Fortune's Algorithm for Voronoi Diagram of given set of points

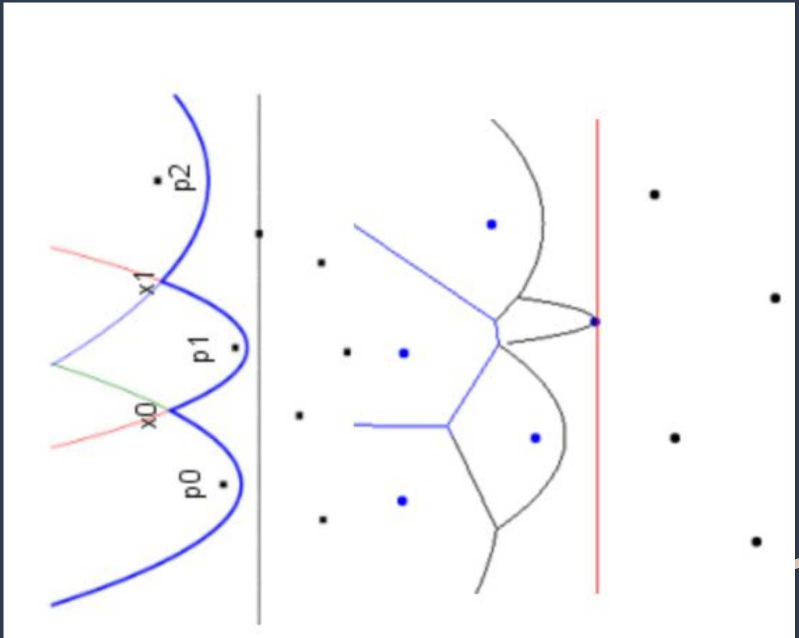
Krishna Pande

Voronoi Diagram

Sites: Distinct central places or points of interest.

Voronoi diagram: The subdivision of the plane into n cells such that a point q lies in the cell corresponding to pi
iff dist(q, pi) < dist(q, pj)
for each pj ∈ P with j != i

Fortune's Algorithm uses the sweep line paradigm to calculate the voronoi diagram for a given set of points.



The algo requires us to maintain a **sweep line** (vertical straight line) points to the left of which have been considered for the voronoi diagram.

The Voronoi diagram above the sweep line is affected by event points below the line
We know the nearest site of a point q if q lies at least as close to a site as it does to the sweep line

This defines a parabola:

Thus, we also need to store a **beach line**. This is a slightly more complicated line formed by sections of parabolas formed by taking the sweep line as directrix and the sites as loci.

Implementation of sweep line and beach line

In the implementation the sweep line moving from left to right is depicted using a priority queue of sites based on their x-coordinate.

We pop the sites from the priority queue and then process them as explained ahead.

Representing the beach line as a doubly linked list is possible but the time to search will then become O(n).

The beach line is stored in the form of a binary tree of arcs. This only changes at two events: Point event and site event from the event queue.

Properties of Beach line

The beach line is an x-monotone curve made up of parabolic arcs. The breakpoints of the beach line lie on Voronoi edges of the final diagram.

– Parabolas are x-monotone, so is the beach line

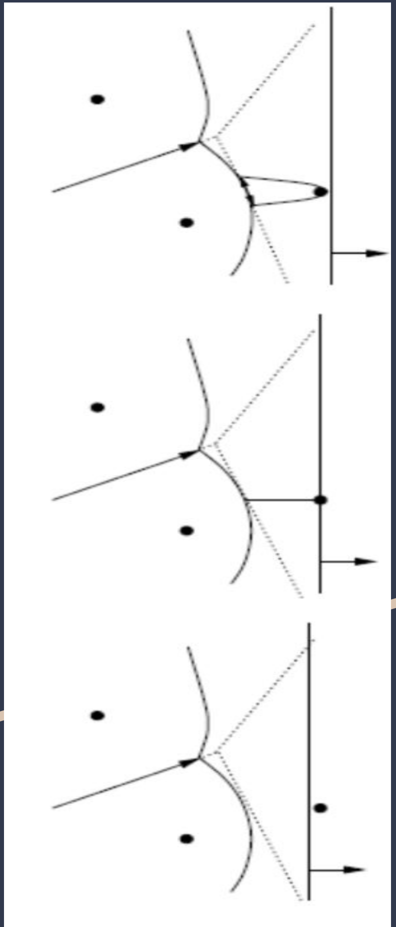
– The breakpoints are equidistant from the two points and the sweep line

The beach line can change combinatorially only with a site event or a circle event

A parabola can contribute up to n arcs on the beach line.

