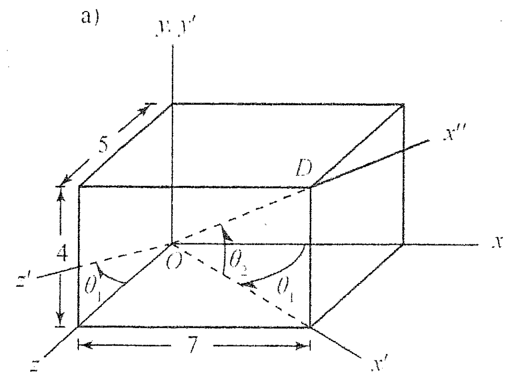
1) Problem 10.12 from Baruh



First find $[I_G]$ using xyz axes parallel to those shown.

From the appendix:

$$I_{Gxx} = \frac{m}{12}(4^2+5^2) = \frac{41m}{12}$$

$$I_{Gyy} = \frac{m}{12}(5^2+7^2) = \frac{74m}{12} \qquad I_{Gxy} = I_{Gxz} = I_{Gyz} = 0$$

$$I_{Gzz} = \frac{m}{12}(4^2+7^2) = \frac{65m}{12}$$

$$[I_G] = \frac{m}{12}\begin{bmatrix} 41 & 0 & 0 \\ 0 & 74 & 0 \\ 0 & 0 & 65 \end{bmatrix}$$

Use parallel axis theorem to find $[I_o]$:

$$\{d\} = \begin{bmatrix} -7/2 & -4/2 & -5/2 \end{bmatrix}^T$$

$$[I_o] = [I_G] + m\left[\{d\}^T\{d\}[1] - \{d\}\{d\}^T\right]$$

From Matlab:

$$[I_o] = m\begin{bmatrix} 13.67 & -7 & -8.75 \\ -7 & 24.67 & -5 \\ -8.75 & -5 & 21.67 \end{bmatrix}$$

Now find the rotation matrix:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \left[R_y(-\theta_1) \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \left[R_{z'}(\theta_2) \right] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\left[R_{z'}(\theta_2) \right] \left[R_y(-\theta_1) \right]}_{[R]} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\theta_1 = \tan^{-1}\left(\frac{5}{7}\right) = 35.54^\circ$$

$$\theta_2 = \tan^{-1}\left(\frac{4}{\sqrt{5^2+7^2}}\right) = 24.94^\circ$$

$$\left[I_o'' \right] = [R][I_o][R]^T$$

From Matlab:

