STRUCTURE OF FINE SELMER GROUPS IN p-ADIC LIE EXTENSIONS

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Abstract. In this paper, the goal is to prove results for fine Selmer groups in Abelian p-adic Lie extensions. A class of examples of elliptic curves are provided where both the Selmer group and the fine Selmer group are trivial in the cyclotomic $\mathbb{Z}_p$-extension of a given number field. The relationship between Conjecture B for CM elliptic curves and the Generalized Greenberg's Conjecture is clarified. Evidence for Conjecture B in previously unknown cases is provided.

1. Introduction

The fine Selmer group of elliptic curves is a module over Iwasawa algebras that is of interest in the arithmetic of elliptic curves. It plays a key role in the formulation of the main conjecture in Iwasawa theory. Coates and the second named author proposed two conjectures, viz. Conjectures A and B [10], on the structure of the fine Selmer groups. While Conjecture A is viewed as a generalization of the classical Iwasawa $\mu = 0$ conjecture to the context of the motive associated to an elliptic curve; Conjecture B is in the spirit of generalising Greenberg's pseudonullity conjecture to elliptic curves. Recently, there has been a renewed interest in studying pseudonull modules over Iwasawa algebras [6]. It is thus natural to investigate Conjecture B, and this article makes progress in that direction. These conjectures have been generalised to fine Selmer groups of ordinary Galois representations associated to modular forms [23]. This article restricts attention to the fine Selmer groups of elliptic curves, with good reduction at a prime p, over Abelian p-adic Lie extensions of the base field. The methods should extend to the broader class of ordinary Galois representations.

Here is a brief outline of the main results in the paper. Let E be an elliptic curve over a number field, F, with good reduction at all the primes of F that lie above an odd prime number, p. Consider an admissible p-adic Lie extension $\mathcal{L}$ of F (see §2 for the precise definition) with Galois group $\mathrm{Gal}(\mathcal{L}/F) =: G_{\mathcal{L}}$. The dual fine Selmer group of E at a prime p over $\mathcal{L}$ is a finitely generated module over the associated Iwasawa algebra (see §2). While Conjecture A asserts that the dual fine Selmer group over the cyclotomic $\mathbb{Z}_p$-extension, F_{cyc}, is finitely generated as a $\mathbb{Z}_p$-module, Conjecture B is an assertion on the structure of the dual fine Selmer group over special

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p-adic Lie extensions that contain the cyclotomic $\mathbb{Z}_p$-extension. It is conjectured that the dual fine Selmer group of E over an admissible p-adic Lie extension is pseudonull as a module over the associated Iwasawa algebra. Here, both conjectures are established for a large class of elliptic curves.

Using a result of Greenberg, a general theorem that gives sufficient conditions for the dual fine Selmer group of E over the cyclotomic $\mathbb{Z}_p$-extension to be finite, is proven. It is shown that these conditions are satisfied for a large class of elliptic curves. Conjecture B can be viewed as an elliptic curve analogue of a conjecture of Greenberg. It is therefore pertinent to investigate the connections between Conjecture B for admissible, (Abelian) p-adic Lie extensions, and Greenberg's conjecture. For CM elliptic curves over an imaginary quadratic field K, the Generalized Greenberg's Conjecture (GGC) is shown to be equivalent to Conjecture B for certain pro-p, p-adic Lie extensions. Consider an elliptic curve with complex multiplication by the ring of integers of E. Conjecture B over the trivializing extension of the associated p-adic Galois representation is then shown to imply GGC for K. This recovers a known result of McCallum [34] (see also [13]). Finally, we prove that Conjecture B holds for special classes of admissible p-adic Lie extensions whenever the dual fine Selmer group over the cyclotomic extension is finite for a CM elliptic curve.

The paper consists of five sections including this introduction. §2 is preliminary in nature; wherein the main objects of study are introduced along with the precise assertions of Conjecture A, Conjecture B, and Generalised Greenberg's Conjecture (GGC). In §3, new evidence for Conjecture A is provided by proving the triviality of the fine Selmer group (as well as the Selmer group) over the cyclotomic extension, for a large class of elliptic curves. In §4, evidence is provided for Conjecture B in the CM case. Also, the relation between Conjecture B for CM elliptic curves and GGC is clarified. Finally, §5 studies the implications of Conjecture B in the context of the Iwasawa Main conjecture and Jannsen's conjecture.

2. Preliminaries

Throughout this article, p will denote an odd prime number, and F will denote an algebraic number field. For a p-adic analytic, pro-p group, G with no elements of order p, the Iwasawa algebra over $\mathbb{Z}_p$, denoted by $\Lambda(G)$, is a left and right Noetherian local ring without zero-divisors [38, Chapter V] (see also [12]). Assume that G is uniform, in the sense of Lazard [30]. This is not a restrictive assumption as every p-adic analytic group contains an open normal subgroup which is uniform. The Iwasawa algebra, $\Lambda(G)$, then comes equipped with a filtration whose associated graded ring, $\mathrm{gr}(\Lambda(G))$, is a commutative Noetherian domain, and is an Auslander regular local domain [52]. All modules over the Iwasawa algebra will be considered as compact left modules, unless otherwise mentioned. The property of being Auslander regular, in particular, affords a dimension theory for finitely generated modules over $\Lambda(G)$. Given a finitely generated $\Lambda(G)$-module M, its dimension is denoted $\dim(M)$. There is a filtered to graded functor gr mapping a finitely generated $\Lambda(G)$-module M to $\mathrm{gr}(M)$, which is a finitely generated module over $\mathrm{gr}(\Lambda(G))$. This functor preserves the dimension, so that $\dim(M)$ is equal to the Krull dimension of $\mathrm{gr}(M)$. If d is the dimension of G considered as a p-adic analytic manifold, then the dimension of $\Lambda(G)$ is $d+1$, and $\mathrm{gr}(\Lambda(G))$ is a polynomial ring in $d+1$ variables over the finite field, $\mathbb{F}_p$ [52].

2.1. A finitely generated $\Lambda(G)$-module M is **torsion** (resp. **pseudonull**) if $\dim(M) \leqslant \dim\big(\Lambda(G)\big) - 1$ (resp. $\dim(M) \leqslant \dim\big(\Lambda(G)\big) - 2$). Equivalently, a finitely generated torsion $\Lambda(G)$-module M is torsion (resp. pseudonull) if

$$E^i_G(M) := \mathrm{Ext}^i_{\Lambda(G)}\big(M, \Lambda(G)\big) = 0 \qquad \text{for } i = 0 \text{ (resp. } i = 0,\ 1). \tag{1}$$

Let M be an Abelian group and n be a positive integer. Write M_{p^n} for the subgroup of elements of M annihilated by p^n. Put

$$M_{p^\infty} := \bigcup_{n \geqslant 1} M_{p^n}, \quad T_p(M) := \varprojlim M_{p^n}.$$

For a discrete p-primary (resp. compact pro-p) Abelian group M, its Pontryagin dual defined as

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p),$$

is a compact (resp. discrete) module over the Iwasawa algebra. Let G be a profinite group and W (resp. M) be a discrete (resp. compact) G-module. The profinite cohomology groups (resp. homology groups) of W (resp. M) play an important role in the study of modules over Iwasawa algebras and are denoted $H^i(G, W)$ (resp. $H_i(G, M)$). The subgroup of elements of W fixed by G is denoted W^G, and M_G is the largest quotient of M on which G acts trivially. These are respectively the zero-th cohomology group $H^0(G, W)$ and homology group $H_0(G, W)$. Consider the field extension L/K with profinite Galois group $G = \mathrm{Gal}(L/K)$. Let W (resp. M) be a discrete (resp. compact) G-module. The cohomology groups (resp. homology groups) are then denoted simply by $H^i(L/K, W)$ (resp. $H_i(L/K, M)$). When G is the absolute Galois group of a field K, the cohomology groups (resp. homology groups) are denoted $H^i(K, W)$ (resp. $H_i(K, M)$).

2.2. Let E be an elliptic curve defined over a number field, F. Let F_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F and S be a finite set of primes of F containing the primes above p, the primes of bad reduction of E, and the Archimedean primes. In short, we write $S \supseteq S_p \cup S_{bad} \cup S_\infty$, where the notation S_p, S_{bad}, and S_∞ are self-explanatory. Let F_S be the maximal extension of F unramified outside S and set $G_S(F) = \mathrm{Gal}(F_S/F)$. Let $F_S(p)$ be its maximal pro-p quotient. For any (finite or infinite) extension $\mathcal{L}/F$ contained in F_S, let $G_S(\mathcal{L})$ denote the Galois group $\mathrm{Gal}(F_S/\mathcal{L})$. Throughout the paper, the focus is on (S-)admissible p-adic Lie extensions $\mathcal{L}/F$.

