

NON-VANISHING MODULO p OF HECKE L -VALUES OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. Let p and q be two distinct odd primes. Let K be an imaginary quadratic field with class number 1 over which p and q are both split. Let E/K be an elliptic curve with complex multiplication by \mathcal{O}_K and has good reduction at the primes above p and q . Let ψ be the Hecke character attached to E . We show that for a Zariski dense set of finite-order characters φ over K of q -power conductor, the algebraic part of the L -value $L(\overline{\varphi\psi^k}, k)$ is not divisible by p for $k = 1, \dots, p-1$.

1. INTRODUCTION

Let p and q be two distinct odd primes. It is a classical problem to study the divisibility of the algebraic part of (Hecke) L -values by a given prime p as one varies the (Hecke) characters of q -power conductor. For Dirichlet L -values, such questions were studied by L. Washington in [Was75, Was78]. He showed that for almost all Dirichlet characters of q -power conductor, the algebraic parts of their L -values are coprime to p . As an application, he proved that the p -part of the class number stabilizes in cyclotomic \mathbb{Z}_q -extensions of abelian number fields. Washington's results have been extended to the case of (finite) product of cyclotomic \mathbb{Z}_{q_i} -extensions of abelian number fields (for distinct primes q_i with $q_i \neq p$) by E. Friedman in [Fri82].

In [Sin87], W. Sinnott introduced the idea of relating non-vanishing of such L -values modulo p to Zariski density (modulo p) of special points of the algebraic variety underlying the L -values. Using this machinery, J. Lamplugh generalized Washington's theorem to *split prime* \mathbb{Z}_q -extensions of imaginary quadratic fields in [Lam15]. Let K be an imaginary quadratic field such that $q\mathcal{O}_K = \mathfrak{q}\mathfrak{q}^*$ with $\mathfrak{q} \neq \mathfrak{q}^*$, then the *split prime* \mathbb{Z}_q -extension of K is one where only one of \mathfrak{q} or \mathfrak{q}^* is ramified.

In [Hid04, Hid07], H. Hida studied analogous questions for anticyclotomic characters. He proved that under certain hypotheses, the algebraic parts of the L -values of "almost all" anticyclotomic characters of q -power conductor over a CM field are non-zero mod p . Here, "almost all" means "Zariski dense" after identifying the characters with a product of the multiplicative group (see Remark 5.2). This has been generalized by M.-L. Hsieh [Hsi12] to include self-dual characters. In the case where the CM field is an imaginary quadratic field, T. Finis has proved this result in more generality and has determined precisely the p -adic valuations of the algebraic parts of anticyclotomic Hecke characters of q -power conductor; see [Fin06]. More recently, A. Burungale showed in [Bur16] that these results may be extended to Hida families of anticyclotomic characters.

We study a generalization of the aforementioned results on anticyclotomic characters to Hecke characters (not necessarily anticyclotomic) of q -power conductor over an imaginary quadratic field of class number 1. More precisely, we prove a 2-variable version of [Lam15, Theorem 6.9].

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Theorem A (Theorem 6.8). *Let K be an imaginary quadratic field of class number 1 over which p and q are both split. Let E/K be an elliptic curve with CM by \mathcal{O}_K and has good ordinary reduction at primes above p and q . Let ψ be the Hecke character attached to E of conductor \mathfrak{f} coprime to pq . Let $F = K(E_{\mathfrak{f}q})$ and $F_{\infty} = \bigcup_{n \geq 1} K(E_{\mathfrak{f}q^n})$ be the \mathbb{Z}_q^2 -extension over F . Let π be the uniformizer of the local field $\mathbb{Q}_p(E_{\mathfrak{f}q})$. Then, for a Zariski dense set of finite-order characters φ of $\text{Gal}(F_{\infty}/F)$,*

$$\text{ord}_{\pi} \left(L^{(\text{alg})} \left(\overline{\varphi \psi^k} \right) \right) = 0.$$

Outline of the Proof. The proof of Theorem A follows closely the line of argument of [Lam15, Theorem 6.9]. It consists of the following ingredients:

- (1) Establish a theory of Gamma transform of "elliptic function measures" on \mathbb{Z}_q^2 , which are measures that arise from a rational function on an elliptic curve.
- (2) Show that the π -adic valuations of the aforementioned Gamma transforms have the same p -adic valuation for almost all finite characters on \mathbb{Z}_q^2 .
- (3) Show that by choosing an appropriate elliptic function measure arising from a rational function on a CM elliptic curve, the Gamma transforms of this measure is related to the special values of L -series that we are interested.
- (4) Show that the π -adic valuation discussed in (2) is zero.

Step (1) is carried out in §3. We follow the strategy of Lamplugh in [Lam15, §3], where the theory for elliptic function measures on \mathbb{Z}_q was developed. To execute (2), we use a lemma of Hida on the Zariski density of characters on \mathbb{Z}_q^d from [Hid04] to prove a result on the algebraic independence of functions on elliptic curves with positive characteristic. In particular, we prove Theorem 4.6 which is a two-variable version of [Lam15, Theorem 4.9]. Next, we prove Theorem 5.1 which completes step (2) of the proof of Theorem A. The corresponding 1-variable version of this theorem was proved in [Lam15, §5]. The construction of the elliptic function measure of step (3) is discussed in Section 6.2; this arises from the same rational function on the CM elliptic curve E utilized in [Lam15, §6.3]. The link between the Gamma transforms of this elliptic function measure and the L -values of interest is given by Lemma 6.4. In step (2), we see that the π -adic valuation mentioned above is in fact given by the valuation of the rational function (see Definition 3.6). Lamplugh showed that this valuation is zero ([Lam15, Lemma 6.7]), which allows us to conclude step (4).

Remark 1.1. While we deduce Theorem A using Lamplugh's techniques and rely crucially on [Lam15, Lemma 6.7], our result is strictly stronger than the one-variable analogue [Lam15, Theorem 6.9]. Indeed, after identifying the characters of $\text{Gal}(F_{\infty}/F)$ with a subset of $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$, the Zariski closure of the set of characters given by loc. cit. is only one copy of $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}$. In particular, it is not Zariski dense in $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$.

Remark 1.2. Using an argument similar to the one presented in [Lam15, §7], Theorem A combined with the Iwasawa main conjecture (proved by Rubin) should show that the p -part of the class groups over a \mathbb{Z}_q^2 -tower is "generically zero". However, it does not seem to be enough to give a generalization of [Lam15, Theorem 7.10] in our setting, unless we replace "almost all" by "all but finitely many".

We conclude by discussing some follow-up questions that we would like to study in the future.

- If we restrict our attention to an anticyclotomic \mathbb{Z}_q -extension of K , the aforementioned result of Finis should allow us to study the growth of the p -part of the class groups in the anticyclotomic extension. Note that unlike [Lam15, Theorem 1.3], we do not expect the

p -part of the class groups to always stabilize since the corresponding Hecke L -values are not always generically zero modulo π .

- Similar to how we build on Lamplugh's results to obtain Theorem A, it may be possible to prove a similar result for Hecke characters of q -power conductor over general CM fields, relying on results of Hida and Hsieh on anticyclotomic characters.
- It may also be interesting to generalize Theorem A to the setting of Hida families, utilizing ideas of Burungale developed in [Bur16].

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2. BASIC NOTIONS

Let K be a fixed imaginary quadratic field with class number 1. Let E be an elliptic curve defined over K with complex multiplication by \mathcal{O}_K , i.e., $\mathcal{O}_K \simeq \text{End}(E)$. Let $q \geq 5$ be a prime number that splits in K , i.e.,

$$q\mathcal{O}_K = \mathfrak{q}\mathfrak{q}^* \text{ with } \mathfrak{q} \neq \mathfrak{q}^*.$$

Suppose that E has good ordinary reduction at both primes above q .

Since K has class number 1, its ring of integers is a principal ideal domain. For any integral ideal $\mathfrak{a} = \langle a \rangle$ in \mathcal{O}_K , we write $E_{\mathfrak{a}}$ to denote

$$\ker([a] : E \rightarrow E).$$

We write μ_K to denote the set of roots of unity in K and w_K to denote the size of this set. If \mathfrak{a} is an integral ideal of \mathcal{O}_K , write $w_{\mathfrak{a}}$ to denote the number of roots of unity congruent to 1 modulo \mathfrak{a} .

