GROWTH OF FINE SELMER GROUPS IN INFINITE TOWERS

DEBANJANA KUNDU

Abstract. In this paper, we study the growth of fine Selmer groups in two cases. First, we show that in an unramified p-class field tower the growth of the fine Selmer group is unbounded. This tower is non-Abelian and non p-adic analytic. Second, we study the growth in fine Selmer ranks in $\mathbb{Z}_p^d$ towers. We show that the growth of the fine Selmer group is unbounded in such towers as well. We obtain a sufficient condition to prove the $\mu = 0$ Conjecture for cyclotomic $\mathbb{Z}_p$-extensions. We show that in certain non-cyclotomic $\mathbb{Z}_p$-towers, the μ-invariant of the fine Selmer group can be arbitrarily large.

1. Introduction

Let E be an elliptic curve defined over a number field F. The Mordell-Weil theorem says that the group of rational points of E, denoted $E(F)$, is finitely generated. Selmer groups play a crucial role in studying the rational points on elliptic curves. In [17], Mazur introduced the Iwasawa theory of Selmer groups. He described the growth of the p-primary part of the Selmer group in $\mathbb{Z}_p$-towers and showed that in the "ordinary" case the growth is controlled.

In [6], Coates and Sujatha introduced the study of a certain subgroup of the Selmer group, called the fine Selmer group. They showed that these subgroups have stronger finiteness properties than the classical Selmer group.

Under some conditions, the (fine) Selmer ranks can stay bounded when moving up the $\mathbb{Z}_p$-tower. In [8], Greenberg studied the growth of the p-primary component of the Selmer group in *any* p-adic analytic Galois extension. In [20], Murty and Ouyang studied the growth of $\mathfrak{p}$-Selmer ranks for Abelian varieties with CM, in p-class field tower of F. By the work of Boston and Hajir, we know that this tower defines an infinite Galois extension of F whose Galois group is not p-adic analytic. Hence, growth in these towers is not covered by the general theory developed by Greenberg. Murty-Ouyang showed that the Selmer rank in such a tower is unbounded. We prove that in fact the fine Selmer rank in such a tower is unbounded.

In the 1980's in a series of papers, Cuoco and Monsky studied the growth of class groups in $\mathbb{Z}_p^d$-towers when $d > 1$. In Section 4, we study the growth of the p-rank of fine Selmer group in multiple $\mathbb{Z}_p$-extensions. We prove this growth is unbounded in general. When $d = 1$, we obtain a sufficient condition to prove the Classical $\mu = 0$ Conjecture. In a specific $\mathbb{Z}_p^2$-tower, it is possible to provide an explicit expression. Finally, in Section 5 we study the growth of fine Selmer groups in non-cyclotomic $\mathbb{Z}_p$-extensions. We prove an analogue of Iwasawa's theorem [10, Theorem 1] for fine Selmer groups and show that there exist non-cyclotomic $\mathbb{Z}_p$-extensions where the fine Selmer can have arbitrarily large μ-invariant.

Date: November 11, 2019.

2010 *Mathematics Subject Classification.* Primary 11R23.

Key words and phrases. Iwasawa Theory, Fine Selmer groups, Class groups, p-rank.

2. Preliminaries

2.1. **Notation.** Let $\varphi : A \to B$ be a homomorphism of Abelian groups (resp. modules). Set $M[\varphi]$ to denote $\ker(\varphi)$. When $A = B$, set $M_\varphi = \varinjlim_n M[\varphi^n]$.

Definition 2.1. *For an Abelian group G, its p-rank $r_p(G) := \dim_{\mathbb{Z}/p\mathbb{Z}} G[p]$. For an $\mathcal{O}$-module M, denote its $\mathfrak{p}$-primary part by $M_{\mathfrak{p}}$. Its $\mathfrak{p}$-rank, $r_{\mathfrak{p}}(M) = \dim_{\mathbb{Z}/p\mathbb{Z}} M_{\mathfrak{p}}[\mathfrak{p}]$.*

2.2. **Review of Fine Selmer Group.** Let K be an imaginary quadratic field and $\mathfrak{p}$ be an unramified prime in K. Let F/K be any finite Galois extension and E be an elliptic curve defined over it. Let $S = S(F)$ be a finite set of primes in F, which contains the Archimedean primes, the primes of bad reduction of E and the primes above $\mathfrak{p}$. Denote by F_S the maximal algebraic extension of F unramified outside S and set $G_S(F) = \mathrm{Gal}(F_S/F)$.

Define the $\mathfrak{p}$-fine Selmer group as

$$R^S_{\mathfrak{p}}(E/F) := \ker\left(H^1\left(G_S(F), E[\mathfrak{p}]\right) \to \bigoplus_{v \in S} H^1\left(F_v, E[\mathfrak{p}]\right)\right).$$

Analogously, define $R^S_{\mathfrak{p}^n}(E/F)$ for any $n \geq 1$. Taking direct limits, define the $\mathfrak{p}^\infty$-fine Selmer group (also called the $\mathfrak{p}$-primary fine Selmer group),

$$R_{\mathfrak{p}^\infty}(E/F) := \varinjlim_m R^S_{\mathfrak{p}^m}(E/F)$$

where the limit is with respect to the maps induced by $E[\mathfrak{p}^m] \hookrightarrow E[\mathfrak{p}^{m+1}]$. An equivalent definition is the following,

$$R_{\mathfrak{p}^\infty}(E/F) = \ker\left(H^1(G_S(F), E[\mathfrak{p}^\infty]) \to \bigoplus_{v \in S} H^1(F_v, E[\mathfrak{p}^\infty])\right).$$

There are analogous definitions for every finite extension, F'/F. When $F_\infty \subset F_S$, such that $\mathrm{Gal}(F_\infty/F)$ is a p-adic Lie group

$$R_{\mathfrak{p}^\infty}(E/F_\infty) := \varinjlim R_{\mathfrak{p}^\infty}(A/F').$$

where the inductive limit is over finite extensions of F contained in F_∞.

Let $p \neq 2$. For any elliptic curve E/F, define $R^S_p(E/F)$, $R_{p^\infty}(E/F)$, $R_{p^\infty}(E/F_\infty)$ analogously as in [25].

2.3. **Split Prime $\mathbb{Z}_p$-Extension.** With the above set up, further let $p \neq 2$ split as $\mathfrak{p}\bar{\mathfrak{p}}$ in K. Let E/F be an elliptic curve with good reduction at p and CM by $\mathcal{O}_K$

Remark 2.2. The hypotheses, $\mathrm{End}_F(E)$ is the maximal order of K, involves no real loss of generality since every E/F with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_F(E)$ isomorphic to K is isogenous over F to one with this property.