$$\boxed{\left[I_o'' \right] = m \begin{bmatrix} 4.46 & 0.79 & 0.86 \\ 0.79 & 28.30 & -0.40 \\ 0.86 & -0.40 & 27.24 \end{bmatrix}}$$

```
% Problem 10.12 from Baruh, Applied Dynamics

clear;
clc;

% Ig (without m)
Ig = 1/12*[41 0 0; 0 74 0; 0 0 65]

% Io from parallel axis theorem (without m)
d = [-7/2 -4/2 -5/2]'
Io = Ig + d'*d*eye(3) - d*d'

% Rotation matrix
th1 = atand(5/7)
th2 = atand(4/sqrt(5^2+7^2))
R = Rz(th2)*Ry(-th1)

% Io'' from rotation matrix (without m)
Iopp = R*Io*R'
```

```
Ig =

    3.4167         0         0
         0    6.1667         0
         0         0    5.4167


d =

   -3.5000
   -2.0000
   -2.5000


Io =

   13.6667   -7.0000   -8.7500
   -7.0000   24.6667   -5.0000
   -8.7500   -5.0000   21.6667


th1 =

   35.5377


th2 =

   24.9380


R =

    0.7379    0.4216    0.5270
   -0.3431    0.9068   -0.2451
   -0.5812         0    0.8137


Iopp =

    4.4611    0.7931    0.8578
    0.7931   28.2979   -0.3988
    0.8578   -0.3988   27.2410

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```
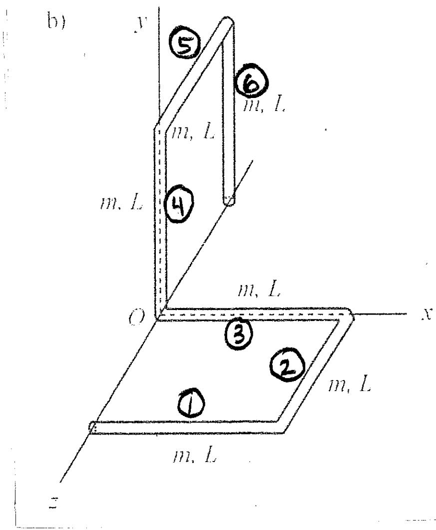

2) Problem 10.10 from Baruh

$m = 0.5$ kg, $L = 0.72$ m



First find the inertia matrix of each bar about its own center of mass. Then use the parallel axis theorem to find it about point O.

Bar	I_{xx}	I_{yy}	I_{zz}	d_x	d_y	d_z
1	0	$\frac{mL^2}{12}$	$\frac{mL^2}{12}$	$-\frac{L}{2}$	0	$-L$
2	$\frac{mL^2}{12}$	$\frac{mL^2}{12}$	0	$-L$	0	$-\frac{L}{2}$
3	0	$\frac{mL^2}{12}$	$\frac{mL^2}{12}$	$-\frac{L}{2}$	0	0
4	$\frac{mL^2}{12}$	0	$\frac{mL^2}{12}$	0	$-\frac{L}{2}$	0
5	$\frac{mL^2}{12}$	$\frac{mL^2}{12}$	0	0	$-L$	$\frac{L}{2}$
6	$\frac{mL^2}{12}$	0	$\frac{mL^2}{12}$	0	$-\frac{L}{2}$	L

From Matlab:

$$\boxed{[I_O] = \begin{bmatrix} 1.123 & 0 & -0.259 \\ 0 & 1.123 & 0.259 \\ -0.259 & 0.259 & 0.864 \end{bmatrix} \text{ kg}\cdot\text{m}^2}$$

```
% Problem 10.10 from Baruh, Applied Dynamics

clear;
clc;

m = 0.5;
L = 0.72;

% Bar #1
Ixx(1) = 0;
Iyy(1) = 1/12*m*L^2;
Izz(1) = 1/12*m*L^2;
d(:,1) = [-L/2 0 -L]';

% Bar #2
Ixx(2) = 1/12*m*L^2;;
Iyy(2) = 1/12*m*L^2;
Izz(2) = 0;
d(:,2) = [-L 0 -L/2]';

% Bar #3
Ixx(3) = 0;
Iyy(3) = 1/12*m*L^2;
Izz(3) = 1/12*m*L^2;
d(:,3) = [-L/2 0 0]';

% Bar #4
Ixx(4) = 1/12*m*L^2;
Iyy(4) = 0;
Izz(4) = 1/12*m*L^2;
d(:,4) = [0 -L/2 0]';

% Bar #5
Ixx(5) = 1/12*m*L^2;;
Iyy(5) = 1/12*m*L^2;
Izz(5) = 0;
d(:,5) = [0 -L L/2]';

% Bar #6
Ixx(6) = 1/12*m*L^2;
Iyy(6) = 0;
Izz(6) = 1/12*m*L^2;
d(:,6) = [0 -L/2 L]';

% Put together inertia matrices into a single 6x3x3 array, where each "row"
% of the array corresponds to an entire inertia matrix
for i = 1:6,
    I(i,1:3,1:3) =zeros(3,3);
    I(i,1,1) = Ixx(i);
    I(i,2,2) = Iyy(i);
    I(i,3,3) = Izz(i);
```

```
end

% Use parallel axis theorem to transfer to origin
for i = 1:6,
    Itemp = squeeze(I(i,1:3,1:3));
    Iotemp = Itemp + m*(d(:,i)'*d(:,i)*eye(3) - d(:,i)*d(:,i)');
    Io(i,1:3,1:3) = Iotemp(1:3,1:3);
end

