NON-VANISHING MODULO p OF HECKE L-VALUES OVER IMAGINARY QUADRATIC FIELDS

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Abstract. Let p and q be two distinct odd primes. Let K be an imaginary quadratic field over which p and q are both split. Let $E_{/K}$ be an elliptic curve with complex multiplication by $\mathcal{O}_K$ which has good reduction at the primes above p and q. Let φ be the Hecke character over K attached to E. We show under certain technical hypotheses that for a Zariski dense set of finite-order characters κ over K of q-power conductor, the algebraic part of the L-value $L(\overline{\kappa\rho\varphi^k}, k)$ is not divisible by p for $k = 1, \dots, p-1$, where ρ is a character on the Hilbert class group of K.

1. Introduction

Let p and q be two distinct odd primes. It is a classical problem to study the divisibility of the algebraic part of (Hecke) L-values by a given prime p as one varies the (Hecke) characters of q-power conductor. For Dirichlet L-values, such questions were studied by L. Washington in [Was75, Was78]. He showed that for almost all Dirichlet characters of q-power conductor, the algebraic parts of their L-values are coprime to p. As an application, he proved that the p-part of the class number stabilizes in cyclotomic $\mathbb{Z}_q$-extensions of abelian number fields. Washington's results have been extended to the case of (finite) product of cyclotomic $\mathbb{Z}_{q_i}$-extensions of abelian number fields (for distinct primes q_i with $q_i \neq p$) by E. Friedman in [Fri82].

In [Sin87], W. Sinnott introduced the idea of relating non-vanishing of such L-values modulo p to Zariski density (modulo p) of special points of the algebraic variety underlying the L-values. Using this machinery, J. Lamplugh generalized Washington's theorem to *split prime* $\mathbb{Z}_q$-extensions of imaginary quadratic fields in [Lam15]. Let K be an imaginary quadratic field such that $q\mathcal{O}_K = \mathfrak{q}\mathfrak{q}^*$ with $\mathfrak{q} \neq \mathfrak{q}^*$, then the *split prime* $\mathbb{Z}_q$-extension of K is one where only one of $\mathfrak{q}$ or $\mathfrak{q}^*$ is ramified.

In [Hid04, Hid07], H. Hida studied analogous questions for anticyclotomic characters. He proved that under certain hypotheses, the algebraic parts of the L-values of "almost all" anticyclotomic characters of q-power conductor over a CM field are non-zero mod p. Here, "almost all" means "Zariski dense" after identifying the characters with a product of the multiplicative group (see Remark 5.2). This has been generalized by M.-L. Hsieh [Hsi12] to include self-dual characters. In the case where the CM field is an imaginary quadratic field, T. Finis has proved this result in more generality and has determined precisely the p-adic valuations of the algebraic parts of anticyclotomic Hecke characters of q-power conductor; see [Fin06]. More recently, A. Burungale showed in [Bur16] that these results may be extended to Hida families of anticyclotomic characters.

We study a generalization of the aforementioned results on anticyclotomic characters to Hecke characters (not necessarily anticyclotomic) of q-power conductor over an imaginary quadratic field. More precisely, we prove a 2-variable version of [Lam15, Theorem 6.9].

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Theorem A (Theorem 6.14)**.** *Let K be an imaginary quadratic field over which p and q are both split and H be its Hilbert class field. Suppose that q does not divide the class number of K and that both the prime ideals above q are principal in K. Let $E_{/H}$ be an elliptic curve with CM by $\mathcal{O}_K$ which has good reduction at primes above p and q. Let φ be the Hecke character over K attached to E of conductor $\mathfrak{f}$. Let $\mathfrak{g}$ denote a principal integral ideal of K that is divisible by $\mathfrak{f}$ and coprime to pq. Let $F = H(E_{\mathfrak{g}q})$ and $F_\infty = \bigcup_{n\geq 1} H(E_{\mathfrak{g}q^n})$ be the $\mathbb{Z}_q^2$-extension over F. Let π be the uniformizer of the local field F_v where $v|p$. Let ρ be a character of $\mathrm{Gal}(H/K)$ satisfying the technical hypothesis (6.10), then, for a Zariski dense set of finite-order characters κ of $\mathrm{Gal}(F_\infty/F)$,*

$$\mathrm{ord}_\pi\left(L^{(\mathrm{alg})}\left(\overline{\kappa\rho\varphi^k}\right)\right) = 0$$

for $k = 1, 2, \ldots, p-1$.

Remark 1.1. *If $p \nmid \#\,\mathrm{Gal}(H/K)$, then it is easy to show (see Remark 6.9) that there exists at least one ρ such that the technical hypothesis is satisfied.*

Outline of the Proof. The proof of Theorem A follows closely the line of argument of [Lam15, Theorem 6.9]. It consists of the following ingredients:

(1) Establish a theory of Gamma transform of "elliptic function measures" on $\mathbb{Z}_q^2$, which are measures that arise from a rational function on an elliptic curve.
(2) Show that the π-adic valuations of the aforementioned Gamma transforms have the same p-adic valuation for almost all finite characters on $\mathbb{Z}_q^2$.
(3) Show that by choosing an appropriate elliptic function measure arising from a rational function on a CM elliptic curve, the Gamma transforms of this measure is related to the special values of L-series that we are interested.
(4) Show that the π-adic valuation discussed in (2) is zero.

Step (1) is carried out in Section 3. We follow the strategy of Lamplugh in [Lam15, Section 3], where the theory for elliptic function measures on $\mathbb{Z}_q$ was developed. To execute (2), we use a lemma of Hida on the Zariski density of characters on $\mathbb{Z}_q^d$ from [Hid04] to prove a result on the algebraic independence of functions on elliptic curves with positive characteristic. In particular, we prove Theorem 4.6, which is a two-variable version of [Lam15, Theorem 4.9]. Next, we prove Theorem 5.1, which completes step (2) of the proof of Theorem A. The corresponding 1-variable version of this theorem was proved in [Lam15, Section 5]. The construction of the elliptic function measure of step (3) is discussed in Section 6.3; this is a generalization of the rational function on the CM elliptic curve E utilized in [Lam15, Section 6.3] and crucially uses the work of E. de Shalit [dS87]. The link between the Gamma transforms of this elliptic function measure and the L-values of interest is given by Lemma 6.10. In step (2), we see that the π-adic valuation mentioned above is in fact given by the valuation of the rational function (see Definition 3.6). Using ideas of the proof of [Lam15, Lemma 6.7] in the one-variable case, we show in Lemma 6.13 that this valuation is zero; this allows us to conclude step (4).

Remark 1.2. *While Theorem A is deduced using Lamplugh's techniques developed in* [Lam15]*, our result is strictly stronger than the one-variable analogue* [Lam15, Theorem 6.9]. *Indeed, after identifying the characters of $\mathrm{Gal}(F_\infty/F)$ with a subset of $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$, the Zariski closure of the set of characters given by* loc. cit. *is one copy of $\mathbb{G}_{m/\overline{\mathbb{Q}}_q}$. In particular, it is not Zariski dense in $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$.*

Furthermore, the class number of K is assumed to be 1 *in* [Lam15]*, whereas Theorem A only assumes that q does not divide the class number of K.*

Remark 1.3. *Using an argument similar to the one presented in* [Lam15, Section 7]*, we expect that Theorem A combined with the Iwasawa main conjecture (proved by K. Rubin) should show that the p-part of the class groups over a $\mathbb{Z}_q^2$-tower is "generically zero". However, it does not seem to be enough to give a generalization of* [Lam15, Theorem 7.10] *in our setting, unless we replace "almost all" by "all but finitely many".*

We conclude by discussing some follow-up questions.

- In our forthcoming work [KL22], we study the growth of the p-part of the class groups in the anticyclotomic $\mathbb{Z}_p$-extension making use of the aforementioned result of Finis. Note that unlike [Lam15, Theorem 1.3], we do not expect the p-part of the class groups to always stabilize since the corresponding Hecke L-values are not always generically zero modulo π.
- Similar to how we build on Lamplugh's results to obtain Theorem A, it may be possible to prove a similar result for Hecke characters of q-power conductor over general CM fields, relying on results of Hida and Hsieh on anticyclotomic characters.
- It may also be interesting to generalize Theorem A to the setting of Hida families, utilizing ideas of Burungale developed in [Bur16].

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2. Basic Notions

Let K be a fixed imaginary quadratic field and H denote its Hilbert class field. Throughout, we assume that q is coprime to the class number of K. Let E be an elliptic curve defined over H with complex multiplication by $\mathcal{O}_K$, i.e., $\mathcal{O}_K \simeq \mathrm{End}(E)$. Let $q \geq 5$ be a prime number that splits in K, i.e.,

$$q\mathcal{O}_K = \mathfrak{q}\mathfrak{q}^* \text{ with } \mathfrak{q} \neq \mathfrak{q}^*.$$

Suppose that E has good reduction at primes above q.

For any integral ideal $\mathfrak{a}$ in $\mathcal{O}_K$, we write $E_\mathfrak{a}$ to denote

$$\ker\left(\mathfrak{a} : E \to E\right).$$

We write μ_K to denote the set of roots of unity in K and w_K to denote the size of this set.

We fix a different prime p satisfying the following properties

(i) $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$ in K with $\mathfrak{p} \neq \mathfrak{p}^*$ and $\gcd(p, 6q) = 1$.
(ii) $E_{/H}$ has good reduction at primes above p.

3. Distributions and measures on $\mathbb{Z}_q^2$

The goal of this section is to generalize the notion of Gamma transform from [Sin87] and elliptic function measures studied in [Lam15, Section 3.2] to the two-variable setting.

Let E be an elliptic curve as before and $k/\mathbb{Q}_p$ be a finite unramified extension containing $\mathbb{Q}_p(E_{\mathfrak{f}q})$. Set $J = k(\mu_{q^\infty})$. This is the unramified $\mathbb{Z}_q$-extension of k (since $\mu_q \subset k$ by assumption). Let $\mathfrak{O}$

denote the ring of integers of J. Fix a uniformizer π of k and let ord_π denote the normalized valuation map

$$\mathrm{ord}_\pi : J \to \mathbb{Z} \cup \{\infty\}.$$

Definition 3.1. *Let α be a J-valued* distribution *on $\mathbb{Z}_q^2$, i.e., α is a finitely additive function on the set of compact open subsets of $\mathbb{Z}_q^2$ with values in J.*

(i) *Given any $c = (c_1, c_2) \in (\mathbb{Z}_q^\times)^2$, define $\alpha \circ c$ to be the distribution given by $\alpha \circ c(X) = \alpha(cX)$ for all open compact subsets X of $\mathbb{Z}_q^2$.*

(ii) *The* Fourier transform *of α is defined to be*

$$\begin{aligned}\hat{\alpha} : \mu_{q^\infty}^2 &\to J\\ (\zeta_1, \zeta_2) &\mapsto \int_{(x,y)\in\mathbb{Z}_q^2} \zeta_1^x \zeta_2^y d\alpha(x, y).\end{aligned}$$

(iii) *Given a finite character χ on $(\mathbb{Z}_q^\times)^2$ with values in J, we define* Leopoldt's Γ-transform *as*

$$\Gamma_\alpha(\chi) = \int_{\mathbb{Z}_q^2} \chi d\alpha,$$

where we extend χ to $\mathbb{Z}_q^2$ by sending all elements not inside $(\mathbb{Z}_q^\times)^2$ to zero.

