GROWTH OF p-PARTS OF IDEAL CLASS GROUPS AND FINE SELMER GROUPS IN $\mathbb{Z}_q$-EXTENSIONS WITH $p \neq q$

DEBANJANA KUNDU AND ANTONIO LEI

Abstract. Fix two distinct odd primes p and q. We study "$p \neq q$" Iwasawa theory in two different settings.
(1) Let K be an imaginary quadratic field of class number 1 such that both p and q split in K. We show that under appropriate hypotheses, the p-part of the ideal class groups is bounded over finite subextensions of an anticyclotomic $\mathbb{Z}_q$-extension of K.
(2) Let F be a number field and let $A_{/F}$ be an abelian variety with $A[p] \subseteq A(F)$. We give sufficient conditions for the p-part of the fine Selmer groups of A over finite subextensions of a $\mathbb{Z}_q$-extension of F to stabilize.

1. Introduction

Let $F/\mathbb{Q}$ be an algebraic number field and F_∞/F be a Galois extension with Galois group isomorphic to the additive group $\mathbb{Z}_q$ of q-adic integers. For each integer $n \geq 0$, there is a unique subfield F_n/F of degree q^n. Let $h(F_n)$ be the class number of F_n. K. Iwasawa showed that if q^{e_n} is the highest power of q dividing $h(F_n)$, then there exist integers λ, μ, ν independent of n, such that $e_n = \mu q^n + \lambda n + \nu$ for $n \gg 0$. On the other hand, in [Was75, Was78], L. C. Washington proved that for distinct primes p and q, the p-part of the class number stabilizes in the *cyclotomic* $\mathbb{Z}_q$-extension of an abelian number field. Washington's results have been extended to other $\mathbb{Z}_q$-extensions where primes are finitely decomposed. In particular, J. Lamplugh proved the following in [Lam15]: if p, q are distinct primes ≥ 5 that split in an imaginary quadratic field K of class number 1 and F/K is a prime-to-p abelian extension which is also unramified at p, then the p-class group stabilizes in the $\mathbb{Z}_q$-extension of F which is unramified outside precisely one of the primes above q.

Throughout this article, p and q will denote two distinct odd primes and K is an imaginary quadratic field of class number 1 in which both p and q split. We write $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ and $q\mathcal{O}_K = \mathfrak{q}\bar{\mathfrak{q}}$.

Given an ideal $\mathfrak{h}$ of $\mathcal{O}_K$, we write $\mathscr{R}(\mathfrak{h})$ for the ray class field of K of conductor $\mathfrak{h}$. Consider the infinite extension $\mathscr{R}(\mathfrak{g}q^\infty) = \bigcup_{n\geq 1} \mathscr{R}(\mathfrak{g}q^n)$, where $\mathfrak{g}$ is a fixed ideal of $\mathcal{O}_K$ coprime to q. We have the isomorphism

$$\mathrm{Gal}(\mathscr{R}(\mathfrak{g}q^\infty)/K) \cong \Delta \times \mathbb{Z}_q^2,$$

where $\Delta = \mathrm{Gal}(\mathscr{R}(\mathfrak{g}q)/K)$. We consider the anticyclotomic $\mathbb{Z}_q$-extension $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/\mathscr{R}(\mathfrak{g}q)$, i.e., the subextension of $\mathscr{R}(\mathfrak{g}q^\infty)/\mathscr{R}(\mathfrak{g}q)$ corresponding to the -1-eigensubspace of the complex conjugation in $\mathrm{Gal}(\mathscr{R}(\mathfrak{g}q^\infty)/\mathscr{R}(\mathfrak{g}q))$. We set $F = \mathscr{R}(\mathfrak{g}q)$ and write F_n to denote the unique subextension of $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}$ such that $\mathrm{Gal}(F_n/\mathscr{R}(\mathfrak{g}q)) \cong \mathbb{Z}/q^n\mathbb{Z}$.

