IWASAWA INVARIANTS OF ABELIAN VARIETIES IN EXTENSIONS OF NUMBER FIELDS

LEA BENEISH, TIMO KELLER, AND DEBANJANA KUNDU

Abstract. We establish a Kida-type formula for the behaviour of the Iwasawa invariants of abelian varieties in finite Galois p-extensions of number fields on the algebraic, and for the invariants of modular abelian varieties on the analytic side. This generalizes results of Hachimori–Matsuno and Matsuno for elliptic curves, respectively.

1. Introduction

The classical Riemann-Hurwitz formula describes the relationship of the Euler characteristics of two Riemann surfaces when one is a ramified covering of the other. Suppose $\pi : R_1 \to R_2$ is an n-fold covering of compact, connected Riemann surfaces and g_1, g_2 are their respective genera. The classical Riemann–Hurwitz formula is the statement

$$2g_1 - 2 = (2g_2 - 2)n + \sum \left(e(P_2) - 1\right),$$

where the sum is over all points P_2 on R_2 and $e(P_2)$ denotes the ramification index of P_2 for the covering π. There is a generalization of this Riemann–Hurwitz formula to finite separable morphisms of smooth projective geometrically connected curves over an arbitrary field provided that the ramification is tame, see [Har83, IV §2]. An analogue of the above formula for CM fields was proven by Y. Kida in [Kid80] and is referred to as Kida's formula. This formula describes the change of the classical Iwasawa λ-invariants in a p-extension in terms of the degree and the ramification index. In [Iwa81], K. Iwasawa proved this formula using the theory of Galois cohomology for extensions of $\mathbb{Q}$ which are not necessarily finite. More precisely, he proved the following theorem

Theorem 1.1. [Iwa81, Theorem 6] Let $p \geqslant 2$ and F be a number field. Let F_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F and $\mathcal{L}/F_{\text{cyc}}$ be a cyclic extension of degree p, unramified at every infinite place of F_{cyc}. Assume that the classical μ-invariant, denoted by $\mu(F_{\text{cyc}})$ is 0. Then

$$\lambda(\mathcal{L}) = p\lambda\left(F_{\text{cyc}}\right) + \sum_{w \nmid p} \Big(e\left(w \mid v\right) - 1\Big) + (p-1)\left(h_2 - h_1\right)$$

where w ranges over all non-p places of $\mathcal{L}$, h_i is the rank of the abelian group $\mathrm{H}^i(\mathcal{L}/F_{\text{cyc}}, E_{\mathcal{L}})$, and $E_{\mathcal{L}}$ is the group of all units of $\mathcal{L}$.

The Selmer group plays a crucial role in the study of rational points of abelian varieties (in particular, elliptic curves). In [Maz72], B. Mazur initiated the study of the growth of the p-primary part of the Selmer group in the cyclotomic $\mathbb{Z}_p$-extension of number fields. In [HM99], Y. Hachimori–K. Matsuno proved an analogue of Kida's formula for Selmer groups of *elliptic curves* in p-extensions

Date: September 6, 2022.

2020 *Mathematics Subject Classification.* 11R23 (primary).

Key words and phrases. Iwasawa invariant, abelian varieties, abelian varieties which are not modular.

TK was supported by the Deutsche Forschungsgemeinschaft (DFG), Projektnummer STO 299/18-1, AOBJ: 667349 while working on this article. DK acknowledges the support of a PIMS postdoctoral fellowship.

of the cyclotomic $\mathbb{Z}_p$-extension of a number field under the assumption that the associated μ-invariant is 0. In [PW06], R. Pollack–T. Weston proved a similar statement for Selmer groups of a general class of Galois representations including the case of p-ordinary Hilbert modular forms and p-supersingular modular forms. Recently, M. F. Lim has made progress towards proving a Kida-type formula for Galois representations without any hypothesis on the μ-invariant (see [Lim21]).

The scope of this article is to prove a Kida-type formula for abelian varieties which are not necessarily modular (see Theorem 3.1). Even though the strategy of our proof is the same as that of [HM99], the calculation of the local Herbrand quotients is significantly different from the elliptic curve case and uses the theory of Néron models. The complications arise because one does not have a product decomposition of (the special fiber of) the Néron model into its good reduction, multiplicative and unipotent part, but only a filtration. Hence, our proof differs significantly from [HM99] and is much more involved. As an application of this algebraic result, we provide an upper bound on the growth of the Mordell–Weil rank of an abelian variety at each finite layer of a uniform p-adic Lie extensions in Corollary 4.3.

For the sake of completeness, we also address the analogue of the Kida-type formula for p-adic L-functions of *modular* abelian varieties. There are several constructions of p-adic L-functions for newforms with non-rational coefficients, see for example [AV75]. As explained in *op. cit.*, it is reasonable to expect that $L(\mathsf{A}, T)$ can be written as a product of local p-adic L-functions $L_p(f^\sigma, T)$ associated to the Galois conjugates f^σ of f; the key difficulty is to find a canonical normalization of $L_p(f^\sigma, T)$. In [BMS16], the authors show that there is in fact a natural way to pick a set of Shimura periods, which allows the construction of the p-adic L-function attached to $\mathsf{A}/\mathbb{Q}$ in the case where p is a prime of good ordinary reduction.

In Section 5 we define the (analytic) Iwasawa invariants associated with the p-adic L-function for a modular abelian variety defined over $\mathbb{Q}$. In Theorem 5.6, we prove a Kida-type formula for the associated analytic λ-invariants. The strategy is inspired by the work of Matsuno in [Mat00] but involves more sophisticated calculations since we are no longer in the dimension 1 case. The algebraic and analytic Kida-type formulae coincide for modular abelian varieties; this may be seen as evidence towards the Iwasawa Main Conjecture in this general setting.

Outlook: From a computation perspective, it would be interesting to provide explicit examples. In particular, it might be interesting to find a modular form f corresponding to an abelian variety A satisfying the hypotheses of [GV00, Corollary 3.14]: i.e., f is a modular form, it is p-distinguished, and p-ordinary satisfying $\mu_f^{\mathrm{an}} = \mu_f^{\mathrm{alg}} = 0$. Applying our theorems in the algebraic and the analytic case will show that the respective λ invariants are equal in every p-power extension. This will ensure that for every modular form $g \equiv f \pmod{p}$, the Iwasawa invariants $\mu_g^{\mathrm{an}} = \mu_g^{\mathrm{alg}} = 0$ (and similarly, for the λ invariants).

Organization: Including the introduction, the article has five sections. Section 2 is preliminary in nature. We discuss the basics of Iwasawa theory of abelian varieties and facts about Néron models of abelian varieties. In Section 3, we prove a Kida-type formula for the algebraic λ-invariant of abelian varieties. Using this result, in Section 4 we prove results towards the asymptotic growth of Mordell–Weil ranks of abelian varieties in non-commutative p-adic Lie extensions. Finally, in Section 5, we prove an an analytic Kida-type formula for modular abelian varieties.

2. Preliminaries

Let F be any number field and A/F be an abelian variety defined over F. Fix an algebraic closure $\overline{F}$ of F and write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. For a given integer m, set $\mathsf{A}[m]$

to be the Galois module of all m-torsion points in $\mathsf{A}(\overline{F})$. If v is a prime in F, we write F_v for the completion of F at v. The main object of interest is the Selmer group.

Definition 2.1. For any integer $m \geqslant 2$, the *m-Selmer group* is defined as follows

$$\mathrm{Sel}_m(\mathsf{A}/F) = \ker\left(\mathrm{H}^1\left(G_F, \mathsf{A}[m]\right) \longrightarrow \prod_v \mathrm{H}^1\left(G_{F_v}, \mathsf{A}\right)[m]\right).$$

This m-Selmer group fits into the following short exact sequence

$$0 \longrightarrow \mathsf{A}(F)/m \longrightarrow \mathrm{Sel}_m(\mathsf{A}/F) \longrightarrow Ш(\mathsf{A}/F)[m] \longrightarrow 0. \tag{2.1}$$

Here, $Ш(\mathsf{A}/F)$ is the *Shafarevich–Tate group* which is conjecturally finite. Throughout this article, we will assume the finiteness of the Shafarevich–Tate group, wherever required.

2.1. **Recollections from Iwasawa theory.** For details, we refer the reader to standard texts in Iwasawa theory (e.g. [Was97, Chapter 13]). Let p be a fixed prime. Consider the *cyclotomic* $\mathbb{Z}_p$-extension of $\mathbb{Q}$, denoted by $\mathbb{Q}_{\mathrm{cyc}}$. Set $\Gamma := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \simeq \mathbb{Z}_p$. The *Iwasawa algebra* $\Lambda = \Lambda(\Gamma)$ is the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. Fix a topological generator γ of Γ; there is the following isomorphism of rings

$$\begin{aligned} \Lambda &\xrightarrow{\sim} \mathbb{Z}_p[\![T]\!] \\ \gamma &\mapsto 1+T. \end{aligned}$$

Let M be a cofinitely generated cotorsion Λ-module. The *Structure Theorem of* Λ-*modules* asserts that the Pontryagin dual of M, denoted by $M^\vee$, is pseudo-isomorphic to a finite direct sum of cyclic Λ-modules. In other words, there is a map of Λ-modules

$$M^\vee \longrightarrow \left(\bigoplus_{i=1}^{s} \Lambda/(p^{m_i})\right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(h_j(T))\right)$$

with finite kernel and cokernel. Here, $m_i > 0$ and $h_j(T)$ is a distinguished polynomial (i.e., a monic polynomial with non-leading coefficients divisible by p). The *characteristic ideal* of $M^\vee$ is (up to a unit) generated by the *characteristic element*,

$$f_M^{(p)}(T) := p^{\sum_i m_i} \prod_j h_j(T).$$

The μ-invariant of M is defined as the power of p in $f_M^{(p)}(T)$. More precisely,

$$\mu(M) = \mu_p(M) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=1}^{s} m_i & \text{if } s > 0. \end{cases}$$

The λ-invariant of M is the degree of the characteristic element, i.e.

$$\lambda(M) = \lambda_p(M) := \sum_{j=1}^{t} \deg h_j(T).$$

These are the algebraic Iwasawa invariants but we suppress the superscript 'alg', if it is clear from the context.

Let A/F be an abelian variety with good *ordinary* reduction at all primes above p. We shall assume throughout this article that the prime p is odd. Let S be a finite set of primes in F containing the primes above p, the primes of bad reduction of A and the archimedean primes. Let F_S be the maximal algebraic extension of F which is unramified at the primes outside S. Set $\mathsf{A}[p^\infty]$

to be the Galois module of all p-power torsion points in $\mathsf{A}(\overline{F})$. For a prime $v \in S$ and any finite extension L/F contained in the unique cyclotomic $\mathbb{Z}_p$-extension of F (denoted by F_{cyc}), write

$$J_v(\mathsf{A}/L) = \bigoplus_{w|v} \mathrm{H}^1\left(G_{L_w}, \mathsf{A}\right)[p^\infty]$$

where the direct sum is over all primes w of L lying above v. Then, the *p-primary Selmer group over* F is defined as follows

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/F) := \ker\left\{\mathrm{H}^1\left(\mathrm{Gal}\left(F_S/F\right), \mathsf{A}[p^\infty]\right) \longrightarrow \bigoplus_{v\in S} J_v(\mathsf{A}/F)\right\}.$$

It is easy to see that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F) = \varinjlim_n \mathrm{Sel}_{p^n}(\mathsf{A}/F)$, see for example [CS00, §1.7]. By taking direct limits of (2.1), the p-primary Selmer group fits into a short exact sequence

$$0 \longrightarrow \mathsf{A}(F)\otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}_{p^\infty}(\mathsf{A}/F) \longrightarrow Ш(\mathsf{A}/F)[p^\infty] \longrightarrow 0. \tag{2.2}$$

Next, define

$$J_v(\mathsf{A}/F_{\mathrm{cyc}}) = \varinjlim_L J_v(\mathsf{A}/L)$$

where L ranges over finite extensions contained in F_{cyc} and the inductive limit is taken with respect to the restriction maps. The *p-primary Selmer group over* F_{cyc} is defined as follows

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) := \ker\left\{\mathrm{H}^1\left(\mathrm{Gal}\left(F_S/F_{\mathrm{cyc}}\right), \mathsf{A}[p^\infty]\right) \longrightarrow \bigoplus_{v\in S} J_v(\mathsf{A}/F_{\mathrm{cyc}})\right\}.$$

When $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is a cofinitely generated cotorsion Λ-module, in view of the Structure Theorem of Λ-modules, we can define the algebraic μ and λ-invariants, which we denote as $\mu(\mathsf{A}/F_{\mathrm{cyc}})$ and $\lambda(\mathsf{A}/F_{\mathrm{cyc}})$, respectively.