Since $\mathfrak{p}$ is a split prime above p in $K/\mathbb{Q}$, it follows

$$E[\mathfrak{p}] \simeq \mathcal{O}_K/\mathfrak{p}\mathcal{O}_K \simeq \mathbb{Z}/p\mathbb{Z}.$$

Therefore, as Abelian groups,

$$E[\mathfrak{p}^\infty] = \bigcup_{n \geq 1} E[\mathfrak{p}^n] \simeq \mathbb{Z}_p.$$

We recall the construction of the split prime $\mathbb{Z}_p$-extension from [5]. The natural way to define this extension is via points of finite order on E. Set $\widetilde{L} = F(E[\mathfrak{p}^\infty])$. The action of $\mathrm{Gal}(\widetilde{L}/F)$ on $E[\mathfrak{p}^\infty]$ gives a canonical injection

$$\chi_\infty : \mathrm{Gal}(\widetilde{L}/F) \hookrightarrow \mathbb{Z}_p^\times .$$

The image is of finite index in $\mathbb{Z}_p^\times$. We have the decomposition

$$\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$$

where μ_{p-1} is the group of $p-1$-th roots of unity. Via χ_∞, there is a corresponding decomposition of $\mathrm{Gal}(\widetilde{L}/F) = \Delta \times \Gamma'$. The image of Δ is a subgroup of μ_{p-1} and that of Γ' is a subgroup of $1 + p\mathbb{Z}_p$. The fixed field of Δ is a $\mathbb{Z}_p$ extension L/F, whose Galois group can be identified with Γ'.

By class field theory, there is a unique $\mathbb{Z}_p$-extension of K unramified outside of $\mathfrak{p}$. Denote this by K_∞, and call it the *split prime $\mathbb{Z}_p$-extension of K*. By the classical theory of CM, $L = FK_\infty$. Therefore, L/F is unramified outside the set of primes above $\mathfrak{p}$ and each prime above $\mathfrak{p}$ is ramified in L/F.

2.4. **Trivializing Extension.** Let $p \neq 2, 3$. Keep the above set up. Further, set

$$F = K(E[p]),\ F_\infty = K(E[p^\infty]),\ G = \mathrm{Gal}(F_\infty/F),\ \mathcal{G}_\infty = \mathrm{Gal}(F_\infty/K)$$

F_∞ is the *trivializing extension of F*. By the Weil pairing, F_∞ contains the cyclotomic $\mathbb{Z}_p$-extension of F, denoted F_{cyc}.

G is a pro-p group isomorphic to $\mathbb{Z}_p^2$. By the theory of CM, $\mathcal{G}_\infty = G \times \Delta$ where $\Delta \simeq \mathrm{Gal}(F/K)$. Note Δ is a finite Abelian group, and $p \nmid |\Delta|$ because p does not ramify in K by assumption. It is known that $F_\infty = F\widetilde{K}$, where $\widetilde{K}$ is the unique $\mathbb{Z}_p^2$-extension of K.

3. Growth of Fine Selmer Groups in Infinite Class Field Towers

In this section, our notations are consistent with [20]. Consider the setting of Section 2.3. Let F/K be any finite Galois extension where $\mathfrak{p}$ is unramified, and E be an elliptic curve defined over F with CM by $\mathcal{O}_K$. Let S be a finite set of primes in F containing the Archimedean primes, primes above p, and the primes of bad reduction of E.

Definition 3.1. *Set $H_S(F)$ to be the p-Hilbert S-class field of F. It is the maximal Abelian unramified p-extension of F such that all primes in S split completely.*

By class field theory, $\mathrm{Gal}\big(H_S(F)/F\big) \simeq \mathrm{Cl}_S(F)$, the S-ideal class group of F. Set F_∞^S to be the maximal unramified p-extension of F such that all primes in S split completely. Set $\Gamma = \Gamma_F = \mathrm{Gal}(F_\infty^S/F)$ and write $\{\Gamma_n\}_{n\geq 0}$ for its derived series. For each $n \geq 0$, the fixed field F_n corresponding to Γ_n is the p-Hilbert S-class field of F_{n-1}. Further, set $L_n = F_nL$ for every $n \geq 0$. Observe L_n/F_n is the unique $\mathbb{Z}_p$-extension inside $\widetilde{L_n} = F_n\left(E[\mathfrak{p}^\infty]\right)$, which is unramified outside $\mathfrak{p}$.

For the ease of the reader, we draw the field diagram.

$$
\begin{tikzcd}[column sep=small]
 & & & & & \widetilde{L_n} & & \\
 & & & & L_n \arrow[ur, dash, "\Delta"] \arrow[dr, dash] & & \widetilde{L} & \\
F_\infty^S \arrow[dr, dash] & & & & & L \arrow[ur, dash, "\Delta"] \arrow[dr, dash] & & K(E[\mathfrak{p}^\infty]) \\
 & F_n \arrow[uurrr, dash, "\Gamma'_n"] \arrow[dr, dash] & & & & & K_\infty \arrow[ur, dash] & \\
 & & F \arrow[uurrr, dash, "\Gamma'"] \arrow[dr, dash] & & & & & \\
 & & & K \arrow[uurrr, dash] \arrow[d, dash] & & & & \\
 & & & \mathbb{Q} & & & &
\end{tikzcd}
$$

For each $n \geq 0$ there are the following isomorphisms

$$\mathrm{Gal}(\widetilde{L_n}/L_n) \simeq \mathrm{Gal}(\widetilde{L}/L) = \Delta; \quad \mathrm{Gal}(\widetilde{L_n}/F_n) \simeq \Delta \times \Gamma'_n.$$

Let S_∞ denote the set of Archimedean primes. Define

$$\delta := \begin{cases} 0 & \text{if } \mu_p \subseteq F \\ 1 & \text{otherwise.} \end{cases}$$

Set $r_1(F)$ and $r_2(F)$ to mean the real and complex places of F respectively. In [7], Golod and Shafarevich showed that if the following inequality holds,

$$r_p\left(\mathrm{Cl}_S(F)\right) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + \left|S \setminus S_\infty\right|}$$

then Γ_F is infinite. Stark posed the following natural question.

Question. *Does $r_p\left(\mathrm{Cl}_S\left(F_n\right)\right)$ tend to infinity as n tends to infinity?*

For any pro-p group G, $\mathbb{Z}/p\mathbb{Z}$ is a G-module with trivial G-action. In particular, take $G = \Gamma_n$, the n-th term of the derived series. By class field theory and finiteness of $\mathrm{Cl}(F_n)$ (and hence of $\mathrm{Cl}_S(F_n)$) we know

$$\begin{aligned} r_p\left(\mathrm{Cl}_S(F_n)\right) &= r_p\left(\mathrm{Cl}_S(F_n)/p\right) \\ &= \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^1\left(\mathrm{Gal}(H_S(F_n)/F_n), \mathbb{Z}/p\mathbb{Z}\right)\right) \\ &= \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^1\left(\Gamma_n, \mathbb{Z}/p\mathbb{Z}\right)\right) \\ &= \dim_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Hom}\left(\Gamma_n, \mathbb{Z}/p\mathbb{Z}\right). \end{aligned}$$

This is the number of minimal generators of Γ_n. By a result of Lubotzky and Mann [14, Theorem A], the question posed by Stark is equivalent to asking whether Γ_F is p-adic analytic. The following result was shown independently by Boston, Hajir and Matar [2], [9], and [15].

