ON THE NODAL LINES OF RANDOM AND DETERMINISTIC LAPLACE EIGENFUNCTIONS

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Abstract. In the present survey we present some of the recent results concerning the geometry of nodal lines of random Gaussian eigenfunctions (in case of spectral degeneracies) or wavepackets and related issues. The most fundamental example, where the spectral degeneracy allows us to consider random eigenfunctions (i.e. endow the eigenspace with Gaussian probability measure), is the sphere, and the corresponding eigenspaces are the spaces of spherical harmonics; this model is the primary focus of the present survey. The list of results presented is, by no means, complete.

1. Introduction

Nodal patterns (first described by Ernest Chladni in 18th century) appear in many problems in engineering, physics and the natural sciences: they describe the sets that remain stationary during vibrations, hence their importance in such diverse areas as the musical instrument industry, mechanical structures, earthquake study and other areas. They also arise in the study of wave propagation, and in astrophysics; this is a very active and rapidly developing research area.

1.1. **Some basic notation.** Let $(\mathcal{M}, g)$ be a compact Riemannian surface (for example S^2, the two dimensional unit sphere equipped with the round metric), and Δ be the Laplace-Beltrami operator on $\mathcal{M}$. We are interested in the eigenvalues λ and the corresponding eigenfunctions ϕ of $-\Delta$, so that

$$\Delta\phi + \lambda\phi = 0.$$

In case $\mathcal{M}$ has a boundary, we impose either the Dirichlet boundary condition $\phi|_{\partial M} \equiv 0$, or the Neumann boundary condition

$$\frac{\partial \phi}{\partial \nu}|_{\partial \mathcal{M}} \equiv 0,$$

or any mixture of the conditions above. The general spectral theory states that there is a complete orthonormal basis of $L^2(\mathcal{M})$ which consists of eigenfunctions, i.e. we may choose a sequence of functions

$$\{\phi_j : \mathcal{M} \rightarrow \mathbb{R}\}_{j=1}^{\infty}$$

and corresponding nondecreasing sequence of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ so that the orthonormal set $\{\phi_j\}$ spans the whole of $L^2(\mathcal{M})$. Note that we allow multiple eigenvalues i.e. spectral degeneracies.

Let $\phi : \mathcal{M} \rightarrow \mathbb{R}$ be any real valued function. The nodal line of ϕ is its zero set

$$\phi^{-1}(0) = \{x \in \mathcal{M} : \phi(x) = 0\}.$$

In principle, a nodal line might have self-intersections[1]; however generically it is a smooth 1-dimensional curve with components homeomorphic to either the circle ("closed component") or an interval (in this case it must intersect the boundary; it is called an "open component"). We are interested in the geometry of the nodal lines of ϕ_j as $j \to \infty$. The most basic aspect of the nodal line is, of course, its length; let us denote l_j to be the length of the nodal line of ϕ_j. In this survey we will only consider the 2-dimensional case; however most of the results presented extend for higher dimensions.

1.2. **Yau's conjecture and Berry's RWM.** Yau conjectured [23, 24] that for any smooth $\mathcal{M}$, l_j are commensurable to $\sqrt{\lambda_j}$ for any *smooth* metric g in the sense that there exist two constants $c(\mathcal{M},g), C(\mathcal{M},g)$ so that

$$c(\mathcal{M},g)\sqrt{\lambda_j} \le l_j \le C(\mathcal{M},g)\sqrt{\lambda_j} \tag{1}$$

for every $j \ge 1$. The lower bound was proved by Bruning and Gromes [9] and Bruning [8] for the planar case. Donnelly and Fefferman [12] finally settled Yau's conjecture for real analytic metrics. However, in its full generality, Yau's conjecture is still open.

In his seminal work [3], Berry argued that the high energy behaviour of the eigenfunctions should be universal, at least for "generic" chaotic surfaces $\mathcal{M}$ (for example, any negatively curved surface, or any ergodic billiard). He proposed to compare an eigenfunction with eigenvalue λ to a "typical" instance of an isotropic, monochromatic random wave with wavenumber

$$k = \sqrt{\lambda}$$

(nowadays called Berry's Random Wave Model - RWM). A 1-dimensional version of the random wave was used by Rice in order to investigate the likelihood of a given signal to exceed a level. Longuet-Higgins generalized Rice's model to 2-dimensional plane to describe the movement of the sea and ocean waves.