Remark 2.2. The Λ-cotorsionness hypothesis is a conjecture of B. Mazur [Maz72] and is known to be true for *modular* abelian varieties defined over $\mathbb{Q}$ by the work of K. Kato [Kat04, Theorem 14.4].

2.2. **Facts about Néron models of abelian varieties.**

We will briefly review some key facts from the theory of Néron models of abelian varieties following [Art86, BLR90]. The results in this section are stated in the context and notation which we use in the rest of the paper; however, note that they hold more generally.

To fix notation, let k be an ℓ-adic local field with ring of integers $\mathcal{O}_k$, and let k_n be the nth layer of the cyclotomic $\mathbb{Z}_p$ extension k_{cyc} of k (with $\ell \neq p$).

Definition 2.3. Let A be an abelian variety over k. The Néron model $\mathscr{A}$ of A is a smooth commutative group scheme over k such that for any smooth morphism $\mathcal{S} \to \mathcal{O}_k$, the natural map of abelian groups $\mathrm{Hom}_{\mathcal{O}_k}(\mathcal{S}, \mathscr{A}) \to \mathrm{Hom}_k(\mathcal{S} \times_{\mathcal{O}_k} k, \mathsf{A})$ is a bijection. This condition is known as the *Néron mapping property.*

It is a theorem of Néron [Nér64] (see also, [Art86, Theorem 1.2]) that the Néron model of A/k exists. Since $\mathcal{O}_{k_n}/\mathcal{O}_k$ is étale, in particular smooth, it follows from the Néron mapping property that $\mathsf{A}(k_n) = \mathscr{A}(\mathcal{O}_{k_n})$.

Let κ denote the residue field of k of characteristic ℓ ($\neq p$). We will also later make use of the following result concerning the kernel of the reduction from the $\mathcal{O}_k$-points of $\mathscr{A}$ (equal to the k-points of A) to the κ-points of $\mathscr{A}$.

Theorem 2.4. Let k be as above with maximal ideal $\mathfrak{m}$. Let A be a d-dimensional abelian variety with associated formal group $\hat{\mathsf{A}}$. The kernel of the reduction $\mathsf{A}(k) = \mathscr{A}(\mathcal{O}_k) \to \mathscr{A}(\kappa)$ is $\hat{\mathsf{A}}(\mathfrak{m}) \cong \mathfrak{m}^d$. Moreover, this latter group is a pro-ℓ group.

Proof. For the first assertion see [HS00, Theorem C.2.6]. The second statement is proven in [Ser06, Corollary 2, p. 118]. ☐

We find it useful to consider the Néron models of abelian varieties when we are interested in different types of reduction. For elliptic curves, we have the notion of good reduction, multiplicative reduction, and additive reduction which can be generalized to abelian varieties using the theory of Néron models. First, we recall the following theorem of C. Chevalley [Che60]:

Theorem 2.5 (Chevalley)**.** For a perfect field k, every smooth connected k-group G is an extension of an abelian variety A by a smooth connected affine k-group N:

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathsf{A} \longrightarrow 1.$$

Let A be an abelian variety over k with Néron model $\mathscr{A}$. Denote by $\mathscr{A}_0$ the special fiber of $\mathscr{A}$ and by $\mathscr{A}_0^\circ$ its identity component. Applying Chevalley's theorem to $\mathscr{A}_0^\circ$, we have

$$1 \longrightarrow H \longrightarrow \mathscr{A}_0^\circ \longrightarrow B \longrightarrow 1,$$

where B is an abelian variety and H is a smooth connected affine k-group. The group H contains a maximal torus T such that $U := H/T$ is a unipotent group variety (where $H = T \oplus U$ is split because H is commutative), $B := \mathscr{A}_0^\circ/H$ is an abelian variety, and $\pi_0^{\text{ét}}(\mathscr{A}/\kappa) = \mathscr{A}_0/\mathscr{A}_0^\circ$ is a finite étale group scheme.

Definition 2.6. With notation as above, we say

- A has *good reduction* if $\mathscr{A}_0^\circ \simeq B$ (or $\mathscr{A}_0 \simeq B$).
- A has *semi-stable reduction* if $H = T$, or in other words $\mathscr{A}_0^\circ$ has no unipotent part.
- A has *anisotropic reduction* (or *unipotent reduction* or *additive reduction*) if $\mathscr{A}_0^\circ \simeq U$ (and T is trivial).

3. Algebraic Kida-Type Formulae for Abelian Varieties

In this section, we prove a generalization of [HM99, Theorem 3.1] to abelian varieties. Throughout this discussion, we fix a number field F and assume that A/F is a d-dimensional abelian variety with good ordinary reduction at a prime $p > 2d+1$. We further suppose that the p-primary Selmer group $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion, see Remark 2.2. Since p is fixed throughout this discussion, we drop it from the subscript.

The main theorem in this section is the following which relates the λ-invariant of the Selmer group of A/F to that of the base-change curve A/L where L/F is a finite p-power Galois extension.

Theorem 3.1 (algebraic Kida-type formula for abelian varieties)**.** Let L/F be a finite Galois extension of degree a power of p. Let A/F be a d-dimensional abelian variety with good ordinary reduction at p. Suppose that $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion and that associated the μ-invariant, denoted by $\mu(\mathsf{A}/F_{\mathrm{cyc}})$, is zero. Then, $\mathrm{Sel}(\mathsf{A}/L_{\mathrm{cyc}})$ is a cotorsion $\Lambda(\Gamma)$-module with $\mu(\mathsf{A}/L_{\mathrm{cyc}}) = 0$. Furthermore, the respective λ-invariants, denoted by $\lambda(\mathsf{A}/F_{\mathrm{cyc}})$ and $\lambda(\mathsf{A}/L_{\mathrm{cyc}})$ satisfy the following formula:

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + \sum_{w \in P_1} c_s(w)\left(e(w) - 1\right) + 2\sum_{w \in P_2} c_g(w)\left(e(w) - 1\right).$$

Here, $e(w)$ is the ramification degree in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$, and the sets P_1, P_2 are the sets of primes in L_{cyc} defined as

$$P_1 = \{w \nmid p : \ \mathsf{A} \text{ has non-trivial split toric multiplicative reduction part at } w\},$$
$$P_2 = \{w \nmid p : \ \mathsf{A} \text{ has non-trivial good reduction part at } w \text{ and } \mathsf{A}(L_{\mathrm{cyc},w})[p] \neq 0\}.$$

The constant $c_s(w)$ is the dimension of the split toric part of the special fiber $\tilde{\mathscr{A}}$ of the Néron model $\mathscr{A}$ of A at w, and $c_g(w)$ is the dimension of the maximal abelian variety subquotient of $\tilde{\mathscr{A}}$.

We often call the maximal abelian variety subquotient of the Néron model or its fibers the "good reduction part", the (split) toric reduction subquotient, the "(split) toric part", etc.

Strategy of proof: We begin by observing that to prove this theorem, it suffices to consider the special case $G = \mathrm{Gal}(L/F) \simeq \mathbb{Z}/p\mathbb{Z}$. For details of this reduction step, see [HM99, Lemma 3.2].

Since we suppose that $\mu(\mathsf{A}/F_{\mathrm{cyc}}) = 0$, the same argument as [HM99, Corollary 3.4] implies that $\mathrm{Sel}(\mathsf{A}/L_{\mathrm{cyc}})$ is a cotorsion $\Lambda(\Gamma)$-module with $\mu(\mathsf{A}/L_{\mathrm{cyc}}) = 0$.

The rest of the argument for this proof proceeds exactly as in [HM99, Section 4], which we omit. In particular, we have the following analogue of (4.3) in *op. cit.*

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + \sum_{w \in T'_{\mathrm{cyc}}} b_w\,(p-1)\,.$$

Here, we use the notation introduced in *op. cit.* and write T'_{cyc} to denote the (finite) subset of primes in L_{cyc} which lie above the primes of bad reduction of A *and do not split* in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$ (equivalently, in L/F). The integer b_w is the maximum power of p dividing the Herbrand quotient of the p-primary torsion points of $\mathsf{A}(L_{\mathrm{cyc},w})$, i.e.,

$$b_w := \mathrm{ord}_p\left(h_G\left(\mathsf{A}(L_{\mathrm{cyc},w})[p^\infty]\right)\right).$$

The only place where our proof differs from the original proof, is in the calculation these integers b_w. In the remainder of this section, we will carry out the computation for b_w.

We compute these b_w in four steps.

Step 1: compute the p^∞-torsion of the connected component of the identity of the special fiber of the Néron model $\mathscr{A}$ of A over the residue field κ_{cyc}, see Section 3.1.
Step 2: compute the Herbrand quotients over the residue field, see Section 3.2.
Step 3: show that this is the same as the Herbrand quotient over $F_{\mathrm{cyc},v}$, see Section 3.3.
Step 4: compare the Herbrand quotient over $F_{\mathrm{cyc},v}$ to the Herbrand quotient over $L_{\mathrm{cyc},w}$ and compute b_w, see Section 3.4.

3.1. **p^∞-torsion of abelian varieties.** Let ℓ be a prime number distinct from p. Let k be an ℓ-adic local field, and let k_n be the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension k_{cyc} of k. This is the unique unramified $\mathbb{Z}_p$-extension of k. Indeed, any $\mathbb{Z}_p$-extension of k must be at most tamely ramified because the wild inertia group is an ℓ-group and $p \neq \ell$. By local class field theory, any $\mathbb{Z}_p$-extension of an ℓ-adic field is unramified. As before, we denote the ring of integers of k_n by $\mathcal{O}_{k_n}$ and its finite residue field by κ_n.

Recall that for an abelian variety A/k, there exists a smooth group scheme $\mathscr{A}/\mathcal{O}_k$ called the Néron model of A/k, such that $\mathsf{A}(k_n) = \mathscr{A}(\mathcal{O}_{k_n})$. Further, we have that the special fiber $\mathscr{A}_0$ has a filtration by smooth commutative group schemes,

$$0 \subseteq T \subseteq H \subseteq \mathscr{A}_0^\circ \subseteq \mathscr{A}_0.$$

Here, $\mathscr{A}_0^\circ$ is the connected component of the identity of $\mathscr{A}_0$, T is a torus, $U := H/T$ is a unipotent group variety (where $H = T \oplus U$ is split because H is commutative), $B := \mathscr{A}_0^\circ/H$ is an abelian variety, and $\pi_0^{\text{ét}}(\mathscr{A}/\kappa) = \mathscr{A}_0/\mathscr{A}_0^\circ$ is a finite étale group scheme. We can decompose the torus T into a split (resp. non split) part, which we denote by $\mathbb{G}_m^{c_s(w)}$ (resp. T_{an}).

Since we would like to compute the Herbrand quotient of $\mathsf{A}[p^\infty]$, it will suffice to compute the Herbrand quotients of B, $\mathbb{G}_m^{c_s(w)}$, and $H = T_{\text{an}} \oplus U$. In the following proposition, we compute the p^∞ torsion of B, H, and $\mathbb{G}_m^{c_s(w)}$ over κ_{cyc}.

Proposition 3.2. Let ℓ and p be distinct primes, and $k/\mathbb{Q}_\ell$ be a finite extension such that μ_p is contained in k, with finite residue field κ of characteristic $\ell \neq p$. Let $k_{\text{cyc}} = k(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of k. Let A be a d-dimensional abelian variety over k.

(1) For the good reduction part, which we denote by B,
$$B[p^\infty](\kappa_{\text{cyc}}) \simeq \begin{cases} B[p^\infty](\overline{\kappa}_{\text{cyc}}) & \text{if } B[p](\kappa) \neq 0, \\ 0 & \text{if } B[p](\kappa) = 0, \end{cases}$$
In the first case, $\mathrm{Gal}(\kappa(B[p^\infty])/\kappa) = \mathbb{Z}_p$.

(2) Let $T^{\text{split}} \cong \mathbb{G}_m^{c_s(w)}$ be the split toric part of $\mathscr{A}_0^\circ$. Then
$$T^{\text{split}}[p^\infty](\kappa_{\text{cyc}}) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{c_s(w)}.$$

(3) Let $H = T_{\text{an}} \oplus U$ be the anisotropic (i.e., non-split toric and unipotent) part of $\mathscr{A}_0^\circ$. here too. Then $H[p^\infty](\kappa_{\text{cyc}})$ is finite. If $p > 2d+1$ or if $H = T_{\text{an}}$, then
$$H[p^\infty](\kappa_{\text{cyc}}) = 0.$$

Proof. First, we make the following observation: since k contains μ_p, the order of the residue field of k_n is congruent to $1 \pmod p$.

