GROWTH OF p-PARTS OF IDEAL CLASS GROUPS AND FINE SELMER GROUPS IN $\mathbb{Z}_q$-EXTENSIONS WITH $p \neq q$

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Abstract. Fix two distinct odd primes p and q. We study "$p \neq q$" Iwasawa theory in two different settings.
(1) Let K be an imaginary quadratic field of class number 1 such that both p and q split in K. We show that under appropriate hypotheses, the p-part of the ideal class groups is bounded over finite subextensions of an anticyclotomic $\mathbb{Z}_q$-extension of K.
(2) Let F be a number field and let $A_{/F}$ be an abelian variety with $A[p] \subseteq A(F)$. We give sufficient conditions for the p-part of the fine Selmer groups of A over finite subextensions of a $\mathbb{Z}_q$-extension of F to stabilize.

1. Introduction

Let $F/\mathbb{Q}$ be an algebraic number field and F_∞/F be a Galois extension with Galois group isomorphic to the additive group $\mathbb{Z}_q$ of q-adic integers. For each integer $n \geq 0$, there is a unique subfield F_n/F of degree q^n. Let $h(F_n)$ be the class number of F_n. K. Iwasawa showed that if q^{e_n} is the highest power of q dividing $h(F_n)$, then there exist integers λ, μ, ν independent of n, such that $e_n = \mu q^n + \lambda n + \nu$ for $n \gg 0$. On the other hand, in [Was75, Was78], L. C. Washington proved that for distinct primes p and q, the p-part of the class number stabilizes in the *cyclotomic* $\mathbb{Z}_q$-extension of an abelian number field. Washington's results have been extended to other $\mathbb{Z}_q$-extensions where primes are finitely decomposed. In particular, J. Lamplugh proved the following in [Lam15]: if p, q are distinct primes ≥ 5 that split in an imaginary quadratic field K of class number 1 and F/K is a prime-to-p abelian extension which is also unramified at p, then the p-class group stabilizes in the $\mathbb{Z}_q$-extension of F which is unramified outside precisely one of the primes above q. There have also been speculations by J. Coates on the size of the whole class group in a cyclotomic tower; see [Coa12], especially the discussion in §3 and Conjecture D.

Let p and q be two distinct odd primes and K an imaginary quadratic field of class number 1 in which both p and q split. We write $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$ and $q\mathcal{O}_K = \mathfrak{q}\overline{\mathfrak{q}}$. Given an ideal $\mathfrak{h}$ of $\mathcal{O}_K$, we write $\mathscr{R}(\mathfrak{h})$ for the ray class field of K of conductor $\mathfrak{h}$. In the first half of this article, we study the growth of the p-part of the ideal class group in a $\mathbb{Z}_q$-anticyclotomic tower. This generalizes [Lam15, Theorem 1.3], where the stability of the p-part of the class numbers $\mathscr{R}(\mathfrak{g}\mathfrak{q}^n)$ is studied. More precisely, we prove the following result.

Theorem A. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{r}$ be a fixed ideal of $\mathcal{O}_K$ coprime to pq such that $\mathfrak{r}$ is a product of split primes*[1]*. Let $\mathcal{F} = \mathscr{R}(\mathfrak{r}q)$. We assume that $p \nmid [\mathcal{F} : K]$. Let $\mathscr{R}(\mathfrak{r}q^\infty)^{\mathrm{ac}}/\mathcal{F}$ denote the anticyclotomic $\mathbb{Z}_q$-extension and write $\mathcal{F}_n$ for the unique subextension of $\mathscr{R}(\mathfrak{r}q^\infty)^{\mathrm{ac}}/\mathcal{F}$ whose degree is q^n. Then there exists an integer N such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(\mathcal{F}_n)) = \mathrm{ord}_p(h(\mathcal{F}_N)).$$

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[1] In this article, a split prime of K refers to a prime ideal of $\mathcal{O}_K$ that lies above a rational prime that splits in K.

The hypothesis on $\mathfrak{r}$ being a product of split primes is crucial for the use of a theorem of H. Hida, which guarantees the non-vanishing modulo p of the algebraic L-values of anticyclotomic characters factoring through $\mathscr{R}(\mathfrak{r}q^\infty)^{\mathrm{ac}}$ (see Theorem 3.2). To prove Theorem A, we link this non-vanishing to the stabilization of the p-class groups via the (p-adic) Iwasawa main conjecture proved by K. Rubin [Rub91]. Our strategy is inspired by the work of Lamplugh [Lam15], which we outline below.

In §2, we introduce an auxiliary elliptic curve $E_{/K}$ with CM by $\mathcal{O}_K$ such that the conductor $\mathfrak{f}$ of its Hecke character is a product of split primes in K with $p \nmid [\mathscr{R}(\mathfrak{f}) : K]$. Let $\mathfrak{g} = \mathrm{lcm}(\mathfrak{f}, \mathfrak{r})$. By a result of Lamplugh, when the algebraic L-value of certain Hecke character is nonzero modulo p, the corresponding modules of local p-adic units and elliptic units over an extension generated by $E[\mathfrak{p}^\infty]$ coincide after taking appropriate isotypic components (see Theorem 4.2 for the precise statement). Combining this with Hida's theorem, we prove in Theorem 4.3 that the p-primary Galois modules featured in the Iwasawa main conjecture stabilize in the anticyclotomic $\mathbb{Z}_q$-extension $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/\mathscr{R}(\mathfrak{g}q)$. This can be translated into a statement on p-class groups, proving a special case of Theorem A, where the ideal $\mathfrak{r}$ is divisible by $\mathfrak{f}$ (see Theorem 4.4). To complete the proof of Theorem A, we bound the p-class groups over the tower $\mathscr{R}(\mathfrak{r}q^\infty)^{\mathrm{ac}}/\mathscr{R}(\mathfrak{r}q)$ by those over $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/\mathscr{R}(\mathfrak{g}q)$.

In the second half of the article, we prove a general statement (see Theorem 5.3) which shows that in certain $\mathbb{Z}_q$-extensions of a number field F, the growth of the p-part of the class group is closely related to that of the p-primary *fine* Selmer group of an abelian variety $A_{/F}$. This subgroup of the classical p-primary Selmer group is denoted by $\mathrm{Sel}_0(A/F)$, and is obtained by imposing stronger vanishing conditions at primes above p (the precise definition is reviewed in §5.1). The following result is an application of the aforementioned theorem to the growth of the p-part of fine Selmer group of a fixed abelian variety A over a $\mathbb{Z}_q$-tower (which is not necessarily anticyclotomic).

Theorem B. *Let p and q be distinct odd primes. Let F be any number field and $A_{/F}$ be an abelian variety such that $A[p] \subseteq A(F)$. Let F_∞/F be a $\mathbb{Z}_q$-extension where the primes above q and the primes of bad reduction of A are finitely decomposed. If there exists $N \geq 0$ such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)),$$

then there exists an integer $N' \geq N$ such that for all $n \geq N'$, there is an isomorphism

$$\mathrm{Sel}_0(A/F_n) \simeq \mathrm{Sel}_0(A/F_{N'}).$$

In particular, Theorem B applies to the setting studied by Washington [Was75, Was78]. Finally, we remark that unlike what we have found for fine Selmer groups in Theorem B, it has been shown by T. Dokchitser and V. Dokchitser that the p-part of the Tate–Shafarevich group of an abelian variety in a $\mathbb{Z}_q$-tower can be unbounded; see [DD15, Example 1.5].

