\,\hat{b}_3$

$\vec{\omega} = \dot{\phi}(-\cos\theta\,\hat{b}_1 + \sin\theta\,\hat{b}_2) + \dot{\theta}\,\hat{b}_3$

$\vec{\omega} = -\dot{\phi}\cos\theta\,\hat{b}_1 + \dot{\phi}\sin\theta\,\hat{b}_2 + \dot{\theta}\,\hat{b}_3$




$$\{\omega\} = \begin{bmatrix} -\dot{\phi}\cos\theta \\ \dot{\phi}\sin\theta \\ \dot{\theta} \end{bmatrix}$$

$$[I_G] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix} = \frac{mL^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{rot} = \frac{1}{2}\frac{mL^2}{12}\begin{bmatrix} -\dot{\phi}\cos\theta & \dot{\phi}\sin\theta & \dot{\theta} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\dot{\phi}\cos\theta \\ \dot{\phi}\sin\theta \\ \dot{\theta} \end{bmatrix}$$

$$T_{rot} = \frac{mL^2}{24}\begin{bmatrix}-\dot\phi\cos\theta & \dot\phi\sin\theta & \dot\theta\end{bmatrix}\begin{bmatrix}0\\ \dot\phi\sin\theta\\ \dot\theta\end{bmatrix}$$

$$T_{rot} = \frac{mL^2}{24}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right)$$

$$T_{arm} = T_{trans} + T_{rot}$$

$$= \frac{mL^2}{8}\dot\theta^2 + \frac{m}{2}\left(a+\frac{L}{2}\sin\theta\right)^2\dot\phi^2 + \frac{mL^2}{24}\dot\phi^2\sin^2\theta$$

$$+\frac{mL^2}{24}\dot\theta^2$$

$$= \frac{mL^2}{6}\dot\theta^2 + \frac{m}{2}a^2\dot\phi^2 + \frac{m}{2}\cancel{2}\frac{aL}{\cancel{2}}\sin\theta\,\dot\phi^2$$

$$+\frac{m}{2}\frac{L^2}{4}\sin^2\theta\,\dot\phi^2 + \frac{mL^2}{24}\dot\phi^2\sin^2\theta$$

$$T_{arm} = \frac{mL^2}{6}\dot\theta^2 + \frac{ma^2}{2}\dot\phi^2 + \frac{maL}{2}\dot\phi^2\sin\theta + \frac{mL^2}{6}\dot\phi^2\sin^2\theta$$

For the shaft and horizontal arm:

$$I_o = \frac{ma^2}{3}$$

$$T_{shaft/arm} = \frac{1}{2}I_o\dot\phi^2 = \frac{ma^2}{6}\dot\phi^2$$

Total:

$$L = \frac{mL^2}{6}\dot\theta^2 + \frac{2ma^2}{3}\dot\phi^2 + \frac{maL}{2}\dot\phi^2\sin\theta + \frac{mL^2}{6}\dot\phi^2\sin^2\theta$$

$$+\frac{mgL}{2}\cos\theta$$

$\underline{k=1 \ (q_1 = \phi)}$

$$\frac{\partial L}{\partial \dot\phi} = \frac{4ma^2}{3}\dot\phi + maL\dot\phi\sin\theta + \frac{mL^2}{3}\dot\phi\sin^2\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{4ma^2}{3}\ddot\phi + maL\ddot\phi\sin\theta + maL\dot\phi\dot\theta\cos\theta + \frac{mL^2}{3}\ddot\phi\sin^2\theta$$
$$+ \frac{2mL^2}{3}\dot\phi\dot\theta\sin\theta\cos\theta$$

$$\frac{\partial L}{\partial \phi} = 0$$

$$\boxed{\left[\frac{4ma^2}{3} + maL\sin\theta + \frac{mL^2}{3}\sin^2\theta\right]\ddot\phi + \left[maL + \frac{2mL^2}{3}\sin\theta\right]\dot\phi\dot\theta\cos\theta = 0}$$

$\underline{k=2 \ (q_2 = \theta)}$

$$\frac{\partial L}{\partial \dot\theta} = \frac{mL^2}{3}\dot\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{mL^2}{3}\ddot\theta$$

$$\frac{\partial L}{\partial \theta} = \frac{maL}{2}\dot\phi^2\cos\theta + \frac{mL^2}{3}\dot\phi^2\sin\theta\cos\theta - \frac{mgL}{2}\sin\theta$$

$$\boxed{\frac{mL^2}{3}\ddot\theta - \left[\frac{maL}{2} + \frac{mL^2}{3}\sin\theta\right]\dot\phi^2\cos\theta + \frac{mgL}{2}\sin\theta = 0}$$