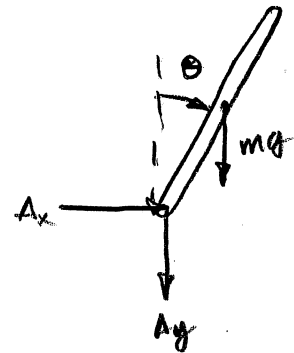
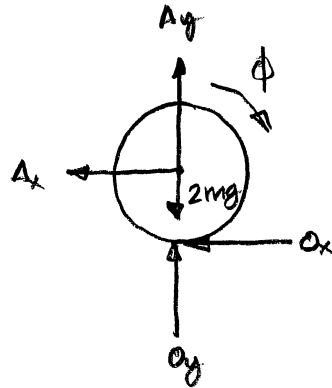
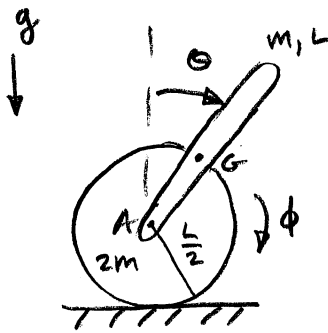


1)



Disk:

$$[M_O = I_O \ddot{\phi}]$$

$$-A_x \frac{L}{2} = \left[ \underbrace{\frac{1}{2} (2m) \left(\frac{L}{2}\right)^2}_{I_A} + \underbrace{(2m) \left(\frac{L}{2}\right)^2}_{md^2} \right] \ddot{\phi}$$

$$-\frac{A_x L}{2} = \left( \frac{mL^2}{4} + \frac{2mL^2}{4} \right) \ddot{\phi} \Rightarrow -\frac{A_x L}{2} = \frac{3mL^2}{4} \ddot{\phi}$$

$$\Rightarrow \ddot{\phi} = -\frac{2A_x}{3mL} \quad (1)$$

Rod:

$$[M_G = I_G \ddot{\theta}]$$

$$-\frac{A_x L}{2} \cos \theta - \frac{A_y L}{2} \sin \theta = \frac{mL^2}{12} \ddot{\theta}$$

$$\frac{mL}{6} \ddot{\theta} = -A_x \cos \theta - A_y \sin \theta \quad (2)$$

Need two more equations. Use Newton's 2nd law on rod:

$$\vec{r}_G = \left(x + \frac{L}{2} \sin \theta\right) \hat{i} + \frac{L}{2} \cos \theta \hat{j}$$

$$\dot{\vec{r}}_G = \left(\dot{x} + \frac{L}{2} \dot{\theta} \cos \theta\right) \hat{i} - \frac{L}{2} \dot{\theta} \sin \theta \hat{j}$$

$$\ddot{\vec{r}}_G = \left(\ddot{x} + \frac{L}{2} \ddot{\theta} \cos \theta - \frac{L}{2} \dot{\theta}^2 \sin \theta\right) \hat{i} - \frac{L}{2} \ddot{\theta} \sin \theta \hat{j} - \frac{L}{2} \dot{\theta}^2 \cos \theta \hat{j}$$

But  $x = \frac{L}{2} \phi \Rightarrow \ddot{x} = \frac{L}{2} \ddot{\phi}$

and  $\dot{\theta} = \dot{\phi} = 0$

$$\ddot{\vec{r}}_G = \left( \frac{L}{2} \ddot{\phi} + \frac{L}{2} \ddot{\theta} \cos \theta \right) \hat{x} - \frac{L}{2} \ddot{\theta} \sin \theta \hat{y}$$

$$[\vec{F} = m \ddot{\vec{r}}_G]$$

$$\hat{x}: A_x = \frac{mL}{2} \ddot{\phi} + \frac{mL}{2} \ddot{\theta} \cos \theta \quad (3)$$

$$\hat{y}: -A_y - mg = -\frac{mL}{2} \ddot{\theta} \sin \theta \quad (4)$$

From (2):

$$\ddot{\theta} = -\frac{6A_x}{mL} \cos \theta - \frac{6A_y}{mL} \sin \theta \quad (5)$$

