Research Statement — Debanjana Kundu

My research lies at the intersection of number theory, algebra, arithmetic algebraic geometry and complex analysis; more specifically I study the arithmetic of elliptic curves. These are precisely the genus 1 algebraic curves with rational points, and hence land in a sweet spot for mathematical enquiry. The arithmetic of simpler, genus 0, algebraic curves has been studied for centuries. On the other hand, higher genus curves are less accessible. A major application was seen in Wiles' proof of Fermat's Last Theorem.

My primary focus is on Iwasawa theory. It is an area of number theory that emerged from the foundational work of Kenkichi Iwasawa in the 1950s. In Mazur (1972), the Iwasawa theory of Selmer groups of elliptic curves was introduced. The main goal is to study growth of arithmetic objects, like Galois modules over infinite towers of number fields. The central theme is that it is often hard to study objects like class groups or Selmer groups in isolation; but in 'nice' families, they are more amenable since their properties stabilize.

A key observation is that a part of this growth exhibits regularity which can be described in terms of values of meromorphic functions known as L-functions, such as the Riemann zeta function. Through such descriptions, Iwasawa theory unveiled intricate links between algebraic, geometric, and analytic objects of arithmetic nature. The existence of such links is a common theme in many areas within arithmetic geometry. So Iwasawa theory has found itself to be a subject of continued great interest. The literature on Iwasawa theory is often technical, but the underlying ideas possess an undeniable beauty!

Iwasawa theory is heavily used in work on the Birch Swinnerton-Dyer (BSD) conjecture, a Clay Math Millennium Problem. The Iwasawa invariants associated to elliptic curves epitomize their arithmetic and Iwasawa theoretic properties. Moreover, there is a deep relationship between the behaviour of Iwasawa invariants and the (p-adic) Birch and Swinnerton-Dyer formula. The first positive result to be proved in this direction, the Coates-Wiles theorem that analytic rank 0 implies algebraic rank 0 for elliptic curves over $\mathbb{Q}$ with complex multiplication (CM), was shown using Iwasawa theory. More generally, almost all the results on BSD that we now have (thanks to Kolyvagin, Rubin, Kato, Perrin-Riou, Kobayashi, etc.) use the machinery of Iwasawa theory.

In Coates and Sujatha (2005), the fine Selmer group was defined. It is a subgroup of the Selmer group with stronger finiteness properties. They formulated a conjecture (Conjecture A) parallel to the Classical Iwasawa $\mu = 0$ Conjecture that the fine Selmer group over the cyclotomic extension mimics precisely the growth of the class group, and is a finitely generated $\mathbb{Z}_p$-module. Selmer groups are often known to not satisfy this property. Given a rank 0 elliptic curve, I have shown that for density 1 primes, the fine Selmer group is trivial in cyclotomic extensions; thereby verifying Conjecture A. Lim and Murty (2016) showed that growth of p-ranks of fine Selmer group and ideal class groups in a such a tower is of the same order. Using these ideas, I have studied growth questions in other infinite Abelian and non-Abelian towers (see Kundu (2020b), Kundu (2020c), and Kundu and Ray (2021b)). In joint work with R. Sujatha, I have made explicit the relationship between the Generalized Greenberg's Conjecture and the Pseudonullity Conjecture of Coates-Sujatha. Using analytic techniques, I improved results of Lim-Murty and Cesnavicius (2017) on growth of fine Selmer groups in degree p-extensions (see Kundu (2020a)). Iwasawa (1981) showed that class groups of number fields are a good analogue of the p-power division points of the Jacobian variety of an algebraic curve and its λ-invariant is an analogue of the genus of the curve. Combining this with the recently understood similarity in the growth pattern of class groups and fine Selmer groups, I obtained a Riemann-Hurwitz type formula for λ-invariants of fine Selmer groups (see Kundu (2021)).

Control Theorems play a key role in Iwasawa theory. In particular, they allow us to extract information about the Selmer group and provide an invaluable approach towards the study of the Birch and Swinnerton-Dyer Conjecture. In Kundu and Lim (2021), we study Control Theorems for fine Selmer groups in *arbitrary* p-adic Lie extensions. This allows us to deduce properties of the fine Selmer group over an infinite tower from the fine Selmer groups at the finite layers (and vice versa).

More recently I have started studying questions at the intersection of arithmetic statistics and Iwasawa theory. The area of arithmetic statistics concerns the behaviour of number theoretic objects in families, and offers a probabilistic model that seeks to explain numerous phenomena in the statistical behaviour of Selmer groups. In Kundu and Ray (2021a), we show that there is promise in the analysis of the average behaviour of Iwasawa invariants. In Kundu et al. (2022), have also explored related but more intricate questions in non-commutative extensions.

Given a Diophantine equation, such as an elliptic curve $E : y^2 = x^3 + Ax + B$ where A and B are integers and $4A^3 + 27B^2 \neq 0$, a natural question to ask is whether the equation has any *integer solutions*. Is the solution-set finite or infinite? Another question of interest is whether the solution-set changes if we look for solutions in more general rings (such as integers of number fields). Using tools from Iwasawa theory and analytic number theory, in joint work with L. Beneish and A. Ray, we show that given an elliptic curve with a finite number of solutions over $\mathbb{Z}$, there are *infinitely many* number fields where the elliptic curve does not gain any new solutions. Questions on Diophantine stability are known to have applications to Hilbert's tenth problem; an area I have recently started to explore.

Questions in number theory can often be elementary to state, but challenging to solve. Success often relies on establishing relationships with ostensibly unconnected areas of mathematics. One such connection is provided by the Langlands Program, which conceptualizes a deep structural relationship between number theory and representation theory. There have been major advances made in recent years, however the Langlands Program remains largely conjectural. Though only proven in special cases, the Langlands functoriality conjectures have become a cornerstone of modern mathematics. The origins of these conjectures were in the area of automorphic forms, which lies at the intersection of number theory, representation theory, and harmonic analysis on Lie groups. The theory developed to understand and prove cases of the conjectures has become crucial to understanding these subjects, and has reached beyond them, leading to important applications in areas such as topology, algebraic geometry and mathematical physics. The trace formula is a powerful tool to study the analytic behaviour of automorphic L-functions. It is one of the main techniques in attacking Langlands' Functoriality Conjecture and related problems like the "Beyond Endoscopy" idea introduced in Langlands (2004).

The strategy of beyond endoscopy is a two-step process. The first step is to isolate via the trace formula the (packets of) cuspidal automorphic representations whose L-functions (for a representation of the dual group) have a pole at $s = 1$. The second step involves a comparison of this data for two different groups and aims to determine functorial transfers. The first step was completed in Altuğ (2015) where he worked with the group $\mathrm{GL}_2(\mathbb{Q})$. In a joint project with M. Emory, M. Espinosa Lara and T. A. Wong, we are utilizing this method to generalize the first step to $\mathrm{GL}_2(K)$ where K is a number field. This project began out of the Functoriality and the Trace Formula workshop at AIM in December 2017.

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