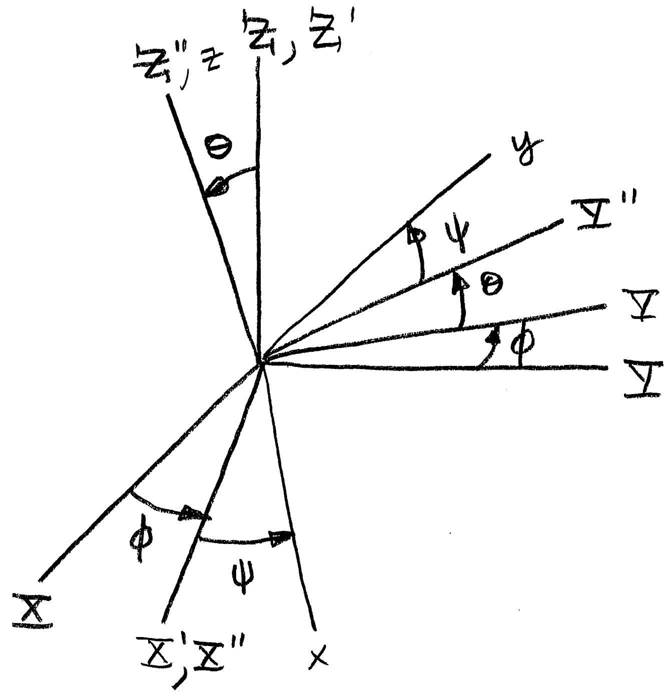
1) Problem 2.20 from Baruh



$$[R_1] = [R_{Z_1}(\phi)] = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[R_2] = [R_{X'}(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$$

$$[R_3] = [R_{Z_1''}(\psi)] = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{[R] = [R_3][R_2][R_1] = \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & s\phi c\psi + c\phi c\theta s\psi & s\theta s\psi \\ -c\phi s\psi - s\phi c\theta c\psi & -s\phi s\psi + c\phi c\psi c\theta & s\theta c\psi \\ s\phi s\theta & -c\phi s\theta & c\theta \end{bmatrix}}$$

$$\phi = 30^\circ, \quad \theta = 45^\circ, \quad \psi = -60^\circ$$

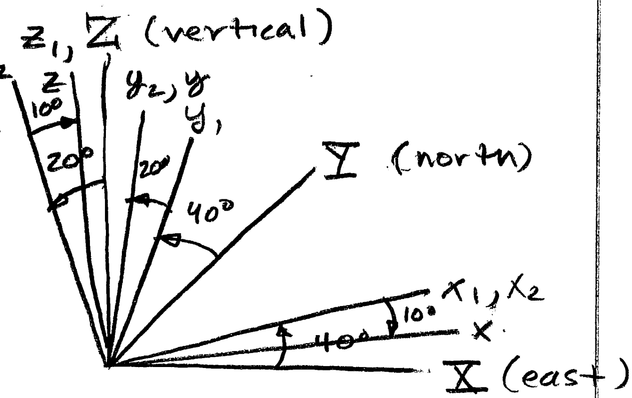
$$\boxed{[R] = \begin{bmatrix} 0.7392 & -0.2803 & -0.6124 \\ 0.5732 & 0.7392 & 0.3536 \\ 0.3536 & -0.6124 & 0.7071 \end{bmatrix}}$$

$$\{{}^{xyz}r\} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

$$\{{}^{XYZ}r\} = [R]^T \{{}^{xyz}r\} = \boxed{\begin{bmatrix} -1.976 \\ 3.619 \\ 4.123 \end{bmatrix}}$$

2)

Find the frame attached to the aircraft using a sequence of rotations



First, rotate about Z by $40°$ to get heading, resulting in new frame $x_1 y_1 z_1$:

$$[x_1\ y_1\ z_1]^T = [R_z(40°)][X\ Y\ Z]^T$$

Next, rotate about x_1 by $20°$ for climb, resulting in new $x_2 y_2 z_2$ frame:

$$[x_2\ y_2\ z_2]^T = [R_x(20°)][x_1\ y_1\ z_1]^T$$

Finally, rotate about y_2 by $10°$ to get bank, resulting in final xyz frame:

$$[x\ y\ z]^T = [R_y(10°)][x_2\ y_2\ z_2]$$

Combining rotations gives:

$$[x\ y\ z]^T = [R_y(10°)][R_x(20°)][R_z(40°)][X\ Y\ Z]^T$$

$$[x\ y\ z]^T = [R][X\ Y\ Z]^T$$

$$[R] = \begin{bmatrix} 0.7162 & 0.6785 & -0.1632 \\ -0.6040 & 0.7198 & 0.3420 \\ 0.3495 & -0.1464 & 0.9254 \end{bmatrix}$$

