Research Proposal — Debanjana Kundu

Broadly, I study the arithmetic of elliptic curves; a branch of mathematics, at the intersection of number theory, algebra, arithmetic algebraic geometry and complex analysis. Elliptic curves are precisely the genus 1 algebraic curves with rational points. The arithmetic of simpler, genus 0, algebraic curves has been studied for centuries. On the other hand, higher genus curves are far less accessible. Being of genus 1, elliptic curves land in a sweet spot for mathematical enquiry. A major application was seen in the proof of Fermat's Last Theorem by Andrew Wiles.

My focus is on the Iwasawa theory of elliptic curves. It is the study of arithmetic objects, like Galois modules over infinite towers of number fields. The central theme is that it is often hard to study objects like class groups or Selmer groups in isolation; but in 'nice' families, they are easy to describe since their properties stabilize. The classical theory started with the work of Kummer on sizes of class groups of cyclotomic fields and his results on regular primes. The p-torsion in the class group of $\mathbb{Q}(\mu_p)$ (the field generated over $\mathbb{Q}$ by the p-th roots of unity) had been identified as the main obstruction to the direct proof of Fermat's last theorem. This motivated Iwasawa to ask questions about the structure of modules over infinite Galois extensions of $\mathbb{Q}(\mu_p)$. Consider the tower of number fields: $\mathbb{Q} = \mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \cdots \subset \mathbb{Q}_{cyc} = \bigcup \mathbb{Q}_n$ where $\mathbb{Q}_n$ is the unique subfield of $\mathbb{Q}(\mu_{p^n})$ with $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$. By infinite Galois theory, $\Gamma := \mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}) \simeq \mathbb{Z}_p$, the additive group of p-adic integers. This is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. One can similarly construct the cyclotomic $\mathbb{Z}_p$-extension for any number field, K. For totally real fields, Leopoldt conjectured that the only $\mathbb{Z}_p$-extension is the one constructed above. Else, there are infinitely many different $\mathbb{Z}_p$-extensions of K. Iwasawa (1959) proved that if p^{e_n} is the exact power of p dividing the class number of K_n, there exist integers $\lambda, \mu, \nu \geq 0$ independent of n such that for n sufficiently large, $e_n = \lambda n + \mu p^n + \nu$. He conjectured that for cyclotomic $\mathbb{Z}_p$ extensions, $\mu = 0$ (Classical Conjecture). Ferrero and Washington (1979) proved this for all Abelian number fields. Later, Sinnott (1984) gave a different proof using p-adic L-functions. For imaginary quadratic fields, Coates and Wiles defined the split-prime $\mathbb{Z}_p$-extension via points of finite order on an elliptic curve, and observed that it behaved similar to the cyclotomic $\mathbb{Z}_p$-extension. They conjectured that the μ-invariant is 0 in such $\mathbb{Z}_p$-extensions for every Abelian extension of an imaginary quadratic field. When the base field has class number 1, this was proven by Schneps (1987). Using similar techniques, Anwesh Ray and myself, we are in progress of resolving the Coates-Wiles conjecture.

In the study of rational points on elliptic curves, the Selmer group plays an important role. In Mazur (1972), the Iwasawa theory of Selmer groups was introduced. He described the growth of the size of the p-primary part of the Selmer group in $\mathbb{Z}_p$-towers. The nature of growth was similar to the classical one.

Recently, Coates and Sujatha (2005) defined the fine Selmer group, a subgroup of the Selmer group with stronger finiteness properties than the classical one. They conjectured (Conjecture A) that unlike the Selmer group, the fine Selmer group over the cyclotomic extension is a finitely generated $\mathbb{Z}_p$-module, ie its $\mu = 0$. Lim and Murty (2016) showed that growth of p-rank of fine Selmer group in a cyclotomic tower is determined by growth of the p-rank of ideal class groups, and vice versa. Using these ideas, I proved that for any number field, K, Conjecture A for an elliptic curve over K_{cyc} is equivalent to the Classical Conjecture for K_{cyc}. Therefore, given K, Conjecture A holds for every elliptic curve over K_{cyc}, if we produce one example for which it holds. For density 1 primes, I have constructed elliptic curves with finite fine Selmer groups in the cyclotomic extension. Conjecture A holds for such elliptic curves, and this provides non-trivial evidence for the Classical Conjecture. With R. Sujatha, we have provided new theoretical evidence for the Coates-Sujatha Pseudo-nullity conjecture. I have generalized some results of Murty and Ouyang (2006) on the growth of fine Selmer groups of CM elliptic curves in infinite unramified pro-p extensions. Using analytic techniques, I improved results of Lim-Murty and Cesnavicius (2017) on growth of fine Selmer groups in degree p-extensions. I spent time working on the growth of λ-invariants of fine Selmer groups and have obtained a Riemann-Hurwitz type formula.

The structure of the fine Selmer group is yet to be fully understood. My recent observations suggest that it would be important to study the p-adic L-functions associated to fine Selmer groups. Understanding them, and then using ideas similar to Schneps and Sinnot, might allow one to fully resolve Conjecture A.

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