% Sum contributions of individual bars at origin
IO = squeeze(sum(Io,1))
```

```
IO =

    1.1232         0   -0.2592
         0    1.1232    0.2592
   -0.2592    0.2592    0.8640

>>
```

3)

$$[I] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & -4 \\ 0 & -4 & 10 \end{bmatrix}$$

$$\left|[I] - \lambda[1]\right| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 10-\lambda & -4 \\ 0 & -4 & 10-\lambda \end{vmatrix}$$

$$= (5-\lambda)(10-\lambda)(10-\lambda) - (-4)(-4)(5-\lambda)$$

$$= (5-\lambda)\left[(10-\lambda)^2 - 16\right] = (5-\lambda)(\lambda^2 - 20\lambda + 84)$$

$$= (5-\lambda)(\lambda-14)(\lambda-6) = 0$$

$$\boxed{\begin{matrix} \lambda_1 = 5 \\ \lambda_2 = 6 \\ \lambda_3 = 14 \end{matrix}} \Rightarrow \boxed{[I'] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 14 \end{bmatrix}}$$

$$\left([I] - \lambda[1]\right)\{u\} = 0$$

$$\begin{bmatrix} 5-\lambda_i & 0 & 0 \\ 0 & 10-\lambda_i & -4 \\ 0 & -4 & 10-\lambda_i \end{bmatrix} \begin{bmatrix} u_{1i} \\ u_{2i} \\ u_{3i} \end{bmatrix} = 0$$

$(5-\lambda_i)u_{1i} = 0$ ①

$(10-\lambda_i)u_{2i} - 4u_{3i} = 0$ ②

$-4u_{2i} + (10-\lambda_i)u_{3i} = 0$ ③

$i=1: \lambda_1 = 5$

①: $(5-5)u_{11} = 0 \Rightarrow u_{11}$ can be anything

$$\left.\begin{array}{ll} ②: \; 5u_{21} - 4u_{31} = 0 \Rightarrow u_{21} = \frac{4}{5}u_{31} \\ ③: \; -4u_{21} + 5u_{31} = 0 \Rightarrow u_{21} = \frac{5}{4}u_{31} \end{array}\right\} \Rightarrow u_{21} = u_{31} = 0$$

$$u_{11}^2 + u_{21}^2 + u_{31}^2 = u_{11}^2 = 1 \Rightarrow u_{11} = \pm 1$$

$$\boxed{\{u_1\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$i=2: \lambda_2 = 6$

①: $u_{12} = 0$

$$\left.\begin{array}{ll} ②: \; 4u_{22} - 4u_{32} = 0 \Rightarrow u_{22} = u_{32} \\ ③: \; -4u_{22} + 4u_{32} = 0 \Rightarrow u_{22} = u_{32} \end{array}\right\} \text{Not independent}$$