Substitute into (3):

$$A_x = \frac{mL}{2} \ddot{\phi} + \frac{mL}{2} \left( -\frac{6A_x}{mL} \cos \theta - \frac{6A_y}{mL} \sin \theta \right) \cos \theta$$

$$A_x = \frac{mL}{2} \ddot{\phi} - 3A_x \cos^2 \theta - 3A_y \sin \theta \cos \theta$$

$$A_x (1 + 3 \cos^2 \theta) + 3A_y \sin \theta \cos \theta = \frac{mL}{2} \ddot{\phi} \quad (6)$$

Substitute (5) into (4):

$$-A_y - mg = +\frac{mL}{2} \left( +\frac{6A_x}{mL} \cos \theta + \frac{6A_y}{mL} \sin \theta \right) \sin \theta$$

$$-A_y - mg = 3A_x \sin \theta \cos \theta + 3A_y \sin^2 \theta$$

$$3A_x \sin \theta \cos \theta + A_y (1 + 3 \sin^2 \theta) = -mg$$

$$A_y = \frac{-mg - 3A_x \sin \theta \cos \theta}{1 + 3 \sin^2 \theta}$$

Substitute into (6):

$$A_x (1 + 3 \cos^2 \theta) + 3 \left( \frac{-\cancel{m}g - 3A_x \sin \theta \cos \theta}{1 + 3 \sin^2 \theta} \right) \sin \theta \cos \theta = \frac{mL}{2} \ddot{\phi}$$

$$A_x (1 + 3 \cos^2 \theta) - \frac{3mg \sin \theta \cos \theta}{1 + 3 \sin^2 \theta} - \frac{9A_x \sin^2 \theta \cos^2 \theta}{1 + 3 \sin^2 \theta} = \frac{mL}{2} \ddot{\phi}$$

Use numerical values:  $\theta = 30^\circ$

$$3.25 A_x - 0.742 mg - 0.9643 A_x = \frac{mL}{2} \ddot{\phi}$$

$$2.2857 A_x = 0.742 mg + \frac{mL}{2} \ddot{\phi}$$

$$A_x = 0.3248 mg + 0.21875 mL \ddot{\phi}$$

Substitute into ①:

$$\ddot{\phi} = -\frac{2}{3mL} \left[ 0.3248 mg + 0.21875 mL \ddot{\phi} \right]$$

$$\ddot{\phi} [1 + 0.14583] = -0.2165 \frac{g}{L}$$

$$\boxed{\ddot{\phi} = -0.189 \frac{g}{L}}$$

2) Problem 11.13 from Baruh.

The angular velocity of the pendulum is:

$$\vec{\omega} = -\Omega \cos \theta \hat{b}_1 + \Omega \sin \theta \hat{b}_2 + \dot{\theta} \hat{b}_3$$

Its inertia matrix is:

$$[I_G] = \frac{1}{12} mL^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The moments about G are:

$$\vec{M}_G = M_1 \hat{b}_1 + (M_2 + \frac{B_3 L}{2}) \hat{b}_2 - \frac{B_2 L}{2} \hat{b}_3$$

$$\dot{\vec{\omega}} = -\Omega \dot{\theta} \sin \theta \hat{b}_1 + \Omega \dot{\theta} \cos \theta \hat{b}_2 + \ddot{\theta} \hat{b}_3$$

Euler's equations:

$$I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = M_1$$

$$(0) - \Omega \dot{\theta} \sin \theta - (\frac{1}{12} mL^2 - \frac{1}{12} mL^2) (\Omega \sin \theta) (\dot{\theta}) = M_1$$

$$\Rightarrow M_1 = 0$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_1 \omega_3 = M_2$$

$$\frac{1}{12} mL^2 \Omega \dot{\theta} \cos \theta - \frac{1}{12} mL^2 (-\Omega \cos \theta) (\dot{\theta}) = M_2 + \frac{B_3 L}{2}$$

$$\frac{1}{6} mL^2 \Omega \dot{\theta} \cos \theta = M_2 + \frac{B_3 L}{2} \quad (1)$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 = M_3$$

