Abstract The main goal of this article is to provide more evidence on the relationship between the Classical Iwasawa $\mu = 0$ Conjecture and the $\mu = 0$ Conjecture for fine Selmer groups (Conjecture A). We give sufficient conditions to prove the Classical $\mu = 0$ Conjecture that improves upon previously known results. Furthermore, this method proves isogeny invariance of Conjecture A in previously unknown cases. We also provide a class of examples for which Conjecture A holds independent of the Classical $\mu = 0$ Conjecture.

Keywords Iwasawa Theory, Fine Selmer groups, Conjecture A, Classical $\mu = 0$ Conjecture

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Relating the Classical $\mu = 0$ Conjecture with Coates-Sujatha Conjecture A

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1 Introduction

Classical Iwasawa theory is concerned with the growth of class groups in towers of number fields. In [7], Iwasawa showed that in a $\mathbb{Z}_p$-extension of a number field F, the growth of the p-part of the class group is regular. In particular,

Theorem. *There exist constant non-negative integers λ and μ and a constant integer ν such that for large enough n,*

$$|A_n| = p^{\mu p^n + \lambda n + \nu}$$

where A_n is the class group of F_n, the n-th layer in the tower.

Further, the following conjecture was made.

Classical Conjecture. *For the cyclotomic $\mathbb{Z}_p$-extension F_{cyc}/F, $\mu = 0$.*

The conjecture is known to hold for Abelian extensions $F/\mathbb{Q}$ (see [4], [15]).

In [10], Mazur introduced the Iwasawa theory of Selmer groups of Abelian varieties and described the growth of the size of the p^∞-Selmer group in $\mathbb{Z}_p$-extensions F_∞/F. For an Abelian variety A/F, the dual Selmer group over F_∞, denoted by $X(A/F_\infty)$, is a finitely generated $\Lambda(\Gamma)$-module; here $\Gamma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$ and $\Lambda(\Gamma)$ is the associated Iwasawa algebra. However, $X(A/F_\infty)$ is *not always* $\Lambda(\Gamma)$-torsion, depending on the reduction type at p.

When $X(A/F_\infty)$ is $\Lambda(\Gamma)$-torsion, it affords a structure theorem like in the classical case. But an analogue of the classical conjecture is known to be false. For the cyclotomic extension of $\mathbb{Q}$, there are examples of elliptic curves where the associated μ-invariant is positive at a prime of good ordinary reduction.

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In [3], Coates and Sujatha studied a subgroup of the Selmer group, called the *fine Selmer group*. They made the following conjecture.

Conjecture A. *Let p be an odd prime and E be an elliptic curve over a number field, F. When $F_\infty = F_{\rm cyc}$ the Pontryagin dual of the fine Selmer group, denoted $Y(E/F_{\rm cyc})$, is a finitely generated $\mathbb{Z}_p$-module i.e. $Y(E/F_{\rm cyc})$ is $\Lambda(\Gamma)$-torsion and the associated μ-invariant, $\mu(Y(E/F_{\rm cyc})) = 0$.*

A priori, a relation between Galois modules coming from class groups and those coming from elliptic curves is not obvious. However, there is growing evidence that in a cyclotomic tower, p^∞-fine Selmer group and p^∞-class group have similar growth patterns [2], [3], [9]. We provide more evidence for this relationship by proving the following theorems.

Theorem 11 *Assume either*

(i) F contains μ_p or
(ii) F is a Galois extensions of odd degree.

In either of the cases, suppose the Classical Conjecture is true. For E an elliptic curve over F with $E(F)[p] \neq 0$, Conjecture A holds for $Y(E/F_{\rm cyc})$.

Remark

(i) [3, Corollary 3.6] is a particular case of part (i). Indeed, this follows from the following two well known facts. First, if $F/\mathbb{Q}$ is an Abelian extension, so is $F(\mu_p)/\mathbb{Q}$ and the Classical Conjecture holds. Second, for finite extensions of number fields L/F, Conjecture A for $Y(E/L_{\rm cyc})$ implies Conjecture A for $Y(E/F_{\rm cyc})$.
(ii) Theorem 11(ii) holds for all totally real fields, $F/\mathbb{Q}$. The proof simplifies with the assumption $F/\mathbb{Q}$ is a Galois extension of odd degree, but the idea for the general proof is identical.
(iii) Tools from Galois cohomology are used to provide new evidence for the Classical Conjecture in Section 4. By an application of Theorem 11, this provides new evidence for Conjecture A.

