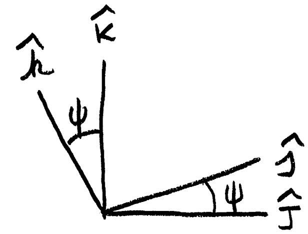
1) Robotic Sander

$\vec{\omega}_{AD} = \dot{\phi}\,\hat{I}$



$\vec{\omega}_{DB} = \dot{\psi}\,\hat{I}$

$\vec{\omega}_B = \omega\hat{k}$

$\vec{\omega} = \vec{\omega}_{DB} + \vec{\omega}_B = \dot{\psi}\hat{I} + \omega\hat{k}$

But $\hat{i} = \hat{I}$.

$\hat{j} = \cos\psi\hat{J} + \sin\phi\hat{K}$

$\hat{k} = -\sin\psi\hat{J} + \cos\psi\hat{K}$

$\vec{\omega} = \dot{\psi}\hat{I} + \omega(-\sin\psi\hat{J} + \cos\psi\hat{K})$

$\vec{\omega} = \dot{\psi}\hat{I} - \omega\sin\psi\hat{J} + \omega\cos\psi\hat{K}$

But $\dot{\psi} = -0.3$ rad/s

$\omega = 1500\left(\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 157.08$ rad/s

$\psi = 60°$

$$\boxed{\vec{\omega} = -0.3\hat{I} - 136.04\hat{J} + 78.54\hat{K} \text{ rad/s}}$$

$\vec{\alpha} = \dot{\vec{\omega}} = \cancel{\ddot{\psi}}^{0}\hat{I} - \cancel{\dot{\omega}}^{0}\sin\psi\hat{J} - \omega\dot{\psi}\cos\psi\hat{J} + \cancel{\dot{\omega}}^{0}\cos\psi\hat{K} - \omega\dot{\psi}\sin\psi\hat{K}$

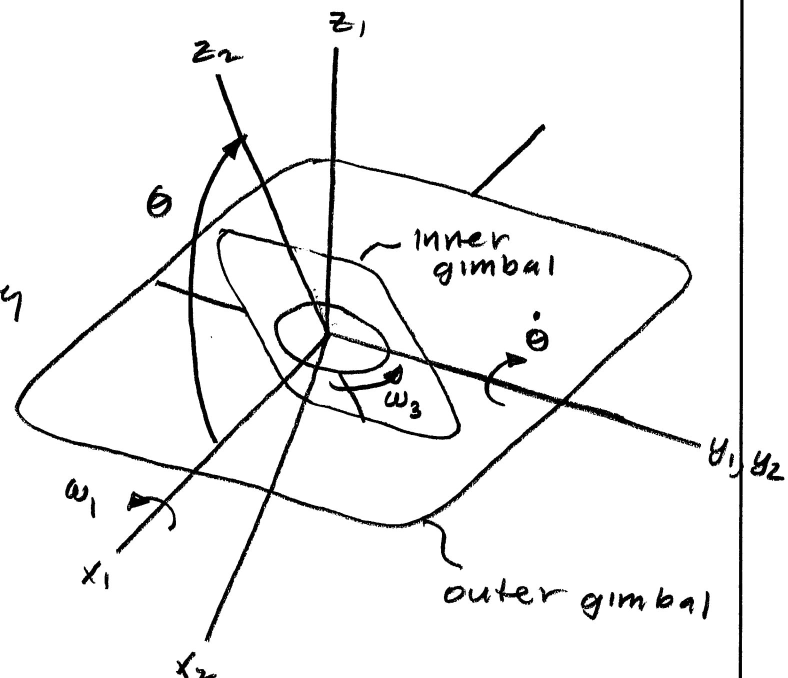
$\vec{\alpha} = -\omega\dot{\psi}\cos\phi\hat{J} - \omega\dot{\psi}\sin\phi\hat{K}$

$$\boxed{\vec{\alpha} = 23.56\hat{J} + 40.81\hat{K} \text{ rad/s}}$$

2) Gyroscope

$x_1y_1z_1$ are attached to the outer gimbal

$x_2y_2z_2$ are attached to the inner gimbal



The total angular velocity of the flywheel is:

$$\vec{\omega} = \vec{\omega}_o + \vec{\omega}_i + \vec{\omega}_f$$

$\vec{\omega}_f$: flywheel's angular velocity about its own axis

$\vec{\omega}_i$: Angular velocity of inner gimbal

$\vec{\omega}_o$: Angular velocity of outer gimbal

$$\vec{\omega}_o = \omega_1 \hat{i}_1$$

$$\vec{\omega}_i = -\dot{\theta}\,\hat{j}_2 = -\dot{\theta}\,\hat{j}_1$$

$$\vec{\omega}_f = \omega_3 \hat{k}_2 = \omega_3 (\cos\theta\,\hat{i}_1 + \sin\theta\,\hat{k}_1)$$

$$\vec{\omega} = (\omega_1 + \omega_3 \cos\theta)\hat{i}_1 - \dot{\theta}\hat{j}_1 + \omega_3 \sin\theta\,\hat{k}_1$$