(iv) *We call α a* measure *on $\mathbb{Z}_q^2$ if the image of α has bounded values with respect to ord_π.*

Lemma 3.2. *Suppose that χ is a finite-order character on $(\mathbb{Z}_q^\times)^2$ factoring through $(\mathbb{Z}/q^m)^\times \times (\mathbb{Z}/q^n)^\times$, then*

$$\Gamma_\alpha(\chi) = \tau(\chi) \sum_{\underline{x}\in\mathbb{Z}/q^m\times\mathbb{Z}/q^n} \chi^{-1}(\underline{x})\hat{\alpha}(\underline{\zeta}^{\underline{x}}),$$

where $\underline{\zeta} = (\zeta_m, \zeta_n)$ with ζ_m and ζ_n being primitive p^m-th and p^n-th roots of unity respectively, and $\tau(\chi)$ is the Gauss sum *of χ defined by*

$$\tau(\chi) = \frac{1}{q^{m+n}} \sum_{(x_1,x_2)\in\mathbb{Z}/q^m\times\mathbb{Z}/q^n} \chi(x_1, x_2)\zeta_m^{-x_1}\zeta_n^{-x_2}.$$

Proof. See [Sin87, proof of Proposition 2.2, equation (2.6)] (or [Lam14, proof of Lemma 2.2.3]). □

Lemma 3.3. *A distribution α on $\mathbb{Z}_q^2$ is uniquely determined by its Fourier transform $\hat{\alpha}$.*

Proof. The characteristic function on the open subset $U_{a,b} := (a+q^m\mathbb{Z}_q)\times(b+q^n\mathbb{Z}_q)$ of $\mathbb{Z}_q^2$ satisfies

$$\mathbf{1}|_{U_{a,b}} = \frac{1}{q^{m+n}} \sum_{(\zeta_1,\zeta_2)\in\mu_{q^m}\times\mu_{q^n}} \zeta_1^{-a}\zeta_2^{-b}\chi_{(\zeta_1,\zeta_2)},$$

where $\chi_{(\zeta_1,\zeta_2)} : \mathbb{Z}_q^2 \mapsto J$ is the character sending (x, y) to $\zeta_1^x\zeta_2^y$. In particular, we see that $\alpha\left(U_{a,b}\right)$ is a linear combination of $\hat{\alpha}(\zeta_1, \zeta_2)$. Since the subsets $U_{a,b}$ form a basis of open compact sets of $\mathbb{Z}_q^2$, α is uniquely determined by $\hat{\alpha}$. □

For the rest of the article, we fix an isomorphism of groups $\delta : (\mu_{q^\infty})^2 \xrightarrow{\sim} E_{q^\infty}$.

Definition 3.4. *A J-valued distribution α on $\mathbb{Z}_q^2$ is an* elliptic function measure *for our fixed elliptic curve E (with respect to δ) if there exists a rational function $R \in J(E)$ such that for almost all $\underline{\zeta} \in (\mu_{q^\infty})^2$, we have*

$$\hat{\alpha}(\underline{\zeta}) = R(\delta(\underline{\zeta})).$$

Lemma 3.5. *Let $f \in \mathfrak{O}[x,y]$ such that the image of f in $J(E)$ is non-zero. Then, there exists a unique integer $n \geq 0$ such that*

$$\mathrm{ord}_\pi(f(Q)) \geq n \ \forall Q \in E_{q^\infty} \setminus \{0\}$$

with equality holding for almost all $Q \in E_{q^\infty}$.

Proof. The proof of [Lam15, Lemma 3.2] goes through in verbatim on replacing $E_{\mathfrak{q}^\infty}$ by E_{q^∞}. $\square$

This lemma allows us to define a valuation on $J(E)$.

Definition 3.6. *Given an $R \in J(E)$. If $R \neq 0$, we define* $\mathrm{ord}_\pi(R)$ *to be the integer n such that* $\mathrm{ord}_\pi(R(Q)) = n$ *for almost all $Q \in E_{q^\infty}$. If $R = 0$, we set* $\mathrm{ord}_\pi(R) = \infty$.

By Lemma 3.5, if α is an elliptic function measure, then it is in fact a measure (not just a distribution) since the values of α are linear combinations of $\hat{\alpha} = R \circ \delta$ as we have seen in the proof of Lemma 3.3 and $\frac{1}{q^{m+n}} \in \mathfrak{O}^\times$ (as $p \neq q$).

Note that for any given rational function $R \in J(E)$, we can define a J-valued measure attached to R as given by the following lemma:

Lemma 3.7. *Let $R \in J(E)$ be a rational function. There exists a unique measure α on $\mathbb{Z}_q^2$ such that the Fourier transform $\hat{\alpha}$ coincides with $R \circ \delta$. In other words, α is an elliptic function measure associated to R in the sense of Definition 3.4.*

Proof. By the proof of Lemma 3.3, we may define a measure α satisfying

$$\alpha\left((a + q^m\mathbb{Z}_q) \times (b + q^n\mathbb{Z}_q)\right) = \frac{1}{q^{m+n}} \sum_{(\zeta_1,\zeta_2) \in \mu_{q^m} \times \mu_{q^n}} \zeta_1^{-a}\zeta_2^{-b} R \circ \delta(\zeta_1, \zeta_2).$$

It follows from direct calculations that α is additive and that $\hat{\alpha} = R \circ \delta$. $\square$

We now show how Gamma transforms behave under Galois actions. This will be utilized in subsequent sections. Let us define the following homomorphisms of groups

$$\begin{aligned}
\chi_\mu &: \mathrm{Gal}(J/k) \hookrightarrow \mathrm{Aut}(\mu_{q^\infty})^2 \cong (\mathbb{Z}_q^\times)^2, \\
\chi_E &: \mathrm{Gal}(J/k) \hookrightarrow \mathrm{Aut}(E_{\mathfrak{q}^\infty}) \times \mathrm{Aut}(E_{\overline{\mathfrak{q}}^\infty}) \cong (\mathbb{Z}_q^\times)^2.
\end{aligned}$$

Note that $\chi_\mu = \chi_{\mathrm{cyc}} \times \chi_{\mathrm{cyc}}$, where χ_{cyc} is the cyclotomic character.

Definition 3.8. *An elliptic function measure α for E is said to be* defined over k, *if $\hat{\alpha} = R \circ \delta$ for a rational function $R \in k(E)$.*

Lemma 3.9. *Suppose that α is an elliptic function measure defined over k. Then, for almost all finite-order characters κ of $(\mathbb{Z}_q^\times)^2$ and for all $\sigma \in \mathrm{Gal}(J/k)$, we have*

$$\Gamma_\alpha(\kappa)^\sigma = \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))} \Gamma_\alpha(\kappa^\sigma).$$

Proof. It follows from Lemma 3.2 that

$$\Gamma_\alpha(\kappa)^\sigma = \tau(\kappa)^\sigma \sum_{\underline{x} \in \mathbb{Z}/q^m \times \mathbb{Z}/q^n} \kappa^{-1}(\underline{x})^\sigma \hat{\alpha}(\underline{\zeta}^{\underline{x}})^\sigma.$$

We have

$$\begin{aligned}
\tau(\kappa)^\sigma &= \frac{1}{q^{m+n}} \sum_{(x_1,x_2)\in \mathbb{Z}/q^m\times\mathbb{Z}/q^n} \kappa(x_1,x_2)^\sigma (\zeta_m^{-x_1}\zeta_n^{-x_2})^\sigma \\
&= \frac{1}{q^{m+n}} \sum_{(x_1,x_2)\in \mathbb{Z}/q^m\times\mathbb{Z}/q^n} \kappa(x_1,x_2)^\sigma \zeta_m^{-\chi_{\mathrm{cyc}}(\sigma)x_1}\zeta_n^{-\chi_{\mathrm{cyc}}(\sigma)x_2} \\
&= \frac{\kappa^\sigma(\chi_{\mathrm{cyc}}(\sigma),\chi_{\mathrm{cyc}}(\sigma))^{-1}}{q^{m+n}} \sum_{(x_1,x_2)\in \mathbb{Z}/q^m\times\mathbb{Z}/q^n} \kappa(x_1,x_2)^\sigma \zeta_m^{-x_1}\zeta_n^{-x_2} \\
&= \kappa^\sigma(\chi_\mu(\sigma))^{-1}\tau(\kappa^\sigma).
\end{aligned}$$

Since α is an elliptic function measure, we have

$$\hat{\alpha}(\underline{\zeta}^{\underline{x}})^\sigma = R(\delta(\underline{\zeta}^{\underline{x}})^\sigma) = R(\delta(\underline{\zeta}^{\chi_E(\sigma)\underline{x}})) = \hat{\alpha}(\underline{\zeta}^{\chi_E(\sigma)\underline{x}})$$

for some $R \in k(E)$. Therefore, combining these equations gives

$$\begin{aligned}
\Gamma_\alpha(\kappa)^\sigma &= \kappa^\sigma(\chi_\mu(\sigma))^{-1}\tau(\kappa^\sigma) \sum_{\underline{x}\in \mathbb{Z}/q^m\times\mathbb{Z}/q^n} \kappa^{-1}(\underline{x})^\sigma \hat{\alpha}(\underline{\zeta}^{\chi_E(\sigma)\underline{x}}) \\
&= \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))}\tau(\kappa^\sigma) \sum_{\underline{x}\in \mathbb{Z}/q^m\times\mathbb{Z}/q^n} \kappa^{-1}(\underline{x})^\sigma \hat{\alpha}(\underline{\zeta}^{\underline{x}}) \\
&= \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))}\Gamma_\alpha(\kappa^\sigma)
\end{aligned}$$

where the last equality follows from Lemma 3.2 applied to κ^σ. □

4. Algebraic Independence Results

The main result of this section is Theorem 4.6, where we prove an algebraic independence result of functions on E_{q^∞} taking values in a *finite field* whose characteristic is distinct from q. The first step is Theorem 4.2, which is an analogue of [Sin87, Proposition 3.1] (and also [Lam15, Theorem 4.5]). This step involves proving an algebraic independence result of functions on E_{q^∞} taking values in a *general field*, $\mathcal{F}$. Let E be an elliptic curve as fixed in Section 2. We suppose that E can be considered as a curve over the field $\mathcal{F}$ (for example, the residue field of H modulo a prime ideal). Suppose that $q > 3$ is a rational prime that splits in $\mathcal{O}_K$ and $\mathrm{char}(\mathcal{F}) \neq q$. This result essentially says that endomorphisms in $\mathrm{End}(E_{\mathfrak{q}^\infty}) \times \mathrm{End}(E_{\mathfrak{q}^{*\infty}})$ which are independent over $\mathrm{End}_{\mathcal{F}}(E)$, are algebraically independent.

The following lemma is required for the proof of Theorem 4.2.

Lemma 4.1. *Let $\Phi_1, \dots, \Phi_s$ be non-trivial morphisms from E^n to E of the form*

$$\Phi_j : (P_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_{ij}(P_i)$$

where $\alpha_{ij} \in \mathrm{End}_{\mathcal{F}}(E)$ for all $1 \le i \le n$ and $1 \le j \le s$. Suppose that the only relation of the kind $\alpha\Phi_k = \beta\Phi_\ell$ for $\alpha, \beta \in \mathrm{End}_{\mathcal{F}}(E)$ and $k \neq \ell$, is when $\alpha = \beta = 0$. If $r_1, \dots, r_s \in \mathcal{F}(E)$ with $\sum_{j=1}^s r_j \circ \Phi_j = 0$, then each r_j is a constant function.

Proof. See [Lam15, Proposition 4.4]. □

Theorem 4.2. *Let $\mathcal{F}$ be any field as above, and E an elliptic curve defined over $\mathcal{F}$ such that $\mathrm{End}_{\mathcal{F}}(E) \simeq \mathcal{O}_K$. Suppose that $\underline{\eta}_1, \dots, \underline{\eta}_s \in \mathrm{End}(E_{\mathfrak{q}^\infty}) \times \mathrm{End}(E_{\mathfrak{q}^*\infty})$ are such that $\alpha\underline{\eta}_k = \beta\underline{\eta}_\ell$ for $k \neq \ell$ and some $\alpha, \beta \in \mathrm{End}_{\mathcal{F}}(E)$ only when $\alpha = \beta = 0$. Consider the function*

$$R = \sum_{j=1}^{s} r_j \circ \underline{\eta}_j : E_{\mathfrak{q}^\infty} \times E_{\mathfrak{q}^*\infty} \to \overline{\mathcal{F}}$$

where $r_j \in \mathcal{F}(E)$ and $\overline{\mathcal{F}}$ denotes an algebraic closure of $\mathcal{F}$. If $R(Q) = 0$ for all $Q \in E_{\mathfrak{q}^\infty} \times E_{\mathfrak{q}^*\infty}$, then all r_j's are constant functions.