A converse of Theorem 11 is true. Given a number field F, we show that the Classical Conjecture holds for $F_{\rm cyc}/F$, if there exists *one* elliptic curve E/F, with $E(F)[p] \neq 0$ for which Conjecture A holds. This improves upon previously known results in this direction [2].

Theorem 12 *Let E be an elliptic curve defined over the number field F. Let p be any odd prime. Further assume that $E(F)[p] \neq 0$. If Conjecture A holds for $Y(E/F_{\rm cyc})$, then the Classical Conjecture holds for $F_{\rm cyc}/F$.*

Theorems 11 and 12 prove isogeny invariance of Conjecture A in some previously unknown cases.

Corollary 13 *Let F be a number field contianing μ_p or be a Galois extension of odd degree over $\mathbb{Q}$. Let E and E' be isogenous elliptic curves such that both E and E' have non-trivial p-torsion points over F. Then, Conjecture A holds for $Y(E/F_{\rm cyc})$ if and only if Conjecture A holds for $Y(E'/F_{\rm cyc})$.*

Remark All statements hold for Abelian varieties of dimension d. The property of cyclotomic $\mathbb{Z}_p$-extensions required in the proofs is that primes in a certain finite set decompose finitely. The theorems are stated for elliptic curves over the cyclotomic $\mathbb{Z}_p$-extension as the original conjectures are in this setting.

2 Preliminaries

Throughout this paper, F will denote a number field and p an odd prime.

Let A/F be a d-dimensional Abelian variety and S be a finite set of primes of F containing the Archimedean primes, primes above p, and primes where A has bad reduction. Fix an algebraic closure $\overline{F}/F$ and set F_S to be the maximal subfield of $\overline{F}$ containing F which is unramified outside S. Denote the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ by G_F and the Galois group $\mathrm{Gal}(F_S/F)$ by $G_S(F)$.

Definitions of p^∞-Selmer group and p^∞-fine Selmer group are as in [18].

Definition 21 *The p-**fine Selmer group**, with respect to the finite set S, is*

$$R_S(A[p]/F) := \ker\left(H^1(G_S(F), A[p]) \to \bigoplus_{v\in S} H^1(F_v, A[p])\right). \tag{1}$$

*The p^∞-**fine Selmer group** is*

$$R(A/F) := \ker\left(H^1(G_S(F), A[p^\infty]) \to \bigoplus_{v\in S} H^1(F_v, A[p^\infty])\right). \tag{2}$$

The definition of $R(A/F)$ is independent of S. For a $\mathbb{Z}_p$-extension F_∞/F,

$$R(A/F_\infty) = \varinjlim_L R(A/L)$$

where the inductive limit is over all finite extensions L/F contained in F_∞.

Recall, the Pontryagin dual of a discrete p-primary (resp. compact pro-p) Abelian group is a compact (resp. discrete) module over the assoiated Iwasawa algebra. For G a profinite group and M a G-module, M^G is the subgroup of elements fixed by G and M_G is the largest quotient of M with trivial G action.

Definition 22 *For an Abelian group N, its p-**rank** is the $\mathbb{Z}/p\mathbb{Z}$-dimension of $N[p]$, denoted by $r_p(N)$. If G is a pro-p group, write $h_i(G) = r_p\left(H^i(G, \mathbb{Z}/p)\right)$.*

Lemma 23 *[9, Lemma 3.1] Let G be a pro-p group and M be a discrete G-module which is co-finitely generated over $\mathbb{Z}_p$. If $h_1(G)$ is finite, $r_p\left(H^1(G, M)\right)$ is finite. Furthermore, the following inequalities hold*

$$\begin{aligned} h_1(G) r_p(M^G) - r_p\left((M/M^G)^G\right) &\leq r_p\left(H^1(G, M)\right) \\ &\leq h_1(G)(\mathrm{corank}_{\mathbb{Z}_p}(M) + \log_p\left(\left|M/M_{div}\right|\right) \end{aligned}$$