We will fix a different prime p satisfying the following properties

- (i) $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$ in K with $\mathfrak{p} \neq \mathfrak{p}^*$ and $\gcd(p, 6q) = 1$.
- (ii) E/K has good ordinary reduction at primes above p .

3. DISTRIBUTIONS AND MEASURES ON \mathbb{Z}_q^2

The goal of this section is to generalize the notion of Gamma transform from [Sin87] and elliptic function measures studied in [Lam15, §3.2] to the two-variable setting.

Let E be an elliptic curve of conductor \mathfrak{f} as before and k/\mathbb{Q}_p be a finite unramified extension containing $\mathbb{Q}_p(E_{\mathfrak{f}q})$. Set $J = k(\mu_{q^\infty})$. This is the unramified \mathbb{Z}_q -extension of k (since $\mu_q \subset k$ by assumption). Let \mathfrak{O} denote the ring of integers of J . Fix a uniformizer π of k and let ord_π denote the normalized valuation map

$$\text{ord}_\pi : J \rightarrow \mathbb{Z} \cup \{\infty\}.$$

Definition 3.1. Let α be a J -valued distribution on \mathbb{Z}_q^2 , i.e., α is a finitely additive function on the set of compact open subsets of \mathbb{Z}_q^2 with values in J .

- (i) Given any $c = (c_1, c_2) \in (\mathbb{Z}_q^\times)^2$, define $\alpha \circ c$ to be the distribution given by $\alpha \circ c(X) = \alpha(cX)$ for all open compact subsets X of \mathbb{Z}_q^2 .

(ii) The *Fourier transform* of α is defined to be

$$\begin{aligned}\hat{\alpha} : \mu_{q^\infty}^2 &\rightarrow J \\ (\zeta_1, \zeta_2) &\mapsto \int_{(x,y) \in \mathbb{Z}_q^2} \zeta_1^x \zeta_2^y d\alpha(x, y).\end{aligned}$$

(iii) Given a finite character χ on $(\mathbb{Z}_q^\times)^2$ with values in J , we define *Leopoldt's Γ -transform* as

$$\Gamma_\alpha(\chi) = \int_{\mathbb{Z}_q^2} \chi d\alpha,$$

where we extend χ to \mathbb{Z}_q^2 by sending all elements not inside $(\mathbb{Z}_q^\times)^2$ to zero.

(iv) We call α a *measure* on \mathbb{Z}_q^2 if the image of α has bounded values with respect to ord_π .

Lemma 3.2. *Suppose that χ is a finite-order character on $(\mathbb{Z}_q^\times)^2$ such that χ is constant on the cosets of $q^m \mathbb{Z}_q \times q^n \mathbb{Z}_q$, then*

$$\Gamma_\alpha(\chi) = \tau(\chi) \sum_{\underline{x} \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \chi^{-1}(\underline{x}) \hat{\alpha}(\underline{\zeta}^{\underline{x}}),$$

where $\underline{\zeta} = (\zeta_m, \zeta_n)$ with ζ_m and ζ_n being primitive p^m -th and p^n -th roots of unity respectively, and $\tau(\chi)$ is the Gauss sum of χ defined by

$$\tau(\chi) = \frac{1}{q^{m+n}} \sum_{(x_1, x_2) \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \chi(x_1, x_2) \zeta_m^{-x_1} \zeta_n^{-x_2}.$$

Proof. See [Sin87, proof of Proposition 2.2, equation (2.6)] (or [Lam14, proof of Lemma 2.2.3]). \square

Lemma 3.3. *A distribution α on \mathbb{Z}_q^2 is uniquely determined by its Fourier transform $\hat{\alpha}$.*

Proof. The characteristic function on the open subset $U_{a,b} := (a + q^m \mathbb{Z}_q) \times (b + q^n \mathbb{Z}_q)$ of \mathbb{Z}_q^2 satisfies

$$\mathbf{1}_{U_{a,b}} = \frac{1}{q^{m+n}} \sum_{(\zeta_1, \zeta_2) \in \mu_{q^m} \times \mu_{q^n}} \zeta_1^{-a} \zeta_2^{-b} \chi_{(\zeta_1, \zeta_2)},$$

where $\chi_{(\zeta_1, \zeta_2)} : \mathbb{Z}_q^2 \rightarrow J$ is the character sending (x, y) to $\zeta_1^x \zeta_2^y$. In particular, we see that $\alpha(U_{a,b})$ is a linear combination of $\hat{\alpha}(\zeta_1, \zeta_2)$. Since the subsets $U_{a,b}$ form a basis of open compact sets of \mathbb{Z}_q^2 , α is uniquely determined by $\hat{\alpha}$. \square

For the rest of the article, we fix an isomorphism of groups $\delta : (\mu_{q^\infty})^2 \xrightarrow{\sim} E_{q^\infty}$.

Definition 3.4. A J -valued distribution α on \mathbb{Z}_q^2 is an *elliptic function measure* for our fixed elliptic curve E (with respect to δ) if there exists a rational function $R \in J(E)$ such that for almost all $\underline{\zeta} \in (\mu_{q^\infty})^2$, we have

$$\hat{\alpha}(\underline{\zeta}) = R(\delta(\underline{\zeta})).$$

Lemma 3.5. *Let $f \in \mathfrak{O}[x, y]$ such that the image of f in $J(E)$ is non-zero. Then, there exists a unique integer $n \geq 0$ such that*

$$\text{ord}_\pi(f(Q)) \geq n \quad \forall Q \in E_{q^\infty} \setminus \{0\}$$

with equality holding for almost all $Q \in E_{q^\infty}$.

Proof. The proof of [Lam15, Lemma 3.2] goes through in verbatim on replacing E_{q^∞} by E_{q^∞} . \square

This lemma allows us to define a valuation on $J(E)$.

Definition 3.6. Given an $R \in J(E)$. If $R \neq 0$, we define $\text{ord}_\pi(R)$ to be the integer n such that $\text{ord}_\pi(R(Q)) = n$ for almost all $Q \in E_{q^\infty}$. If $R = 0$, we set $\text{ord}_\pi(R) = \infty$.

By Lemma 3.5, if α is an elliptic function measure, then it is in fact a measure (not just a distribution) since the values of α are linear combinations of $\hat{\alpha} = R \circ \delta$ as we have seen in the proof of Lemma 3.3 and $\frac{1}{q^{m+n}} \in \mathfrak{D}^\times$ (as $p \neq q$).

For any given rational function $R \in J(E)$, we can define a J -valued measure attached to R .

Lemma 3.7. *Let $R \in J(E)$ be a rational function. There exists a unique measure α on \mathbb{Z}_q^2 such that the Fourier transform $\hat{\alpha}$ coincides with $R \circ \delta$. In other words, α is an elliptic function measure associated to R in the sense of Definition 3.4.*

Proof. By the proof of Lemma 3.3, we may define a measure α satisfying

$$\alpha((a + q^m \mathbb{Z}_q) \times (b + q^n \mathbb{Z}_q)) = \frac{1}{q^{m+n}} \sum_{(\zeta_1, \zeta_2) \in \mu_{q^m} \times \mu_{q^n}} \zeta_1^{-a} \zeta_2^{-b} R \circ \delta(\zeta_1, \zeta_2).$$

It follows from direct calculations that α is additive and that $\hat{\alpha} = R \circ \delta$. \square

Proposition 3.8. *If α is a J -valued elliptic function measure for E on \mathbb{Z}_q^2 , then so is $[\phi]\alpha$ for any locally constant function $\phi : \mathbb{Z}_q^2 \rightarrow J$. Furthermore, if ϕ takes values in \mathfrak{D} , then*

$$\text{ord}_\pi([\phi]\alpha) \geq \text{ord}_\pi(\alpha).$$

Proof. See [Lam14, Proposition 2.2.7]. The proof generalizes verbatim to our situation. \square

We now show how Gamma transforms behave under Galois actions. This will be utilized in the subsequent section. Let us define the following homomorphisms of groups

$$\begin{aligned} \chi_\mu &: \text{Gal}(J/k) \hookrightarrow \text{Aut}(\mu_{q^\infty})^2 \cong (\mathbb{Z}_q^\times)^2, \\ \chi_E &: \text{Gal}(J/k) \hookrightarrow \text{Aut}(E_{q^\infty}) \times \text{Aut}(E_{\bar{q}^\infty}) \cong (\mathbb{Z}_q^\times)^2. \end{aligned}$$

Note that $\chi_\mu = \chi_{\text{cyc}} \times \chi_{\text{cyc}}$, where χ_{cyc} is the cyclotomic character.

