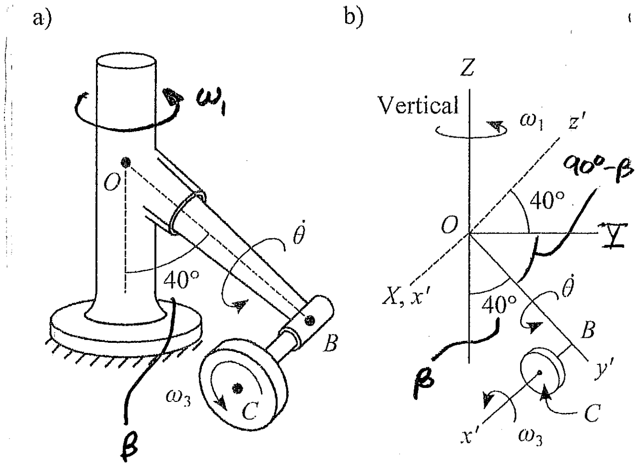
1) Problem 2.28 from Baruh



Find the total angular velocity of the disk at $t=3$ s in terms of the ~~XYZ~~ frame, which is attached to the vertical shaft.

$$\vec{\omega} = \omega_1 \hat{K} + \dot{\theta}\hat{j}' + \omega_3 \hat{i}'$$

where the $x'y'z'$ frame is attached to the arm.

Find the relationship between XYZ and $x'y'z'$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [R]\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [R_{y'}(\theta)]\,[R_X(-(90°-\beta))]\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = [R_y(\theta)][R_X(\beta-90°)]\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

But $\theta = \frac{\pi}{20}\cos 2t$, $\quad \theta(t=3) = 0.1508 \text{ rad} = 8.6415°$

$\dot{\theta} = -\frac{\pi}{10}\sin 2t$, $\quad \dot{\theta}(t=3) = 0.0878$ rad/s

$\beta = 40° = 0.6981$ rad

$\omega_1 = 0.5$ rad/s

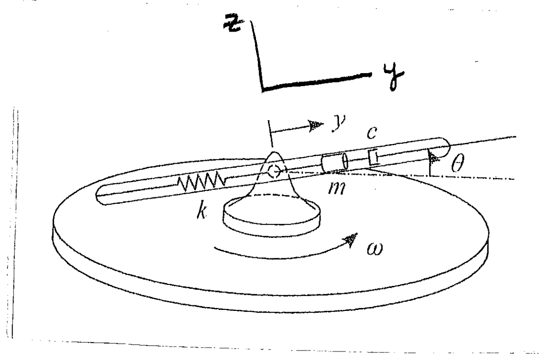
$\omega_3 = 7$ rad/s

$$[R] = \begin{bmatrix} 0.9886 & -0.1151 & -0.0966 \\ 0 & 0.6428 & -0.7660 \\ 0.1503 & 0.7573 & 0.6355 \end{bmatrix}$$

$$\{\omega\} = \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix} + [R]^T \begin{bmatrix} \omega_3 \\ \dot{\theta} \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 6.9205 \\ -0.7493 \\ -0.2433 \end{bmatrix} \text{rad/s}}$$

$$\boxed{\vec{\omega} = 6.92\hat{I} - 0.75\hat{J} - 0.24\hat{K}}$$

2) Problem 2.40 from Baruh.



$\omega = 0.2 \text{ rad/s}$

$y = 40\text{cm} = 0.4 \text{ m}$

$\dot{y} = -30\text{cm/s} = -0.3\text{m/s}$

$\ddot{y} = -4\text{ cm/s}^2 = -0.04\text{m/s}^2$

$\theta(t) = \frac{\pi}{6}\sin 2t \;;\; \theta(t=\pi) = 0$

$\dot{\theta}(t) = \frac{\pi}{3}\cos 2t, \;\; \dot{\theta}(t=\pi) = \frac{\pi}{3}$

$\ddot{\theta}(t) = -\frac{2\pi}{3}\sin 2t, \;\; \ddot{\theta}(t=\pi) = 0$

$\vec{\omega} = \dot{\theta}\hat{\imath} + \omega\sin\theta\hat{\jmath} + \omega\cos\theta\hat{k} = \frac{\pi}{3}\hat{\imath} + 0.2\hat{k}$

$\vec{r} = y\hat{\jmath} = 0.4\hat{\jmath}$

$\vec{v} = (\dot{\vec{r}})_{rel} + \vec{\omega}\times\vec{r}$

$(\dot{\vec{r}})_{rel} = \dot{y}\hat{\jmath} = -0.3\hat{\jmath}$

$\vec{\omega}\times\vec{r} = (\dot{\theta}\hat{\imath} + \omega\sin\theta\hat{\jmath} + \omega\cos\theta\hat{k})\times y\hat{\jmath}$