Lemma 24 *[9, Lemma 3.2] Consider an exact sequence of co-finitely generated Abelian groups,*

$$W \to X \to Y \to Z.$$

Then

$$\left|r_p(X) - r_p(Y)\right| \le 2r_p(W) + r_p(Z).$$

Definition 25 *The* p***-Hilbert*** S***-class field*** *of* F*, denoted* $H_S(F)$*, is the maximal Abelian unramified* p*-extension of* F *in which all primes in* S *split completely. By class field theory, the Galois group* $\mathrm{Gal}(H_S(F)/F) = \mathrm{Cl}_S(F)$*, is the* S*-class group.*

Lemma 26 *[9, Lemma 5.2, 5.3] Let* F_{cyc}/F *be the cyclotomic* $\mathbb{Z}_p$*-extension and* F_n *be the subfield of* F_{cyc} *such that* $[F_n : F] = p^n$*. Let* A *be an Abelian variety over* F *and* S *be as defined before. Then*

$$\left|r_p\left(\mathrm{Cl}(F_n)\right) - r_p\left(\mathrm{Cl}_S(F_n)\right)\right| = O(1), \tag{3}$$

$$\left|r_p\left(R_S(A[p]/F_n)\right) - r_p\left(R(A/F_n)\right)\right| = O(1). \tag{4}$$

3 Proof of the Main Results

We first prove Corollary 13. It follows from the main theorems in Section 1.

Proof (Proof of Corollary 13) Let F be a number field that contains μ_p or $F/\mathbb{Q}$ be an odd degree Galois extension. Let E be an elliptic curve isogenous to E' over F with the additional property that both $E(F)[p], E'(F)[p]$ are non-trivial. WLOG if Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ then by Theorem 12 the Classical Conjecture holds for F_{cyc}/F. Now by Theorem 11 Conjecture A holds for $Y(E'/F_{\mathrm{cyc}})$. This proves the corollary.

3.1 Proof of Theorem 12

Theorem 12 follows from the following lemma when A is an elliptic curve.

Lemma 31 *Let* F_{cyc}/F *be the cyclotomic* $\mathbb{Z}_p$*-extension and* F_n *be the subfield of* F_{cyc} *such that* $[F_n : F] = p^n$*. Let* A *be a* d*-dimensional Abelian variety over* F *and* S *be as defined before. Assume* $A(F)[p] \neq 0$*. Then for some positive constant* k_1 *that depends on* $A(F)[p]$*,*

$$k_1 r_p\left(\mathrm{Cl}_S(F_n)\right) \le r_p\left(R(A/F_n)\right) + O(1) \tag{5}$$

Proof For the ease of notation, set $H_n = H_S(F_n)$ and $H_{n,w} = H_S(F_n)_w$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & R(A/F_n) & \to & H^1(G_S(F_n), A[p^\infty]) & \to & \bigoplus_{v_n} H^1(F_{n,v_n}, A[p^\infty]) \\
& \downarrow r_n & & \downarrow f_n & & \downarrow \gamma_n \\
0 \to & R(A/H_n) & \to & H^1(G_S(H_n), A[p^\infty]) & \to & \bigoplus_{v_n} \bigoplus_{w|v_n} H^1(H_{n,w}, A[p^\infty])
\end{array}$$

Here v_n runs over all primes in $S(F_n)$, the finite set of primes in F_n that lie above the primes in S. Observe

$$\ker \gamma_n = \bigoplus_{v_n} \ker \gamma_{n,v_n}.$$

Each $\ker \gamma_{n,v_n} = H^1(G_{n,v_n}, A(H_{n,v_n})[p^\infty])$ where G_{n,v_n} is the decomposition group of $G_n := \mathrm{Gal}(H_n/F_n)$. By definition of p-Hilbert S-class field, $G_{n,v_n} = 1$. So, $\ker \gamma_n = \operatorname{coker} \gamma_n = 0$.

