Introduction to Higher Coleman Theory:

Suppose F is a number fld, there are two imp objects when studying class groups — (1) $G_F = \mathrm{Gal}(\bar{F}/F)$

(2) Adeles of F, $\mathbb{A}_F$.

The Langlands Program gives a correspondence between these objects:

$\{G_F$-repns with $\mathbb{Q}_p$-coeff unram outside fin many primes, de Rham at primes$/p\}$ $\leftrightarrow$ $\{$automorphic, algebraic, cuspidal repns of $GL_n(\mathbb{A}_F)\}$

not functorial

[for $n=1$, this is class field theory]

On the LHS, there are congruence mod primes.

Ques: The study of congruences between objects on the RHS is dealt with in Coleman theory

In order to study congruences between aut repns, one realizes them in the coherent coho in Shimura var (GSp_{2g}, $g \geq 1$)

If $k = (k_1, \ldots, k_g) \in \mathbb{Z}^g$, generically the aut. forms can be found in exactly one coh. degree

Coleman Theory: The study of p-adic congruences between modular forms in H^0 of Shimura var

Higher Coleman Theory: congruences between modular forms that appear in higher cohomology gps.

(new theory!)

Lecture 1: p-adic modular forms à la Serre

Modular forms / $\mathbb{C}$: A modular form of wt $k \in \mathbb{Z}$ (and level 1) is a holomorphic funcn $f: \mathfrak{h} \to \mathbb{C}$ st

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

(ii) (holo at ∞) it admits a q-expansion
$$f(z) = \sum_{n \geq 0} a_n q^n \qquad (\text{where } q = e^{2\pi i z})$$

We identify f with its q-expansion. We say f is defined over $A \subseteq \mathbb{C}$ if $f \in A[[q]]$ (A = ring)

Define $M_k(A)$ = space of modular forms / A, weight k
$$M(A) = \bigoplus_{k \in \mathbb{Z}} M_k(A)$$

Example: [normalized] Eisenstein Series
$$E_{2k} = 1 + 2 \cdot \frac{2k}{B_{2k}} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n \in M_{2k}(\mathbb{Q}) \quad \text{if } k \geq 2$$

B_{2k}: Bernoulli number

$\sigma_{2k-1}(n) = \sum_{d \mid n} d^{2k-1}$

$$P = E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n \text{ "}\in\text{" } M_2(\mathbb{Z})$$

not true due to convergence issues but will be a p-adic mod form

$$Q = E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \in M_4(\mathbb{Z})$$
$$R = E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n \in M_6(\mathbb{Z})$$

$$\Delta = \frac{Q^3 - R^2}{1728} = q \prod_{n \geq 1} (1 - q^n)^{24} \in M_{12}(\mathbb{Z})$$

(Thm) $M(\mathbb{C}) = \mathbb{C}[Q, R] \simeq \mathbb{C}[X, Y]$ (X, Y are formal var of wt 4, 6 resp.)

(Thm') Let $k \in \mathbb{Z}_{\geq 4}$ even and let $\dim M_k(\mathbb{C}) - 1 =: d$.

Choose $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ st (i) $4\alpha + 6\beta \equiv k \pmod{12}$

(ii) $4\alpha + 6\beta \leq 14$ ($\neq 12$?)

Then, $\{ g_j = \Delta^j Q^\alpha R^{2(d-j)+\beta} : 0 \leq j \leq d \}$ is an integral basis of $M_k(\mathbb{C})$; ie $M_k(A) = \bigoplus_{0 \leq j \leq d} A \cdot g_j$, $A \subseteq \mathbb{C}$

Idea: There are the right no. of g_j's. By construction,

$$g_j = q^j + O(q^{j+1}) \in \mathbb{Z}[[q]].$$

In particular, $M(A) = A[Q, R, \Delta]$ with relations ,,

§ Algebraic Theory (Modular forms mod p)

Let p be a fixed prime; $\bar{\cdot}$ to denote reduction mod p.