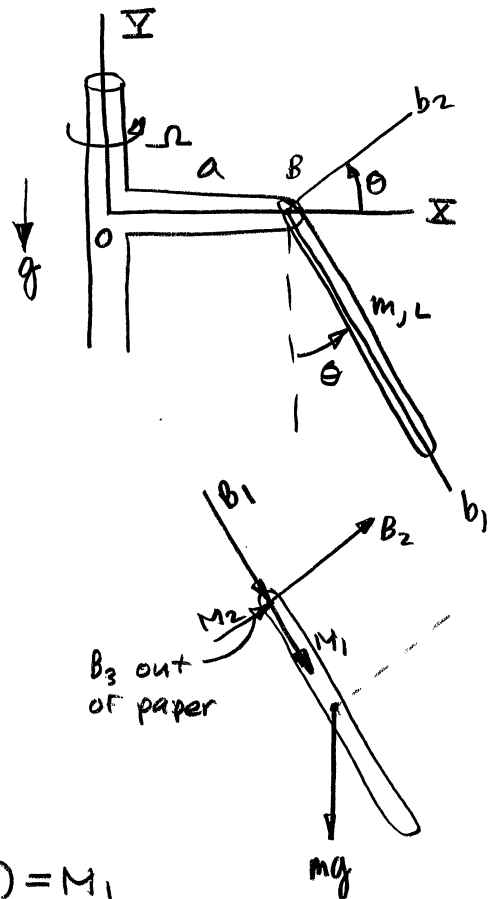
$$\frac{1}{12} mL^2 \ddot{\theta} - (0 - \frac{1}{12} mL^2) (-\Omega \cos \theta) (\Omega \sin \theta) = -\frac{B_2 L}{2}$$

$$\frac{1}{12} mL^2 \ddot{\theta} - \frac{1}{12} mL^2 \Omega^2 \sin \theta \cos \theta = -\frac{B_2 L}{2} \quad (2)$$

We now have four unknown reactions ( $M_2, B_1, B_2, B_3$ ), plus we need an equation of motion.

We only have two equations ((1) and (2)), and we need five.

$\Rightarrow$  Use Newton's 2<sup>nd</sup> Law



$$\vec{r}_G = a\hat{I} + \frac{L}{2}\hat{b}_1 = a(\sin\theta\hat{b}_1 + \cos\theta\hat{b}_2) + \frac{L}{2}\hat{b}_1$$

$$\vec{r}_G = \left(a\sin\theta + \frac{L}{2}\right)\hat{b}_1 + a\cos\theta\hat{b}_2$$

$$\dot{\vec{r}}_G = a\dot{\theta}\cos\theta\hat{b}_1 + \left(a\sin\theta + \frac{L}{2}\right)\dot{\hat{b}}_1 - a\dot{\theta}\sin\theta\hat{b}_2 + a\cos\theta\dot{\hat{b}}_2$$

$$\dot{\hat{b}}_1 = \vec{\omega} \times \hat{b}_1 = \Omega\sin\theta\hat{b}_2 \times \hat{b}_1 + \dot{\theta}\hat{b}_3 \times \hat{b}_1 = \dot{\theta}\hat{b}_2 - \Omega\sin\theta\hat{b}_3$$

$$\dot{\hat{b}}_2 = \vec{\omega} \times \hat{b}_2 = -\Omega\cos\theta\hat{b}_1 \times \hat{b}_2 + \dot{\theta}\hat{b}_3 \times \hat{b}_2 = -\dot{\theta}\hat{b}_1 - \Omega\cos\theta\hat{b}_3$$

$$\begin{aligned}\dot{\hat{b}}_3 &= \vec{\omega} \times \hat{b}_3 = -\Omega\cos\theta\hat{b}_1 \times \hat{b}_3 + \Omega\sin\theta\hat{b}_2 \times \hat{b}_3 \\ &= \Omega\sin\theta\hat{b}_1 + \Omega\cos\theta\hat{b}_2\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}_G &= a\ddot{\theta}\cos\theta\hat{b}_1 + \left(a\sin\theta + \frac{L}{2}\right)(\ddot{\theta}\hat{b}_2 - \Omega\sin\theta\hat{b}_3) - a\dot{\theta}\sin\theta\hat{b}_2 \\ &\quad + a\cos\theta(-\dot{\theta}\hat{b}_1 - \Omega\cos\theta\hat{b}_3)\end{aligned}$$