By inflation-restriction, $\ker(f_n) = H^1(G_n, A(H_n)[p^\infty])$ and by diagram chasing, one obtains $\ker(f_n) \hookrightarrow R(A/F_n)$. Thus,

$$r_p\left(H^1(G_n, A(H_n)[p^\infty])\right) \leq r_p\left(R(A/F_n)\right).$$

Combining this with Lemma 23 gives the following inequality

$$h_1(G_n) r_p\left(A(F_n)[p^\infty]\right) - 2d \leq r_p\left(R(A/F_n)\right). \tag{6}$$

By definition of S-class group, $\mathrm{Gal}(H_n/F_n) = \mathrm{Cl}_S(F_n)$. So

$$\begin{aligned}
h_1(G_n) &= h_1\left(\mathrm{Gal}(H_n/F_n)\right) \\
&= r_p\left(\mathrm{Cl}_S(F_n)/p\right) \\
&= r_p\left(\mathrm{Cl}_S(F_n)\right)
\end{aligned}$$

where the last equality follows from the finiteness of the S-class group. Also,

$$\begin{aligned}
r_p\left(A(F_n)[p^\infty]\right) &\geq r_p\left(A(F)[p^\infty]\right) \\
&= r_p\left(A(F)[p]\right).
\end{aligned}$$

From Equation 6 and the above discussion it follows,

$$r_p\left(A(F)[p]\right) r_p\left(\mathrm{Cl}_S(F_n)\right) \leq r_p\left(R(A/F_n)\right) + O(1). \tag{7}$$

This proves the lemma as the hypothesis forces $r_p\left(A(F)[p]\right) \neq 0$.

We now provide a proof of Theorem 12.

Proof By an application of [17, Lemma 13.20], Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ if and only if $r_p\left(R(E/F_n)\right) = O(1)$. In other words, Conjecture A holds if and only if the p-rank remains bounded in the cyclotomic tower.

By hypothesis, Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ so $r_p\left(R(E/F_n)\right) = O(1)$. Also by hypothesis, $E(F)[p] \neq 0$. Equation 7 implies $r_p\left(\mathrm{Cl}_S(F_n)\right)$ is bounded independent of n. By Equation 3, so is $r_p\left(\mathrm{Cl}(F_n)\right)$.

This is enough to prove the Classical Conjecture. Indeed, the Classical Conjecture holds for F_{cyc}/F if and only if $r_p\left(\mathrm{Cl}(F_n)/p\right)$ is bounded independent of n [17, Proposition 13.23]. Since class groups are finite, it follows that $r_p\left(\mathrm{Cl}(F_n)\right) = r_p\left(\mathrm{Cl}(F_n)/p\right)$. Thus, the Classical Conjecture is equivalent to $r_p\left(\mathrm{Cl}(F_n)\right)$ being independent of n. This finishes the proof.

3.2 Proof of Theorem 11

Recall the following well-known facts from [8]. If L/F is a p-power Galois extension, $\mu(F_{\rm cyc}/F) = 0$ implies $\mu(L_{\rm cyc}/L) = 0$. On the other hand, for any extension L/F, $\mu(L_{\rm cyc}) = 0$ implies $\mu(F_{\rm cyc}) = 0$.
Case (i) Suppose $F \supset \mu_p$.
Since $E(F)[p] \neq 0$, it follows from the Weil pairing that $F(E[p])/F$ is either trivial or cyclic of order p. This is precisely the situation of [9, Theorem 5.5]. There is nothing left to prove.
Case(ii) Suppose $F \not\supset \mu_p$ and F is a Galois extensions of odd degree.
Theorem 11 follows from the following inequlity where k_2 is a positive constant,

$$r_p\left(R(A/F_n)\right) \leq k_2 r_p\left(\mathrm{Cl}(F_n)\right) + O(1). \tag{8}$$

Indeed, the Classical Conjecture is equivalent to $r_p\left(\mathrm{Cl}(F_n)\right)$ being bounded independent of n [17, Proposition 13.23]. If the Classical Conjecture holds, Equation 8 for an elliptic curve, E, says $r_p\left(R(E/F_n)\right)$ is bounded independent of n. Therefore Conjecture A holds.