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In the first half of this article, we study the growth of the p-part of the ideal class group of F_n as $n \to \infty$. This generalizes [Lam15, Theorem 1.3], where the stability of the p-part of the class numbers $\mathscr{R}(\mathfrak{g}\mathfrak{q}^n)$ is studied. More precisely, we prove the following result.

Theorem A. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ coprime to pq such that $\mathfrak{g}$ is a product of split primes*[1] *and $w_{\mathfrak{g}} = 1$, where $w_{\mathfrak{g}}$ denotes the number of roots of unity in K that are congruent to 1 modulo $\mathfrak{g}$. Let $F = \mathscr{R}(\mathfrak{g}q)$ be a prime-to-p extension of K and $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F$ be the anticyclotomic $\mathbb{Z}_q$-extension. Then, there exists an integer N such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)).$$

To prove this theorem, we will make use of a special case of a theorem of H. Hida, which guarantees the non-vanishing modulo p of certain algebraic L-functions (see Theorem 2.2). Using a result of Lamplugh (see Theorem 3.1), we show in Theorem 3.2 that certain p-primary Galois modules stabilize in the said anticyclotomic $\mathbb{Z}_q$-extension. Finally, Theorem A follows by an application of the Iwasawa Main Conjecture, which is known in this setting by the work of K. Rubin [Rub91].

In the second half of the article, we prove a general statement (see Theorem 4.3) which shows that in certain $\mathbb{Z}_q$-extensions of a number field F, the growth of the p-part of the class group is closely related to that of the p-primary *fine* Selmer group of an abelian variety $A_{/F}$. This subgroup of the classical p-primary Selmer group is denoted by $\mathrm{Sel}_0(A/F)$, and is obtained by imposing stronger vanishing conditions at primes above p (the precise definition is reviewed in §4.1). The following result is an application of the aforementioned theorem to the growth of the p-part of fine Selmer group of a fixed abelian variety A over a $\mathbb{Z}_q$-tower (which is not necessarily anticyclotomic).

Theorem B. *Let p and q be distinct odd primes. Let F be any number field and $A_{/F}$ be an abelian variety such that $A[p] \subseteq A(F)$. Let F_∞/F be a $\mathbb{Z}_q$-extension where the primes above q and the primes of bad reduction of A are finitely decomposed. If there exists $N \gg 0$ such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)),$$

then

$$\mathrm{Sel}_0(A/F_n) = \mathrm{Sel}_0(A/F_N).$$

In particular, Theorem B applies to the setting studied by Washington [Was75, Was78].

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2. A result of Hida on L-values of anticyclotomic Hecke characters

Throughout this section, K is a fixed imaginary quadratic field of class number one. We review a special case of a result of Hida from [Hid07] that will play a crucial role in our proof of Theorem A.

Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ that is a product of split primes of K coprime to pq with $w_{\mathfrak{f}} = 1$. Let $E_{/K}$ be an elliptic curve with CM by $\mathcal{O}_K$ such that its corresponding Hecke character ψ is of conductor $\mathfrak{f}$ (such an E exists by [dS87, Lemma on p. 41]; see also Appendix A).

[1] In this article, a split prime of K refers to a prime ideal of $\mathcal{O}_K$ that lies above a rational prime that splits in K.

Definition 2.1. Let $\mathfrak{h}$ be any integral ideal of K. Let ϵ be any Hecke character of K. The *imprimitive L-function* of ϵ modulo $\mathfrak{h}$ is defined as follows

$$\begin{aligned} L_{\mathfrak{h}}(\epsilon, s) &= \prod_{\gcd(\nu,\mathfrak{h})=1} \left(1 - \frac{\epsilon(\nu)}{(N\nu)^s}\right)^{-1} \\ &= \sum_{\gcd(\mathfrak{a},\mathfrak{h})=1} \frac{\epsilon(\mathfrak{a})}{(N\mathfrak{a})^s}, \end{aligned}$$

where the product runs over *prime ideals* of K coprime to $\mathfrak{h}$, and sum is taken over *integral ideals* coprime to $\mathfrak{h}$.