Differentiate to get angular acceleration:

$$\vec{\alpha} = \dot{\vec{\omega}} = (\dot{\omega}_1 + \dot{\omega}_3 \cos\theta - \omega_3 \dot{\theta} \sin\theta)\hat{i}_1 + (\omega_1 + \omega_3\cos\theta)\dot{\hat{i}}_1$$
$$- \ddot{\theta}\hat{j}_1 - \dot{\theta}\dot{\hat{j}}_1 + \dot{\omega}_3 \sin\theta\,\hat{k}_1 + \omega_3\dot{\theta}\cos\theta\,\hat{k}_1 + \omega_3 \sin\theta\,\dot{\hat{k}}_1$$

where $\dot{\hat{i}}_1 = \vec{\omega}_o \times \hat{i}_1 = 0 \qquad \dot{\hat{j}}_1 = \vec{\omega}_o \times \hat{j}_1 = \omega_1 \hat{k}_1$

$$\dot{\hat{k}}_1 = \vec{\omega}_o \times \hat{k}_1 = -\omega_1 \hat{j}_1$$

and $\dot{\theta} = \dot{\omega}_3 = 0$ (from problem statement)

$$\vec{\alpha} = \dot{\omega}_1 \hat{i}_1 + \ddot{\theta} \hat{j}_1 - \omega_1 \omega_3 \sin\theta \hat{j}_1 = \dot{\omega}_1 \hat{i}_1 - (\ddot{\theta} + \omega_1 \omega_3 \sin\theta) \hat{j}_1$$

$\dot{\omega}_1 = -1.8 \text{ rad/s}^2$

$\ddot{\theta} = 3 \text{ rad/s}^2$

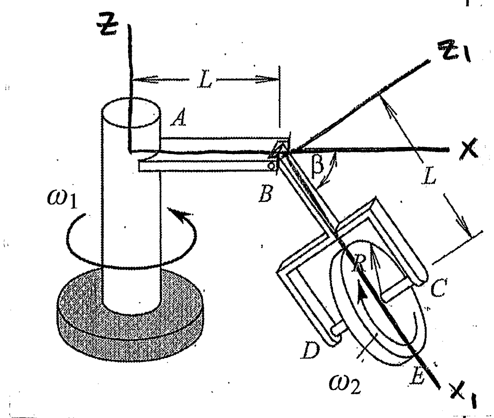
$\omega_1 = 3 \text{ rad/s}$

$\omega_3 = 5000 \text{ rpm} = 523.6 \text{ rad/s}$

$\theta = 75°$

$$\boxed{\vec{\alpha} = -1.8 \hat{i}_1 - 1520.3 \hat{j}_1 \text{ rad/s}^2}$$

3) Find the velocity of E when it is at its lowest point.

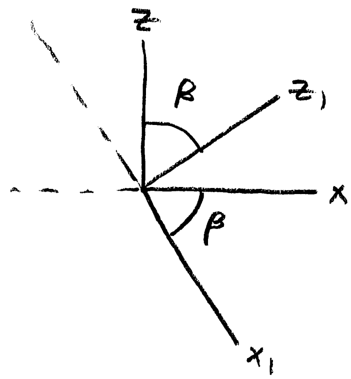


Measure its position from an inertial point. I choose A.

$$\vec{r} = L\hat{\imath} + (L + R\cos\phi)\hat{\imath}_1 - R\sin\phi\,\hat{\jmath}_1$$