Theorem 3.2. *Let F be a number field. If the following inequality holds*

$$r_p\left(\mathrm{Cl}_S(F)\right) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + \left|S \setminus S_\infty\right|},$$

then Γ_F is not p-adic analytic.

The main result we prove in this section is the following.

Theorem 3.3. *Let F be a number field containing the imaginary quadratic field K. Let $p \neq 2,\ 3$ be a prime that splits in K as $\mathfrak{p}\bar{\mathfrak{p}}$ such that $\mathfrak{p}$ is unramified in F/K. Suppose $E(F)[\mathfrak{p}] \neq 0$. Let S be a finite set of primes in F containing the Archimedean primes, primes above $\mathfrak{p}$ and primes where E has bad reduction. Assume that F satisfies the Golod-Shafarevich inequality*

$$r_p\left(\mathrm{Cl}_S(F)\right) \geq 2 + 2\sqrt{r_1(F) + r_2(F) + \delta + \left|S \setminus S_\infty\right|}.$$

F_∞^S be the maximal unramified non-constant pro-p extension of F where primes in S split completely and F_n be the n-th layer of the class field tower F_∞^S/F. Then the $\mathfrak{p}$-rank of the fine Selmer group of E/F_n becomes arbitrarily large as n tends to ∞.

Remark 3.4.
1. Theorem 3.3 is true for any d-dimensional CM Abelian variety.
2. The result of Lim-Murty [12, Theorem 6.2] crucially needs $E(F)[\mathfrak{p}] \neq 0$ (else, the inequality they establish is vacuous). Theorem 3.3 holds without assuming $E(F)[\mathfrak{p}] \neq 0$. By making this assumption we ensure $L_n = \widetilde{L_n}$, which simplifies notation considerably. Note Theorem 3.3 is stronger and implies this result of Lim-Murty.
3. Theorem 3.3 is the fine Selmer variant of [20, Theorem 4]. Our result implies the result of Murty-Ouyang. Indeed, we show the fine Selmer rank (hence the Selmer rank) is unbounded in the p-Hilbert S-class field tower (hence in the p-Hilbert class field tower).

3.1. **Proof of Theorem 3.3.** The proof is similar to [20, Theorem 4]. We begin by recording some lemma.

Lemma 3.5. *[20, Lemma 5] Let G be a pro-p group and M be a discrete p-primary G-module. Then $M = 0$ if and only if $H^0(G, M) = M^G = 0$.*

Either $E(F)[\mathfrak{p}]$ is trivial or it is all of $E[\mathfrak{p}]$. Therefore, Lemma 3.5 implies there are only two possibilities for $E(L_n)[\mathfrak{p}]$. We record this in the following lemma

Lemma 3.6.
(1) If $E(F)[\mathfrak{p}]$ is trivial, then $L_n \neq \widetilde{L_n}$ and $E(L_n)[\mathfrak{p}] = 0$ for every $n \geq 0$.
(2) If $E[\mathfrak{p}] \subset E(F)$, then $L_n = \widetilde{L_n}$. For every $n \geq 0$, $E(L_n)[\mathfrak{p}] = E[\mathfrak{p}^\infty]$.

Lemma 3.7. *The intersection of F_∞^S and L is a finite extension of F. In particular, if $E[\mathfrak{p}] \subseteq E(F)$ then F_∞^S and L are disjoint over F.*

Proof. The intersection $F_\infty^S \cap L$ is an Abelian extension of F. Observe, the maximal Abelian quotient of F_∞^S/F is the p-Hilbert S-class field F_1/F. Thus, $F_\infty^S \cap L \subset F_1$. This proves the first part.

Moreover, if $E[\mathfrak{p}] \subset E(F)$, then $L = \widetilde{L}$. Note L/F is totally ramified over $\mathfrak{p}$, but F_∞^S/F is totally unramified. So the two extensions are disjoint. □

By hypothesis, $L_n = \widetilde{L_n}$. Let M_n^S be the maximal Abelian pro-p unramified extension of L_n such that all primes of $S(L_n)$ split completely. Here, $S(L_n)$ is the (finite) set of primes in L_n which are above the primes in the finite set S. Finiteness of the $S(L_n)$ is a consequence of primes of $S(F_n)$ being finitely decomposed in the split prime $\mathbb{Z}_p$ extension L_n/F_n. Set $\mathfrak{X}_n^S = \mathrm{Gal}(M_n^S/L_n)$. Using standard Iwasawa theory techniques, $\mathfrak{X}_n^S$ is a $\mathbb{Z}_p[\Delta][[T]]$-module.

Lemma 3.8. *$\mathbb{Z}_p$-corank of $R_{\mathfrak{p}^\infty}(E/L_n) \to \infty$ as $n \to \infty$.*

Proof. Recall the following equality [22, 6.1]

$$R_{\mathfrak{p}^\infty}(E/L_n) = \operatorname{Hom}(\mathfrak{X}_n^S, E[\mathfrak{p}^\infty]).$$

Thus, to prove the lemma it suffices to show the $\mathbb{Z}_p[\Delta]$-rank of $\mathfrak{X}_n^S$ tends to ∞.

F_{n+1}/F_n is an Abelian extension where primes in $S(F_n)$ split completely and L_n/F_n is an Abelian extension unramified outside $\mathfrak{p}$. Their compositum is L_{n+1}. The (finitely many) primes in $S(L_n)$ split completely in L_{n+1}. Therefore, L_{n+1}/L_n is a subextension of M_n^S/L_n. We have

$$\operatorname{Gal}(L_{n+1}/L_n) = \operatorname{Gal}(F_{n+1}/F_{n+1} \cap L_n) = \operatorname{Gal}(F_{n+1}/F_n)$$

where the last equality follows from hypothesis. By Theorem 3.2, the p-rank of $\operatorname{Gal}(F_{n+1}/F_n)$ approaches ∞. The same holds for the p-rank of $\operatorname{Gal}(L_{n+1}/L_n)$. $\square$

Remark 3.9. $(\mathfrak{X}_n^S)_{\Gamma_n'}$ contains $\operatorname{Gal}(L_{n+1}/L_n)$. Therefore by Lemma 3.8, $\mathfrak{p}$-rank of $R_{\mathfrak{p}^\infty}(E/L_n)^{\Gamma_n'}$ is unbounded as n tends to ∞.

The fine-version of Mazur's Control Theorem was proven independently by Rubin [22, chapter VII] and Wuthrich [25].

Theorem. *(Control Theorem) The following map*

$$s_n : R_{\mathfrak{p}^\infty}(E/F_n) \to R_{\mathfrak{p}^\infty}(E/L_n)^{\Gamma_n'}$$

induced by the natural restriction is a pseudo-isomorphism with a finite kernel and cokernel whose orders are bounded as $n \to \infty$.

The control theorem and Remark 3.9 prove that $\mathfrak{p}$-rank of $R_{\mathfrak{p}^\infty}(E/F_n)$ is unbounded. This finishes the proof of Theorem 3.3.