(1) First, consider the case that $B(\kappa)[p] = 0$. By an application of the topological Nakayama Lemma (see [NSW13, Corollary 5.2.18 (ii)]) we know that $B[p](\kappa_{\text{cyc}}) = 0$. In particular, $B[p](\kappa_n) = 0$, for each n. It is now elementary to observe that $B[p^\infty](\kappa_n) = 0$, as well. Therefore,
$$B[p^\infty](\kappa_{\text{cyc}}) = \varinjlim_n B[p^\infty](\kappa_n) = 0.$$
We now consider the case that $B(\kappa)[p] \neq 0$. Let $g = \dim B$. The image of
$$\rho_{B,p^n} : \mathrm{Gal}_\kappa \longrightarrow \mathrm{Aut}(B[p^n](\overline{\kappa}))$$
is contained in $\mathrm{GSp}_{2g}(\mathbb{Z}_p/p^n)$ for all $n \geqslant 1$ and $n = \infty$ (with $\mathbb{Z}_p/p^\infty$ understood as $\mathbb{Z}_p$). Since $\mathrm{char}\,\kappa = \ell (\neq p)$, it follows that $\dim_{\mathbb{F}_p} B[p](\overline{\kappa}) = 2g$ and there exists the alternating Weil pairing. Moreover, the Weil pairing
$$B[p](\overline{\kappa}) \times B[p](\overline{\kappa}) \longrightarrow \mu_p(\kappa)$$
is Galois equivariant because $B[p]/\kappa$ is étale since $p \neq \ell$. Since $\mu_p \subseteq k^\times$ and $p \neq \ell$, $\mu_p(\kappa)$ is a *trivial* Galois module.

Since $p \neq \ell$, $B[p]/\kappa$ is étale and self-dual with respect to Cartier duality, i.e.,
$$\mathrm{Hom}(B[p], \mu_p) = \mathrm{Hom}(B[p], \mathbb{Z}/p) \simeq B[p].$$
The category of finite étale group schemes over κ is equivalent to the category of finite $\mathrm{Gal}(\overline{\kappa}/\kappa) \simeq \hat{\mathbb{Z}}$-modules. Being commutative and p-torsion corresponds to the subcategory of $\mathbb{F}_p$-vector spaces with an action by the étale fundamental group. Consequently, the category of finite étale p-torsion group schemes over κ is equivalent to the category of finite dimensional $\mathbb{F}_p$-vector spaces together with an automorphism commuting with the morphisms in the category.

Let $B[p]$ correspond to an $\mathbb{F}_p$-vector space V with an automorphism τ.
First we note that to show $\kappa(B[p])/\kappa$ is a p-extension, it suffices to show that all eigenvalues of τ over $\overline{\mathbb{F}}_p$ are equal to 1. This is because if there is an element of $\mathrm{Gal}(\kappa(B[p])/\kappa)$ of order

prime to p then it has a non-trivial semisimple part, and so it must have an eigenvalue not equal to 1.
Claim: All eigenvalues of τ over $\overline{\mathbb{F}}_p$ are equal to 1.
Justification: Under the equivalence of categories of finite étale $\mathbb{F}_p$-vector space and group schemes with finite dimensional $\mathbb{F}_p$-vector spaces with $\mathrm{Gal}(\overline{\kappa}/\kappa)$-action, the group $\mathrm{Hom}(B[p], \mathbb{Z}/p)$ can be identified with $\mathrm{Hom}(V, \mathbb{F}_p)$ together with an automorphism τ'. Consider the linear forms $\phi : V \to \mathbb{F}_p$ with $\phi \circ \tau = \phi$ (i.e., trivial action on $\mathbb{F}_p$) and $\tau'(\phi) = \phi \circ \tau = \phi$. If v, is an eigenvector for τ with eigenvalue λ, pick a linear form ϕ not vanishing on v. Then

$$0 \neq \phi(v) = \phi\left(\tau(v)\right) = \phi(\lambda v) = \lambda\phi(v),$$

which implies that $\lambda = 1$. This completes the proof of the claim.

We have shown that $\kappa(B[p])/\kappa$ is a p-extension. Note that

$$\ker\left(\mathrm{GSp}_{2g}(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GSp}_{2g}(\mathbb{F}_p)\right) \hookrightarrow \mathrm{Mat}_{2g\times 2g}(\mathbb{F}_p)$$

is a pro-p group. Hence, $\kappa(B[p^\infty])/\kappa$ is a pro-p extension. Further, since κ has absolute Galois group $\hat{\mathbb{Z}}$, we know that

$$\mathrm{Gal}(\kappa(B[p^\infty])/\kappa) = \mathbb{Z}_p = \mathrm{Gal}(\kappa_{\mathrm{cyc}}/\kappa).$$

Hence, $B(\kappa_{\mathrm{cyc}})[p^\infty] = B(\overline{\kappa}_{\mathrm{cyc}})[p^\infty]$.

(2) Recall that we denote the split part of the torus, written as T^{split}, by $\mathbb{G}_m^{c_s(w)}$; in other words, $\mathbb{G}_m^{c_s(w)} \simeq (\mu_{p^\infty})^{c_s(w)}$ as p-divisible groups. Thus, we have $\mathbb{G}_m^{c_s(w)}(\kappa_{\mathrm{cyc}}) = (\mu_{p^\infty})^{c_s(w)}(\kappa_{\mathrm{cyc}})$. Because κ_{cyc} is the residue field of k_{cyc}, which is by definition $k(\mu_{p^\infty})$, it contains all of the p^n roots of unity. It follows that

$$(\mu_{p^\infty})(\kappa_{\mathrm{cyc}}) \simeq \varinjlim \mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Q}_p/\mathbb{Z}_p.$$

Thus, $(\mu_{p^\infty})^{c_s(w)}(\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{c_s(w)}$.

(3) Let α, $c_s(w)$, and $c_g(w)$ be the dimensions of the unipotent, toric and abelian parts of the special fiber of the Néron model. For the *anisotropic part*, it is proven in [CX08, Main Theorem (b)] that the prime-to-ℓ torsion part of $\#H(k)$ is finite and bounded above by an absolute constant $C = C(\alpha, c_s(w), c_g(w), \ell)$. Therefore, if $p \gg 0$, then $H(k)[p^\infty] = 0$ and by the topological Nakayama's Lemma we know that $H(k_{\mathrm{cyc}})[p^\infty] = 0$, as well.

In the *non-split toric case*, $\alpha = c_g(w) = 0$ and the Néron special fiber does not contain a copy of $\mathbb{G}_m$. The non-split toric part T_{an} corresponds to a twist of a totally split torus $\mathbb{G}_m^n$ by an element f in

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{\kappa}/\kappa), \mathrm{GL}_n(\mathbb{Z})\right) = \mathrm{Hom}_{\mathrm{cont}}\left(\mathrm{Gal}(\overline{\kappa}/\kappa), \mathrm{GL}_n(\mathbb{Z})\right).$$

Since $\mathrm{Gal}(\overline{\kappa}/\kappa)$ is pro-cyclic, this is the same as twisting by a homomorphism

$$f\colon\ \mathrm{Gal}(\kappa'/\kappa) \longrightarrow \mathrm{GL}_n(\mathbb{Z})$$

with κ'/κ a finite extension. Or, equivalently by a torsion element g in $\mathrm{GL}_n(\mathbb{Z})$. Since T_{an} has no split sub-torus, g fixes only the zero element of $\mathbb{Z}^n$. As a Galois module, $T_{\mathrm{an}}(\overline{\kappa}) \simeq (\overline{\kappa}^\times)^n$ but it is twisted by an action of g.
Claim: The $\mathrm{Gal}(\overline{\kappa}/\kappa)$-invariants of $T_{\mathrm{an}}(\overline{\kappa})$ have no p-torsion. In other words, every element of $(\overline{\kappa}^\times)^n$ of order exactly p is not fixed by g.
Justification: Since $\mu_p \subseteq \kappa$, therefore

$$(\overline{\kappa}^\times)^n[p] \simeq (\mathbb{Z}/p)^n$$

where $(\overline{\kappa}^\times)^n[p]$ has the standard $\mathrm{Gal}(\overline{\kappa}/\kappa)$-action. But, we have noted above that in our case $(\overline{\kappa}^\times)^n[p]$ is further twisted by a character which does not fix any element except 0.

Therefore, the twisted $(\overline{\kappa}^\times)^n$ has no p-torsion. In other words, $T_{\text{an}}[p](\kappa) = 0$. This implies that $T_{\text{an}}[p^\infty](\kappa) = 0$.

In the *purely unipotent case*, we have that $c_s(w) = c_g(w) = 0$. By [CX08, Main Theorem (c)] it is known that $U(k)[p^\infty] = 0$ for primes $p > 2d+1$. An application of the topological Nakayama's Lemma implies that $U(k_{\text{cyc}})[p^\infty] = 0$.

When $p \leqslant 2d+1$, it is still possible to show that $U(k_{\text{cyc}})[p^\infty]$ is finite. To prove the claim, it is enough to show that when $c_s(w) = c_g(w) = 0$ and $n \geqslant 0$, the p^∞-torsion part of $\mathsf{A}(k_n)$ is independent of n. For each n, let κ_n denote the residue field of k_n. By Theorem 2.4, we know that there exists an exact sequence

$$0 \longrightarrow \hat{\mathsf{A}}(\mathfrak{m}) \longrightarrow \mathsf{A}(k_n) \longrightarrow \mathscr{A}(\kappa_n),$$

such that $\hat{\mathsf{A}}(\mathfrak{m})[p^\infty]$ is trivial. Further, a result of H. W. Lenstra and F. Oort [LJO85, Theorem 1.15] implies that if $c_s(w) = c_g(w) = 0$ then $\pi_0^{\text{ét}}(\mathscr{A}/\kappa_n)[p^\infty]$ has order bounded by 2^{2d}. In particular, this quantity is independent of n. Therefore, for our purposes it is now enough to show that $\mathscr{A}^0_{\kappa_n}$ has order prime to p, which is true in this setting (see for example, [Mil08, p. 141 proof of Theorem 3.5]). □

3.2. **Herbrand quotients over the residue field.** Throughout this section, we suppose that $G \simeq \mathbb{Z}/p\mathbb{Z}$. We make an observation that $\mathsf{A}[p^\infty](\kappa_{\text{cyc}})$ is a direct sum of copies of $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial G-action. In this section, we compute the Herbrand quotient $h_G\left(\mathsf{A}[p^\infty](\kappa_{\text{cyc}})\right)$.

Definition 3.3. Let M be a divisible $\mathbb{Z}_p[G]$-module of cofinite type. The *Herbrand quotient* is defined as follows

$$h_G(M) = \frac{\#\mathrm{H}^2(G, M)}{\#\mathrm{H}^1(G, M)}.$$

Lemma 3.4. With notation as above, $\operatorname{ord}_p\left(h_G(\mathbb{Q}_p/\mathbb{Z}_p)\right) = -1$.

Proof. First, we simplify the denominator. Since the G-action is trivial,

$$\mathrm{H}^1(G, \mathbb{Q}_p/\mathbb{Z}_p) = \operatorname{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p) = (\tfrac{1}{p}\mathbb{Z}/\mathbb{Z}).$$

In particular,

$$\#\mathrm{H}^1(G, \mathbb{Q}_p/\mathbb{Z}_p) = p.$$

Next, we simplify the numerator. Since G is a cyclic group, we know that

$$\mathrm{H}^2(G, \mathbb{Q}_p/\mathbb{Z}_p) = \hat{\mathrm{H}}^0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p).$$

Here, we have used the notation $\hat{\mathrm{H}}^0(\mathbb{Z}/p, \mathbb{Q}_p/\mathbb{Z}_p)$ for the 0-th Tate cohomology group, also called the norm residue group (see [NSW13, p. 21]). Now, by definition of the 0-th Tate cohomology group

$$\hat{\mathrm{H}}^0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p) = (\mathbb{Q}_p/\mathbb{Z}_p)^{\mathbb{Z}/p\mathbb{Z}}/N_G(\mathbb{Q}_p/\mathbb{Z}_p) = (\mathbb{Q}_p/\mathbb{Z}_p)/p(\mathbb{Q}_p/\mathbb{Z}_p),$$

where N_G is the image of the norm map. Finally, we know that $\mathbb{Q}_p/\mathbb{Z}_p$ is a divisible group; hence, $\mathbb{Q}_p/\mathbb{Z}_p = p\mathbb{Q}_p/\mathbb{Z}_p$. Therefore,

$$\#\mathrm{H}^2(G, \mathbb{Q}_p/\mathbb{Z}_p) = 1.$$

The result is now immediate. □

Recall that α, $c_s(w)$, and $c_g(w)$ are the dimensions of the unipotent, toric and abelian parts of the special fiber of the Néron model.