Definition. *An (S-)**admissible** p-adic Lie extension is a Galois extension, $\mathcal{L}/F$, satisfying the following conditions:*

- *The group $G_{\mathcal{L}} = \mathrm{Gal}\big(\mathcal{L}/F\big)$ is a pro-p, p-adic Lie group with no elements of order p.*
- *The field $\mathcal{L}$ contains the cyclotomic $\mathbb{Z}_p$-extension, F_{cyc}.*
- *The field $\mathcal{L}$ is contained in F_S.*

It follows from the Weil pairing that $F_\infty := \bigcup_{n\geqslant 1} F(E_{p^n})$, contains F_{cyc} and is contained in F_S. The Galois group $G_{F_\infty} = \mathrm{Gal}(F_\infty/F)$, has no p-torsion if $p \geqslant 5$ and contains an open, normal, pro-p subgroup. In fact, the extension $F_\infty/F(E_p)$ is always pro-p and hence an admissible extension. Recall that if E has no complex multiplication, then G_{F_∞} is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$; whereas in the CM case, G_{F_∞} contains an open subgroup which is Abelian and isomorphic to $\mathbb{Z}_p^2$.

2.3. Consider the compact, pro-p, p-adic Lie group $G_{\mathcal{L}} = \mathrm{Gal}(\mathcal{L}/F)$, with associated Iwasawa algebra $\Lambda(G_{\mathcal{L}})$. The main interest will be in the modules over $\Lambda(G_{\mathcal{L}})$ that arise in Iwasawa theory, such as the Selmer group and the fine Selmer group. For any finite extension L/F, define

$$J_v(L) = \bigoplus_{w|v} H^1\left(L_w, E\right)(p), \qquad \text{and} \qquad K_v(L) = \bigoplus_{w|v} H^1\left(L_w, E_{p^\infty}\right)$$

where the sum is taken over all primes w of L lying above v. Taking direct limits, then define

$$J_v(\mathcal{L}) = \varinjlim_L J_v(L), \qquad \text{and} \qquad K_v(\mathcal{L}) = \varinjlim_L K_v(L)$$

where L varies over finite sub-extensions of $\mathcal{L}/F$. Over F, the p-**primary Selmer group** and p-**primary fine Selmer group** are defined as follows

$$\begin{aligned} 0 \longrightarrow \quad & \mathrm{Sel}\left(E/F\right) \longrightarrow \quad H^1\left(G_S(F), E_{p^\infty}\right) \longrightarrow \quad \bigoplus_{v\in S} J_v(F),\\ 0 \longrightarrow \quad & R\left(E/F\right) \longrightarrow \quad H^1\left(G_S\left(F\right), E_{p^\infty}\right) \longrightarrow \quad \bigoplus_{v\in S} K_v(F). \end{aligned}$$

Given an admissible p-adic Lie extension $\mathcal{L}$, define $\mathrm{Sel}(E/\mathcal{L}) = \varinjlim_L \mathrm{Sel}(E/L)$ and $R(E/\mathcal{L}) = \varinjlim_L R(E/L)$. It can then be shown that

$$\mathrm{Sel}\left(E/\mathcal{L}\right) \cong \ker\left(H^1\left(G_S\left(\mathcal{L}\right), E_{p^\infty}\right) \longrightarrow \bigoplus_{v\in S} J_v(\mathcal{L})\right),$$

$$R\left(E/\mathcal{L}\right) \cong \ker\left(H^1\left(G_S\left(\mathcal{L}\right), E_{p^\infty}\right) \longrightarrow \bigoplus_{v\in S} K_v(\mathcal{L})\right).$$

For a p-adic Lie extension $\mathcal{L}$, the dual Selmer (resp. dual fine Selmer)group is written as $\mathfrak{X}(E/\mathcal{L})$ (resp. $\mathfrak{Y}(E/\mathcal{L})$). They are compact $\Lambda(G_{\mathcal{L}})$-modules.

2.4. Let T denote any finitely generated $\mathbb{Z}_p$-module, endowed with a continuous action of $G_S(F)$. Consider the i-th **Iwasawa cohomology modules**

$$\mathcal{Z}_S^i\left(T/\mathcal{L}\right) = \varprojlim_L H^i\left(G_S(L), T\right), \text{ for } i = 0,\ 1,\ 2, \tag{2}$$

where L ranges over all finite extensions of F contained in $\mathcal{L}$ and the projective limit is taken with respect to the corestriction maps. By [10, Lemma 2.1], $\mathcal{Z}_S^0(T/\mathcal{L}) = 0$. In this article, we consider $T = \mathbb{Z}_p(1) = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ or $T = T_p(E) = \varprojlim_n E_{p^n}$. It is known that $\mathcal{Z}_S^2\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is $\Lambda(G_{\mathcal{L}})$-torsion. On the other hand, $\mathcal{Z}_S^2\left(T_p(E)/\mathcal{L}\right) = \mathcal{Z}_S^2(E/\mathcal{L})$ is $\Lambda(G_{\mathcal{L}})$-torsion if and only if $H^2\left(G_S(\mathcal{L}), E_{p^\infty}\right) = 0$ (see [10, Lemma 3.1]).

For $i \geqslant 0$ and $T = \mathbb{Z}_p(1)$, choose S to be a finite set of places of F containing the primes above p. Let $O_L[1/S]$ be the subring of L consisting of all elements which are integral at any finite place of L not lying over S, and H^i_{et} be the étale cohomology. An equivalent definition is the following (see [25, §2.2 p. 552])

$$\mathcal{Z}^i_S\left(\mathbb{Z}_p(1)/\mathcal{L}\right) = \varprojlim_L H^i_{et}\left(O_L[1/S], \mathbb{Z}_p(1)\right) \tag{3}$$

where L ranges over all finite extensions of F contained in $\mathcal{L}$. The dual fine Selmer group of $\mathbb{Z}_p(1)$ (with respect to the set S) is defined as follows (see [25, §2.4])

$$\mathfrak{Y}_S(\mathbb{Z}_p(1)/\mathcal{L}) = \varprojlim_L \operatorname{Pic}\left(O_L[1/S]\right)_{p^\infty}.$$

There is another well-known isomorphism (see [9, (1.2)])

$$\mathfrak{Y}_S(\mathbb{Z}_p(1)/\mathcal{L}) \simeq \operatorname{Gal}\left(M\left(\mathcal{L}\right)/\mathcal{L}\right), \tag{4}$$

where $M(\mathcal{L})$ is the maximal Abelian, pro-p unramified extension of $\mathcal{L}$ with the primes above p splitting completely. Since the dual fine Selmer group is independent of the choice of S, it is dropped from the notation.

2.5. For $\mathcal{L} = F_{\mathrm{cyc}}$, consider the dual fine Selmer group associated to an elliptic curve, $\mathfrak{Y}(E/F_{\mathrm{cyc}})$. Conjecture A is the following assertion.

Conjecture A (cf. [10])**.** Let E be an elliptic curve defined over F and p be an odd prime. Then $\mathfrak{Y}(E/F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module.

This conjecture is closely related to the classical Iwasawa $\mu = 0$ conjecture.

Theorem ([10, Theorem 3.4])**.** *Let E be an elliptic curve defined over a number field, F. Suppose that $F(E_{p^\infty})/F$ is pro-p. Then Conjecture A for E over F_{cyc} is equivalent to the classical Iwasawa $\mu = 0$ conjecture.*

The dimension theory for modules over Auslander regular rings provides another equivalent definition for pseudonull modules as described below (see [54, Proposition 5.4]).

Definition. *Let F be a number field and $\mathcal{L}/F$ be an S-admissible p-adic Lie extension. Set $H_{\mathcal{L}} = \operatorname{Gal}\left(\mathcal{L}/F_{\mathrm{cyc}}\right)$. A finitely generated $\Lambda(G_{\mathcal{L}})$-module which is also finitely generated as a $\Lambda(H_{\mathcal{L}})$-module, is in fact a torsion $\Lambda(G_{\mathcal{L}})$-module. A $\Lambda(G_{\mathcal{L}})$-module is pseudonull if and only if it is $\Lambda(H_{\mathcal{L}})$-torsion.*

Conjecture B is inspired by the following conjecture of Greenberg which is referred to as GGC.