Observe that by Equations 3 and 4, Theorem 11 follows from the following variant of Equation 8,

$$r_p\left(R_S(A[p]/F_n)\right) \leq k_2 r_p\left(\mathrm{Cl}_S(F_n)\right) + O(1). \tag{9}$$

Define $R_S(A(F_n)[p]/F_n)$ by replacing $A[p]$ with $A(F_n)[p]$ in Equation 1. $G_S(F_n)$ acts trivially on $A(F_n)[p]$; hence it is possible to relate $R_S(A(F_n)[p]/F_n)$ with $\mathrm{Cl}_S(F_n)$ and similarly their p-ranks. Since the Galois action is trivial,

$$H^1(G_S(F_n), A(F_n)[p]) = \mathrm{Hom}(G_S(F_n), A(F_n)[p])$$

and there are similar identifications for the local cohomology groups. It follows

$$R_S(A(F_n)[p]/F_n) = \mathrm{Hom}(\mathrm{Cl}_S(F_n), A(F_n)[p]) \simeq \mathrm{Cl}_S(F_n)[p]^{r_p\left(A(F_n)[p]\right)}$$

where the isomorphism is as Abelian groups. This gives the following inequality of p-ranks for d-dimensional Abelian varieties

$$\begin{aligned} r_p\left(R_S(A(F_n)[p]/F_n)\right) &= r_p\left(A(F_n)[p]\right) r_p\left(\mathrm{Cl}_S(F_n)\right) \\ &\leq 2d r_p\left(\mathrm{Cl}_S(F_n)\right). \end{aligned}$$

Note that Equation 9 follows from the above inequality provided the p-ranks of $R_S(A[p]/F_n)$ and $R_S(A(F_n)[p]/F_n)$ have the same order of growth in $F_{\rm cyc}$. This is the content of the following lemma. This completes the proof of Theorem 11.

Lemma 32 *Let $F/\mathbb{Q}$ be an odd degree Galois extension. Let A be an Abelian variety over F of dimension d. Let $F_{\rm cyc}/F$ be the cyclotomic $\mathbb{Z}_p$-extension and suppose the Classical Conjecture holds for $F_{\rm cyc}/F$. Let S be as defined before. Assume $A(F)[p] \neq 0$. Then*

$$\left| r_p\left(R_S(A(F_n)[p]/F_n)\right) - r_p\left(R_S(A[p]/F_n)\right) \right| = O(1). \tag{10}$$

Proof Set $B_n = A(F_n)[p]$. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 \to & R_S(B_n/F_n) & \to & H^1(G_S(F_n), B_n) & \to & \bigoplus_{v_n} H^1(F_{n,v_n}, B_n) \\ & \downarrow s_n & & \downarrow f_n & & \downarrow g_n \\ 0 \to & R(A[p]/F_n) & \to & H^1(G_S(F_n), A[p]) & \to & \bigoplus_{v_n} H^1(F_{n,v_n}, A[p]) \end{array}$$

where v_n runs over all the primes in the finite set $S(F_n)$.

Since $B_n = A(F_n)[p] \subset A[p]$, there is a short exact sequence

$$0 \to B_n \to A[p] \to M \to 0.$$

Taking the $G_S(F_n)$-cohomology of this sequence, $\ker(f_n) = H^0(G_S(F_n), M)$. $|M|$ is finite and bounded, thus $r_p\left(\ker(f_n)\right) = O(1)$ and $r_p\left(\ker(s_n)\right) = O(1)$. A similar argument for the local cohomology gives $r_p\left(\ker(g_n)\right) = O(1)$.

