1. Introduction

My primary research is on the Iwasawa theory of elliptic curves. I use tools from Galois cohomology, module theory, algebraic and analytic number theory, and arithmetic statistics to answer questions on the structure and growth of Selmer groups, fine Selmer groups, and class groups in infinite (and finite) field extensions.

Organization: In §2, basic definitions and notations are introduced. In §3, I discuss some of my results on growth questions of *fine* Selmer groups. These results indicate that the fine Selmer group "interpolates" between the class group and the Selmer group. In §4, I explain some of my results which lie at the intersection of arithmetic statistics and Iwasawa theory.

2. Basic Definitions and Notations

Let p be a fixed prime. Consider the *cyclotomic* $\mathbb{Z}_p$-extension of $\mathbb{Q}$, denoted by $\mathbb{Q}_{\mathrm{cyc}}$. Set $\Gamma := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \simeq \mathbb{Z}_p$. The *Iwasawa algebra* $\Lambda = \Lambda(\Gamma)$ is the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. Fix a topological generator γ of Γ; there is the following isomorphism of rings

$$\begin{aligned}\Lambda &\xrightarrow{\sim} \mathbb{Z}_p[\![T]\!]\\ \gamma &\mapsto 1+T.\end{aligned}$$

Let M be a cofinitely generated cotorsion Λ-module. The *Structure Theorem of* Λ*-modules* asserts that the Pontryagin dual of M, denoted by $M^\vee$, is pseudo-isomorphic to a finite direct sum of cyclic Λ-modules. In other words, there is a map of Λ-modules

$$M^\vee \longrightarrow \left(\bigoplus_{i=1}^{s} \Lambda/(p^{m_i})\right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(h_j(T))\right)$$

with finite kernel and cokernel. Here, $m_i > 0$ and $h_j(T)$ is a distinguished polynomial (i.e., a monic polynomial with non-leading coefficients divisible by p). The *characteristic ideal* of $M^\vee$ is (up to a unit) generated by the *characteristic element*,

$$f_M^{(p)}(T) := p^{\sum_i m_i} \prod_j h_j(T).$$

The μ-invariant of M is defined as the power of p in $f_M^{(p)}(T)$. More precisely,

$$\mu(M) = \mu_p(M) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=1}^{s} m_i & \text{if } s > 0. \end{cases}$$

The λ-invariant of M is the degree of the characteristic element, i.e.

$$\lambda(M) = \lambda_p(M) := \sum_{j=1}^{t} \deg h_j(T).$$

3. Research Focus I: Fine Selmer Groups

The notion of *fine Selmer group* was formally introduced by J. Coates and R. Sujatha in [4] even though it had been studied by K. Rubin [35] and B. Perrin-Riou [32, 33], under various guises in the late 80's and early 90's. This is a subgroup of the classical Selmer group obtained by imposing stronger vanishing conditions at primes above p (the precise definition is reviewed in §3.1 below). A deep result of K. Kato shows that the fine Selmer group of an elliptic curve over $\mathbb{Q}_{\mathrm{cyc}}$ is always Λ-cotorsion, regardless of whether the elliptic curve $E_{/\mathbb{Q}}$ is ordinary at p or not, a fact that is *not* true for classical Selmer groups. The fine Selmer group is a fundamental object in the study of Iwasawa theory and plays a crucial role in the reformulation of the Iwasawa Main Conjecture for elliptic curves without p-adic L-functions (see [14, Conjecture 12.10] and [41, Conjecture 7]).

3.1. **Definition of fine Selmer groups.** Suppose F is a number field. Throughout, $E_{/F}$ is a fixed elliptic curve. Fix a finite set S of primes of F containing p, the primes dividing the conductor of E, as well as the archimedean primes. Denote by F_S, the maximal algebraic extension of F unramified outside S. For every (possibly infinite) extension L/F contained in F_S, write $G_S(L) = \mathrm{Gal}\left(F_S/L\right)$. Write $S(L)$ for the set of primes of L above S. If L is a finite extension of F and w is a place of L, write L_w for its completion at w; when L/F is infinite, it is the union of completions of all finite sub-extensions of L.