4. Growth of p-Fine Selmer Ranks in $\mathbb{Z}_p^d$-Extensions

From here on, the focus is to study growth of fine Selmer groups in Abelian pro-p, p-adic analytic extensions. In this section, we prove results on the rank growth in $\mathbb{Z}_p^d$-extensions of a number field. In Theorem 4.2, we obtain a precise formula for the rank growth of fine Selmer groups in a specific $\mathbb{Z}_p^2$ extension. In the general setting, we can only prove that the p-rank growth is unbounded (see Theorem 4.8). Our results are a refinement of the result of Lim-Murty [12, Proposition 5.1].

4.1. **Notation:** Consider a $\mathbb{Z}_p^d$-extension F_∞/F. Set $\Sigma = \operatorname{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^d$. Further, write $\Sigma_n = \Sigma^{p^n}$ and $F_n = F_\infty^{\Sigma_n}$. Here, F_n/F is the unique $(\mathbb{Z}/p^n\mathbb{Z})^d$-extension of F inside F_∞. The Iwasawa algebra Λ_d, is the completed group ring of Σ over $\mathbb{Z}_p$; let $\overline{\Lambda}_d$ be the completion (with respect to the powers of the augmentation ideal) of the $\mathbb{Z}/p\mathbb{Z}$ group ring of Σ. We have,

$$\Lambda_d \simeq \mathbb{Z}_p[[T_1, T_2, \dots, T_d]]; \quad \overline{\Lambda}_d \simeq \mathbb{Z}/p\mathbb{Z}[[T_1, T_2, \dots, T_d]].$$

Let $M(F_\infty)$ (resp. $M_S(F_\infty)$) be the maximal Abelian unramified pro-p extension of F_∞ (resp. the maximal Abelian unramified pro-p extension of F_∞ such that primes in S are totally split). The Iwasawa-Greenberg module (resp. S-Iwasawa-Greenberg module) denote $X := \operatorname{Gal}(M(F_\infty)/F_\infty)$ (resp. $X_S := \operatorname{Gal}(M_S(F_\infty)/F_\infty)$). Let $\overline{X}$ (resp. $\overline{X_S}$) be the reduction mod p of the Iwasawa-Greenberg module (resp. S-Iwasawa-Greenberg module). Define a (resp. a') to be the height of the local ring $\overline{\Lambda}_d/\operatorname{Ann}\overline{X}$ (resp. $\overline{\Lambda}_d/\operatorname{Ann}\overline{X_S}$); so $1 \leq a, a' \leq d$.

4.2. **Trivialising Extension of an Elliptic Curves with CM.** Consider the setting of Section 2.4. The main result we will prove is the following.

Proposition 4.1. *Let E be an elliptic curve over the imaginary quadratic field K with CM by $\mathcal{O}_K$. Set $F = K(E[p])$ and F_∞ be the trivialising extension. Then*

$$\Big| r_p\left(R(E/F_n)\right) - 2r_p\left(\mathrm{Cl}(F_n)\right) \Big| = O(1)$$

where F_n/F is the unique $(\mathbb{Z}/p^n\mathbb{Z})^2$-extension of F contained in F_∞.

Proof. The proof is identical to that of [13, Theorem 5.1] provided the finite primes $S_f(F)$ are finitely decomposed in F_∞/F. Recall, primes above p ramify, and all ramified primes in this extension are finitely decomposed.

By hypothesis, E attains good reduction at all primes of F [23, Lemma 2] or [24]. The definition of $R(E/F_n)$ is independent of S, so choose S to be $S_p \cup S_\infty$. The condition of finite decomposition is met, and the proposition follows. □

Using the main result in [19], there is the following interesting corollary.

Theorem 4.2. *Let E be an elliptic curve over an imaginary quadratic field K, with CM by $\mathcal{O}_K$. Set $F = K(E[p])$ and F_∞ be the trivialising extension. Then for F_n and a as defined before,*

$$r_p\Big(R\left(E/F_n\right)\Big) = cp^{an} + O(p^{(a-1)n}).$$

When $a = 1$, c is an integer and when $a = 2$, $c = \mathrm{rank}_{\overline{\Lambda}_2}(\overline{X})$.

Proof. Consider [19, Theorem 1.9] when $d = 2$. Monsky proves that

$$r_p\left(\mathrm{Cl}(F_n)\right) = cp^{an} + O\left(p^{(a-1)n}\right).$$

When $a = 1$, $c \in \mathbb{Z}$; when $a = 2$, $c = \mathrm{rank}_{\overline{\Lambda}_2}(\overline{X})$. The theorem follows. □

Remark 4.3. When $a = 1$, using [19, Theorem 1.18] in fact,

$$r_p\Big(R\left(E/F_n\right)\Big) = cp^{an} + \delta_n$$

where δ_n is a periodic function of n for large n.

4.3. **General $\mathbb{Z}_p^d$ Extension.** Using a variant of Monsky's result in [19], we show that the fine Selmer rank growth is unbounded in a general $\mathbb{Z}_p^d$-extension.

Lemma 4.4. *[13, Lemma 3.2] Let G be any pro-p group and M be a discrete G-module that is cofinitely generated over $\mathbb{Z}_p$. Define $h_1(G) := r_p\Big(\left(H^1(G, \mathbb{Z}/p\mathbb{Z})\right)\Big)$. $h_i(G)$ is finite then $r_p\left(\left(H^1(G, M)\right)\right)$ is finite. Furthermore*

$$\begin{aligned} h_1(G) r_p(M^G) - r_p\Big((M/M^G)^G\Big) &\leq r_p\Big(H^1(G, M)\Big) \\ &\leq h_1(G)\Big(\mathrm{corank}_{\mathbb{Z}_p}(M) + \log_p(\big|M/M_{\mathrm{div}}\big|\Big). \end{aligned}$$

Lemma 4.5. *Let A be a d-dimensional Abelian variety defined over a number field F. Consider a finite set of primes S containing $S_p \cup S_{bad} \cup S_\infty$. Suppose $A(F)[p] \neq 0$. Then*

$$r_p\Big(R\left(A/F\right)\Big) \geq r_p\Big(R_p^S\left(A/F\right)\Big) \geq r_p\left(\mathrm{Cl}_S\left(F\right)\right) r_p\left(A\left(F\right)[p]\right) - 2d. \tag{1}$$

Proof. Let $H_S(F) = H$ be the p-Hilbert S-class field of F. Denote the Galois group, $\mathrm{Gal}(H_S(F)/F) = \mathrm{Cl}_S(F)$ by $\mathcal{G}$. We have

$$\begin{aligned} r_p\left(\mathrm{Cl}_S(F)\right) &= r_p\left(\mathrm{Cl}_S(F)/p\,\mathrm{Cl}_S(F)\right)\\ &= \dim_{\mathbb{Z}/p\mathbb{Z}}\left(H^1(\mathcal{G}, \mathbb{Z}/p\mathbb{Z})\right)\\ &= h_1(\mathcal{G}). \end{aligned}$$