Generalized Greenberg's Conjecture ([18, Conjecture 3.5])**.** Let F be any number field and p be any prime. Let $\widetilde{F}$ denote the compositum of all $\mathbb{Z}_p$-extensions of F and $\widetilde{L}$ denote the pro-p Hilbert class field of $\widetilde{F}$. Then $\widetilde{X} = \operatorname{Gal}\left(\widetilde{L}/\widetilde{F}\right)$ is a pseudonull module over the ring $\widetilde{\Lambda} = \mathbb{Z}_p[\![\operatorname{Gal}(\widetilde{F}/F)]\!]$.

For evidence towards GGC (theoretical and/or computational) see [35, 34, 41, 13, 48, 14, 50]. Conjecture B concerns the phenomenon of certain arithmetic Iwasawa modules for p-adic Lie extensions of dimension at least 2 being much smaller than intuitively expected.

Conjecture B (cf [10]). Let $E_{/F}$ be an elliptic curve such that $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ is finitely generated as a $\mathbb{Z}_p$-module. Let $\mathcal{L}/F$ be an admissible p-adic Lie extension and $G_{\mathcal{L}}$ be a pro-p p-adic Lie group of dimension strictly greater than 1. Then $\mathfrak{Y}\left(E/\mathcal{L}\right)$ is a pseudonull $\Lambda(G_{\mathcal{L}})$-module.

2.6. The notion of p-rational number fields was introduced in [39]. We refer the reader to [15, Theorem IV.3.5] for a detailed discussion.

Definition. *Let F be a number field, p be an odd prime, and S_p be the set of primes above p. Let F_{S_p} be the maximal p-ramified extension of F and $F_{S_p}(p)$ be its maximal pro-p quotient. Set $\mathcal{G}_{S_p}(F) = \mathrm{Gal}\left(F_{S_p}(p)/F\right)$. If $\mathcal{G}_{S_p}(F)$ is free pro-p, then F is called a* ***p-rational number field****.*

Some examples of p-rational fields include:

(i) the field of rational numbers, $\mathbb{Q}$.
(ii) imaginary quadratic fields such that p does not divide the class number.
(iii) Abelian fields $\mathbb{Q}(\mu_{p^n})$, where p is a regular prime (i.e. p does not divide the class number of $\mathbb{Q}(\mu_p)$) and $n \geqslant 1$.
(iv) number fields F containing μ_p with the property that $\#S_p(F) = 1$ and p does not divide the class number of F.

In [19], p-rational number fields have been studied by Greenberg, wherein he explains heuristic reasons to believe that given a number field F, it *should be* p-rational for all primes outside a set of density 0. In [2], Barbulescu and Ray provide examples of *non-Abelian* p-rational number fields.

3. Fine Selmer Groups in the Cyclotomic Extension

The results in this section provide evidence for Conjecture A. First, we prove that for density one ordinary primes, the Selmer group is trivial over the cyclotomic $\mathbb{Z}_p$-extension for rank 0 elliptic curves with good ordinary reduction at primes above p. Next, we provide evidence for Conjecture A for a class of elliptic curves defined over p-rational number fields.

3.1. **Trivial Fine Selmer Groups in the Cyclotomic Tower.** Fix a number field F and an odd prime p. Let $\Gamma = \mathrm{Gal}\left(F_{\mathrm{cyc}}/F\right)$, where F_{cyc} is the cyclotomic $\mathbb{Z}_p$-extension of F. Let E be an elliptic curve over F with good ordinary reduction at all primes of F that lie above p. If the p-primary Selmer group of an elliptic curve $E_{/F}$, denoted by $\mathrm{Sel}(E/F)$ is finite, it follows from Mazur's Control Theorem, that the cyclotomic Selmer group $\mathrm{Sel}\left(E/F_{\mathrm{cyc}}\right)$ is $\Lambda(\Gamma)$-cotorsion [17, Corollary 4.9]. This happens for example, if $E_{/F}$ is a rank 0 elliptic curve.

Let $f_E(T)$ be the characteristic polynomial generating the characteristic ideal of $\mathfrak{X}\left(E/F_{\mathrm{cyc}}\right)$. If $\mathrm{Sel}(E/F)$ is finite, then $f_E(0) \neq 0$ (see [16]).

At a prime v, the reduction of E modulo v is denoted $\widetilde{E_v}$; it is a curve over the residue field, f_v. Recall the notion of an anomalous prime as defined in [32, §1(b)]. A prime v is called *anomalous* if p divides $\left|\widetilde{E}_v(f_v)\right|$. Let c_v be the Tamagawa number and $c_v^{(p)}$ be the highest power of p dividing it.

Theorem 3.1 extends results of Greenberg [16, Proposition 5.1] and Wuthrich [56, §9] to base fields other than $\mathbb{Q}$, and provides evidence for Conjecture A for a large class of elliptic curves over a general number field.

Theorem 3.1. *Let F be a number field and $E_{/F}$ be a rank 0 elliptic curve. Assume that the Shafarevich-Tate group of $E_{/F}$ is finite. Then, the Selmer group,* $\mathrm{Sel}\left(E/F_{\mathrm{cyc}}\right)$*, is trivial for all primes of good ordinary reduction outside a set of density zero. In particular, Conjecture A holds for $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ for all primes of good ordinary reduction outside a set of density zero.*

Proof. With the setting as in the theorem, it is known from [16, §4] that

$$f_E(0) \sim \left(\prod_{v\ bad} c_v^{(p)}\right)\left(\prod_{v\mid p}\left|\widetilde{E}_v(f_v)_p\right|^2\right)\left|\mathrm{Sel}(E/F)\right| \Big/ \left|E(F)_p\right|^2 \tag{5}$$

where $a \sim b$ for $a,\ b \in \mathbb{Q}_p^\times$ means that a, b have the same p-adic valuation. Consider an elliptic curve E with the following four properties:

(i) $E_{/F}$ is a rank 0 elliptic curve with no p-torsion points over F.
(ii) E has good ordinary non-anomalous reduction at primes above p.
(iii) the p-primary part of the Shafarevich-Tate group, $\text{Ш}(E/F)_{p^\infty}$ is trivial.
(iv) p does not divide the Tamagawa number c_v, where v is a bad prime.

It follows from (5) that for such an elliptic curve, $f_E(0)$ is a unit.

If E is a rank 0 elliptic curve with $E(F)_p = 0$, then $\mathrm{Sel}(E/F) = \text{Ш}(E/F)_{p^\infty}$. When $f_E(0)$ is a unit, we know that $\mathfrak{X}\left(E/F_{\mathrm{cyc}}\right)$ (and hence $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$) is finite. Equivalently, both $\mathrm{Sel}\left(E/F_{\mathrm{cyc}}\right)$ and $R\left(E/F_{\mathrm{cyc}}\right)$ are finite. When $E(F)_p = 0$, it is further known that $\mathfrak{X}\left(E/F_{\mathrm{cyc}}\right)$ has no non-trivial finite Λ-submodules (see [20]). Thus, $\mathfrak{X}\left(E/F_{\mathrm{cyc}}\right)$ must be trivial, if it is finite. Since $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ must also be trivial, we conclude that Conjecture A holds for $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ when $E_{/F}$ is an elliptic curve satisfying (i)-(iv).

To complete the proof, it remains to show that given a number field F, there exist elliptic curves $E_{/F}$ with properties (i)-(iv). Given a rank zero elliptic curve, it now suffices to show that the remaining properties are also satisfied. For a given elliptic curve $E_{/F}$, by a result of Merel, it is known that for all but finitely many primes $E(F)_p = 0$. Further, since the primes of bad reduction are finite, the curve has good reduction at primes above a rational prime p, for all but finitely many p. A Chebotarev density argument, for density 1 ordinary primes, then yields that E has non-anomalous reduction at p (see [36]). Assuming the finiteness of the Shafarevich-Tate group, it is plain that condition (iii) holds away from a finite set of primes. Finally, observe that the Tamagawa number is finite and bounded at the places of bad reduction. Thus, given an elliptic curve, condition (iv) holds away from a finite set of primes. This completes the proof of the theorem. ∎

Remark 3.2. (1) It is shown in [33, Corollary 1.11] that given any number field F there exist 'many' rank zero elliptic curves over F. Therefore, the theorem is not empty.

(2) For a fixed elliptic curve over F, it is possible that $f_E(0)$ is a unit when $E(F)_p$ is non-trivial; for example if there is anomalous reduction at the prime p. It is also possible that such elliptic curves have $\mathrm{Sel}\left(E/F_{\mathrm{cyc}}\right) = 0$ (see [8, Theorem 3.11 and Remark 3.12(v)]). In [8, Chapter 5], it is shown that elliptic curves with these properties exist. Therefore, there are some elliptic curves that are excluded by our theorem. This raises the following natural question.

Question. Let E be a rank 0 elliptic curve defined over F. Is $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ trivial for all but finitely many primes?