Theorem 3.5. With notation introduced above,

$$\operatorname{ord}_p\left(h_G\left(\mathsf{A}[p^\infty](\kappa_{\text{cyc}})\right)\right) = -2c_g(w) - c_s(w).$$

Proof. As before, let B denote the good reduction part, $\mathbb{G}_m^{c_s(w)}$ be the totally split part, T_{an} be the anisotropic toric part and U be the unipotent part. From Proposition 3.2, one has the p^∞-torsion of each component. By applying Lemma 3.4, we obtain their respective Herbrand quotients. More precisely:

- Since $B[p^\infty](\kappa_{\text{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{2c_g(w)}$, the Herbrand quotient $h_G(B[p^\infty](\kappa_{\text{cyc}})) = p^{-2c_g(w)}$.
- Since $\mathbb{G}_m^{c_s(w)}[p^\infty](\kappa_{\text{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{c_s(w)}$, the Herbrand quotient $h_G(\mathbb{G}_m^{c_s(w)}[p^\infty](\kappa_{\text{cyc}}) = p^{-c_s(w)}$.
- Since $T_{\text{an}}[p^\infty](\kappa_{\text{cyc}}) = 0$, it follows that $h_G(T_{\text{an}}[p^\infty](\kappa_{\text{cyc}})) = 1$.
- Since $U[p^\infty] = 0$, once again $h_G(U[p^\infty](\kappa_{\text{cyc}})) = 1$.

The result is now immediate since the Herbrand quotient is additive in exact sequences. □

3.3. **Herbrand quotients over F_v.** In this section, we are able to go from the Herbrand quotient over the residue field κ_{cyc} to the Herbrand quotient over $F_{\text{cyc},v}$. We begin with the following result.

Proposition 3.6. With notation as above,

$$h_G\left(\mathsf{A}[p^\infty]\left(F_{\text{cyc},v}\right)\right) = h_G\left(\mathscr{A}[p^\infty]\left(\kappa_{\text{cyc}}\right)\right).$$

Proof. Recall that $G = \mathbb{Z}/p\mathbb{Z}$ for our purpose. To prove the assertion it suffices to show that

$$\mathrm{H}^i\left(G, \mathsf{A}[p^\infty](F_{\text{cyc},v})\right) = \mathrm{H}^i\left(G, \mathscr{A}[p^\infty](\kappa_{\text{cyc}})\right) \quad \text{for } i = 1, 2.$$

For ease of notation, we write $M_{F,\text{cyc}} = \mathsf{A}[p^\infty](F_{\text{cyc},v})$ and $M_{\kappa,\text{cyc}} = \mathscr{A}[p^\infty](\kappa_{\text{cyc}})$. Further writing $M_{F,n} = \mathsf{A}[p^n](F_{\text{cyc},v})$ and $M_{\kappa,n} = \mathscr{A}[p^n](\kappa_{\text{cyc}})$ we note that

$$M_{F,\text{cyc}} = \varinjlim_n M_{F,n} \text{ and } M_{\kappa,\text{cyc}} = \varinjlim_n M_{\kappa,n}.$$

Since G is a finite group, we know that

$$\mathrm{H}^n(G, M_{F,\text{cyc}}) = \varinjlim_n \mathrm{H}^n(G, M_{F,n}) \text{ and } \mathrm{H}^n(G, M_{\kappa,\text{cyc}}) = \varinjlim_n \mathrm{H}^n(G, M_{\kappa,n})$$

for all $n \geqslant 1$. Thus to show $h_G(\mathsf{A}[p^\infty](F_{\text{cyc},v})) = h_G(\mathscr{A}[p^\infty](\kappa_{\text{cyc}}))$ it suffices to show that $\mathsf{A}[p^n](F_v^{\text{nr}}) = \mathscr{A}[p^n](\kappa_{\text{cyc}})$ for all $n \geqslant 1$.

By [Con, Lemma 1.3], the quasi-finite étale group scheme $E := \mathscr{A}[p^n]/\mathcal{O}_k$ splits into $E = E_f \coprod E_\eta$ with $E_\eta(\overline{\kappa}) = \varnothing$, and E_f is finite étale. In order to show the statement of the corollary, it suffices to show that $E(F_{\text{cyc},v}) = E(\kappa_{\text{cyc}})$. But note that [Con, Lemma 1.3] asserts that $E_f(\overline{\kappa}) = E_f(\overline{F}_v)^{I_v}$, so $E_f(\kappa_{\text{cyc}}) = E_f(F_{\text{cyc},v})$. Thus, it remains to show that $E_\eta(\kappa_{\text{cyc}}) = E_\eta(F_{\text{cyc},v})$. Since E_η has empty special fiber by definition, we have $E_\eta(\overline{\kappa}) = \varnothing$ and so $E_\eta(\kappa_{\text{cyc}}) = \varnothing$. We are now required to show that $E_\eta(F_v^{\text{nr}}) = \varnothing$.

We proceed by contradiction. Suppose $E_\eta(F_v^{\text{nr}}) \neq \varnothing$, then there exists a section x such that $x \in E_\eta(F_v^{\text{nr}})$. This $x \in E_\eta(F_v^{\text{nr}}) \subset E(F_v^{\text{nr}}) \subset \mathsf{A}(F_v^{\text{nr}})$ gives rise to a map from $\operatorname{Spec}(F_v^{\text{nr}}) \to E_\eta \subset \mathsf{A}$. By the Néron mapping property the section x extends uniquely to a map $\overline{x}$ from $\operatorname{Spec}(\mathcal{O}_{F_v}^{\text{nr}}) \to E_\eta \subset \mathscr{A}$ and thus $E_\eta(\overline{\kappa}) \neq \varnothing$, which is a contradiction. So, we have that $E_\eta(F_v^{\text{nr}}) = \varnothing$.

Now we have shown that $E(F_v^{\text{nr}}) = E(\kappa_{\text{cyc}})$, which means that $\mathsf{A}[p^n](F_v^{\text{nr}}) = \mathscr{A}[p^n](\kappa_{\text{cyc}})$ for all $n \geqslant 1$. The statement of the corollary follows. □

3.4. **Herbrand quotients over L_w.** The main result of this section is Theorem 3.7 which compares the Herbrand quotients of $\mathsf{A}[p^\infty](F_{\text{cyc},v})$ and $\mathsf{A}[p^\infty](L_{\text{cyc},w})$.

We now state and prove the main result of this section.

Theorem 3.7. With notation as before,

$$\mathrm{ord}_p\left(h_G\left(\mathsf{A}[p^\infty]\left(F_{\mathrm{cyc},v}\right)\right)\right) = \mathrm{ord}_p\left(h_G\left(\mathsf{A}[p^\infty]\left(L_{\mathrm{cyc},w}\right)\right)\right) = -c_s(w) - 2c_g(w).$$

Proof. We remind the reader of the setting we are in. We have a degree-p Galois extension of number fields, L/F. Let v be a prime above ℓ $(\neq p)$ in F and let π_{F_v} be the uniformizer of the local field F_v. Consider a prime $w \mid v$ in L such that

$$L_w = F_v(\sqrt[p]{\pi_{F_v}}).$$

This extension L_w/F_v is tamely ramified. We write F_{cyc}/F (resp. $F_{\mathrm{cyc},v}/F_v$) for the cyclotomic $\mathbb{Z}_p$-extension of number fields (resp. local fields), and similarly for L. For ease of notation, write

$$G = \mathrm{Gal}(L/F) \simeq \mathrm{Gal}(L_{\mathrm{cyc}}/F_{\mathrm{cyc}}) \simeq \mathrm{Gal}(L_{\mathrm{cyc},w}/F_{\mathrm{cyc},v}) \simeq \mathbb{Z}/p\mathbb{Z}.$$

Set $F^{(p)}_{\mathrm{cyc},v}$ (resp. $L^{(p)}_{\mathrm{cyc},w}$) to denote the maximal pro-p extension of $F_{\mathrm{cyc},v}$ (resp. $L_{\mathrm{cyc},w}$). Then

$$F^{(p)}_{\mathrm{cyc},v} = F_{\mathrm{cyc},v}(\sqrt[p^\infty]{\pi_{F_v}}) = L^{(p)}_{\mathrm{cyc},w}. \tag{3.1}$$

Here, we have used the fact that $\pi_{F_v} = \pi_{F_{\mathrm{cyc},v}}$ since $F_{\mathrm{cyc},v}/F_v$ is an unramified extension. The Galois group $\mathrm{Gal}(F^{(p)}_{\mathrm{cyc},v}/F_{\mathrm{cyc},v}) \simeq \mathbb{Z}_p(1)$ is the maximal pro-p quotient of the absolute Galois group $G_{F_{\mathrm{cyc},v}} = \mathrm{Gal}(\overline{F_{\mathrm{cyc},v}}/F_{\mathrm{cyc},v})$ and can be canonically identified with the p-Sylow subgroup. In particular,

$$\mathrm{Gal}(F^{(p)}_{\mathrm{cyc},v}/F_{\mathrm{cyc},v}) \simeq \mathcal{I}^{(p)}_{F_{\mathrm{cyc},v}}.$$

Here, $\mathcal{I}(F) := \mathcal{I}^{(p)}_{F_{\mathrm{cyc},v}}$ is the maximal pro-p inertia subgroup of $G_{F_{\mathrm{cyc},v}}$. Note that $\mathcal{I}(L)$ can be defined analogously and is a subgroup of $\mathcal{I}(F)$ isomorphic to $p\mathbb{Z}_p(1)$.

Recall Grothendieck's ℓ-adic monodromy theorem [FO08, Corollary 1.25]. It says that

$$\rho : G_{F_v} \longrightarrow \mathrm{Aut}\left(T_p(\mathsf{A})\right)$$

is potentially semistable, i.e., when $p \neq \ell$, the inertia subgroup of Gal_{F_v} acts quasi-unipotently. Now, as A is defined over F_v, Grothendieck's monodromy theorem applies to ρ and we may consider the restriction to $G_{F_{\mathrm{cyc},v}}$.

Let g be a topological generator of $\mathcal{I}(F)$ (written multiplicatively), i.e. g^p is a topological generator of $\mathcal{I}(L)$. Now since $\rho(g)$ is quasi-unipotent, we can split $\rho(g) = g_s g_u$ into a semisimple part g_s and unipotent part g_u such that $g_s g_u = g_u g_s$ where g_s is of finite order. Let F' be a finite extension of F_v such that $\rho(\mathcal{I}(F')) = g_u$ where $\mathcal{I}(F')$ is defined as above. In other words, F' is the extension of F_v, which trivializes the semi-simple part of $\rho(g)$. Let F'' be the compositum of F' and $F_{\mathrm{cyc},v}$, and define L'' analogously.

Now we claim that

$$h_G\left(\mathsf{A}[p^\infty]\left(F_{\mathrm{cyc},v}\right)\right) = h_G\left(\mathsf{A}[p^\infty]\left(F''\right)\right).$$

To see this, consider the exact sequence

$$0 \longrightarrow \mathsf{A}[p^\infty](F_{\mathrm{cyc},v}) \longrightarrow \mathsf{A}[p^\infty](F'') \longrightarrow V \longrightarrow 0.$$

Consider the inclusion $\iota : \mathsf{A}[p^\infty](F_{\mathrm{cyc},v}) \to \mathsf{A}[p^\infty](F'')$ and the trace $\mathrm{tr} := \mathrm{tr}_{F''/F_{\mathrm{cyc},v}}$ in the opposite direction. They satisfy $\mathrm{tr} \circ \iota = [F'' : F_{\mathrm{cyc},v}] =: \deg < \infty$. By [Xio20, Theorem 1], we have the following exact sequence

$$0 \longrightarrow \mathrm{coker}(\iota) \longrightarrow \mathrm{coker}(\mathrm{tr} \circ \iota) = \mathsf{A}[p^\infty](F_{\mathrm{cyc},v})/[\deg]\mathsf{A}[p^\infty](F_{\mathrm{cyc},v}).$$

To show that $\mathrm{coker}\,\iota$ is finite, it is enough to show that $\mathsf{A}[p^\infty](F_{\mathrm{cyc},v})/\deg \mathsf{A}[p^\infty](F_{\mathrm{cyc},v})$ is finite. This follows because $\mathsf{A}[p^\infty](\overline{F}_{\mathrm{cyc},v})$ is a subgroup of $(\mathbb{Q}_p/\mathbb{Z}_p)^{2g}$, its subgroup $\mathsf{A}[p^\infty](F_{\mathrm{cyc},v})$ has

Pontryagin dual a quotient of $\mathbb{Z}_p^{2g}$, so by the classification of finitely generated $\mathbb{Z}_p$-modules, is of the form $\mathbb{Z}_p^n$ plus a finite order part. Hence $\mathsf{A}[p^\infty](F_{\mathrm{cyc},v}) = (\mathbb{Q}_p/\mathbb{Z}_p)^n$ plus something finite, therefore its quotient is finite. This implies that $h_G(V) = 1$, from which the claim follows. A similar argument shows that

$$h_G\left(\mathsf{A}[p^\infty]\left(L_{\mathrm{cyc},w}\right)\right) = h_G\left(\mathsf{A}[p^\infty]\left(L''\right)\right).$$