$\vec{\omega}\times\vec{r} = y\dot{\theta}\hat{k} - y\omega\cos\theta\hat{\imath}$

$\vec{v} = -y\omega\cos\theta\hat{\imath} + \dot{y}\hat{\jmath} + y\dot{\theta}\hat{k}$

$\vec{v} = -(0.4)(0.2)(1)\hat{\imath} - 0.3\hat{\jmath} + (0.4)(\frac{\pi}{3})\hat{k}$

$$\boxed{\vec{v} = -0.08\hat{\imath} - 0.3\hat{\jmath} + 0.419\hat{k} \text{ m/s}}$$

$\vec{a} = (\dot{\vec{v}})_{rel} + \vec{\omega}\times\vec{v}$

$(\dot{v})_{rel} = (-\dot{y}\omega\cos\theta + y\omega\dot{\theta}\sin\theta)\hat{\imath} + \ddot{y}\hat{\jmath} + (\dot{y}\dot{\theta} + y\ddot{\theta})\hat{k}$

$(\dot{v})_{rel} = (-(-0.3)(0.2)(1) + (0.4)(0.2)(\frac{\pi}{3})(0))\hat{\imath} - 0.04\hat{\jmath}$
$\quad + ((-0.3)(\frac{\pi}{3}) + (0.4)(0))\hat{k}$

$(\dot{v})_{rel} = 0.06\hat{\imath} - 0.04\hat{\jmath} - 0.314\hat{k}$

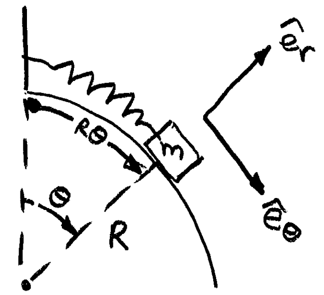
$$\vec{\omega} \times \vec{v} = \left(\frac{\pi}{3}\hat{\imath} + 0.2\hat{k}\right) \times \left(-0.08\hat{\imath} - 0.3\hat{\jmath} + 0.419\hat{k}\right)$$

$$= -\frac{(0.3)\pi}{3}\hat{k} - \frac{0.419\pi}{3}\hat{\jmath} - (0.2)(0.08)\hat{\jmath} + (0.2)(0.3)\hat{\imath}$$

$$= -0.314\hat{k} - 0.455\hat{\jmath} + 0.06\hat{\imath}$$

$$\boxed{\vec{a} = 0.12\hat{\imath} - 0.495\hat{\jmath} - 0.628\hat{k} \ \ m/s^2}$$

3) Mass sliding over disk:



Kinematics:

$\vec{\omega} = \dot{\theta}\,\hat{e}_z$

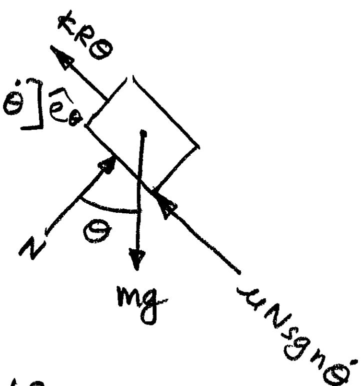
$\Rightarrow \dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta, \quad \dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r, \quad \dot{\hat{e}}_z = 0$

$\vec{r} = R\hat{e}_r$

$\dot{\vec{r}} = R\dot{\hat{e}}_r = R\dot{\theta}\,\hat{e}_\theta$

$\ddot{\vec{r}} = R\ddot{\theta}\,\hat{e}_\theta + R\dot{\theta}\,\dot{\hat{e}}_\theta = -R\dot{\theta}^2\,\hat{e}_r + R\ddot{\theta}\,\hat{e}_\theta$

Kinetics:



$\vec{F} = [N - mg\cos\theta]\,\hat{e}_r + [mg\sin\theta - kR\theta - \mu N \operatorname{sgn}\dot{\theta}]\,\hat{e}_\theta$