$$\begin{aligned}\ddot{\vec{r}}_G &= (a\ddot{\theta}\cos\theta - a\dot{\theta}\sin\theta)\hat{b}_1 + \left(a\dot{\theta}\sin\theta + \frac{L}{2}\ddot{\theta} - a\dot{\theta}\sin\theta\right)\hat{b}_2 \\ &\quad + \left(-a\Omega\sin^2\theta - \frac{L}{2}\Omega\sin\theta - a\Omega\cos^2\theta\right)\hat{b}_3 \\ &= \frac{L}{2}\ddot{\theta}\hat{b}_2 - \left(\frac{L}{2}\Omega\sin\theta + a\Omega\right)\hat{b}_3\end{aligned}$$

$$\ddot{\vec{r}}_G = \frac{L}{2}\ddot{\theta}\hat{b}_2 - \Omega\left(\frac{L}{2}\sin\theta + a\right)\hat{b}_3$$

$$\begin{aligned}\ddot{\vec{r}}_G &= \frac{L}{2}\ddot{\theta}\hat{b}_2 + \frac{L}{2}\ddot{\theta}(-\dot{\theta}\hat{b}_1 - \Omega\cos\theta\hat{b}_3) - \Omega\left(\frac{L}{2}\dot{\theta}\cos\theta\right)\hat{b}_3 \\ &\quad - \Omega\left(\frac{L}{2}\sin\theta + a\right)(\Omega\sin\theta\hat{b}_1 + \Omega\cos\theta\hat{b}_2)\end{aligned}$$

$$\begin{aligned}&= \left[-\frac{L}{2}\ddot{\theta}^2 - \Omega^2\left(\frac{L}{2}\sin\theta + a\right)\sin\theta\right]\hat{b}_1 + \left[\frac{L}{2}\ddot{\theta}' - \Omega^2\left(\frac{L}{2}\sin\theta + a\right)\cos\theta\right]\hat{b}_2 \\ &\quad + \underbrace{\left[-\frac{L}{2}\ddot{\theta}\Omega\cos\theta - \frac{L}{2}\ddot{\theta}\Omega\cos\theta\right]\hat{b}_3}_{-\Omega\ddot{\theta}\Omega\cos\theta\hat{b}_3}\end{aligned}$$

$$[\vec{F} = m\ddot{\vec{r}}_G]$$

$$\hat{b}_1: B_1 + mg\cos\theta = m\left[-\frac{L}{2}\ddot{\theta}^2 - \Omega^2\left(\frac{L}{2}\sin\theta + a\right)\sin\theta\right]$$

$$\hat{b}_2: B_2 - mg\sin\theta = m\left[\frac{L}{2}\ddot{\theta}' - \Omega^2\left(\frac{L}{2}\sin\theta + a\right)\cos\theta\right]$$

$$\hat{b}_3: B_3 = -mL\dot{\theta}\Omega\cos\theta$$

$$B_1 = -m \left[ g\cos\theta + \frac{L}{2}\dot{\theta}^2 + \Omega^2 \left( \frac{L}{2}\sin\theta + a \right) \sin\theta \right]$$

$$B_2 = m \left[ g\sin\theta + \frac{L}{2}\ddot{\theta} - \Omega^2 \left( \frac{L}{2}\sin\theta + a \right) \cos\theta \right]$$

From ②:

$$\begin{aligned} \frac{1}{12}mL^2\ddot{\theta} - \frac{1}{12}mL^2\Omega^2\sin\theta\cos\theta &= -\frac{KL}{2} \left[ g\sin\theta + \frac{L}{2}\ddot{\theta} \right. \\ &\quad \left. - \Omega^2 \left( \frac{L}{2}\sin\theta + a \right) \cos\theta \right] \end{aligned}$$

$$\frac{2L}{3}\ddot{\theta} + g\sin\theta - \frac{L}{6}\Omega^2\sin\theta\cos\theta - \frac{L}{2}\Omega^2\sin\theta\cos\theta - a\Omega^2\cos\theta = 0$$

$$\left[ \frac{2L}{3}\ddot{\theta} + g\sin\theta - \frac{2L}{3}\Omega^2\sin\theta\cos\theta - a\Omega^2\cos\theta = 0 \right] \text{ EOM}$$

or

$$\frac{L}{3}\ddot{\theta} - \Omega^2 \left( \frac{L}{3}\sin\theta + \frac{a}{2} \right) \cos\theta + \frac{g}{2}\sin\theta = 0$$