Define: (1) $M_k(\mathbb{F}_p) = \{ \bar{f} \in \mathbb{F}_p[[q]] : f \in M_k(\mathbb{Z}_{(p)}) \}$

(2) $M(\mathbb{F}_p) = \sum_{k \in \mathbb{Z}} M_k(\mathbb{F}_p)$

no longer a dir sum

If $p = 2, 3$, $\bar{Q} = \bar{R} = 1 \Rightarrow$ modular forms mod p

$$M(\mathbb{F}_p) = \mathbb{F}_p[\bar{\Delta}] \simeq \mathbb{F}_p[T].$$

Assume $p \geq 5 \Rightarrow p \nmid 1728 \Rightarrow M(\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[Q, R]$

$$\simeq \mathbb{Z}_{(p)}[X, Y]$$

We have, $M(\mathbb{Z}_{(p)}) \simeq \mathbb{Z}_{(p)}[X, Y] \twoheadrightarrow \mathbb{F}_p[X, Y] \twoheadrightarrow M(\mathbb{F}_p)$

$$\phi(Q, R) \mapsto \phi(X, Y) \longmapsto \bar{\phi}(X, Y) \longmapsto \bar{\phi}(\bar{Q}, \bar{R})$$

We need to study the kernel of the last arrow

(1) Serre's differential operator $\theta = q \frac{d}{dq}$

Thm (Ramanujan) (1) $f \in M_k(\mathbb{C}) \Rightarrow (12\theta - kP) f \in M_{k+2}(\mathbb{C})$

(2) $(12\theta - P)P = -Q$; $(12\theta - 6P)R = -6Q^2$

$(12\theta - 4P)Q = -4R$; $(12\theta - 12P)\Delta = 0$

Define: $\partial \circlearrowright \mathcal{M}(\mathbb{C})$ by $\partial|_{\mathcal{M}_k(\mathbb{C})} = 12\theta - kP$

On $\mathbb{Z}_{(p)}[X,Y]$ (resp. $\mathbb{F}_p[X,Y]$), $\partial X = -4Y$

$\partial Y = -6X^2$

for $f \in \mathcal{M}_k(\mathbb{Z}_{(p)})$, write $\partial \bar{f} = \overline{\partial f} \in \mathcal{M}_{k+2}(\mathbb{F}_p)$

(2) Congruences:

Thm (1) $p-1 \mid 2k \Rightarrow pB_{2k} \in \mathbb{Z}_{(p)}$

$pB_{2k} \equiv -1 \bmod p$ (Clausen-von Staudt)

(2) $p-1 \nmid 2k \Rightarrow \frac{B_{2k}}{2k} \in \mathbb{Z}_{(p)}$; $\frac{B_{2k}}{2k} \equiv \frac{B_{2k+(p-1)}}{2k+(p-1)} \bmod p$ (Kummer)

Cor: $\bar{E}_{p-1} = 1$, $\bar{E}_{p+1} = \bar{P}$

Define: Let $A, B \in \mathbb{Z}_{(p)}[X,Y]$ st $A(Q,R) = E_{p-1}$; $B(Q,R) = E_{p+1}$

Lemma: (1) $\partial\bar{A} = \bar{B}$ and $\partial\bar{B} = -X\bar{A}$

(2) The polynomial $\bar{A}$ has no repeated factors in $\mathbb{F}_p[X,Y]$

Thm: $\mathcal{M}(\mathbb{F}_p) \simeq \mathbb{F}_p[X,Y]/(\bar{A}-1)$

Idea: $\dim \mathbb{F}_p[X,Y] = 2$ and $\mathfrak{a} = \ker\big(\mathbb{F}_p[X,Y] \to M(\mathbb{F}_p)\big)$

$\Phi(X,Y) \mapsto \Phi(\bar{Q},\bar{R})$

is prime of height $1 \supseteq (\bar{A}-1)$

By lemma, one checks that $\bar{A}-1$ is prime. //

§ p-adic modular forms

fix $p \geq 3$; for $f \in \mathbb{Q}_p[\![q]\!]$, write $v_p(f) = \inf_{n \geq 0}(v_p(a_n(f)))$

(Thm) Let $f \in M_k(\mathbb{Q})$ and $\tilde{f} \in M_{\tilde{k}}(\mathbb{Q})$. If $f \neq 0$;

$v_p(f - \tilde{f}) \geq v_p(f) + m \Rightarrow k \equiv \tilde{k} \pmod{(p-1)p^{m-1}}$

The condn $v_p(f-\tilde{f}) \geq v_p(f)+m$ says, f & $\tilde{f}$ are 'close'.