$[\vec{F} = m\ddot{\vec{r}}]$

$\hat{e}_r: \; N - mg\cos\theta = -mR\dot{\theta}^2$

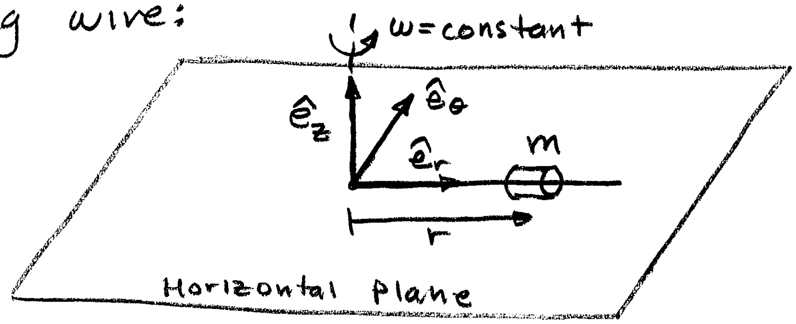
$\hat{e}_\theta: \; mg\sin\theta - kR\theta - \mu N \operatorname{sgn}\dot{\theta} = mR\ddot{\theta}$

From $\hat{e}_r$ equation, $N = mg\cos\theta - mR\dot{\theta}^2$

Substitute into $\hat{e}_\theta$ equation;

$$\boxed{\begin{gathered} mg\sin\theta - kR\theta - \mu[mg\cos\theta - mR\dot{\theta}^2]\operatorname{sgn}\dot{\theta} = mR\ddot{\theta} \\ \text{or} \\ mR\ddot{\theta} + kR\theta - mg\sin\theta + \mu m(g\cos\theta - R\dot{\theta}^2)\operatorname{sgn}\dot{\theta} = 0 \\ \text{or} \\ \ddot{\theta} + \frac{k}{m}\theta - \frac{g}{R}\sin\theta + \frac{\mu}{R}(g\cos\theta - R\dot{\theta}^2)\operatorname{sgn}\dot{\theta} = 0 \end{gathered}}$$

4) Bead on rotating wire:



Kinematics:

$\vec{r} = r\hat{e}_r$

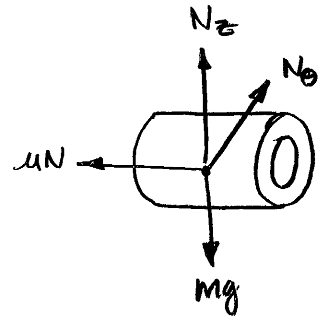
$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$

$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\omega\hat{e}_\theta$

$\ddot{\vec{r}} = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\omega\hat{e}_\theta + r\omega\dot{\hat{e}}_\theta$

$\ddot{\vec{r}} = (\ddot{r} - r\omega^2)\hat{e}_r + 2\dot{r}\omega\hat{e}_\theta$

Kinetics:



$\vec{F} = -\mu N\hat{e}_r + N_\theta\hat{e}_\theta + (N_z - mg)\hat{e}_z$

$[\vec{F} = m\ddot{\vec{r}}]$

① $\hat{e}_r$: $-\mu N = m(\ddot{r} - r\omega^2)$

② $\hat{e}_\theta$: $N_\theta = 2m\dot{r}\omega$

③ $\hat{e}_z$: $N_z - mg = 0 \Rightarrow N_z = mg$

But $N = \sqrt{N_\theta^2 + N_z^2}$ ④

Substitute ② and ③ into ④: $N = \sqrt{4m^2\dot{r}^2\omega^2 + m^2g^2}$

$N = m\sqrt{4\dot{r}^2\omega^2 + g^2}$

From ①:

$-\mu \cancel{m}\sqrt{4\dot{r}^2\omega^2 + g^2} = \cancel{m}(\ddot{r} - r\omega^2)$

$\boxed{\ddot{r} = r\omega^2 - \mu\sqrt{4\dot{r}^2\omega^2 + g^2}}$ This is the equation of motion

$\omega = 10$ rad/s, $\mu = 0.5$, $r(0) = 5\text{cm} = 0.05\text{ m}$, $\dot{r}(0) = 0$