There are several ways to construct the ensemble of random waves associated with wavenumber k. One way to do it is consider summations of type[2]

$$u_{k;J}(x) = \frac{1}{\sqrt{J}}\Re\left(\sum_{j=1}^{J} e^{i(k\langle\theta_j,x\rangle+\phi_j)}\right),$$

on $\mathbb{R}^2$, where θ_j are random directions drawn uniformly on the unit circle, and $\phi_j \in [0,2\pi)$ are the random phases. One would like to define the random wave $u_k(x)$ on $\mathbb{R}^2$ as the limiting ensemble

$$u_k(x) = \lim_{J\to\infty} u_{k;J}(x);$$

which should converge in distribution. Another mathematically rigorous way to define the wavenumber k Random Wave is to identify it as the unique Gaussian isotropic random field (ensemble of functions) with covariance function

$$r_{RWM}(x,y) = J_0(k|x-y|), \tag{2}$$

[1] For example the eigenfunction $\phi(x,y) = \sin(10\pi x)\sin(20\pi y)$, defined on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, having eigenvalue $\lambda = 500\pi^2$.

where J_0 is the usual Bessel function.

Since, according to the RWM, the random waves model the high-energy eigenvalues, the nodal lines of random waves should also model the nodal lines of honest eigenfunctions. This approach allows us to study local quantities like the nodal length, boundary intersections and intersections with a test curve etc.

Suppose, for example, we are interested in the nodal length on the torus. Then we are to choose a representative planar domain $U \subseteq \mathbb{R}^2$ (e.g. a rectangle with the same aspect ratio and area as the torus), and study the distribution of $\mathcal{Z}_{U;\sqrt{\lambda}}$, the nodal length of random wave with wavenumber $k = \sqrt{\lambda}$ inside U. It is easy to compute the expected length to be of order of magnitude

$$\mathbb{E}[\mathcal{Z}_{U;\sqrt{\lambda}}] \sim const \cdot \sqrt{\lambda}|U|,$$

where $|U|$ is the area of $|U|$, and Berry argued [4] that the variance should be of order

$$\mathrm{Var}\left(\mathcal{Z}_{U;\sqrt{\lambda}}\right) \sim const \cdot |U| \log \lambda. \tag{3}$$

1.3. **Bogomolny and Schmit's percolation model.** The RWM, however, does not help if one is interested in making predictions regarding the more subtle (and arguably, more interesting) aspects of the nodal structures such as the number of nodal domains[3], their size distribution, the size of the largest nodal domain, the inner radius etc. For this purpose an elegant independent bond percolation-like model was introduced by Bogomolny and Schmit [5]. According to this model, the nodal domains should correspond to the clusters connected by open bonds, and the nodal line corresponds to these clusters' boundaries [7]. Let $\nu_j = N(\phi_j)$ be the number of nodal domains of ϕ_j, and $N(u_{\sqrt{\lambda}}; U)$ the number of nodal domains of random wave $u_{\sqrt{\lambda};U}$ on U. The main criticism against this model is that the independence assumption ignores all the dependencies that occur between the bonds; to try to justify the independence assumption, Bogomolny and Schmit [6] apply a heuristic principle, the so-called Harris criterion.

By the classical Courant Nodal Domain Theorem (see e.g. [10]), $\nu_j \leq j$, and Pleijel [19] asymptotically improved the latter to

$$\limsup_{j\to\infty} \frac{\nu_j}{j} \leq 0.691\ldots.$$

On the other hand, no nontrivial lower bound for ν_j could be found, since one may find a sequence of energy levels λ_{j_k} on the torus (say), so that the corresponding eigenfunctions would have only 2 nodal domains. Bogomolny and Schmit [5] used the general percolation theory to predict that

$$N(u_{\sqrt{\lambda}}; U)$$

[2]In reality the summation is slightly more complicated than the one presented - see e.g. [5], (1).