[Proof by induction]

Define: for $m \in \mathbb{Z}_{\geq 1}$, $W_m = \mathbb{Z}/_{p-1} \times \mathbb{Z}/_{p^{m-1}} \simeq (\mathbb{Z}/p^m)^\times$

$$W = \varprojlim W_m = \mathbb{Z}/_{p-1} \times \mathbb{Z}_p \simeq \mathbb{Z}_p^\times$$

(Group of p-adic weights)

One often identifies W with the characters of $\mathbb{Z}_p^\times$ ie

$$W \simeq \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$
$$k \mapsto (x \mapsto x^k)$$

Define: p-adic modular forms

$f \in \mathbb{Q}_p[\![q]\!]$ st $\exists f_i \in M_{k_i}(\mathbb{Q})$, $i \in \mathbb{Z}_{\geq 1}$ with $v_p(f-f_i) \xrightarrow[i\to\infty]{} \infty$

If $k = \lim_{i\to\infty} k_i \in W$, we say f has weight k

$M_k(\mathbb{Q}_p)$ = Space of p-adic modular forms of wt k.

Example: $p=3$, $\bar{Q}=1$

$$\frac{1}{Q} = \lim_{i\to\infty} \frac{Q^{p^i}}{Q} \in M_{-4}(\mathbb{Q}_p)$$

Thm: Let $f_i \in M_{k_i}(\mathbb{Q})$, $i \in \mathbb{Z}_{\geq 1}$. If

(i) $a_n(f_i) \xrightarrow[i\to\infty]{} a_n \in \mathbb{Q}_p$ $\quad \forall n \geq 1$

(ii) $k_i \xrightarrow[i\to\infty]{} k \in W \setminus \{0\}$.

Then, $\exists\, a_0 = \lim_{i\to\infty} a_0(f_i) \in \mathbb{Q}_p$ and $f = \sum_{n\geq 0} a_n q^n \in M_k(\mathbb{Q}_p)$

Idea: By the previous thm applied to $g \in M_k(\mathbb{Q}_p)$ and $\tilde{g} = a_0(g) \in M_0(\mathbb{Q}_p)$

[The weights are far $\Rightarrow$ modular forms are far]

Since $k \not\equiv 0 \pmod{W_{m+1}} \Rightarrow v_p(a_0(g)) + m \geq \inf_{n\geq 0}\{v_p(a_n(g))\}$

$\Rightarrow$ convergence of a_0 is forced by that of a_n's $\forall n > 0$ //

We can now construct Eisenstein Series of p-adic weights.

Example: Take $k_i \in 2\mathbb{Z}_{\geq 2}$ st $|k_i| \underset{i\to\infty}{\longrightarrow} \infty$ in $\mathbb{R}$ and $k_i \to k \in 2W$. Then, $\sigma_{k_i-1}(n) = \sum d^{k_i-1} \underset{i\to\infty}{\longrightarrow} \sum_{\substack{d|n \\ p\nmid d}} d^k = \sigma^*_{k-1}(n)$

$$\frac{B_{k_i}}{2k_i} E_{k_i} = \frac{B_{k_i}}{2k_i} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n \underset{i\to\infty}{\longrightarrow} \Big(\underbrace{\lim \frac{B_{k_i}}{2k_i}}_{\text{special value of Kubota-Leopoldt}}\Big) E^*_k$$

§ Hecke Operators

Define: Let $f = \sum_{n\geq 0} a_n(f) q^n \in \mathbb{Q}_p[[q]]$

$$f|U_p = \sum_{n\geq 0} a_{np}(f) q^n, \quad f|V_p = \sum_{n\geq 0} a_n(f) q^{np}$$

$$f|_k T_\ell = \sum_{n\geq 0} a_{n\ell}(f) q^n + \ell^{k-1} \sum_{n\geq 0} a_n(f) q^{n\ell}$$

(where ℓ is another prime & $k \in W$)

Thm: (1) The operators T_ℓ (for ℓ prime) act on $M_k(\mathbb{Z}_{(p)})$ $k\in\mathbb{Z}$.