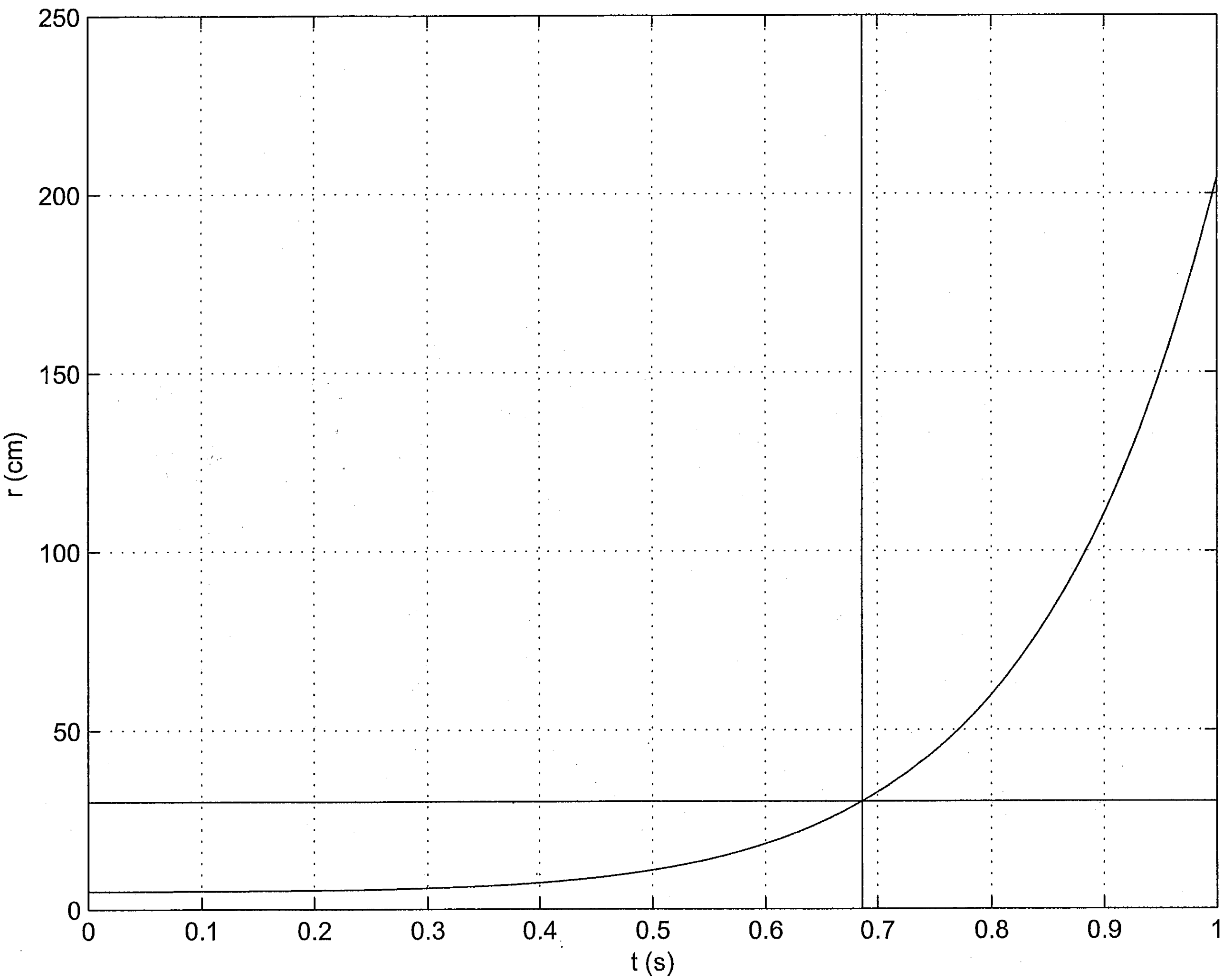
$\dot{r} = v$

$\dot{v} = r\omega^2 - \mu\sqrt{4v^2\omega^2 + g^2}$

I used Matlab's 'ode45' function to solve. See attached files.

The time at which it reaches 30 cm is

$\boxed{t = 0.6868\text{ s}}$

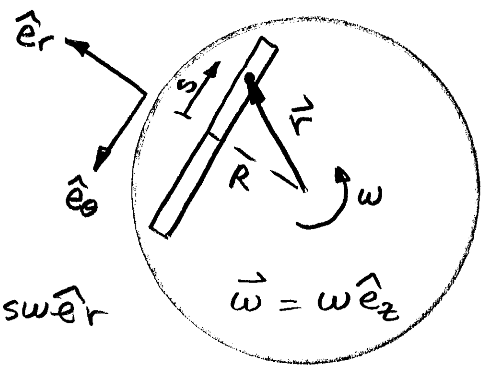

250
200
150
100
50
0
r (cm)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t (s)

```
% This m-file numerically solves for the radial position of a bead↙
released
% from rest on a rotating wire.

clear;
clc;

% Setup model parameters
global omega mu g;
omega = 10; % Angular velocity of wire
mu = 0.5;   % Friction coefficient
g = 9.81;   % Gravity

% Initial conditions
y0 = [0.05 0];

% Time vector
t = 0:.00001:1;

% Use ode45 to solve equation of motion
[t,y] = ode45('bead_wire_func', t, y0);

% Extract position and velocity
r = y(:,1);
v = y(:,2);
clear y;

% Plot results
plot(t,r*100,'r');
xlabel('t (s)');
ylabel('r (cm)');
grid on;
axis([0 1 0 250]);

% Add a line at 30 cm
hold on;
line([0 1],[30 30]);

% Find when it reaches 30 cm
i = find(r < 30/100);
t1 = t(length(i))
clear i;
```

```
% Add a vertical line at that time
hold on;
line([t1 t1],[0 250]);
```

```
unction ydot = bead_wire_func(t,y)

global omega mu g;

ydot = [y(2); y(1)*omega^2 - mu*sqrt(4*y(2)^2*omega^2 + g^2)];
```