Definition. Let L/F be an algebraic extension. The *p-primary fine Selmer group of E* over L is defined as

$$\mathrm{Sel}_0(E/L) = \ker\left(H^1\left(G_S\left(L\right), E[p^\infty]\right) \to \bigoplus_{v\in S} H^1\left(L_v, E[p^\infty]\right)\right).$$

Similarly, the *p-fine Selmer group of E* over L is defined as

$$\mathrm{Sel}_0(E[p]/L) = \ker\left(H^1\left(G_S\left(L\right), E[p]\right) \to \bigoplus_{v\in S} H^1\left(L_v, E[p]\right)\right).$$

Equivalently, the p-primary fine Selmer group is a subgroup of the classical Selmer group with additional vanishing conditions at $v|p$. More precisely,

$$0 \to \mathrm{Sel}_0\left(E/F\right) \to \mathrm{Sel}\left(E/F\right) \to \bigoplus_{v|p} E(F_v)\otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

It is easy to observe that if F_∞/F is an infinite extension,

$$\mathrm{Sel}_0\left(E/F_\infty\right) = \varinjlim_L \mathrm{Sel}_0\left(E/L\right), \quad \mathrm{Sel}_0\left(E[p]/F_\infty\right) = \varinjlim_L \mathrm{Sel}_0\left(E[p]/L\right),$$

where the inductive limits are taken with respect to the restriction maps and L runs over all finite extensions of F contained in F_∞.

3.2. $\mu = 0$ **Conjecture for fine Selmer groups.** In [30], B. Mazur initiated the study of Iwasawa theory of classical Selmer groups of elliptic curves. Even over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$, when the Selmer group is Λ-cotorsion (i.e., at an odd prime p of good *ordinary* reduction), he constructed examples of elliptic curves where the μ-invariant of the Selmer group is *positive*. Thus, providing evidence that class groups and Selmer groups often differ in growth behaviour in infinite extensions. On the other hand, when $E_{/F}$ is an elliptic curve with good ordinary reduction at p and the residual Galois representation $\overline{\rho}: \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(\mathbb{F}_p)$ is *irreducible*, R. Greenberg has conjectured that the associated μ-invariant of the Selmer group over F_{cyc} is trivial (see [8, Conjecture 1.11]).

Motivated by the *classical Iwasawa $\mu = 0$ conjecture* for the *cyclotomic* $\mathbb{Z}_p$-extension and the observation that the growth behaviour of the fine Selmer group parallels that of the class group [4, Lemma 3.8], Coates–Sujatha formulated the following conjecture. Henceforth, this will be referred to as *Conjecture A*.

Conjecture ([4, Conjecture A])**.** *Let p be an odd prime and $E_{/F}$ be an elliptic curve. Over F_{cyc}/F, the fine Selmer group $\mathrm{Sel}_0(E/F_{\mathrm{cyc}})$ is a cofinitely generated $\mathbb{Z}_p$-module. Equivalently, $\mathrm{Sel}_0(E/F_{\mathrm{cyc}})$ is Λ-cotorsion and the associated μ-invariant, denoted by $\mu_{\mathrm{fine}}(E/F_{\mathrm{cyc}})$, is 0.*

Much of my research has been driven by trying to understand this conjecture. Even though we are far from proving this conjecture in full generality, I have made some modest contributions towards this (see Corollary 4.3). In my doctoral thesis, I provided a large class of examples where the conjecture is true (see [19]). More precisely,

Theorem 3.1. *Let E be a rank 0 elliptic curve defined over the number field, F. Assume finiteness of the Shafarevich-Tate group. Then, for* density 1 primes *of good ordinary reduction the p-part of the Selmer group (hence p-part of the fine Selmer group) is trivial over F_{cyc}. In particular, Conjecture A holds for density 1 good ordinary primes.*

I have now extended this result to higher rank elliptic curves and primes of good supersingular reduction (see §4).