Definition 3.9. An elliptic function measure α for E is said to be *defined over k* , if $\hat{\alpha} = R \circ \delta$ for a rational function $R \in k(E)$.

Lemma 3.10. *Suppose that α is an elliptic function measure defined over k . Then, for almost all finite-order characters κ of $(\mathbb{Z}_q^\times)^2$ and for all $\sigma \in \text{Gal}(J/k)$, we have*

$$\Gamma_\alpha(\kappa)^\sigma = \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))} \Gamma_\alpha(\kappa^\sigma).$$

Proof. It follows from Lemma 3.2 that

$$\Gamma_\alpha(\kappa)^\sigma = \tau(\kappa)^\sigma \sum_{\underline{x} \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa^{-1}(\underline{x})^\sigma \hat{\alpha}(\underline{\zeta}^\sigma).$$

We have

$$\begin{aligned}
\tau(\kappa)^\sigma &= \frac{1}{q^{m+n}} \sum_{(x_1, x_2) \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa(x_1, x_2)^\sigma (\zeta_m^{-x_1} \zeta_n^{-x_2})^\sigma \\
&= \frac{1}{q^{m+n}} \sum_{(x_1, x_2) \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa(x_1, x_2)^\sigma \zeta_m^{-\chi_{\text{cyc}}(\sigma)x_1} \zeta_n^{-\chi_{\text{cyc}}(\sigma)x_2} \\
&= \frac{\kappa^\sigma(\chi_{\text{cyc}}(\sigma), \chi_{\text{cyc}}(\sigma))^{-1}}{q^{m+n}} \sum_{(x_1, x_2) \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa(x_1, x_2)^\sigma \zeta_m^{-x_1} \zeta_n^{-x_2} \\
&= \kappa^\sigma(\chi_\mu(\sigma))^{-1} \tau(\kappa^\sigma).
\end{aligned}$$

Since α is an elliptic function measure, we have

$$\hat{\alpha}(\underline{\zeta}^\sigma)^\sigma = R(\delta(\underline{\zeta}^\sigma))^\sigma = R(\delta(\underline{\zeta}^{\chi_E(\sigma)}))^\sigma = \hat{\alpha}(\underline{\zeta}^{\chi_E(\sigma)})^\sigma$$

for some $R \in k(E)$. Therefore, combining these equations gives

$$\begin{aligned}
\Gamma_\alpha(\kappa)^\sigma &= \kappa^\sigma(\chi_\mu(\sigma))^{-1} \tau(\kappa^\sigma) \sum_{\underline{x} \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa^{-1}(\underline{x})^\sigma \hat{\alpha}(\underline{\zeta}^{\chi_E(\sigma)})^\sigma \\
&= \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))} \tau(\kappa^\sigma) \sum_{\underline{x} \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa^{-1}(\underline{x})^\sigma \hat{\alpha}(\underline{\zeta}^\sigma)^\sigma \\
&= \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))} \Gamma_\alpha(\kappa^\sigma)
\end{aligned}$$

where the last equality follows from Lemma 3.2 applied to κ^σ . \square

4. ALGEBRAIC INDEPENDENCE RESULTS

In this section, we prove an algebraic independence result of functions on E_{q^∞} taking values in a *finite field* whose characteristic is distinct from q (see Theorem 4.6). First, we prove an analogue of [Sin87, Proposition 3.1] (and also [Lam15, Theorem 4.5]), i.e., an algebraic independence result (Theorem 4.2) of functions on E_{q^∞} taking values in a *general field*, \mathcal{F} . Let \mathcal{F} be any field and E be any elliptic curve defined over \mathcal{F} already fixed E in §2. Suppose that $q > 3$ is a rational prime that splits in \mathcal{O}_K and $\text{char}(\mathcal{F}) \neq q$. This result essentially says that endomorphisms in $\text{End}(E_{q^\infty}) \times \text{End}(E_{q^*\infty})$ which are independent over $\text{End}_{\mathcal{F}}(E)$, are algebraically independent.

The following lemma will be required for the proof of Theorem 4.2.

Lemma 4.1. *Let Φ_1, \dots, Φ_s be non-trivial morphisms from E^n to E of the form*

$$\Phi_j : (P_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_{ij}(P_i)$$

where $\alpha_{ij} \in \text{End}_{\mathcal{F}}(E)$ for all $1 \leq i \leq n$ and $1 \leq j \leq s$. Suppose that the only relation of the kind $\alpha \Phi_k = \beta \Phi_\ell$ for $\alpha, \beta \in \text{End}_{\mathcal{F}}(E)$ and $k \neq \ell$, is when $\alpha = \beta = 0$. If $r_1, \dots, r_s \in \mathcal{F}(E)$ with $\sum_{j=1}^s r_j \circ \Phi_j = 0$, then each r_j is a constant function.

Proof. See [Lam15, Proposition 4.4]. \square

Theorem 4.2. *Let \mathcal{F} be any field as above, and E be an elliptic curve defined over \mathcal{F} such that $\text{End}_{\mathcal{F}}(E) \simeq \mathcal{O}_K$. Suppose that $\eta_1, \dots, \eta_s \in \text{End}(E_{q^\infty}) \times \text{End}(E_{q^{*\infty}})$ are such that $\alpha \eta_k = \beta \eta_\ell$ for $k \neq \ell$ and some $\alpha, \beta \in \text{End}_{\mathcal{F}}(E)$ only when $\alpha = \beta = 0$. Consider the function*

$$R = \sum_{j=1}^s r_j \circ \eta_j : E_{q^\infty} \times E_{q^{*\infty}} \rightarrow \overline{\mathcal{F}}$$

where $r_j \in \mathcal{F}(E)$ and $\overline{\mathcal{F}}$ denotes an algebraic closure of \mathcal{F} . If $R(Q) = 0$ for all $Q \in E_{q^\infty} \times E_{q^{*\infty}}$, then all r_j 's are constant functions.

Proof. We recall that $\text{End}(E_{q^\infty}) \simeq \mathcal{O}_q$ and $\text{End}(E_{q^{*\infty}}) \simeq \mathcal{O}_{q^*}$ and $\text{End}_{\mathcal{F}}(E) \simeq \mathcal{O}_K$. Consider a free \mathcal{O}_K submodule A of $\mathcal{O}_q \times \mathcal{O}_{q^*}$ of rank n that contains η_j for $1 \leq j \leq s$. Let $\{\underline{\varepsilon}_i\}_{i=1}^n$ be a \mathcal{O}_K basis of A . Then, there exist $\alpha_{ij} \in \mathcal{O}_K$ such that

$$\eta_j = \sum_{i=1}^n \alpha_{ij} \underline{\varepsilon}_i.$$

Define the map

$$\iota : E_{q^\infty} \times E_{q^{*\infty}} = E_{q^\infty} \rightarrow E^n \text{ given by } Q \mapsto (\underline{\varepsilon}_i Q)_{i=1}^n.$$

For each $1 \leq j \leq s$, denote the morphism

$$\Phi_j : E^n \rightarrow E; \quad (P_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_{ij} P_i.$$

We have assumed that

$$\sum_{j=1}^s r_j \circ \Phi_j(Q) = 0 \quad \text{for all } Q \in \iota(E_{q^\infty} \times E_{q^{*\infty}}) \subseteq E^n.$$

Hence, the above equality must hold for all Q in the Zariski closure of $\iota(E_{q^\infty} \times E_{q^{*\infty}})$. It follows from basic facts about Zariski closed subgroups of E^n (see [Sch87, Lemma 1 and 3]) that either the Zariski closure of $\iota(E_{q^\infty})$ is E^n or there exist $\alpha_i \in \mathcal{O}_K$ (not all zero) such that

$$\sum_{i=1}^n \alpha_i \underline{\varepsilon}_i(Q) = 0 \text{ for all } Q \in E_{q^\infty}.$$

If the latter holds, it means that $\sum_{i=1}^n \alpha_i \underline{\varepsilon}_i = 0$. However, this contradicts the fact that $\underline{\varepsilon}_1, \dots, \underline{\varepsilon}_n$ is a basis for A . Thus, the Zariski closure of $\iota(E_{q^\infty})$ is E^n . Lemma 4.1 implies that each r_i is a constant function. \square

To prove the main result in this section, we need a strengthened version of Theorem 4.2. This is achieved by combining the following Diophantine approximation result with a special case of a lemma due to Hida, which we record below.