But $\hat{\imath} = \cos\beta\,\hat{\imath}_1 + \sin\beta\,\hat{k}_1$

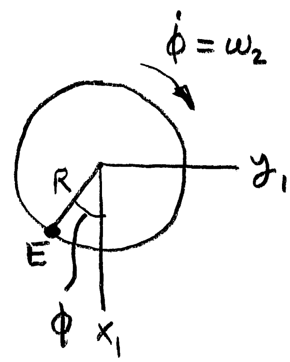
$$\vec{r} = (L + L\cos\beta + R\cos\phi)\hat{\imath}_1 - R\sin\phi\,\hat{\jmath}_1 + L\sin\beta\,\hat{k}_1$$



$$\vec{\omega}_1 = \omega_1\hat{k} = \omega_1(-\sin\beta\,\hat{\imath}_1 + \cos\beta\,\hat{k}_1)$$

$$\vec{\omega}_1 = -\omega_1\sin\beta\,\hat{\imath}_1 + \omega_1\cos\beta\,\hat{k}_1$$

$$\vec{\omega}_2 = -\omega_2\hat{k}_1$$



The $x_1y_1z_1$ frame rotates with an angular velocity $\vec{\omega}_1$

Find derivatives of unit vectors:

$$\dot{\hat{\imath}}_1 = \vec{\omega}_1 \times \hat{\imath}_1 = (-\omega_1\sin\beta\,\hat{\imath}_1 + \omega_1\cos\beta\,\hat{k}_1) \times \hat{\imath}_1$$

$$\dot{\hat{\imath}}_1 = \omega_1\cos\beta\,\hat{\jmath}_1$$

$$\dot{\hat{\jmath}}_1 = \vec{\omega}_1 \times \hat{\jmath}_1 = (-\omega_1\sin\beta\,\hat{\imath}_1 + \omega_1\cos\beta\,\hat{k}_1) \times \hat{\jmath}_1$$

$$\dot{\hat{\jmath}}_1 = -\omega_1\sin\beta\,\hat{k}_1 - \omega_1\cos\beta\,\hat{\imath}_1$$

$$\dot{\hat{k}}_1 = \vec{\omega}_1 \times \hat{k}_1 = (-\omega_1\sin\beta\,\hat{\imath}_1 + \omega_1\cos\beta\,\hat{k}_1) \times \hat{k}_1$$

$$\dot{\hat{k}}_1 = \omega_1\sin\beta\,\hat{\jmath}_1$$

Now find $\dot{\vec{r}}$, remembering that β is a constant

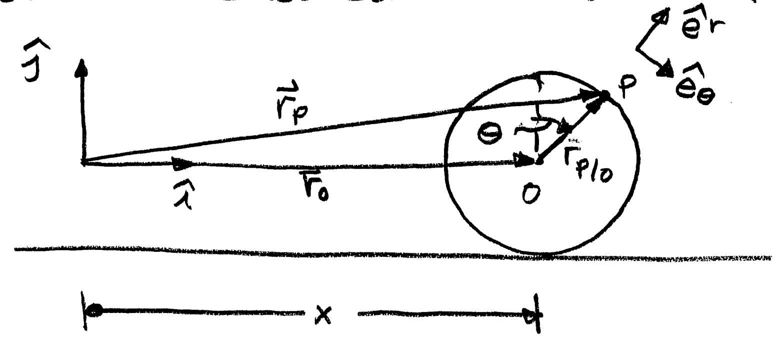
$$\dot{\vec{r}} = (-R\dot{\phi}\sin\phi)\hat{i}_1 + (L + L\cos\beta + R\cos\phi)\dot{\hat{i}}_1$$
$$- R\dot{\phi}\cos\phi\,\hat{j}_1 - R\sin\phi\,\dot{\hat{j}}_1 + L\sin\beta\,\dot{\hat{k}}_1$$

Substitute derivatives, $\dot{\phi} = \omega_2$, and $\phi = 0$ (E at bottom):

$$\dot{\vec{r}} = (L + R + L\cos\beta)\,\omega_1\cos\beta\,\hat{j}_1 - R\omega_2\,\hat{j}_1 + L\sin\beta\,(\omega_1\sin\beta\,\hat{j}_1)$$
$$\dot{\vec{r}} = \left[L\omega_1\cos\beta + R\omega_1\cos\beta + L\omega_1\cos^2\beta - R\omega_2 + L\omega_1\sin^2\beta\right]\hat{j}_1$$

$$\boxed{\dot{\vec{r}} = \left[(L + (L+R)\cos\beta)\,\omega_1 - R\omega_2\right]\hat{j}_1}$$

4) The motion must be measured relative to a fixed point.



$$\vec{\omega} = \dot{\theta}\,\hat{e}_z$$

$$\vec{r}_p = \vec{r}_o + \vec{r}_{p/o} = x\hat{\imath} + r\hat{e}_r$$

But $x = r\theta$ (the distance the wheel has rolled)