3.3. **Close relationship with class groups.** In [28], M.F. Lim and V.K. Murty showed for the first time that the class groups of number fields and fine Selmer groups are closely related in some finite extensions of number fields and in $\mathbb{Z}_p$-extensions (other than the cyclotomic one) where primes are finitely decomposed.

3.3.1. **Arbitrarily large μ-invariant.** In my doctoral thesis, I explored similar questions in $\mathbb{Z}/p\mathbb{Z}$-extensions, non p-adic analytic extensions, and other p-adic Lie extensions. A particular question I was interested in studying was whether the growth properties of fine Selmer groups mimics that of class groups even in those p-adic Lie extensions where primes *are not finitely decomposed*. K. Iwasawa had shown in [13] the existence of $\mathbb{Z}_p$-extensions of certain number fields where the classical μ-invariant can be made *arbitrarily large*. In [17], I proved that an analogous result holds for fine Selmer groups by comparing p-ranks of fine Selmer groups and class groups in towers of number fields, and observing that the μ-invariant is closely related to the p-rank. More precisely,

Theorem 3.2. *Let $F = \mathbb{Q}(\zeta_p)$ be the cyclotomic field of p-th roots of unity for $p > 2$. Let $E_{/F}$ be an elliptic curve such that $E(F)[p] \neq 0$. Given an integer $N \geq 1$, there exists a cyclic Galois extension L/F of degree p and a* non-cyclotomic *$\mathbb{Z}_p$-extension L_∞/L such that $\mu_{\mathrm{fine}}(E/L_\infty) \geq N$.*

In [18], I *developed a strategy* to show that the (generalized) μ-invariant of fine Selmer groups can be arbitrarily large in extensions where the (generalized) μ-invariant associated to the class group is arbitrarily large. Using results of [10], I provided *explicit examples* of commutative and non-commutative p-adic Lie extensions with arbitrarily large (generalized) μ-invariant of fine Selmer groups. A striking feature in *all* these examples is that there are infinitely many primes which are infinitely decomposed in these extensions. This raises the following question:

Question 3.3. Should one expect that for any $\mathbb{Z}_p$-extension, where primes are finitely decomposed the classical Iwasawa μ-invariant (i.e., associated to class groups) is 0? More generally, if F_∞/F is a (uniform) pro-p p-adic Lie extension where primes are finitely decomposed, is the (generalized) μ-invariant trivial?

3.3.2. **Anticyclotomic $\mathbb{Z}_p$-extension.** In [29, Conjecture B], A. Matar extended Conjecture A to the *anticyclotomic* $\mathbb{Z}_p$-extension, K^{ac}, of an imaginary quadratic field, K. He provided computational evidence for the same when the mod-p representation of E is *irreducible*. In contrast, when the residual representation in *reducible*, I proved the following result in [22] which again underlines the relationship between class groups and fine Selmer groups.

Theorem 3.4. *Let E be an elliptic curve defined over an imaginary quadratic field K. Assume that*

(i) $E(K)[p] \neq 0$ and
(ii) the Heegner hypothesis is satisfied.

Then the classical (anticyclotomic) Iwasawa μ-invariant, $\mu(K^{\mathrm{ac}}/K) = 0$ if and only if $\mathrm{Sel}_0(E/K^{\mathrm{ac}})$ is a cotorsion Λ-module with $\mu_{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$.

3.3.3. $p \neq q$ **Iwasawa theory.** In [39, 40], L.C. Washington proved that for distinct primes p and q, the p-part of the class number stabilizes in the *cyclotomic* $\mathbb{Z}_q$-extension of an abelian number field. These results were extended by J. Lamplugh in [25] to other $\mathbb{Z}_q$-extensions where primes are finitely decomposed. More precisely, if p, q are distinct primes ≥ 5 that split in an imaginary quadratic field K of class number 1 and F/K is a prime-to-p abelian extension which is unramified at p, then the p-class group stabilizes in the $\mathbb{Z}_q$-extension of F which is unramified outside precisely one of the primes above q. Using a theorem of H. Hida on the non-vanishing modulo p of algebraic L-functions, I have extended these results to a class of *anticyclotomic* $\mathbb{Z}_p$-extensions in joint work with A. Lei [20].