Lemma 4.3. *Given $\underline{\beta}_1, \dots, \underline{\beta}_d \in \mathcal{O}_q \times \mathcal{O}_{q^*}$ for any integer $d \geq 1$, and a positive constant $c \leq 1$, there exists an integer N such that for all $n \geq N$, there exist algebraic integers $b_1, \dots, b_d \in \mathcal{O}_K$ and a unit $u \in \mathcal{O}_q^\times \times \mathcal{O}_{q^*}^\times$ satisfying*

$$v_{\mathfrak{p}}(u \underline{\beta}_i - b_i) \geq n \text{ for } \mathfrak{p} \in \{q, q^*\} \text{ and} \\ N_{K/\mathbb{Q}}(b_i) < c \cdot q^{2n}.$$

Proof. See [Lam14, Lemma 2.3.9]. \square

Lemma 4.4. *Let r be a positive integer. Let $X = \bigcup_{i=1}^k X_i$ be a proper subset of $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$ such that*

- (i) *X is Zariski closed.*
- (ii) *For each i , there exists a closed subscheme Y_i that is stable under $t \mapsto t^{p^{rn}}$ for all $n \in \mathbb{Z}$, such that $X_i = \zeta Y_i$ for certain $\zeta \in \mu_{q^\infty}^2$;*

There exists P , which is a p^r -power, and an infinite sequence of integers $0 < n_1 < n_2 < \dots$ such that for all $j \geq 1$,

$$\Xi_j \cap X = \emptyset,$$

where Ξ_j is defined by

$$\left\{ \left(\frac{P^x}{q^{n_j}}, \frac{P^y}{q^{n_j}} \right) \mod \mathbb{Z}_q^2 : x, y \in \mathbb{Z} \right\} \subset (\mathbb{Q}_q/\mathbb{Z}_q)^2$$

after identifying $\mu_{q^\infty}^2$ with $(\mathbb{Q}_q/\mathbb{Z}_q)^2$ under an appropriate choice of basis.

Proof. See [Hid04, Lemma 3.4]. □

Remark 4.5. By the proof of the above lemma, $P \equiv 1 \mod q$. Writing $P = 1 + q^v u$, where $q \nmid u$, we have that $|\Xi_j| = q^{2(n_j - v)}$.

Theorem 4.6. *Let \mathbb{F} be a finite field. Suppose that $\eta_1, \dots, \eta_s \in \text{End}(E_{q^\infty}) \times \text{End}(E_{q^*\infty})$ are such that the only relation of the kind $\alpha \eta_k = \beta \eta_\ell$ for $k \neq \ell$ and $\alpha, \beta \in \text{End}_{\mathbb{F}}(E)$ is when $\alpha = \beta = 0$. Consider the function*

$$R = \sum_{i=1}^s r_i \circ \eta_i : E_{q^\infty} \times E_{q^*\infty} \rightarrow \overline{\mathbb{F}}$$

where $r_i \in \mathbb{F}(E)$. We identify E_{q^∞} with $\mu_{q^\infty}^2 \subset \mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$. Then either $\{Q \in E_{q^\infty} : R(Q) \neq 0\}$ is Zariski dense in $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$ or R is identically zero. In the latter case, all r_i 's are constant functions.

Proof. Suppose that $\{Q \in E_{q^\infty} : R(Q) \neq 0\}$ is not Zariski dense and that R is not identically zero. We take P to be a large enough p -power so that R is defined over \mathbb{F}_P (the finite field of cardinality P). Let X be the Zariski closure of $\{Q \in E_{q^\infty} : R(Q) \neq 0\}$ in $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$. Then, X is a proper subset of $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$ and $X^P \subseteq X$.

Let $\log_q : \widehat{\mathbb{G}}_{m/\overline{\mathbb{Q}}_q}^2 \rightarrow \widehat{\mathbb{G}}_{a/\overline{\mathbb{Q}}_q}^2$ be the q -adic logarithm map. We decompose $\log_q(X)$ into a finite union of closed subsets, each of which is stable under the multiplication by P . This allows us to write X as a finite union of closed subschemes X_i of the form ζY_i , where $\zeta \in \mu_{q^\infty}^2$ and Y_i is stable under $t \mapsto t^P$. Therefore, Lemma 4.4 applies. In particular, there exists a sequence integers $0 < n_1 < n_2 < \dots$ and a collection of subsets Ξ_j of q^{n_j} -torsion points in E_{q^∞} on which R vanishes, with $|\Xi_j| = q^{2(n_j - v)}$ for some fixed integer v .

Define

$$\delta := \max_{1 \leq i \leq s} \deg(r_i).$$

We apply Lemma 4.3 to η_1, \dots, η_s and $c = \frac{1}{q^{2v} \cdot s \cdot \delta}$. There exists an integer N such that for all $n_j \geq N$, there are algebraic integers $b_1, \dots, b_s \in \mathcal{O}_K$ and $u \in \mathcal{O}_q^\times \times \mathcal{O}_{q^*}^\times$ (depending on n_j) satisfying

$$v_{\mathfrak{p}}(u \eta_i - b_i) \geq n_j \text{ for } \mathfrak{p} \in \{\mathfrak{q}, \mathfrak{q}^*\} \text{ and} \\ \left| N_{K/\mathbb{Q}}(b_i) \right| < c \cdot q^{2n_j}.$$

In particular, the rational function

$$R_{n_j} := \sum_{i=1}^s r_i \circ b_i \in \mathbb{F}(E)$$

agrees with $R \circ u$ on $E_{q^{n_j}}$. In particular, it vanishes on Ξ_j . Moreover,

$$\deg(R_{n_j}) \leq \sum_{i=1}^s \delta \cdot N_{K/\mathbb{Q}}(b_i) < s\delta \cdot c \cdot q^{2n_j} = q^{2n_j-2v} = |\Xi_j|.$$

Therefore, $R_{n_j} = 0$ and thus R is zero on $E_{q^{n_j}}$. But n_j can be arbitrarily large. This implies that R is identically zero, which is a contradiction. This concludes the first assertion of the theorem. The last assertion follows immediately from Theorem 4.2. \square

5. A THEOREM ON TWO-VARIABLE GAMMA TRANSFORMS

The purpose of this section is to prove a two-variable version of [Lam15, Theorem 5.1] (which in turn generalizes a result of Sinnott [Sin87, Theorem 3.1]). Our proof utilizes crucially Theorem 4.6 from the previous section. Recall notations and definitions introduced in Sections 2 and 3.

Theorem 5.1. *Let α be an elliptic function measure for E defined over k on \mathbb{Z}_q^2 that is supported on $(\mathbb{Z}_q^\times)^2$, and satisfies $\alpha \circ \omega = \alpha$ for all $\omega \in \mu_K^2$. Let R denote the corresponding rational function (so that $\hat{\alpha} = R \circ \delta$ as in Definition 3.4), and let $n = \text{ord}_\pi(R)$ (as in Definition 3.6). Then for a Zariski dense set of finite-order characters κ of $(1 + q\mathbb{Z}_q)^2$, we have*

$$\text{ord}_\pi(\Gamma_\alpha(\kappa)) = n.$$

Remark 5.2. We view $\text{Hom}(\mathbb{Z}_q^2, \mu_{q^\infty})$ as a subset of $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$ by sending κ to $(\kappa(1,0), \kappa(0,1))$. A set of finite-order characters is called *Zariski dense* if its image in $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}^2$ is a dense subset under the Zariski topology.

The following lemma is a key technical ingredient in the proof of Theorem 5.1.