An analogous statement can be proven in the supersingular case as well.

Theorem 3.3. *Let $E_{/F}$ be a rank 0 elliptic curve. Assume that the Shafarevich-Tate group of $E_{/F}$ is finite. Then, Conjecture A holds for $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ for all but finitely many primes of good supersingular reduction.*

Proof. When $E_{/F}$ is an elliptic curve and p is a prime of supersingular reduction, it is well-known that the Selmer group is *not* $\Lambda(\Gamma)$-cotorsion. However, there is a notion of $\pm$ Selmer groups in this setting, denoted by $\mathrm{Sel}^{\pm}\left(E/F_{\mathrm{cyc}}\right)$. These are well-known to be $\Lambda(\Gamma)$-cotorsion, see [28]. Therefore, the Structure Theorem holds and one can define the characteristic polynomial $f_E^{\pm}(T)$. Moreover, the fine Selmer group is a subgroup of the $\pm$ Selmer groups. Therefore, to prove the theorem it suffices to show that $\mathrm{Sel}^{\pm}\left(E/F_{\mathrm{cyc}}\right)$, is trivial for all primes of good supersingular reduction.

In [31, Theorem 5.15], it is shown that for rank 0 elliptic curves over F,

$$f_E^{\pm}(0) \sim |\,\mathrm{Sel}^{\pm}(E/F)| \prod_{v \text{ bad}} c_v^{(p)}.$$

Just as before, to show that $\mathrm{Sel}^{\pm}\left(E/F_{\mathrm{cyc}}\right)$ is trivial we need to show that $f_E^{\pm}(T) \sim 1$. Since $\mathrm{Sel}^{\pm}(E/F)$ is finite, there are only finitely many primes p which can divide this quantity. Also, there are finitely many primes of bad reduction of E. The Tamagawa number is finite and bounded at the places of bad reduction. Therefore, for all but finitely many primes $p \nmid c_v$. □

Combining Theorems 3.1 and 3.3, the next result is immediate.

Theorem 3.4. *Let F be a number field and $E_{/F}$ be a rank 0 elliptic curve. Assume that the Shafarevich-Tate group of $E_{/F}$ is finite. Then, Conjecture A holds for $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ for all primes outside a set of density zero.*

Let us now turn to the class of p-rational number fields (see §2.6).

3.2. **Conjecture A over p-Rational Number Fields.** Let F be a number field, and S be a finite set of primes of F *containing* the primes above p and the Archimedean primes. The weak Leopoldt conjecture for F is the assertion

$$H^2\left(\mathrm{Gal}\left(F_S/F_{\mathrm{cyc}}\right), \mathbb{Q}_p/\mathbb{Z}_p\right) = 0. \tag{6}$$

It holds for the cyclotomic extension of a number field (see [38, Theorem 10.3.25]). If (6) holds for a finite set S as mentioned above, it also holds for the set $S = \Sigma = S_p \cup S_\infty$ (see [38, Theorem 11.3.2]). Therefore, the weak

Leopoldt Conjecture is independent of the choice of S, as long as S contains Σ. Now, fix $S = \Sigma$. An equivalent formulation of the classical Iwasawa $\mu = 0$ conjecture for F is the assertion that $\mathcal{G}_\Sigma\left(F_{\mathrm{cyc}}\right) = \mathrm{Gal}\left(F_\Sigma(p)/F_{\mathrm{cyc}}\right)$ is a free pro-p group (see [38, Theorem 11.3.7]). Moreover, a pro-p group G is free if and only if its p-cohomological dimension $\mathrm{cd}_p(G) \leqslant 1$ (see [47, p. 23], [38, Corollary 3.5.17]). Combining these with a standard fact in Galois cohomology of pro-p groups (see [47, Chapter I, §4, Proposition 21]), one obtains the following equivalent formulation: the classical Iwasawa $\mu = 0$ conjecture holds for F if and only if

$$H^2\left(\mathcal{G}_\Sigma\left(F_{\mathrm{cyc}}\right), \mathbb{Z}/p\mathbb{Z}\right) = 0. \tag{7}$$

The interested reader is referred to [38, Theorem 11.3.7] for a broader discussion. The following result is easily deduced from these results, and a proof is included for the sake of completeness.

Theorem 3.5. *(1) Let F be a p-rational number field. The classical Iwasawa $\mu = 0$ conjecture holds for F.*

(2) Let F be a p-rational number field containing μ_p. Suppose that E is an elliptic curve over F such that $E(F)_p \neq 0$. Then Conjecture A holds, equivalently $\mathfrak{Y}(E/F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module.

Proof. (1) Since $p \neq 2$, we replace S_p by Σ in the definition of p-rational fields. By definition, if F is p-rational, $\mathcal{G}_\Sigma(F) := \mathrm{Gal}\left(F_\Sigma(p)/F\right)$ has p-cohomological dimension at most 1; hence

$$H^2\left(\mathcal{G}_\Sigma(F), \mathbb{Z}/p\mathbb{Z}\right) = 0.$$

Since $\mathcal{G}_\Sigma(F_{\mathrm{cyc}}) = \mathrm{Gal}(F_\Sigma(p)/F_{\mathrm{cyc}})$ is a closed normal subgroup of $\mathcal{G}_\Sigma(F)$, we have

$$\mathrm{cd}_p\left(\mathcal{G}_\Sigma(F_{\mathrm{cyc}})\right) \leqslant \mathrm{cd}_p\left(\mathcal{G}_\Sigma(F)\right) \leqslant 1.$$

Thus $H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), \mathbb{Z}/p\mathbb{Z}) = 0$, and the result follows from (7).

(2) Since $F \supseteq \mu_p$ and $E(F)_p \neq 0$ by assumption, the Weil pairing ensures that $F(E_p)/F$ is either trivial or of degree p. Thus, $F(E_{p^\infty})/F$ is pro-p. The theorem follows from part (1) and [10, Theorem 3.4]. □

4. Pseudonullity Conjectures for Elliptic Curves with Complex Multiplication

In this section, we provide evidence for Conjecture B for elliptic curves with complex multiplication when p is a prime of good ordinary reduction. Next, we clarify the relationship between the Generalized Greenberg's Conjecture and Conjecture B for CM elliptic curves. Both these conjectures pertain to the pseudonullity of certain Iwasawa modules. Even though Conjecture B was proposed as a generalization of GGC, the precise formulation of this relationship is rather intricate and uses the Powerful Diagram introduced by Jannsen [22] and later also studied by Ochi-Venjakob [40].

4.1. **Finite Fine Selmer Group at the Cyclotomic Level.** Let E be a CM elliptic curve over a number field F and p be an odd prime of good ordinary reduction. Writing $F_\infty = F(E_{p^\infty})$, the Galois group $\mathrm{Gal}(F_\infty/F)$ contains an open subgroup which is Abelian and isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Assume that $G = \mathrm{Gal}(F_\infty/F)$ is pro-p, and set $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$.

Let $\mathcal{G}$ be a p-adic Lie group, and M be a finitely generated $\Lambda(\mathcal{G})$-module. Then, the module $M_{\mathcal{G}} := H_0(\mathcal{G}, M)$ is a finitely generated $\mathbb{Z}_p$-module.

Lemma 4.1. *With the setting as above, the following natural map is a pseudo-isomorphism, i.e. it has a finite kernel and cokernel,*

$$\mathfrak{Y}\left(E/F_\infty\right)_H \to \mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right).$$

Proof. Let L be a finite extension of F contained in F_S. For each $v \in S$, write $W_v(L) = \bigoplus_{w|v} E(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. We have the maps

$$r_{\mathrm{cyc}} : \mathrm{Sel}(E/F_{\mathrm{cyc}}) \longrightarrow \bigoplus_{v|p} W_v(F_{\mathrm{cyc}}),$$
$$r_\infty : \mathrm{Sel}(E/F_\infty) \longrightarrow \bigoplus_{v|p} W_v(F_\infty)$$

where $W_v(F_{\mathrm{cyc}})$ (resp. $W_v(F_\infty)$) is the inductive limit of $W_v(L)$ with respect to the restriction map as L ranges over all finite extensions of F contained in F_{cyc} (resp. F_∞). Write $C(F_{\mathrm{cyc}})$ (resp. $C(F_\infty)$) for the image of r_{cyc} (resp. r_∞). Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & R(E/F_{\mathrm{cyc}})_p \arrow[r] \arrow[d, "\alpha"] & \mathrm{Sel}(E/F_{\mathrm{cyc}})_p \arrow[r] \arrow[d, "\beta"] & C(F_{\mathrm{cyc}})_p \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & R(E/F_\infty)_p^H \arrow[r] & \mathrm{Sel}(E/F_\infty)_p^H \arrow[r] & C(F_\infty)_p^H &
\end{tikzcd}$$

Note that β is an isomorphism (see [42, Lemma 1.1(i) and Lemma 1.3]). Therefore $\ker(\beta)$ and $\mathrm{coker}(\beta)$ are trivial; hence $\ker(\alpha) = 0$. By Shapiro's Lemma, $\ker(\gamma) = \bigoplus_{v|p} H^1\left(H_v, E(F_{\infty,v})_{p^\infty}\right)$, which is known to be finite (see [11]). Therefore, by the snake lemma, $\mathrm{coker}(\alpha)$ must be finite. $\square$

Since E is an elliptic curve with complex multiplication, both G and H are Abelian. In particular, $\Lambda(H) \simeq \mathbb{Z}_p[\![T]\!]$.