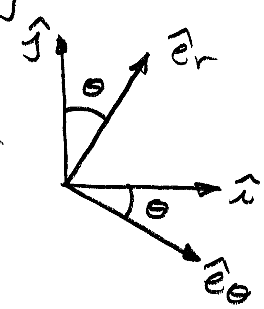
$$\vec{r}_p = r\theta\hat{\imath} + r\hat{e}_r$$

$$\dot{\vec{r}}_p = r\dot{\theta}\hat{\imath} + r\dot{\hat{e}}_r = r\dot{\theta}\hat{\imath} + r\vec{\omega}\times\hat{e}_r = r\dot{\theta}\hat{\imath} + r\dot{\theta}\,\hat{e}_z\times\hat{e}_r$$

$$\dot{\vec{r}}_p = r\dot{\theta}\hat{\imath} + r\dot{\theta}\hat{e}_\theta$$

$$\ddot{\vec{r}}_p = r\ddot{\theta}\hat{\imath} + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta = r\ddot{\theta}\hat{\imath} + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

Now transform $\hat{\imath}$ into the rotating frame:



$$\hat{\imath} = \cos\theta\,\hat{e}_\theta + \sin\theta\,\hat{e}_r$$

$$\ddot{\vec{r}}_p = r\ddot{\theta}\cos\theta\,\hat{e}_\theta + r\ddot{\theta}\sin\theta\,\hat{e}_r + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

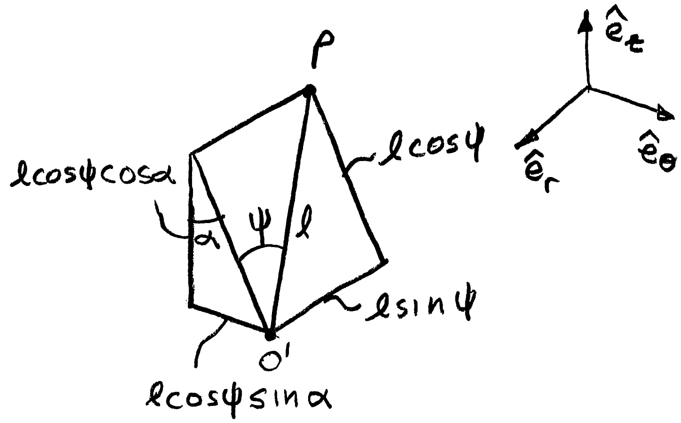
$$\ddot{\vec{r}}_p = \left[r\ddot{\theta}\sin\theta - r\dot{\theta}^2\right]\hat{e}_r + r\ddot{\theta}\left[1+\cos\theta\right]\hat{e}_\theta$$

But $\omega = \alpha t = \dot{\theta} \Rightarrow \ddot{\theta} = \alpha \Rightarrow \theta = \frac{\alpha t^2}{2}$

$$\ddot{\vec{r}}_p = \left[r\alpha\sin\frac{\alpha t^2}{2} - r\alpha^2 t^2\right]\hat{e}_r + r\alpha\left[1+\cos\frac{\alpha t^2}{2}\right]\hat{e}_\theta$$

$$\boxed{\ddot{\vec{r}}_p = r\alpha\left[\sin\frac{\alpha t^2}{2} - \alpha t^2\right]\hat{e}_r + r\alpha\left[1+\cos\frac{\alpha t^2}{2}\right]\hat{e}_\theta}$$

5) $\vec{\omega} = \frac{v}{R}\hat{e}_z$



Find derivatives:

$$\dot{\hat{e}}_r = \frac{v}{R}\hat{e}_z \times \hat{e}_r = \frac{v}{R}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = \frac{v}{R}\hat{e}_z \times \hat{e}_\theta = -\frac{v}{R}\hat{e}_r$$

$$\dot{\hat{e}}_z = 0$$

$$\vec{r}_P = \vec{r}_{O'} + \vec{r}_{P/O'}$$

$$\vec{r}_{O'} = R\hat{e}_r$$

$$\dot{\vec{r}}_{O'} = R\dot{\hat{e}}_r = R\frac{v}{R}\hat{e}_\theta = v\hat{e}_\theta$$

$$\ddot{\vec{r}}_{O'} = v\dot{\hat{e}}_\theta = -\frac{v^2}{R}\hat{e}_r$$

$$\vec{r}_{P/O'} = -l\sin\psi\,\hat{e}_r - l\cos\psi\sin\alpha\,\hat{e}_\theta + l\cos\psi\cos\alpha\,\hat{e}_z$$