$$u_{12}^2 + u_{22}^2 + u_{32}^2 = 2u_{22}^2 = 1 \Rightarrow u_{22}^2 = \frac{1}{2} \Rightarrow u_{22} = \pm\frac{\sqrt{2}}{2}$$

$$\boxed{\{u_2\} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}}$$

$i=3: \lambda_3 = 14$

①: $u_{13} = 0$

$$\left.\begin{array}{ll} ②: \; -4u_{23} - 4u_{33} = 0 \Rightarrow u_{23} = -u_{33} \\ ③: \; -4u_{23} - 4u_{33} = 0 \Rightarrow u_{23} = -u_{33} \end{array}\right\} \text{Not independent}$$

$$u_{13}^2 + u_{23}^2 + u_{33}^2 = 2u_{23}^2 = 1 \Rightarrow u_{23}^2 = \frac{1}{2} \Rightarrow u_{23} = \pm\frac{\sqrt{2}}{2}$$

$$\boxed{\{u_3\} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}}$$

$$[R] = \begin{bmatrix} \{u_1\}^T \\ \{u_2\}^T \\ \{u_3\}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Check $|R| = -\frac{1}{2} - \frac{1}{2} = -1 \Rightarrow$ Not a right-handed rotation matrix

For $i=2$, choose $u_{22} = u_{32} = -\frac{\sqrt{2}}{2}$

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Check $|R| = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow$ OK

4)

From a previous problem,

$$[I_o] = m \begin{bmatrix} 13.67 & -7 & -8.75 \\ -7 & 24.67 & -5 \\ -8.75 & -5 & 21.67 \end{bmatrix}$$

From Matlab,

$$[I_o'] = m \begin{bmatrix} 4.40 & 0 & 0 \\ 0 & 27.16 & 0 \\ 0 & 0 & 28.44 \end{bmatrix}$$

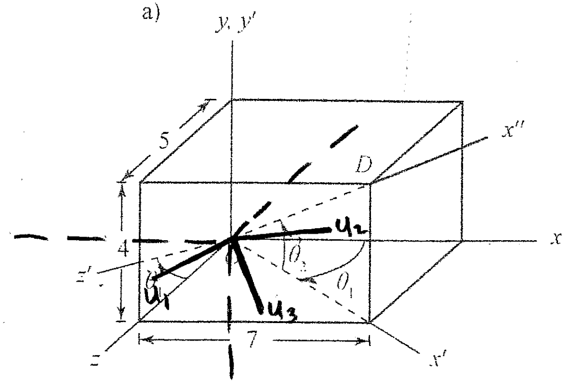
$$[R] = \begin{bmatrix} -0.771 & -0.391 & -0.504 \\ 0.623 & -0.292 & -0.726 \\ 0.136 & -0.873 & 0.468 \end{bmatrix}$$

Note that this [R] is not unique. There are several variations with different signs.

From [R],

$$\{u_1\} = \begin{bmatrix} -0.771 \\ -0.391 \\ -0.504 \end{bmatrix} \qquad \{u_2\} = \begin{bmatrix} 0.623 \\ -0.292 \\ -0.726 \end{bmatrix} \qquad \{u_3\} = \begin{bmatrix} 0.136 \\ -0.873 \\ 0.468 \end{bmatrix}$$

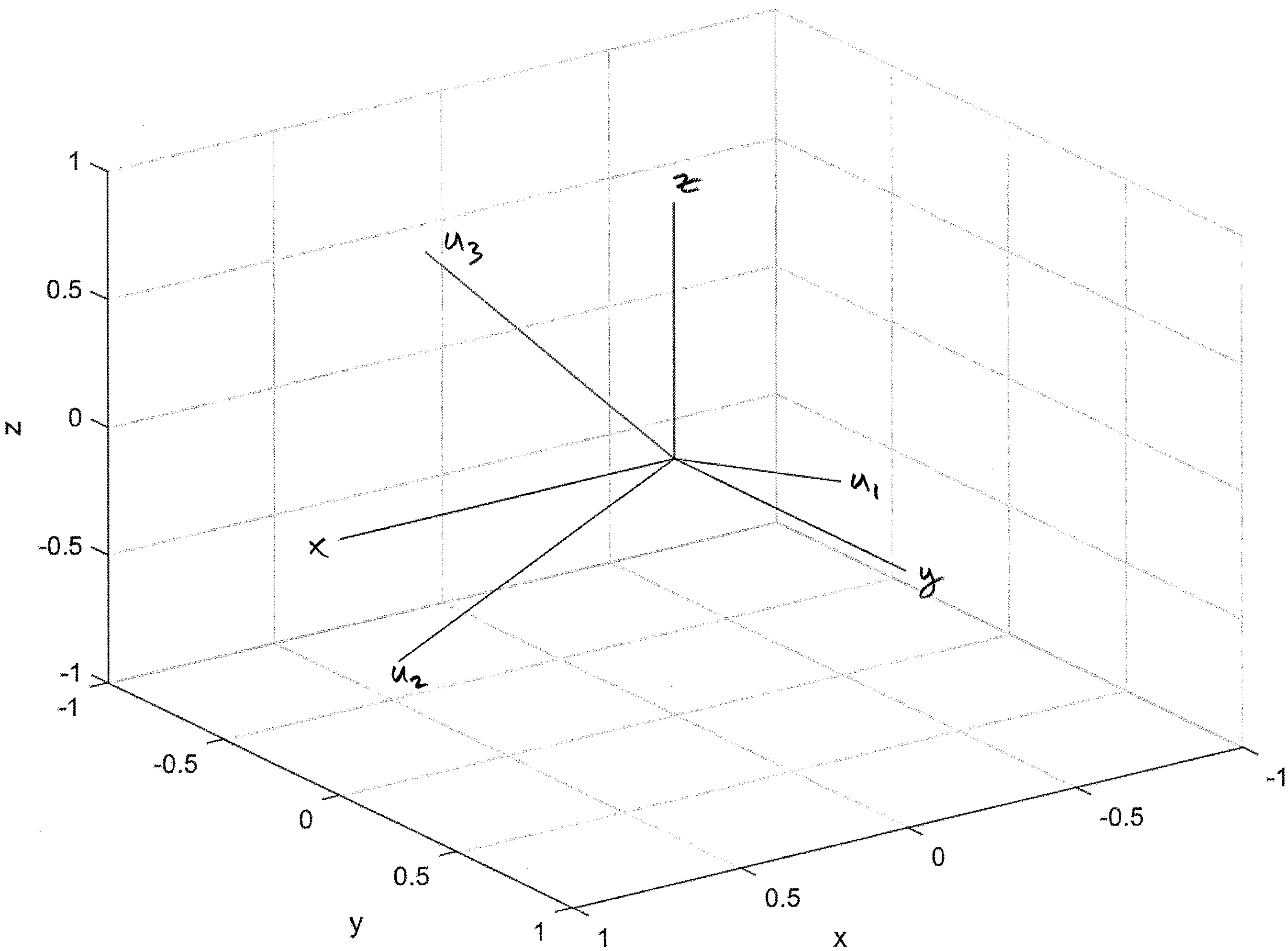
a)