Lemma 4.2. *Let M be a finitely generated $\Lambda(H)$-module. If M_H is finite, then M is a pseudonull $\Lambda(G)$-module.*

Proof. If M_H is finite, the higher homology groups, $H_i(H, M)$, are trivial for all $i > 0$ [46, Chapter IV, Appendix II]. Since $\Lambda(H)$ is a regular local ring, the rank of a module is equal to its homological rank (see [21]). Therefore, $\Lambda(H)$-rank of M is 0 and M is $\Lambda(H)$-torsion. The lemma follows from the definition of pseudonullity (see §2.5). $\square$

The main theorem of this section is stated below.

Theorem 4.3. *With the set up as above, if $\mathfrak{Y}(E/F_{\mathrm{cyc}})$ is finite, then $\mathfrak{Y}(E/F_\infty)$ is a pseudonull $\Lambda(G)$-module.*

Proof. By Lemma 4.1, if $\mathfrak{Y}(E/F_{\mathrm{cyc}})$ is finite, then $\mathfrak{Y}\left(E/F_\infty\right)_H$ is finite. The theorem now follows from Lemma 4.2. □

We record another class of examples where it is easy to show that Conjecture B holds. In fact more is true; the dual fine Selmer group is *trivial* over the trivializing extension.

Theorem 4.4. *Let E be an elliptic curve over a number field F which has complex multiplication by an order in an imaginary quadratic field. Suppose that $p > 3$ is a regular prime which is also a prime of good ordinary reduction for E. Further suppose that F_∞/F is a pro-p extension with Galois group, G. Then, $\mathfrak{Y}(E/F_\infty)$ is the trivial group. In particular, Conjecture B holds.*

Proof. By Poitou–Tate duality (see for example [10, Equation 43]),

$$\mathfrak{Y}\left(E/F_\infty\right) \hookrightarrow \mathcal{Z}^2_S\left(T_pE/F_\infty\right).$$

To prove the theorem, it suffices to show that $\mathcal{Z}^2(T_p(E)/F_\infty)$ is trivial. By Nekovar's spectral sequence (see [37, Proposition 8.4.8.3])

$$\mathcal{Z}^2_S\left(T_p(E)/F_\infty\right)_G \simeq \mathcal{Z}^2_S\left(T_p(E)/F\right) = H^2\left(G_S(F), T_p(E)\right).$$

When p is a regular prime of good ordinary reduction, it is proven in [55, Corollary 2] that $H^2\left(G_S(F), T_p(E)\right) = 0$. By Nakayama's Lemma, it is immediate that $\mathcal{Z}^2_S\left(T_p(E)/F_\infty\right) = 0$. This completes the proof. □

Remark 4.5.
(1) It is possible to generalize Theorem 4.3 using results in §4.2 to prove that under the same assumptions, $\mathfrak{Y}\left(E/\mathcal{L}\right)$ is a pseudonull $\Lambda\left(G_{\mathcal{L}}\right)$-module where $\mathcal{L}/F$ is an Abelian S-admissible p-adic Lie extension *containing* F_∞.
(2) It is necessary to point out that for a fixed prime p, Theorem 3.1 does not guarantee that $\mathfrak{Y}\left(E/F_{\mathrm{cyc}}\right)$ is trivial (or finite) for an elliptic curve $E_{/F}$ with CM. This is because, in the proof of Theorem 3.1 it was required that the elliptic curve *does not* contain any points of p-torsion over F. However, in proving Theorem 4.3, it is assumed that F_∞/F is a pro-p extension; hence F might contain p-torsion points.

4.2. Conjecture B and the Generalized Greenberg's Conjecture.

We begin by proving a general result (see Theorem 4.6) which holds for all elliptic curves, not just those with complex multiplication. Using this result, it is possible to prove that Conjecture B for CM elliptic curves is a generalization of GGC (see Theorem 4.10).

Let $E_{/F}$ be an elliptic curve, and $\mathcal{L}$ be an S-admissible p-adic Lie extension containing the trivializing extension, F_∞. Since $G_S(\mathcal{L})$ acts trivially on E_{p^∞}, Conjecture B for $\mathfrak{Y}(E/\mathcal{L})$ has an equivalent formulation in terms of pseudonullity of a Galois extension of $\mathcal{L}$ (see [10, p. 827]). Recall the following isomorphism from §2.4,

$$\mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \simeq \mathrm{Gal}\left(M\left(\mathcal{L}\right)/\mathcal{L}\right).$$

The following statement is the pseudonullity conjecture for the Tate motive, $\mathbb{Z}_p(1)$, over a p-adic Lie extension containing the trivializing extension.

Conjecture. *Let $E_{/F}$ be an elliptic curve, and $\mathcal{L}$ be an S-admissible, p-adic Lie extension over F such that $G_S(\mathcal{L})$ acts trivially on E_{p^∞}. Then $\mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is a pseudonull $\Lambda(G_{\mathcal{L}})$-module.*

Let $L(\mathcal{L})/\mathcal{L}$ be the maximal Abelian unramified pro-p extension of $\mathcal{L}$. Set $\mathrm{Gal}\left(L(\mathcal{L})/\mathcal{L}\right) = X_{\mathrm{nr}}^{\mathcal{L}}$. We prove that the for an S-admissible p-adic Lie extension $\mathcal{L}/F$ containing F_∞, pseudonullity of the Iwasawa module $X_{\mathrm{nr}}^{\mathcal{L}}$ is equivalent to the pseudonullity of a quotient module.

Theorem 4.6. *Let $\mathcal{L}$ be an S-admissible p-adic Lie extension containing F_∞. The following statements are equivalent*

(1) $X_{\mathrm{nr}}^{\mathcal{L}}$ is pseudonull.
(2) Conjecture B holds for $\mathfrak{Y}(E/\mathcal{L})$.
(3) $\mathfrak{Y}(\mathbb{Z}_p(1)/\mathcal{L})$ is pseudonull.

Further, if p is the only ramified prime in $\mathcal{L}$ then the following statement is equivalent to the above ones.

(4) $\mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is pseudonull.

Proof. When $\mathcal{L}$ contains the trivializing extension, the equivalence of statements (2) and (3) is discussed in [9, p. 328]. We briefly sketch a proof here. Recall that (see discussion in [10, p. 825])

$$\mathfrak{Y}\left(E/F_\infty\right) \simeq \mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right) \otimes T_p(E)^\dagger.$$

Here $G_{\mathcal{L}} = \mathrm{Gal}(\mathcal{L}/F)$ acts diagonally on the tensor product, and $T_p(E)^\dagger$ is $T_p(E)$ as a $\mathbb{Z}_p$-module with a $G_{\mathcal{L}}$-action induced by the $G_S(F)$-action (where this latter action makes sense as F_∞ is the trivialising extension of E_{p^∞}). Therefore, $\mathfrak{Y}\left(E/F_\infty\right)$ is pseudonull if and only if $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right)$ is pseudonull (see [53, Proposition 2.12]).

