HW #3 — MEEN 537 — Dustan Kraus

1a ⇒ 4-2) $S(a)p = a \times p$ ⇐ Verify by direct calculation $\quad a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$S(a)\,p = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} -a_z p_y + a_y p_z \\ a_z p_x - a_x p_z \\ -a_y p_x + a_x p_y \end{bmatrix}$$

they're the same!

$$a \times p = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ p_x & p_y & p_z \end{vmatrix} = \begin{bmatrix} a_y p_z - a_z p_y \\ -a_x p_z + a_z p_x \\ a_x p_y - p_x a_y \end{bmatrix}$$

1b → 4-10) $e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$ Given $S \in so(3)$, show $e^S \in SO(3)$

$e^A e^B = e^{A+B}$, $AB = BA$, $\det(e^A) = e^{Tr(A)}$ $\quad so(3)$ = skew symmetric

for $SO(3)$ 1) $R^T = R^{-1}$ 2) columns & rows are orthogonal 3) Each column & row is a unit vector 4) $\det R = 1$

1) $e^S e^{S^T} = e^{S+S^T} = e^0 = I$ ✓ means $(e^S)^T = (e^S)^{-1}$:)

4) $\det(e^S) = e^{Tr(S)}$ but $Tr(S) = 0 \Rightarrow e^{Tr(S)} = e^0 = 1$ ✓

1c → 4-13) $R = R_{z,\psi} R_{y,\theta} R_{z,\phi}$ show $\frac{d}{dt}R = S(\omega)R$ where

$\omega = \{c_\psi s_\theta \dot\phi - s_\psi \dot\theta\}\hat{i} + \{s_\psi s_\theta \dot\phi + c_\psi \dot\theta\}\hat{j} + \{\dot\psi + c_\theta \dot\phi\}\hat{k}$

from eq. 2.26 on pg 54 but switch ψ & ϕ

$$R = \begin{bmatrix} c_\psi c_\theta c_\phi - s_\psi s_\phi & -c_\psi c_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta \\ s_\psi c_\theta c_\phi + c_\psi s_\phi & -s_\psi c_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta \\ -s_\theta c_\phi & s_\theta s_\phi & c_\theta \end{bmatrix}$$

from pg. 122 eq. 4.5

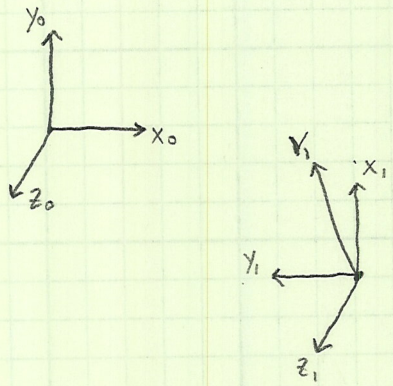
$$S(\omega) = \begin{bmatrix} 0 & -\dot\psi - c_\theta \dot\phi & s_\psi s_\theta \dot\phi + c_\psi \dot\theta \\ \dot\psi + c_\theta \dot\phi & 0 & -c_\psi s_\theta \dot\phi + s_\psi \dot\theta \\ -s_\psi s_\theta \dot\phi - c_\psi \dot\theta & c_\psi s_\theta \dot\phi - s_\psi \dot\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix}$$

☆ See matlab attached

1d → 4-15)

$$H_1^0 = \begin{array}{cccc} x_1^0 & y_1^0 & z_1^0 & \\ \end{array}$$
$$H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad V_1^1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

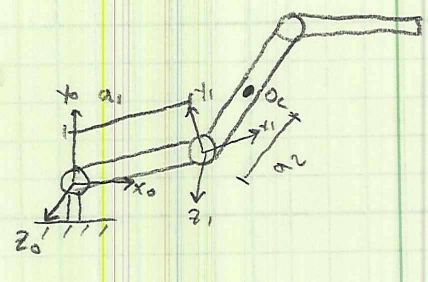
what is V_1^0 ?



by intuition: $\boxed{V_1^0 = [-1, 3, 0]^T}$

$$V_1^0 = H_1^0 V_1^1 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

1e → 4-16) Compute O_c & derive manipulator Jacobian matrix



$$\boxed{O_c = \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_1 + a_2 S_{12} \\ 0 \end{bmatrix}}$$