Lemma 5.3. *Let α be an elliptic function measure as in the statement of Theorem 5.1. Define*

$$\beta = \sum_{\eta} (\alpha \circ \eta)|_{(1+q\mathbb{Z}_q)^2},$$

where η runs over a set of representatives for $(\mu_{q-1}/\mu_K)^2$ and $\alpha \circ \eta$ is defined as in Definition 3.1(i). For each $y = (y_1, y_2) \in \mu_{q-1}^2$, we write

$$\beta_y = \beta|_{y_1(1+q^M\mathbb{Z}_q) \times y_2(1+q^M\mathbb{Z}_q)},$$

where $M \geq 1$ is the integer such that $\mu_{q^\infty} \cap k = \mu_{q^M}$. Let $\kappa = (\kappa_1, \kappa_2)$ be a finite-order character of $(1 + q\mathbb{Z}_q)^2$. Suppose that there exist integers $m, n \geq M$ satisfying

$$\ker(\kappa_1) = 1 + q^{m+M}\mathbb{Z}_q, \quad \ker(\kappa_2) = 1 + q^{n+M}\mathbb{Z}_q.$$

Let $\underline{\zeta} = (\zeta_1, \zeta_2) \in \mu_{q^\infty}$ such that

$$\zeta_1^{q^m} = \kappa_1(1 + q^m), \quad \zeta_2^{q^n} = \kappa_2(1 + q^n).$$

Then $\Gamma_\beta(\kappa) \in \pi\mathfrak{D}$ if and only if $\hat{\beta}_y(\underline{\zeta}^{y^{-1}}) \in \pi\mathfrak{D}$ for all $y \in \mu_{q-1}^2$.

Proof. Suppose that $\Gamma_\beta(\kappa) \in \pi\mathfrak{D}$. Let $\sigma \in \text{Gal}(J/k)$ and $\underline{\xi} \in \mu_{q^\infty}^2$. Recall from the proof of Lemma 3.10 that

$$\hat{\alpha}(\underline{\xi})^\sigma = \hat{\alpha}(\underline{\xi}^{\chi_E(\sigma)}).$$

Since Fourier transform is additive, we have equivalently

$$\hat{\beta}(\underline{\xi})^\sigma = \hat{\beta}(\underline{\xi}^{\chi_E(\sigma)}).$$

Furthermore, Lemma 3.10 asserts that

$$\Gamma_\beta(\kappa)^\sigma = \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))} \Gamma_\beta(\kappa^\sigma).$$

Thus, $\text{ord}_\pi(\Gamma_\beta(\kappa^\sigma))$ is independent of $\sigma \in \text{Gal}(J/k)$ because $\Gamma_\beta(\kappa) \in \pi\mathfrak{D}$ by assumption and κ takes values in the group of roots of unity. In particular, $\Gamma_\beta(\kappa^\sigma) \in \pi\mathfrak{D}$ for all $\sigma \in \text{Gal}(J/k)$ under our hypothesis that $\Gamma_\beta(\kappa) \in \pi\mathfrak{D}$.

Let $N = \max(m, n)$. Write k_{N-1} to denote the $(N-1)$ -th layer of the \mathbb{Z}_q -extension J/k , and set $H = \text{Gal}(k_{N-1}/k)$. Let $y \in (1 + q\mathbb{Z}_q)^2$. We have

$$\begin{aligned} \sum_{\sigma \in H} \kappa^\sigma(y)^{-1} \Gamma_\beta(\kappa^\sigma) &= \sum_{\sigma \in H} \kappa^\sigma(y)^{-1} \int_{(1+q\mathbb{Z}_q)^2} \kappa^\sigma(x) d\beta(x) \\ &= \sum_{\sigma \in H} \int_{(1+q\mathbb{Z}_q)^2} \kappa^\sigma(x/y) d\beta(x) \\ &= \int_{(1+q\mathbb{Z}_q)^2} \text{Tr}_{k_{N-1}/k} \circ \kappa(x/y) d\beta(x) \\ &= q^{N-1} \int_{y_1(1+q^m\mathbb{Z}_q) \times y_2(1+q^n\mathbb{Z}_q)} \kappa(x/y) d\beta(x). \end{aligned}$$

Note that q^{N-1} is a unit in \mathfrak{D} (since $q \neq p$). Therefore, $\Gamma_\beta(\kappa) \in \pi\mathfrak{D}$ implies that

$$\int_{y_1(1+q^m\mathbb{Z}_q) \times y_2(1+q^n\mathbb{Z}_q)} \kappa(x/y) d\beta(x) \in \pi\mathfrak{D}.$$

Let $x = (x_1, x_2) = y(1 + q^m z_1, 1 + q^n z_2) = (y_1(1 + q^m z_1), y_2(1 + q^n z_2))$, where $z_1, z_2 \in \mathbb{Z}_q$. Then

$$\kappa(x/y) = \kappa(1 + q^m z_1, 1 + q^n z_2) = \kappa((1 + q^m)^{z_1}, (1 + q^n)^{z_2}) = \zeta_1^{x_1/y_1-1} \zeta_2^{x_2/y_2-1}.$$

Thus, we deduce that

$$\int_{y_1(1+q^m\mathbb{Z}_q) \times y_2(1+q^n\mathbb{Z}_q)} \zeta_1^{x_1/y_1} \zeta_2^{x_2/y_2} d\beta(x) \in \pi\mathfrak{D}.$$

Replacing y by $yt = (y_1 t_1, y_2 t_2)$ and (ζ_1, ζ_2) by $(\zeta_1^{t_1}, \zeta_2^{t_2})$ for any $t = (t_1, t_2) \in (1 + q^M \mathbb{Z}_q)^2$. Hence, summing over $t \in (1 + q\mathbb{Z}_q)^2 / (1 + q^m \mathbb{Z}_q) \times (1 + q^n \mathbb{Z}_q)$, we deduce that

$$\hat{\beta}_y(\zeta_y^{-1}) = \int_{y(1+q\mathbb{Z}_q)^2} \zeta_1^{x_1/y_1} \zeta_2^{x_2/y_2} d\beta(x) \in \pi\mathfrak{D}.$$

The converse follows from Lemma 3.2 and the fact that $\tau(\kappa)$ is a π -adic unit (which is a consequence of the fact that its conductor is coprime to p). \square

Proof of Theorem 5.1. Without loss of generality, we assume that $n = \text{ord}_\pi(R) = 0$. Let β be as defined in the statement of Lemma 5.3, and w_K denote the number of elements in μ_K (which is coprime to $p > 3$). We have

$$\frac{1}{w_K^2} \Gamma_\alpha(\kappa) = \Gamma_\beta(\kappa).$$

Let us write

$$\alpha_{\eta y} = \alpha|_{\eta_1 y_1(1+q^M \mathbb{Z}_q) \times \eta_2 y_2(1+q^M \mathbb{Z}_q)}$$

for $\eta = (\eta_1, \eta_2) \in \mu_{q-1}^2$ and $y = (y_1, y_2) \in (1 + q\mathbb{Z}_q)^2$. Note that $\alpha_{\eta y}$ is an elliptic function measure since it is a restriction of α . Furthermore, we write $R_{\eta y}$ for the rational function on E attached to $\alpha_{\eta y}$ (meaning that $\hat{\alpha}_{\eta y} = R_{\eta y} \circ \delta$ as functions on $\mu_{q^\infty}^2$). As can be seen in the proof of Lemma 3.7, $R_{\eta y}$ takes values in \mathfrak{D} . Let $\tilde{R}_{\eta y}$ denote the function $R_{\eta y}$ modulo π .

Suppose that the set of characters κ with $\text{ord}_\pi(\Gamma_\alpha(\kappa)) = 0$ is not Zariski dense. Note that for all κ , we have $\text{ord}_\pi(\Gamma_\alpha(\kappa)) = \text{ord}_\pi(\Gamma_\beta(\kappa))$ by Lemma 3.5 and the fact that $p \nmid w_K$. Equivalently, the set of characters κ such that $\Gamma_\beta(\kappa) \notin \pi\mathfrak{D}$ is not Zariski dense. By Lemma 5.3, the set of elements $Q \in E_{q^\infty}$ such that

$$\sum_{\eta \in (\mu_{q-1}/\mu_K)^2} \tilde{R}_{\eta y}([\eta^{-1}] \circ Q) \neq 0$$

is not Zariski dense.