3)

$$\vec{\omega} = \dot{\theta} \hat{j} + \Omega \hat{b}_3$$

$$\vec{\omega} = \dot{\theta} \sin \phi \hat{b}_1 + \dot{\theta} \cos \phi \hat{b}_2 + \Omega \hat{b}_3$$

$$\dot{\vec{\omega}} = [\ddot{\theta} \sin \phi + \dot{\theta} \Omega \cos \phi] \hat{b}_1 + [\ddot{\theta} \cos \phi - \dot{\theta} \Omega \sin \phi] \hat{b}_2$$

$$I_1 = \frac{1}{4} m R^2 \quad I_2 = \frac{1}{4} m R^2 \quad I_3 = \frac{1}{2} m R^2$$

$$M_1 = M_x \cos \phi + M_y \sin \phi$$

$$M_2 = M_y \cos \phi - M_x \sin \phi$$

$$M_3 = 0$$

Euler's equations:

$$\frac{1}{4} m R^2 (\ddot{\theta} \sin \phi + \dot{\theta} \Omega \cos \phi) + \frac{1}{4} m R^2 (\dot{\theta} \cos \phi) (\Omega) = M_x \cos \phi + M_y \sin \phi$$

$$\frac{1}{4} m R^2 \ddot{\theta} \sin \phi + \frac{m R^2}{2} \dot{\theta} \Omega \cos \phi = M_x \cos \phi + M_y \sin \phi \quad (1)$$

$$\frac{1}{4} m R^2 (\ddot{\theta} \cos \phi - \dot{\theta} \Omega \sin \phi) - \frac{1}{4} m R^2 (\dot{\theta} \sin \phi) (\Omega) = M_y \cos \phi - M_x \sin \phi$$

$$\frac{m R^2}{4} \ddot{\theta} \cos \phi - \frac{m R^2}{2} \dot{\theta} \Omega \sin \phi = M_y \cos \phi - M_x \sin \phi \quad (2)$$

$$\Rightarrow M_x = -\frac{m R^2}{4} \ddot{\theta} \frac{\cos \phi}{\sin \phi} + \frac{m R^2}{2} \dot{\theta} \Omega + M_y \frac{\cos \phi}{\sin \phi}$$

$$\text{Plug into (1): } \Rightarrow M_y = \frac{m R^2}{4} \ddot{\theta}$$

$$\Rightarrow M_x = \frac{m R^2}{2} \dot{\theta} \Omega$$

To find forces, use Newton's 2nd Law:

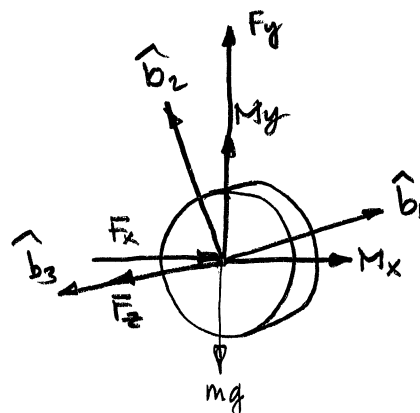
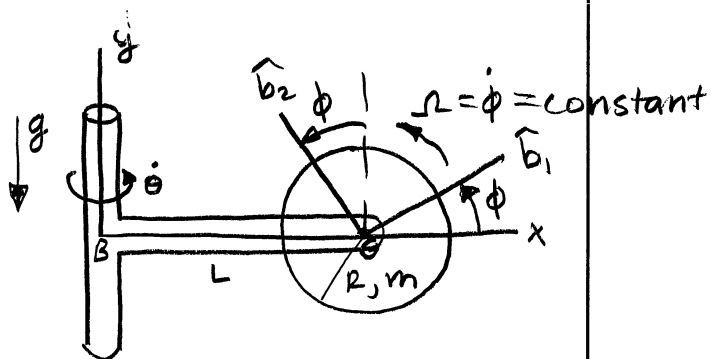
$$\vec{r}_G = L \hat{x} \Rightarrow \ddot{\vec{r}}_G = -L \ddot{\theta} \hat{k} - L \dot{\theta}^2 \hat{x}$$