[3]The nodal domains are the connected components of the complement of the nodal line.

should be asymptotically Gaussian, with mean and variance proportional to

$$|U| \cdot \left(\sqrt{\lambda}\right)^2 = |U| \cdot \lambda.$$

More strikingly, in their later paper, Bogomolny et al. [7] argue that since, according to the recent developments in the percolation theory (see e.g. Smirnov [21]), the "interface" (cluster boundaries) should converge to SLE_6, the distribution of the largest[4]component of the nodal line (rather than its length) should converge to SLE_6 as well.

1.4. **Equidistribution conjecture.** It is conjectured[5] that on any chaotic surface $\mathcal{M}$, the nodal lines are asymptotically equidistributed in $\mathcal{M}$, so that, in particular, the nodal length is asymptotic to

$$l_j \sim c_{\mathcal{M}} \cdot \sqrt{\lambda_j} \tag{4}$$

for some constant $c_{\mathcal{M}} > 0$ (this refines Yau's conjecture (1)). Despite the fact that heuristically, (4) follows from the RWM (it follows for example from the results mentioned in the end of Section 1.2), this conjecture seems extremely difficult or even out of reach by the present analytic methods, and it seems highly unlikely that it is going to be settled in the near future.

However, some information could be inferred from the completely integrable case, even though the picture that emerges here is very different. For instance, one may use the spectrum degeneracy of the standard torus to easily construct sequences of eigenfunctions $\phi_{n_{1;j}}$ and $\phi_{n_{2;j}}$, $i = 1, 2$, $j = 1, 2, \ldots$, so that

$$\text{length}\left(\phi_{n_{i;j}}^{-1}(0)\right) \sim c_i \cdot \sqrt{\lambda_{n_j}},$$

$i = 1, 2$, with $c_1 \neq c_2$; one may obtain such sequences on the sphere only slightly modifying the same argument. One way to infer some information is use the following heuristic principle:

Principle 1.1 ("Word exchangeability"). *Any property satisfied by* generic *eigenfunctions on (all) completely integrable manifolds is also satisfied by* all *eigenfunctions on a* generic *chaotic manifold.*

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[4]According to the percolation theory, there exists exactly one component that "covers" the whole domain; this is the only macroscopic component of the nodal line.

[5]In a recent survey by S. Nonnenmacher [17], this conjecture was attributed to S. Zelditch.

and encouraging collaboration and new ideas' exchange, and also for the generous financial support.

2. Some results

2.1. Spherical harmonics.

It is well known that the eigenvalues E of the Laplacian

$$\Delta f + Ef = 0$$

on the 2-dimensional sphere $\mathcal{S}^2$ are all the numbers of the form

$$E_n = n(n+1), \tag{5}$$

where n is an integer. The corresponding eigenspace is the space $\mathcal{E}_n$ of spherical harmonics of degree n; its dimension is

$$\mathcal{N}_n = 2n+1.$$

Given an integer n, we fix an $L^2(\mathcal{S}^2)$ orthonormal basis of $\mathcal{E}_n$

$$\eta_1^n(x),\, \eta_2^n(x), \dots, \eta_{2n+1}^n(x),$$

giving an identification $\mathcal{E}_n \cong \mathbb{R}^{\mathcal{N}_n}$. For further reading on the spherical harmonics we refer the reader to [1], chapter 9.

2.2. Random models.

In case of spectral degeneracy, such as the sphere or the torus, we may consider a *random eigenfunction* lying inside an eigenspace. For the sphere, we define it as

$$f_n(x) = \sqrt{\frac{2}{\mathcal{N}_n}} \sum_{k=1}^{\mathcal{N}_n} a_k \eta_k^n(x), \tag{6}$$

where a_k are standard Gaussian $N(0,1)$ i.i.d. That is, we use the identification

$$\mathcal{E}_n \cong \mathbb{R}^{\mathcal{N}_n}$$

to endow the space $\mathcal{E}_n$ with Gaussian probability measure υ as

$$d\upsilon(f_n) = e^{-\frac{1}{2}\|\vec{a}\|^2} \frac{da_1 \cdot \ldots \cdot da_{\mathcal{N}_n}}{(2\pi)^{\mathcal{N}_n/2}},$$

where $\vec{a} = (a_i) \in \mathbb{R}^{\mathcal{N}_n}$ are as in (6).