Now, by Lemma 23

$$\begin{aligned} \Big| r_p\left(R_S(B_n/F_n)\right) - r_p\left(R_S(A[p]/F_n)\right)\Big| &\leq 2r_p\left(\ker(s_n)\right) + r_p\left(\operatorname{coker}(s_n)\right) \\ &= r_p(\operatorname{coker}(s_n)) + O(1). \end{aligned}$$

It is therefore left to prove $r_p\left(\operatorname{coker}(s_n)\right) = O(1)$. This is done by proving the p-ranks of $\operatorname{coker}(f_n) = H^1(G_S(F_n), M)$ and $\operatorname{coker}(g_n) = \bigoplus_{v_n} H^1(F_{n,v_n}, M)$ have the same order of growth.

First we estimate $r_p\left(\operatorname{coker}(f_n)\right)$. Note $\operatorname{coker}(f_n)$ is finite because M is finite [12, Theorem 8.3.20]. Tracing through the proof of the said theorem, one observes that Classical Conjecture for F_{cyc}/F implies $r_p\left(\operatorname{coker}(f_n)\right)$ is equal to the p-rank of $\mathcal{O}_{n,S}^\times/(\mathcal{O}_{n,S}^\times)^p$ where $\mathcal{O}_{n,S}^\times$ is the notation for the S-units of F_n. In a cyclotomic extension, primes are finitely decomposed so $|S|$ is bounded by an absolute constant. By hypothesis, F is a Galois extension of odd degree, so $F_n/\mathbb{Q}$ is a totally real extension. By the S-units analogue of Dirichlet's Unit Theorem,

$$r_p\left(\operatorname{coker}(f_n)\right) = [F_n : \mathbb{Q}] + O(1).$$

We now estimate $r_p\left(\operatorname{coker}(g_n)\right)$ using [14, Lemma 2, Chapter II §5]. As before, $\operatorname{coker}(g_n)$ is finite. It is known that primes above p are totally ramified in the cyclotomic extension and all other finite primes of S are finitely decomposed. Therefore,

$$\begin{aligned} r_p\left(\operatorname{coker}(g_n)\right) &= \sum_{v_n | p} r_p\left(F_{n,v_n}^\times/(F_{n,v_n}^\times)^p\right) \\ &= \sum_{v_n|p} [F_{n,v_n} : \mathbb{Q}_p] + O(1) \\ &= [F_n : \mathbb{Q}] + O(1). \end{aligned}$$

The second equality follows from [11, Proposition II.5.7]. This finishes the proof of the lemma.

4 Illustrating the Results with Examples

In this section, we show that the Classical $\mu = 0$ Conjecture holds for p-rational number fields. This allows us to provide evidence for Conjecture A.

Let F be a number field. Let S be a finite set of primes of F *containing* the primes above p and the Archimedean primes. The weak Leopoldt conjecture in the classical setting is the assertion

$$H^2(\mathrm{Gal}(F_S/F_{\mathrm{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p) = 0. \tag{11}$$

It holds for the cyclotomic extension of a number field [12, Theorem 10.3.25]. If Equation 11 holds for a finite set S as mentioned above, it holds for the set $S = \Sigma = S_p \cup S_\infty$ where S_p is the set of primes of F above p and S_∞ are the Archimedean primes [12, Theorem 11.3.2]. Therefore, the weak Leopoldt Conjecture is independent of the choice of S. From here on, fix $S = \Sigma$.

Let F_{S_p} denote the maximal p-ramifed extension of F. Consider the Galois group $\mathrm{Gal}(F_{S_p}/F)$ and let $\mathcal{G}_{S_p}(F)$ be its maximal pro-p quotient.

Definition 41 *[13] Let F be a number field, p be an odd prime. F is called p-**rational** if and only if $\mathcal{G}_{S_p}(F)$ is pro-p-free.*

For examples of non-Abelian p-rational fields see [1]. Given F, it is conjectured to be p-rational for all primes outside a set of Dirichlet density 0 [6].

The following theorem is well-known.