(2) The operators, U_p, V_p, T_ℓ for $\ell \neq p$ act on $M_k(\mathbb{Q}_p)$, $k \in W$.

(3) The operators T_ℓ commute among themselves & with U_p, V_p

We are usually interested in simultaneous eigenvectors (called eigenforms). We can also consider $\theta = q\frac{d}{dq}$, which increases weight by 2.

Since $T_p \equiv U_p \bmod p$, we get $U_p \circlearrowright M(\mathbb{F}_p)$ with the foll "contracting property".

Thm: If $k > p+1$, then $M_k(\mathbb{F}_p) \xrightarrow{U_p} M_{\tilde{k}}(\mathbb{F}_p)$, $\tilde{k} < k$

Cor: for $p \geq 5$, let $[a] \in 2\mathbb{Z}/(p-1)\mathbb{Z}$ and define

$$M_{[a]}(\mathbb{F}_p) = \bigcup_{k\in[a]} M_k(\mathbb{F}_p).$$

$\exists!$ decomposition $M_{[a]}(\mathbb{F}_p) = S \oplus N$ st $U_p \circlearrowright S$ invertibly
$U_p \circlearrowright N$ nilpotently

§ Compact Operators on Banach Spaces

Let X be (an orthonormalizable) $\mathbb{Q}_p$-Banach space

$\exists (e_i)_{i\in I} \subseteq X$ st every $x \in X$ admits a unique

$x = \sum_{i\in I} x_i e_i$ st $|x| = \operatorname{Sup}\{|x_i|_p\}$, $\lim_{i\to\infty} x_i = 0$

Write $\mathcal{L}(X)$ = Cont. linear endo of X with $\|U\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{|Ux|}{|x|}$

Define: $U \in \mathcal{L}(X)$ is compact if it is the limit of operators of finite rank in $\mathcal{L}(X)$. [Write $\mathcal{C}(X)$ = compact operators on X]

Given a compact operator, we can define Fredholm determinant

$$\det(1-tU) \in \mathbb{Q}_p[\![t]\!]$$

- assume up to scaling that $\|U\| \leq 1$, so that $U \curvearrowright X_0 = \{x \in X : |x| \leq 1\}$
- for $n \geq 1$, by defn of compact, the image of $U|_{X_0/p^n}$ is $\subseteq$ in finite free $\mathbb{Z}/p^n$-module, Y_n. Then, $\exists \det(1-tU|_{Y_n}) \in (\mathbb{Z}/p^n)[t]$
- take $\varprojlim_n$

Proposition: $\det(1-tU)$ is entire

Thm (Riesz decomposition) Let $a \in \mathbb{Q}_p^\times$ be a zero of order h of $\det(1-tU)$. $\exists!$ decomposition

$X = S(a) \oplus N(a)$ st $(1-aU) \curvearrowright S(a)$ invertibly and

$(1-aU) \curvearrowright N(a)$ nilpotently.

Moreover, $\dim_{\mathbb{Q}_p}(N(a)) = h$.

Rk: $N(a) = \ker(1-aU)^h$ = U-eigenspace of eigenvalue a^{-1}.

its elts are generalized eigenvectors of slope $\alpha = -v_p(a)$

(a slope of the Newton polygon of $\det(1-tU)$)

Cor: Let $Q(t) \in \mathbb{Q}_p[t]$ irreducible and $Q(0) = 1$. $\exists!$ decomposn

$X = S(Q) \oplus N(Q)$ with $Q(U) \curvearrowright S(Q)$ inv, $Q(U) \curvearrowright N(Q)$ nilp; $\dim_{\mathbb{Q}_p}(N(Q)) < \infty$.

Fix $h \in \mathbb{R}$. There are only finitely many Q's as above with $v_p(Q) = v_p(\text{"root of } Q\text{"}) \leq h$.

Define $X^{(\leq h)} = \bigoplus_{v_p(Q) \leq h} N(Q)$

We get slope $\leq h$ decomposn, $X = X^{\leq h} \oplus X^{>h}$

fact: $M_k(\mathbb{Q}_p)$ are a $\mathbb{Q}_p$-Banach sp

[But, U_p is not compact, this leads to the next step].