Note that υ is invariant with respect to the orthonormal basis for $\mathcal{E}_n$. Moreover, the Gaussian *random field* f_n^m is isotropic in the sense that for every $x_1, \dots x_l \in \mathcal{S}^2$ and every orthogonal $R \in O(3)$,

$$(f_n(Rx_1), \dots, f_n(Rx_l)) \overset{d}{=} (f_n(x_1), \dots, f_n(x_l))\,. \tag{7}$$

There exists yet another way to define f_n: it is the centered Gaussian isotropic random field with covariance function

$$r_n(x,y) := \mathbb{E}[f_n(x) \cdot f_n(y)] = P_n(\cos d(x,y)), \tag{8}$$

where P_n are the well-known Legendre polynomial of degree n, and d is the spherical distance. The Legendre polynomials admit Hilb's asymptotics

$$P_n(\cos(\varphi)) \approx \sqrt{\frac{\varphi}{\sin\varphi}} J_0\left((n+\frac{1}{2})\varphi \right), \tag{9}$$

i.e. almost identical to RWM (2), up to the "correction factor" $\sqrt{\frac{\varphi}{\sin\varphi}}$. This factor seems to "keep a trace" or "remember" about the geometry of the sphere.

For generic manifolds there are no spectral degeneracies, so that we should consider linear Gaussian combinations of individual eigenfunctions ("wavepackets"). The two most accepted models are the so-called *long energy window* and *short energy window.* In the long window case we consider Gaussian combinations of eigenfunctions with eigenvalue lying in the window $[0,\lambda]$ for $\lambda\rightarrow\infty$

$$f_\lambda^L(x)=\frac{1}{\sqrt{N_{\mathcal{M}}(\lambda)}}\sum_{\sqrt{\lambda_j}\le\sqrt{\lambda}}a_j\phi_j(x),$$

where $x\in\mathcal{M}$, and

$$N_{\mathcal{M}}(\lambda)=\#\{\lambda_j\le\lambda\}$$

is the spectral function; the reason we took the square root of the eigenvalues is that it makes it more convenient to write the short energy window random function

$$f_\lambda^S(x)=\frac{1}{\sqrt{N_{\mathcal{M}}((\sqrt{\lambda}+1)^2)-N_{\mathcal{M}}(\lambda)}}\sum_{\sqrt{\lambda}\le\sqrt{\lambda_j}\le\sqrt{\lambda}+1}a_j\phi_j(x).$$

The short energy model[6] is considered more significant, as it is more representative of the individual eigenfunctions [7]; however working with the long energy window is relatively easier. The spectral prefactor in the definition of $f^{L,S}$ was introduced to make the expected L^2-norm unity.

As usual, for any random variable X, we denote its expectation $\mathbb{E}X$. For example, with the normalization factor in (6), for every n *fixed* point $x\in\mathcal{S}^2$, one has

$$\mathbb{E}[f_n(x)^2]=\frac{\mathcal{S}^2}{\mathcal{N}_n}\sum_{k=1}^{\mathcal{N}_n}\eta_k^n(x)^2=1, \tag{10}$$

a simple corollary from the Addition Theorem (see [1]).

Any characteristic $X(L)$ of the nodal line

$$f_n^{-1}(0)=\{x\in\mathcal{S}^2:\ f_n(x)=0\}$$

is a random variable. The most natural characteristic of the nodal line of f_n is, of course, its length $\mathcal{Z}(f_n)$. One may then study the distribution of the random variable $\mathcal{Z}(f_n)$ for a random Gaussian $f_n\in\mathcal{E}_n$, as $n\rightarrow\infty$. It is also natural to consider the number $N(f_n)$ of the nodal domains of f_n, i.e. the connected components of

$$\mathcal{S}^2\setminus f_n^{-1}(0);$$

[6]The window $[\lambda,\lambda+1]$ may be replaced by $[\lambda,\lambda+a]$ for any constant $a>0$.

[7]Much like the random trigonometric polynomials $\sum\limits_{n=N}^{N+\sqrt{N}}a_n\cos(nt)$ on $[0,2\pi]$ is more representative of $\cos(Nt)$ than $\sum\limits_{n=1}^{N}a_n\cos(nt)$, where in both summations a_n are standard Gaussian i.i.d.; for example it possesses asymptotically the same number of zeros.

its distribution should be consistent to the one predicted by Bogomolny and Schmit based on their percolation model (see Section 1.3 above).