The first equality follows from finiteness of the S-class group. The second equality is the definition of the S-class group. By Lemma 4.4, to prove Inequality 1, it suffices to prove

$$r_p\left(R\left(A/F\right)\right) \geq r_p\left(R_p^S\left(A/F\right)\right) \geq r_p\left(H^1(\mathcal{G}, A(H)[p])\right). \tag{2}$$

To prove this inequality, consider the following diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_p(A/F) & \longrightarrow & H^1(G_S(F),\ A[p]) & \longrightarrow & \bigoplus_{v\in S} H^1(F_v,\ A[p]) \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & R_p(A/H) & \longrightarrow & H^1(G_S(H),\ A[p]) & \longrightarrow & \bigoplus_{v\in S}\bigoplus_{w\mid v} H^1(H_w,\ A[p]) \end{array}$$

The vertical maps are given by restriction maps. Write $\gamma = \oplus_v \gamma_v$ where

$$\gamma_v : H^1\left(F_v,\ A[p]\right) \to \bigoplus_{w\mid v} H^1\left(H_w,\ A[p]\right).$$

By the inflation-restriction sequence, $\ker \gamma_v = H^1(\mathcal{G}_v,\ A[p])$ where $\mathcal{G}_v$ is the decomposition group of $\mathcal{G}$ at v. By the definition of the p-Hilbert S-class field, all primes in $S(F)$ split completely in H_S; hence $\mathcal{G}_v = 1$. Thus, $\ker\gamma$ is trivial.

By inflation-restriction, $\ker\beta = H^1\left(\mathcal{G},\ A(H)[p]\right)$. By digram chase,

$$H^1\left(\mathcal{G}, A\left(H\right)[p]\right) \hookrightarrow R_p(A/F).$$

This implies

$$r_p\left(R_p^S(A/F)\right) \geq r_p\left(H^1\left(\mathcal{G},\ A\left(H\right)[p]\right)\right).$$

The result follows. □

Remark 4.6. Lemma 4.5 holds at every layer F_n/F in the $\mathbb{Z}_p^d$-tower.

Repeating the argument as in the original paper of Monsky, it is possible to prove the following variant of [19, Theorem 1.9].

Theorem 4.7. *With notation introduced at the start of the section, there is a positive real constant c, such that*

$$r_p\left(\mathrm{Cl}_S(F_n)\right) = cp^{a'n} + O\left(p^{(a'-1)n}\right).$$

The fact that $c > 0$, follows from [18, Corollary to Theorem 1.8].

Theorem 4.8. *Let A be an Abelian variety defined over F, p be an odd prime such that $A(F)[p] \neq 0$. Let F_∞/F be a $\mathbb{Z}_p^d$ extension such that $\overline{X_S}$ is infinite. Let F_n/F be the n-th layer of this tower. Then as $n \to \infty$, $r_p\left(R(A/F_n)\right) \to \infty$.*

Proof. By Theorem 4.7, if $\overline{X_S}$ is infinite then $r_p\left(\mathrm{Cl}_S(F_n)\right)$ approaches infinity as n approaches infinity. The conclusion follows from Lemma 4.5. □

Remark 4.9. Theorem 4.8 is a fine Selmer variant of [12, Proposition 5.1]. Unlike the result of Lim-Murty, we do not need to impose any restrictions on the reduction type at p. This comes at a cost that we need a smaller group $\overline{X_S}$ to be infinite.

4.4. **Growth in $\mathbb{Z}_p$-Extensions.** We focus on the case $d = 1$. Let F be a number field and F_∞/F be *any* $\mathbb{Z}_p$-extension with $\Gamma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$. There is an isomorphism $\Lambda(\Gamma) \simeq \mathbb{Z}_p[[T]]$ given by $\gamma - 1 \mapsto T$, where γ is a topological generator of Γ. By the structure theorem for a finitely generated $\Lambda(\Gamma)$-module M, there is a pseudo-isomorphism

$$M \to \Lambda(\Gamma)^r \oplus \bigoplus_{i=1}^{s} \Lambda(\Gamma)\Big/\left(p^{m_i}\right) \oplus \bigoplus_{j=1}^{t} \Lambda(\Gamma)\Big/\left(f_j^{l_j}\right)$$

where s, t are finite, m_i, $l_j > 0$, and each f_j is a distinguished polynomial. If $r = 0$, define $\mu(M) := \sum_{i=1}^{s} m_i$.

Conjecture (Classical Iwasawa $\mu = 0$ Conjecture)**.** *Let F_{cyc}/F be the cyclotomic $\mathbb{Z}_p$-extension and $X(F_{\mathrm{cyc}})$ be the associated Iwasawa module. Then $\mu(X(F_{\mathrm{cyc}})) = 0$.*

Lemma 4.10. *[12, Lemma 5.3] Let M be a finitely generated $\Lambda(\Gamma)$-module and $w_n = (1+T)^{p^n} - 1$. For $n \gg 0$,*

$$r_p\left(M\Big/(p, w_n)\,M\right) = \left(r(M) + s(M)\right)p^n + O(1)$$

The Pontryagin dual of the fine Selmer group is denoted $Y(A/F_\infty)$. Also,

$$Y(A/F_\infty)\Big/pY(A/F_\infty) \simeq \left(R(A/F_\infty)[p]\right)^\vee.$$

We prove a simple variant of Inequality 1, which follows from the next lemma.

Lemma 4.11. *[13, Lemma 5.2] Let F_∞ be any $\mathbb{Z}_p$ extension of F and F_n be a subfield of F_∞ such that $[F_n : F] = p^n$. Let $S = S_p \cup S_\infty$. Then*

$$\left| r_p\left(\mathrm{Cl}(F_n)\right) - r_p\left(\mathrm{Cl}_S(F_n)\right)\right| = O(1).$$

Proof. The only finite primes in S are the primes above p, they are finitely decomposed in any F_∞/F. □

Proposition 4.12. *Let A/F be an Abelian variety and p be an odd prime. Suppose A has good reduction everywhere over F and $A(F)[p] \neq 0$. Let F_∞ be a $\mathbb{Z}_p$-extension of F and F_n be a subfield of F_∞ such that $[F_n : F] = p^n$. Then*

$$r_p\left(R(A/F_n)\right) \geq r_p\left(\mathrm{Cl}(F_n)\right) r_p\left(A(F_n)[p]\right) + O(1) \tag{3}$$

Proof. The definition of $R(A/F_n)$ is independent of S, choose $S = S(F_n)$ to be precisely the set of Archimedean primes and the primes above p. The proposition follows from Lemma 4.5 upon noting that Lemma 4.11 applies in this case. □

Using Proposition 4.12 we give the fine Selmer variant of a theorem of Lim-Murty [12, Theorem 5.6] and Česnavičius [3, Proposition 7.1].