Proof. We recall that $\mathrm{End}(E_{\mathfrak{q}^\infty}) \simeq \mathcal{O}_{\mathfrak{q}}$, $\mathrm{End}(E_{\mathfrak{q}^*\infty}) \simeq \mathcal{O}_{\mathfrak{q}^*}$ and $\mathrm{End}_{\mathcal{F}}(E) \simeq \mathcal{O}_K$. Consider a free $\mathcal{O}_K$ submodule A of $\mathcal{O}_{\mathfrak{q}} \times \mathcal{O}_{\mathfrak{q}^*}$ of rank n that contains $\underline{\eta}_j$ for $1 \leq j \leq s$. Let $\{\underline{\varepsilon}_i\}_{i=1}^n$ be an $\mathcal{O}_K$-basis of A. Then, there exist unique $\alpha_{ij} \in \mathcal{O}_K$ such that

$$\underline{\eta}_j = \sum_{i=1}^{n} \alpha_{ij}\underline{\varepsilon}_i.$$

Define the map

$$\iota : E_{\mathfrak{q}^\infty} \times E_{\mathfrak{q}^*\infty} = E_{q^\infty} \to E^n \text{ given by } Q \mapsto (\underline{\varepsilon}_i Q)_{i=1}^n .$$

For each $1 \leq j \leq s$, denote the morphism

$$\Phi_j : E^n \to E; \qquad (P_i)_{i=1}^n \mapsto \sum_{i=1}^{n} \alpha_{ij} P_i.$$

We have assumed that

$$\sum_{j=1}^{s} r_j \circ \Phi_j(\mathcal{Q}) = 0 \qquad \text{for all } \mathcal{Q} \in \iota(E_{\mathfrak{q}^\infty} \times E_{\mathfrak{q}^*\infty}) \subseteq E^n.$$

Hence, the above equality must hold for all $\mathcal{Q}$ in the Zariski closure of $\iota(E_{\mathfrak{q}^\infty} \times E_{\mathfrak{q}^*\infty})$. It follows from basic facts about Zariski closed subgroups of E^n (see [Sch87, Lemmas 1 and 3]) that either the Zariski closure of $\iota(E_{q^\infty})$ is E^n or there exist $\alpha_i \in \mathcal{O}_K$ (not all zero) such that

$$\sum_{i=1}^{n} \alpha_i\underline{\varepsilon}_i(Q) = 0 \text{ for all } Q \in E_{q^\infty}.$$

If the latter holds, it means that $\sum_{i=1}^n \alpha_i\underline{\varepsilon}_i = 0$. However, this contradicts the fact that $\underline{\varepsilon}_1, \dots, \underline{\varepsilon}_n$ is a basis for A. Thus, the Zariski closure of $\iota(E_{q^\infty})$ is E^n. Lemma 4.1 implies that each r_i is a constant function. □

To prove the main result in this section, we need a strengthened version of Theorem 4.2. This is achieved by combining the following Diophantine approximation result (Lemma 4.3) with a special case of a lemma due to Hida (Lemma 4.4), which we record below.

Lemma 4.3. *Given $\underline{\beta}_1, \dots, \underline{\beta}_d \in \mathcal{O}_{\mathfrak{q}} \times \mathcal{O}_{\mathfrak{q}^*}$ for any integer $d \geq 1$, and a positive constant $c \leq 1$, there exists an integer N such that for all $n \geq N$, there exist algebraic integers $b_1, \dots, b_d \in \mathcal{O}_K$ and a unit $u \in \mathcal{O}_{\mathfrak{q}}^\times \times \mathcal{O}_{\mathfrak{q}^*}^\times$ satisfying*

$$v_{\mathfrak{p}}(u\underline{\beta}_i - b_i) \geq n \text{ for } \mathfrak{p} \in \{\mathfrak{q}, \mathfrak{q}^*\} \text{ and}$$
$$N_{K/\mathbb{Q}}(b_i) < c \cdot q^{2n}.$$

Proof. See [Lam14, Lemma 2.3.9]. □

Lemma 4.4. *Let r be a positive integer. Let $X = \bigcup_{i=1}^k X_i$ be a proper subset of $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$ such that*

(i) *X is Zariski closed.*
(ii) *For each i, there exists a closed subscheme Y_i that is stable under $t \mapsto t^{p^{rn}}$ for all $n \in \mathbb{Z}$, such that $X_i = \underline{\zeta} Y_i$ for certain $\underline{\zeta} \in \mu^2_{q^\infty}$;*

There exists P, which is a p^r-power, and an infinite sequence of integers $0 < n_1 < n_2 < \cdots$ such that for all $j \geq 1$,

$$\Xi_j \cap X = \emptyset,$$

where Ξ_j is defined by

$$\left\{ \left(\frac{P^x}{q^{n_j}}, \frac{P^y}{q^{n_j}} \right) \mod \mathbb{Z}_q^2 : x, y \in \mathbb{Z} \right\} \subset (\mathbb{Q}_q/\mathbb{Z}_q)^2$$

after identifying $\mu^2_{q^\infty}$ with $(\mathbb{Q}_q/\mathbb{Z}_q)^2$ under an appropriate choice of basis.

Proof. See [Hid04, Lemma 3.4]. □

Remark 4.5. *On studying he proof of the above lemma, we see that $P \equiv 1 \mod q$. If we write $P = 1 + q^v u$, where $q \nmid u$, then $|\Xi_j| = q^{2(n_j - v)}$.*

Theorem 4.6. *Let $\mathbb{F}$ be a finite field. Suppose that $\underline{\eta}_1, \ldots, \underline{\eta}_s \in \mathrm{End}(E_{\mathfrak{q}^\infty}) \times \mathrm{End}(E_{\mathfrak{q}^*\infty})$ are such that the only relation of the kind $\alpha \underline{\eta}_k = \beta \underline{\eta}_\ell$ for $k \neq \ell$ and $\alpha, \beta \in \mathrm{End}_{\mathbb{F}}(E)$ is when $\alpha = \beta = 0$. Consider the function*

$$R = \sum_{i=1}^s r_i \circ \underline{\eta}_i : E_{\mathfrak{q}^\infty} \times E_{\mathfrak{q}^*\infty} \to \overline{\mathbb{F}}$$

where $r_i \in \mathbb{F}(E)$. We identify E_{q^∞} with $\mu^2_{q^\infty} \subset \mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$. Then either $\{Q \in E_{q^\infty} : R(Q) \neq 0\}$ is Zariski dense in $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$ or R is identically zero. In the latter case, all r_i's are constant functions.

Proof. Suppose that $\{Q \in E_{q^\infty} : R(Q) \neq 0\}$ is not Zariski dense and that R is not identically zero. We take P to be a large enough p-power so that R is defined over $\mathbb{F}_P$ (the finite field of cardinality P). Let X be the Zariski closure of $\{Q \in E_{q^\infty} : R(Q) \neq 0\}$ in $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$. Then, X is a proper subset of $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$ and $X^P \subseteq X$.

Let $\log_q : \widehat{\mathbb{G}}^2_{m/\overline{\mathbb{Q}}_q} \to \widehat{\mathbb{G}}^2_{a/\overline{\mathbb{Q}}_q}$ be the q-adic logarithm map. We decompose $\log_q(X)$ into a finite union of closed subsets, each of which is stable under the multiplication by P. This allows us to write X as a finite union of closed subschemes X_i of the form $\underline{\zeta} Y_i$, where $\underline{\zeta} \in \mu^2_{q^\infty}$ and Y_i is stable under $t \mapsto t^P$. Therefore, Lemma 4.4 applies. In particular, there exists a sequence of integers $0 < n_1 < n_2 < \cdots$ and a collection of subsets Ξ_j of q^{n_j}-torsion points in E_{q^∞} on which R vanishes, with $|\Xi_j| = q^{2(n_j - v)}$ for some fixed integer v.

Define

$$\delta := \max_{1 \leq i \leq s} \deg(r_i).$$

We apply Lemma 4.3 to $\underline{\eta}_1, \ldots, \underline{\eta}_s$ and $c = \dfrac{1}{q^{2v} \cdot s \cdot \delta}$. There exists an integer N such that for all $n_j \geq N$, there are algebraic integers $b_1, \ldots, b_s \in \mathcal{O}_K$ and $u \in \mathcal{O}_{\mathfrak{q}}^\times \times \mathcal{O}_{\mathfrak{q}^*}^\times$ (depending on n_j) satisfying

$$v_{\mathfrak{p}}(u\underline{\eta}_i - b_i) \geq n_j \text{ for } \mathfrak{p} \in \{\mathfrak{q}, \mathfrak{q}^*\} \text{ and}$$
$$\left| N_{K/\mathbb{Q}}(b_i) \right| < c \cdot q^{2n_j}.$$

In particular, the rational function

$$R_{n_j} := \sum_{i=1}^{s} r_i \circ b_i \in \mathbb{F}(E)$$

agrees with $R \circ u$ on $E_{q^{n_j}}$. Thus, it vanishes on Ξ_j. Moreover,

$$\deg(R_{n_j}) \leq \sum_{i=1}^{s} \delta \cdot N_{K/\mathbb{Q}}(b_i) < s\delta \cdot c \cdot q^{2n_j} = q^{2n_j - 2v} = |\Xi_j|.$$

Therefore, $R_{n_j} = 0$ and thus R is zero on $E_{q^{n_j}}$. But n_j can be arbitrarily large. This implies that R is identically zero, which is a contradiction. This concludes the first assertion of the theorem. The last assertion follows immediately from Theorem 4.2. $\square$

5. A theorem on two-variable Gamma transforms

The purpose of this section is to prove a two-variable version of [Lam15, Theorem 5.1] (which in turn generalizes a result of Sinnott [Sin87, Theorem 3.1]). Our proof utilizes crucially Theorem 4.6 from the previous section. Throughout, we use the same notation introduced in Sections 2 and 3.

Theorem 5.1. *Let α be an elliptic function measure for E defined over k on $\mathbb{Z}_q^2$ that is supported on $(\mathbb{Z}_q^\times)^2$, and satisfies $\alpha \circ \omega = \alpha$ for all $\omega \in \mu_K^2$. Let R denote the corresponding rational function (so that $\hat{\alpha} = R \circ \delta$ as in Definition 3.4), and let $n = \mathrm{ord}_\pi(R)$ (as in Definition 3.6). Then for a Zariski dense set of finite-order characters κ of $(1 + q\mathbb{Z}_q)^2$, we have*

$$\mathrm{ord}_\pi \big(\Gamma_\alpha(\kappa)\big) = n.$$

Remark 5.2. *We view $\mathrm{Hom}(\mathbb{Z}_q^2, \mu_{q^\infty})$ as a subset of $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$ by sending κ to $(\kappa(1,0), \kappa(0,1))$. A set of finite-order characters is called* Zariski dense *if its image in $\mathbb{G}^2_{m/\overline{\mathbb{Q}}_q}$ is a dense subset under the Zariski topology.*

The following lemma is a key technical ingredient of the proof of Theorem 5.1.

Lemma 5.3. *Let α be an elliptic function measure as in the statement of Theorem 5.1. Define*

$$\beta = \sum_{\eta} (\alpha \circ \eta)\,|_{(1+q\mathbb{Z}_q)^2},$$

where η runs over a set of representatives for $(\mu_{q-1}/\mu_K)^2$ and $\alpha \circ \eta$ is defined as in Definition 3.1(i). For each $y = (y_1, y_2) \in \mu_{q-1}^2$, we write

$$\beta_y = \beta|_{y_1(1+q^M\mathbb{Z}_q)\times y_2(1+q^M\mathbb{Z}_q)},$$

where $M \geq 1$ is the integer such that $\mu_{q^\infty} \cap k = \mu_{q^M}$. Let $\kappa = (\kappa_1, \kappa_2)$ be a finite-order character of $(1 + q\mathbb{Z}_q)^2$. Suppose that there exist integers $m, n \geq M$ satisfying

$$\ker(\kappa_1) = 1 + q^{m+M}\mathbb{Z}_q, \quad \ker(\kappa_2) = 1 + q^{n+M}\mathbb{Z}_q.$$

Let $\underline{\zeta} = (\zeta_1, \zeta_2) \in \mu_{q^\infty}$ such that

$$\zeta_1^{q^m} = \kappa_1(1 + q^m), \quad \zeta_2^{q^n} = \kappa_2(1 + q^n).$$

Then $\Gamma_\beta(\kappa) \in \pi\mathfrak{O}$ if and only if $\hat{\beta}_y(\underline{\zeta}^{y^{-1}}) \in \pi\mathfrak{O}$ for all $y \in \mu_{q-1}^2$.