Therefore, it suffices to prove that

$$h_G\left(\mathsf{A}[p^\infty]\left(F''\right)\right) = h_G\left(\mathsf{A}[p^\infty]\left(L''\right)\right).$$

Consider the short exact sequence

$$0 \longrightarrow \mathsf{A}[p^\infty](F'') \longrightarrow \mathsf{A}[p^\infty](L'') \longrightarrow Q \longrightarrow 0.$$

Since the Herbrand quotient of a finite module is trivial and $T_p(-)$ is left exact, it suffices to show that Q is finite. Note that Q is a finitely generated p-primary abelian group since so are $\mathsf{A}[p^\infty](F'')$ and $\mathsf{A}[p^\infty](L'')$. Therefore, Q is finite if and only if $T_p(Q) = 0$. To prove the claim, we are left to show that

$$T_p\left(\mathsf{A}(F'')\right) = T_p\left(\mathsf{A}(L'')\right).$$

For $F''^{(p)}$ (resp. $L''^{(p)}$) the maximal pro-p extension of F'' (resp. L''), we have that

$$T_p\left(\mathsf{A}(F'')[p^\infty]\right) = T_p\left(\mathsf{A}(F''^{(p)})[p^\infty]^{\mathcal{I}(F)}\right),$$

and similarly for $T_p\left(\mathsf{A}(L'')[p^\infty]\right)$, we are left to show that

$$T_p\left(\mathsf{A}(F''^{(p)})[p^\infty]^{\mathcal{I}(F)}\right) = T_p\left(\mathsf{A}(L''^{(p)})[p^\infty]^{\mathcal{I}(L)}\right).$$

Equivalently,

$$T_p\left(\mathsf{A}(F''^{(p)})[p^\infty]\right)^{\mathcal{I}(F)} = T_p\left(\mathsf{A}(L''^{(p)})[p^\infty]\right)^{\mathcal{I}(L)}.$$

This follows from the fact that the sequential projective limit $(\mathsf{A}(F''^{(p)})[p^n])_{n\geqslant 0}$ commutes with the equalizer of id and $\rho(g)$ with g a topological generator of $\mathcal{I}$. Since g is a topological generator of $\mathcal{I}(F)$ (written multiplicatively), g^p is a topological generator of $\mathcal{I}(L)$. We will show that

$$\left(T_p\left(\mathsf{A}(F''^{(p)})\right)\right)^{\rho(g)} = \left(T_p\left(\mathsf{A}(L''^{(p)})\right)\right)^{\rho(g^p)} = \left(T_p\left(\mathsf{A}(F''^{(p)})\right)\right)^{\rho(g^p)}.$$

Here, we have used (3.1) for the last equality.

Let g be a unipotent automorphism of a finite dimensional $\mathbb{Q}_p$ vector space. We are left to show that the fixed space of g equals that of g^p. By the Jordan canonical form, g is a direct sum of Jordan blocks J_n. So, without loss of generality,

$$g = J_n = (1 + N),$$

with N nilpotent, which fixes exactly a one dimensional subspace, corresponding to the upper left entry 1 in J_n. Moreover,

$$g^p = 1 + \sum_{i=1}^{p} \binom{p}{i} N^i$$

has the same fixed space using the fact that all binomial coefficients are non-zero because $\mathbb{Q}_p$ has characteristic 0. □

This completes the proof of Theorem 3.1.

3.5. **Special case: modular abelian varieties.** When $\mathsf{A}/\mathbb{Q}$ is a d-dimensional *modular* abelian variety (i.e., of GL_2-type), the result is more precise. This is because for modular abelian varieties, it is not possible to have mixed reduction type. More precisely, if A is associated to a newform $f \in S_2(\Gamma_0(N))$ then the reduction type is

- purely good i.e., $p \nmid N$ *or*
- purely semistable i.e., $p \mid N$ and $\left|a_p(f)\right| = 1$ *or*
- purely unipotent i.e., $p \mid N$ and $a_p(f) = 0$.

Theorem 3.8 (algebraic Kida-type formula for modular abelian varieties)**.** Let L/F be a finite Galois extension of degree a power of p. Let A/F be a d-dimensional abelian variety which is the base extension of a modular abelian variety over $\mathbb{Q}$. Assume it has good ordinary reduction at primes above p.

Suppose that $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion and that $\mu(\mathsf{A}/F_{\mathrm{cyc}}) = 0$. Then, $\mathrm{Sel}(\mathsf{A}/L_{\mathrm{cyc}})$ is a cotorsion $\Lambda(\Gamma)$-module with $\mu(\mathsf{A}/L_{\mathrm{cyc}}) = 0$. Furthermore, the respective λ-invariants, denoted by $\lambda(\mathsf{A}/F_{\mathrm{cyc}})$ and $\lambda(\mathsf{A}/L_{\mathrm{cyc}})$ satisfy the following formula:

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\sum_{w \in P_1}\big(e(w) - 1\big) + 2d\sum_{w \in P_2}\big(e(w) - 1\big)\,.$$

Here, $e(w)$ is the ramification degree in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$, and the sets P_1, P_2 are the sets of primes in L_{cyc} defined as

$$P_1 = \{w \nmid p : \ \mathsf{A} \text{ has split toric reduction at } w\},$$
$$P_2 = \{w \nmid p : \ \mathsf{A} \text{ has good reduction at } w \text{ and } \mathsf{A}(L_{\mathrm{cyc},w})[p] \neq 0\}.$$

We will see in Section 5 that Theorem 3.8 has the same shape as the analytic Kida-type formula for modular abelian varieties.

4. Applications of the algebraic Kida-type formula

We begin by recording the following standard lemma relating the λ-invariant of the Selmer group of an elliptic curve to its rank is well-known but we include it for the sake of completeness.

Lemma 4.1. Let A/F be an abelian variety with good ordinary reduction at all primes above p and assume that $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})$ is Λ-cotorsion. Then, $\lambda(\mathsf{A}/F_{\mathrm{cyc}}) \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F))$.

Proof. Let r_p denote the $\mathbb{Z}_p$-corank of $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})^{\Gamma}$. Since $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})$ is cotorsion, r_p is finite. Consider the following short exact sequence

$$0 \longrightarrow \mathsf{A}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}(\mathsf{A}/F) \longrightarrow Ш(\mathsf{A}/F)[p^{\infty}] \longrightarrow 0.$$

We deduce that

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}(\mathsf{A}/F) \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F)), \tag{4.1}$$

with equality precisely when $Ш(\mathsf{A}/F)[p^{\infty}]$ is finite. It follows from the structure theory of Λ-modules that $\lambda(\mathsf{A}/F_{\mathrm{cyc}}) \geqslant r_p$. Hence, it suffices to show that $r_p \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F))$. This is indeed the case, since Mazur's Control Theorem (see for example [Gre01, Theorem 4.1]) asserts that there is a natural map

$$\mathrm{Sel}(\mathsf{A}/F) \longrightarrow \mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})^{\Gamma}$$

with finite kernel and cokernel. From (4.1), we see that $r_p \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F))$ and the result follows. $\square$

In [CH01, Proposition 6.9], J. Coates and S. Howson used the result of Hachimori–Matsuno to study the asymptotic growth of λ-invariants at each finite layer of a certain (non-commutative) p-adic Lie extension arising from geometry. Unfortunately, using the Kida-type formula provides no indication as to whether the growth in the size of the dual Selmer group in a p-adic Lie extension is due to a large Mordell–Weil group or a large Tate–Shafarevich group. On the other hand, using the theory of Verma modules, M. Harris proved a lower bound for the Mordell–Weil rank of an abelian variety (with good ordinary reduction at p in the) at each finite layer of the trivializing extension of A, see [Har79].

Growth of Mordell–Weil ranks in pro-p p-adic Lie extensions. Using Theorem 3.1, we can extend the results of Coates–Howson to abelian varieties and all uniform pro-p p-adic Lie extensions. In the case of elliptic curves, these results have been generalized to pro-p p-adic Lie extension in [Ray22]. We consider the following setup. Let F be a given number field and F_∞ be any uniform pro-p p-adic Lie extension of dimension D, containing the cyclotomic $\mathbb{Z}_p$-extension of F. We further suppose that only finitely many primes ramify in F_∞/F.[1] Set $G = \mathrm{Gal}(F_\infty/F)$ and write $G_n = G^{p^n}$, i.e., the group generated by the p^n-th powers of elements in G. Further, we denote by $F_{(n)}$ the fixed field of G_n. This gives a filtration

$$F \subset F_{(1)} \subset \ldots \subset F_{(n)} \subset \ldots.$$

Observe that by construction, $\mathrm{Gal}(F_{(n)}/F_{(n-1)}) \cong (\mathbb{Z}/p\mathbb{Z})^D$. We can then consider the cyclotomic $\mathbb{Z}_p$-extensions of each $F_{(n)}$ to get the filtration

$$F_{\mathrm{cyc}} \subset F_{(1),\mathrm{cyc}} \subset \ldots \subset F_{(n),\mathrm{cyc}} \subset \ldots.$$

Since $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ has dimension $D-1$ and it follows that $\mathrm{Gal}(F_{(n),\mathrm{cyc}}/F_{\mathrm{cyc}}) \cong (\mathbb{Z}/p\mathbb{Z})^{D-1}$. Consider an abelian variety A/F of dimension d and fix a prime p of good ordinary reduction of A. For each finite extension $F_{(n)}/F$, we define the sets $P_1(F_{(n)})$ and $P_2(F_{(n)})$ as in Theorem 3.1. Note that the primes in $P_i(F_{(n)})$ lie above the primes in $P_i(F)$ and each of these sets is a collection of finitely many primes. Finally, to simplify notation, for both $i = 1, 2$ set

$$\mathfrak{p}_i(F) = \#P_i(F).$$

Proposition 4.2. With notation as above,

$$\lambda(\mathsf{A}/F_{(n),\mathrm{cyc}}) \leqslant p^{n(D-1)}\left(\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\mathfrak{p}_1(F)\right) - p^{n(D-2)}\left(d\mathfrak{p}_1(F) + 2d\mathfrak{p}_2(F)\right).$$

Proof. In particular, we get

$$\sum_{w \in P_1(F_{(n)})} c_s(w)\left(e(w)-1\right) = \sum_{v \in P_1(F)} \left(\sum_{w|v} c_s(w)\left(e(w)-1\right)\right) \leqslant d \sum_{v \in P_1(F)} \left(\sum_{w|v}\left(e(w)-1\right)\right),$$
$$\sum_{w \in P_2(F_{(n)})} 2c_g(w)\left(e(w)-1\right) = \sum_{v \in P_2(F)} \left(\sum_{w|v} c_g(w)\left(e(w)-1\right)\right) \leqslant 2d \sum_{v \in P_2(F)} \left(\sum_{w|v}\left(e(w)-1\right)\right).$$

Here we have used the fact that both $c_s(w)$ and $c_g(w)$ are always less than or equal to the dimension d of the abelian variety. However, since $F_{(n)}/F$ is a Galois extension, the ramification index $e(w)$

[1]Such p-adic Lie extensions are abundant and arise naturally in number theory.

must be the same for all $w \mid v$. Therefore,

$$\begin{aligned}\sum_{w|v}\big(e(w)-1\big) &= \sum_{w|v} e(w) - \sum_{w|v} 1\\ &= [F_{(n),\mathrm{cyc}} : F_{\mathrm{cyc}}] - \sum_{w|v} 1\\ &= p^{n(D-1)} - \sum_{w|v} 1.\end{aligned}$$

Now, observe that for each $v \in P_i(F)$, we know that

$$\sum_{w|v} 1 = \#\{w \mid v\} = \frac{[F_{(n),\mathrm{cyc}} : F_{\mathrm{cyc}}]}{e(w)}.$$

Since $w \nmid p$, and our extension is pro-p, it follows that the prime w must be tamely ramified in the extension. But, recall that tame inertia is cyclic; hence $e(w) \leqslant p^n$. Thus, we conclude that

$$\sum_{w|v}\big(e(w)-1\big) \leqslant p^{n(D-1)} - p^{n(D-2)}.$$

It now follows immediately from Theorem 3.1 that

$$\lambda(\mathsf{A}/F_{(n),cyc}) \leqslant p^{n(D-1)}\left(\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\mathfrak{p}_1(F)\right) - p^{n(D-2)}\left(d\mathfrak{p}_1(F) + 2d\mathfrak{p}_2(F)\right).$$

The proof of the proposition is now complete. □

An application of Lemma 4.1 yields an upper bound on the Mordell–Weil rank of the abelian variety at the n-th layer of a uniform pro-p p-adic Lie extension. More precisely, we have the following result.