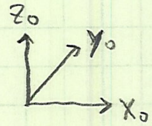
$$J(q) = \begin{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_1 + a_2 S_{12} \\ 0 \end{bmatrix} \right) & \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 C_{12} \\ a_2 S_{12} \\ 0 \end{bmatrix} \right) & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\boxed{J(q) = \begin{bmatrix} -a_1 S_1 - a_2 S_{12} & -a_2 S_{12} & 0 \\ a_1 C_1 + a_2 C_{12} & a_2 C_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

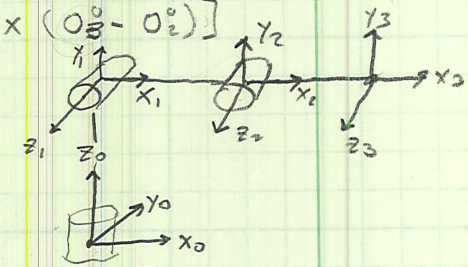
1f → 4.17) find J_{11} of ex. 4.9 & show agrees w/ eqn. 4.98 show det. agrees w/ 4.99 eq.

pg 145

Using new axes for O_0



$$J_{11} = \left[z_0^0 \times (O_3^0 - O_0^0) \quad z_1^0 \times (O_3^0 - O_1^0) \quad z_2^0 \times (O_3^0 - O_2^0) \right]$$



DH-Parameters:

	θ	d	a	α
1	q_1	a_1	0	$\pi/2$
2	q_2	0	a_2	0
3	q_3	0	a_3	0

from eq. 3.10

$$A_1 = T_1^0 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = T_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = T_3^2 = \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & z_1^0 & O_1^0 \end{array}$$

$$T_3^0 = A_1A_2A_3 = T_2^0 T_3^2 = \begin{bmatrix} C_1C_2 & -S_2C_1 & S_1 & a_2C_1C_2 \\ S_1C_2 & -S_1S_2 & -C_1 & a_2S_1C_2 \\ S_2 & C_2 & 0 & a_2S_2+a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & a_3C_3 \\ S_3 & C_3 & 0 & a_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ z_2^0 & O_2^0 \end{array}$$

trig ident.
$C(a+b) = C_aC_b - S_aS_b$
$S(a+b) = S_aC_b + C_aS_b$

$$T_3^0 = \begin{bmatrix} \vdots & \vdots & S_1 & a_3C_1C_2C_3 - a_3C_1S_2S_3 + a_2C_1C_2 \\ \vdots & \vdots & -C_1 & a_3S_1C_2C_3 - a_3S_1S_2S_3 + a_2S_1C_2 \\ \vdots & \vdots & 0 & a_3C_2S_3 + a_2S_2 + a_1 + a_3S_2C_3 \\ \vdots & \vdots & 0 & 1 \end{bmatrix} \quad O_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ z_3^0 & O_3^0 \end{array}$$

⇒ simplified via trig

$$J_{11} = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2C_1C_2 + a_3C_1C_{23} \\ a_2S_1C_2 + a_3S_1C_{23} \\ a_1 + a_2S_2 + a_3S_{23} \end{bmatrix} \quad \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_2C_1C_2 + a_3C_1C_{23} \\ a_2S_1C_2 + a_3S_1C_{23} \\ a_2S_2 + a_3S_{23} \end{bmatrix} \quad \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a_3C_1C_{23} \\ a_3S_1C_{23} \\ a_3S_{23} \end{bmatrix} \right]$$

$$S(a)b = a \times b$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ C_1 & S_1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -S_1 \\ C_1 & S_1 & 0 \end{bmatrix}$$

$$\boxed{J_{11} = \begin{bmatrix} -a_2S_1C_2 - a_3S_1C_{23} & -a_2C_1S_2 - a_3C_1S_{23} & -a_3C_1S_{23} \\ a_2C_1C_2 + a_3C_1C_{23} & -a_2S_1S_2 - a_3S_1S_{23} & -a_3S_1S_{23} \\ 0 & a_2C_2 + a_3C_{23} & a_3C_{23} \end{bmatrix}} \longleftarrow$$

$$\det(T_{11}) = -a_2 a_3 [a_2 c_2 s_3 - a_3 s_2 + a_3 s_2 c_3^2 + a_3 c_2 c_3 s_3]$$