```
% Problem 10.12 from Baruh, Applied Dynamics

clear;
clc;

% Ig (without m)
Ig = 1/12*[41 0 0; 0 74 0; 0 0 65]

% Io from parallel axis theorem (without m)
d = [-7/2 -4/2 -5/2]'
Io = Ig + d'*d*eye(3) - d*d'

% Find principal moments of inertia (without m)
% and principal axes
[Rt,Ip] = eig(Io)
R = Rt'
det(R)

% Change sign of first row of R
R(1,:) = -R(1,:)
det(R)

% Plot axes
figure(1);
hold on;
for i=1:3,
    plot3([0,R(i,1)],[0,R(i,2)],[0,R(i,3)])
end
plot3([0,1],[0,0],[0,0],'k');
plot3([0,0],[0,1],[0,0],'k');
plot3([0,0],[0,0],[0,1],'k');
grid on;
axis([-1 1 -1 1 -1 1]);
xlabel('x');
ylabel('y');
zlabel('z');
```

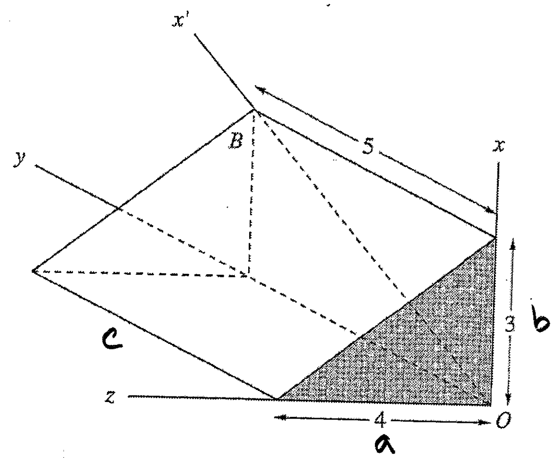
```
Ig =