The beach line has at most 2n – 1 arcs

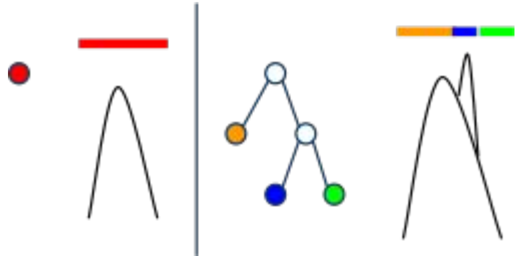
Site Event



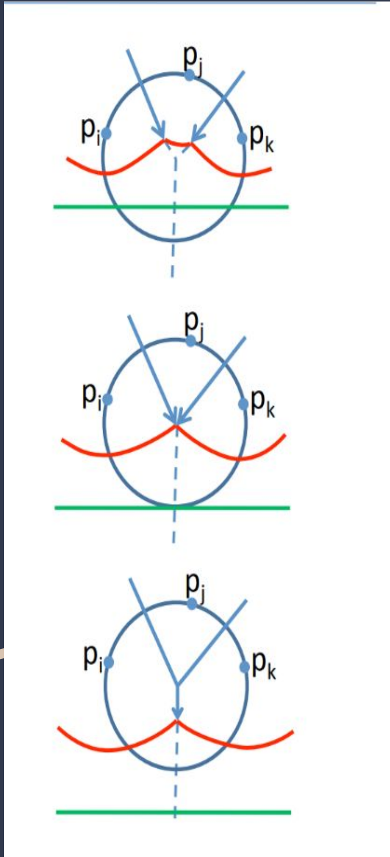
Correspond to adding a new edge to the diagram
These are our predetermined events

- A new parabola is added to the beach line
- New break points begin to trace out the same new edge
- Check the new triple of consecutive arcs for any potential circle events
- Add to event queue if any circle event is found

When we add a new arc into the tree (when a sweep line rolls over the new site), the "right leaf" (the arc under the site being added) splits into two half-arcs and a new arc is added between them. The "right leaf" becomes an inner node and gets 2 children. Left child represents a left half-arc of a previous arc. Right child is a subtree, which contains a newly added arc on the left and right half-arc of previous arc on the right.



Circle Event



For each consecutive triple pi, pj, and pk on the beach line we compute the circumcircle of these events. If the lower endpoint of the circle lies below the sweep line, we create a vertex event whose y-coordinate is the y-coordinate of the bottom endpoint of the circumcircle.

When the sweep line reaches a circle event, a parabola disappears from the beach line

Removing an arc is also very easy. Its parent represents the left or the right edge, some "higher predecessor" is the next edge. If currently removed arc is the left child, we replace the predecessor by the right child and vice versa. We attach the new edge to the "higher predecessor".

The center of this circle is added as a vertex of the Voronoi diagram

Time complexity

Theorem: For
$n \geq 3$
$n \geq 3$ sites, Voronoi diagram contains at most
$2n-5$
$2n-5$ vertices and
$3n-6$
$3n-6$ edges.
Proof:
For sites lying in the line, it is evidently true.
Let's suppose, that sites are not in the line.
Let's denote these variables:
V
V: number of vertices in Voronoi diagram
E
E: number of edges
N
N: number of inner faces = number of sites
From Euler's formula we know, that:

$V-E+N=2$
Because Voronoi diagram contains infinite edges, let's create a new vertex "infinity" and connect all these edges to it. Now it is a planar graph.
$(V+1)-E+N=2$
In Voronoi graph we know, that every vertex has a degree at least 3 (including "infinity" vertex). An edge si between two vertices, so
$3*(V+1) \leq 2*E$
By a simple algebraic modification, we get a previous theorem.
Primitive operations, such as searching for an item in the tree or in the queue, removing from the queue, all it can be done in O(log(n)). At each event, we do a constant number of these primitive operations.

The number of site events is N, circle events are at most $2N-5$. At each of them, we do c primitive operations. Thus, final complexity is O(n∗log(n)).