

Iwasawa Theory of Fine Selmer Groups

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IWASAWA THEORY OF CLASS GROUPS

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Iwasawa Theory

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A \mathbb{Z}_p -extension of F is a Galois extension F_∞/F such that

$$F_\infty = \bigcup_n F_n$$

with each F_n/F a cyclic extension, $\text{Gal}(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

Example $F = \mathbb{Q}$

Consider the tower

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where \mathbb{Q}_n is the unique subfield of $\mathbb{Q}(\zeta_{p^n})$ such that $\text{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n\mathbb{Z}$.

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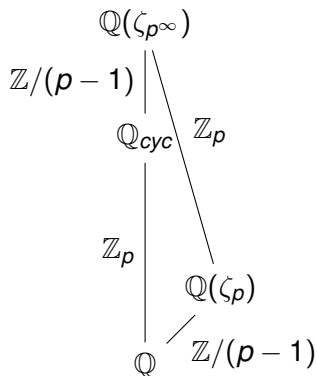
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$$\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{Z}/(p-1) \simeq (1 + p\mathbb{Z}_p) \times \mathbb{Z}/(p-1)$$

Field Diagram



For a number field F , the cyclotomic \mathbb{Z}_p -extension always exists.

$$F_{cyc} = F \cdot \mathbb{Q}_{cyc}.$$

Leopoldt Conjecture

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$$[F : \mathbb{Q}] = r_1 + 2r_2.$$

Conjecture (Leopoldt Conjecture)

Let F be a number field. Then F admits $r_2 + 1$ independent \mathbb{Z}_p -extensions.

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Brumer proved the Leopoldt Conjecture for Abelian extensions F/\mathbb{Q} .

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One of them is the cyclotomic \mathbb{Z}_p -extension. The other is the anti-cyclotomic \mathbb{Z}_p -extension.

The *anti-cyclotomic* \mathbb{Z}_p -extension (denoted F_{ac}/F) is the unique \mathbb{Z}_p -extension of F which is Galois over \mathbb{Q} but not Abelian over \mathbb{Q} .

Field Diagram

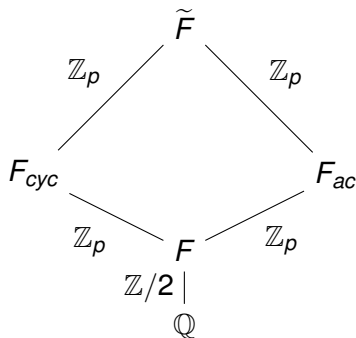
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Equivalent Formulation of the Leopoldt Conjecture

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Theorem

The Leopoldt Conjecture is equivalent to the following assertion:

$$H^2(\mathrm{Gal}(F_{\{p\}}/F), \mathbb{Q}_p/\mathbb{Z}_p) = 0$$

Some Properties of \mathbb{Z}_p -Extensions

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- 2 For $F_\infty = F_{\text{cyc}}$, the extension ramifies at *every prime* $\mathfrak{p} \mid p$.
- 3 $\mathbb{Q}_{\text{cyc}}/\mathbb{Q}$ is totally ramified at p .

Iwasawa algebra

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$$\begin{aligned} \Lambda(\Gamma) &= \mathbb{Z}_p[[\Gamma]] \xrightarrow{\sim} \mathbb{Z}_p[[T]] \\ \gamma &\mapsto 1 + T \end{aligned}$$

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M is finite (equivalently has Krull dimension 0).

Structure Theorem: Iwasawa and Serre

Theorem

Let M be a finitely generated $\Lambda(\Gamma)$ -module. Then

$$M \sim \Lambda^r \oplus \left(\bigoplus_{i=1}^t \Lambda / p_i^{n_i} \right) \oplus \left(\bigoplus_{j=1}^s \Lambda / f_j^{m_j} \right)$$

where f_j are distinguished polynomials in $\mathbb{Z}_p[T]$.

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The λ -invariant is defined

$$\lambda(M) = \sum_{j=1}^s m_j \deg(f_j).$$

Theorem (Iwasawa)

Let F_∞/F be a \mathbb{Z}_p -extension and let e_n be the integer so that $p^{e_n} \parallel h_n$ where h_n is the order of the class group of F_n . There exist integers $\lambda, \mu \geq 0$ and ν such that

$$e_n = \lambda n + \mu p^n + \nu$$

for all n sufficiently large where λ, μ, ν are all independent of n .