Theorem 42 *[12, Theorem 11.3.7] The Classical $\mu = 0$ Conjecture holds for F_{cyc} if and only if $\mathcal{G}_\Sigma(F_{\mathrm{cyc}}) = \mathrm{Gal}(F_\Sigma(p)/F_{\mathrm{cyc}})$ is a free pro-p group.*

Definition 43 *[14, Page 23] A pro-p group G is free if and only if its p-cohomological dimension $\mathrm{cd}_p(G) \leq 1$.*

By a standard fact in Galois cohomology of pro-p groups [14, Chapter I, Section 4, Proposition 21], an equivalent formulation of Theorem 42 is the following: the Classical $\mu = 0$ Conjecture holds for F_{cyc} if and only if

$$H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), \mathbb{Z}/p\mathbb{Z}) = 0. \tag{12}$$

Corollary 44 *Let F be a p-rational number field. The Classical $\mu = 0$ Conjecture holds for F_{cyc}.*

Proof Since $p \neq 2$, we replace S_p by Σ in the definition of p-rational fields. For p-rational number fields, $\mathcal{G}_\Sigma(F) = \mathrm{Gal}(F_\Sigma(p)/F)$ has p-cohomological dimension at most 1. Equivalently,

$$H^2(\mathcal{G}_\Sigma(F), \mathbb{Z}/p\mathbb{Z}) = 0.$$

Since $\mathcal{G}_\Sigma(F_{\mathrm{cyc}}) = \mathrm{Gal}(F_\Sigma(p)/F_{\mathrm{cyc}})$ is a closed normal subgroup of $\mathcal{G}_\Sigma(F)$, by [14, Proposition 14]

$$\mathrm{cd}_p\left(\mathcal{G}_\Sigma(F_{\mathrm{cyc}})\right) \leq \mathrm{cd}_p\left(\mathcal{G}_\Sigma(F)\right) \leq 1.$$

Thus,

$$H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), \mathbb{Z}/p\mathbb{Z}) = 0.$$

By Equation 12, the result follows.

This provides new evidence for Conjecture A.

Corollary 45 *Let F be a p-rational number field such that either*

(i) $F \supseteq \mu_p$ *or*
(ii) F *is a totally real number field.*

Suppose E is an elliptic curve over F with $E(F)[p] \neq 0$. Then Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$

Proof This follows from Theorem 11 (and the remark following it) along with Corollary 44.

In some cases, Conjecture A holds *independent* of the Classical Conjecture. For this, we need the following equivalent formulation of Conjecture A and plays a key role in this section. It was established independently by Greenberg [5] and Sujatha [16].

Proposition 46 *Assume the weak Leopoldt conjecture for elliptic curves holds, i.e. $H^2(G_S(F_{\mathrm{cyc}}), E[p^\infty]) = 0$. Then Conjecture A for $Y(E/F_{\mathrm{cyc}})$ is equivalent to the assertion*

$$H^2(G_S(F_{\mathrm{cyc}}), E[p]) = 0.$$

Proposition 47 *Let F be a p-rational field and E be an elliptic curve with good reduction everywhere over F (or bad reduction at primes above p) such that $E[p] \subset E(F)$. Then Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$.*

Proof Choose $S = \Sigma = S_p \cup S_\infty$. By p-rationality of F and the isomorphism of the inflation map [12, Corollary 10.4.8], it follows

$$H^2(\mathcal{G}_\Sigma(F), E[p]) = H^2(G_\Sigma(F), E[p]) = 0. \tag{13}$$

By Hochschild-Serre Spectral sequence we have the following exact sequence [12, Page 119]

$$H^2(\mathcal{G}_\Sigma(F), E[p]) \to H^0(\Gamma, H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), E[p])) \to 0,$$

where $\Gamma = \mathrm{Gal}(F_{\mathrm{cyc}}/F)$. The first term is 0, thus $H^0\left(\Gamma, H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), E[p])\right)$ is trivial. Since $H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), E[p])$ is a discrete module, it must be 0. Once again by the isomorphism of the inflation map,

$$0 = H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), E[p]) = H^2(G_S(F_{\mathrm{cyc}}), E[p]).$$

By Proposition 46, Conjecture A holds for $Y(E/F_{cyc})$.

The author has been unable to weaken the hypothesis $E[p] \subseteq F$, in proving Conjecture A independent of the Classical Conjecture.

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