Theorem 4.13. *Let $p \neq 2$. Let A be an Abelian variety defined over F with good reduction everywhere over F and $A(F)[p] \neq 0$. Let F_∞ be any $\mathbb{Z}_p$-extension of F. Then*

$$r\left(Y(A/F_\infty)\right) + s\left(Y(A/F_\infty)\right) \geq s\left(X(F_\infty)\right) r_p\left(A\left(F_\infty\right)[p]\right)$$

where $X(F_\infty)$ is the Iwasawa module over the $\mathbb{Z}_p$-extension F_∞/F.

Proof. Set $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$. Consider the following commutative diagram with the vertical maps given by restriction

$$\begin{array}{ccccccc}
0 & \to & R(A/F_n) & \to & H^1(G_S(F_n), A[p^\infty]) & \to & \bigoplus_{v_n} H^1(F_{n,v_n}, A[p^\infty]) \\
 & & \Big\downarrow r_n & & \Big\downarrow f_n & & \Big\downarrow \gamma_n \\
0 & \to & R(A/F_\infty)^{\Gamma_n} & \to & H^1(G_S(F_\infty), A[p^\infty])^{\Gamma_n} & \to & \left(\varinjlim_n \bigoplus_{v_n} H^1\left(F_{n,v_n}, A[p^\infty]\right)\right)^{\Gamma_n}
\end{array}$$

Note that $r_p\left(\ker(f_n)\right) \leq 2d$. Thus $\ker(r_n)$ has bounded p-rank. Using Lemma 4.10,

$$\begin{aligned}
\Big(r\left(Y(A/F_\infty)\right) + s\left(Y(A/F_\infty)\right)\Big)\, p^n &= r_p\left(R(A/F_n)\right) + O(1) \\
&\geq r_p\left(\mathrm{Cl}(F_n)\right) r_p\left(A(F_n)[p]\right) + O(1)
\end{aligned}$$

where the last inequality follows from Proposition 4.12. There exists n_0, such that for $n \geq n_0$

$$\mathrm{Cl}(F)\Big/ p\,\mathrm{Cl}(F) \to X(F_\infty)\Big/\left(p, \tfrac{w_n}{w_{n_0}}\right) \to 0.$$

The kernel of this map is bounded independent of n [21, Lemma 11.1.5]. Since $X(F_\infty)$ is *always* a finitely generated *torsion* $\Lambda(\Gamma)$-module, by Lemma 4.10

$$r_p\left(\mathrm{Cl}(F_n)\right) = s(X(F_\infty))p^n + O(1).$$

The result follows since $r_p(A(F_\infty)[p]) = r_p(A(F_n)[p])$. □

Remark 4.14.
(1) Theorem 4.13 can be proved without assuming A has good reduction everywhere over F. In this case one obtains,
$$r\left(Y(A/F_\infty)\right) + s\left(Y(A/F_\infty)\right) \geq s\left(X_S(F_\infty)\right) r_p\left(A(F_\infty)[p]\right).$$
(2) In the cyclotomic tower, all primes are finitely decomposed, hence Lemma 4.11 holds without the assumption $S = S_p \cup S_\infty$. For such a tower we may drop the hypothesis of *good reduction everywhere* from Theorem 4.13. Thus, for the cyclotomic $\mathbb{Z}_p$-extension, an immediate corollary is the following: if there exists *one* elliptic curve with $A(F)[p] \neq 0$ such that Coates-Sujatha Conjecture A holds (i.e. $r\left(Y(A/F_{\mathrm{cyc}})\right) = s\left(Y(A/F_{\mathrm{cyc}})\right) = 0$) [6], then the Classical $\mu = 0$ Conjecture holds for F_{cyc}/F.

Corollary 4.15. *Let $p \neq 2$. Given an Abelian variety A/F, there exists a finite extension L/F and a $\mathbb{Z}_p$-extension L_∞/L, such that $Y(A/L_\infty)$ is not $\Lambda(\Gamma)$-torsion or $\mu(Y(A/L_\infty)) > 0$.*

Proof. By a result of Iwasawa [10, Theorem 1], given a fixed positive integer N, there exists a finite-extension F'/F and a $\mathbb{Z}_p$-extension F'_∞/F, such that $\mu\left(X\left(F'_\infty\right)\right)$ is positive; in fact $\mu\left(X(F'_\infty)\right) > N$. Thus, $s\left(X(F'_\infty)\right)$ is positive.

Consider a finite extension L/F' such that A has good reduction everywhere over L and $A(L)[p] \neq 0$. Consider $L_\infty = LF'_\infty$, this is a $\mathbb{Z}_p$-extension of L. We know $\mu\left(X(L_\infty)\right) \geq \mu\left(X(F'_\infty)\right)$ [10]. Thus,

$$\begin{aligned}
r\left(Y(A/L_\infty)\right) + \mu\left(Y(A/L_\infty)\right) &\geq r\left(Y(A/L_\infty)\right) + s\left(Y(A/L_\infty)\right) \\
&\geq s\left(X(L_\infty)\right) r_p(A(L_\infty)[p]) \\
&\geq s\left(X(F'_\infty)\right) r_p(A(L_\infty)[p]) \\
&> 0.
\end{aligned}$$

If $r\left(Y(A/L_\infty)\right)$ is positive, then $Y(A/L_\infty)$ is not $\Lambda(\Gamma)$-torsion. Else, $s\left(Y(A/L_\infty)\right)$ is positive, hence the μ-invariant is positive. With this the proof is complete. $\square$

Remark 4.16.
(1) The Abelian variety analogue of the weak Leopoldt Conjecture is believed to be true for all $\mathbb{Z}_p$-extensions of a number field F. By [16, Theorem 2.2], this is equivalent to $Y(A/F_\infty)$ is $\Lambda(\Gamma)$-torsion; thus conjecturally $r\left(Y(A/F_\infty)\right)$ should always be 0. However, there is little unconditional evidence towards this claim for an anti-cyclotomic $\mathbb{Z}_p$-extension. Conditional on the *Heegner hypothesis*, Bertolini proved the elliptic curve analogue of the weak Leopoldt Conjecture for the anti-cyclotomic extension of an imaginary quadratic field [1].
(2) Corollary 4.15 can be obtained as a corollary of Theorem 4.7 with $d = 1$; i.e. without any hypothesis on the reduction type. However, this requires we know $\overline{X_S}$ is infinite (equivalently $s(X_S) > 0$).

5. Arbitrarily Large μ-Invariants of Fine Selmer Groups in Non-Cyclotomic $\mathbb{Z}_p$-Extensions

We begin by recalling a theorem proven by Iwasawa [10, Theorem 1].

Theorem 5.1 (Iwasawa's Theorem for Anti-Cyclotomic Extensions)**.** *Let F be the cyclotomic field of p-th or 4-th roots of unity according as $p > 2$ or $p = 2$. For any given integer $N \geq 1$, there exists a cyclic extension L/F of degree p and a $\mathbb{Z}_p$-extension L_∞/F_∞ such that*

$$\mu\left(X\left(L_\infty\right)\right) \geq N.$$

Remark 5.2. Theorem 5.1 is true for *any* field F which has an anti-cyclotomic $\mathbb{Z}_p$-extension F_{ac}/F. In particular, it is true for all CM fields.