Fix an integral ideal $\mathfrak{g}$ of K, which is divisible by $\mathfrak{f}$, relatively prime to pq, and such that only split primes of K divide $\mathfrak{g}$. Note that we have $w_{\mathfrak{g}} = 1$ since $\mathfrak{f}|\mathfrak{g}$ and $w_{\mathfrak{f}} = 1$. Let $F = \mathscr{R}(\mathfrak{g}q)$ be the *ray class field* of K of conductor $\mathfrak{g}q$ and write $\Delta = \mathrm{Gal}(F/K)$. Set $F_\infty = \bigcup_{n\geq 1} \mathscr{R}(\mathfrak{g}q^n)$; this is a $\mathbb{Z}_q^2$-extension of F. We fix an isomorphism

$$\mathrm{Gal}(F_\infty/K) \simeq \mathrm{Gal}(F/K) \times \mathrm{Gal}(K_\infty/K) = \Delta \times \mathbb{Z}_q^2.$$

Let ϵ be a character of $\mathrm{Gal}(F_\infty/K)$. For our purpose, ϵ will be of the form $\overline{\varphi\psi^k}$, where φ is a finite-order character and k is an integer between 1 and $p-1$. Denote by $L(\epsilon, s)$ the *primitive Hecke L-function* of ϵ. Recall that the imprimitive (or partial) L-function differs from the primitive (or classical) L-function by a finite number of Euler factors. We can further define the *primitive algebraic Hecke L-function*,

$$L^{\mathrm{alg}}(\overline{\varphi\psi^k}) = L^{\mathrm{alg}}_{\mathfrak{h}}(\epsilon) := \frac{L(\epsilon, k)}{\Omega_\infty^k} = \frac{L\left(\overline{\varphi\psi^k}N^{-k}, 0\right)}{\Omega_\infty^k}.$$

Here, Ω_∞ denotes a complex period for $E_{/\mathbb{C}}$. Similarly, given an integral ideal $\mathfrak{h}$ of K, we define the *imprimitive algebraic Hecke L-function*,

$$L^{\mathrm{alg}}_{\mathfrak{h}}(\overline{\varphi\psi^k}) = L^{\mathrm{alg}}(\epsilon) := \frac{L_{\mathfrak{h}}(\epsilon, k)}{\Omega_\infty^k} = \frac{L_{\mathfrak{h}}\left(\overline{\varphi\psi^k}N^{-k}, 0\right)}{\Omega_\infty^k}.$$

In what follows, we say that a Hecke character ϵ of K is of *infinity type* (a, b) if its infinity component sends x to $x^a\overline{x}^b$. Under this convention, ψ has infinity type $(-1, 0)$, whereas the norm map N is of infinity type $(-1, -1)$. Thus, the Hecke character $\overline{\psi^k}N^{-k}$ is of infinity type $(k, 0)$.

Henceforth, we fix a prime $v|\mathfrak{p}$ of F and an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_p}$ so that v is sent into the maximal ideal of $\mathcal{O}_{\overline{\mathbb{Q}_p}}$. This allows us to consider $L^{\mathrm{alg}}_{\mathfrak{h}}(\overline{\varphi\psi^k})$ as elements of $\overline{\mathbb{Q}_p}$. Throughout, π will be a fixed uniformizer of F_v and we write ord_π for the valuation on $\overline{\mathbb{Q}_p}$ normalized so that $\mathrm{ord}_\pi(\pi) = 1$.