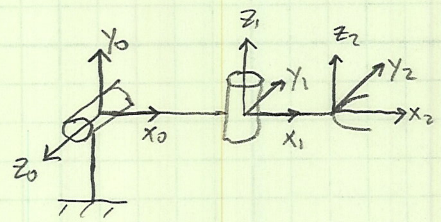
$$= -a_2 a_3 [a_2 c_2 s_3 + a_3 s_2(-1 + c_3^2) + a_3 c_2 c_3 s_3]$$

$$= -a_2 a_3 [a_2 c_2 s_3 - a_3 s_2 s_3^2 + a_3 c_2 c_3 s_3]$$

$$= -a_2 a_3 s_3 [a_2 c_2 - a_3 s_2 s_3 + a_3 c_2 c_3] \quad \xi \quad c_{23} = c_2 c_3 - s_2 s_3$$

$$\boxed{\det(T_{11}) = -a_2 a_3 s_3 [a_2 c_2 + a_3 c_{23}]} \longleftarrow$$

2a)



	Θ	d	a	α
1	θ_1	0	a_1	$-\pi/2$
2	θ_2	0	a_2	0

$$_n J(q) = \begin{bmatrix} Z_0^2 \times (O_2^2 - O_0^2) & Z_1^2 \times (O_2^2 - O_1^2) \\ Z_0^2 & Z_1^2 \end{bmatrix} = \begin{bmatrix} Z_0^2 \times (-O_0^2) & Z_1^2 \times (-O_1^2) \\ Z_0^2 & Z_1^2 \end{bmatrix}$$

$$Z_0^2 = R_0^2 Z_0^0 \qquad O_1^2 = T_0^2 \, O_1^0 = (T_2^0)^{-1} O_2^1$$

$$Z_1^2 = R_0^2 Z_1^0 \qquad O_0^2 = T_0^2 \, O_0^0 = (T_2^0)^{-1} O_0^0$$

see matlab

2b) See matlab

Contents

Problem 1c - 4-13

Create Variables

```
syms psi theta phi psidot thetadot phidot

% Create R
Rz_psi = [...
    cos(psi),   -sin(psi),  0;...
    sin(psi),   cos(psi),   0;...
    0       ,   0       ,   1];
Ry_th = [...
    cos(theta),     0,  sin(theta);...
    0         ,     1,  0;...
    -sin(theta),    0,  cos(theta)];
Rz_phi = [...
    cos(phi),   -sin(phi),  0;...
    sin(phi),   cos(phi),   0;...
    0       ,   0       ,   1];
R = Rz_psi*Ry_th*Rz_phi;

% Create Skew Symmetric Matrix
w = [cos(psi)*sin(theta)*phidot - sin(psi)*thetadot;...
    sin(psi)*sin(theta)*phidot + cos(psi)*thetadot;...
    psidot + cos(theta)*phidot];

S_w = [0, -w(3), w(2);...
    w(3), 0, -w(1);...
    -w(2), w(1), 0]; %equation 4.5 on pg 122

% Find Derivative using S*R
dR_dt_skew = simplify(S_w*R);

% Find Derivative using symbolic derivative
dR_dt_sym = diff(R,psi)*psidot + diff(R,theta)*thetadot +...
    diff(R,phi)*phidot;
dR_dt_sym = simplify(dR_dt_sym);

% Compare the two derivatives
check_equality = simplify(dR_dt_sym - dR_dt_skew)
```

```
check_equality =

[ 0, 0, 0]
[ 0, 0, 0]
```

```
[ 0, 0, 0]
```

Problem 1f - 4-17 Check

```
clear all
close all
clc
syms th1 th2 th3 a1 a2 a3
A1 = calc_A(th1, a1, 0, 90);
A2 = calc_A(th2, 0, a2, 0);
A3 = calc_A(th3, 0, a3, 0);

T2_0 = A1*A2;
T2_0 = simplify(T2_0);
T3_0 = A1*A2*A3;
T3_0 = simplify(T3_0);

J11 = simplify([cross([0;0;1],T3_0(1:3,4)),...
    cross([sin(th1); -cos(th1); 0], T3_0(1:3,4)-A1(1:3,4)),...
    cross([sin(th1); -cos(th1); 0], T3_0(1:3,4) - T2_0(1:3,4))]);

det = simplify(det(J11));
```