The close relationship between fine Selmer groups and class groups in $\mathbb{Z}_p$-towers raises the following natural question: does an analogue of Iwasawa's theorem for anti-cyclotomic $\mathbb{Z}_p$-extensions [10, Theorem 1] hold for fine Selmer groups, i.e. can the μ-invariant associated with a fine Selmer group be arbitrarily large in an anti-cyclotomic $\mathbb{Z}_p$-extension. We answer this question in the affirmative.

Theorem 5.3. *Let F be the cyclotomic field of p-th roots of unity for $p > 2$. Let A/F be an Abelian variety of dimension d such that $A(F)[p] \neq 0$. Suppose the analogue of the weak Leopoldt conjecture holds. Given integer $N \geq 1$, there exists a cyclic Galois extension L/F of degree p and a $\mathbb{Z}_p$-extension L_∞/F_∞ such that*

$$\mu\Big(Y\left(A/L_\infty\right)\Big) \geq N.$$

5.1. Review of Iwasawa's Result.

In proving Theorem 5.1, Iwasawa crucially used a result of Chevalley [4] on *ambiguous class number formula*.

Definition 5.4. *Let F be a number field and L/F be a cyclic $\mathbb{Z}/p\mathbb{Z}$-extension. Let $G = \mathrm{Gal}(L/F) = \langle\sigma\rangle$. An ideal class $[\mathfrak{a}] \in \mathrm{Cl}(L)$ is called*

- ***ambiguous*** *if $[\mathfrak{a}]^\sigma = [\mathfrak{a}]$, i.e. there exists an element $\alpha \in L^\times$ such that $\mathfrak{a}^{\sigma-1} = (\alpha)$.*
- ***strongly ambiguous*** *if $\mathfrak{a}^{\sigma-1} = (1)$.*

The subgroup of the class group $\mathrm{Cl}(L)$ consisting of ambiguous ideal classes (resp. strongly ambiguous ideal classes) is denoted by $\mathrm{Am}(L/F)$ (resp. $\mathrm{Am}_{\mathrm{st}}(L/F)$).

Theorem (Ambiguous Class Number Formula)**.** *The number of ambiguous ideal classes is given by*

$$\# \operatorname{Am}(L/F) = h(F) \times \frac{p^{T-1}}{[E_F : E_F \cap NL^\times]} \tag{4}$$

$$\# \operatorname{Am}_{\mathrm{st}}(L/F) = h(F) \times \frac{p^{T-1}}{[E_F : NE_L]} \tag{5}$$

where $h(F)$ is the class number of the base field F, T is the number of ramified primes, E_F is the unit group of F, and $E_F \cap NL^\times$ is the subgroup of units that are norms of elements of L. Moreover, the above two formulas are equivalent.

Proof. See [4] or [11, Theorem 1]. □

The following variant of the ambiguous class number formula can be proved very similar to [11, Theorem 1].

Proposition 5.5. *Let F be a number field and L/F be a cyclic $\mathbb{Z}/p\mathbb{Z}$-extension with σ a generator of the Galois group $G = \operatorname{Gal}(L/F)$. Let D be the degree of the extension $L/\mathbb{Q}$ and T be the number of primes of F that ramify in L. Then,*

$$r_p\left(\operatorname{Am}_{\mathrm{st}}\left(L/F\right)\right) \geq T - D.$$

5.2. **Proof of Theorem 5.3.** The proof can be divided into four main steps. The first three are the same as that of Theorem 5.1.

Step 1: Let F_{ac}/F be an anti-cyclotomic $\mathbb{Z}_p$-extension. Let F_+ be the maximal totally real subfield of F. The anti-cyclotomic $\mathbb{Z}_p$-extension satisfies a special property: if F_n^{ac} is the n-th layer of the $\mathbb{Z}_p$-tower, then F_n^{ac} is Galois over F_+. Furthermore, $G_n = \operatorname{Gal}(F_n^{\mathrm{ac}}/F_+)$ is the dihedral group of order $2p^n$.

Step 2: Let $\mathfrak{l}_+ \nmid p$ be a prime ideal of F_+ which is inert in F, and be the unique prime ideal of F above $\mathfrak{l}_+$. Note $\mathfrak{l}_+$ is unramified in F_n^{ac}. Using group theoretic properties of the dihedral group and class field theory, it can be shown that $\mathfrak{l}$ is *totally split* in F_n^{ac} [10]. This holds for every n, therefore $\mathfrak{l}$ is totally split in F_{ac}/F. By Chebotarev density theorem, there are infinitely many prime ideals $\mathfrak{l}_+$ in F_+ which are inert in F. Thus, there are infinitely many prime ideals in F which split completely in F_{ac}/F.

Step 3: Choose prime ideals $\mathfrak{l}_1, \dots, \mathfrak{l}_t$, $t \geq 1$, in F which are prime to p and are totally split in F_{ac}/F. We know from Step 2 that there are infinitely many such primes. Let η be a non-zero element of F which is divisible exactly by the first power of $\mathfrak{l}_i$ for $1 \leq i \leq t$. Set

$$L = F\left(\sqrt[p]{\eta}\right); \qquad L_\infty = LF_{\mathrm{ac}}.$$

Note $F_{\mathrm{ac}} \cap L = F$ and L_∞/L is a $\mathbb{Z}_p$-extension. Let L_n be the n-th layer of the $\mathbb{Z}_p$ tower L_∞/L, then L_n/F_n is a cyclic extension of degree p. More precisely,

$$L_n = F_n^{\mathrm{ac}}\left(\sqrt[p]{\eta}\right), \qquad n \geq 0.$$

We need to prove a technical lemma. Recall that the definition of the p-primary fine Selmer group is independent of the choice of S, set $S = S(L) = S_p \cup S_{bad} \cup S_\infty$. The subset of finite primes of S be S_f. Further, set $|S_f| = s_0$

Lemma 5.6. *Let F be the cyclotomic field of p-th roots of unity for $p > 2$. Let A/F be an Abelian variety of dimension d such that $A(F)[p] \neq 0$. Suppose the* *analogue of the weak Leopoldt conjecture holds. Let L/F be a $\mathbb{Z}/p\mathbb{Z}$ extension as constructed in Theorem 5.1. Then,*

$$r_p\left(R(A/L_n)\right) \geq r_p\left(A\left(L_n\right)[p]\right)\Big(s\left(X(L_\infty)\right)p^n\Big) - 4ds_0p^n + c$$

where c is a constant.

Proof. The proof follows the same steps as Proposition 4.12. Since primes in this $\mathbb{Z}_p$-extension L_∞/L no longer satisfy the condition that primes are finitely decomposed, our analysis will be more intricate.