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2. Finding auxiliary CM elliptic curves

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field of class number 1. As discussed in the introduction, we shall work with an auxiliary CM elliptic curve $E_{/K}$ in order to prove Theorem A. Recall that the imaginary quadratic fields of class number 1 are precisely the following

$$\mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-2}),\ \mathbb{Q}(\sqrt{-3}),\ \mathbb{Q}(\sqrt{-7}),\ \mathbb{Q}(\sqrt{-11}),\ \mathbb{Q}(\sqrt{-19}),\ \mathbb{Q}(\sqrt{-43}), \mathbb{Q}(\sqrt{-67}),\ \mathbb{Q}(\sqrt{-163}).$$

For each choice of K, we shall write down an explicit elliptic curve $E_{/K}$ such that

(a) E has CM by $\mathcal{O}_K$;
(b) If $\mathfrak{f}$ denotes the conductor of the Hecke character ψ attached to E, then $\mathfrak{f}$ is only divisible by split primes of K;

(c) The rational primes dividing $[\mathscr{R}(\mathfrak{f}) : K]$ are either $2, 3$ or primes that are non-split in K.

We remark that condition (c) ensures that the prime p in the statement of Theorem A does not divide $[\mathscr{R}(\mathfrak{f}) : K]$.

If $E_{/K}$ is an elliptic curve with CM by an order $\mathcal{O}$ in K, the j-invariant $j(E)$ is an integer in this case, so E must be a twist of the base extension of an elliptic curve $A_{/\mathbb{Q}}$. For $d > 3$, A is uniquely determined (up to isomorphism over K) by the condition that it has CM by $\mathcal{O}_K$ and its base change to K has bad reduction at the ramified prime $\mathfrak{P} = (\sqrt{-d})$. For $d = 1, 2,$ and 3 there are several choices for the elliptic curve over $\mathbb{Q}$ (see [CP19, Remark 3.1]).

When $d > 3$, it follows from [CP19, Theorem 3.3] that if we twist $A_{/K}$ by a character corresponding to $K(\sqrt{\alpha})$ where $\alpha = \mathfrak{P}\mathfrak{Q}$ such that $\mathfrak{Q}$ is a prime of K distinct from $\mathfrak{P}$ satisfying $\mathfrak{Q} \equiv u^2\sqrt{-d} \mod 4\mathcal{O}_K$ for some $u \in \mathcal{O}_K$, then the twisted elliptic curve (over K) has good reduction everywhere except at $\mathfrak{Q}$. Therefore, for our purposes, it is enough to find such $\mathfrak{Q}$ that is a split prime in K. Indeed, we may choose $r \in \mathbb{Z}$ such that $(4r + \sqrt{-d})(4r - \sqrt{-d}) = 16r^2 + d$ is an odd rational prime. Such r exists for all possible values of d. For example, we may take r to be 1 when $d = 43, 67, 163$. Then $4r + \sqrt{-d}$ is a split prime of K and $4r + \sqrt{-d} \equiv 1^2\sqrt{-d} \mod 4\mathcal{O}_K$. In particular, we may apply [CP19, Theorem 3.3] with $\mathfrak{Q} = (4r + \sqrt{-d})$ and $u = 1$, resulting in a CM curve E satisfying properties (a) and (b) above.

When $d < 43$, we find $E_{/K}$ by inspection using the data available on [LMF23]. In all our examples below, $E_{/K}$ has bad reduction at one or two split primes which are coprime to 6. In particular, the conductor of $E_{/K}$ is given by square of the product of the bad prime(s), whereas the conductor $\mathfrak{f}$ of the Hecke character ψ attached to E is given by the product of the bad prime(s) (see [ST68, Theorem 12]). The ray class group $\mathrm{Gal}(\mathscr{R}(\mathfrak{f})/K)$ (and hence $[\mathscr{R}(\mathfrak{f}) : K]$) is computed using MAGMA [BCP97].

d	$A_{/\mathbb{Q}}$	base change curve $A_{/K}$	twisted curve $E_{/K}$	bad prime(s) of $E_{/K}$	$[\mathscr{R}(\mathfrak{f}) : K]$
1	64.a4	2.0.4.1-256.1-CMa1	2.0.4.1-25.1-CMa1	$2 + \sqrt{-1}$	1
2	256.d1	2.0.8.1-1024.1-CMb1	2.0.8.1-9.3-CMa1	$1 - \sqrt{-2}$	1
3	27.a4	2.0.3.1-81.1-CMa1	2.0.3.1-2401.3-CMa1	$2 \pm \sqrt{-3}$	6
7	49.a4	2.0.7.1-49.1-CMa1	2.0.7.1-1849.1-CMa1	$6 - \sqrt{-7}$	21
11	121.b2	2.0.11.1-121.1-CMa1	2.0.11.1-9.1-CMa1	$\frac{-1-\sqrt{-11}}{2}$	1
19	361.a2	2.0.19.1-361.1-CMa1	2.0.19.1-49.3-CMa1	$\frac{-1+\sqrt{-19}}{2}$	3
43	1849.b2	2.0.43.1-1849.1-CMa1	$\alpha = -43 + 4\sqrt{-43}$	$4 + \sqrt{-43}$	29
67	4489.b2	2.0.67.1-4489.1-CMa1	$\alpha = -67 + 4\sqrt{-67}$	$4 + \sqrt{-67}$	41
163	26569.a2	curve not in database	$\alpha = -163 + 4\sqrt{-163}$	$4 + \sqrt{-163}$	89

3. A result of Hida on L-values of anticyclotomic Hecke characters

Throughout this section and the next, K is a fixed imaginary quadratic field of class number 1. We fix an elliptic curve $E_{/K}$ with CM by $\mathcal{O}_K$ as given in §2. Recall that ψ denotes the Hecke character over K with conductor $\mathfrak{f}$ attached to E.

We review a special case of a result of Hida from [Hid07] that will play a crucial role in our proof of Theorem A.

Definition 3.1. Let $\mathfrak{h}$ be any integral ideal of K and let ϵ be any Hecke character of K. The $\mathfrak{h}$-*imprimitive* L-*function* of ϵ is defined as follows

$$\begin{aligned} L_{\mathfrak{h}}(\epsilon, s) &= \prod_{\gcd(\nu,\mathfrak{h})=1} \left(1 - \frac{\epsilon(\nu)}{(N\nu)^s}\right)^{-1} \\ &= \sum_{\gcd(\mathfrak{a},\mathfrak{h})=1} \frac{\epsilon(\mathfrak{a})}{(N\mathfrak{a})^s}, \end{aligned}$$

where the product runs over *prime ideals* ν of K coprime to $\mathfrak{h}$, and sum is taken over *integral ideals* $\mathfrak{a}$ coprime to $\mathfrak{h}$.