$$[\vec{F} = m \ddot{\vec{r}}_G]$$

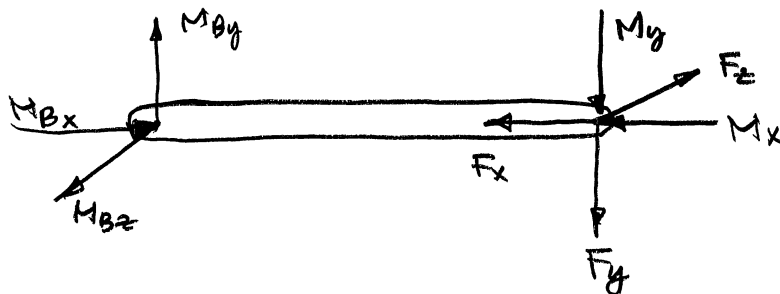
$$\hat{x}: F_x = -m L \dot{\theta}^2$$

$$\hat{j}: F_y - mg = 0 \Rightarrow F_y = mg$$

$$\hat{k}: F_z = -m L \ddot{\theta}$$



Now draw a FBD of the rod



$$M_{Bx} - M_{Ax} = 0$$

$$M_{Bx} = \frac{mR^2}{2} \ddot{\theta}$$

$$M_{By} - M_{Ay} + F_z L = 0$$

$$M_{By} = \frac{mR^2}{4} \ddot{\theta} + mL^2 \ddot{\theta}$$

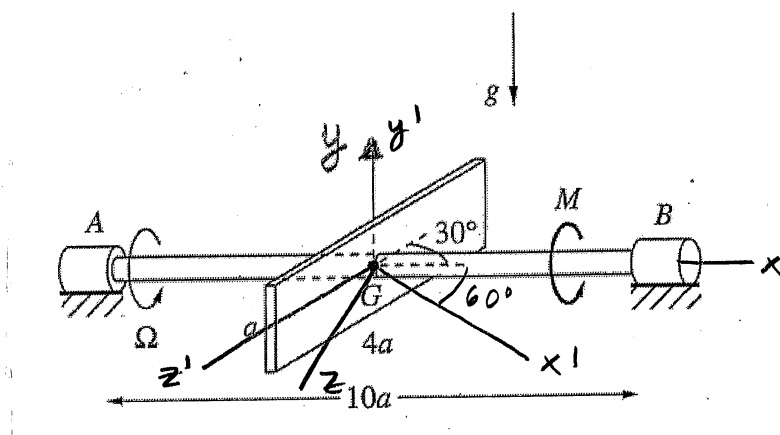
$$M_{By} = m \left( \frac{R^2}{4} + L^2 \right) \ddot{\theta}$$

$$M_{Bz} - F_g L = 0$$

$$M_{Bz} = mgL$$



4)



The principal (primed) axes are shown. We know the inertia matrix in the primed frame:

$$[I'] = \frac{m}{12} \begin{bmatrix} 16a^2 + a^2 & 0 & 0 \\ 0 & 16a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} = \frac{ma^2}{12} \begin{bmatrix} 17 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix relating unprimed and primed frames is:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [R_y(-60^\circ)] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos 60 & 0 & \sin 60 \\ 0 & 1 & 0 \\ -\sin 60 & 0 & \cos 60 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[I] = [R][I'] [R]^T \Rightarrow [I] = [R]^T [I'] [R]$$

$$[I] = ma^2 \begin{bmatrix} 0.4167 & 0 & 0.5774 \\ 0 & 1.333 & 0 \\ 0.5774 & 0 & 1.0833 \end{bmatrix}$$

$$\Rightarrow I_{xx} = 0.4167 ma^2$$

$$I_{yy} = 1.333 ma^2$$

$$I_{zz} = 1.0833 ma^2$$

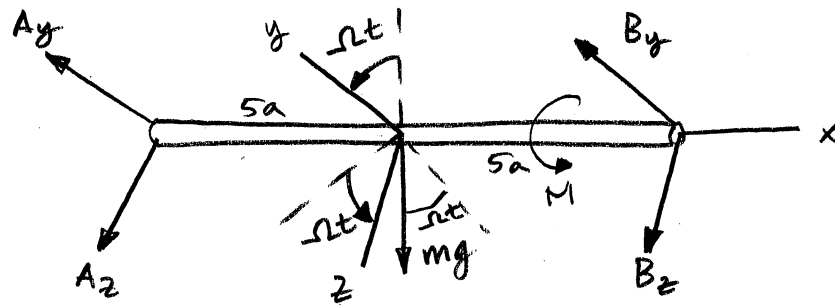
$$I_{xz} = -0.5774 ma^2$$



$$\vec{\omega} = \Omega \hat{x}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = 0$$