Step a: Consider the following short exact sequence for all n [21, Lemma 10.3.12],

$$\mathbb{Z}^{\left|S_f(L_n)\right|} \to \mathrm{Cl}(L_n) \xrightarrow{\alpha_n} \mathrm{Cl}_S(L_n) \to 0.$$

Comparing p-ranks in this short exact sequence [13, Lemma 3.2], we obtain

$$\Big|r_p\left(\mathrm{Cl}\left(L_n\right)\right) - r_p\left(\mathrm{Cl}_S\left(L_n\right)\right)\Big| \leq 2\left|S_f(L_n)\right| \leq 2s_0p^n.$$

Step b: Lemma 4.5 applied to the number field L_n yields

$$r_p\Big(R_p^S\left(A/L_n\right)\Big) \geq r_p\left(\mathrm{Cl}_S\left(L_n\right)\right)r_p\left(A\left(L_n\right)[p]\right) - 2d.$$

Therefore, we obtain

$$\begin{aligned}
r_p\Big(R\left(A/L_n\right)\Big) &\geq r_p\Big(R_p^S\left(A/L_n\right)\Big) \\
&\geq r_p\left(\mathrm{Cl}\left(L_n\right) - 2s_0p^n\right)r_p\left(A\left(L_n\right)[p]\right) - 2d \\
&\geq r_p\left(\mathrm{Cl}\left(L_n\right)\right)r_p\left(A\left(L_n\right)[p]\right) - 4ds_0p^n - 2d.
\end{aligned}$$

By the Structure Theorem (in particular Lemma 4.10),

$$r_p\left(\mathrm{Cl}\left(L_n\right)\right) = s\left(X(L_\infty)\right)p^n + O(1).$$

Plugging this back into the above inequality, proves the lemma. ☐

Step 4: We are assuming that the analogue of the weak Leopoldt Conjecture holds for the $\mathbb{Z}_p$-extension L_∞/L where $L_\infty = LF_{\mathrm{ac}}$. Now,

$$\begin{aligned}
s\left(Y(A/L_\infty)\right)p^n &= r_p\Big(R\left(A/L_n\right)\Big) + O(1) \\
&\geq r_p\left(A\left(L_n\right)[p]\right)\Big(s\left(X(L_\infty)\right)p^n\Big) - 4ds_0p^n + O(1) \\
&\geq r_p\left(A\left(L_n\right)[p]\right)\left(t - p(p-1) - 4ds_0\right)p^n + O(1)
\end{aligned}$$

The first equality follows from Lemma 4.10. The last line follows from Proposition 5.5 by observing that at least tp^n primes of F_n ramify in L_n (by construction) and $[L_n : \mathbb{Q}] = p(p-1)p^n$. By *Step 2* we know that t can be chosen to be arbitrarily large. Therefore, given $N \geq 1$ there exists L/F such that

$$s\left(Y(A/L_\infty)\right) \geq N.$$

Since $\mu\left(Y(A/L_\infty)\right) \geq s\left(Y(A/L_\infty)\right)$, the theorem follows.

Remark 5.7. The condition A/F is an Abelian variety such that $A(F)[p] = 0$ is a mild condition. We can base change to a finite extension F'/F such that $A(F')[p] \neq 0$. Theorem 5.3 can then be stated (and proved) in terms of F'.

Acknowledgement

The author thanks Kumar Murty for his constant support and encouragement. The author is grateful to R. Sujatha, Antonio Lei, Ahmed Matar, and Sudhanshu Shekhar for helpful discussions. Much thanks to all past and present members of the GANITA Lab.

References

[1] M. Bertolini. Iwasawa theory for elliptic curves over imaginary quadratic fields. *Journal de théorie des nombres de Bordeaux*, 13(1):1–25, 2001.
[2] N. Boston. Some cases of the Fontaine-Mazur conjecture. *J. Number Theory*, 42(3):285–291, 1992.
[3] K. Česnavičius. Selmer groups and class groups. *Compositio Mathematica*, 151(3):416–434, 2015.
[4] C. Chevalley. Sur la théorie du corps de classes dans les corps finis et les corps locaux. 1934.
[5] J. Coates. Infinite descent on elliptic curves with complex multiplication. In *Arithmetic and geometry*, pages 107–137. Springer, 1983.
[6] J. Coates and R. Sujatha. Fine selmer groups of elliptic curves over p-adic Lie extensions. *Math. Ann.*, 331(4):809–839, 2005.
[7] E. S. Golod and I. R. Shafarevich. On the class field tower. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 28(2):261–272, 1964.
[8] R. Greenberg. Introduction to Iwasawa theory for elliptic curves. arithmetic algebraic geometry (park city, ut, 1999), 407–464. *IAS/Park City Math. Ser*, 9, 1999.
[9] F. Hajir. On the growth of p-class groups in p-class field towers. *J. Algebra*, 188(1):256–271, 1997.
[10] K. Iwasawa. On the μ-invariants of $\mathbb{Z}_\ell$-extensions, number theory. *Algebraic Geometry and Commutative Algebra*, pages 1–11, 1973.
[11] F. Lemmermeyer. The ambiguous class number formula revisited. *J. Ramanujan Math. Soc*, 28:415–421, 2013.
[12] M. F. Lim and V. K. Murty. Growth of Selmer groups of CM abelian varieties. *Canadian J. Math.*, 67(3):654–666, 2015.
[13] M. F. Lim and V. K. Murty. The growth of fine selmer groups. *J. Ramanujan Math. Soc. 31, no. 1, 79–94*, 2016.
[14] A. Lubotzky and A. Mann. Powerful p-groups. II. p-adic analytic groups. *J. Algebra*, 105(2):506–515, 1987.
[15] A. Matar. Selmer groups and generalized class field towers. *Int. J. Number Theory*, 8(04):881–909, 2012.
[16] A. Matar. On the Λ-cotorsion subgroup of the Selmer group. *Asian Journal of Mathematics*, to appear.
[17] B. Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18(3-4):183–266, 1972.
[18] P. Monsky. The Hilbert-Kunz function. *Math. Annalen*, 263(1):43–49, 1983.
[19] P. Monsky. p-ranks of class groups in $\mathbb{Z}_p^d$-extensions. *Math. Ann.*, 263(4):509–514, 1983.
[20] V. K. Murty and Y. Ouyang. The growth of Selmer ranks of an abelian variety with complex multiplication. *Pure and Applied Mathematics Quarterly*, 2(2):539–555, 2006.

[21] J. Neukirch, A. Schmidt, and K. Wingberg. Cohomology of number fields, 2008.

[22] K. Rubin. *Euler Systems.(AM-147)*, volume 147. Princeton University Press, 2014.

[23] K. Rubin et al. Tate–shafarevich groups of elliptic curves with complex multiplication. In *Algebraic Number Theory—in honor of K. Iwasawa*, pages 409–419. Mathematical Society of Japan, 1989.

[24] J.-P. Serre and J. Tate. Good reduction of abelian varieties. *Ann. of Math.(2)*, 88(492-517):2, 1968.

[25] C. Wuthrich. *The fine Selmer group and height pairings.* PhD thesis, University of Cambridge, 2004.

Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Room 6290, Toronto, Ontario, Canada, M5S 2E4

Email address: dkundu@math.utoronto.ca