We now prove the equivalence of statements (1) and (3). Let $X_S^{\mathcal{L}}$ denote the Galois group of the maximal Abelian unramified outside S pro-p extension over $\mathcal{L}$. In our setting, pseudonullity of $X_{\mathrm{nr}}^{\mathcal{L}}$ is equivalent to $X_S^{\mathcal{L}}$ being torsion-free (see [53, Theorem 4.9] and also [53, p. 32])). Therefore, to prove the theorem it is enough to show that

$$X_S^{\mathcal{L}} \text{ is torsion-free } \Leftrightarrow \mathfrak{Y}(\mathbb{Z}_p(1)/\mathcal{L}) \text{ is pseudonull.} \tag{8}$$

It is known that (see [53, §4.1.1]),

$$X_S^{\mathcal{L}} = H^1\left(G_S(\mathcal{L}), \mathbb{Q}_p/\mathbb{Z}_p\right)^\vee \simeq \mathrm{Gal}\left(F_S/\mathcal{L}\right)^{ab}_{p^\infty}.$$

The following exact sequence is well-known and comes from the rightmost column of the Powerful Diagram (see [40, Lemma 4.5])

$$0 \to X_S^{\mathcal{L}} \to Y_S^{\mathcal{L}} \to J^{\mathcal{L}} \to 0.$$

The terms appearing in the above short exact sequence are defined as follows. Let $\mathcal{G}$ be the maximal pro-p part of $\mathrm{Gal}(F_S/F)$ and $\mathcal{H}$ be the maximal pro-p part of $\mathrm{Gal}(F_S/\mathcal{L})$. Set $I(\mathcal{G})$ to denote the augmentation ideal. Define

$$Y_S^{\mathcal{L}} = \left(I\left(\mathcal{G}\right) \otimes \mathbb{Z}_p(1)\right)_{\mathcal{H}} \qquad \textit{and} \qquad J^{\mathcal{L}} = \ker\left(\left(\Lambda\left(\mathcal{G}\right) \otimes \mathbb{Z}_p(1)\right)_{\mathcal{H}} \to (A^\vee)_{\mathcal{H}}\right).$$

Since $J^{\mathcal{L}}$ has no non-zero torsion submodules, it follows that

$$X_S^{\mathcal{L}} \text{ is torsion-free} \Leftrightarrow Y_S^{\mathcal{L}} \text{ is torsion-free.} \tag{9}$$

By the Poitou–Tate sequence, pseudonullity of the dual fine Selmer groups, $\mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$, is equivalent to the pseudonullity of the Iwasawa cohomology module $\mathcal{Z}_S^2(\mathbb{Z}_p(1)/\mathcal{L})$. By (8) and (9), the proof is complete if

$$Y_S^{\mathcal{L}} \text{ is torsion-free} \Leftrightarrow \mathcal{Z}_S^2\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \text{ is pseudonull.}$$

This is precisely [53, Proposition 2.16].

Finally, we prove the equivalence of (3) and (4) for p-adic Lie extensions containing F_∞ where the only ramified prime is p. It is known that $\mathfrak{Y}_S\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \simeq \mathfrak{Y}_{S_p}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is always true [25, §2.5 p. 554]. Under the additional assumption, it follows from the Poitou–Tate sequence that

$$\mathfrak{Y}_{S_p}\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \hookrightarrow \mathcal{Z}_{S_p}^2\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \to \bigoplus_{v\in S_p} \mathbb{Z}_p \to \mathbb{Z}_p \to 0.$$

Thus, $\mathfrak{Y}_{S_p}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is pseudonull if and only if $\mathcal{Z}_{S_p}^2\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is pseudonull. □

Remark 4.7. Suppose that F contains μ_p and $\widetilde{F}$ is the compositum of all $\mathbb{Z}_p$-extensions of F. Then, GGC (see §2.5) is equivalent to $X_S^{\widetilde{F}}$ being torsion-free (see, for example [53, p. 32]). It follows from the proof of (1) $\Leftrightarrow$ (3) in the above theorem that GGC is equivalent to the pseudonullity of $\mathfrak{Y}(\mathbb{Z}_p(1)/\widetilde{F})$.

From here on, we consider elliptic curves with complex multiplication. Let K be an imaginary quadratic field and $\mathcal{O}_K$ be its ring of integers. Let $E_{/K}$ be an elliptic curve with CM by $\mathcal{O}_K$. Set

$$F = K(E_p), \quad F_\infty = K(E_{p^\infty}), \quad G = G_{F_\infty} = \mathrm{Gal}(F_\infty/F), \quad \mathcal{G}_\infty = \mathrm{Gal}(F_\infty/K).$$

Note that $G \simeq \mathbb{Z}_p^2$. Set $\widetilde{K}$ (resp. $\widetilde{F}$) to be the compositum of all $\mathbb{Z}_p$-extensions of K (resp. F). Since the Leopoldt conjecture is (trivially) true for (imaginary) quadratic fields, $\widetilde{K}$ is the unique $\mathbb{Z}_p^2$ Galois extension of K. For the rest of this section, we make the following assumption.

Hypothesis. *The prime $p \geqslant 3$ is unramified in K.*

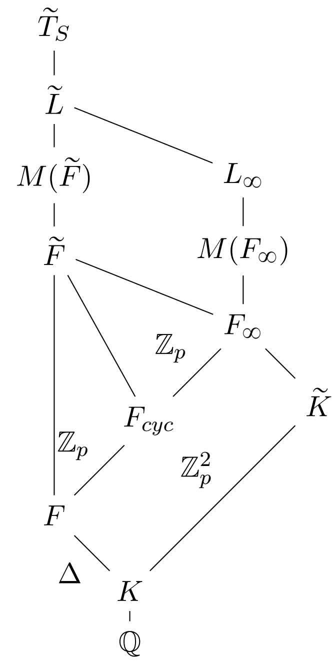
By the theory of complex multiplication, $\mathcal{G}_\infty = G \times \Delta$ where $\Delta \simeq \mathrm{Gal}(F/K)$ is a finite Abelian group. Since p does not ramify in K, we know $p \nmid |\Delta|$.

The theory of complex multiplication also guarantees that $F_\infty = F\widetilde{K}$. Thus, the trivializing extension F_∞ contains F_{cyc} and is an admissible p-adic Lie extension. Note that $F_\infty \subseteq \widetilde{F}$.

Denote by $\widetilde{L}$ (resp. L_∞) the maximal Abelian unramified pro-p-extension of $\widetilde{F}$ (resp. F_∞). Denote by $\widetilde{T}_S$ the maximal Abelian extension of $\widetilde{F}$ unramified outside S. Set the notation

$$\widetilde{X} = X_{\mathrm{nr}}^{\widetilde{F}} = \mathrm{Gal}(\widetilde{L}/\widetilde{F}), \quad X_{\mathrm{nr}}^{F_\infty} = \mathrm{Gal}(L_\infty/F_\infty), \quad X_S^{\widetilde{F}} = \mathrm{Gal}(\widetilde{T}_S/\widetilde{F}). \tag{10}$$

For convenience, the field diagram of the set up is drawn below.



For a p-adic Lie extension $\mathcal{L}/F$, write $G_{\mathcal{L}} = \mathrm{Gal}(\mathcal{L}/F)$. For most parts of the discussion, $\mathcal{L} = F_\infty$ or $\widetilde{F}$. Let $M(\mathcal{L})$ be the maximal unramified Abelian p-extension of $\mathcal{L}$ such that all primes above p in $\mathcal{L}$ split completely. Recall the statement of the Generalized Greenberg's Conjecture for $\widetilde{F}/F$.

Conjecture (GGC for $\widetilde{F}/F$)**.** *With notation as above, $X_{\mathrm{nr}}^{\widetilde{F}}$ is a pseudonull $\Lambda(G_{\widetilde{F}})$-module.*

The following results will be required to relate this conjecture to the pseudonullity of the fine Selmer group. The first lemma assures pseudonullity over a larger tower, once it holds for a non-trivial quotient.

Lemma 4.8 (Pseudonullity Lifting Lemma)**.** *Let $n \geqslant 3$, and $\mathcal{F}/\mathbb{Q}$ be a finite Galois extension containing μ_p. Let $\widetilde{\mathcal{F}}$ be the compositum of all $\mathbb{Z}_p$-extensions of $\mathcal{F}$. Let $\mathrm{Gal}(\widetilde{\mathcal{F}}/\mathcal{F}) \simeq \mathbb{Z}_p^n$ and $\mathcal{F}' \subset \widetilde{\mathcal{F}}$ be such that $\mathrm{Gal}(\mathcal{F}'/\mathcal{F}) \simeq \mathbb{Z}_p^d$ for some $2 \leqslant d < n$. If $X_{\mathrm{nr}}^{\mathcal{F}'}$ is pseudonull then GGC holds for $\widetilde{\mathcal{F}}/\mathcal{F}$.*

Proof. This lemma is a special case of [1, Theorem 12]. Since $\mathcal{F}$ contains μ_p, the technical conditions in the mentioned theorem are satisfied by [29, Theorem 3.2] or [1, Remark 15]. $\square$

The next result studies pseudonullity of Galois modules under base change.