Applying Theorem 4.6, it follows that $\tilde{R}_{\eta y}$ is a constant function. Let $c_{\eta y}$ denote a constant of \mathfrak{D} lifting $\tilde{R}_{\eta y}$ and let δ_0 denote the Dirac measure of \mathbb{Z}_q^2 concentrated at $(0, 0)$. By definition, the Fourier transform $\hat{\delta}_0$ sends all $\zeta \in \mu_{q^\infty}$ to 1. Therefore, the Fourier transform of $\alpha_{\eta y} - c_{\eta y}\delta_0$ takes values in $\pi\mathfrak{D}$. In particular,

$$\text{ord}_\pi(\alpha_{\eta y} - c_{\eta y}\delta_0) > 0.$$

However, if we restrict the measure $\alpha_{\eta y} - c_{\eta y}\delta_0$ to $(\mathbb{Z}_q^\times)^2$, it agrees with $\alpha_{\eta y}$. Thus,

$$\text{ord}_\pi(\alpha_{\eta y}) = \text{ord}_\pi(\alpha_{\eta y} - c_{\eta y}\delta_0) > 0.$$

This contradicts our hypothesis that $\text{ord}_\pi(R) = 0$. \square

6. PROOF OF THEOREM A

In this section we will apply Theorem 5.1 to obtain information on special values of L -functions and prove Theorem A in the introduction. We will keep the notation introduced in Section 2. Let K be a fixed imaginary quadratic field with class number 1, and E/K be an elliptic curve with complex multiplication by \mathcal{O}_K with good reduction at q . Let ω_E denote the Neron differential for E/K and $\mathcal{L} = \Omega_\infty \mathcal{O}_K$ be its period lattice. Here Ω_∞ is determined up to a root of unity in K . Let $\psi = \psi_{E/K}$ be the Hecke character of E/K of conductor \mathfrak{f} . It is known that \mathfrak{f} is divisible precisely by the primes of bad reduction of E/K (see [ST68, Theorem 12]). Recall that we may identify ψ with the natural \mathbb{Z}_q^\times -valued characters of $\text{Gal}(\overline{K}/K)$ arising via the Galois action on E_{q^∞} , i.e.,

$$\psi : \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(E_{q^\infty}) \xrightarrow{\sim} \mathcal{O}_{K,q}^\times \simeq \mathbb{Z}_q^\times.$$

The complex conjugate of this Hecke character, denoted by $\overline{\psi}$, can then be identified with the analogous map below, as explained in [Rub92, p. 325],

$$\psi^* : \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(E_{q^*\infty}) \xrightarrow{\sim} \mathcal{O}_{K,q^*}^\times \simeq \mathbb{Z}_q^\times.$$

6.1. Review on L -functions.

Definition 6.1. Let \mathfrak{h} be any integral ideal of K . Let ϵ be any Hecke character of K with conductor dividing some power of \mathfrak{h} . The *imprimitive L -function* of ϵ modulo \mathfrak{h} is defined as follows

$$L_{\mathfrak{h}}(\epsilon, s) = \sum_{\gcd(\mathfrak{a}, \mathfrak{h})=1} \frac{\epsilon(\mathfrak{a})}{(N\mathfrak{a})^s}.$$

Fix an integral ideal \mathfrak{g} of K which is divisible by \mathfrak{f} and is relatively prime to q . Let $F = \mathcal{R}(\mathfrak{g}q)$ be the *ray class field* of K of conductor $\mathfrak{g}q$ and write $\Delta = \text{Gal}(F/K)$. Since $\mathfrak{f} \mid \mathfrak{g}$, we have

$$F = \mathcal{R}(\mathfrak{g}q) = K(E_{\mathfrak{g}q}).$$

Set $F_{\infty} = \bigcup_{n \geq 1} \mathcal{R}(\mathfrak{g}q^n)$; this is a \mathbb{Z}_q^2 -extension of F . We fix an isomorphism

$$\text{Gal}(F_{\infty}/K) \simeq \text{Gal}(F/K) \times \text{Gal}(K_{\infty}/K) = \Delta \times \mathbb{Z}_q^2.$$

Let ϵ be a character of $\text{Gal}(F_{\infty}/K)$. For our purpose, ϵ will be of the form $\varphi\psi^k$, where φ is a finite-order character and k is an integer between 1 and $p-1$. Denote by $L(\epsilon, s)$ the *primitive Hecke L -function* of ϵ . Recall that the imprimitive (or partial) L -function differs from the primitive (or classical) L -function by a finite number of Euler factors. We can further define the *primitive algebraic Hecke L -function*,

$$L^{(\text{alg})}(\varphi\psi^k) = L^{(\text{alg})}(\epsilon) = L^{(\text{alg})}(\epsilon; k) := \frac{L(\epsilon, k)}{\Omega_{\infty}^k}.$$

Recall that p is a rational prime satisfying the following hypotheses

- (i) $p = \mathfrak{p}\mathfrak{p}^*$ in K with $\mathfrak{p} \neq \mathfrak{p}^*$ and $\gcd(p, 6q) = 1$.
- (ii) E/K has good (ordinary) reduction at primes above p .

As before, set π to be the uniformizer of the local field k , which is a finite unramified extension of \mathbb{Q}_p containing $\mathbb{Q}_p(E_{\mathfrak{f}q})$.

Set $F_{m,n} = \mathcal{R}(\mathfrak{g}q^{m+1}\mathfrak{q}^{*n+1})$ be the m, n -th layer of the \mathbb{Z}_q^2 extension F_{∞}/F . This is the ray class field of conductor $\mathfrak{g}q^{m+1}\mathfrak{q}^{*n+1}$. Let φ be a character of $\text{Gal}(F_{m,n}/K)$. Set $\mathfrak{h} = \mathfrak{g}q^{m+1}\mathfrak{q}^{*n+1}$ in Definition 6.1. The imprimitive L -function of $\varphi\psi^k$ modulo \mathfrak{h} is

$$L_{\mathfrak{h}}(\overline{\varphi\psi^k}, s) = \sum_{\tau \in \text{Gal}(F_{m,n}/K)} \overline{\varphi}(\tau) \sum_{\tau_{\mathfrak{b}}=\tau} \frac{\overline{\psi^k}(\mathfrak{b})}{(N\mathfrak{b})^s}$$

where $\gcd(\mathfrak{b}, \mathfrak{h}) = 1$ and $\tau_{\mathfrak{b}} = (\mathfrak{b}, F_{m,n}/K)$ is the Artin symbol. For ease of notation, we write

$$\sum_{\tau_{\mathfrak{b}}=\tau} \frac{\overline{\psi^k}(\mathfrak{b})}{(N\mathfrak{b})^s} = L_{\mathfrak{h}}(\overline{\psi^k}, s, \tau).$$

The (primitive and imprimitive) L -functions we have discussed so far only converge on some right half-plane. However, they have analytic continuations to the entire complex plane. This will allow us to relate $L^{(\text{alg})}(\overline{\varphi\psi^k})$ to Gamma transforms of certain elliptic function measures.

6.2. A rational function with a canonical divisor. We review the construction of a rational function on E from [Lam15, §6.3]. This allows us to carry out step (3) outlined in the introduction.

Let \mathfrak{h} be *any* integral ideal divisible by \mathfrak{f} and $\mathfrak{a} \in \text{End}(E)$ be coprime to $6\mathfrak{h}$. Define the rational function $\zeta_{\mathfrak{a}}$ on E by

$$\zeta_{\mathfrak{a}}(P) = \prod_Q (x(P) - x(Q))^{-1}$$

where Q runs over a set of representatives of $E_{\mathfrak{a}} \setminus \{0\} \pmod{\pm 1}$. There exists a constant $c(\mathfrak{a}) \in K^{\times}$ such that the function

$$\gamma_{\mathfrak{a}}(P) := c(\mathfrak{a})\zeta_{\mathfrak{a}}(P)$$

has the property that for all $\beta \in \text{End}(E)$ with $\gcd(\beta, \mathfrak{a}) = 1$,

$$(1) \quad \gamma_{\mathfrak{a}}(\beta(P)) = \prod_{R \in \ker(\beta)} \gamma_{\mathfrak{a}}(P \oplus R).$$

For any integer $k \geq 1$, define the k -th derivative operator on $\mathbb{C}(E)$ by

$$D^k(f) = \frac{d^k}{dz^k} \log f(z).$$

Consider the complex analytic isomorphism of complex Lie groups

$$\xi(z, \mathcal{L}) : \mathbb{C}/\mathcal{L} \xrightarrow{\sim} E(\mathbb{C}) \text{ given by } \xi(z, \mathcal{L}) = (\wp(z, \mathcal{L}), \wp'(z, \mathcal{L}))$$

where \wp is the Weierstrass \wp -function and \wp' is the corresponding derivative.