In the xyz frame, the accelerometers measure

$$\{a\} = \begin{bmatrix} 0 \\ 0.5g \\ -2g \end{bmatrix}$$

In the earth frame, this corresponds to

$$\{a\} = [R]^T \begin{bmatrix} 0 \\ 0.5g \\ -2g \end{bmatrix} = \begin{bmatrix} -1.0011g \\ 0.6527g \\ -1.6798g \end{bmatrix}$$

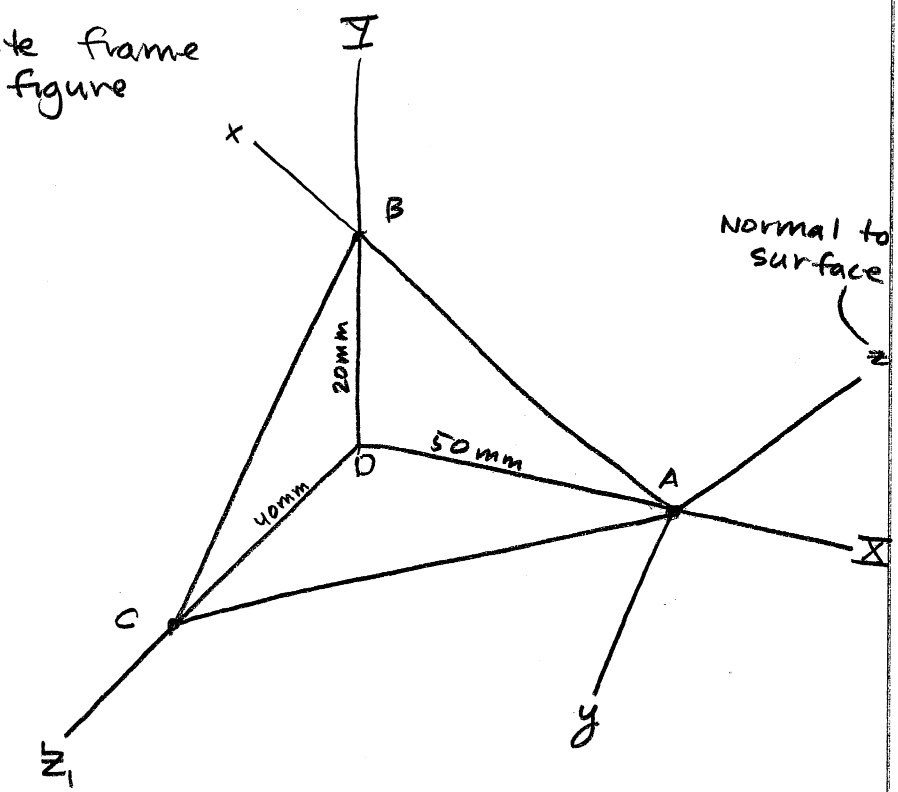
North-south: $\boxed{a_Y = 0.6527g = 6.403 \text{ m/s}^2}$ north

East-west: $\boxed{a_X = -1.0011g = -9.8205 \text{ m/s}^2}$ west

Vertical: $\boxed{a_Z = -1.6798g = -16.4791 \text{ m/s}^2}$ down

3)

The new coordinate frame is shown in the figure



First find $\hat{i}$ unit vector:

$$\hat{i} = \hat{e}_{B/A} = \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|} = \frac{-50\hat{I} + 20\hat{J}}{\sqrt{50^2 + 20^2}} = -0.9285\hat{I} + 0.3714\hat{J}$$

Next find $\hat{k}$ unit vector using cross product:

$$\hat{k} = \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{|\vec{r}_{B/A} \times \vec{r}_{C/A}|} = \frac{(-50\hat{I} + 20\hat{J}) \times (-50\hat{I} + 40\hat{K})}{|\vec{r}_{B/A} \times \vec{r}_{C/A}|}$$

$$\hat{k} = 0.337\hat{I} + 0.842\hat{J} + 0.421\hat{K}$$

Finally, $\hat{j}$ can be found from cross product

$$\hat{j} = \hat{k} \times \hat{i} = -0.1564\hat{I} - 0.3910\hat{J} + 0.907\hat{K}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} -0.9285 & 0.3714 & 0 \\ -0.1564 & -0.3910 & 0.907 \\ 0.337 & 0.842 & 0.421 \end{bmatrix} \begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix}$$

$$\boxed{[R] = \begin{bmatrix} -0.9285 & 0.3714 & 0 \\ -0.1564 & -0.3910 & 0.907 \\ 0.337 & 0.842 & 0.421 \end{bmatrix}}$$