Iwasawa's Conjecture: Setting up the Diagram

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Let F_∞/F be *any* \mathbb{Z}_p -extension.

M_n/F_n is the maximal unramified p -extension.

$$H_n = \text{Gal} \left(M_n^{ab} / F_n \right) = p - \text{Hilbert class field of } F_n$$

$$\mathcal{H} = \varprojlim_n H_n$$

$$\mathcal{L} = \varprojlim_n M_n^{ab}$$

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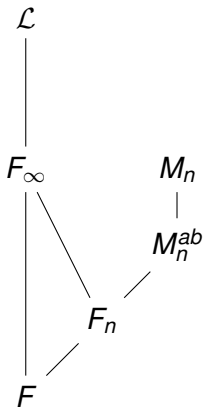
M'_n/F_n is the maximal unramified p -extension completely decomposed at all primes above p .

$$H'_n = \text{Gal} \left(M_n'^{ab} / F_n \right)$$

$$\mathcal{H}' = \varprojlim_n H'_n$$

$$\mathcal{L}' = \varprojlim_n M'_n$$

Field Diagram



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Therefore, by the Structure Theorem, one can define the μ, λ -invariants.

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Iwasawa proved that when $F = \mathbb{Q}$, $\mu = \lambda = \nu = 0$.

More generally, this holds when F/\mathbb{Q} is an Abelian extension by the work of Ferrero-Washington (1979).

Another proof was given by Sinnott (1984).

Iwasawa's Conjecture: Equivalent Formulation

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Theorem

Iwasawa $\mu = 0$ Conjecture is equivalent to the following two assertions combined:

- 1 $H^2(\text{Gal}(F_{\{p\}}/F), \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and
- 2 $H^2(\text{Gal}(F_{\{p\}}/F), \mathbb{Z}/p) = 0$.

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It is known for primes less than 163 million (2008) and in particular is known for all *regular primes*.

A generalization due to Greenberg

Conjecture (Greenberg(1971, 1976))

Let F be a totally real field and F_{cyc}/F be the cyclotomic \mathbb{Z}_p -extension. Then

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In particular, X_{nr} is finite.

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This conjecture was further generalized (2001). We will study this conjecture in the next few slides.

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This can be identified with the maximal Abelian unramified p -extension of \tilde{F} . It is a $\Lambda(\mathcal{G})$ -module where $\mathcal{G} = \text{Gal}(\tilde{F}/F) \simeq \mathbb{Z}_p^d$. Here, $d \leq r_1 + r_2 - 1$ (equality iff the Leopoldt Conjecture is true).

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Conjecture (Pseudonullity Conjecture)

\mathcal{A} is pseudonull, equivalently

$$\dim \mathcal{A} \leq d - 1.$$

IWASAWA THEORY OF ELLIPTIC CURVES

Selmer Groups of Elliptic Curves

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The *Selmer group* of E/L for a finite Galois extension L/F contained in F_S is given by the exact sequence

$$0 \rightarrow \text{Sel}(E/L) \rightarrow H^1(\text{Gal}(F_S/L), E_{p^\infty}) \xrightarrow{\lambda_L} \bigoplus_{v \in S} J_v(E_{p^\infty}/L)$$

where

$$J_v(E_{p^\infty}/L) = \bigoplus_{w|v} H^1(L_w, E)(p).$$

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FINE SELMER GROUPS

We define

$$R(E/L) := \ker \left(H^1(\mathrm{Gal}(F_S/L), E_{p^\infty}) \rightarrow \bigoplus_{v \in S} K_v^1(E_{p^\infty}/L) \right)$$

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Taking direct limits as before, define

$$R(E/F_\infty) := \varinjlim_L R(E/L)$$

where L runs over all finite extensions of F contained in F_∞ .

Some important sequences

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$$0 \rightarrow E(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa} \text{Sel}(E/L) \xrightarrow{\lambda} \text{III}(E/L)(p) \rightarrow 0$$

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$$0 \rightarrow R(E/L) \rightarrow \text{Sel}(E/L) \rightarrow \bigoplus_{w|p} E(L_w)_{p^\infty} \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

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The Pontryagin dual of a p -primary module M is defined as

$$M^\vee = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p).$$

CONJECTURES

Conjecture of Mazur

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By a deep result of Kato, this is now known in some cases such as when $F = \mathbb{Q}$ or an Abelian extension.