Problem 2a

```
close all
clear all
clc
syms th1 th2 a1 a2
A1 = calc_A(th1, 0, a1, -90);
A2 = calc_A(th2, 0, a2, 0);
T2_0 = simplify(A1*A2);
z0_0 = [0;0;1];
z1_0 = A1(1:3, 3);
o2_0 = T2_0(1:3,4);
o1_0 = A1(1:3,4);
J2_0 = simplify([cross(z0_0,o2_0),cross(z1_0,o2_0-o1_0);...
    z0_0, z1_0]); %jacobian at point 2 in frame 0
% i) Find jacobian at point 2 in frame 2 by direct calculation
o2_1 = A2(1:3,4);
o0_0 = [0;0;0];
T0_2 = T2_0^-1;
o0_2 = simplify(T0_2*[o0_0;1]);
o0_2 = o0_2(1:3);
o1_2 = simplify(T0_2*[o1_0;1]);
o1_2 = o1_2(1:3);
R0_2 = simplify(T2_0(1:3, 1:3).');
z0_2 = simplify(R0_2*z0_0);
z1_2 = simplify(R0_2*z1_0);
J2_2_direct = simplify([cross(z0_2,-o0_2), cross(z1_2,-o1_2); z0_2, z1_2])

% ii) Find jacobian at point 2 in frame 2 using rotation matrix
R2_0 = T2_0(1:3, 1:3); % p0 = Rn_0*pn
R0_2 = R2_0.'; %if I only use ', conj appears
```

```
mat_0 = [0,0,0;0,0,0;0,0,0];
J2_2_rot = simplify([R0_2, mat_0;mat_0, R0_2]*J2_0)
```

```
J2_2_direct =

[                 0,  0]
[                 0, a2]
[ a1 + a2*cos(th2),  0]
[         -sin(th2),  0]
[         -cos(th2),  0]
[                 0,  1]


J2_2_rot =

[                 0,  0]
[                 0, a2]
[ a1 + a2*cos(th2),  0]
[         -sin(th2),  0]
[         -cos(th2),  0]
[                 0,  1]
```

Problem 2b

```
close all
clear all
clc
syms th1 th2 a1 a2
A1 = calc_A(th1, 0, a1, -90);
A2 = calc_A(th2, 0, a2, 0);
T2_0 = simplify(A1*A2);
z0_0 = [0;0;1];
z1_0 = A1(1:3, 3);
o2_0 = T2_0(1:3,4);
o1_0 = A1(1:3,4);
J2_0 = simplify([cross(z0_0,o2_0),cross(z1_0,o2_0-o1_0);...
    z0_0, z1_0]); %jacobian at point 2 in frame 0

J2_0_i = subs(J2_0,[th1 th2 a1 a2],[0 pi/4 1 1]);
J2_0_ii = subs(J2_0,[th1 th2 a1 a2],[0 pi/2 1 1]);
J2_0_iii = subs(J2_0,[th1 th2 a1 a2],[pi/4 pi/4 1 1]);
J2_0_iv = subs(J2_0,[th1 th2 a1 a2],[0 0 1 1]);
J2_0_v = subs(J2_0,[th1 th2 a1 a2],[0 0 1 1]);

Fi = [-1;0;0;0;0;0];
Fii = [-1;0;0;0;0;0];
Fiii = [-1;-1;0;0;0;0];
Fiv = [0;0;1;0;0;0];
Fv = [1;0;0;0;0;0];

Tau_i = double(J2_0_i.'*Fi)
Tau_ii = double(J2_0_ii.'*Fii)
Tau_iii = double(J2_0_iii.'*Fiii)
```

```
Tau_iv = double(J2_0_iv.'*Fiv)
Tau_v = double(J2_0_v.'*Fv)

%If I switch to reaction forces, then the signs switch
```

```
Tau_i =

         0
    0.7071


Tau_ii =

     0
     1


Tau_iii =

     0
     1


Tau_iv =

     0
    -1


Tau_v =

     0
     0
```

Published with MATLAB® R2014b