Theorem 2.2 (Hida)**.** *For all but finitely many characters φ that factor through $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}$, we have*

$$\mathrm{ord}_\pi\left(L^{\mathrm{alg}}_{(q)}(\overline{\varphi\psi^k})\right) = 0.$$

Proof. For each φ, we have $\overline{\varphi} = \phi\eta$, where ϕ is a character of Δ and η is a character of $\mathrm{Gal}(\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F)$. We may further decompose ϕ into $\phi'\nu^{-1}$, where ν is a character of $\mathrm{Gal}(F/\mathscr{R}(\mathfrak{g}))$ and ϕ' is a character of $\mathrm{Gal}(\mathscr{R}(\mathfrak{g})/K)$. We have the field diagram:

$$\begin{tikzcd}
\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}} \arrow[dd, dash] \arrow[dd, dash, bend right, "\mathbb{Z}_q"'] & & & \\
{} \arrow[rrr, dashed, dash] & & & \eta \\
\mathscr{R}(\mathfrak{g}q) = F \arrow[dddd, dash] \arrow[dddd, dash, bend right, "\Delta"'] \arrow[dr, dash] & & & \\
& {} \arrow[rr, dashed, dash] & & \nu^{-1} \\
& \mathscr{R}(\mathfrak{g}) \arrow[dd, dash] & & \\
& {} \arrow[rr, dashed, dash] & & \phi' \\
K \arrow[d, dash] & & & \\
\mathbb{Q} & & &
\end{tikzcd}$$

We take the CM field M in [Hid07] to be the imaginary quadratic field K. We take the CM type Σ there to be the one that corresponds to the infinity type $(1,0)$ and $\kappa = 0$. Then the infinity type of the character λ in *op. cit.* becomes

$$k\Sigma + 0(1-c) = k(1,0) + (0,0) - (0,0) = (k,0).$$

The condition (M1) in [Hid07, Theorem 4.3] does not hold since $K/\mathbb{Q}$ is not unramified. Hence, we can apply the aforementioned theorem with λ and χ^{-1} taken to be $\overline{\psi^k} N^{-k}\phi'$ and η, respectively. $\square$

Remark 2.3 ([Lam14, proof of Theorem 3.1.9]). Let $\mathfrak{g}$ be a fixed ideal as before. Fix an ideal $\mathfrak{h}$ of $\mathcal{O}_K$ which is coprime to $\mathfrak{p}$ and divisible by $\mathfrak{g}q$. Recall that the algebraic L-function of $\overline{\varphi\psi^k}$ modulo $\mathfrak{h}$ is given by

$$L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\varphi\psi^k}) = \frac{L_{\mathfrak{h}}(\overline{\varphi\psi^k}, k)}{\Omega_\infty^k}.$$

Then, for almost all characters of $\mathrm{Gal}\left(\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F\right) \cong \mathbb{Z}_q$, we have that

$$\mathrm{ord}_\pi\left(L_{(q)}^{(\mathrm{alg})}(\overline{\varphi\psi^k})\right) = \mathrm{ord}_\pi\left(L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\varphi\psi^k})\right).$$

This follows from the observation that for a given prime ideal $\mathfrak{a}$ of K that is coprime to q, for almost all characters η,

$$\mathrm{ord}_\pi\left(1 - \frac{\overline{\varphi\psi^k}(\mathfrak{a})}{(N\mathfrak{a})^k}\right) = 0$$

since η sends $\mathfrak{a}$ to a q-power roots of unity, which are distinct modulo π.

3. Consequences on class groups

In this section, we will make use of Theorem 2.2 to study the growth of the p-part of the class group in an anticyclotomic $\mathbb{Z}_q$-extension. Let us introduce the necessary notation. Throughout, $p \nmid 6q$ is a fixed prime that is split in K and $E_{/K}$ is a fixed CM elliptic curve as in the previous section (with Hecke character ψ whose conductor is $\mathfrak{f}$). Let K_0 be a finite abelian extension of K such that p is unramified in K_0 and $p \nmid [K_0 : K]$. Fix a prime $\mathfrak{p}$ of K lying above p. Set $L = K_0(E_{\mathfrak{p}})$

and $L_\infty = L(E_{\mathfrak{p}^\infty})$. Let $\Delta = \mathrm{Gal}(L/K)$ and $\Gamma = \mathrm{Gal}(L_\infty/L) \simeq \mathbb{Z}_p$. Let $\mathcal{G} = \mathrm{Gal}(L_\infty/K) \cong \Delta \times \Gamma$ and $\Lambda = \mathbb{Z}_p[\![\mathcal{G}]\!]$.