Proof. Suppose that $\Gamma_\beta(\kappa) \in \pi\mathfrak{O}$. Let $\sigma \in \mathrm{Gal}(J/k)$ and $\underline{\xi} \in \mu_{q^\infty}^2$. Recall from the proof of Lemma 3.9 that

$$\hat{\alpha}(\underline{\xi})^\sigma = \hat{\alpha}(\underline{\xi}^{\chi_E(\sigma)}).$$

Since Fourier transform is additive, we have equivalently

$$\hat{\beta}(\underline{\xi})^\sigma = \hat{\beta}(\underline{\xi}^{\chi_E(\sigma)}).$$

Furthermore, Lemma 3.9 asserts that

$$\Gamma_\beta(\kappa)^\sigma = \frac{\kappa^\sigma(\chi_E(\sigma))}{\kappa^\sigma(\chi_\mu(\sigma))}\Gamma_\beta(\kappa^\sigma).$$

Thus, $\mathrm{ord}_\pi\left(\Gamma_\beta(\kappa^\sigma)\right)$ is independent of $\sigma \in \mathrm{Gal}(J/k)$ because $\Gamma_\beta(\kappa) \in \pi\mathfrak{O}$ by assumption and κ takes values in the group of roots of unity. In particular, $\Gamma_\beta(\kappa^\sigma) \in \pi\mathfrak{O}$ for all $\sigma \in \mathrm{Gal}(J/k)$ under our hypothesis that $\Gamma_\beta(\kappa) \in \pi\mathfrak{O}$.

Let $N = \max(m, n)$. Write k_{N-1} to denote the $(N-1)$-th layer of the $\mathbb{Z}_q$-extension J/k, and set $H = \mathrm{Gal}(k_{N-1}/k)$. Let $y \in (1 + q\mathbb{Z}_q)^2$. We have

$$\begin{aligned}
\sum_{\sigma\in H} \kappa^\sigma(y)^{-1}\Gamma_\beta(\kappa^\sigma) &= \sum_{\sigma\in H} \kappa^\sigma(y)^{-1} \int_{(1+q\mathbb{Z}_q)^2} \kappa^\sigma(x)d\beta(x)\\
&= \sum_{\sigma\in H} \int_{(1+q\mathbb{Z}_q)^2} \kappa^\sigma(x/y)d\beta(x)\\
&= \int_{(1+q\mathbb{Z}_q)^2} \mathrm{Tr}_{k_{N-1}/k} \circ\kappa(x/y)d\beta(x)\\
&= q^{N-1} \int_{y_1(1+q^m\mathbb{Z}_q)\times y_2(1+q^n\mathbb{Z}_q)} \kappa(x/y)d\beta(x).
\end{aligned}$$

Note that q^{N-1} is a unit in $\mathfrak{O}$ (since $q \neq p$). Therefore, $\Gamma_\beta(\kappa) \in \pi\mathfrak{O}$ implies that

$$\int_{y_1(1+q^m\mathbb{Z}_q)\times y_2(1+q^n\mathbb{Z}_q)} \kappa(x/y)d\beta(x) \in \pi\mathfrak{O}.$$

Let $x = (x_1, x_2) = y(1 + q^m z_1, 1 + q^n z_2) = (y_1(1 + q^m z_1), y_2(1 + q^n z_2))$, where $z_1, z_2 \in \mathbb{Z}_q$. Then

$$\kappa(x/y) = \kappa(1 + q^m z_1, 1 + q^n z_2) = \kappa((1 + q^m)^{z_1}, (1 + q^n)^{z_2}) = \zeta_1^{x_1/y_1-1}\zeta_2^{x_2/y_2-1}.$$

Thus, we deduce that

$$\int_{y_1(1+q^m\mathbb{Z}_q)\times y_2(1+q^n\mathbb{Z}_q)} \zeta_1^{x_1/y_1}\zeta_2^{x_2/y_2}d\beta(x) \in \pi\mathfrak{O}.$$

If we replace y by $yt = (y_1t_1, y_2t_2)$ and (ζ_1, ζ_2) by $(\zeta_1^{t_1}, \zeta_2^{t_2})$ for any $t = (t_1, t_2) \in (1 + q^M\mathbb{Z}_q)^2$, the same containment holds. Hence, summing over $t \in (1+q^M\mathbb{Z}_q)^2/(1+q^m\mathbb{Z}_q)\times(1+q^n\mathbb{Z}_q)$, we deduce that

$$\hat{\beta}_y(\underline{\zeta}^{y^{-1}}) = \int_{y(1+q\mathbb{Z}_q)^2} \zeta_1^{x_1/y_1}\zeta_2^{x_2/y_2}d\beta(x) \in \pi\mathfrak{O}.$$

The converse follows from Lemma 3.2 and the fact that the Gauss sum $\tau(\kappa)$ is a π-adic unit (which is a consequence of the fact that its conductor is coprime to p). □

Proof of Theorem 5.1. Without loss of generality, we assume that $n = \mathrm{ord}_\pi(R) = 0$. Let β be as defined in the statement of Lemma 5.3, and w_K denote the number of elements in μ_K (which is coprime to $p > 3$). We have

$$\frac{1}{w_K^2}\Gamma_\alpha(\kappa) = \Gamma_\beta(\kappa).$$

Let us write

$$\alpha_{\eta y} = \alpha|_{\eta_1 y_1(1+q^M\mathbb{Z}_q)\times\eta_2 y_2(1+q^M\mathbb{Z}_q)}$$

for $\eta = (\eta_1, \eta_2) \in \mu_{q-1}^2$ and $y = (y_1, y_2) \in (1 + q\mathbb{Z}_q)^2$. Note that $\alpha_{\eta y}$ is an elliptic function measure since it is a restriction of α. Furthermore, we write $R_{\eta y}$ for the rational function on E attached to $\alpha_{\eta y}$ (meaning that $\hat{\alpha}_{\eta y} = R_{\eta y} \circ \delta$ as functions on $\mu_{q^\infty}^2$). As can be seen in the proof of Lemma 3.7, $R_{\eta y}$ takes values in $\mathfrak{O}$. Let $\tilde{R}_{\eta y}$ denote the function $R_{\eta y}$ modulo π.

Suppose that the set of characters κ with $\mathrm{ord}_\pi(\Gamma_\alpha(\kappa)) = 0$ is not Zariski dense. Note that for all κ, we have $\mathrm{ord}_\pi(\Gamma_\alpha(\kappa)) = \mathrm{ord}_\pi(\Gamma_\beta(\kappa))$ by Lemma 3.5 and the fact that $p \nmid w_K$. Equivalently, the set of characters κ such that $\Gamma_\beta(\kappa) \notin \pi\mathfrak{O}$ is not Zariski dense. By Lemma 5.3, the set of elements $Q \in E_{q^\infty}$ such that

$$\sum_{\eta\in(\mu_{q-1}/\mu_K)^2} \tilde{R}_{\eta y}([\eta^{-1}] \circ Q) \neq 0$$

is not Zariski dense.

Applying Theorem 4.6, it follows that $\tilde{R}_{\eta y}$ is a constant function. Let $c_{\eta y}$ denote a constant of $\mathfrak{O}$ lifting $\tilde{R}_{\eta y}$ and let δ_0 denote the Dirac measure of $\mathbb{Z}_q^2$ concentrated at $(0, 0)$. By definition, the Fourier transform $\hat{\delta}_0$ sends all $\underline{\zeta} \in \mu_{q^\infty}$ to 1. Therefore, the Fourier transform of $\alpha_{\eta y} - c_{\eta y}\delta_0$ takes values in $\pi\mathfrak{O}$. In particular,

$$\mathrm{ord}_\pi(\alpha_{\eta y} - c_{\eta y}\delta_0) > 0.$$

However, if we restrict the measure $\alpha_{\eta y} - c_{\eta y}$ to $(\mathbb{Z}_q^\times)^2$, it agrees with $\alpha_{\eta y}$. Thus,

$$\mathrm{ord}_\pi(\alpha_{\eta y}) = \mathrm{ord}_\pi(\alpha_{\eta y} - c_{\eta y}\delta_0) > 0.$$

This contradicts our hypothesis that $\mathrm{ord}_\pi(R) = 0$. □

6. Proof of Theorem A

In this section we apply Theorem 5.1 to study π-adic valuations of special values of L-functions and prove Theorem A stated in the introduction.

6.1. Notation on ray class fields and CM elliptic curves.

We keep the notation introduced in Section 2. Recall that K is a fixed imaginary quadratic field, and H is its Hilbert class field.

Definition 6.1. *Let $\mathfrak{a}$ be an integral ideal of K.*

- *We write $\mathscr{R}(\mathfrak{a})$ for the ray class field of K with conductor $\mathfrak{a}$.*
- *Given another ideal $\mathfrak{b}$ of K which is coprime to $\mathfrak{a}$, we write $(\mathfrak{b}, \mathscr{R}(\mathfrak{a})) \in \mathrm{Gal}(\mathscr{R}(\mathfrak{a})/K)$ for the Artin symbol of $\mathfrak{b}$.*
- *Given a character ρ on $\mathrm{Gal}(\mathscr{R}(\mathfrak{a})/K)$, we shall write $\rho(\mathfrak{b})$ and $\rho\big((\mathfrak{b}, \mathscr{R}(\mathfrak{a}))\big)$ interchangeably.*

Let $E_{/H}$ be an elliptic curve with complex multiplication by $\mathcal{O}_K$ with good reduction at the primes above p and q. Let ω_E denote the Néron differential for $E_{/H}$ and $\mathcal{L} = \Omega_\infty\mathcal{O}_K$ be its period lattice. Here Ω_∞ is determined up to a root of unity in K. Let $\psi = \psi_{E_{/H}}$ be the Hecke character over H attached to $E_{/H}$. We write

$$\psi_{E_{/H}} = \varphi \circ N_{H/K},$$

where $N_{H/K}$ denotes the norm map and φ is a Hecke character over K of infinity type $(1,0)$. We write $\mathfrak{f}$ for the conductor of φ (note that $\mathfrak{f}$ is coprime to pq).

Given an ideal $\mathfrak{b}$ of K coprime to $\mathfrak{f}$, there exists $\Lambda(\mathfrak{b}) \in H^\times$ such that

$$\mathcal{L}_{\mathfrak{b}} = \Lambda(\mathfrak{b})\mathfrak{b}^{-1}\mathcal{L} \tag{6.1}$$

is the lattice associated with $E^{(\mathfrak{b},H)}$, as given by [dS87, (16) on p. 42] (see also [GS81, Définition, p. 198]). For simplicity, we shall write $E^{(\mathfrak{b})}$ for the CM elliptic curve $E^{(\mathfrak{b},H)}$ and denote by

$$\lambda(\mathfrak{b}) : E \to E^{(\mathfrak{b})}$$

the unique isogeny given by [dS87, (15) on p. 42].

Consider the complex analytic isomorphism of complex Lie groups

$$\xi_{\mathfrak{b}} : \mathbb{C}/\mathcal{L}_{\mathfrak{b}} \xrightarrow{\sim} E^{(\mathfrak{b})}(\mathbb{C}) \text{ given by } \xi_{\mathfrak{b}}(z) = \left(\wp(z, \mathcal{L}_{\mathfrak{b}}), \wp'(z, \mathcal{L}_{\mathfrak{b}})\right), \tag{6.2}$$

where $\wp$ is the Weierstrass $\wp$-function and $\wp'$ is the corresponding derivative. We have the Weierstrass equation

$$y^2 = 4x^3 - g_2(\mathcal{L}_{\mathfrak{b}})x - g_3(\mathcal{L}_{\mathfrak{b}}) \tag{6.3}$$

describing $E^{(\mathfrak{b})}$.

When $\mathfrak{b} = \mathcal{O}_K$, we shall write ξ_1 in place of $\xi_{\mathcal{O}_K}$. We recall the following relation:

$$\xi_{\mathfrak{b}}\left(\Lambda(\mathfrak{b})z\right) = \lambda(\mathfrak{b})(\xi_1(z)) \tag{6.4}$$

as discussed in [dS87, commutative diagram (21) on p. 43] and [GS81, Proposition 4.10].