2.3. **Some generic results.** Berard [2], and subsequently Zelditch [28] found that the expected nodal length of the long energy window random functions is

$$\mathbb{E}\left[\mathcal{Z}(f_\lambda^L)\right] \sim const \cdot \sqrt{\lambda},$$

consistent with Yau. Zelditch [28] also extended this result to the short energy window case

$$\mathbb{E}\left[\mathcal{Z}(f_\lambda^S)\right] \sim const \cdot \sqrt{\lambda}.$$

In addition, for g real analytic, Zelditch [28] considered the complexified manifold $(\mathcal{M}_{\mathbb{C}}, g_{\mathbb{C}})$ (whose projection on $\Im z = 0$ is (M, g)), the analytic continuations $\phi_j^{\mathbb{C}}$, and the corresponding random combinations $f_\lambda^{\mathbb{C};L,S}$, defined analogously to the real random combinations $f_\lambda^{L,S}$. In this case the zeros are isolated points in $\mathcal{M}_{\mathbb{C}}$; Zelditch [28] found that their expected number is again proportional to $\sqrt{\lambda}$.

Toth and Wigman [22] considered the number of boundary intersections

$$\mathcal{I}\left(f_\lambda^{L,S}\right)$$

of the nodal line of $f_\lambda^{L,S}$ in case $\mathcal{M}$ is a generic billiard (i.e. a planar oval with a smooth boundary), or equivalently (up to the factor 2), the number of open components. They found the correct order of magnitude for the expected number of intersections to be

$$\mathbb{E}\left[\mathcal{I}\left(f_\lambda^{L,S}\right)\right] \sim const|\partial\mathcal{M}|\sqrt{\lambda},$$

where $|\partial\mathcal{M}|$ is the boundary length of $\mathcal{M}$, and the constants differ in the long and short window cases. This result is consistent to both Yau's conjecture, random wave model, and the resulting interpretation of the boundary trace as approximating trigonometric polynomials; the asymptotics depends only on the boundary length of the billiard, notably independent of its shape.

2.4. **Number of nodal domains.** Nazarov-Sodin [18] found the correct order of magnitude for the expected number of nodal domains of random spherical harmonics, and established an exponential decay result for deviations from the mean.

Theorem 2.1 (Nazarov-Sodin [18])**.** *There exists a constant $a > 0$ so that the expected number of nodal domains is asymptotic to*

$$\mathbb{E}[N(f_n)] = an^2 + o(n^2). \tag{11}$$

Moreover, for every $\epsilon > 0$, there exist two constants $c(\epsilon), C(\epsilon) > 0$, so that

$$\Pr\left(\left|\frac{N(f_n)}{n^2} - a\right| > \epsilon\right) \leq C(\epsilon)e^{-c(\epsilon)n}. \tag{12}$$

The result (11) on the expected number of nodal domains is of more general nature: it extends to a wide range of sequences of random fields (to appear in a paper by Nazarov and Sodin). For example, rather than taking a random element lying in a single spherical harmonics space, one may superpose elements from several spaces). However, unlike the rapid decay (12) in the particular case of spherical harmonics, in the more general situation, Nazarov-Sodin's result does not prescribe the rate of decay of the tails of the distribution. Instead, they prove the weaker statement: for any $\epsilon > 0$

$$\lim_{n\to\infty} \Pr\left(\left| \frac{N(f_n)}{n^2} - a \right| > \epsilon \right) = 0.$$

One disadvantage of the results above and the method of their proofs is the fact that the constant $a > 0$, whose existence is established, remains mysterious and completely open; one cannot establish the dependence of a on the underlying random field. As an example, one may contrast f_n to g_n, a superposition of random spherical harmonics of degree $\le n$. A generalized version of Nazarov-Sodin's Theorem implies the existence of a constant so that

$$\mathbb{E}[N(g_n)] \sim bn^2.$$

Theorem 2.1 and its generalization do not shed a light on the the relation between a and b.

Though the expectation result is consistent with the percolation model, Theorem 2.1 gives us no clue what would be the result for the variance. However, it provides us with a very strong rate of decay (namely exponential); the decay results are usually complementary to the variance results. The authors also proved that the prescribed rate of decay cannot be improved, so that the exponential decay they establish, is of the correct order of magnitude.