Following [Rub91], we write $\overline{\mathcal{C}}(L_\infty)$ (resp. $U(L_\infty)$) for the inverse limits over all finite sub-extensions inside L_∞ of the completion of the elliptic units (resp. local principal units) at $\mathfrak{p}$.

Fix an ideal $\mathfrak{h}$ of $\mathcal{O}_K$ which is coprime to $\mathfrak{p}$, is divisible by $\mathfrak{f}$, and is such that $K_0 \subset K(E_{\mathfrak{h}})$. Let μ_K be the group of roots of unity of K and $\lambda \in \mathcal{O}_K \setminus \mu_K$ such that $\lambda \equiv 1 \mod \mathfrak{h}$ with $(\lambda, 6\mathfrak{h}\mathfrak{p}) = 1$. We let $\sigma_{(\lambda)} \in \mathrm{Gal}(K_0/K)$ denote the Artin symbol associated to λ.

We further decompose Δ as $H \times I$, where $H = \mathrm{Gal}(K_0/K)$ and $I = \mathrm{Gal}(K(E_{\mathfrak{p}})/K)$. Here, I is the inertia subgroup at $\mathfrak{p}$ inside Δ. Let $\theta_{\mathfrak{p}}$ denote the canonical character given by the Galois action on $E_{\mathfrak{p}^\infty}$ restricted to I. Given a character χ of Δ, we write it as $\varphi\theta_{\mathfrak{p}}^k$, where φ is a character of H and $1 \leq k \leq p-1$. We have the following diagram:

$$\begin{tikzcd}
L_\infty = L(E_{\mathfrak{p}^\infty}) \arrow[dd, dash] \arrow[dd, dash, bend right, "\mathbb{Z}_p"'] & & \\
& & \\
L = K_0(E_{\mathfrak{p}}) \arrow[dr, dash] \arrow[dd, dash] \arrow[dd, dash, bend right, "\Delta"'] & & \theta_{\mathfrak{p}}^k \\
& K_0 & \\
K \arrow[ur, dash] \arrow[d, dash] & & \varphi \\
\mathbb{Q} & &
\end{tikzcd}$$

Let $\mathcal{Z}$ denote the decomposition subgroup at $\mathfrak{p}$ inside $\mathcal{G}$. Recall that π is the uniformizer of $\mathcal{O}_{K_{\mathfrak{p}}}$. Before proceeding, we need to introduce the notion of an *anomalous prime.* In the sense of Mazur (see [Maz72, p. 186]), a prime $v|p$ is called *anomalous* if $\widetilde{E}(\kappa_v)[p] \neq 0$, where κ_v is the residue field. In our current setting, this can be related to counting the p-power roots of unity in the residue field (see [Lam14, pp. 70–71]). Indeed, if $v|p$ is a prime above p in L_∞ then

$$\#\widetilde{E}(\kappa_v) = \left(\pi^f - 1\right)\left(\overline{\pi}^f - 1\right)$$

where f is the residue degree of the extension K_0/K. It follows that

$$M := \mathrm{ord}_p\left(\#\widetilde{E}(\kappa_v)\right) = \mathrm{ord}_p\left(\left(\pi^f - 1\right)\left(\overline{\pi}^f - 1\right)\right) = \mathrm{ord}_\pi\left(\left(\pi^f - 1\right)\left(\overline{\pi}^f - 1\right)\right) = \mathrm{ord}_\pi\left(\overline{\pi}^f - 1\right).$$

Therefore, an equivalent definition of a prime being anomalous is $M = \mathrm{ord}_\pi\left(\overline{\pi}^f - 1\right) \geq 1$. Since $\gcd(p, |\Delta|) = 1$, the action of $\Delta \cap \mathcal{Z}$ on $\mu_{p^\infty}(L_{\infty,v}) = \mu_{p^M}$ gives a $\mathbb{Z}_p$-valued character which we denote by $\chi_{\mathrm{cyc}} : \Delta \cap \mathcal{Z} \to \mu_{p-1} \subseteq \mathbb{Z}_p^\times$. We have the following theorem.