$\vec{r}_{D/A} = -50\hat{I}$ mm

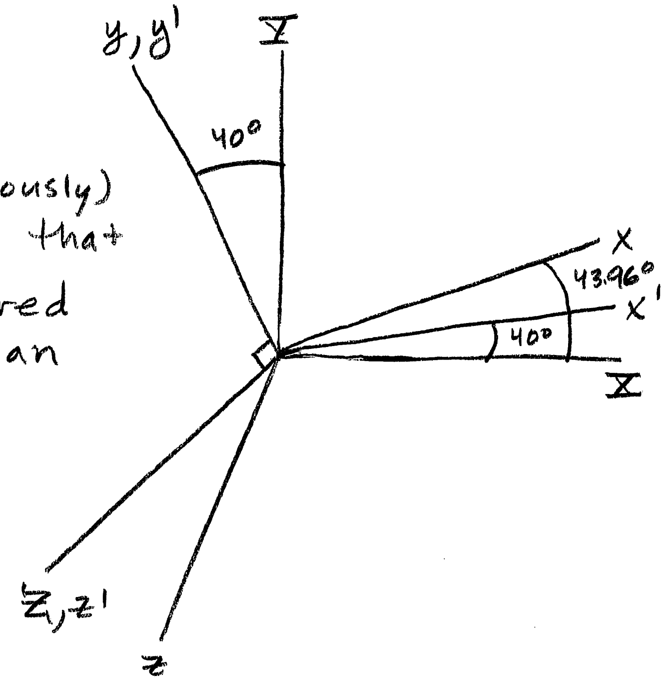
so

$$\{r_{D/A}\} = [R]\begin{bmatrix} -50 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 46.42 \\ 7.82 \\ -16.84 \end{bmatrix} \text{ mm}}$$

or

$$\vec{r}_{D/A} = 46.42\hat{\imath} + 7.82\hat{\jmath} - 16.84\hat{k}$$

4) Find the rotation matrix relating XYZ_1 to xyz



Z_1 is $\perp$ to both Y (obviously) and y, which suggests that an initial rotation occurred about the Z_1 axis by an angle of 40°, resulting in a $x'y'z'$ frame with $y' = y$ and $z' = Z_1$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [R_1] \begin{bmatrix} X \\ Y \\ Z_1 \end{bmatrix}$$

$$[R_1] = [R_z(40°)]$$

The next (and final rotation) must then be a rotation about the y' axis by an unknown angle θ.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [R_2] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$[R_2] = [R_y(\theta)]$$

$$[R] = [R_2][R_1] = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c40 & s40 & 0 \\ -s40 & c40 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta c40 & c\theta s40 & -s\theta \\ -s40 & c40 & 0 \\ s\theta c40 & s\theta s40 & c\theta \end{bmatrix}$$

$$[R] = \begin{bmatrix} 0.766c\theta & 0.643c\theta & -s\theta \\ -0.643 & 0.766 & 0 \\ 0.766s\theta & 0.643s\theta & c\theta \end{bmatrix}$$

To complete [R] we need to find θ.

We know $\theta_{11} = 43.96^\circ \Rightarrow l_{11} = 0.7198$

From the 1,1 element of [R], we know that

$l_{11} = 0.766\,c\theta = 0.7198 \Longrightarrow \theta = 20^\circ$

$$\boxed{[R] = \begin{bmatrix} 0.7198 & 0.604 & -0.342 \\ -0.643 & 0.766 & 0 \\ 0.262 & 0.2198 & 0.9397 \end{bmatrix}}$$

Alternate approach (which I haven't tried):