Theorem 3.5. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ coprime to p such that $\mathfrak{g}$ is a product of split primes which precisely divide the conductor of an elliptic curve over K with complex multiplication by $\mathcal{O}_K$. Let $F = \mathscr{R}(\mathfrak{g}q)$ be the ray class field of conductor $\mathfrak{g}q$ and $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F$ be the anticyclotomic $\mathbb{Z}_q$-extension. Then, there exists a positive integer N such that $\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N))$ for all $n \geq N$, where F_n is the n-th layer of the anticyclotomic $\mathbb{Z}_q$-extension.*

Even in this $p \neq q$ setting, it is possible to relate the growth of the p-part of the class group to the p-part of fine Selmer group of a fixed elliptic curve E over a $\mathbb{Z}_q$-tower. More precisely,

Theorem 3.6. *Let p and q be distinct odd primes. Let F be any number field and $E_{/F}$ be an elliptic curve such that $E(F)[p] \neq 0$. Let F_∞/F be* any *$\mathbb{Z}_q$-extension such that the primes above q and the primes of bad reduction of E are* finitely decomposed. *If there exists $N \gg 0$ such that $\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N))$ for all $n \geq N$, then $\mathrm{Sel}_0(E/F_n) = \mathrm{Sel}_0(E/F_N)$.*

In particular, the above theorem applies to the setting studied by Washington [39, 40].

3.4. **Close relationship with classical Selmer groups.** Since the fine Selmer group is a subgroup of the classical Selmer group, unsurprisingly these two arithmetic objects often show some similarity in their growth behaviour.

3.4.1. **Control Theorems.** In [30], Mazur conjectured that the *classical* Selmer group $\mathrm{Sel}(E/F_{\mathrm{cyc}})$ is Λ-cotorsion, and also provided the first theoretical evidence towards the same. Using the *Control Theorem*, he verified the conjecture when $\mathrm{Sel}(E/F)$ is finite. This condition is satisfied precisely when the Shafarevich–Tate group over F is finite and the elliptic curve $E_{/F}$ has Mordell–Weil rank 0. Till date, this conjecture is known only when E is an elliptic curve over $\mathbb{Q}$ and F is an abelian extension of $\mathbb{Q}$; see [14].

Via the *Iwasawa Main Conjecture*, the Selmer group $\mathrm{Sel}(E/F_{\mathrm{cyc}})$ can be related to a p-adic L-function. Therefore, Mazur's Control Theorem provides a channel to extract information on $\mathrm{Sel}(E/F)$ from the main conjecture thereby providing an invaluable approach towards studying the Birch and Swinnerton-Dyer Conjecture (see [14, 34, 38]). The Control Theorem connects the Selmer groups at the finite layers with that over the infinite tower, allowing one to deduce properties of this arithmetic object over the infinite tower from those at the finite layers, and vice versa.

In joint work with M.F. Lim [21], I proved Control Theorems for fine Selmer groups. More precisely,

(a) established estimates on the $\mathbb{Z}_p$-coranks of the kernel and cokernel of the restriction maps

$$r_{F_\infty/F'} : \mathrm{Sel}_0(E/F') \longrightarrow \mathrm{Sel}_0(E/F_\infty)^{\mathrm{Gal}(F_\infty/F')}$$

for a p-adic Lie extension F_∞/F with intermediate sub-fields F'/F.
(b) showed how the module theoretic structure of $\mathrm{Sel}_0(E/F_\infty)$ determines the growth of $\mathbb{Z}_p$-coranks of $\mathrm{Sel}_0(E/F')$ in intermediate sub-fields F'.
(c) obtained sharper results by specializing to three cases of p-adic Lie extensions: $\mathbb{Z}_p^d$-extensions, multi-false-Tate extensions, and the trivializing extension obtained by adjoining to F all the p-power division points of the elliptic curve, E. In each of these cases, it is possible to show (under appropriate assumptions) that the kernel and cokernel of the restriction map are *finite*, and also establish growth estimates for their orders.