Lemma 6.2. *Let \mathfrak{h} denote an integral ideal of K divisible by \mathfrak{f} . Suppose that P_0 is a primitive \mathfrak{h} -division point on E . Then there exists an element $\sigma \in \text{Gal}(\mathcal{R}(\mathfrak{h})/K)$ and $\zeta \in \mu_K$ such that*

$$P_0 = \zeta (\xi(\rho)^{\sigma})$$

for $\rho \in \mathfrak{h}^{-1}\mathcal{L} \setminus \mathcal{L}$. With this notation,

$$D^k(\gamma_{\mathfrak{a}})(P_0) = (-1)^{k-1}(k-1)! \left(N(\mathfrak{a}) - \psi(\mathfrak{a})^k \tau_{\mathfrak{a}} \right) \frac{L_{\mathfrak{h}}(\overline{\psi^k}, k, \sigma)}{(\zeta \rho)^k}$$

where $\tau_{\mathfrak{a}} = (\mathfrak{a}, \mathcal{R}(\mathfrak{h})/K)$.

Proof. See [Lam14, Lemma 3.1.4] or [Lam15, Lemma 6.4]. \square

From here on, we take \mathfrak{h} to be $\mathfrak{g}\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$, where \mathfrak{g} is an ideal of K divisible by \mathfrak{f} and coprime to q . Let $F_{m,n} = \mathcal{R}(\mathfrak{h})$. Let θ be a character of $\text{Gal}(\mathcal{R}(\mathfrak{g})/K)$ and V be a primitive \mathfrak{g} -division point. For each $k \geq 1$, define the rational function on E

$$\vartheta_{\mathfrak{a},V}^{\theta,k}(P) := \sum_{\tau \in \text{Gal}(\mathcal{R}(\mathfrak{g})/K)} \theta^{-1}(\tau) (D^k \gamma_{\mathfrak{a}})(P \oplus V^{\tau}).$$

Let χ be a primitive character of $\text{Gal}(F_{m,n}/\mathcal{R}(\mathfrak{g}))$ and set $\varphi = \theta\chi$ to be a character of $\text{Gal}(F_{m,n}/K)$.

Theorem 6.3. *Let V be a primitive \mathfrak{g} -division point and $Q_{m,n}$ be a primitive $\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$ -division point on E . There exists $\sigma \in \text{Gal}(F_{m,n}/K)$ such that*

$$\sum_{\tau \in \text{Gal}(F_{m,n}/\mathcal{R}(\mathfrak{g}))} \chi^{-1}(\tau) \vartheta_{\mathfrak{a},V}^{\theta,k}(Q_{m,n}^{\tau}) = (-1)^{k-1}(k-1)! \left(N(\mathfrak{a}) - \psi(\mathfrak{a})^k \varphi(\tau_{\mathfrak{a}}) \right) \varphi(\sigma) \frac{L_{\mathfrak{h}}(\overline{\varphi\psi^k}, k)}{(\zeta \rho_{m,n})^k}.$$

Proof. Using the definition of $\vartheta_{\mathfrak{a},V}^{\theta,k}$ and the fact that $\varphi = \theta\chi$, we have that

$$\sum_{\tau \in \text{Gal}(F_{m,n}/\mathcal{R}(\mathfrak{g}))} \chi^{-1}(\tau) \vartheta_{\mathfrak{a},V}^{\theta,k}(Q_{m,n}^\tau) = \sum_{\tau \in \text{Gal}(F_{m,n}/K)} \varphi^{-1}(\tau) (D^k \gamma_{\mathfrak{a}}) ((V \oplus Q_{m,n})^\tau).$$

Let g be a generator of \mathfrak{g} , and set

$$(2) \quad \rho_{m,n} = \frac{\Omega_\infty}{g\psi(\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1})}.$$

This is a primitive $\mathfrak{g}\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$ -division point in \mathbb{C}/\mathcal{L} . It follows from Lemma 6.2, there exist $\sigma \in \text{Gal}(F_{m,n}/K)$ and $\zeta \in \mu_K$ such that

$$(V \oplus Q_{m,n})^\tau = \zeta \left(\xi(\rho_{m,n})^{\sigma\tau} \right).$$

Once again, applying Lemma 6.2, we see that

$$\sum_{\tau} \varphi^{-1}(\tau) (D^k \gamma_{\mathfrak{a}}) ((V \oplus Q_{m,n})^\tau) = (-1)^{k-1} (k-1)! \left(N(\mathfrak{a}) - \psi(\mathfrak{a})^k \tau_{\mathfrak{a}} \right) \sum_{\tau} \varphi^{-1}(\tau) \frac{L_{\mathfrak{h}}(\overline{\psi^k}, k, \sigma\tau)}{(\zeta \rho_{m,n})^k}.$$

The result follows from Definition 6.1. \square

6.3. Gamma transforms and L -values. In the previous section, we constructed the rational function $\vartheta_{\mathfrak{a},V}^{\theta,k} \in J(E)$. To prove Theorem A, let k be an integer between 1 and $p-1$. Let $F = \mathcal{R}(\mathfrak{g}\mathfrak{q})$ as before. Let φ be a finite-order character of $\text{Gal}(F_\infty/K)$. Note that we may write $\varphi = \theta\phi\kappa$ where θ is a character of $\text{Gal}(\mathcal{R}(\mathfrak{g})/K)$, ϕ is a character of $\text{Gal}(F/\mathcal{R}(\mathfrak{g}))$, and κ is a finite-order character of $\text{Gal}(F_\infty/F) \simeq \mathbb{Z}_q^2$. We draw a field diagram to clarify the setting.

$$\begin{array}{ccc} \mathcal{R}(\mathfrak{q}\mathfrak{q}^\infty) = F_\infty & & \\ \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right. & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & \kappa \\ \mathbb{Z}_q^2 & & \\ \left. \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) & & \\ \mathcal{R}(\mathfrak{g}\mathfrak{q}) = F & & \\ \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right. & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & \phi \\ \Delta & \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} & \mathcal{R}(\mathfrak{g}) \\ \left. \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) & & \theta \\ K & & \\ | & & \\ \mathbb{Q} & & \end{array}$$

We fix θ , ϕ , and a non-trivial ideal $\mathfrak{a} \in \text{End}(E)$ coprime to $6q\mathfrak{p}\mathfrak{g}$. Then we can associate with $\vartheta_{\mathfrak{a},V}^{\theta,k}$ an elliptic function measure, α on \mathbb{Z}_q^2 via Lemma 3.7. This α depends on the choice of \mathfrak{a} , V , θ , and k . We further define $\alpha^* = \alpha|_{(\mathbb{Z}_q^\times)^2}$.

We now relate the Gamma transform of α^* to special values of imprimitive algebraic L -functions.