Corollary 4.3. With notation introduced above,

$$\operatorname{rank}_{\mathbb{Z}}(\mathsf{A}/F_{(n)}) \leqslant p^{n(D-1)}\left(\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\mathfrak{p}_1(F)\right) - p^{n(D-2)}\left(d\mathfrak{p}_1(F) + 2d\mathfrak{p}_2(F)\right).$$

5. Analytic Kida-type formula for modular abelian varieties

In Iwasawa theory, there is a general philosophy that (an appropriate analogue of) the Iwasawa Main Conjecture should relate the structure of of the arithmetic object (such as the Selmer group) to an associated p-adic L-function. More precisely, the invariants of the p-adic L-function associated with a modular abelian variety should be equal to the invariants of the characteristic polynomial attached to the Selmer group of the said abelian variety over the cyclotomic $\mathbb{Z}_p$-extension. For modular abelian varieties, we proved a Kida-type formula in Theorem 3.8. The Iwasawa Main Conjecture predicts that the analytic Iwasawa invariants satisfy the same formula. In Theorem 5.6, we prove that the analytic λ-invariants indeed satisfy the same Kida-type formula upon base-change. One may treat this as evidence towards the Iwasawa Main Conjecture in this setting.

In this section, we focus on the case of good *ordinary* reduction. In this special case, there is a known construction of p-adic L-functions attached to *modular* abelian varieties, see [BMS16]. We will assume that $f(z) = \sum_{n\geqslant 1} a_n e^{2\pi i n z}$ is an element of $S_2(\Gamma_0(N))$. Let $\mathbb{Q}(f)$ be the totally real number field $\mathbb{Q}(\ldots, a_n, \ldots)$. Write G_f to denote the set of embeddings $\sigma : \mathbb{Q}(f) \hookrightarrow \mathbb{C}$. For each $\sigma \in G_f$, we write f^σ to denote the conjugate of f by σ. Let A_f be the modular abelian variety associated to a newform f. It is simple over $\mathbb{Q}$, of dimension equal $[\mathbb{Q}(f) : \mathbb{Q}]$ and has $\mathbb{Q}$-endomorphism ring the coefficient ring of f, which is an order in the totally real number field $\mathbb{Q}(f)$. Moreover, its endomorphism algebra is the total ring of fractions of the endomorphism ring, with the endomorphisms defined over $\mathbb{Q}$.

For a fixed choice of Shimura periods $\Omega^{\pm}_{f^\sigma}$ (see [BMS16, Theorem 2.2] and the text after Remark 2.6 on how to compute *canonical* periods), the *plus* and *minus* modular symbol maps associated to f is defined as below:

$$[\ ]^+_f : \mathbb{Q} \longrightarrow \mathbb{Q}(f); \quad r \mapsto [r]^+_f := -\frac{\pi i}{\Omega^+_f}\left(\int_r^{i\infty} f(z)dz + \int_{-r}^{i\infty} f(z)dz\right)$$
$$[\ ]^-_f : \mathbb{Q} \longrightarrow \mathbb{Q}(f); \quad r \mapsto [r]^-_f := \frac{\pi i}{\Omega^-_f}\left(\int_r^{i\infty} f(z)dz - \int_{-r}^{i\infty} f(z)dz\right)$$

In fact, the codomain can be taken to be $\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]$ by linear extension. For any integer m prime to p, we write

$$\mathbb{Z}_{p,m} := \varprojlim_n (\mathbb{Z}/p^n m\mathbb{Z}) \simeq (\mathbb{Z}/m\mathbb{Z}) \times \mathbb{Z}_p,$$
$$\mathbb{Z}^\times_{p,m} := \varprojlim_n (\mathbb{Z}/p^n m\mathbb{Z})^\times \simeq (\mathbb{Z}/pm\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p).$$

Henceforth, we fix a prime $\mathfrak{p}$ of $\mathbb{Q}(f)$ above p. In other words, we fix an embedding

$$\iota : \mathbb{Q}(f) \hookrightarrow \mathbb{Q}(f)_{\mathfrak{p}}.$$

We often omit ι in the notation. If we consider an element of $\mathbb{Q}(f)$ as a p-adic number, we implicitly mean its image under ι. We assume throughout that p is a prime of good ordinary reduction of the modular abelian variety A_f, i.e., $p \nmid N_f$ and $\iota(a_p(f^\sigma))$ is a p-adic unit for all $\sigma \in G_f$.

To form the p-adic L-function, we take the product over all Galois conjugates f^σ with $\sigma \in G_f$ and the embeddings of $a_n(f^\sigma)$ under ι. Using the modular symbols maps, we can define two measures $\nu^{\pm}_{f,m,\alpha}$ on $\mathbb{Z}^\times_{p,m}$, which depend on the the unit root of the polynomial, i.e., a root which is a unit in the ring of integers $\mathbb{Z}(f)_{\mathfrak{p}}$ of the completion $\mathbb{Q}(f)_{\mathfrak{p}}$,

$$h(x) := x^2 - a_p(f)x + p \in \mathbb{Q}(f)_{\mathfrak{p}}[x].$$

The ordinary condition imposed on A_f guarantees that the polynomial $h(x)$ has a unique unit root $\alpha \in \mathbb{Z}(f)^\times_{\mathfrak{p}}$. These measures $\nu^{\pm}_{f,m,\alpha}$ on $\mathbb{Z}^\times_{p,m}$ with values in $\mathbb{Q}(f)_{\mathfrak{p}}$ are defined as follows:

$$\nu^{\pm}_{f,m,\alpha}(a + p^n m\mathbb{Z}_{p,m}) = \frac{1}{\alpha^n}\left[\frac{a}{p^n m}\right]^{\pm}_f - \frac{1}{\alpha^{n+1}}\left[\frac{a}{p^{n-1}m}\right]^{\pm}_f$$

for $n \geqslant 1$, $a \in (\mathbb{Z}/p^n m\mathbb{Z})^\times$. Let $\ell \neq p$ be a prime of good reduction. The characteristic polynomial of Frob_ℓ with $\ell \neq p$ under the p-adic Galois representation ρ_{f,p^∞} is given by

$$\prod_{\sigma \in G_f}\left(x^2 - a_\ell(f^\sigma)x + \ell\right) = \prod_{\sigma \in G_f}\left(x^2 - \sigma(a_\ell(f))x + \ell\right).$$

Indeed, this is because for each $\sigma : \mathbb{Q}(f) \hookrightarrow \mathbb{C}$, we have by definition that

$$a_\ell(f^\sigma) = \sigma(a_\ell(f)).$$

Let $\langle\ \rangle : \mathbb{Z}^\times_{p,m} \to 1 + p\mathbb{Z}_p$ be the natural projection map and let $\kappa : \Gamma \to 1 + p\mathbb{Z}_p$ denote the cyclotomic character. Observe that κ takes values in $1 + p\mathbb{Z}_p \subseteq \mathbb{Z}^\times_p$ because Γ is a p-extension. Given $x \in \mathbb{Z}^\times_{p,m}$, define

$$t : \mathbb{Z}^\times_{p,m} \longrightarrow \mathbb{Z}_p \text{ by } x \mapsto \log_\gamma\langle x\rangle$$

where γ is a topological generator of Γ. Here, the p-adic logarithm is uniquely defined on the principal units $1 + p\mathbb{Z}_p$. Let χ be a character of $\mathbb{Z}^\times_{p,m}$ of finite order with values in $\overline{\mathbb{Q}}^\times_p$. It factors through $(\mathbb{Z}/p^n m)^\times$ for some $n \geqslant 0$. We consider χ as a Dirichlet character defined on $\mathbb{Z}$ of conductor dividing $p^n m$ and extend it by 0 on integers not coprime to $p^n m$. Define

$$G_{p,m}(f,\chi,T) := \int_{\mathbb{Z}_{p,m}^\times} \chi(x)(1+T)^{t(x)} d\nu^+_{f,m,\alpha}(x).$$

The p-adic L-function of f associated to χ is then defined in the following way

$$\mathcal{L}_{p,m}(f,\chi,s) := G_{p,m}(f,\chi,\kappa(\gamma)^{s-1}-1).$$

Next, following [BMS16, Section 2.3], we write

$$G_{p,m}(\mathsf{A}_f,\chi,T) := \prod_{\sigma\in G_f} G_{p,m}(f^\sigma,\chi,T)$$

and the corresponding p-adic L-function

$$\mathcal{L}_{p,m}(\mathsf{A}_f,\chi,s) := \prod_{\sigma\in G_f} \mathcal{L}_{p,m}(f^\sigma,\chi,s) = \prod_{\sigma\in G_f} G_{p,m}(f^\sigma,\chi,\kappa(\gamma)^{s-1}-1).$$

Let $S_{\mathrm{add}} := S_{\mathrm{add}}(f)$ denote the set of prime numbers at which the modular abelian variety A_f has bad non-toric reduction (i.e., $p \mid N$ and $a_p = 0$). For the remainder of this discussion we will make the following assumption.

(Add) Over any finite Galois extension $F/\mathbb{Q}$, the modular abelian variety A_f of dimension d has bad non-toric reduction at all primes lying above S_{add}.

The above hypothesis holds for example when $p > 2d+1$, see [ST68, p. 498]. Suppose that $F/\mathbb{Q}$ is disjoint from $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$. Under **(Add)**, we can define the p-adic L-function $G_p(\mathsf{A}_f/F,T)$ by the following formula

$$G_p\left(\mathsf{A}_f/F,(1+T)^{p^n}-1\right) = \prod_{\chi\in\widehat{\mathrm{Gal}(F/\mathbb{Q})}} G_p(\mathsf{A}_f,\chi,T).$$

The assumption on the behaviour of the additive places ensures that the p-adic L-function interpolates the values of the complex L-function at $s=1$. The fact that such a power series exists uniquely is guaranteed by [Was97, Lemma 13.39]. Let us also assume the following hypothesis for the remainder of this section:

(Int) For all $r \in \mathbb{Q}$, $[r]^\pm_f \in \mathbb{Z}(f)_\mathfrak{p}$.

We use this hypothesis to ensure that $G_p(\mathsf{A}_f/F,T)$ is a power series contained in $\mathbb{Z}_p[\![T]\!]$. This allows us to appeal to the Weierstrass Preparation Theorem, for example. Note that **(Int)** implies that $\nu^+_{f,m,\alpha}$ takes integral values. Since $G_{p,m}(f,\chi,T)$ is defined as an integral over an integral power series with respect to an integral measure, it is also integral. We associate the μ-invariant and λ-invariant to $G_p(f,\chi,T)$, which are denoted by $\mu_{f,m}(\chi)$ and $\lambda_{f,m}(\chi)$, respectively. More precisely,

$$\lambda_{f,m}(\chi) := \lambda\left(G_{p,m}(f,\chi,T)\right), \quad \mu_{f,m}(\chi) := \mu\left(G_{p,m}(f,\chi,T)\right).$$

For each conjugate f^σ, we may define $\lambda_{f^\sigma,m}(\chi)$ and $\mu_{f^\sigma,m}(\chi)$ in an analogous way. Thus,

$$\lambda_{\mathsf{A}_f,m}(\chi) := \lambda\left(G_{p,m}(\mathsf{A}_f,\chi,T)\right) = \sum_\sigma \lambda_{f^\sigma,m}(\chi);$$
$$\lambda_{\mathsf{A}_f}(F) := \lambda\left(G_p(\mathsf{A}_f/F,T)\right) = \sum_\chi \lambda_{\mathsf{A}_f,m}(\chi);$$
$$\mu_{\mathsf{A}_f,m}(\chi) := \mu\left(G_{p,m}(\mathsf{A}_f,\chi,T)\right) = \sum_\sigma \mu_{f^\sigma,m}(\chi);$$
$$\mu_{\mathsf{A}_f}(F) := \mu\left(G_p(\mathsf{A}_f/F,T)\right) = \sum_\chi \mu_{\mathsf{A}_f,m}(\chi).$$

Lemma 5.1. Let ℓ be a prime number coprime to pm and let $\phi : \mathbb{Z}_{p,m\ell}^\times \to \mathbb{Z}_{p,m}^\times$ be the natural projection map. For any open compact set $U \subseteq \mathbb{Z}_{p,m}^\times$,

$$\nu^+_{f,m\ell,\alpha}\left(\phi^{-1}(U)\right) = a_\ell \nu^+_{f,m,\alpha}(U) - \nu^+_{f,m,\alpha}(\ell^{-1}U) - \nu^+_{f,m,\alpha}(\ell U).$$

Proof. See [Mat00, Lemma 2.2]. We point out to the reader that the proof given there works for arbitrary coefficient rings of f. ☐

Now, for each prime ℓ and any (possibly infinite) algebraic extension $\mathcal{F}/\mathbb{Q}$, set $g_{\mathcal{F}}(\ell)$ to denote the number of primes above $\mathcal{F}$ lying above ℓ (if it is finite). As before, let χ be a character of $\mathbb{Z}_{p,m}^\times$ of finite order considered as a Dirichlet character and π be a prime element of $\mathbb{Z}_p[\chi]$, the finite extension of $\mathbb{Z}_p$ obtained by adjoining all values of χ. For each $\sigma \in G_f$, define an integer $g_\chi^\sigma(\ell)$ as below:

(1) If $\chi(\ell) \neq 0$ and $\ell \nmid N$:
$$g_\chi^\sigma(\ell) = \begin{cases} 0 & \text{if } a_\ell(f^\sigma) \not\equiv \chi(\ell) + \chi^{-1}(\ell) \pmod{\pi} \\ g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) & \text{if } a_\ell(f^\sigma) \equiv \chi(\ell) + \chi^{-1}(\ell),\ a_\ell(f^\sigma) \not\equiv \pm 2 \pmod{\pi} \\ 2g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) & \text{else.} \end{cases}$$

(2) If $\chi(\ell) \neq 0$ and $\ell \mid N$:
$$g_\chi^\sigma(\ell) = \begin{cases} 0 & \text{if } a_\ell(f^\sigma) \not\equiv \chi^{-1}(\ell) \pmod{\pi} \\ g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) & \text{if } a_\ell(f^\sigma) \equiv \chi^{-1}(\ell) \pmod{\pi}. \end{cases}$$

(3) If $\chi(\ell) = 0$, then $g_\chi^\sigma(\ell) = 0$.