2.5. **Nodal length of random spherical harmonics.** It is a standard application of the Kac-Rice formula (see e.g. [11]) to compute the expected nodal length of random spherical harmonics [2]

$$\mathbb{E}\left[\mathcal{Z}(f_n)\right] = c \cdot \sqrt{E_n}, \tag{13}$$

where

$$c = \sqrt{2}\pi,$$

(see also [16] and [25]). Our main concern in this pursuit is the subtle question of the variance.

Based on the natural scaling of the sphere (e.g. the relation to the Legendre polynomials, in particular (8)), we conjectured [25], that

$$\mathrm{Var}(\mathcal{Z}(f_n)) \sim const \cdot n.$$

Surprisingly, the variance turned out to be much smaller, due to an unexpected cancellation ("Berry's cancellation phenomenon"). We derived the following asymptotics for the nodal length variance, improving the earlier bounds of Neuheisel [16] and Wigman [25]:

Theorem 2.2 (Wigman [26])**.** *As $n \to \infty$, one has*

$$\mathrm{Var}\left(\mathcal{Z}(f_n)\right) = \frac{65}{32}\log n + O(1). \tag{14}$$

Note that the leading constant $\frac{65}{32}$ in (2.5) is different[8]from the one predicted by Berry for the RWM (see (3)). Our explanation for this discrepancy is the nontrivial local geometry of the sphere. It seems reasonable that for a generic chaotic surface, the nodal length variance for the short window

Gaussian random combinations f_λ^S should be logarithmic; the leading constant is then an artifact of the local geometry. One of our central goals is to find this dependency explicitly, namely, given a Riemannian surface, $(\mathcal{M}, g)$, compute $a = a(\mathcal{M}, g)$, so that

$$\operatorname{Var}\left(\mathcal{Z}(f_\lambda^S)\right) \sim a \log \lambda.$$

(if the logarithmic prediction is indeed correct).

Theorem 2.2 implies that the series

$$\sum_{n=1}^{\infty} \operatorname{Var}\left(\frac{\mathcal{Z}(f_n)}{\mathbb{E}[\mathcal{Z}(f_n)]}\right)$$

of variances of the normalized length

$$\frac{\mathcal{Z}(f_n)}{\mathbb{E}[\mathcal{Z}(f_n)]},$$

is convergent. Together with the Borel-Cantelli Lemma, it implies that for independently chosen f_n,

$$\lim_{n\to\infty} \frac{\mathcal{Z}(f_n)}{\sqrt{E_n}} = c,$$

almost surely, where $c > 0$ is the same constant as in (13).

The same problem may be also considered on the standard 2-dimensional torus $\mathcal{T} = \mathbb{R}^2/\mathbb{Z}^2$. Here the eigenvalues are of the form

$$E_n^{\mathcal{T}} = 4\pi^2 n,$$

where n is an integer, expressible as a sum of two integer squares, and the corresponding eigenspace is spanned[9]by functions $\cos(2\pi\langle\lambda, x\rangle)$ and $\sin(2\pi\langle\lambda, x\rangle)$, where $\lambda \in \mathbb{Z}^2$ with $\|\lambda\|^2 = n$, are all the lattice points lying on the circle of radius $\sqrt{n}$; its dimension is $r_2(n)$, the number of representations of n as a sum of two squares. The Gaussian random eigenfunction is a stationary random field with the covariance function

$$r_n^{\mathcal{T}}(x) = \frac{1}{\mathcal{N}} \sum_{\|\lambda\|^2 = n} \cos(2\pi\langle\lambda, x\rangle). \tag{15}$$

It is again standard to compute that the expected nodal length of this ensemble to be proportional to $\sqrt{E_n^{\mathcal{T}}}$ [20], and we are interested in the asymptotic behaviour of the variance again. This question was initially considered by Rudnick and Wigman [20]; however it got only a partial answer then. An (almost) complete answer will be given in the forthcoming paper Krishnapur-Kurlberg-Wigman [13].