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*Suppose E/F is an elliptic curve with good ordinary reduction at all primes above p . Then $X(E/F_{\text{cyc}})$ is a finitely generated **torsion** $\Lambda(\Gamma)$ -module.*

By a deep result of Kato, this is now known in some cases such as when $F = \mathbb{Q}$ or an Abelian extension.

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For dual Selmer groups of elliptic curves over \mathbb{Q} , we therefore have the structure theorem. But there are lots of examples of elliptic curves with *positive* μ -invariant.

Analogue of the Weak Leopoldt Conjecture

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Let E/F be an elliptic curve and p be an odd prime. For any \mathbb{Z}_p -extension F_∞/F ,

$$H^2(F_S/F_\infty, E_{p^\infty}) = 0.$$

Equivalently, the (dual) fine Selmer group is $\Lambda(\Gamma)$ -torsion.

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Equivalently, the (dual) fine Selmer group is $\Lambda(\Gamma)$ -torsion.

The equivalence of the two statements was shown by Perrin-Riou.

Conjecture A

Conjecture (Coates-S.)

Let E be an elliptic curve over F and p be an odd prime. $Y(E/F_{\text{cyc}})$ is finitely generated as a \mathbb{Z}_p -module.

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Equivalently, the elliptic curve analogue of the weak Leopoldt Conjecture holds and $\mu(Y(E/F_{\text{cyc}})) = 0$.

This conjecture is to be viewed as an analogue of Iwasawa's $\mu = 0$ Conjecture for the case of elliptic curves.

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Conjecture (Coates-S.)

Suppose Conjecture A holds for E/F_{cyc} and G has dimension strictly larger than 1 as a p -adic Lie group, then $Y(E/F_\infty)$ is a pseudonull $\Lambda(G)$ -module.

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This echoes Greenberg's pseudonullity conjecture in the context of elliptic curves.

RECENT RESULTS

Recent Evidence towards Conjecture A

Theorem (K.-S.)

Let F be a number field and E be an elliptic curve of rank 0 over F . Assume that the Shafarevich-Tate group of E/F is finite. Varying over primes of good ordinary reduction, $\text{Sel}(E/F_{\text{cyc}})(p)$ is trivial for all primes outside a set of density 0.

In particular, Conjecture A holds for $Y(E/F_{\text{cyc}})$.

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In particular, Conjecture A holds for $Y(E/F_{\text{cyc}})$.

This result was first proven for $F = \mathbb{Q}$ by Greenberg. To extend this to the general number fields case it was necessary to use an effective Chebotarev density result of Kumar Murty.

Evidence for Conjecture B: CM case

Theorem (K.-S.)

Let E be a CM elliptic curve defined over a number field F and p be an odd prime of good ordinary reduction.

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Let $F_\infty = F(E_{p^\infty})$. In this case, $\text{Gal}(F_\infty/F)$ contains an open subgroup which is Abelian and isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Further, assume $G = \text{Gal}(F_\infty/F)$ is pro- p .

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If $Y(E/F_{\text{cyc}})$ is finite, $Y(E/F_\infty)$ is a pseudonull $\Lambda(G)$ -module.

Evidence for Conjecture B: non-CM case for regular primes

Theorem (K.-S.)

Let E be an elliptic curve defined over \mathbb{Q} . Set $F = \mathbb{Q}(\mu_p)$ such that p is a regular prime. Then Conjecture B is true for $Y(E/\mathbb{Q}(E_{p^\infty}))$.

Specific Example

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It was shown in Coates-S. (2005) that either Conjecture B holds or there is no point of infinite order over F_∞ . The latter possibility was not ruled out in the intervening years, despite advances in computational methods.

Our theorem settles this example theoretically.

Relating Greenberg's Conjecture with Conjecture B

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Relating Greenberg's Conjecture with Conjecture B

Theorem (K.-S.)

Let E/F be an elliptic curve and p be a fixed odd prime. Let $\mathcal{L} = F_\infty = F(E_{p^\infty})$ or \tilde{F} be an admissible extension of F . Then $X_{nr}^\mathcal{L}$ is pseudonull (i.e. Greenberg's Conjecture holds) if and only if Conjecture B holds for $Y(E/\mathcal{L})$.