Lemma 4.9 (Pseudonullity Shifting Down Lemma)**.** *Let $\mathcal{F}$ be a number field and $\mathcal{F}'/\mathcal{F}$ be a $\mathbb{Z}_p^d$-extension. Suppose $\mathcal{F}_1/\mathcal{F}$ is a finite extension and $\mathcal{K} = \mathcal{F}' \cdot \mathcal{F}_1$. If $X_{\mathrm{nr}}^{\mathcal{K}}$ is a pseudonull module, then $X_{\mathrm{nr}}^{\mathcal{F}'}$ is a pseudonull module.*

Proof. For a proof, see [27, Theorem 3.1(1)]. $\square$

We now prove that Conjecture B is indeed a generalization of GGC.

Theorem 4.10. *If there exists one CM elliptic curve $E_{/K}$, such that $\mathfrak{Y}\left(E/K\left(E_{p^\infty}\right)\right)$ is pseudonull, then GGC holds for K, $K(\mu_p)$, and $K(E_p)$.*

Proof. Let $E_{/K}$ be an elliptic curve with complex multiplication. It is known that E has good reduction everywhere over $F = K(E_p)$ (see [44, Lemma 2]). Choose S to be precisely the set of Archimedean primes and the primes above p. Suppose that Conjecture B holds for $\mathfrak{Y}\left(E/K\left(E_{p^\infty}\right)\right)$; equivalently, Conjecture B holds for the fine Selmer group associated with the Tate motive, $\mathfrak{Y}\left(\mathbb{Z}_p(1)/K\left(E_{p^\infty}\right)\right)$ (see Theorem 4.6).

We first show that GGC holds for K. Let $\widetilde{K}$ denote the compositum of all $\mathbb{Z}_p$-extensions of K. Since K is an (imaginary) quadratic extension, $\mathrm{Gal}(\widetilde{K}/K) \simeq \mathbb{Z}_p^2$ and $\widetilde{K}$ is the unique $\mathbb{Z}_p^2$-extension of K. By the theory of complex multiplication, $F_\infty = K(E_{p^\infty}) = F\widetilde{K}$. By hypothesis, $\mathfrak{Y}(\mathbb{Z}_p(1)/F_\infty)$ is pseudonull; therefore an application of Lemma 4.9 with $\mathcal{K} = F_\infty = F\widetilde{K}$, shows that pseudonullity of $X_{\mathrm{nr}}^{F_\infty}$ can be *shifted down* to pseudonullity of $X_{\mathrm{nr}}^{\widetilde{K}}$. This is GGC for the imaginary quadratic field K.

Next, we show that GGC holds for $K(E_p)$. By Theorem 4.6, the fine Selmer group $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right)$ is pseudonull if and only if $X_{\mathrm{nr}}^{F_\infty}$ is pseudonull. Using Lemma 4.8 with $\mathcal{F} = F = K(E_p)$ and $\mathcal{F}' = F_\infty$, pseudonullity of $X_{\mathrm{nr}}^{F_\infty}$ can be *lifted* to $X_{\mathrm{nr}}^{\widetilde{F}}$ where $\widetilde{F} \supset F_\infty$. Thus, GGC holds for $\widetilde{F}/F$.

Finally, we show that GGC also holds for $K(\mu_p)$. It is well known that $K \subseteq K(\mu_p) \subseteq K(E_p)$. There is a $\mathbb{Z}_p^2$- extension of $K(\mu_p)$, say K'_∞, such that

$$K'_\infty = K(\mu_p)\widetilde{K} \quad \text{and} \quad F_\infty = FK'_\infty.$$

Applying Lemma 4.9 with $\mathcal{K} = F_\infty = FK'_\infty$, pseudonullity of $X_{\mathrm{nr}}^{F_\infty}$ can be *shifted down* to that of $X_{\mathrm{nr}}^{K'_\infty}$. Next by Lemma 4.8, pseudonullity can be *lifted* to the compositum of $\mathbb{Z}_p$-extensions of $K(\mu_p)$, i.e. GGC holds for $K(\mu_p)$. $\square$

Remark 4.11.
(1) Even though there is growing evidence towards GGC when K is an imaginary quadratic field, the authors are not aware of results which can be used to prove GGC for finite (or even Abelian) extensions of K in such cases. Thus, Theorem 4.10 provides evidence towards GGC in previously unknown cases.
(2) Imitating the proof of Theorem 4.10 it can be shown that if there exists one CM elliptic curve $E_{/\mathbb{Q}}$ such that Conjecture B holds for $\mathfrak{Y}\left(E/\mathbb{Q}\left(E_{p^\infty}\right)\right)$, then GGC holds for $\mathbb{Q}(\mu_p)$. Recall that GGC is trivially true for $\mathbb{Q}(\mu_p)$ when p is a regular prime. When p is irregular, there is limited evidence towards this conjecture [34, 13, 48].

We end this section with the observation mentioned in Remark 4.5(1).

Theorem 4.12. *Let E be an elliptic curve with CM, and p be a prime of good ordinary reduction such that $\mathfrak{Y}(E/F_{\mathrm{cyc}})$ be finite. Let $\mathcal{L}/F$ be an Abelian, S-admissible p-adic Lie extension containing the trivializing extension, F_∞. Then Conjecture B holds for $\mathfrak{Y}(E/\mathcal{L})$.*

Proof. In this setting, $\mathfrak{Y}(E/F_\infty)$ is a pseudonull $\Lambda(G)$-module by Theorem 4.3. Applying Theorem 4.6, the Galois group $X_{\mathrm{nr}}^{F_\infty}$ is pseudonull. By the Pseudonullity Lifting Lemma, $X_{\mathrm{nr}}^{\mathcal{L}}$ is pseudonull for any multi-$\mathbb{Z}_p$ extension, $\mathcal{L}/F$, which contains the trivializing extension. □

5. Applications

In this section, we study the implications of Conjecture B to the four term exact sequence studied in proving the Main Conjecture.

5.1. **The (Cyclotomic) Main Conjecture.** For simplicity, assume that the base field is $\mathbb{Q}$. The cyclotomic Main Conjecture for the p-primary Selmer group, $\mathrm{Sel}(E/\mathbb{Q}_{\mathrm{cyc}})$, is the following statement.

Conjecture. *Let $E_{/\mathbb{Q}}$ be an elliptic curve with good ordinary or multiplicative reduction at p. The Pontryagin dual of the Selmer group, denoted $\mathfrak{X}(E/\mathbb{Q}_{\mathrm{cyc}})$, is a torsion $\Lambda(\Gamma)$-module. Furthermore, its characteristic ideal is generated by a p-adic L-function $\mathcal{L}_p\left(E/\mathbb{Q}_{\mathrm{cyc}}\right)$ in $\Lambda(\Gamma)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. If E_p is an irreducible* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-representation, then $\mathcal{L}_p\left(E/\mathbb{Q}_{\mathrm{cyc}}\right)$ is in* $\Lambda(\Gamma)$.

As a consequence of global duality, we have the following short exact sequence of torsion $\Lambda(\Gamma)$-modules

(11)
$$0 \to \frac{\mathcal{Z}_S^1\left(E/\mathbb{Q}_{\mathrm{cyc}}\right)}{\langle Z\rangle} \to \frac{\mathcal{Z}_{S,p}^1\left(E/\mathbb{Q}_{\mathrm{cyc},p}\right)}{\langle Z_p\rangle} \to \mathfrak{X}\left(E/\mathbb{Q}_{\mathrm{cyc}}\right) \to \mathfrak{Y}\left(E/\mathbb{Q}_{\mathrm{cyc}}\right) \to 0.$$

Here, $\mathcal{Z}_S^1\left(E/\mathbb{Q}_{\mathrm{cyc}}\right)$ is the compact Iwasawa cohomology group as defined in (2) and $\mathcal{Z}_{S,p}^1\left(E/\mathbb{Q}_{\mathrm{cyc},p}\right)$ is the the local Iwasawa cohomology group which is defined analogously. Here, Z is the zeta element constructed inside $\mathcal{Z}_S^1\left(E/\mathbb{Q}_{\mathrm{cyc}}\right)$ (see [24]) and $\langle Z\rangle$ is the $\Lambda(\Gamma)$-submodule generated by this element. Under the natural functorial map, the image of $\langle Z\rangle$ generates a submodule of the local Iwasawa cohomology group, denoted by $\langle Z_p\rangle$.

The Coleman isomorphism yields the following identification

$$\mathrm{Col}\colon \mathcal{Z}_{S,p}^1\left(E/\mathbb{Q}_{\mathrm{cyc},p}\right) \simeq \Lambda(\Gamma).$$

Kato showed that $\mathrm{Col}(Z_p)$ is precisely the p-adic L-function appearing in the statement of the Main Conjecture.