Even though we have an explicit expression (15) for the covariance function, no analogue of (9) is known for the asymptotic long-range behaviour of $r_n^{\mathcal{T}}$ (recall (8)). As a replacement, we cope with some subtle issues of the arithmetics of lattice points lying on a circle. Here as well we observed

[8]Since f_n is odd for odd n and even for even n, the nodal lines are invariant w.r.t. the involution $x \mapsto -x$. Therefore the natural planar domain to compare would be one of area of a hemisphere rather than of the full sphere.

[9]Note the invariance w.r.t. $\lambda \mapsto -\lambda$, so that we need to factor the set of lattice points by $\pm$.

the "arithmetic Berry's cancellation", a phenomenon of different appearance but similar nature to "Berry's cancellation phenomenon".

2.6. **Level exceeding.** Let us define the *spherical harmonics level exceeding* measure as follows: for all $z \in (-\infty, \infty)$,

$$\Phi_n(z) := \int_{S^2} \mathbb{1}(f_n(x) \le z) dx, \tag{16}$$

where $\mathbb{1}(\cdot)$ is, as usual, the indicator function which takes value one if the condition in the argument is satisfied, zero otherwise. In words, the function $\Phi_n(z)$ provides the (random) measure of the set where the eigenfunction lie below the value z. For example, the value of $\Phi_n(z)$ at $z = 0$ is related to the so-called defect

$$\mathcal{D}_n := \text{meas}\left(f_n^{-1}(0,\infty)\right) - \text{meas}\left(f_n^{-1}(-\infty,0)\right)$$

by the straightforward transformation

$$\mathcal{D}_n = 4\pi - 2\Phi_n(0).$$

Of course, $4\pi - \Phi_n(z)$ provides the area of the excursion set

$$\mathcal{A}_n(z) := \{x : f_n(x) > z\}.$$

Clearly, for all $z \in \mathbb{R}$,

$$\mathbb{E}\left[\Phi_n(z)\right] = 4\pi\Phi(z),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian. The following lemma (see Marinucci-Wigman [14]) deals with the variance of $\Phi_n(z)$ as $n \to \infty$.

Lemma 2.3. *For every $z \in \mathbb{R}$,*

$$\text{Var}(\Phi_n(z)) = z^2\phi(z)^2 \cdot \frac{1}{n} + O_z\left(\frac{\log n}{n^2}\right),$$

where ϕ is the standard Gaussian probability density function.

In particular, for $z \ne 0$, Lemma 2.3 gives the asymptotic form of the variance as $n \to \infty$. In contrast, for $z = 0$ (this case corresponds to the defect), this yields only a "o"-bound and one needs to work harder to obtain a precise estimate; we do so in the forthcoming paper:

Theorem 2.4 (Marinucci-Wigman [15])**.** *As $n \to \infty$, the defect variance is asymptotic to*

$$\text{Var}(\mathcal{D}_n) \sim C \cdot \frac{1}{n^2},$$

where $C > 0$ is some constant.

In light of Lemma 2.3 it is then natural to normalize $\Phi_n(z)$ and define the *spherical harmonics empirical process* by

$$G_n(z) := \sqrt{n}\left[\int_{S^2} \mathbb{1}\left(f_n(x) \le z\right) dx - \{4\pi \times \Phi(z)\}\right] \tag{17}$$

for $n = 1, 2, ..., z \in (-\infty, \infty)$.

In [14] we proved the following result:

Theorem 2.5 (Marinucci-Wigman [14]). *(The Uniform Central Limit Theorem) As $n \to \infty$, the process $G_n(z)$ converges in distribution to $G_\infty(z)$, where $G_\infty(z)$ is the mean zero, degenerate Gaussian process on $\mathbb{R}$ given by*

$$G_\infty(z) = z\phi(z)Z$$

with $Z \sim N(0,1)$ standard Gaussian random variable.

This result, in particular, implies the full asymptotic dependence of $G_n(z)$ for different values of z, as $n \to \infty$. See the next Section for some explanation to this phenomenon.