6.2. Review on L-functions.

Definition 6.2. *Let $\mathfrak{h}$ be any integral ideal of K. Let ϵ be any Hecke character of K with conductor dividing some power of $\mathfrak{h}$. The* imprimitive L-function *of ϵ modulo $\mathfrak{h}$ is defined as follows*

$$L_{\mathfrak{h}}(\epsilon, s) = \sum_{\gcd(\mathfrak{a},\mathfrak{h})=1} \frac{\epsilon(\mathfrak{a})}{(N\mathfrak{a})^s}.$$

Fix an integral ideal $\mathfrak{g}$ of K which is divisible by $\mathfrak{f}$ and is relatively prime to pq. Let $F = \mathscr{R}(\mathfrak{g}q)$ and write $\Delta = \mathrm{Gal}(F/K)$. Since $\mathfrak{f} \mid \mathfrak{g}$, we have

$$F = \mathscr{R}(\mathfrak{g}q) = K\left(j(E), h(E_{\mathfrak{g}q})\right) = H\left(x(E_{\mathfrak{g}q})\right).$$

Here h denotes a Weber function and we may choose this to be the x-coordinates on a Weierstrass model for the elliptic curve. Set $F_\infty = \bigcup_{n\geq 1} \mathscr{R}(\mathfrak{g}q^n)$; this is a $\mathbb{Z}_q^2$-extension of F. Writing K_∞ for the unique $\mathbb{Z}_q^2$-extension of K, we fix an isomorphism

$$\mathrm{Gal}(F_\infty/K) \simeq \mathrm{Gal}(F/K) \times \mathrm{Gal}(F_\infty/F) \simeq \mathrm{Gal}(F/K) \times \mathrm{Gal}(K_\infty/K) = \Delta \times \mathbb{Z}_q^2.$$

Let ϵ be a Hecke character over K. For our purpose, ϵ will be of the form $\upsilon\varphi^k$, where υ is a finite character of $\mathrm{Gal}(F_\infty/K)$ and $k \geq 1$ is an integer. Denote by $L(\epsilon, s)$ the *primitive Hecke L-function* of ϵ. Recall that the imprimitive (or partial) L-function differs from the primitive (or classical) L-function by a finite number of Euler factors. We can further define the *primitive algebraic Hecke L-function*,

$$L^{(\mathrm{alg})}(\upsilon\varphi^k) = L^{(\mathrm{alg})}(\epsilon) := (k-1)!\frac{L(\epsilon, k)}{\Omega_\infty^k}.$$

Recall that p is a rational prime satisfying the following hypotheses

(i) $p = \mathfrak{p}\mathfrak{p}^*$ in K with $\mathfrak{p} \neq \mathfrak{p}^*$ and $\gcd(p, 6q) = 1$.

(ii) $E_{/H}$ has good reduction at primes above p.

As before, set π to be the uniformizer of the local field k, which is a finite unramified extension of $\mathbb{Q}_p$ containing $\mathbb{Q}_p(E_{\mathfrak{f}q})$.

Let $F_{m,n} = \mathscr{R}(\mathfrak{g}\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1})$ be the m,n-th layer of the $\mathbb{Z}_q^2$ extension F_∞/F. Let υ be a character of $\mathrm{Gal}\left(F_{m,n}/K\right)$. Set $\mathfrak{h} = \mathfrak{g}\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$ in Definition 6.2. The imprimitive L-function of $\upsilon\varphi^k$ modulo $\mathfrak{h}$ can be written as

$$L_{\mathfrak{h}}\left(\overline{\upsilon\varphi^k}, s\right) = \sum_{\tau\in\mathrm{Gal}(F_{m,n}/K)} \overline{\upsilon}(\tau) \sum_{(\mathfrak{b},F_{m,n})=\tau} \frac{\overline{\varphi^k}(\mathfrak{b})}{(N\mathfrak{b})^s},$$

where the second sum runs over integral ideals $\mathfrak{b}$ of $\mathcal{O}_K$ such that $\gcd(\mathfrak{b},\mathfrak{h}) = 1$. We define the following partial imprimitive L-functions:

Definition 6.3. *Let $\tau \in \mathrm{Gal}(F_{m,n}/K)$, we define*

$$L_{\mathfrak{h}}\left(\overline{\varphi^k}, s, (\mathfrak{b}, F_{m,n}/K)\right) = \sum_{\substack{\mathfrak{b}\trianglelefteq\mathcal{O}_K \\ (\mathfrak{b},F_{m,n})=\tau \\ \gcd(\mathfrak{b},\mathfrak{h})=1}} \frac{\overline{\varphi^k}(\mathfrak{b})}{(N\mathfrak{b})^s}.$$

In particular, we have

$$L_{\mathfrak{h}}\left(\overline{\upsilon\varphi^k}, s\right) = \sum_{\tau\in\mathrm{Gal}(F_{m,n}/K)} \overline{\upsilon}(\tau) L_{\mathfrak{h}}(\overline{\varphi^k}, s, \tau).$$

Remark 6.4. *The (primitive and imprimitive) L-functions we have discussed so far only converge on some right half-plane. However, they have analytic continuations to the entire complex plane. This allows us to relate $L^{(\mathrm{alg})}\left(\overline{\upsilon\varphi^k}\right)$ to Gamma transforms of certain elliptic function measures.*

6.3. **A rational function with a canonical divisor.** The goal of this section is to generalize the construction of a rational function on a CM elliptic curve from [Lam15, Section 6.3]. In particular, we remove the hypothesis that K is of class number one. This allows us to carry out step (3) outlined in the introduction.

Throughout this section, $\mathfrak{h} = \mathfrak{g}\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$ and $F_{m,n} = \mathscr{R}(\mathfrak{h})$. Recall that q is assumed to be coprime to the class number of K. Furthermore, $\mathfrak{g}$, $\mathfrak{q}$ and $\mathfrak{q}^*$ are assumed to be principal ideals. There is a decomposition of Galois groups

$$\mathrm{Gal}(F_{m,n}/K) = \mathrm{Gal}(F_{m,n}/H) \times \mathrm{Gal}(H/K). \tag{6.5}$$

Definition 6.5. *Given an element $\sigma \in \mathrm{Gal}(F_{m,n}/K)$, we write σ' and σ'' for its images in $\mathrm{Gal}(F_{m,n}/H)$ and $\mathrm{Gal}(H/K)$ under the decomposition (6.5), respectively.*

Note that $\sigma \mapsto \sigma''$ is the restriction map from $F_{m,n}$ to H. Furthermore, if $\sigma = (\mathfrak{c}, F_{m,n})$, then σ'' is nothing but $(\mathfrak{c}, H)$.

Let $\mathfrak{b}$ be an integral ideal of K that is coprime to $\mathfrak{f}$. Let $\mathfrak{a} \in \mathrm{End}\left(E^{(\mathfrak{b})}\right)$ be coprime to $6\mathfrak{h}$. Define the rational function $\zeta_{\mathfrak{b},\mathfrak{a}}$ on $E^{(\mathfrak{b})}$ by

$$\zeta_{\mathfrak{b},\mathfrak{a}}(P) = \prod_{Q} \left(x(P) - x(Q)\right)^{-1} \tag{6.6}$$

where Q runs over a set of representatives of $E_{\mathfrak{a}}^{(\mathfrak{b})} \setminus \{0\} \pmod{\pm 1}$. There exists a constant $c(\mathfrak{b},\mathfrak{a}) \in H^\times$ such that the function

$$\gamma_{\mathfrak{b},\mathfrak{a}}(P) := c(\mathfrak{b},\mathfrak{a})\zeta_{\mathfrak{b},\mathfrak{a}}(P)$$

has the property that for all $\beta \in \mathrm{End}\left(E^{(\mathfrak{b})}\right)$ with $\gcd(\beta,\mathfrak{a}) = 1$,

$$\gamma_{\mathfrak{b},\mathfrak{a}}(\beta(P)) = \prod_{R\in\ker(\beta)} \gamma_{\mathfrak{b},\mathfrak{a}}(P \oplus R)$$

(see [Coa91, Appendix]).

For any integer $k \geq 1$, define the k-th logarithmic derivative operator on $\mathbb{C}(E^{(\mathfrak{b})})$ by

$$D^k(f) = \frac{d^k}{dz^k}\log f(z),$$

where z is a complex variable after identifying $E^{(\mathfrak{b})}$ with $\mathbb{C}/\mathcal{L}_{\mathfrak{b}}$ via $\xi_{\mathfrak{b}}$ as given by (6.4).

Lemma 6.6. *There exists $\sigma \in \mathrm{Gal}(F_{m,n}/H)$, $\zeta \in \mu_K$, and $\rho \in \mathfrak{h}^{-1}\mathcal{L}\setminus\mathcal{L}$ such that P_0 is a primitive $\mathfrak{h}$-division point on E given by*

$$P_0 = \zeta\left(\xi_1(\rho)^\sigma\right).$$

Let $\mathfrak{c}$ be an ideal coprime to $\mathfrak{h}$ such that $(\mathfrak{c},F_{m,n})' = \sigma$. We have

$$D^k(\gamma_{\mathfrak{c},\mathfrak{a}})\circ\lambda(\mathfrak{c})(P_0) = (-1)^{k-1}(k-1)!\left(N(\mathfrak{a}) - \Lambda(\mathfrak{a})^k(\mathfrak{a},F_{m,n})\right)\frac{\varphi(\mathfrak{c})^k}{\Lambda(\mathfrak{c})^k}\cdot\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{c},F_{m,n})\right)}{(\zeta\rho)^k}.$$

Proof. By Class Field Theory, we have

$$(\mathcal{O}_K/\mathfrak{h})^\times/\mu_K \simeq \mathrm{Gal}(F_{m,n}/H) \hookrightarrow \mathrm{Aut}(E[\mathfrak{h}]).$$

It follows that $\mathrm{Aut}(E[\mathfrak{h}])$ is generated by the image of $\mathrm{Gal}(F_{m,n}/H)$ and μ_K. The first assertion now follows just as in the proof of [Lam14, Lemma 3.1.4] or [Lam15, Lemma 6.4].

In the appendix, we prove in (A.5) that

$$D^k(\gamma_{\mathfrak{c},\mathfrak{a}})(\xi_{\mathfrak{c}}(\Lambda(\mathfrak{c})\rho)) = (-1)^{k-1}\left(N(\mathfrak{a}) - \Lambda(\mathfrak{a})^k(\mathfrak{a},F_{m,n})\right)\frac{\varphi(\mathfrak{c})^k}{\Lambda(\mathfrak{c})^k}\cdot\frac{L_{\mathfrak{h}}\left(\overline{\varphi}^k,k,(\mathfrak{c},F_{m,n})\right)}{\rho^k}.$$

Therefore, it follows from the definition of $\gamma_{\mathfrak{c},\mathfrak{a}}$ that

$$D^k(\gamma_{\mathfrak{c},\mathfrak{a}})\left(\zeta\cdot\xi_{\mathfrak{c}}(\Lambda(\mathfrak{c})\rho)\right) = (-1)^{k-1}\left(N(\mathfrak{a}) - \Lambda(\mathfrak{a})^k(\mathfrak{a},F_{m,n})\right)\frac{\varphi(\mathfrak{c})^k}{\Lambda(\mathfrak{c})^k}\cdot\frac{L_{\mathfrak{h}}\left(\overline{\varphi}^k,k,(\mathfrak{c},F_{m,n})\right)}{(\zeta\rho)^k}$$

and the result now follows from (6.4). □

Define

$$\rho_{m,n} = \frac{\Omega_\infty}{g\nu^{m+1}\nu^{*n+1}} \in \mathbb{C}^\times,$$

where g, ν, ν^* are fixed generators of $\mathfrak{g}$, $\mathfrak{q}$ and $\mathfrak{q}^*$ respectively. Then $\xi_1(\rho_{m,n})$ is a primitive $\mathfrak{h}$-division point of E (since $\mathfrak{h} = \mathfrak{g}\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$).

Let V (respectively $Q_{m,n}$) be a fixed primitive $\mathfrak{g}$-division (respectively $\mathfrak{q}^{m+1}\mathfrak{q}^{*n+1}$-division) point on E. By Lemma 6.6, there exist $\zeta \in \mu_K$ and $\sigma_0 = (\mathfrak{c}_0, F_{m,n})'$, where $\mathfrak{c}_0$ is an ideal of K, coprime to $\mathfrak{h}$, depending on V and $Q_{m,n}$, such that

$$V \oplus Q_{m,n} = \zeta(\xi_1(\rho_{m,n})^{\sigma_0}).$$

Without loss of generality, we may assume that $\sigma_0''=1$.