$$\dot{\vec{r}}_{P/O'} = -l\dot{\psi}\cos\psi\,\hat{e}_r - l\sin\psi\,\dot{\hat{e}}_r + l\dot{\psi}\sin\psi\sin\alpha\,\hat{e}_\theta - l\cos\psi\sin\alpha\,\dot{\hat{e}}_\theta$$
$$- l\dot{\psi}\sin\psi\cos\alpha\,\hat{e}_z$$

$$\dot{\vec{r}}_{P/O'} = -l\dot{\psi}\cos\psi\,\hat{e}_r - \frac{vl}{R}\sin\psi\,\hat{e}_\theta + l\dot{\psi}\sin\psi\sin\alpha\,\hat{e}_\theta + \frac{vl}{R}\cos\psi\sin\alpha\,\hat{e}_r$$
$$- l\dot{\psi}\sin\psi\cos\alpha\,\hat{e}_z$$

$$\dot{\vec{r}}_{P/O'} = \left[\frac{vl}{R}\cos\psi\sin\alpha - l\dot{\psi}\cos\psi\right]\hat{e}_r + \left[l\dot{\psi}\sin\psi\sin\alpha - \frac{vl}{R}\sin\psi\right]\hat{e}_\theta$$
$$- l\dot{\psi}\sin\psi\cos\alpha\,\hat{e}_z$$

$$\ddot{\vec{r}}_{P/O'} = \left[-\frac{vl}{R}\dot{\psi}\sin\psi\sin\alpha - l\ddot{\psi}\cos\psi + l\dot{\psi}^2\sin\psi\right]\hat{e}_r$$
$$+ \left[\frac{vl}{R}\cos\psi\sin\alpha - l\dot{\psi}\cos\psi\right]\frac{v}{R}\hat{e}_\theta$$
$$+ \left[l\ddot{\psi}\sin\psi\sin\alpha + l\dot{\psi}^2\cos\psi\sin\alpha - \frac{vl}{R}\dot{\psi}\cos\psi\right]\hat{e}_\theta$$
$$- \left[l\dot{\psi}\sin\psi\sin\alpha - \frac{vl}{R}\sin\psi\right]\frac{v}{R}\hat{e}_r$$
$$- l\ddot{\psi}\sin\psi\cos\alpha\,\hat{e}_z - l\dot{\psi}^2\cos\psi\cos\alpha\,\hat{e}_z$$

$$\ddot{\vec{r}}_{P/O'} = \left[-\frac{2vl}{R}\dot{\psi}\sin\psi\sin\alpha - l\ddot{\psi}\cos\psi + l\dot{\psi}^2\sin\psi + \frac{v^2 l}{R^2}\sin\psi\right]\hat{e}_r$$

$$+ \left[\frac{v^2 l}{R^2}\cos\psi\sin\alpha - \frac{2vl}{R}\dot{\psi}\cos\psi + l\ddot{\psi}\sin\psi\sin\alpha + l\dot{\psi}^2\cos\psi\sin\alpha\right]\hat{e}_\theta$$

$$- l\cos\alpha\left[\ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi\right]\hat{e}_z$$

Total acceleration:

$$\boxed{\begin{aligned}\ddot{\vec{r}}_P &= \ddot{\vec{r}}_{O'} + \ddot{\vec{r}}_{P/O'} \\ &= \left[-\frac{2vl}{R}\dot{\psi}\sin\psi\sin\alpha - l\ddot{\psi}\cos\psi + l\dot{\psi}^2\sin\psi + \frac{v^2 l}{R^2}\sin\psi - \frac{v^2}{R}\right]\hat{e}_r \\ &+ \left[\frac{v^2 l}{R^2}\cos\psi\sin\alpha - \frac{2vl}{R}\dot{\psi}\cos\psi + l\ddot{\psi}\sin\psi\sin\alpha + l\dot{\psi}^2\cos\psi\sin\alpha\right]\hat{e}_\theta \\ &- l\cos\alpha\left[\ddot{\psi}\sin\psi + \dot{\psi}^2\cos\psi\right]\hat{e}_z\end{aligned}}$$

Where

$$\psi = \psi_0 \sin\beta t$$

$$\dot{\psi} = \psi_0 \beta\cos\beta t$$

$$\ddot{\psi} = -\psi_0\beta^2\sin\beta t$$