3.4.2. **Fine Selmer groups and duality.** In [7], Greenberg established a criteria for when two finitely generated Λ-modules are pseudo-isomorphic. This result has been used to show that the Selmer groups of ordinary (resp. non-ordinary) representations satisfy a functional equation in [7] (resp. [15, 1, 26]). A key ingredient in all these works is that the local Selmer conditions at p are *exact annihilator of each other*. Unfortunately, this is *not true* in the case of fine Selmer groups, since the local conditions at p are trivial. In joint work with J. Hatley, A. Lei, and J. Ray [11], I investigated the link between fine Selmer groups of weight k modular forms and its dual. More precisely, let f be a weight $k(\geq 2)$ modular form and $\overline{f}$ be the conjugate modular form. Then,

(a) for an integer i, several control theorems were proven for the fine Selmer groups of $f(i)$ and $\overline{f}(k-i)$.
(b) via global duality and global Euler characteristic formulae, it could be shown that the criteria established by Greenberg can be reinterpreted in terms of growth conditions on the localization maps.
(c) under hypotheses which could be verified computationally, using the control theorems it was shown that the growth conditions on certain localization maps suffice to study the relation between the fine Selmer groups of $f(i)$ and $\overline{f}(k-i)$ over $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$. In particular, if certain naturally arising growth conditions of localization maps are satisfied, the μ-invariants of $\mathrm{Sel}_0(f(i)/\mathbb{Q}_{\mathrm{cyc}})$ and $\mathrm{Sel}_0(\overline{f}(k-i)/\mathbb{Q}_{\mathrm{cyc}})$ are equal.

4. Research Focus II: Arithmetic Statistics and Iwasawa Theory

In a series of articles (with collaborators), I am exploring questions at the intersection of arithmetic statistics and Iwasawa theory. The main goal is to understand the variation of the Iwasawa invariants as the triple (E, F, p) varies such that $E_{/F}$ has good reduction at p. More precisely, I focus on studying the following interrelated problems.

Question 4.1. (i) For a fixed elliptic curve $E_{/F}$, how do the Iwasawa invariants vary as p varies over all odd primes p at which E has good reduction?
(ii) For a fixed prime p and fixed number field F, how do the Iwasawa invariants vary as E varies over all elliptic curves (with good reduction at p)?
(iii) Fix an elliptic curve $E_{/\mathbb{Q}}$ with good reduction at p. How do the Iwasawa invariants of $E_{/F}$ vary when F varies over a family of number fields?

4.1. **Iwasawa invariants in $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$.** In [23], I started exploring questions at the intersection of arithmetic statistics and Iwasawa theory using the *Euler characteristic*. The Euler characteristic is defined as an alternating product of Galois cohomology groups. By the work of P. Schneider (see [36, 37]) and B. Perrin-Riou (see [31]), this invariant is known to be given by the *p-adic BSD formula* for primes of good ordinary reduction. Thus, it captures information about the size of the *Tate–Shafarevich group*, the *Tamagawa number*, the *anomalous primes*, and the (global) *torsion points* of the elliptic curve; thereby providing information about the Iwasawa invariants.