The next two lemmas will allow us to prove the main theorem.

Lemma 5.2. Let χ and ψ be characters of $\mathbb{Z}_{p,m}^\times$ of finite order. Assume that ψ is of p-power order and that $\mu_{\mathsf{A}_f,m}(\chi) = 0$. Then,

$$\lambda_{\mathsf{A}_f,m}(\chi\psi) = \lambda_{\mathsf{A}_f,m}(\chi) \text{ and } \mu_{\mathsf{A}_f,m}(\chi\psi) = 0.$$

Proof. We begin with the observation that the assumption $\mu_{\mathsf{A}_f,m}(\chi) = 0$ implies that

$$\mu_{f^\sigma,m}(\chi) = 0 \text{ for all } \sigma.$$

Let π be a prime element of $\mathbb{Z}_p[\chi\psi]$. Since ψ has p-power order, it follows that $\psi(x) \equiv 1 \pmod{\pi}$ for any $x \in \mathbb{Z}_{p,m}^\times$ because the only roots of unity of p-power order in a field of characteristic p are 1. Hence, we know that for all σ,

$$G_{p,m}(f^\sigma, \chi\psi, T) \equiv G_{p,m}(f^\sigma, \chi, T) \pmod{\pi}$$

Therefore, we have that $\mu_{f^\sigma,m}(\chi\psi) = 0$ for each σ. Moreover,

$$\lambda_{f^\sigma,m}(\chi) = \lambda_{f^\sigma,m}(\chi\psi).$$

The additivity of λ and μ proves the claim. More precisely,

$$\begin{aligned} G_{p,m}(\mathsf{A}_f, \chi\psi, T) = \prod_\sigma G_{p,m}(f^\sigma, \chi\psi, T) &\equiv \prod_\sigma G_{p,m}(f^\sigma, \chi, T) \pmod{\pi} \\ &\equiv G_{p,m}(\mathsf{A}_f, \chi, T) \pmod{\pi}. \end{aligned}$$

Hence our assumption $\mu_{\mathsf{A}_f,m}(\chi) = 0$ immediately implies that $\mu_{\mathsf{A}_f,m}(\chi\psi) = 0$. Also, using the definition of the λ-invariant we see from the above equivalence that

$$\lambda_{\mathsf{A}_f,m}(\chi\psi) = \sum_\sigma \lambda_{f^\sigma,m}(\chi\psi) = \sum_\sigma \lambda_{f^\sigma,m}(\chi) = \lambda_{\mathsf{A}_f,m}(\chi). \qquad \square$$

Lemma 5.3. Let χ be a character of $\mathbb{Z}_{p,m}^\times$ of finite order and ℓ be a prime number that is coprime to pm. Assume that the order of χ is prime to p. Then

$$\lambda_{\mathsf{A}_f,m\ell}(\chi) = \lambda_{\mathsf{A}_f,m}(\chi) + \sum_\sigma g_\chi^\sigma(\ell) \text{ and } \mu_{\mathsf{A}_f,m\ell}(\chi) = \mu_{\mathsf{A}_f,m}(\chi).$$

Sketch of proof. This proof is a generalization of [Mat00, Lemma 3.3]. Recall that for any $a, b \in \mathbb{Z}_{p,m}^\times$

$$t(ab) = t(a) + t(b).$$

Writing

$$G_{p,m\ell}(\mathsf{A}_f, \chi, T) = \prod_{\sigma \in G_f} \left(\int_{\mathbb{Z}_{p,m\ell}^\times} \chi(x)(1+T)^{t(x)} d\mu^+_{f^\sigma,m\ell,\alpha^\sigma} \right)$$

and using Lemma 5.1, we conclude that

$$\begin{aligned} G_{p,m\ell}(\mathsf{A}_f, \chi, T) &= \prod_\sigma \left(a_\ell(f^\sigma) - \chi^{-1}(\ell)(1+T)^{-t(\ell)} - \chi(\ell)(1+T)^{t(\ell)} \right) G_{p,m}(f^\sigma, \chi, T) \\ &= G_{p,m}(\mathsf{A}_f, \chi, T) \times \prod_\sigma \left(a_\ell(f^\sigma) - \chi^{-1}(\ell)(1+T)^{-t(\ell)} - \chi(\ell)(1+T)^{t(\ell)} \right). \end{aligned}$$

For ease of notation, define

$$h_\ell(\mathsf{A}_f, \chi, T) := \prod_\sigma h_\ell(f^\sigma, \chi, T) := \prod_\sigma \left(a_\ell(f^\sigma) - \chi^{-1}(\ell)(1+T)^{-t(\ell)} - \chi(\ell)(1+T)^{t(\ell)} \right).$$

Then

$$\begin{aligned} \lambda_{\mathsf{A}_f,m\ell}(\chi) &= \lambda_{\mathsf{A}_f,m}(\chi) + \lambda(h_\ell(\mathsf{A}_f, \chi, T)) = \lambda_{\mathsf{A}_f,m}(\chi) + \sum_\sigma \lambda(h_\ell(f^\sigma, \chi, T)), \\ \mu_{\mathsf{A}_f,m\ell}(\chi) &= \mu_{\mathsf{A}_f,m}(\chi) + \mu(h_\ell(\mathsf{A}_f, \chi, T)) = \mu_{\mathsf{A}_f,m}(\chi) + \sum_\sigma \mu(h_\ell(f^\sigma, \chi, T)). \end{aligned}$$

Next, write $t(\ell) = up^a$ with $a \geqslant 0$ and $u \in \mathbb{Z}_p^\times$, and set $\sigma_\ell \in \Gamma$ to be the Frobenius element of ℓ. Since $\kappa(\gamma_0)^{t(\ell)} = \langle \ell \rangle = \kappa(\sigma_\ell)$, it follows that the decomposition group of ℓ in Γ is p^a. Therefore, $p^a = g_{\mathbb{Q}_{\mathrm{cyc}}(\ell)}$. The calculation of the μ-invariant and the λ-invariant of $h_\ell(A_{f^\sigma}, \chi, T)$ is identical to that of Matsuno. In particular, $\mu(h_\ell(f^\sigma, \chi, T)) = 0$ and $\lambda(h_\ell(f^\sigma, \chi, T)) = g_\chi^\sigma(\ell)$. This completes the proof of the lemma. $\square$

Lemma 5.4. Let $F/\mathbb{Q}$ be an abelian extension such that $p \nmid [F:\mathbb{Q}]$ and Hypothesis **(Add)** is satisfied. Let $\ell \equiv 1 \pmod{p}$ be prime and v be any prime of F above ℓ. Then with the same notation as before, for each $\sigma \in G_f$ and writing q_v for the cardinality of the residue field of F_v,

$$\sum_{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})}} g_\chi^\sigma(\ell) = \begin{cases} 2g_{F_{\mathrm{cyc}}}(\ell) & \text{if } \ell \nmid N \text{ and } a_\ell(f^\sigma) \text{ is a } v\text{-adic unit} \\ g_{F_{\mathrm{cyc}}}(\ell) & \text{if } \ell \mid N \text{ and } a_\ell(f^\sigma) = a_\ell(f) = +1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.5. The hypothesis that $p \nmid [F:\mathbb{Q}]$ ensures the following:

(1) For each χ, we have that the order of $\chi(\ell) \mod \pi$ coincides with the order of $\chi(\ell)$.
(2) Since $F/\mathbb{Q}$ is a Galois extension which is disjoint from $\mathbb{Q}_{\mathrm{cyc}}$, it follows from Galois theory that the number of primes in F_{cyc} above ℓ is given by

$$g_{F_{\mathrm{cyc}}}(\ell) = g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) g_F(\ell).$$

Proof. First suppose that ℓ is a prime of good reduction, i.e., $\ell \nmid N$. Let $\mathfrak{p} \mid p$ be a prime that lies above p. Since we are assuming that $\ell \equiv 1 \pmod{p}$, it follows that $\ell \equiv 1 \pmod{\mathfrak{p}}$. For each $\sigma \in G_f$, the characteristic polynomial of Frob_ℓ is

$$x^2 - a_\ell(f^\sigma)x + 1 \pmod{\mathfrak{p}}.$$

Denote one of its roots in $\overline{\mathbb{F}}_{\mathfrak{p}}$ by α^σ. Since the constant term is $+1$, the other root is $(\alpha^\sigma)^{-1}$. For ease of notation set

$$\delta_{\chi,\alpha^\sigma} = \begin{cases} 1 & \text{if } \chi(\ell) \equiv \alpha^\sigma \pmod{\pi} \\ 0 & \text{otherwise,} \end{cases} ; \quad \delta_{\chi,\alpha^{\sigma,-1}} = \begin{cases} 1 & \text{if } \chi(\ell) \equiv (\alpha^\sigma)^{-1} \pmod{\pi} \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\alpha^\sigma = \alpha^{\sigma,-1} \Longleftrightarrow a_\ell(f^\sigma) \equiv \pm 2 \pmod{\mathfrak{p}},$$

by the definition of $g_\chi^\sigma(\ell)$ introduced in the discussion after Lemma 5.1, we know that

$$g_\chi^\sigma(\ell) = g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) \left(\delta_{\chi,\alpha^\sigma} + \delta_{\chi,\alpha^{\sigma,-1}} \right).$$

It follows that

$$\sum_{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})}} g_\chi^\sigma(\ell) = g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) \sum_{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})}} \left(\delta_{\chi,\alpha^\sigma} + \delta_{\chi,\alpha^{\sigma,-1}} \right).$$

Recall from [Was97, Theorem 3.7] that

$$\#\{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})} : \chi(\ell) = 1\} = g_F(\ell)$$

Writing r to denote the residue degree of ℓ in $F/\mathbb{Q}$, one concludes from the above discussion that

$$\sum_{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})}} g_\chi^\sigma(\ell) = \begin{cases} 2g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) g_F(\ell) & \text{if } (\alpha^\sigma)^r = 1 \\ 0 & \text{if } (\alpha^\sigma)^r \neq 1 \end{cases}$$

But notice that $(\alpha^\sigma)^r$ is one of the eigenvalues of the action of the Frobenius of $\ell \pmod{\mathfrak{p}}$. Hence, just as in the proof of [Mat00, Theorem 3.4], we have that

$$\begin{aligned} (\alpha^\sigma)^r = 1 &\Longleftrightarrow \alpha^\sigma = (\alpha^\sigma)^{-r} = 1 \\ &\Longleftrightarrow a_\ell(f^\sigma) \equiv 2 \pmod{\mathfrak{p}} \\ &\Longleftrightarrow \ell + 1 - a_\ell(f^\sigma) \equiv 0 \pmod{\mathfrak{p}}. \end{aligned}$$

Here, we have used the fact that $\ell \equiv 1 \pmod{\mathfrak{p}}$.

Next, suppose that $\ell \mid N$. Define

$$\alpha^\sigma := a_\ell(f^\sigma) \pmod p.$$

Note that α^σ is independent of σ because $a_\ell(f) \in \{\pm 1, 0\}$. Once again, by the definition introduced in the discussion after Lemma 5.1 we know that

$$\sum_{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})}} g_\chi^\sigma(\ell) = g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) \sum_{\chi \in \widehat{\mathrm{Gal}(F/\mathbb{Q})}} \delta_{\chi, \alpha^\sigma, -1} = \begin{cases} g_{\mathbb{Q}_{\mathrm{cyc}}}(\ell) g_F(\ell) & \text{if } (\alpha^\sigma)^r = 1 \\ 0 & \text{if } (\alpha^\sigma)^r \neq 1. \end{cases}$$

But observe that when the reduction type is additive (or unipotent) then $\alpha^\sigma = 0$. Whereas, if the reduction type is split toric (resp. non-split toric) then $\alpha^\sigma = 1$ (resp. -1). We know that when $\alpha^\sigma = -1$, to get that $(\alpha^\sigma)^r = 1$ we must have that r is even. This is equivalent to the reduction type being split toric. ☐

We now state and prove the main theorem of this section. The following result is a Kida-type formula for p-adic L-functions associated with modular abelian varieties.