2.7. **Nodal line vs. Level curves.** Interestingly, the behaviour of level curves $f_n^{-1}(L)$ for $L \neq 0$ is very different compared to the behaviour of nodal lines. Let $\mathcal{Z}^L(f_n)$ be the length of the level curve $f_n^{-1}(L)$. It is standard to compute the expected length, using the Kac-Rice formula

$$\mathbb{E}[\mathcal{Z}^L(f_n)] = c_1 e^{-L^2/2}\sqrt{E_n}$$

consistent with the nodal case $L = 0$. However, unlike the nodal lines, level length variance is asymptotic to [27]

$$\mathrm{Var}(\mathcal{Z}^L(f_n)) \sim c_2 L^4 e^{-L^2} \cdot n; \tag{18}$$

it is also interesting to observe the fact that the leading term depends on L^4 (a priori, the dependence on L should be symmetric w.r.t. $L \mapsto -L$, however we would rather expect L^2; its dependence cancels out - another obscure cancellation related to this problem).

Moreover, the length of the level curves is asymptotically fully correlated, in the sense that[10]

$$\rho(\mathcal{Z}^{L_1}(f_n), \mathcal{Z}^{L_2}(f_n)) = 1 - o_{n\to\infty}(1). \tag{19}$$

Let us relate between the latter and the setup of Theorem 2.5. One may express $\Phi_n(z)$ in terms of the level lengths using

$$\Phi_n(z) = \int_{-\infty}^{z} \mathcal{Z}^L(f_n)dL. \tag{20}$$

Intuitively, the asymptotic degeneracy of $\Phi_n(z)$ for different values of z is then an artifact of the asymptotic full dependence (19) of the individual values in the integrand on the RHS of (20).

One possible explanation for the phenomenon (19) is the following conjecture, due to Mikhail Sodin (see Marinucci-Wigman [15] for further reading on this conjecture). For $x \in \mathcal{S}^2$ and $L \in \mathbb{R}$ let

$$\mathcal{Z}_x^L = \mathcal{Z}_x^L(f_n)$$

(the "local length") be the (random) length of the unique component[11]of $f_n^{-1}(L)$ that contains x inside (or 0, if f_n does not cross the level L).

[10]For two random variables X, Y the correlation is defined as $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$; $|\rho| \leq 1$ measures the linear correlation between X and Y.

[11]We should assume that $f_n(x) \neq L$; the latter is satisfied almost surely.

Conjecture 2.6 (M. Sodin)**.** *The local lengths are asymptotically fully dependent in the sense that for every $x \in \mathcal{S}^2$ and $L_1, L_2 \in \mathbb{R}$*

$$\rho\left(\mathcal{Z}_x^{L_1}, \mathcal{Z}_x^{L_2}\right) = 1 - o_{n\to\infty}(1).$$

Intuitively, it should be clear that Conjecture 2.6 implies (19) (and thus also the asymptotic degeneracy of the level exceeding measure via (20)), since $\mathcal{Z}^L$ is some summation of $\mathcal{Z}_x^L$ over some x on the sphere.

All the results above team up nicely together while dealing with the corresponding questions in case of *Gaussian subordinated* random fields[12]: let $G : \mathbb{R} \to \mathbb{R}$ be a (possibly nonlinear) nice function and define the random field g_n by

$$g_n(x) = G(f_n(x));$$

g_n is Gaussian subordinated. We are interested in the nodal length of g_n

$$\mathcal{Z}_{g_n} = \text{length}(g_n^{-1}(0)),$$

so that if $z_1, \ldots, z_k$ are all the zeros[13]of G, then it is obvious that

$$\mathcal{Z}_{g_n} = \sum_{i=1}^{k} \mathcal{Z}_n^{z_i}.$$

Therefore, the expected nodal length of g_n is

$$\mathbb{E}[\mathcal{Z}_{g_n}] = c_1 \sum_{i=1}^{k} e^{-L^2/2}\sqrt{E_n}$$

for some explicitly given $c_1 > 0$.

It is not difficult to see that (18) together with (19) gives an elegant and compact asymptotic result (for $n \to \infty$) for the nodal length variance of g_n as

$$\text{Var}[\mathcal{Z}_{g_n}] \sim c_2 \left(\sum_{i=1}^{k} e^{-z_i^2/2} z_i^2\right)^2 \cdot n,$$

with some explicit $c_2 > 0$, provided that $z_i \neq 0$ for at least one index i.

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[13]We do allow infinitely many zeros of G; in this case it is easy to modify the formulas to follow.

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