Find resultant moment vector:



$$\vec{M} = M\hat{i} + 5a(A_z - B_z)\hat{j} + 5a(B_y - A_y)\hat{k}$$

Special case for fixed-axis rotation:

$$x: M_x = I_{xx}\alpha_x \Rightarrow M = 0$$

$$y: M_y = -I_{xy}\alpha_x + I_{xz}\omega_x^2$$

$$5a(A_z - B_z) = -0.5774 ma^2 \Omega^2$$

$$A_z - B_z = -0.11548 ma \Omega^2 \quad (1)$$

$$z: M_z = -I_{xz}\alpha_x - I_{xy}\omega_x^2$$

$$5a(B_y - A_y) = 0 \Rightarrow A_y = B_y \quad (2)$$

Need more equations, sum forces.

$$[\vec{F} = 0]$$

$$y: A_y + B_y - mg \cos \Omega t = 0$$

$$A_y + B_y = mg \cos \Omega t \quad (3)$$

$$z: A_z + B_z + mg \sin \Omega t = 0$$

$$A_z + B_z = -mg \sin \Omega t \quad (4)$$

From ② and ③:

$$2A_y = mg \cos \Omega t \Rightarrow A_y = \frac{mg}{2} \cos \Omega t$$

$$\Rightarrow B_y = \frac{mg}{2} \cos \Omega t$$

Add ① and ④:

$$2A_z = -0.11548 m a \Omega^2 - mg \sin \Omega t$$

$$\Rightarrow A_z = -0.05774 m a \Omega^2 - \frac{mg}{2} \sin \Omega t$$

$$A_z = -\frac{\sqrt{3}}{30} m a \Omega^2 - \frac{mg}{2} \sin \Omega t$$

Subtract ① from ④:

$$2B_z = -mg \sin \Omega t + 0.11548 m a \Omega^2$$

$$\Rightarrow B_z = 0.05774 m a \Omega^2 - \frac{mg}{2} \sin \Omega t$$

$$B_z = \frac{\sqrt{3}}{30} m a \Omega^2 - \frac{mg}{2} \sin \Omega t$$

Alternate approach: do everything in the primed frame.

$$\{ \omega \}' = [P] \{ \omega \} = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \Omega \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Omega/2 \\ 0 \\ -\sqrt{3}\Omega/2 \end{bmatrix}$$

$$\{ \alpha \}' = 0$$

$$\{ M \}' = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} M \\ 5a(A_z - B_z) \\ 5a(B_y - A_y) \end{bmatrix}$$

$$\{ M \}' = \begin{bmatrix} M/2 + 5\sqrt{3}/2 a(B_y - A_y) \\ 5a(A_z - B_z) \\ -\sqrt{3}/2 M + 5/2 a(B_y - A_y) \end{bmatrix}$$

Apply Euler's equations:

$$x': I_{xx}' \alpha_x' - (I_{yy}' - I_{zz}') \omega_y' \omega_z' = M_x'$$

$$\Rightarrow M = 0$$

$$y': I_{yy}' \alpha_y' - (I_{zz}' - I_{xx}') \omega_x' \omega_z' = M_y'$$

$$-\left(\frac{ma^2}{12} - \frac{17ma^2}{12}\right) \frac{\Omega}{2} \left(-\frac{\sqrt{3}\Omega}{2}\right) = 5a(A_z - B_z)$$

$$-\frac{16ma^2}{12} \frac{\sqrt{3}\Omega^2}{4} = 5a(A_z - B_z)$$

$$-\frac{\sqrt{3}ma^2\Omega^2}{3} = 5a(A_z - B_z)$$

$$\Rightarrow A_z - B_z = -0.11548 ma\Omega^2 \quad (\text{Same as } \textcircled{1})$$

$$z': I_{zz}' \alpha_z' - (I_{xx}' - I_{yy}') \omega_x' \omega_y' = M_z'$$

$$0 = -\frac{\sqrt{3}}{2} \cancel{M} + \frac{5a}{2} (B_y - A_y)$$

$$\Rightarrow A_y = B_y \quad (\text{same as } \textcircled{2})$$

$\Rightarrow$  We get the same results using the principal axes!