We will keep Hypotheses **(Add)** and **(Int)** introduced earlier in this section.

Theorem 5.6 (analytic Kida-type formula for modular abelian varieties)**.** Let L/F be a finite Galois extension of degree a power of p. Let $\mathsf{A}/\mathbb{Q}$ be a d-dimensional modular abelian variety with good ordinary reduction at p. Suppose that $\mathrm{Sel}(\mathsf{A}/F_{\mathrm{cyc}})$ is a cotorsion Λ-module and that the associated *analytic* μ-invariant is zero. Then, under Hypothesis **(Int)**, $\mu^{\mathrm{an}}(\mathsf{A}/L_{\mathrm{cyc}}) = 0$. Furthermore, the respective *analytic* λ-invariants, denoted by $\lambda^{\mathrm{an}}(\mathsf{A}/F)$ and $\lambda^{\mathrm{an}}(A/L)$ satisfy the following formula:

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d \sum_{w \in P_1} \big(e(w) - 1\big) + 2d \sum_{w \in P_2} \big(e(w) - 1\big).$$

Here, $e(w)$ is the ramification degree in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$, and the sets P_1, P_2 are the sets of primes in L_{cyc} defined as

$$P_1 = \{w \nmid p : \ \mathsf{A} \text{ has split toric reduction at } w\},$$
$$P_2 = \{w \nmid p : \ \mathsf{A} \text{ has good reduction at } w \text{ and } \mathsf{A}(L_{\mathrm{cyc},w})[p] \neq 0\}.$$

Remark 5.7. Without loss of generality, we may prove the theorem when $p \nmid [F : \mathbb{Q}]$. This is because if $p \mid [F : \mathbb{Q}]$, we have a subextension $\mathbb{Q} \subseteq F_1 \subset F$ such that $p \nmid [F_1 : \mathbb{Q}]$ and we can work with the extension F_1 instead. The Kida-type formula will automatically allow us to go from F_1 to F. It suffices to prove the theorem for the special case that $G = \mathrm{Gal}(L/F) \cong \mathbb{Z}/p\mathbb{Z}$ by induction on a composition series of the p-group G.

Proof. Write L' for the maximal subfield of L such that the degree $[L' : \mathbb{Q}]$ is a p-power. Note that $L' \cap F = \mathbb{Q}$ since we assume that $p \nmid [F : \mathbb{Q}]$. Therefore, we can write any character θ of $\mathrm{Gal}(L/\mathbb{Q})$ as a product

$$\theta = \chi\psi$$

where χ is a (Dirichlet) character of $\mathrm{Gal}(F/\mathbb{Q})$ and ψ is a (Dirichlet) character of $\mathrm{Gal}(L'/\mathbb{Q})$. Suppose that $L \cap \mathbb{Q}_{\mathrm{cyc}} = \mathbb{Q}_{(n)}$, i.e., the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Our reduction step ensures that $n \leqslant 1$. Therefore,

$$p^n \lambda_{\mathsf{A}_f}(L) = \sum_\sigma \sum_\psi \sum_\chi \lambda_{f^\sigma}(\chi\psi), \quad \lambda_{\mathsf{A}_f}(F) = \sum_\sigma \sum_\chi \lambda_{f^\sigma}(\chi)$$
$$\mu_{\mathsf{A}_f}(L) = \sum_\sigma \sum_\psi \sum_\chi \mu_{f^\sigma}(\chi\psi), \quad \mu_{\mathsf{A}_f}(F) = \sum_\sigma \sum_\chi \mu_{f^\sigma}(\chi).$$

Let m (resp. m') denote the p'-conductor of χ (resp. $\chi\psi$). Note that $m \mid m'$ and $\frac{m'}{m}$ must be square-free because ψ is a Dirichlet character and hence has conductor which is square-free. By applying Lemmas 5.2 and 5.3, we have that

$$\lambda_{\mathsf{A}_f}(\chi\psi) = \lambda_{\mathsf{A}_f}(\chi) + \sum_{\sigma}\sum_{\ell \mid m'} g_\chi^\sigma(\ell),$$
$$\mu_{\mathsf{A}_f}(\chi\psi) = 0.$$

Combining the above equations we observe that

$$\begin{aligned} p^n\lambda_{\mathsf{A}_f}(L) &= \sum_{\psi}\sum_{\chi}\lambda_{\mathsf{A}_f}(\chi\psi) = \sum_{\psi}\sum_{\chi}\lambda_{\mathsf{A}_f}(\chi) + \sum_{\psi}\sum_{\chi}\sum_{\sigma}\sum_{\ell\mid m'} g_\chi^\sigma(\ell) \\ &= [L' : \mathbb{Q}]\lambda_{\mathsf{A}_f}(F) + \sum_{\ell\neq p}\#\{\psi : \psi(\ell) = 0\}\sum_{\sigma}\sum_{\chi} g_\chi^\sigma(\ell), \\ \mu_{\mathsf{A}_f}(L) &= \sum_{\psi}\sum_{\chi}\mu_{\mathsf{A}_f}(\chi\psi) = 0. \end{aligned}$$

We further notice that $[L' : \mathbb{Q}] = p^n[L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]$ and hence we can compare the ramification indices as follows:

$$e_{L'/\mathbb{Q}}(\ell) = e_{L_{\mathrm{cyc}}/F_{\mathrm{cyc}}}(w).$$

As explained in the proof of [Mat00, Theorem 3.1, p. 91], we can also compute that

$$\#\left\{\psi : \psi(\ell) = 0\right\} = p^n[L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\left(1 - \frac{1}{e_{L_{\mathrm{cyc}}/F_{\mathrm{cyc}}}(w)}\right).$$

On the contrary, Lemma 5.4 asserts that

$$\sum_{\sigma}\sum_{\chi} g_\chi^\sigma(\ell) = \delta_w g_{L_{\mathrm{cyc}}}(\ell)$$

Now we are left to investigate δ_w. Observe that

$$\delta_w = \begin{cases} 2d & \text{if } \ell \nmid N \text{ and } \widetilde{A}(\mathbb{F}_w)[p] \neq 0 \\ d & \text{if } \ell \mid N \text{ and } a_\ell(f) = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

In the case when $\ell \mid N$ and $a_\ell(f) = \pm 1$, we know that for each $\sigma \in G_f$, we have $a_\ell(f^\sigma) = \pm 1$, as well. Since the abelian variety A_f is modular, exactly one of $\alpha, c_g(w)$, and $c_s(w)$ (i.e., the dimension of unipotent, abelian, and toric parts) is non-zero and the equal to $\dim \mathsf{A}_f = [\mathbb{Q}(f) : \mathbb{Q}] = d$. In addition, we know that the dimension of the toric part is non-zero precisely when $a_\ell(f) = \pm 1$.

We point out that

$$\left|\widetilde{A}(\mathbb{F}_w)\right| = \prod_{\sigma\in G_f}\left(q_w + 1 - a_w(f^\sigma)\right).$$

Therefore, $\widetilde{A}(\mathbb{F}_w)[p] \neq 0$ precisely if there exists *at least one* σ such that $q_w + 1 - a_w(f^\sigma) \equiv 0 \pmod p$. The contribution to δ_w is hence

$$2\#\left\{\sigma : \alpha^\sigma \equiv 1 \pmod p\right\} = 2\#\left\{\sigma : q_w + 1 - a_w(f^\sigma) \equiv 0 \pmod p\right\} \leqslant 2d.$$

But since $p = p^\sigma$, the quantity $1 - a_\ell(f^\sigma) + \ell$ is divisible by p for some σ if and only if it is divisible by p for all σ. Thus,

$$\#\left\{\sigma : q_w + 1 - a_w(f^\sigma) \equiv 0 \pmod p\right\} = d.$$

□

References

[Art86] M. Artin. Néron models. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 213–230. Springer, New York, 1986.

[AV75] Yvette Amice and Jacques Vélu. Distributions p-adiques associées aux séries de Hecke. *Astérisque*, 24(25):119–131, 1975.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[BMS16] Jennifer Balakrishnan, J Müller, and William Stein. A p-adic analogue of the conjecture of Birch and Swinnerton-Dyer for modular abelian varieties. *Mathematics of Computation*, 85(298):983–1016, 2016.

[CH01] John Coates and Susan Howson. Euler characteristics and elliptic curves II. *Journal of the Mathematical Society of Japan*, 53(1):175–235, 2001.

[Che60] Claude Chevalley. Une démonstration d'un théoreme sur les groupes algébriques. *J. Math. Pures Appl.(9)*, 39:307–317, 1960.

[Con] Brian Conrad. Classification of quasi-finite etale separated schemes. http://math.stanford.edu/~conrad/vigregroup/vigre03/zmt.pdf.

[CS00] John Coates and Ramdorai Sujatha. *Galois cohomology of elliptic curves*. Narosa, 2000.

[CX08] Pete L Clark and Xavier Xarles. Local bounds for torsion points on abelian varieties. *Canadian J. Math.*, 60(3):532–555, 2008.

[FO08] Jean-Marc Fontaine and Yi Ouyang. Theory of p-adic Galois representations, 2008. Course notes in Tsinghua University, Beijing, 2004.

[Gre01] Ralph Greenberg. Introduction to Iwasawa theory for elliptic curves. In *IAS/Park City Mathematics Series*, volume 9, pages 407–464, 2001.

[GV00] Ralph Greenberg and Vinayak Vatsal. On the Iwasawa invariants of elliptic curves. *Invent. math.*, 142(1):17–63, 2000.

[Har79] Michael Harris. Systematic growth of Mordell–Weil groups of abelian varieties in towers of number fields. *Invent. math.*, 51(2):123–141, 1979.

[Har83] Robin Hartshorne. *Algebraic geometry. Corr. 3rd printing.* Graduate Texts in Mathematics, **52**. New York-Heidelberg-Berlin: Springer-Verlag. xvi + 496 p., 1983.

[HM99] Yoshitaka Hachimori and Kazuo Matsuno. An analogue of Kida's formula for the Selmer groups of elliptic curves. *J. Alg. Geom.*, 8(3):581–601, 1999.

[HS00] Marc Hindry and Joseph H Silverman. *Diophantine geometry: an introduction*, volume 201. Springer-Verlag New York, Inc., 2000.

[Iwa81] Kenkichi Iwasawa. Riemann-Hurwitz formula and p-adic Galois representations for number fields. *Tohoku Mathematical Journal, Second Series*, 33(2):263–288, 1981.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, 295:117–290, 2004.

[Kid80] Yûji Kida. ℓ-extensions of CM-fields and cyclotomic invariants. *J. Number Theory*, 12:519–528, 1980.

[Lim21] Meng Fai Lim. Some remarks on Kida's formula when $\mu \neq 0$. *The Ramanujan J.*, pages 1–18, 2021.

[LJO85] HW Lenstra Jr and F Oort. Abelian varieties having purely additive reduction. *Journal of Pure and Applied Algebra*, 36:281–298, 1985.

[Mat00] Kazuo Matsuno. An analogue of Kida's formula for the p-adic L-functions of modular elliptic curves. *J. Number Theory*, 84(1):80–92, 2000.

[Maz72] Barry Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18(3-4):183–266, 1972.

[Mil08] James S. Milne. Abelian varieties (v2.00), 2008. Available at www.jmilne.org/math/.

[Nér64] André Néron. Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *Inst. Hautes Études Sci. Publ. Math.*, (21):128, 1964.

[NSW13] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Graduate Texts in Mathematics*. Springer, 2013.

[PW06] Robert Pollack and Tom Weston. Kida's formula and congruences. *Doc. Math., Special*, 2006:615–630, 2006.

[Ray22] Anwesh Ray. Asymptotic growth of Mordell–Weil ranks of elliptic curves in noncommutative towers. *Canadian Math. Bull.*, 2022.

[Ser06] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.

[ST68] Jean-Pierre Serre and John Tate. Good reduction of abelian varieties. *Ann. Math.*, pages 492–517, 1968.

[Was97] Lawrence C Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer, 1997.

[Xio20] Rui Xiong. Kernel-cokernel sequence for composition and its applications. *arXiv preprint arXiv:2001.07528*, 2020.

(Beneish) Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley CA, 94720, United States

Email address: leabeneish@math.berkeley.edu

(Keller) Lehrstuhl Mathematik II (Computeralgebra), Universität Bayreuth, Universitätsstrasse 30, 95440 Bayreuth, Germany

Current address: Leibniz Universität Hannover, Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Welfengarten 1, 30167 Hannover, Germany

Email address: keller@math.uni-hannover.de

URL: https://www.timo-keller.de

(Kundu) Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2

Email address: dkundu@math.ubc.ca