Fix an integral ideal $\mathfrak{g}$ of K which is divisible by $\mathfrak{f}$, relatively prime to pq, and such that only split primes of K divide $\mathfrak{g}$. Let $F = \mathscr{R}(\mathfrak{g}q)$ be the *ray class field* of K of conductor $\mathfrak{g}q$ and write $\Delta = \mathrm{Gal}(F/K)$. Set $F_\infty = \bigcup_{n\geq 1} \mathscr{R}(\mathfrak{g}q^n)$; this is a $\mathbb{Z}_q^2$-extension of F. We fix an isomorphism

$$\mathrm{Gal}(F_\infty/K) \simeq \mathrm{Gal}(F/K) \times \mathrm{Gal}(K_\infty/K) = \Delta \times \mathbb{Z}_q^2.$$

Let ϵ be a character of $\mathrm{Gal}(F_\infty/K)$. For our purpose, ϵ will be of the form $\overline{\varphi\psi^k}$, where φ is a finite-order character and k is an integer between 1 and $p-1$. Denote by $L(\epsilon, s)$ the *primitive Hecke L-function* of ϵ. Recall that the imprimitive (or partial) L-function differs from the primitive (or classical) L-function by a finite number of Euler factors. Let $N_{K/\mathbb{Q}}$ denote the norm map. We can further define the *primitive algebraic Hecke L-value*,

$$L^{\mathrm{alg}}(\overline{\varphi\psi^k}) = L^{\mathrm{alg}}(\epsilon) := \frac{L\left(\epsilon, k\right)}{\Omega_\infty^k} = \frac{L\left(\overline{\varphi\psi^k} N_{K/\mathbb{Q}}^{-k}, 0\right)}{\Omega_\infty^k}.$$

Here, Ω_∞ denotes a complex period for $E_{/\mathbb{C}}$. Similarly, given an integral ideal $\mathfrak{h}$ of K, we define the $\mathfrak{h}$-*imprimitive algebraic Hecke L-value*,

$$L_{\mathfrak{h}}^{\mathrm{alg}}(\overline{\varphi\psi^k}) = L_{\mathfrak{h}}^{\mathrm{alg}}(\epsilon) := \frac{L_{\mathfrak{h}}\left(\epsilon, k\right)}{\Omega_\infty^k} = \frac{L_{\mathfrak{h}}\left(\overline{\varphi\psi^k} N_{K/\mathbb{Q}}^{-k}, 0\right)}{\Omega_\infty^k}.$$

Note that L and $L_{\mathfrak{h}}$ differ by the omission of the Euler factors at primes dividing $\mathfrak{h}$.

In what follows, we say that a Hecke character ϵ of K is of *infinity type* (a, b) if its infinity component sends x to $x^a\overline{x}^b$. Under this convention, ψ has infinity type $(-1, 0)$, whereas the norm map $N_{K/\mathbb{Q}}$ is of infinity type $(-1, -1)$. Thus, the Hecke character $\overline{\psi^k} N_{K/\mathbb{Q}}^{-k}$ is of infinity type $(k, 0)$.

Henceforth, we fix a prime $v \mid \mathfrak{p}$ of F and an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_p}$ so that v is sent into the maximal ideal $\mathfrak{m}_{\overline{\mathbb{Q}_p}}$ of $\mathcal{O}_{\overline{\mathbb{Q}_p}}$. This allows us to consider $L_{\mathfrak{h}}^{\mathrm{alg}}(\overline{\varphi\psi^k})$ as elements of $\overline{\mathbb{Q}_p}$. Throughout, π is a fixed uniformizer of F_v and we write ord_π for the valuation on $\overline{\mathbb{Q}_p}$ normalized so that $\mathrm{ord}_\pi(\pi) = 1$.

Theorem 3.2 (Hida)**.** *For all but finitely many characters φ that factor through $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}$, we have*

$$\mathrm{ord}_\pi\left(L_{(q)}^{\mathrm{alg}}(\overline{\varphi\psi^k})\right) = 0.$$

Proof. For each φ, we have $\overline{\varphi} = \phi\eta$, where ϕ is a character of Δ and η is a character of the Galois group $\mathrm{Gal}(\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F)$. We may further decompose ϕ into $\phi'\nu^{-1}$, where ν is a character of $\mathrm{Gal}(F/\mathscr{R}(\mathfrak{g}))$ and ϕ' is a character of $\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/K)$. We have the field diagram:

$$\begin{tikzcd}
\mathscr{R}(\mathfrak{g}q^{\infty})^{\mathrm{ac}} \arrow[dd, dash] \arrow[dd, bend right, dash, "\mathbb{Z}_q"'] & & \\
{} \arrow[rr, dashed, dash] & & \eta \\
\mathscr{R}(\mathfrak{g}q) = F \arrow[dr, dash] \arrow[dd, dash] \arrow[dd, bend right, dash, "\Delta"'] \arrow[rr, phantom, "\text{- - - - - - - -}"] & & \nu^{-1} \\
 & \mathscr{R}(\mathfrak{g}) \arrow[dl, dash] \arrow[r, phantom, "\text{- - - - - - - -}"] & \phi' \\
K \arrow[d, dash] & & \\
\mathbb{Q} & &
\end{tikzcd}$$

We take the CM field M in [Hid07] to be the imaginary quadratic field K. We take the CM type Σ there to be the one that corresponds to the infinity type $(1,0)$ and $\kappa = 0$. Then the infinity type of the character λ in *op. cit.* becomes

$$k\Sigma + 0(1-c) = k(1,0) + (0,0) - (0,0) = (k,0).$$

The condition (M1) in [Hid07, Theorem 4.3] does not hold since $K/\mathbb{Q}$ is not unramified everywhere (it ramifies at the primes dividing the discriminant of K, which is nontrivial). Hence, we can apply the aforementioned theorem with λ and χ^{-1} taken to be $\overline{\psi^k} N^{-k} \phi'$ and η, respectively. $\square$

Remark 3.3 ([Lam14, proof of Theorem 3.1.9]). Let $\mathfrak{g}$ be a fixed ideal as before. Fix an ideal $\mathfrak{h}$ of $\mathcal{O}_K$ which is coprime to $\mathfrak{p}$ and divisible by $\mathfrak{g}q$. Recall that the $\mathfrak{h}$-imprimitive algebraic L-value of $\overline{\varphi\psi^k}$ is given by

$$L^{\mathrm{alg}}_{\mathfrak{h}}(\overline{\varphi\psi^k}) = \frac{L_{\mathfrak{h}}(\overline{\varphi\psi^k}, k)}{\Omega_\infty^k}.$$

Then, for almost all characters of $\mathrm{Gal}\left(\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F\right) \cong \mathbb{Z}_q$, we have that

$$\mathrm{ord}_\pi\left(L^{\mathrm{alg}}_{(q)}(\overline{\varphi\psi^k})\right) = \mathrm{ord}_\pi\left(L^{\mathrm{alg}}_{\mathfrak{h}}(\overline{\varphi\psi^k})\right).$$

This follows from the observation that for a given prime ideal $\mathfrak{a}$ of K that is coprime to q, for almost all characters η,

$$\mathrm{ord}_\pi\left(1 - \frac{\overline{\varphi\psi^k}(\mathfrak{a})}{(N\mathfrak{a})^k}\right) = 0$$

since η sends $\mathfrak{a}$ to a q-power root of unity, and the images of q-power roots of unity under the reduction map on $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ modulo $\mathfrak{m}_{\overline{\mathbb{Q}_p}}$ are distinct.