4.1.1. As a first step, Theorem 3.1 was extended to *higher rank* elliptic curves and to *supersingular primes.*

When E has *supersingular reduction* at p, the p-primary Selmer group, $\mathrm{Sel}(E/\mathbb{Q}_{\mathrm{cyc}})$, is *not* Λ-cotorsion. This makes the analysis of the algebraic structure of the Selmer group particularly difficult. Instead, one considers the *plus and minus* Selmer groups, denoted by $\mathrm{Sel}^{\pm}(E/\mathbb{Q}_{\mathrm{cyc}})$, which were introduced by S. Kobayashi in [16] and are known to be Λ-cotorsion. The Iwasawa invariants $\mu^{\pm}$ and $\lambda^{\pm}$ associated with the $\pm$-Selmer group are defined analogously. Even in the supersingular case, there is sufficient computational evidence suggesting that *often* the associated μ-invariants vanish. Under standard hypotheses on the Shafarevich–Tate group, it is easy to show that the λ-invariant associated to a (Λ-cotorsion) Selmer group is *always* at least as large as the Mordell–Weil rank of E. For rank 0 elliptic curves over $\mathbb{Q}$, the precise $\pm$-Euler characteristic formula (associated to $\pm$-Selmer groups) has been obtained by A. Lei and R. Sujatha in [27]. This recent result was used crucially in proving the following theorem.

Theorem 4.2. *Let $E_{/\mathbb{Q}}$ be a fixed rank 0 elliptic curve. Assume finiteness of the Shafarevich-Tate group over $\mathbb{Q}$. Then, for* all but finitely many *primes primes of good supersingular reduction* $\mathrm{Sel}^{\pm}(E/\mathbb{Q}_{\mathrm{cyc}})$ *is trivial. In particular,* $\mathrm{Sel}_0(E/\mathbb{Q}_{\mathrm{cyc}})$ *is trivial for all but finitely many supersingular primes.*

The final assertion holds because for a prime of supersingular reduction, $\mathrm{Sel}_0(E/\mathbb{Q}_{\mathrm{cyc}})$ is a subgroup of $\mathrm{Sel}^{\pm}(E/\mathbb{Q}_{\mathrm{cyc}})$. Combining Theorems 3.1 and 4.2, the next result is immediate as there are only a finite number of bad primes for E.

Corollary 4.3. *Let $E_{/\mathbb{Q}}$ be a fixed rank 0 elliptic curve. Assume finiteness of the Shafarevich-Tate group over $\mathbb{Q}$. Then, Conjecture A holds for density 1 primes.*

In the *higher rank* setting, answering the question is more difficult. This is because of the presence of the (normalized) p-adic regulator term in the Euler characteristic formula. In any case, the following theorem can be proven.

Theorem 4.4. *Let $E_{/\mathbb{Q}}$ be an elliptic curve such that its Mordell–Weil rank, $r_E \geq 1$. Then there exists an explicitly determined set of good ordinary primes such that $\mu(E/\mathbb{Q}_{\mathrm{cyc}}) = 0$ and $\lambda(E/\mathbb{Q}_{\mathrm{cyc}}) = r_E$.*

Remark 4.5. (i) Numerical data suggests that this explicitly determined set from the above theorem is a *density 1 subset of the set of good ordinary primes.* The obstruction in attaining an unconditional result is due to our limited knowledge on how often the normalized p-adic regulator is a unit.

(ii) An analogue of Theorem 4.4 in the supersingular setting can be proven analogously, provided one assumes the *conjectural* Euler characteristic formula in this setting.

In the direction of Question 4.1(ii), the following result from [23] allows distinguishing between when the λ-invariant is *exactly equal* to the Mordell–Weil rank and when it is *strictly greater* than the rank.

Theorem 4.6. *Let $p \geq 5$ be a fixed prime number. Let $\mathcal{E}(X)$ be the set of isomorphism classes of all elliptic curves over $\mathbb{Q}$ with height $\leq X$. Let $\mathcal{J}(X)$ be the subset of $\mathcal{E}(X)$ containing rank 0 elliptic curves E with good reduction at p, and $\mathcal{Z}(X)$ be a subset for which either of the following hold:*

(i) if E has good ordinary reduction at p, then $\mathrm{Sel}(E/\mathbb{Q}_{\mathrm{cyc}}) = 0$ or
(ii) if E has good supersingular reduction at p, then $\mathrm{Sel}^{\pm}(E/\mathbb{Q}^{\mathrm{cyc}}) = 0$.