5) Speed governor:



$\dot{\hat{e}}_r = \omega \hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\omega \hat{e}_r$

Kinematics:

$\vec{r} = R\hat{e}_r - s\hat{e}_\theta$

$\dot{\vec{r}} = R\dot{\hat{e}}_r - \dot{s}\hat{e}_\theta - s\dot{\hat{e}}_\theta = R\omega\hat{e}_\theta - \dot{s}\hat{e}_\theta + s\omega\hat{e}_r$

$\dot{\vec{r}} = s\omega\,\hat{e}_r + (R\omega - \dot{s})\,\hat{e}_\theta$

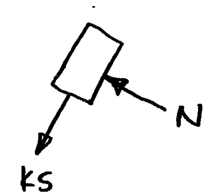
$\ddot{\vec{r}} = \dot{s}\omega\hat{e}_r + s\dot{\omega}\hat{e}_r + s\omega\dot{\hat{e}}_r + (R\dot{\omega} - \ddot{s})\hat{e}_\theta + (R\omega - \dot{s})\dot{\hat{e}}_\theta$

$\ddot{\vec{r}} = \dot{s}\omega\hat{e}_r + s\dot{\omega}\hat{e}_r + s\omega^2\hat{e}_\theta + (R\dot{\omega} - \ddot{s})\hat{e}_\theta - \omega(R\omega - \dot{s})\hat{e}_r$

$\ddot{\vec{r}} = (2\dot{s}\omega + s\dot{\omega} - R\omega^2)\,\hat{e}_r + (s\omega^2 + R\dot{\omega} - \ddot{s})\,\hat{e}_\theta$

Kinetics:

$\vec{F} = N\hat{e}_r + ks\,\hat{e}_\theta$



$[\vec{F} = m\ddot{\vec{r}}]$

$\hat{e}_r:\ N = m(2\dot{s}\omega + s\dot{\omega} - R\omega^2)$

$\hat{e}_\theta:\ ks = m(s\omega^2 + R\dot{\omega} - \ddot{s})$

a) From $\hat{e}_\theta$:

$\boxed{\ddot{s} + \left(\frac{k}{m} - \omega^2\right)s = R\dot{\omega}}$ or $\boxed{\ddot{s} = \left(\omega^2 - \frac{k}{m}\right)s + R\dot{\omega}}$

b) From $\hat{e}_r$:

$\boxed{N = m\left(2\dot{s}\omega + s\dot{\omega} - R\omega^2\right)}$

c) $\omega = 5\sin t$

First, find out how long it takes to complete one revolution:

one revolution

$\omega = \frac{d\theta}{dt} = 5\sin t \Rightarrow \int_0^{2\pi} d\theta = \int_0^t 5\sin t\,dt$

$2\pi = -5\cos t\Big|_0^t = 5(-\cos t + \cos 0) = 5(1 - \cos t)$

$\Rightarrow t = 1.8303\ s$

Now put the equation of motion in the form needed for a Runge-Kutta solution:

$$\dot{s} = v$$

$$\dot{v} = \left(\omega^2 - \frac{k}{m}\right)s + R\dot{\omega}$$

where $\omega = 5\sin t \Rightarrow \dot{\omega} = 5\cos t \Rightarrow \theta = 5(1-\cos t)$