Lemma 6.7. *Let $\sigma=(\mathfrak{b},F_{m,n})\in\mathrm{Gal}(F_{m,n}/K)$ such that $\sigma''=1$. Then $\Lambda(\mathfrak{b})=\varphi(\mathfrak{b})$. Furthermore, for any integral ideal $\mathfrak{c}$ of K that is coprime to $\mathfrak{h}$, we have*

$$\sigma\cdot\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{c},F_{m,n})\right)}{(\zeta\rho_{m,n})^k}=\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{bc},F_{m,n})\right)}{(\zeta\rho_{m,n})^k}.$$

Proof. Since $(\mathfrak{b},H)=1$, the first equality follows from [dS87, (18) on p. 42]. Now, (A.4) in the appendix tells us that

$$\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{c},F_{m,n})\right)}{(\zeta\rho_{m,n})^k}=\frac{\Lambda(\mathfrak{c})^k}{(k-1)!\zeta^k\varphi(\mathfrak{c})^k}E_k\left(\varphi(\mathfrak{c})\rho,\mathcal{L}\right). \tag{6.7}$$

Since σ acts trivially on H, we deduce that

$$\sigma\cdot\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{c},F_{m,n})\right)}{(\zeta\rho_{m,n})^k}=\frac{\Lambda(\mathfrak{c})^k}{(k-1)!\zeta^k\varphi(\mathfrak{c})^k}E_k\left(\varphi(\mathfrak{c})\rho,\mathcal{L}\right)^{(\mathfrak{b},F_{m,n})}=\frac{\Lambda(\mathfrak{c})^k}{(k-1)!\zeta^k\varphi(\mathfrak{c})^k}E_k\left(\varphi(\mathfrak{bc})\rho,\mathcal{L}\right)$$

by [GS81, Théorème 6.2].

On replacing $\mathfrak{c}$ by $\mathfrak{bc}$ in (6.7), we have

$$\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,\tau_{\mathfrak{bc}}\right)}{(\zeta\rho_{m,n})^k}=\frac{\Lambda(\mathfrak{bc})^k}{(k-1)!\zeta^k\varphi(\mathfrak{bc})^k}E_k\left(\varphi(\mathfrak{bc})\rho,\mathcal{L}\right).$$

But

$$\frac{\Lambda(\mathfrak{bc})^k}{\varphi(\mathfrak{bc})^k}=\left(\frac{\Lambda(\mathfrak{b})\Lambda(\mathfrak{c})^{(\mathfrak{b},H)}}{\varphi(\mathfrak{b})\varphi(\mathfrak{c})}\right)^k=\frac{\Lambda(\mathfrak{c})^k}{\varphi(\mathfrak{c})^k}.$$

Hence the result follows. ∎

Following Lemma 6.7, we assume that $\mathfrak{a}$ is chosen so that $(\mathfrak{a},F_{m,n})''=(\mathfrak{a},H)=1$. Furthermore, recall from [GS81, Proposition 4.10] that when $(\mathfrak{c},F_{m,n})''=1$ (so $E^{(\mathfrak{c})}=E$), the isogeny $\lambda(\mathfrak{c})$ sends a $\mathfrak{h}$-torsion P of E to $P^{(\mathfrak{c},F_{m,n})}\in E$. Therefore, by Lemma 6.6, for any given ideals $\mathfrak{b}$ and $\mathfrak{c}$ of K such that $(\mathfrak{b},F_{m,n})'=1$ and $(\mathfrak{c},F_{m,n})''=1$, we have

$$\begin{aligned}&D^k(\gamma_{\mathfrak{b},\mathfrak{a}})\circ\lambda(\mathfrak{b})\left((V\oplus Q_{m,n})^{(\mathfrak{c},F_{m,n})}\right)\\&=(-1)^{k-1}(k-1)!\left(N(\mathfrak{a})-\Lambda(\mathfrak{a})^k(\mathfrak{a},F_{m,n})\right)\left(\frac{\varphi(\mathfrak{b})^k}{\Lambda(\mathfrak{b})^k}\cdot\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{bcc}_0,F_{m,n})\right)}{(\zeta\rho_{m,n})^k}\right)\\&=\frac{(-1)^{k-1}(k-1)!\varphi(\mathfrak{b})^k}{\Lambda(\mathfrak{b})^k}\cdot\frac{N(\mathfrak{a})L_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{bcc}_0,F_{m,n})\right)-\varphi(\mathfrak{a})^kL_{\mathfrak{h}}\left(\overline{\varphi^k},k,(\mathfrak{abcc}_0,F_{m,n})\right)}{(\zeta\rho_{m,n})^k}.\end{aligned} \tag{6.8}$$

Let θ be a character of $\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)$, χ a character of $\mathrm{Gal}(F_{m,n}/\mathscr{R}(\mathfrak{g}))$, ρ a character of $\mathrm{Gal}(H/K)$, and put $\upsilon=\theta\chi\rho$, which we consider as a character of $\mathrm{Gal}(F_{m,n}/K)$ via (6.5).

From now on, we let $\{\mathfrak{b}_i : i \in I\}$ be a set of representatives of integral ideals in K such that $\mathrm{Gal}(H/K) = \{(\mathfrak{b}_i, H) : i \in I\}$. Furthermore, we assume that $\mathfrak{b}_i$ is coprime to $\mathfrak{p}$ for all i. We have

(6.9)
$$\begin{aligned}
&\sum_{\tau\in \mathrm{Gal}(F_{m,n}/\mathscr{R}(\mathfrak{g}))} \chi^{-1}(\tau) \sum_{\delta\in\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)} \theta^{-1}(\delta) \sum_{i\in I} \frac{\rho^{-1}(\mathfrak{b}_i)\Lambda(\mathfrak{b}_i)^k}{\varphi(\mathfrak{b}_i)^k} D^k(\gamma_{\mathfrak{b}_i,\mathfrak{a}}) \circ \lambda(\mathfrak{b}_i)(V^\delta \oplus Q^\tau_{m,n})\\
&= (-1)^{k-1}(k-1)! \sum_{\eta\in\mathrm{Gal}(F_{m,n}/K)} \upsilon^{-1}(\eta) \frac{N(\mathfrak{a})L_{\mathfrak{h}}\left(\overline{\varphi^k}, k, \eta(\mathfrak{c}_0, F_{m,n})\right) - \varphi(\mathfrak{a})^k L_{\mathfrak{h}}\left(\overline{\varphi^k}, k, \eta(\mathfrak{a}\mathfrak{c}_0, F_{m,n})\right)}{(\zeta\rho_{m,n})^k}\\
&= (-1)^{k-1}(k-1)! \frac{N(\mathfrak{a})\upsilon(\sigma_0)L_{\mathfrak{h}}\left(\overline{\varphi^k\upsilon}, k\right) - \varphi(\mathfrak{a})^k\upsilon(\sigma_0\tau_{\mathfrak{a}})L_{\mathfrak{h}}\left(\overline{\varphi^k\upsilon}, k\right)}{(\zeta\rho_{m,n})^k}\\
&= (-1)^{k-1}(k-1)! \left(N(\mathfrak{a}) - \varphi(\mathfrak{a})^k\upsilon(\tau_{\mathfrak{a}})\right)\upsilon(\sigma_0)\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k\upsilon}, k\right)}{(\zeta\rho_{m,n})^k},
\end{aligned}$$

where $\tau_{\mathfrak{a}}$ denotes $(\mathfrak{a}, F_{m,n})$.

The above calculations lead us to define the following rational function on E:

Definition 6.8. *Let $k \in \{1, 2, \dots, p-1\}$ and $\mathfrak{a}$ and ideal of $\mathcal{O}_K$ chosen as above. Let V be a primitive $\mathfrak{g}$-division point of E, ρ a character of $\mathrm{Gal}(H/K)$ and θ a character of $\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)$, we define a rational function on E sending $P \in E$ to*

$$\vartheta^{\rho,\theta,k}_{\mathfrak{a},V}(P) = \sum_{\delta\in\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)} \theta^{-1}(\delta) \sum_{i\in I} \frac{\rho^{-1}(\mathfrak{b}_i)\Lambda(\mathfrak{b}_i)^k}{\varphi(\mathfrak{b}_i)^k} D^k(\gamma_{\mathfrak{b}_i,\mathfrak{a}}) \circ \lambda(\mathfrak{b}_i)(V^\delta \oplus P).$$

6.4. **Gamma transforms and L-values.** In the previous section, we constructed the rational function $\vartheta^{\rho,\theta,k}_{\mathfrak{a},V} \in J(E)$. To prove Theorem A, let k be an integer between 1 and $p-1$. Let $F = \mathscr{R}(\mathfrak{g}q)$ as before. Let ρ be a character of $\mathrm{Gal}(H/K)$ satisfying

$$\mathrm{ord}_\pi\left(\sum_{i\in I}\frac{\rho^{-1}(\mathfrak{b}_i)}{\varphi(\mathfrak{b}_i)^k}\right) = 0. \tag{6.10}$$

Remark 6.9. *Note that the π-adic valuation in (6.10) is always non-negative since $\mathfrak{b}_i$ are coprime to $\mathfrak{p}$. Suppose that $p \nmid [H : K]$, then there exists at least one ρ such that (6.10) holds. Indeed,*

$$\sum_{\rho\in\widehat{\mathrm{Gal}(H/K)}}\sum_{i\in I}\frac{\rho^{-1}(\mathfrak{b}_i)}{\varphi(\mathfrak{b}_i)^k} = [H : K].$$

Therefore, if $\mathrm{ord}_\pi([H : K]) = 0$, then at least one of the summands should have zero π-adic valuation.

Let υ be a finite-order character of $\mathrm{Gal}(F_\infty/K)$ given by $\upsilon = \theta\phi\kappa\rho$ where θ is a character of $\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)$, ϕ is a character of $\mathrm{Gal}(F/\mathscr{R}(\mathfrak{g}))$, and κ is a finite-order character of $\mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_q^2$. We draw a field diagram to clarify the setting.

$$\begin{tikzcd}
\mathscr{R}(\mathfrak{g}q^\infty) = F_\infty \arrow[dd, dash] \arrow[dd, bend right, dash, "\mathbb{Z}_q^2"'] & & \\
{} \arrow[rr, dashed, dash] & & \kappa \\
\mathscr{R}(\mathfrak{g}q) = F \arrow[dddd, dash] \arrow[dr, dash] \arrow[dddddd, bend right, dash, "\Delta"'] & & \\
 & {} \arrow[r, dashed, dash] & \phi \\
 & \mathscr{R}(\mathfrak{g}) \arrow[ddl, dash] & \\
 & {} \arrow[r, dashed, dash] & \theta \\
H \arrow[dd, dash] & & \\
{} \arrow[rr, dashed, dash] & & \rho \\
K \arrow[d, dash] & & \\
\mathbb{Q} & &
\end{tikzcd}$$

We fix θ, ϕ, and a non-trivial ideal $\mathfrak{a} \in \operatorname{End}(E)$ coprime to $6q\mathfrak{p}\mathfrak{g}\prod_{i\in I}\mathfrak{b}_i$. Then we can associate with $\vartheta^{\rho,\theta,k}_{\mathfrak{a},V}$ an elliptic function measure, α on $\mathbb{Z}_q^2$ via Lemma 3.7. This α depends on the choice of $\mathfrak{a}$, V, θ, ρ, and k. We further define $\alpha^* = \alpha|_{(\mathbb{Z}_q^\times)^2}$.

We now relate the Gamma transform of α^* to special values of imprimitive algebraic L-functions.

Lemma 6.10. *For all finite-order characters κ of $\operatorname{Gal}(F_\infty/F)$,*

$$\operatorname{ord}_\pi\left(\Gamma_{\alpha^*}(\phi\kappa)\right) = \operatorname{ord}_\pi\left(\left(N(\mathfrak{a}) - \varphi(\mathfrak{a})^k\upsilon(\mathfrak{a})\right)L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\upsilon\varphi^k})\right),$$

where $\upsilon = \theta\phi\kappa\rho$.