4. Consequences on class groups

We now use Theorem 3.2 to study the growth of the p-part of the class group in an anticyclotomic $\mathbb{Z}_q$-extension. Let us introduce the necessary notation. Throughout, $p \nmid 6q$ is a fixed prime that is split in K and $E_{/K}$ is a fixed CM elliptic curve as in the previous section (with Hecke character ψ whose conductor is $\mathfrak{f}$). Let K_0 be any finite abelian extension of K such that p is unramified in K_0 and $p \nmid [K_0 : K]$ (in the next subsection, we will let K_0 vary inside the anticyclotomic tower $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}$). Fix a prime $\mathfrak{p}$ of K lying above p. Set $L = K_0(E_{\mathfrak{p}})$ and $L_\infty = L(E_{\mathfrak{p}^\infty})$. Let $\Delta = \mathrm{Gal}(L/K)$ and $\Gamma = \mathrm{Gal}(L_\infty/L) \simeq \mathbb{Z}_p$. Let $\mathcal{G} = \mathrm{Gal}(L_\infty/K) \cong \Delta \times \Gamma$ and $\Lambda = \mathbb{Z}_p[\![\mathcal{G}]\!]$.

Following [Rub91], we write $\overline{\mathcal{C}}(L_\infty)$ (resp. $U(L_\infty)$) for the inverse limits over all finite sub-extensions inside L_∞ of the completion of the elliptic units (resp. local principal units) at $\mathfrak{p}$.

Fix an ideal $\mathfrak{h}$ of $\mathcal{O}_K$ which is coprime to $\mathfrak{p}$, is divisible by $\mathfrak{f}$, and is such that $K_0 \subset K(E_{\mathfrak{h}}) = \mathscr{R}(\mathfrak{h})$. Let μ_K be the group of roots of unity of K and $\lambda \in \mathcal{O}_K \setminus \mu_K$ such that $\lambda \equiv 1 \mod \mathfrak{h}$ with $(\lambda, 6\mathfrak{h}\mathfrak{p}) = 1$. We let $\sigma_{(\lambda)} \in \mathrm{Gal}(K_0/K)$ denote the Artin symbol associated to λ.

We further decompose Δ as $H \times I$, where $H = \mathrm{Gal}(K_0/K)$ and $I = \mathrm{Gal}(K_0(E_{\mathfrak{p}})/K_0)$. Here, I is the inertia subgroup at $\mathfrak{p}$ inside Δ. Let $\theta_{\mathfrak{p}}$ denote the canonical character given by the Galois action on $E_{\mathfrak{p}^\infty}$ restricted to I. Given a character χ of Δ, we write it as $\varphi\theta_{\mathfrak{p}}^k$, where φ is a character of H and $1 \leq k \leq p-1$. We have the following diagram:

$$\begin{tikzcd}
L_\infty = L(E_{\mathfrak{p}^\infty}) \arrow[d, dash] \arrow[d, dash, bend right, "\mathbb{Z}_p"'] & & \\
L = K_0(E_{\mathfrak{p}}) \arrow[dr, dash] \arrow[dd, dash] \arrow[dd, dash, bend right, "\Delta"'] & & \theta_{\mathfrak{p}}^k \\
& K_0 \arrow[dl, dash] & \\
K \arrow[d, dash] & & \varphi \\
\mathbb{Q} & &
\end{tikzcd}$$

Before proceeding, we need to introduce the notion of an anomalous prime.

Definition 4.1. Fix a prime p and a number field $\mathcal{K}$. Let v be a prime above p in $\mathcal{K}$ and write κ_v to denote the corresponding residue field. In the sense of Mazur (see [Maz72, p. 186]), E is *anomalous* at v if $\widetilde{E}(\kappa_v)[p] \neq 0$.

Let w be a prime in L_∞ which lies above v. Denote by $\mathcal{Z}$ the decomposition subgroup at $\mathfrak{p}$ inside $\mathcal{G}$. Since $\gcd(p,|\Delta|) = 1$, the action of $\Delta \cap \mathcal{Z}$ on $\mu_{p^\infty}(L_{\infty,w}) = \mu_{p^M}$ gives a $\mathbb{Z}_p$-valued character which we denote by $\chi_{\mathrm{cyc}} : \Delta \cap \mathcal{Z} \to \mu_{p-1} \subseteq \mathbb{Z}_p^\times$.

We now record a theorem of Lamplugh which will be important for our discussion.

Theorem 4.2. *Let $\chi = \varphi\theta_{\mathfrak{p}}^k$ be a character of Δ. When E/K_0 is anomalous at a prime above $\mathfrak{p}$, we assume that $\chi|_{\Delta\cap\mathcal{Z}}$ is not the cyclotomic character. Let $\mathfrak{h}$ and λ be as above. If*

$$\mathrm{ord}_\pi\left(\left(N(\lambda) - \lambda^k\varphi(\sigma_{(\lambda)})\right) \cdot L_{\mathfrak{h}}^{\mathrm{alg}}(\overline{\varphi\psi^k})\right) = 0,$$

then $\overline{\mathcal{C}}(L_\infty)^\chi = U(L_\infty)^\chi$. Here, M^χ denotes the χ-isotypic component of a Λ-module M.

Proof. See [Lam15, Theorem 7.7]. □

4.1. **Variations of class groups.** Let $F = \mathscr{R}(\mathfrak{g}q)$ for some ideal $\mathfrak{g}$ of $\mathcal{O}_K$ such that $\mathfrak{g}$ is divisible by $\mathfrak{f}$, is a product of primes that split in K, p is unramified in F/K, $p \nmid [F : K]$, and is coprime to $\mathfrak{p}q$. Furthermore, we assume that both $\mathfrak{q}$ and $\overline{\mathfrak{q}}$ are tamely ramified in F. Then $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}$ is a $\mathbb{Z}_q$-extension of F, and for integers $n \geq 0$, let F_n/F be the n-th layer of this $\mathbb{Z}_q$-extension. Note that only primes above q ramify in F_n/F, $p \nmid [F_n : K]$ (since $q \neq p$), and $F_n \subseteq \mathscr{R}(\mathfrak{g}q^{n+1})$. Therefore, we may take K_0 and $\mathfrak{h}$ in the previous section to be F_n and $\mathfrak{g}q^{n+1}$, respectively.