    3.4167         0         0
         0    6.1667         0
         0         0    5.4167


d =

   -3.5000
   -2.0000
   -2.5000


Io =

   13.6667   -7.0000   -8.7500
   -7.0000   24.6667   -5.0000
   -8.7500   -5.0000   21.6667


Rt =

    0.7706    0.6225    0.1363
    0.3905   -0.2923   -0.8730
    0.5036   -0.7260    0.4683


Ip =

    4.4016         0         0
         0   27.1570         0
         0         0   28.4415


R =

    0.7706    0.3905    0.5036
    0.6225   -0.2923   -0.7260
    0.1363   -0.8730    0.4683


ans =

   -1.0000


R =

   -0.7706   -0.3905   -0.5036
```

```
    0.6225   -0.2923   -0.7260
    0.1363   -0.8730    0.4683


ans =

    1.0000

>>
```



z
u_3
u_1
x
y
u_2
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1
0.5
0
-0.5
-1

5)



From the appendix, the moments and products of inertia about xyz axes through the center of mass are:

$$I_{xx} = \frac{1}{18} m a^2 + \frac{1}{12} m c^2$$

$$I_{yy} = \frac{1}{18} m (a^2 + b^2)$$

$$I_{zz} = \frac{1}{18} m b^2 + \frac{1}{12} m c^2$$

$$I_{xy} = 0$$

$$I_{yz} = 0$$

$$I_{xz} = -\frac{1}{36} m a b$$

From Matlab,

$$[I_G] = m \begin{bmatrix} 2.97 & 0 & 0.33 \\ 0 & 1.39 & 0 \\ 0.33 & 0 & 2.58 \end{bmatrix}$$

$$[I_O] = m \begin{bmatrix} 11 & -2.5 & -1 \\ -2.5 & 4.17 & -3.33 \\ -1 & -3.33 & 9.83 \end{bmatrix}$$

$$\boxed{[I_O'] = m \begin{bmatrix} 1.84 & 0 & 0 \\ 0 & 11.33 & 0 \\ 0 & 0 & 11.83 \end{bmatrix}}$$

```
% Triangular prism

clear;
clc;

% Dimensions
a = 4;
b = 3;
c = 5;

% Inertia at center of mass
Ixx = 1/18*a^2 + 1/12*c^2;
Iyy = 1/18*(a^2+b^2);
Izz = 1/18*b^2 + 1/12*c^2;
Ixz = -1/36*a*b;
Ig = zeros(3,3);
Ig(1,1) = Ixx;
Ig(2,2) = Iyy;
Ig(3,3) = Izz;
Ig(1,3) = -Ixz;
Ig(3,1) = -Ixz;
Ig

% Parallel axis theorem to get inertia about origin
d = [b/3; c/2; a/3];
Io = Ig + (d'*d*eye(3) -d*d')


% Find eigenvalues and eigenvectors.  C is the direction cosine matrix,
% whose columns contain the eigenvectors
[Rt,Iprime] = eig(Io)

% Find the corresponding rotation matrix
R = Rt';

% Check rotation matrix
det(R)

% Fix rotation matrix so that it is right-handed
R(1,:) = -1*R(1,:);

% Check answer using rotation matrix
Iprime = R*Io*R'
```

```
Ig =

    2.9722         0    0.3333
         0    1.3889         0
    0.3333         0    2.5833


Io =

   11.0000   -2.5000   -1.0000
   -2.5000    4.1667   -3.3333
   -1.0000   -3.3333    9.8333


Rt =

   -0.2816   -0.2738    0.9196
   -0.8727   -0.3254   -0.3641
   -0.3989    0.9051    0.1473


Iprime =

    1.8361         0         0
         0   11.3343         0
         0         0   11.8296


ans =

   -1.0000


Iprime =

    1.8361   -0.0000    0.0000
   -0.0000   11.3343    0.0000
    0.0000    0.0000   11.8296

>>
```