Then,

$$\limsup_{X\to\infty} \frac{\#\mathcal{Z}(X)}{\#\mathcal{E}(X)} \geq \limsup_{X\to\infty} \frac{\#\mathcal{J}(X)}{\#\mathcal{E}(X)} - \epsilon(p).$$

Here, $\epsilon(p)$ is an explicitly determined positive *constant which approaches 0 (quickly) as $p \to \infty$.*

On average, the proportion of elliptic curves over $\mathbb{Z}_p$ with good reduction at p (ordered by height) is $(1 - \frac{1}{p})$, see [5]. By Goldfeld's Conjecture, it is expected that $1/2$ the elliptic curves have rank 1. Therefore, one expects that

$$\limsup_{X\to\infty} \frac{\#\mathcal{J}(X)}{\#\mathcal{E}(X)} = \frac{1}{2}\left(1 - \frac{1}{p}\right).$$

Theorem 4.6 indicates that for a *positive proportion* of elliptic curves $\mathrm{Sel}(E/\mathbb{Q}_{\mathrm{cyc}}) = 0$ the proportion approaches $1/2$ as $p \to \infty$. In [24], it was possible to refine the results and prove that given any integer n, there is an *explicit lower bound* for the density of the set of elliptic curves with good ordinary reduction at p for which $\lambda + \mu \geq n$. This lower bound depends on p (and n), is *strictly positive*, and becomes smaller as p or n become larger. More precisely,

Theorem 4.7. *Let $n > 0$ be an integer and p be an odd prime number. Assume that the Shafarevich–Tate group is finite for all elliptic curves $E_{/\mathbb{Q}}$. The set of elliptic curves $E_{/\mathbb{Q}}$ with good* ordinary *reduction at p* and *the additional property that $\mu + \lambda \geq n$, has* positive density *which can be explicitly determined.*

4.2. **Iwasawa invariants in anticyclotomic $\mathbb{Z}_p$-extensions.** In [12], the goal was to study problems raised in Question 4.1 for rank 0 elliptic curves with good *ordinary* reduction at p over the *anticyclotomic* $\mathbb{Z}_p$-extensions of an *imaginary quadratic field* in both the *definite* and the *indefinite* setting.

4.2.1. *Definite Case: Heegner hypothesis is* not *satisfied.* In this setting, the Selmer group over the anticyclotomic $\mathbb{Z}_p$-extension K^{ac}/K is Λ-cotorsion and hence, the story is somewhat similar to the cyclotomic one. In particular, it is possible to prove an Euler characteristic formula for $\mathrm{Sel}(E/K^{\mathrm{ac}})$.

In the direction of Question 4.1(i), it is possible to prove that for *non-CM elliptic curves*, the exact order of growth for the number of primes at which $\mu = 0$ is closely related to the Lang–Trotter Conjecture.

In response to Question 4.1(ii) it can be shown that for rank 0 elliptic curves the answer is primarily dependent on the variation of Shafarevich–Tate groups, which can be studied via the heuristics of C. Delaunay.

Question 4.1(iii) however is more subtle. This question is largely dependent on the divisibility by p of the order of the Shafarevich–Tate group upon base-change to $\mathbb{Q}(\sqrt{-d})$ (as $d > 0$ varies). Even though it appears difficult to provide (unconditional) theoretical results, there is computational data which suggests that "often" large values of p *do not* divide the order of the Shafarevich–Tate group.