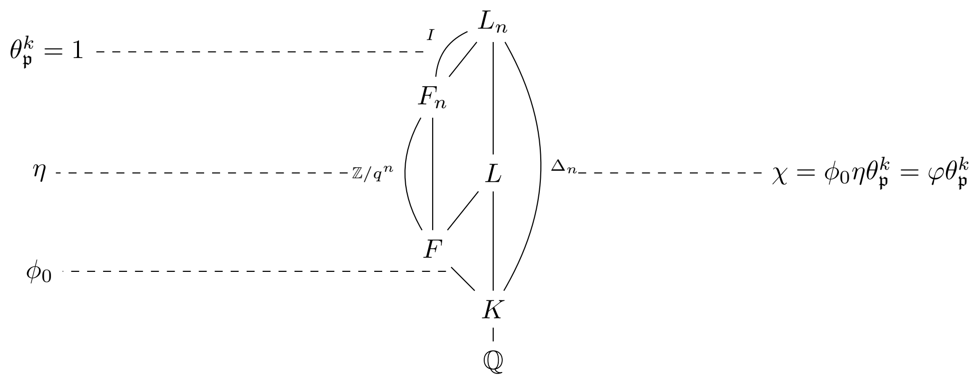
For $n \geq 1$, just as before we define $L_n = F_n(E_{\mathfrak{p}})$, $L_{n,\infty} = F_n(E_{\mathfrak{p}^\infty})$, $\Delta_n = H_n \times I$, $\mathcal{G}_n = \Delta_n \times \Gamma$, $U_{n,\infty} = U(L_{n,\infty})$, etc. Note that $I = \mathrm{Gal}(K_0(E_{\mathfrak{p}})/K_0) \cong \mathrm{Gal}(L_n/F_n)$. Define $X_{n,\infty}$ to be the Galois group of the maximal abelian p-extension of $L_{n,\infty}$ which is unramified outside $\mathfrak{p}$. By global class field theory we have the following four-term exact sequence

$$0 \to \overline{\mathcal{E}}_{n,\infty}/\overline{\mathcal{C}}_{n,\infty} \to U_{n,\infty}/\overline{\mathcal{C}}_{n,\infty} \to X_{n,\infty} \to A_{n,\infty} \to 0. \tag{1}$$

Here, $\overline{\mathcal{E}}_{n,\infty} = \overline{\mathcal{E}}(L_{n,\infty})$ is used to denote the global units of $L_{n,\infty}$. Finally, $A_{n,\infty} = A(L_{n,\infty})$ is the inverse limit of the p-part of the class group for each finite extension of F_n contained inside $L_{n,\infty}$. In other words, $A_{n,\infty}$ can be identified with the Galois group of the maximal abelian unramified p-extension of $L_{n,\infty}$. We now prove the key result which will be required in proving Theorem A.

Theorem 4.3. *There exists an integer $N \geq 0$ such that $X_{n,\infty}^I = X_{N,\infty}^I$ for all $n \geq N$, where M^I denotes the subgroup of M fixed by I.*

Proof. We start by drawing a field diagram for convenience of the reader



Set $\mathcal{O}$ to denote the ring of integers of the unique unramified $\mathbb{Z}_q$-extension of F_v. In other words, $\mathcal{O} = \mathcal{O}_{F_v(\mu_{q^\infty})}$. Let λ be as defined before. We begin our proof with the observation that

$$\begin{aligned}\left(\lambda\overline{\lambda} - \lambda^k \varphi(\sigma_{(\lambda)})\right) \equiv 0 \pmod{\pi\mathcal{O}} &\Leftrightarrow \varphi(\sigma_{(\lambda)}) \equiv \overline{\lambda}\lambda^{1-k} \pmod{\pi\mathcal{O}} \\ &\Leftrightarrow \eta\phi_0(\sigma_{(\lambda)}) \equiv \overline{\lambda}\lambda^{1-k} \pmod{\pi\mathcal{O}} \\ &\Leftrightarrow \eta(\sigma_{(\lambda)}) \equiv \overline{\lambda}\lambda^{1-k}\phi_0^{-1}(\sigma_{(\lambda)}) \pmod{\pi\mathcal{O}}.\end{aligned}$$

Here, we have written $\varphi = \eta\phi_0$, where η is a character of $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/q^n$. Note that η has exact order q^m for some $m \geq 1$. Therefore, $\eta(\sigma_{(\lambda)})$ is a primitive q^m-th root of unity. But, modulo π, the q-power roots of unity are distinct. Therefore, by the same argument outlined in Remark 3.3, we deduce that for almost all η,

$$\mathrm{ord}_\pi\left(N_{K/\mathbb{Q}}(\lambda) - \lambda^k \varphi(\sigma_{(\lambda)})\right) = 0.$$

By Theorem 3.2 and Remark 3.3, one can choose a sufficiently large N such that $\mathrm{ord}_\pi\left(L_{\mathfrak{h}}^{\mathrm{alg}}(\overline{\varphi\psi^k})\right) = 0$ holds for all characters χ of Δ_n with $n \geq N$, which do not factor through Δ_N. Therefore for all such $\chi = \varphi\theta_{\mathfrak{p}}^k$, with φ being a character of H_n and $k = p-1$ (so that its restriction to I is trivial), there exists $\lambda \in \mathcal{O}_K \setminus \mu_K$ such that $\lambda \equiv 1 \mod \mathfrak{h}$ and $(\lambda, 6\mathfrak{h}\mathfrak{p}) = 1$ (where $\mathfrak{h} = \mathfrak{g}q^{n+1}$) satisfying

$$\left(\lambda\overline{\lambda} - \lambda^k\varphi(\sigma_{(\lambda)})\right) \cdot L_{\mathfrak{h}}^{\mathrm{alg}}(\overline{\varphi\psi^k}) \not\equiv 0 \pmod{\pi\mathcal{O}}.$$

Furthermore, recall that the L-value has zero valuation. Therefore, the Euler factor is integral, and we can conclude that

$$\mathrm{ord}_\pi\left(\left(N_{K/\mathbb{Q}}(\lambda) - \lambda^k\varphi(\sigma_{(\lambda)})\right) \cdot L_{\mathfrak{h}}^{\mathrm{alg}}(\overline{\varphi\psi^k})\right) = 0. \tag{2}$$

Note that since the restriction of the character to I is trivial, the hypothesis regarding E/K_0 being anomalous at a prime above $\mathfrak{p}$ in Theorem 4.2 always holds. Taking K_0 to be F_n in the aforementioned theorem, we deduce that

$$U_{n,\infty}^\chi = \overline{\mathcal{C}}_{n,\infty}^\chi$$

for all characters χ of Δ_n that do not factor through Δ_N with $\chi|_I = 1$. This implies

$$U_{n,\infty}^I / \overline{\mathcal{C}}_{n,\infty}^I = U_{N,\infty}^I / \overline{\mathcal{C}}_{N,\infty}^I.$$

Next, via the main conjecture of Iwasawa theory for imaginary quadratic fields (see [Rub91, Theorem 4.1(i)]) we can conclude that there exists an integer $N \geq 0$ such that

$$\mathrm{char}_\Lambda(X_{n,\infty}^I) = \mathrm{char}_\Lambda(X_{N,\infty}^I)$$

for all $n \geq N$. Now, consider the restriction map

$$\pi_{n,N} : X^I_{n,\infty} \twoheadrightarrow X^I_{N,\infty}.$$

Since characteristic ideals are multiplicative in short exact sequences, the kernel of the above surjective map must be finite. However, a theorem of R. Greenberg (see [Gre78, Theorem §1]) ensures that there are no non-trivial finite submodules inside $X^I_{n,\infty}$. This forces the kernel to be trivial, i.e.,

$$X^I_{n,\infty} = X^I_{N,\infty}.$$

The proof of the theorem is now complete. □

We can now state and prove the auxiliary result that will allow us to conclude Theorem A. Our proof follows the proof of [Lam15, Theorem 7.10] very closely. We repeat the statement below for the convenience of the reader.