Lemma 6.4. *For all finite-order characters κ of $\text{Gal}(F_\infty/F)$,*

$$\text{ord}_\pi(\Gamma_{\alpha^*}(\phi\kappa)) = \text{ord}_\pi \left(\left(N(\mathfrak{a}) - \psi(\mathfrak{a})^k \varphi(\tau_{\mathfrak{a}}) \right) L_{\mathfrak{h}}^{(\text{alg})}(\overline{\varphi\psi^k}) \right).$$

Proof. Suppose that κ is a primitive character of $\text{Gal}(F_{m,n}/F)$. Choose $\underline{\zeta} = (\zeta_m, \zeta_n) \in \mu_{q^\infty}^2$ such that ζ_1 (resp. ζ_2) is a q^{m+1} -th (resp. q^{n+1} -th) roots of unity. Set $Q_{m,n} = \delta(\underline{\zeta})$. Write $\chi = \phi\kappa$; this is a finite-order character on

$$\text{Gal}(F_\infty/\mathcal{R}(\mathfrak{g})) \simeq \text{Gal}(K_\infty/K) \simeq \mathbb{Z}_q^2 \times \text{Gal}(K(E_q)/K).$$

Note that $\text{Gal}(K(E_q)/K)$ may be identified with two cyclic groups of order $q-1$. Using Lemma 3.2 in conjunction with Theorem 6.3, yields

$$\begin{aligned} \Gamma_{\alpha^*}(\phi\kappa) &= \tau(\chi) \sum_{g \in \text{Gal}(F_{m,n}/\mathcal{R}(\mathfrak{g}))} \chi^{-1}(g) \vartheta_{\mathfrak{a},V}^{\theta,k}(Q_{m,n}^g) \\ &= (-1)^{k-1} (k-1)! \tau(\chi) \left(N(\mathfrak{a}) - \psi(\mathfrak{a})^k \varphi(\tau_{\mathfrak{a}}) \right) \varphi(\sigma) \frac{L_{\mathfrak{h}}(\overline{\varphi\psi^k}, k)}{(\zeta\rho_{m,n})^k}. \end{aligned}$$

Standard facts about Gauss sums tell us that $\text{ord}_\pi(\tau(\chi)) = 0$ since the conductor of χ is coprime to p . Furthermore, $1 \leq k \leq p-1$, so $(k-1)!$ is a p -adic unit. Finally, as φ is a finite character, $\varphi(\sigma)$ is a root of unity. This completes the proof of the lemma. \square

We now study the factor $N(\mathfrak{a}) - \psi(\mathfrak{a})^k \varphi(\tau_{\mathfrak{a}})$.

Lemma 6.5. *For a Zariski dense set of κ , we have*

$$\text{ord}_\pi \left(N(\mathfrak{a}) - \psi^k(\mathfrak{a}) \varphi(\tau_{\mathfrak{a}}) \right) = 0.$$

Proof. Suppose the contrary. Let $(\zeta_1, \zeta_2) = (\kappa(\tau_{\mathfrak{a},q}), \kappa(\tau_{\mathfrak{a},q^*})) \in \mu_{q^\infty}^2$, where $\tau_{\mathfrak{a},\mathfrak{l}}$ denotes the restriction of $\tau_{\mathfrak{a}}$ to $\text{Gal}(\mathcal{R}(\mathfrak{gl}^\infty)/\mathcal{R}(\mathfrak{g}))$. Then,

$$\text{ord}_\pi \left(N(\mathfrak{a}) - \psi^k(\mathfrak{a}) \varphi(\tau_{\mathfrak{a}}) \right) = 0$$

if and only if

$$N(\mathfrak{a}) \psi(\mathfrak{a})^{-k} (\theta\phi(\tau_{\mathfrak{a}}))^{-1} \not\equiv \zeta_1 \zeta_2 \pmod{\pi\mathfrak{D}}.$$

Note that the left-hand side is independent of κ . In particular, this condition is invariant under the map $(\zeta_1, \zeta_2) \mapsto (\zeta_1, \zeta_2)^{p^r}$, where p^r is the cardinality of the residue field of \mathfrak{D} .

Our assumption that the set of κ satisfying the stated property above is not Zariski dense allows us to apply Lemma 4.4. Let P be the power of p^r given by the said lemma. In particular, under the isomorphism $\mu_{q^\infty}^2 \cong (\mathbb{Q}_q/\mathbb{Z}_q)^2$, there exists an arbitrary large n such that

$$(3) \quad N(\mathfrak{a}) \psi(\mathfrak{a})^{-k} (\theta\phi(\tau_{\mathfrak{a}}))^{-1} \equiv \zeta_1 \zeta_2 \pmod{\pi\mathfrak{D}}$$

for all (ζ_1, ζ_2) which can be identified with $\left(\frac{P^x}{q^n}, \frac{P^y}{q^n} \right)$, where $x, y \in \mathbb{Z}$. In particular, Remark 4.5 tells us that there are $q^{2(n-v)}$ such elements, where $v = \text{ord}_q(P-1)$.

Note that q -power roots of unity modulo $\pi\mathfrak{D}$ are distinct since $p \neq q$. Suppose that the left-hand side of (3) modulo π is a q^m -th root of unity, where $m < n$. Then, for each q^n -th root of unity ζ_1 , there are exactly q^m choices of q^n -th roots of unity ζ_2 such that (3) holds. This gives us at most q^{n+m} choices of $(\zeta_1, \zeta_2) \in \mu_{q^n}^2$. But this is a contradiction as soon as $n+m < 2(n-v)$. \square

Remark 6.6. For a given ideal \mathfrak{a} , denote the Zariski dense set of characters described in Lemma 6.5 by $Z_{\mathfrak{a}}$. This set is defined by the equation

$$f(\mathfrak{a}) := N(\mathfrak{a}) - \psi^k(\mathfrak{a}) \varphi(\tau_{\mathfrak{a}}) \not\equiv 0 \pmod{\pi\mathfrak{D}}.$$

But note that $Z_{\mathfrak{a}} = \bigcup_{m \in \pi\mathfrak{D}} Z_{m,\mathfrak{a}}$ where each $Z_{m,\mathfrak{a}}$ is defined by equation

$$f_m(\mathfrak{a}) := f(\mathfrak{a}) - m \neq 0,$$

as m varies over elements of $\pi\mathfrak{D}$. Since each $Z_{m,\mathfrak{a}}$ is Zariski open, it follows that $Z_{\mathfrak{a}}$ is Zariski open.

We require the following lemma which was used in the proving the main result on vanishing of special values of L -functions in split prime \mathbb{Z}_q -extensions of F (see [Lam15, Theorem 6.9]).

Lemma 6.7. *Suppose that $\mathfrak{a} \subset \mathfrak{D}$ is a non-trivial ideal such that $\gcd(\mathfrak{a}, 6\mathfrak{p}q) = 1$. Then,*

$$\mathrm{ord}_{\pi} \left(\vartheta_{\mathfrak{a},V}^{\theta,k} \right) = 0.$$

Proof. See [Lam15, Lemma 6.7]. □

We conclude this article with the following slightly more general version of Theorem A.

Theorem 6.8. *For a Zariski dense set of characters κ of $\mathrm{Gal}(F_{\infty}/F)$,*

$$\mathrm{ord}_{\pi} \left(L^{(\mathrm{alg})} \left(\overline{\theta\phi\kappa\psi^k} \right) \right) = 0.$$

Proof. An argument similar to Lemma 6.5 suggests that it suffices to prove the theorem for imprimitive values $L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\phi\theta\kappa\psi^k})$ because for almost all finite-order characters κ of $\mathrm{Gal}(F_{\infty}/F)$

$$\mathrm{ord}_{\pi} \left(L^{(\mathrm{alg})} \left(\overline{\theta\phi\kappa\psi^k} \right) \right) = \mathrm{ord}_{\pi} \left(L_{\mathfrak{h}}^{(\mathrm{alg})} \left(\overline{\phi\theta\kappa\psi^k} \right) \right).$$

Indeed, for any prime ideal \mathfrak{r} of K and for almost all characters κ ,

$$\mathrm{ord}_{\pi} \left(1 - \frac{\overline{\psi^k\theta\phi\kappa(\mathfrak{r})}}{N(\mathfrak{r})^k} \right) = 0$$

as the q -power roots of unity modulo $\pi\mathfrak{D}$ are distinct since $p \neq q$.

Lemma 6.7 asserts that $\mathrm{ord}_{\pi} \vartheta_{\mathfrak{a},V}^{\theta,k} = 0$. In particular, the associated elliptic function measure α^* satisfies $\mathrm{ord}_{\pi} \alpha^* = 0$. Therefore, on combining Lemma 6.4 with Theorem 5.1, we deduce that for a Zariski dense set of κ , we have

$$\mathrm{ord}_{\pi} \left(\left(N(\mathfrak{a}) - \psi(\mathfrak{a})^k \varphi(\tau_{\mathfrak{a}}) \right) L_{\mathfrak{h}}^{(\mathrm{alg})} \left(\overline{\theta\phi\kappa\psi^k} \right) \right) = 0.$$

The same argument as in Remark 6.6 shows that this Zariski dense set is also open. Since the intersection of two open dense sets is open dense, there exists a dense set of characters κ with

$$\mathrm{ord}_{\pi} \left(L^{(\mathrm{alg})} \left(\overline{\theta\phi\kappa\psi^k} \right) \right) = 0.$$

□

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