4.2.2. *Indefinite Case: Heegner hypothesis holds.* When the Heegner hypothesis holds, the p-primary Selmer group $\mathrm{Sel}(E/K^{\mathrm{ac}})$ is *not* Λ-cotorsion. The theory in this setting is vastly different. Many of the arguments used in proving the earlier results fail, and their analogues are often false. Importantly, in this setting, there is no known formula for the Euler characteristic of $\mathrm{Sel}(E/K^{\mathrm{ac}})$. This issue can be circumvented by relating this Selmer group to an *auxiliary* Selmer group which is Λ-cotorsion and then using recent progress towards the anticyclotomic Iwasawa Main Conjectures made by A. Burungale–F. Castella–C. H. Kim [3] to obtain an Euler characteristic formula for the auxiliary Selmer group. It appears that answering Questions 4.1(i)-(iii) systematically in the indefinite setting is deeply intertwined with the theory of the *BDP p-adic L-function* and is currently out of reach. However, it was possible to provide some partial answers and supplement the results with computational data.

4.3. **Diophantine stability.** Questions pertaining to rank growth are of much interest to arithmetic geometers. In [2], the aim was to study questions pertaining to Diophantine stability using tools and techniques from Iwasawa theory. To answer questions on rank jump of elliptic curves upon base change, the natural thing to do was to study the growth of a more tractable arithmetic object, i.e., the p-primary Selmer groups upon base change. In particular, the following two questions were investigated using a Kida-type formula for λ-invariants (proven in [9]).

Question 4.8.
(i) Given an elliptic curve $E_{/\mathbb{Q}}$ with trivial p-primary Selmer group, for what proportion of degree-p cyclic extensions does the p-primary Selmer group remain trivial upon base-change.
(ii) Given $p \neq 2, 3$, for how many elliptic curves over $\mathbb{Q}$ does there exist *at least one* $\mathbb{Z}/p\mathbb{Z}$-extension where the p-primary Selmer group remains trivial upon base-change.

The answer to the first question is the following.

Theorem 4.9. *Given an elliptic curve $E_{/\mathbb{Q}}$ and a prime $p \geq 7$ with $\mu(E/\mathbb{Q}_{\mathrm{cyc}}) = \lambda(E/\mathbb{Q}_{\mathrm{cyc}}) = 0$, there are infinitely many $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathbb{Q}$ where the λ-invariant does not increase; in particular, the rank does not jump. Moreover, there are infinitely many $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathbb{Q}$ where the Mordell–Weil group does not grow.*

The assertion on rank growth follows from the fact that the Mordell–Weil rank is *at most* as large as the λ-invariant. The final assertion is an immediate consequence of a recent result of [6] on torsion growth. The proof shows that the λ-invariant does not jump in *many* $\mathbb{Z}/p\mathbb{Z}$-extensions. Unfortunately, the method falls short of proving a positive proportion as predicted by a conjecture of C. David–J. Fearnley–H. Kisilevsky.

A natural follow-up question is when does the p-primary Selmer group grow upon base-change. After establishing a criterion for either the rank to jump, or the order of the Shafarevich–Tate group to increase upon base-change, the next result is proven by exploiting the relationship between λ-invariants and the Euler characteristic formula.

Theorem 4.10. *Let $p \geq 5$ be a fixed prime and $E_{/\mathbb{Q}}$ be an elliptic curve with good ordinary reduction at p. Suppose that the image of the residual representation is surjective,* $\mathrm{Sel}(E/\mathbb{Q})$ *is trivial, and* $\mu(E/\mathbb{Q}_{\mathrm{cyc}}) = \lambda(E/\mathbb{Q}_{\mathrm{cyc}}) = 0$*. Then, there is a set of primes of the form $q \equiv 1 \pmod{p}$ with density at least $\frac{p}{(p-1)^2(p+1)}$ such that the p-primary Selmer group* becomes non-trivial *in the unique $\mathbb{Z}/p\mathbb{Z}$-extension contained in $\mathbb{Q}(\mu_q)$.*

The following theorem answers the second question.

Theorem 4.11. *For a* positive proportion *of rank 0 elliptic curves defined over $\mathbb{Q}$, there exists* at least one *$\mathbb{Z}/p\mathbb{Z}$-extension over $\mathbb{Q}$ disjoint from $\mathbb{Q}_{\mathrm{cyc}}$, such that the p-primary Selmer group upon base-change is trivial.*

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