Theorem 4.4. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ coprime to pq such that $\mathfrak{g}$ is a product of split primes and is divisible by the conductor of an elliptic curve over K with CM by $\mathcal{O}_K$. Let $F = \mathscr{R}(\mathfrak{g}q)$ be a prime-to-p extension of K and $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F$ be the anticyclotomic $\mathbb{Z}_q$-extension. Then, there exists an integer N such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)).$$

Proof. Let the p-class group of F_n (resp. F_N) be denoted by $A(F_n)$ (resp. $A(F_N)$). Since p does not divide $[F_n : F_N]$, we have an injection

$$A(F_N) \hookrightarrow A(F_n). \tag{3}$$

It follows from global class field theory that for all $n \geq 0$, we have the identification

$$A_{n,\infty} \simeq \mathrm{Gal}(M_{n,\infty}/L_{n,\infty}),$$

where $M_{n,\infty}$ is the maximal abelian unramified p-extension of $L_{n,\infty}$. Consider the following diagram

$$\begin{tikzcd} A^I_{N,\infty} \arrow[r] \arrow[d] & A^I_{n,\infty} \arrow[d] \\ A(F_N) \arrow[r] & A(F_n) \end{tikzcd}$$

where the vertical maps are given by restriction and are surjective because the extension $L_{n,\infty}/F_n$ and $L_{N,\infty}/F_N$ are totally ramified at primes above $\mathfrak{p}$. Furthermore, the top horizontal map is surjective by Theorem 4.3 and the exact sequence (1). Therefore, the bottom row is a surjective map as well. When combined with (3), we see that the bottom row is in fact an isomorphism. This completes the proof of the theorem. □

The following lemma allows us to complete the proof of Theorem A via Theorem 4.4.

Lemma 4.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathcal{O}_K$. If $p \nmid [\mathscr{R}(\mathfrak{a}) : K] \cdot [\mathscr{R}(\mathfrak{b}) : K]$, then $p \nmid [\mathscr{R}(\mathrm{lcm}(\mathfrak{a}, \mathfrak{b})) : K]$.*

Proof. Let us write $\mathfrak{a} = \prod \mathfrak{p}_i^{m_i}$, $\mathfrak{b} = \prod \mathfrak{p}_i^{n_i}$, where $\mathfrak{p}_i$ are distinct prime ideals of $\mathcal{O}_K$. Recall that K is of class number 1. By the theory of complex multiplication, if I is an ideal of $\mathcal{O}_K$, we have

$$\mathrm{Gal}(\mathscr{R}(I)/K) = \mathrm{Gal}(K(E[I])/K) \cong (\mathcal{O}_K/I)^\times.$$

Thus, by the Chinese remainder theorem,

$$p \nmid \left|(\mathcal{O}_K/\mathfrak{p}_i^{m_i})^\times\right|, \quad p \nmid \left|(\mathcal{O}_K/\mathfrak{p}_i^{n_i})^\times\right|$$

for all i. As $\mathrm{lcm}(\mathfrak{a}, \mathfrak{b}) = \prod \mathfrak{p}_i^{\max(m_i,n_i)}$, we deduce that

$$p \nmid \left|(\mathcal{O}_K/\,\mathrm{lcm}(\mathfrak{a}, \mathfrak{b}))^\times\right| = [\mathscr{R}(\mathrm{lcm}(\mathfrak{a}, \mathfrak{b})) : K].$$

□

We can now prove Theorem A from the introduction.

Theorem. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{r}$ be a fixed ideal of $\mathcal{O}_K$ coprime to pq such that $\mathfrak{r}$ is a product of split primes. Let $\mathcal{F} = \mathscr{R}(\mathfrak{r}q)$ and write $\mathscr{R}(\mathfrak{r}q^\infty)^{\mathrm{ac}}/\mathcal{F}$ for the anticyclotomic $\mathbb{Z}_q$-extension. Assume that $p \nmid [\mathcal{F} : K]$. Then, there exists an integer N such that for all $n \geq N$,*

$$\operatorname{ord}_p(h(\mathcal{F}_n)) = \operatorname{ord}_p(h(\mathcal{F}_N)).$$

Proof. Let $E_{/K}$ be a CM elliptic curve of conductor $\mathfrak{f}$ such that all the prime divisors of $\mathfrak{f}$ are split in K but the prime divisors (which are ≥ 5) of $[\mathscr{R}(\mathfrak{f}) : K]$ are not split in K. Such elliptic curves exist as we have seen in §2.

Let $\mathfrak{r}$ be any ideal of $\mathcal{O}_K$ and p, q be two distinct primes satisfying the hypotheses in the statement of the theorem. Set $\mathfrak{g} = \operatorname{lcm}(\mathfrak{f}, \mathfrak{r})$ and define $F = \mathscr{R}(\mathfrak{g}q)$. By assumption, $p \nmid [\mathscr{R}(\mathfrak{r}q) : K]$ and we have chosen our auxiliary CM elliptic curve so that $p \nmid [\mathscr{R}(\mathfrak{f}) : K]$. Thus, it follows from Lemma 4.5 that $p \nmid [\mathscr{R}(\mathfrak{g}q) : K]$. Furthermore, both $\mathfrak{f}$ and $\mathfrak{r}$ are only divisible by split primes. Therefore, Theorem 4.4 holds for the ideal $\mathfrak{g}$.

Since $p \nmid [F_n : \mathcal{F}_n]$ and $p \nmid [\mathcal{F}_{n+1} : \mathcal{F}_n]$ for all $n \geq 0$, we have

$$A(\mathcal{F}_n) \hookrightarrow A(F_n), \quad A(\mathcal{F}_n) \hookrightarrow A(\mathcal{F}_{n+1}).$$

Theorem 4.4 asserts that $\operatorname{ord}_p(h(F_n))$ stabilizes as $n \to \infty$. Hence, the same is true for $\operatorname{ord}_p(h(\mathcal{F}_n))$. ∎

5. Asymptotic growth of fine Selmer groups of abelian varieties

5.1. **Definition of fine Selmer groups.** Suppose F is a number field. Throughout, $A_{/F}$ is a fixed abelian variety. We fix a finite set S of primes of F containing p, the primes dividing the conductor of A, as well as the Archimedean primes. We write S_f to denote the set of finite primes. Denote by F_S, the maximal algebraic extension of F unramified outside S. For every (possibly infinite) extension L of F contained in F_S, write $G_S(L) = \operatorname{Gal}(F_S/L)$. Write $S(L)$ for the set of primes of L above S. If L is a finite extension of F and w is a place of L, we write L_w for its completion at w; when L/F is infinite, it is the union of completions of all finite sub-extensions of L.

Definition 5.1. Let L/F be an algebraic extension. The *p-primary fine Selmer group of A* over L is defined as

$$\operatorname{Sel}_0(A/L) = \ker\left(H^1\left(G_S(L), A[p^\infty]\right) \to \bigoplus_{v \in S} H^1\left(L_v, A[p^\infty]\right)\right).$$

Similarly, the *p-fine Selmer group of A* over L is defined as

$$\operatorname{Sel}_0(A[p]/L) = \ker\left(H^1\left(G_S(L), A[p]\right) \to \bigoplus_{v \in S} H^1\left(L_v, A[p]\right)\right).$$

Note that $\operatorname{Sel}_0(A/L)$ is independent of the choice of S, whereas the definition of $\operatorname{Sel}_0(A[p]/L)$ depends on S; see for example [LM16, Lemma 4.1 and p. 86]. Since our main result concerns $\operatorname{Sel}_0(A/L)$, we suppress S from the notation of $\operatorname{Sel}_0(A[p]/L)$ for simplicity.