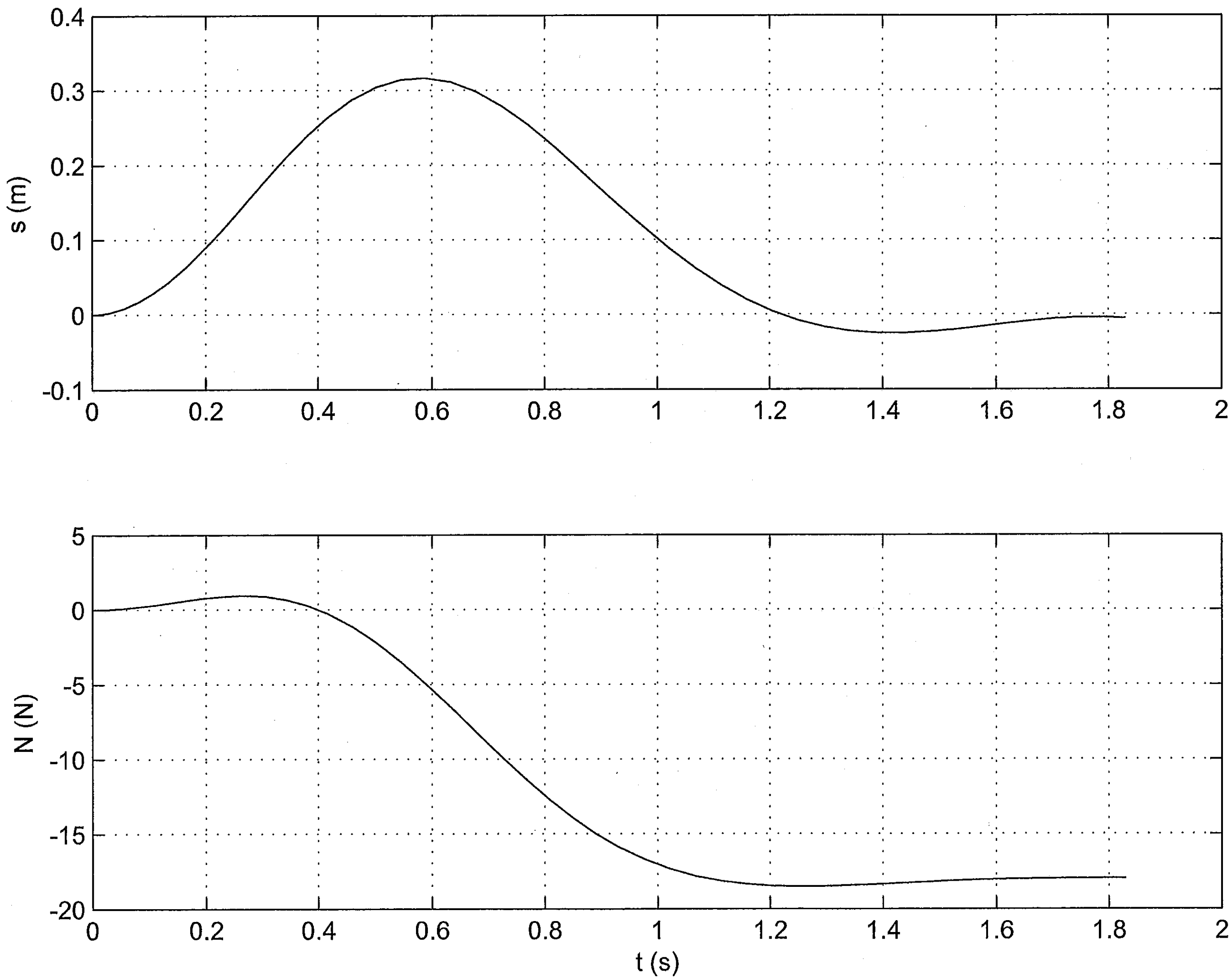
See the attached Matlab files and plots, which use $m = 0.75$ kg, $k = 25$ N/m, $R = 1$ m