Proof. Suppose that κ is a primitive character of $\operatorname{Gal}(F_{m,n}/F)$. Choose $\underline{\zeta} = (\zeta_m, \zeta_n) \in \mu_{q^\infty}^2$ such that ζ_1 (resp. ζ_2) is a q^{m+1}-th (resp. q^{n+1}-th) roots of unity. Set $Q_{m,n} = \delta(\underline{\zeta})$. Write $\chi = \phi\kappa$; this is a finite-order character on

$$\operatorname{Gal}(F_\infty/\mathscr{R}(\mathfrak{g})) \simeq \operatorname{Gal}(K_\infty/K) \times \operatorname{Gal}(F/\mathscr{R}(\mathfrak{g})) \simeq \mathbb{Z}_q^2 \times \operatorname{Gal}(F/\mathscr{R}(\mathfrak{g})).$$

Using Lemma 3.2 in conjunction with (6.9), yields

$$\begin{aligned}
\Gamma_{\alpha^*}(\phi\kappa) &= \tau(\chi)\sum_{g\in\operatorname{Gal}(F_{m,n}/\mathscr{R}(\mathfrak{g}))}\chi^{-1}(g)\vartheta^{\rho,\theta,k}_{\mathfrak{a},V}(Q^g_{m,n}) \\
&= (-1)^{k-1}(k-1)!\tau(\chi)\left(N(\mathfrak{a}) - \varphi(\mathfrak{a})^k\upsilon(\tau_{\mathfrak{a}})\right)\upsilon(\sigma_0)\frac{L_{\mathfrak{h}}\left(\overline{\varphi^k\upsilon}, k\right)}{(\zeta\rho_{m,n})^k}.
\end{aligned}$$

Standard facts about Gauss sums tell us that $\operatorname{ord}_\pi(\tau(\chi)) = 0$ since the conductor of χ is coprime to p. Furthermore, $1 \leq k \leq p-1$, so $(k-1)!$ is a p-adic unit. Finally, as υ is a finite character, $\upsilon(\sigma_0)$ is a root of unity. This completes the proof of the lemma. $\square$

We now study the factor $N(\mathfrak{a})-\varphi(\mathfrak{a})^k\upsilon(\tau_{\mathfrak{a}})$. Recall that $\tau_{\mathfrak{a}}$ denotes $(\mathfrak{a}, F_{m,n})$, and thus depends on m and n, a priori. However, We may regard it as an element of $\mathrm{Gal}(F_\infty/K)$ since the Artin symbols $(\mathfrak{a}, F_{m,n})$ are compatible under restriction as $\mathfrak{h}$ varies over ideals dividing $\mathfrak{g}q^\infty$.

Lemma 6.11. *For a Zariski dense set of κ, we have*

$$\mathrm{ord}_\pi\left(N(\mathfrak{a})-\varphi^k(\mathfrak{a})\upsilon(\tau_{\mathfrak{a}})\right)=0.$$

Proof. Suppose the contrary. Let $(\zeta_1,\zeta_2)=(\kappa(\tau_{\mathfrak{a},\mathfrak{q}}),\kappa(\tau_{\mathfrak{a},\mathfrak{q}^*}))\in\mu_{q^\infty}^2$, where $\tau_{\mathfrak{a},\mathfrak{l}}$ denotes the restriction of $\tau_{\mathfrak{a}}$ to $\mathrm{Gal}(\mathscr{R}(\mathfrak{g}\mathfrak{l}^\infty)/\mathscr{R}(\mathfrak{g}))$. Then,

$$\mathrm{ord}_\pi\left(N(\mathfrak{a})-\varphi^k(\mathfrak{a})\upsilon(\tau_{\mathfrak{a}})\right)=0$$

if and only if

$$N(\mathfrak{a})\varphi(\mathfrak{a})^{-k}(\theta\phi(\tau_{\mathfrak{a}}))^{-1}\not\equiv\zeta_1\zeta_2\mod\pi\mathfrak{O}.$$

Note that the left-hand side is independent of κ. In particular, this condition is invariant under the map $(\zeta_1,\zeta_2)\mapsto(\zeta_1,\zeta_2)^{p^r}$, where p^r is the cardinality of the residue field of $\mathfrak{O}$.

Our assumption that the set of κ satisfying the stated property above is not Zariski dense allows us to apply Lemma 4.4. Let P be the power of p^r given by the said lemma. In particular, under the isomorphism $\mu_{q^\infty}^2\cong(\mathbb{Q}_q/\mathbb{Z}_q)^2$, there exists an arbitrary large n such that

$$N(\mathfrak{a})\varphi(\mathfrak{a})^{-k}\left(\theta\phi(\tau_{\mathfrak{a}})\right)^{-1}\equiv\zeta_1\zeta_2\mod\pi\mathfrak{O}\tag{6.11}$$

for all (ζ_1,ζ_2) which can be identified with $\left(\frac{P^x}{q^n},\frac{P^y}{q^n}\right)$, where $x,y\in\mathbb{Z}$. In particular, Remark 4.5 tells us that there are $q^{2(n-v)}$ such elements, where $v=\mathrm{ord}_q(P-1)$.

Note that q-power roots of unity modulo $\pi\mathfrak{O}$ are distinct since $p\neq q$. Suppose that the left-hand side of (6.11) modulo π is a q^m-th root of unity, where $m<n$. Then, for each q^n-th root of unity ζ_1, there are exactly q^m choices of q^n-th roots of unity ζ_2 such that (6.11) holds. This gives us at most q^{n+m} choices of $(\zeta_1,\zeta_2)\in\mu_{q^n}^2$. But this is a contradiction as soon as $n+m<2(n-v)$. ☐

Remark 6.12. *For a given ideal $\mathfrak{a}$, denote the Zariski dense set of characters described in Lemma 6.11 by $Z_{\mathfrak{a}}$. This set is defined by the equation*

$$f(\mathfrak{a}):=N(\mathfrak{a})-\varphi^k(\mathfrak{a})\upsilon(\tau_{\mathfrak{a}})\not\equiv0\mod\pi\mathfrak{O}.$$

But note that $Z_{\mathfrak{a}}=\bigcup_{m\in\pi\mathfrak{O}}Z_{m,\mathfrak{a}}$ where each $Z_{m,\mathfrak{a}}$ is defined by equation

$$f_m(\mathfrak{a}):=f(\mathfrak{a})-m\neq0,$$

as m varies over elements of $\pi\mathfrak{O}$. Since each $Z_{m,\mathfrak{a}}$ is Zariski open, it follows that $Z_{\mathfrak{a}}$ is Zariski open.

The following lemma, which generalizes [Lam15, Lemma 6.7], will be crucial in our proof of Theorem A.

Lemma 6.13. *Suppose that $\mathfrak{a}\subset\mathcal{O}_K$ is a non-trivial ideal such that $\gcd\left(\mathfrak{a},6q\mathfrak{p}\mathfrak{g}\prod_{i\in I}\mathfrak{b}_i\right)=1$ satisfying $(\mathfrak{a},F_{m,n})''=1$ and $\mathfrak{a}\equiv1\mod\mathfrak{f}$. Then,*

$$\mathrm{ord}_\pi\left(\vartheta_{\mathfrak{a},V}^{\rho,\theta,k}\right)=0.$$

Proof. Let us first recall the following facts proved in [Lam15, proof of Lemma 6.7].

(a) The rational function $D^k(\gamma_{\mathfrak{b},\mathfrak{a}})$ on $E^{(\mathfrak{b})}$ has poles of order k at all the elements of $P \in E_{\mathfrak{a}}^{(\mathfrak{b})} \setminus \{0\}$, with leading coefficient with respect to $z - z_P$ equal to $(-1)^{k-1}(k-1)!$.
(b) Furthermore, $D^k(\gamma_{\mathfrak{b},\mathfrak{a}})$ has a pole of order k at $P = 0$, with leading coefficient with respect to z equal to $(-1)^{k-1}(N(\mathfrak{a}) - 1)$.
(c) The poles described above are the only poles of $D^k(\gamma_{\mathfrak{b},\mathfrak{a}})$.
(d) Let $x_{\mathfrak{b}}$ and $y_{\mathfrak{b}}$ be the functions sending a point $P \in E^{(\mathfrak{b})}$ to its x- and y-coordinates given by the Weierstrass equation (6.3). The only zeros of the function $x_{\mathfrak{b}}(P) - x_{\mathfrak{b}}(R)$ are $P = R$ and $P = \ominus R$. If $x_{\mathfrak{b}}(R) \neq 0$, these are simple zeros and the leading coefficient with respect to $z - z_P$ is given by $y_{\mathfrak{b}}(P)$.

Let $i \in I$. Since $\mathfrak{a}$ is coprime to $\mathfrak{b}_i$, the isogeny $\lambda(\mathfrak{b}_i)$ induces an isomorphism $E_{\mathfrak{a}} \cong E_{\mathfrak{a}}^{(\mathfrak{b}_i)}$. Therefore, by (c), the poles of $D^k(\gamma_{\mathfrak{b}_i,\mathfrak{a}}) \circ \lambda(\mathfrak{b}_i)$ are precisely the elements in $E_{\mathfrak{a}}$. Recall that

$$\vartheta_{\mathfrak{a},V}^{\rho,\theta,k}(P) = \sum_{i \in I} \frac{\rho^{-1}(\mathfrak{b}_i)\Lambda(\mathfrak{b}_i)^k}{\varphi(\mathfrak{b}_i)^k} \sum_{\delta \in \mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)} \theta^{-1}(\delta) D^k(\gamma_{\mathfrak{b}_i,\mathfrak{a}}) \circ \lambda(\mathfrak{b}_i)(V^\delta \oplus P).$$

In particular, the poles of $\vartheta_{\mathfrak{a},V}^{\rho,\theta,k}(P)$ are given by $U \ominus V^\delta$, where $U \in E_{\mathfrak{a}}$ and $\delta \in \mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)$.

Let P be a pole of $D^k(\gamma_{\mathfrak{b}_i,\mathfrak{a}}) \circ \lambda(\mathfrak{b}_i)$. By (6.4), the leading coefficient of $D^k(\gamma_{\mathfrak{b}_i,\mathfrak{a}}) \circ \lambda(\mathfrak{b}_i)$ with respect to $z - z_P$ is that of $D^k(\gamma_{1,\mathfrak{a}})$ multiplied by $\Lambda(\mathfrak{b}_i)^{-k}$, where $\gamma_{1,\mathfrak{a}}$ denotes the rational function on E (so corresponding to the choice of i gives $E^{(\mathfrak{b}_i)} = E$). Consequently, (a) tells us that the leading coefficient of $\vartheta_{\mathfrak{a},V}^{\rho,\theta,k}$ with respect to $z - z_P$, when P is the pole $U \ominus V^\sigma$ where $U \in E_{\mathfrak{a}} \setminus \{0\}$, is given by

$$(-1)^{k-1}(k-1)! \sum_{i \in I} \frac{\rho^{-1}(\mathfrak{b}_i)}{\varphi(\mathfrak{b}_i)^k} \theta^{-1}(\delta),$$

which has π-adic valuation equal to zero by assumption (6.10).

Let $i \in I$, $\delta \in \mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H)$ and $Q \in E_{\mathfrak{a}} \setminus \{0\}$. By (d), the rational functions (on E) given by $x_{\mathfrak{b}_i} \circ \lambda(\mathfrak{b}_i)(P \oplus V^\delta) - x_{\mathfrak{b}_i} \circ \lambda(\mathfrak{b}_i)(Q)$ and $x(P \oplus V^\delta) - x(Q)$ (where x denotes the x-coordinate function on E) have the same zeros. Furthermore, by (6.4), the leading terms of these two rational functions differ by the constant $\Lambda(\mathfrak{b}_i)$. Consequently, these two functions differ by a unit in $\mathfrak{O}$. Therefore, as in [Lam15, proof of Lemma 6.7], we can write

$$\vartheta_{\mathfrak{a},V}^{\rho,\theta,k}(P) = g(P) \prod_{\substack{\delta \in \mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H) \\ Q \in \left(E_{\mathfrak{a}} \setminus \{0\}\right)/\pm 1}} \Big((x(P \oplus V^\delta) - x(Q) \Big)^{-k},$$

where g is a rational function on E belonging to

$$\mathfrak{O}\Big[x\left(\lambda(\mathfrak{b}_i)(P \oplus V^\delta)\right), y\left(\lambda(\mathfrak{b}_i)(P \oplus V^\delta)\right) : i \in I, \delta \in \mathrm{Gal}(\mathscr{R}(\mathfrak{g})/H) \Big].$$

In particular $\mathrm{ord}_\pi(g) \geq 0$.

As has been established in [Lam15, proof of Lemma 6.7], the functions $x(P \oplus V^\delta) - x(Q)$ take values in $\mathfrak{O}^\times$ for almost all P. Furthermore, by comparing leading terms at $P = U \ominus V^\delta$, we deduce that g takes values in $\mathfrak{O}^\times$ at these points. Thus, $\mathrm{ord}_\pi(g) = 0$, which concludes the proof. $\square$

We conclude this article with the following slightly more general version of Theorem A.