It is easy to observe that if F_∞/F is an infinite extension,

$$\operatorname{Sel}_0\left(A/F_\infty\right) = \varinjlim_L \operatorname{Sel}_0\left(A/L\right), \quad \operatorname{Sel}_0\left(A[p]/F_\infty\right) = \varinjlim_L \operatorname{Sel}_0\left(A[p]/L\right),$$

where the inductive limits are taken with respect to the restriction maps and L runs over all finite extensions of F contained in F_∞. Next, we define the notion of p-rank of an abelian group G.

Definition 5.2. Let G be an abelian group. Define the *p-rank* of G as

$$r_p(G) = r_p(G[p]) := \dim_{\mathbb{F}_p}\left(G[p]\right).$$

5.2. **Growth of fine Selmer groups in $\mathbb{Z}_q$-extensions.** In this section, we prove the following theorem which essentially says that the p-part of the class group and the p-primary fine Selmer group have similar growth behaviour in $\mathbb{Z}_q$-extensions. Our result is motivated by [LM16, Section 5].

Theorem 5.3. *Let A be a d-dimensional abelian variety defined over a number field F. Let $S(F)$ be a finite set of primes in F consisting precisely of the primes above q, the primes of bad reduction of A, and the Archimedean primes. Let F_∞/F be a fixed $\mathbb{Z}_q$ extension such that primes in $S_f(F)$ are finitely decomposed in F_∞/F and suppose $[F_n : F] = q^n$. Further suppose that $A[p] \subseteq A(F)$. Then as $n \to \infty$,*

$$\left| r_p\left(\mathrm{Sel}_0\left(A/F_n\right)\right) - 2dr_p\left(\mathrm{Cl}(F_n)\right)\right| = O(1).$$

If $A[p] \subseteq A(F)$, then the action of G_F on $A[p]$ is trivial. Let $A^\vee$ be the dual abelian variety. The action on the dual representation, $A^\vee[p]$ is also trivial. This tells us that $A^\vee[p] \subseteq A^\vee(F)$. Therefore, Theorem 5.3 allows us to deduce the following result.

Corollary 5.4. *With the same hypothesis as in Theorem 5.3*

$$\left| r_p\left(\mathrm{Sel}_0\left(A/F_n\right)\right) - r_p\left(\mathrm{Sel}_0\left(A^\vee/F_n\right)\right)\right| = O(1).$$

To prove Theorem 5.3, we need a few lemmas.

Lemma 5.5. *Consider the following short exact sequence of co-finitely generated abelian groups*

$$P \to Q \to R \to S.$$

Then,

$$\left| r_p\left(Q\right) - r_p\left(R\right)\right| \leq 2r_p\left(P\right) + r_p\left(S\right).$$

Proof. See [LM16, Lemma 3.2]. □

Lemma 5.6. *Let F_∞ be any $\mathbb{Z}_q$-extension of F such that all the primes in $S_f(F)$ are finitely decomposed. Let F_n be the subfield of F_∞ such that $[F_n : F] = q^n$. Then*

$$\left| r_p\left(\mathrm{Cl}(F_n)\right) - r_p\left(\mathrm{Cl}_S(F_n)\right)\right| = O(1).$$

Proof. For each F_n, we write $S_f(F_n)$ for the set of finite primes of F_n above S_f. For each n, we have the following exact sequence

$$\mathbb{Z}^{|S_f(F_n)|} \longrightarrow \mathrm{Cl}(F_n) \xrightarrow{\alpha_n} \mathrm{Cl}_S(F_n) \longrightarrow 0$$

(see [NSW08, Lemma 10.3.12]). Since the class group is always finite, it follows that $\ker(\alpha_n)$ is finite. Also, $r_p\left(\ker(\alpha_n)\right) \leq \left|S_f(F_n)\right|$ and $r_p\left(\ker(\alpha_n)/p\right) \leq \left|S_f(F_n)\right|$. By Lemma 5.5,

$$\left| r_p\left(\mathrm{Cl}(F_n)\right) - r_p\left(\mathrm{Cl}_S(F_n)\right)\right| \leq 2\left|S_f(F_n)\right| = O(1).$$ □

Lemma 5.7. *Let F_∞/F be a $\mathbb{Z}_q$-extension and let F_n be the subfield of F_∞ such that $[F_n : F] = q^n$. Let A be an abelian variety defined over F. Suppose that all primes of $S_f(F)$ are finitely decomposed in F_∞/F. Then*

$$\left| r_p\left(\mathrm{Sel}_0(A[p]/F_n)\right) - r_p\left(\mathrm{Sel}_0(A/F_n)\right)\right| = O(1).$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Sel}_0(A[p]/F_n) & \to & H^1(G_S(F_n), A[p]) & \to & \bigoplus_{v\in S(F_n)} H^1(F_{n,v_n}, A[p]) \\
 & & \downarrow{\scriptstyle s_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \gamma_n} \\
0 & \to & \mathrm{Sel}_0(A/F_n)[p] & \to & H^1(G_S(F_n),\ A[p^\infty])[p] & \to & \bigoplus_{v_n\in S(F_n)} H^1(F_{n,v_n},\ A[p^\infty])[p].
\end{array}$$

Both f_n and γ_n are surjective. Since $A[p] \subset A(F_n)$, the kernel of these maps are given by

$$\ker(f_n) = A(F_n)[p^\infty]/p \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)^{2d},$$
$$\ker(\gamma_n) = \bigoplus_{v_n \in S(F_n)} A(F_{n,v_n})[p^\infty]/p \simeq \bigoplus_{v_n \in S_f(F_n)} \left(\mathbb{Z}/p\mathbb{Z}\right)^{2d},$$

where the last isomorphism follows from our assumption that p is odd.

Observe that $r_p\left(\ker\left(s_n\right)\right) \leq r_p\left(\ker\left(f_n\right)\right) = 2d$ and that $r_p\left(\ker\left(\gamma_n\right)\right) = 2d\left|S_f(F_n)\right|$. By hypothesis, $S_f(F_n)$ is bounded as n varies. It follows from the snake lemma that both $r_p\left(\ker\left(s_n\right)\right)$ and $r_p\left(\operatorname{coker}\left(s_n\right)\right)$ are finite and bounded. Applying Lemma 5.5 to the following exact sequence

$$0 \to \ker(s_n) \to \mathrm{Sel}_0(A[p]/F_n) \to \mathrm{Sel}_0(A/F_n)[p] \to \operatorname{coker}(s_n) \to 0$$

completes the proof. □

Proof of Theorem 5.3. By hypothesis, $A[p] \subseteq A(F)$. Therefore, $A[p] \simeq (\mathbb{Z}/p)^{2d}$. We have

$$H^1\left(G_S(F_n),\ A[p]\right) = \mathrm{Hom}\left(G_S(F_n),\ A[p]\right).$$

There are similar identifications for the local cohomology groups. Thus,

$$\mathrm{Sel}_0\left(A[p]/F_n\right) \ \simeq \mathrm{Hom}\left(\mathrm{Cl}_S(F_n), A[p]\right) \simeq \mathrm{Cl}_S(F_n)[p]^{2d}$$

as abelian groups. Therefore,

$$r_p\left(\mathrm{Sel}_0\left(A[p]/F_n\right)\right) = 2d r_p\left(\mathrm{Cl}_S(F_n)\right).$$

The theorem now follows from Lemmas 5.6 and 5.7. □

Let p^{e_n} be the largest power of p that divides the class number of F_n. If e_n is bounded then it follows (trivially) that the p-rank is bounded. Thus, the following corollary is immediate.