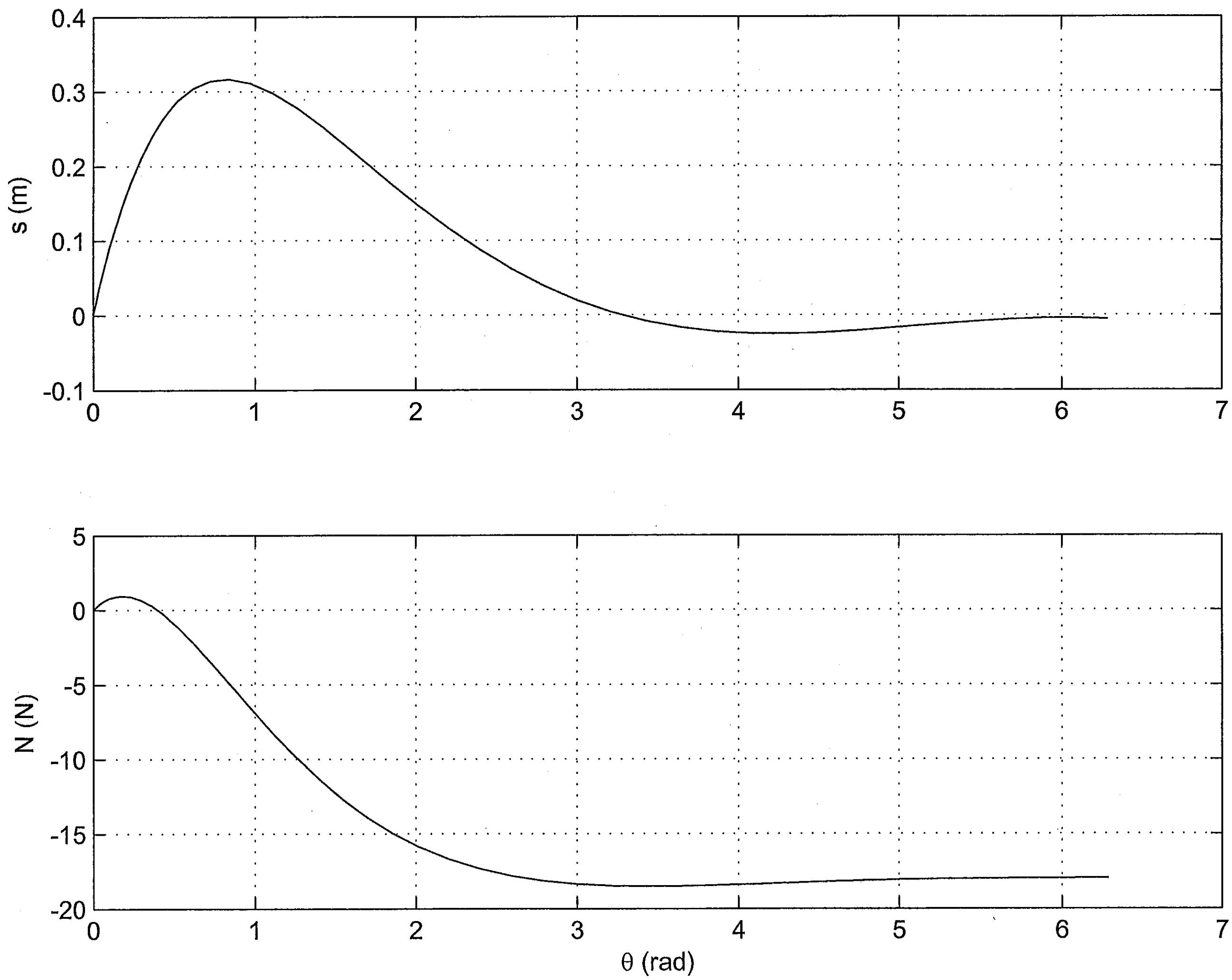
d) From Matlab,

$$\boxed{|s_{max}| = 0.3164 \text{ m}}$$

e) From Matlab,

$$\boxed{|N_{max}| = 18.5185 \text{ N}}$$


s (m)
0.4
0.3
0.2
0.1
0
-0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
N (N)
5
0
-5
-10
-15
-20
t (s)



s (m)
0.4
0.3
0.2
0.1
0
-0.1
0
1
2
3
4
5
6
7
N (N)
5
0
-5
-10
-15
-20
θ (rad)

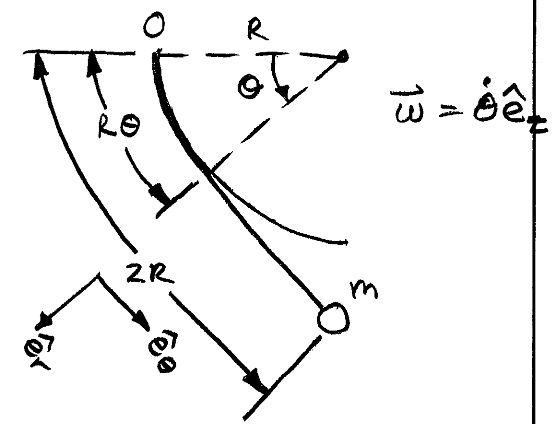
```
% This m-file solves for the motion of the speed governor.

clear;
clc;

% Setup model parameters
global m k R;
m = 0.75;       % Slider mass
k = 25;         % Spring constant
R = 1;          % Distance from center to slot
tf = acos(1 - 2*pi/5)

% Initial conditions
y0 = [0 0];

% Time span
tspan = [0 tf];

% Use ode45 to solve equation of motion
[t,y] = ode45('speed_governor_func', tspan, y0);

% Extract position and velocity
s = y(:,1);
v = y(:,2);
clear y;

%   Calculate theta(t), omega(t), and alpha(t)
theta = 5*(1-cos(t));
omega = 5*sin(t);
omegadot = 5*cos(t);

%   Calculate N(t)
N = m*(2*v.*omega + s.*omegadot - R*omega.^2);

% Plot results vs. time
subplot(2,1,1);
plot(t,s);
grid on;
ylabel('s (m)');
subplot(2,1,2);
plot(t,N);
grid on;
```

```
label('N (N)');
xlabel('t (s)');

% Plot results vs. angle
figure;
subplot(2,1,1);
plot(theta,s);
grid on;
ylabel('s (m)');
subplot(2,1,2);
plot(theta,N);
grid on;
ylabel('N (N)');
xlabel('\theta (rad)');

%   Find the maximum absolute value of s
s_max = max(abs(s))

%   Find the maximum absolute value of N
N_max = max(abs(N))
```

```
function ydot = speed_governor_func(t, y)

global m k R;

omega = 5*sin(t);
omegadot = 5*cos(t);

ydot = [y(2); (omega^2 - k/m)*y(1) + R*omegadot];
```