Theorem 6.14. *Let θ, ϕ, ρ and k be as above. For a Zariski dense set of characters κ of* $\mathrm{Gal}(F_\infty/F)$,

$$\mathrm{ord}_\pi\left(L^{(\mathrm{alg})}\left(\overline{\theta\phi\kappa\rho\varphi^k}\right)\right) = 0.$$

Proof. Let $\upsilon = \theta\phi\kappa\rho$. An argument similar to Lemma 6.11 tells us that it suffices to prove the theorem for imprimitive values $L_{\mathfrak{h}}^{(\mathrm{alg})}\left(\overline{\upsilon\varphi^k}\right)$ because for almost all finite-order characters κ of $\mathrm{Gal}(F_\infty/F)$

$$\mathrm{ord}_\pi\left(L^{(\mathrm{alg})}\left(\overline{\upsilon\varphi^k}\right)\right) = \mathrm{ord}_\pi\left(L_{\mathfrak{h}}^{(\mathrm{alg})}\left(\overline{\upsilon\varphi^k}\right)\right).$$

Indeed, for any prime ideal $\mathfrak{r}$ of K and for almost all characters κ,

$$\mathrm{ord}_\pi\left(1 - \frac{\overline{\varphi^k\upsilon(\mathfrak{r})}}{N(\mathfrak{r})^k}\right) = 0$$

as the q-power roots of unity modulo $\pi\mathfrak{O}$ are distinct since $p \neq q$.

Lemma 6.13 asserts that $\mathrm{ord}_\pi \vartheta_{\mathfrak{a},V}^{\rho,\theta,k} = 0$. In particular, the associated elliptic function measure α^* satisfies $\mathrm{ord}_\pi \alpha^* = 0$. Therefore, on combining Lemma 6.10 with Theorem 5.1, we deduce that for a Zariski dense set of κ, we have

$$\mathrm{ord}_\pi\left(\left(N(\mathfrak{a}) - \varphi(\mathfrak{a})^k\upsilon(\tau_{\mathfrak{a}})\right) L_{\mathfrak{h}}^{(\mathrm{alg})}\left(\overline{\upsilon\varphi^k}\right)\right) = 0.$$

The same argument as in Remark 6.12 shows that this Zariski dense set is also open. Since the intersection of two open dense sets is open dense, there exists a dense set of characters κ with

$$\mathrm{ord}_\pi\left(L^{(\mathrm{alg})}\left(\overline{\upsilon\varphi^k}\right)\right) = 0.$$

□

Appendix A. appendix

In this appendix we carry out a technical calculation required in the proof of Lemma 6.6. For this calculation, we rely heavily on the work of de Shalit in [dS87]. In particular, we express special L-values in terms of logarithmic derivatives of rational functions. We do so by relating both of these quantities to values of Eisenstein series.

A.1. **Relating rational functions to Eisenstein series.** As in the main text, let K be an imaginary quadratic field and H/K be the Hilbert class field of K. Let $E_{/H}$ be a CM elliptic curve with CM by $\mathcal{O}_K$ and $\mathcal{L}$ be the associated lattice. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of K such that $\mathfrak{b}$ is coprime to $6\mathfrak{f}$. With respect to $\mathcal{L}_{\mathfrak{b}}$, we can define an *elliptic function*, denoted by $\Theta(z;\mathcal{L}_{\mathfrak{b}},\mathfrak{a})$, as in [dS87, Chapter II, Section 2.3, (10) on p. 49]. Let $\xi_{\mathfrak{b}}$ be the isomorphism of complex Lie groups defined in (6.2). It follows from [dS87, (16) on p. 54] that for any $z \in \mathbb{C}$ with $P = \xi_{\mathfrak{b}}(z) \in E^{(\mathfrak{b})}$,

$$\Theta(z;\mathcal{L}_{\mathfrak{b}},\mathfrak{a}) = C_{\mathfrak{b},\mathfrak{a}} \cdot \zeta_{\mathfrak{b},\mathfrak{a}}(P)^{12}, \tag{A.1}$$

where $\zeta_{\mathfrak{b},\mathfrak{a}}(P)$ is the rational function introduced in (6.6) and $C_{\mathfrak{b},\mathfrak{a}}$ is some constant that is independent of P and z (the power of 12 appears because the product in (6.6) is taken over $\mathfrak{a}$-torsions modulo ±1, whereas the product in [dS87, (16) on p. 54] is taken over all non-trivial $\mathfrak{a}$-torsions, without modulo ±1).

For each $k \geq 1$, we define the k-th *Eisenstein series* associated to $\mathcal{L}_{\mathfrak{b}}$ as

$$E_k(z, \mathcal{L}_{\mathfrak{b}}) = \sum_{w \in \mathcal{L}_{\mathfrak{b}}}{}' \frac{(\overline{z} + \overline{w})^k}{|z+w|^{2k}}.$$

Here, the sum runs over all $w \in \mathcal{L}_{\mathfrak{b}}$ except possibly $w = -z$ if $z \in \mathcal{L}_{\mathfrak{b}}$. Further, for each integral ideal $\mathfrak{a}$, we can define (see [dS87, (5) on p. 57], where we take j to be 0)

$$E_k(z; \mathcal{L}_{\mathfrak{b}}, \mathfrak{a}) = (N\mathfrak{a}) E_k(z, \mathcal{L}_{\mathfrak{b}}) - E_k(z, \mathfrak{a}^{-1}\mathcal{L}_{\mathfrak{b}}).$$

Let ∂ denote the operator $-\frac{\partial}{\partial z}$. Recall that the operator D defined by $D^k(f) = \frac{d^k}{dz^k} \log f(z)$. From (A.1), we deduce that, for $k \geq 1$,

$$\begin{aligned} 12D^k(\gamma_{\mathfrak{b},\mathfrak{a}})(P) &= (-1)^k \partial^k \log \Theta(z; \mathcal{L}_{\mathfrak{b}}, \mathfrak{a}) \\ &= (-1)^{k-1} 12 E_k(z, \mathcal{L}_{\mathfrak{b}}, \mathfrak{a}) \quad \text{by [dS87, Chapter II, Section 3.1, (7) on p.58].} \end{aligned} \tag{A.2}$$

A.2. **Relating Eisenstein series to rational L-values.** Recall that $\mathfrak{f}$ denotes the conductor of the Hecke character φ. Let $\mathfrak{m}$ be a principal ideal of $\mathcal{O}_K$ such that $\mathfrak{f}|\mathfrak{m}$. Let $\mathfrak{c}$ be another ideal which is coprime to $\mathfrak{m}$. Then for any $\Omega \in \mathbb{C}^\times$ [dS87, Chapter II, Proposition 3.5, p. 62] asserts that

$$E_k\left(\Omega, \mathfrak{c}^{-1}\mathfrak{m}\Omega\right) = (k-1)!\Omega^{-k}\varphi(\mathfrak{c})^k L_{\mathfrak{m}}(\overline{\varphi^k}, k, (\mathfrak{c}, \mathscr{R}(\mathfrak{m}))). \tag{A.3}$$

Let $\alpha \in \mathcal{O}_K$ be a generator of our chosen principal ideal $\mathfrak{m}$. We choose $\Omega \in \mathbb{C}^\times$ in (A.3) to be the period Ω_∞ so that

$$\mathcal{L} = \Omega_\infty \mathcal{O}_K.$$

Let ρ be the primitive $\mathfrak{m}$-division point on $\mathbb{C}/\mathcal{L}$ given by $\rho = \frac{\Omega_\infty}{\alpha}$. Then,

$$\begin{aligned} E_k\left(\Omega_\infty, \mathfrak{c}^{-1}\mathfrak{m}\Omega_\infty\right) &= E_k\left(\rho\alpha, \mathfrak{c}^{-1}\mathfrak{m}\Omega_\infty\right) \\ &= E_k\left(\rho\alpha, \mathfrak{c}^{-1}\mathfrak{m}\mathcal{L}\right) \\ &= \alpha^{-k} E_k\left(\rho, \mathfrak{c}^{-1}\mathcal{L}\right) \quad \text{by [dS87, Chapter II, Proposition 3.3(i)]} \\ &= \alpha^{-k}\Lambda(\mathfrak{c})^k E_k\left(\rho, \mathcal{L}\right)^{(\mathfrak{c}, \mathscr{R}(\mathfrak{m}))} \quad \text{by [dS87, Chapter II, Proposition 3.3(iii)],} \end{aligned}$$

where $\Lambda(\mathfrak{c}) \in H^\times$ is defined as in (6.1).

Combined with (A.3), the above calculation shows that

$$\begin{aligned} (k-1)! L_{\mathfrak{m}}\left(\overline{\varphi^k}, k, (\mathfrak{c}, \mathscr{R}(\mathfrak{m}))\right) &= \frac{\Lambda(\mathfrak{c})^k \Omega_\infty^k}{\alpha^k \varphi(\mathfrak{c})^k} E_k\left(\rho, \mathcal{L}\right)^{(\mathfrak{c}, \mathscr{R}(\mathfrak{m}))} \\ &= \frac{\rho^k \Lambda(\mathfrak{c})^k}{\varphi(\mathfrak{c})^k} E_k\left(\Lambda(\mathfrak{c})\rho, \mathcal{L}_{\mathfrak{c}}\right) \quad \text{by [GS81, Théorème 6.2].} \end{aligned} \tag{A.4}$$

Remark A.1. *In the special case when $H = K$, (i.e., K has class number 1) we know from* [dS87, (18) on p. 42] *that $\Lambda(\mathfrak{c}) = \varphi(\mathfrak{c})$. Moreover, it is also clear in this case that $\psi = \varphi$. Therefore, we obtain*

$$L_{\mathfrak{m}}\left(\overline{\psi^k}, k, (\mathfrak{c}, \mathscr{R}(\mathfrak{m}))\right) = \frac{\rho^k}{(k-1)!} E_k\left(\psi(\mathfrak{c})\rho, \mathcal{L}_{\mathfrak{c}}\right)$$

(c.f. [Lam15, Theorem 6.2]*).*

A.3. **Relating rational functions to L-values.** Our final step is to combine the calculations in the previous two sections to relate the image of the operator D^k applied to our chosen rational function to the k-th Eisenstein series. Let P be an $\mathfrak{m}$-torsion on E. We know from (A.2) that

$$\begin{aligned}
D^k(\gamma_{\mathfrak{c},\mathfrak{a}})(P) &= (-1)^{k-1}E_k(z;\mathcal{L}_{\mathfrak{c}},\mathfrak{a})\\
&= (-1)^{k-1}\Big((N\mathfrak{a})E_k(z,\mathcal{L}_{\mathfrak{c}}) - E_k(z,\mathfrak{a}^{-1}\mathcal{L}_{\mathfrak{c}})\Big)\\
&= (-1)^{k-1}\Big((N\mathfrak{a})E_k(z,\mathcal{L}_{\mathfrak{c}}) - \Lambda(\mathfrak{a})^k E_k(z,\mathcal{L}_{\mathfrak{c}})^{(\mathfrak{a},\mathscr{R}(\mathfrak{m}))}\Big) \quad \text{by [dS87, Ch. II, Prop. 3.3(iii)]}\\
&= (-1)^{k-1}\Big((N\mathfrak{a}) - \Lambda(\mathfrak{a})^k(\mathfrak{a},\mathscr{R}(\mathfrak{m}))\Big)E_k(z,\mathcal{L}_{\mathfrak{c}}).
\end{aligned}$$

Now, if we choose $P = \xi_{\mathfrak{c}}\left(\Lambda(\mathfrak{c})\rho\right)$, then we deduce that

$$D^k(\gamma_{\mathfrak{c},\mathfrak{a}})(P) = (-1)^{k-1}(k-1)!\Big((N\mathfrak{a}) - \Lambda(\mathfrak{a})^k(\mathfrak{a},\mathscr{R}(\mathfrak{m})\Big)\frac{\varphi(\mathfrak{c})^k}{\rho^k\Lambda(\mathfrak{c})^k}L_{\mathfrak{m}}\left(\overline{\varphi^k},k,(\mathfrak{c},\mathscr{R}(\mathfrak{m}))\right), \tag{A.5}$$

which is the formula that is utilized in the proof of Lemma 6.6.

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