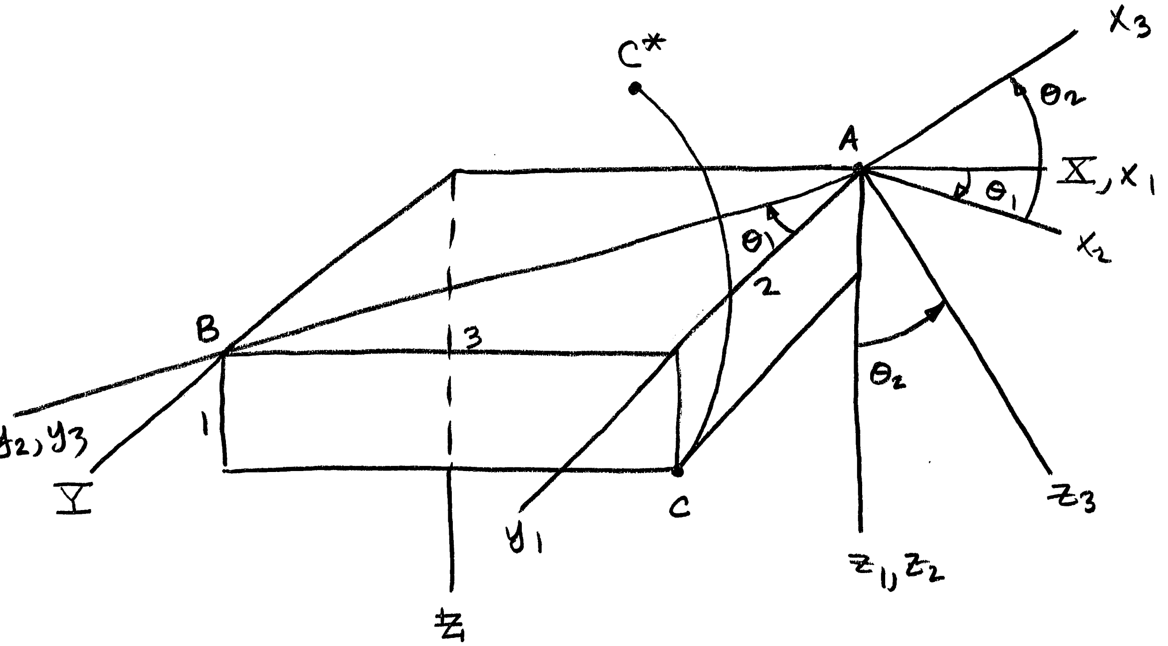
We know $l_{11} = \cos 43.96$

$l_{22} = \cos 40$

$l_{23} = \cos 90$

Use $[R][R]^T = [1]$ and $[R]^T[R] = [1]$ to get equations to solve for the other direction cosines. Need to be careful with signs.

5)



Rotating the box about line AB will cause point C to end up at C^*.

We first need to setup the line about which the box will be rotated. Create an $x_1y_1z_1$ frame at A, as shown. Then create the $x_2y_2z_2$ frame by rotating about the z_1 axis by an angle of $\theta_1 = \tan^{-1}(\frac{3}{2}) = 56.31°$. Note that the box has not rotated yet — we have just created some new frames. What is the location of the original point C expressed in the 2-frame? In frame 1 it was $\{C_1\} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. In frame 2 it will be expressed as $\{C_2\} = [R_1]\{C_1\}$

where $[R_1] = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so, $\{C_2\} = \begin{bmatrix} 1.664 \\ 1.109 \\ 1 \end{bmatrix}$

Now rotate the <u>box</u> about y_2 by an angle $\theta_2 = 45°$.

We have now created a new point C^* and a new frame $(x_3 y_3 z_3)$, as shown. What is the location of the new point C^* in the new 3-frame?

It is the same as the location of the old point C in the 2-frame:

$$\{C_3^*\} = \{C_2\} = \begin{bmatrix} 1.664 \\ 1.109 \\ 1 \end{bmatrix}$$

Now find the location of C^* in the original frame by going backwards;

$$\{C_3^*\} = [R_2]\{C_2^*\} \Rightarrow \{C_2^*\} = [R_2]^T\{C_3^*\}$$

where $[R_2] = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix}$ and $\theta_2 = 45°$

And $\{C_2^*\} = [R_1]\{C_1^*\} \Rightarrow \{C_1^*\} = [R_1]^T\{C_2^*\}$

$$\{C_1^*\} = [R_1]^T[R_2]^T\{C_3^*\}$$

$$\{C_1^*\} = \begin{bmatrix} 0.1219 \\ 2.1828 \\ -0.4696 \end{bmatrix}$$

This is the location of the point C^* expressed in the 1-frame.

Finally, we need to find the location of C^* in the original frame. Frame 1 is related to the original frame by:

$$\{C^*\} = \{C_1^*\} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\{C^*\} = \begin{bmatrix} 3.1219 \\ 2.1828 \\ -0.4696 \end{bmatrix} \text{ m}}$$