6) Pendulum:

$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$



Kinematics:

$$\vec{r} = R\hat{e}_r + (2R - R\theta)\hat{e}_\theta$$

$$\dot{\vec{r}} = R\dot{\hat{e}}_r - R\dot{\theta}\hat{e}_\theta + (2R - R\theta)\dot{\hat{e}}_\theta$$

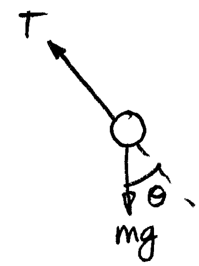
$$\dot{\vec{r}} = R\dot{\theta}\hat{e}_\theta - R\dot{\theta}\hat{e}_\theta - (2R - R\theta)\dot{\theta}\hat{e}_r$$

$$\dot{\vec{r}} = (R\theta - 2R)\dot{\theta}\hat{e}_r$$

$$\ddot{\vec{r}} = R\dot{\theta}^2\hat{e}_r + (R\theta - 2R)\ddot{\theta}\hat{e}_r + (R\theta - 2R)\dot{\theta}\dot{\hat{e}}_r$$

$$\ddot{\vec{r}} = (R\dot{\theta}^2 + R\theta\ddot{\theta} - 2R\ddot{\theta})\hat{e}_r + (R\theta - 2R)\dot{\theta}^2\hat{e}_\theta$$

Kinetics:



$$\vec{F} = mg\sin\theta\,\hat{e}_r + (mg\cos\theta - T)\hat{e}_\theta$$

$$[\vec{F} = m\ddot{\vec{r}}]$$

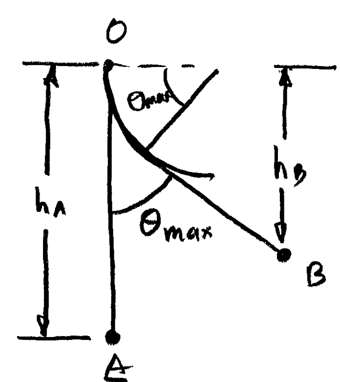
① $\hat{e}_r$: $mg\sin\theta = m(R\dot{\theta}^2 + R\theta\ddot{\theta} - 2R\ddot{\theta})$

② $\hat{e}_\theta$: $mg\cos\theta - T = m(R\theta - 2R)\dot{\theta}^2$

a) From ①:

$$\boxed{(2-\theta)R\ddot{\theta} - R\dot{\theta}^2 + g\sin\theta = 0}$$

b) Use energy methods to find θ_{max}



A = initial position
B = final position
Let $V = 0$ at point O

$$V_A = -mgh_A = -2mgR$$

$$V_B = -mgh_B = -mg[R\sin\theta_{max} + (2R - R\theta_{max})\cos\theta_{max}]$$

$$V_B = -mgR[\sin\theta_{max} + (2 - \theta_{max})\cos\theta_{max}]$$

$T_B = 0$ because @ $\theta_B = \theta_{max}$, $\dot{\theta}_B = 0$

$$T_A = \tfrac{1}{2}m\dot{\vec{r}}_A\cdot\dot{\vec{r}}_A = \tfrac{1}{2}m(R\theta - 2R)^2\dot{\theta}^2\Big|_{\theta=0,\ \dot{\theta}=\left(\frac{g}{2R}\right)^{\frac{1}{2}}}$$

$$T_A = mRg$$

$$T_A + V_A = T_B + V_B \Rightarrow \boxed{\theta_{max} = \frac{\pi}{2}}$$