The philosophy in proving the Main Conjecture is to show that the first and the last term of (11) have the same characteristic power series. Since the characteristic power series is multiplicative in exact sequences, the second and the third term must have the same characteristic power series as well. This is leads to an equivalent formulation of the Main Conjecture.

In a large number of examples considered in the earlier sections, $\mathfrak{X}(E/\mathbb{Q}_{\mathrm{cyc}})$ was trivial. This yields the isomorphism

$$\frac{\mathcal{Z}_S^1\left(E/\mathbb{Q}_{\mathrm{cyc}}\right)}{\langle Z\rangle} \simeq \frac{\mathcal{Z}_{S,p}^1\left(E/\mathbb{Q}_{\mathrm{cyc},p}\right)}{\langle Z_p\rangle}.$$

If the Main Conjecture holds, then the triviality of the dual fine Selmer group implies that the first term is pseudonull (equivalently finite in a $\mathbb{Z}_p$-extension). Note that the second term in (11) is of projective dimension

1, since $\mathcal{Z}^1_{S,p}\left(E/\mathbb{Q}_{\mathrm{cyc},p}\right)$ is a free $\Lambda(\Gamma)$-module and $\langle Z_p\rangle$ is a principal ideal. Therefore, this term has no non-zero finite submodules, whence the first term must be trivial. This shows that if the dual Selmer group is trivial over the cyclotomic extension, all terms in (11) are trivial. In particular, the Euler system generates $\mathcal{Z}^1_S(E/\mathbb{Q}_{\mathrm{cyc}})$.

More generally, consider a commutative p-adic Lie extension $\mathcal{L}/\mathbb{Q}$ of dimension at least 2. Let $\mathcal{L}_{(p)}$ be the localization of $\mathcal{L}$ at p and the other terms be defined as before. We obtain a short exact sequence similar to (11)

$$0 \to \frac{\mathcal{Z}^1_S(E/\mathcal{L})}{\langle Z\rangle} \to \frac{\mathcal{Z}^1_{S,p}(E/\mathcal{L}_{(p)})}{\langle Z_p\rangle} \to \mathfrak{X}(E/\mathcal{L}) \to \mathfrak{Y}(E/\mathcal{L}) \to 0. \tag{12}$$

(See for example [43, 45, 49].) If the Main Conjecture is valid over the extension $\mathcal{L}/\mathbb{Q}$ and Conjecture B holds for $\mathfrak{Y}\left(E/\mathcal{L}\right)$, then by the same argument as before we have that $\frac{\mathcal{Z}^1_S\left(E/\mathcal{L}\right)}{\langle Z\rangle}$ is pseudonull and therefore trivial.

5.2. **Tamagawa Number Conjecture.** The next application relates to isogeny invariance and the Tamagawa Number conjecture (TNC), as considered by Bloch and Kato in [7] (see also [3] and [26]). The context and setting is as in [7, §7]. Consider the motive associated to an elliptic curve E defined over $\mathbb{Q}$ with complex multiplication by the ring of integers in an imaginary quadratic field. Note that $\mathcal{Z}^i_S\left(T_p(E)(1)/\mathbb{Q}\right) = H^i\left(G_S(\mathbb{Q}), T_p(E)(1)\right)$.

Let $G_{\mathcal{L}} = \mathrm{Gal}(\mathcal{L}/\mathbb{Q})$, and φ be the natural map

$$\varphi : \left(\mathcal{Z}^1_S\left(T_p(E)(1)/\mathcal{L}\right)\right)_{G_{\mathcal{L}}} \longrightarrow H^1\left(G_S(\mathbb{Q}), T_p(E)(1)\right).$$

When the Euler system exists, let $\mathfrak{z}$ denote its image under this map, φ.

To keep the discussion simple, suppose that $\mathcal{L}$ is either the cyclotomic extension, $\mathbb{Q}_{\mathrm{cyc}}$, or the trivialising extension, $\mathbb{Q}(E_{p^\infty})$, of a CM elliptic curve E with Galois group $G_{\mathcal{L}}$ which is pro-p.

Morally, the TNC asserts that $\mathcal{Z}^i_S\left(T_p(E)(1)/\mathbb{Q}\right)$ is finite and

$$\#\left(\frac{\mathcal{Z}^1_S\left(T_p(E)(1)/\mathcal{L}\right)}{\mathfrak{z}}\right) = \#\left(\mathcal{Z}^2_S\left(T_p(E)(1)/\mathcal{L}\right)\right). \tag{13}$$

Viewed through the lens of Iwasawa theory, this translates to the assertion that Euler characteristics of the first and last terms in the exact sequence (12), when defined, are finite and equal.

It is shown in [7, Proposition 5.14] that the TNC is invariant under isogeny. Given two isogenous elliptic curves E and E' over $\mathbb{Q}$, the validity of TNC for one implies the same for the other. Recall that over the trivialising extension $\mathbb{Q}(E_{p^\infty})$, Conjecture B implies that $\mathcal{Z}^2_S\left(T_p(E)(1)/\mathbb{Q}(E_{p^\infty})\right)$ is pseudonull; hence its Euler characteristic, whenever defined, is trivial. The same is true for the Euler characteristic of $\mathcal{Z}^2_S\left(T_p(E')(1)/\mathbb{Q}(E_{p^\infty})\right)$.

The zeta element Z (resp. Z') of E (resp. E') in $\mathcal{Z}^1_S\left(T_p(E)(1)/\mathbb{Q}(E_{p^\infty})\right)$ (resp. $\mathcal{Z}^1_S\left(T_p(E')(1)/\mathbb{Q}(E_{p^\infty})\right)$ generates the $\Lambda(\Gamma)$-submodule $\langle Z\rangle$ (resp. $\langle Z'\rangle$). Denote by $\bar{\mathfrak{z}}$ (resp. $\overline{\mathfrak{z}'}$), the image of the Euler system in the cohomology group $H^1\left(\mathbb{Q}_S/\mathbb{Q}, T_p(E)(1)\right)$ (resp. in $H^1\left(\mathbb{Q}_S/\mathbb{Q}, T_p(E')(1)\right)$). For a compact $\Lambda(G)$-module M, let $\chi_G(M)$ denote the G-Euler characteristic of

M, when it is defined. Since Conjecture B is isogeny invariant, when the conjecture holds for E (hence also for E') over $\mathbb{Q}\left(E_{p^\infty}\right)$, we have

$$\chi_G\left(\mathcal{Z}_S^2\left(T_p(E)(1)/\mathbb{Q}(E_{p^\infty})\right)\right) = \chi_G\left(\mathcal{Z}_S^2\left(T_p(E')(1)/\mathbb{Q}(E_{p^\infty})\right)\right) = 1.$$

Further, we also obtain

$$\chi_G\left(\frac{\mathcal{Z}_S^1\left(T_p(E)(1)/\mathbb{Q}(E_{p^\infty})\right)}{\langle Z\rangle}\right) = \chi_G\left(\frac{\mathcal{Z}_S^1\left(T_p(E')(1)/\mathbb{Q}(E_{p^\infty})\right)}{\langle Z'\rangle}\right) = 1.$$

Let $\theta : E \to E'$ be the isogeny map and θ_* be the induced functorial map. Then,

$$\theta_* : \mathcal{Z}_S^1\left(T_p(E)(1)/\mathbb{Q}(E_{p^\infty})\right) \to \mathcal{Z}_S^1\left(T_p(E')(1)/\mathbb{Q}(E_{p^\infty})\right).$$

As explained in §5.1, we have the following equalities of $\Lambda(G)$-modules

$$\langle Z\rangle = \mathcal{Z}_S^1\left(T_p(E)/\mathbb{Q}(E_{p^\infty})\right),$$
$$\langle \theta_*(Z)\rangle \subseteq \langle Z'\rangle = \mathcal{Z}_S^1\left(T_p(E')/\mathbb{Q}(E_{p^\infty})\right).$$

It would be interesting to explicitly understand the precise relationship between the elements $\theta_*(Z)$ and Z'.

Jannsen's conjecture may be viewed as a generalization of the classical weak Leopoldt conjecture (see [22, Conjecture 1]). In particular, Jannsen's conjecture claims that $H^2\left(G_S(F), T_pE(k+1)\right)$ is finite for almost all k. The final application we record is the observation that examples where the fine Selmer group is finite over the cyclotomic extension provides additional evidence for the above mentioned conjecture of Jannsen. Indeed, by a result of Imai, when the fine Selmer group is finite over the cyclotomic extension, the second Iwasawa cohomology group is also finite. The claim is now immediate for almost all twists (see [22, Lemma 8]). We remark that there are results towards Jannsen's conjecture which do not require the finiteness of the fine Selmer group (see [26, 51, 4, 5].)

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