Corollary 5.8. *Let $p \neq q$. Let $F/\mathbb{Q}$ be any finite extension of $\mathbb{Q}$ and F_∞/F be any $\mathbb{Z}_q$-extension of F. Let p^{e_n} be the exact power of p dividing the class number of the n-th intermediate field F_n. Let $A_{/F}$ be an abelian variety such that $A[p] \subseteq A(F)$. If e_n is bounded as $n \to \infty$, then $r_p\left(\mathrm{Sel}_0\left(A/F_n\right)\right)$ is bounded independently of n.*

In addition to Theorem A, there are some other results in the literature where it is known that the p-part of the class group stabilizes in a $\mathbb{Z}_q$-extension (when p, q are distinct primes). These were discussed briefly in the introduction and are recorded here more precisely.

(1) ([Was78, Theorem]) Let $F/\mathbb{Q}$ be an abelian extension of $\mathbb{Q}$ and F_∞/F be the cyclotomic $\mathbb{Z}_q$-extension of F. If p^{e_n} be the exact power of p dividing the class number of the n-th intermediate field F_n, then e_n is bounded as $n \to \infty$.
(2) ([Lam15, Theorem 7.10]) Let p, q be fixed odd distinct primes both ≥ 5, K be an imaginary quadratic field of class number 1 where p and q split, and $E_{/K}$ be an elliptic curves with CM by $\mathcal{O}_K$ and good reduction at p, q. Let K_∞ be the $\mathbb{Z}_q$ extensions of K which is unramified outside $\mathfrak{q}$ (resp. $\overline{\mathfrak{q}}$). Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ such that it is coprime to pq and $F = \mathscr{R}(\mathfrak{g}\mathfrak{q})$ is of degree prime-to-p over F. Then, the p-part of the class number stabilizes in $FK_\infty = \mathscr{R}(\mathfrak{g}\mathfrak{q}^\infty)$. However, since p is assumed to be unramified in F in *loc. cit.*, the hypothesis $A[p] \subseteq A(F)$ in Theorem 5.3 is unlikely to hold. The same can be said regarding the setting studied in Theorem A.

Theorem 5.9. *With notation as above, suppose that the p-rank of the fine Selmer group, denoted by $r_p\left(\mathrm{Sel}_0(A/F_n)\right)$ stabilizes in a $\mathbb{Z}_q$-extension of F. Then there exists $n \geq 0$, such that for all $m \geq n$, the restriction map induces an isomorphism*

$$\mathrm{Sel}_0(A/F_n) \simeq \mathrm{Sel}_0(A/F_m).$$

Proof. The following argument is similar to the one presented in [Lam14, p. 15], where instead of classical Selmer groups, we consider fine Selmer groups. Consider the extension F_m/F_n. Then $[F_m : F_n] = q^{m-n} = t$ (say). The restriction map

$$\mathrm{Gal}\left(\overline{F}/F_n\right) \longrightarrow \mathrm{Gal}\left(\overline{F}/F_m\right)$$

induces the restriction homomorphism

$$\mathrm{res} : \mathrm{Sel}_0(A/F_n) \longrightarrow \mathrm{Sel}_0(A/F_m).$$

Since $\gcd(q,p) = 1$, this maps is an injection. Moreover, we have

$$\mathrm{Sel}_0(A/F_n) \xrightarrow{\mathrm{res}} \mathrm{Sel}_0(A/F_m) \xrightarrow{\mathrm{cores}} \mathrm{Sel}_0(A/F_n) \xrightarrow{t^{-1}} \mathrm{Sel}_0(A/F_n)$$

where $\mathrm{cores} \circ \mathrm{res} = \mathrm{t}$. The composition $\mathrm{res} \circ \mathrm{cores} \circ \mathrm{t}^{-1}$ is the identity map; thus, the exact sequence

$$0 \longrightarrow \mathrm{Sel}_0(A/F_n) \longrightarrow \mathrm{Sel}_0(A/F_m) \longrightarrow \mathrm{Sel}_0(A/F_m) \big/ \mathrm{Sel}_0(A/F_n) \longrightarrow 0$$

is split exact.

Let us write $\mathrm{Sel}_0(A/F_n) = (\mathbb{Q}_p/\mathbb{Z}_p)^{s_n} \oplus T_n$, where $s_n \geq 0$ and T_n is a finite p-group. Then,

$$r_p\left(\mathrm{Sel}_0(A/F_n)\right) = s_n + r_p(T_n).$$

The injection $\mathrm{Sel}_0(A/F_n) \hookrightarrow \mathrm{Sel}_0(A/F_m)$ tells us that $s_m \geq s_n$. If the p-rank $r_p\left(\mathrm{Sel}_0(A/F_n)\right)$ eventually stabilizes it follows that s_n also stabilizes. Denote the cokernel of the injection by $C_{m,n}$. By duality, we have the short exact sequence

$$0 \to C_{m,n}^\vee \to \mathbb{Z}_p^{s_m} \oplus T_m^\vee \to \mathbb{Z}_p^{s_n} \oplus T_n^\vee \to 0.$$

When $s_m = s_n$, $C_{m,n}^\vee$ must be finite. Consequently, the image of $C_{m,n}^\vee$ in $\mathrm{Sel}_0(A/F_n)^\vee$ is contained inside $T_m^\vee$. Furthermore, since the short exact sequence splits, we deduce the isomorphism

$$T_m = T_n \oplus C_{m,n}.$$

As s_n stabilizes, $r_p(T_n)$ also stabilizes. Therefore, $C_{m,n}$ has to be 0 eventually. □

Theorem B is now an immediate corollary of Theorems 5.3 and 5.9.

Corollary 5.10. *Let p, q be distinct odd primes. Let F be any number field and $A_{/F}$ be an abelian variety such that $A[p] \subseteq A(F)$. Let F_∞/F be a $\mathbb{Z}_q$-extension where the primes above q and the primes of bad reduction of A are finitely decomposed. If the p-part of the class group stabilizes, i.e., there exists $N \geq 0$ such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)),$$

then the growth of the p-primary fine Selmer group stabilizes in the $\mathbb{Z}_q$-extension as well, i.e., there exists an integer $N' \geq N$ such that for all $n \geq N'$, the restriction map induces an isomorphism

$$\mathrm{Sel}_0(A/F_n) \simeq \mathrm{Sel}_0(A/F_{N'}).$$

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