Theorem 3.1. *Let $\chi = \varphi\theta_{\mathfrak{p}}^k$ be a character of Δ. When E/K_0 is anomalous at $\mathfrak{p}$, we assume that $\chi|_{\Delta\cap\mathcal{Z}}$ is not the cyclotomic character. Let $\mathfrak{h}$ and λ be as above. If*

$$\mathrm{ord}_\pi\left(\left(N(\lambda) - \lambda^k\varphi(\sigma_{(\lambda)})\right) \cdot L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\varphi\psi^k})\right) = 0,$$

then $\overline{\mathcal{C}}(L_\infty)^\chi = U(L_\infty)^\chi$*. Here,* M^χ *denotes the* χ*-isotypic component of a* Λ*-module* M*.*

Proof. See [Lam15, Theorem 7.7]. □

3.1. **Variations of class groups.** Let $F \subset K(E_{\mathfrak{g}})$ for some ideal $\mathfrak{g}$ of $\mathcal{O}_K$ such that p is unramified in F/K, $p \nmid [F : K]$, and $\mathfrak{g}$ is divisible by $\mathfrak{f}$ and is coprime to $\mathfrak{p}q$. Furthermore, we assume that both $\mathfrak{q}$ and $\overline{\mathfrak{q}}$ are tamely ramified in F. Then $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}$ is a $\mathbb{Z}_q$-extension of F, and for integers $n \geq 0$, let F_n/F be the n-th layer of this $\mathbb{Z}_q$-extension. Note that only primes above q ramify in F_n/F, $p \nmid [F_n : K]$ (since $q \neq p$), and $F_n \subset K(E_{\mathfrak{g}q^{n+1}})$. Therefore, we may take K_0 and $\mathfrak{h}$ in the previous section to be F_n and $\mathfrak{h}_n := \mathfrak{g}q^{n+1}$, respectively.

For $n \geq 1$, just as before we define $L_n = F_n(E_\mathfrak{p})$, $L_{n,\infty} = F_n(E_{\mathfrak{p}^\infty})$, $\Delta_n = H_n \times I$, $\mathcal{G}_n = \Delta_n \times \Gamma$, $U_{n,\infty} = U(L_{n,\infty})$, etc. Note that $I = \mathrm{Gal}(K_0(E_\mathfrak{p})/K_0) \cong \mathrm{Gal}(L_n/F_n)$. Define $X_{n,\infty}$ to be the Galois group of the maximal abelian p-extension of $L_{n,\infty}$ which is unramified outside $\mathfrak{p}$. By global class field theory we have the following four-term exact sequence

$$0 \to \overline{\mathcal{E}}_{n,\infty}/\overline{\mathcal{C}}_{n,\infty} \to U_{n,\infty}/\overline{\mathcal{C}}_{n,\infty} \to X_{n,\infty} \to A_{n,\infty} \to 0. \tag{1}$$

Here, $\overline{\mathcal{E}}_{n,\infty} = \overline{\mathcal{E}}(L_{n,\infty})$ is used to denote the global units of $L_{n,\infty}$. Finally, $A_{n,\infty} = A(L_{n,\infty})$ is the inverse limit of the p-part of the class group for each finite extension of F_n contained inside $L_{n,\infty}$. In other words, $A_{n,\infty}$ can be identified with the Galois group of the maximal abelian unramified p-extension of $L_{n,\infty}$. We now prove the key result which will be required in proving Theorem A.

Theorem 3.2. *There exists an integer* $N \geq 0$ *such that* $X^I_{n,\infty} = X^I_{N,\infty}$ *for all* $n \geq N$*, where* M^I *denotes the isotypic component of* M *corresponding to the trivial character of* I*.*

Proof. To prove the theorem, it suffices to show that for $N \gg 0$,

$$\mathrm{char}_\Lambda(X^I_{n,\infty}) = \mathrm{char}_\Lambda(X^I_{N,\infty})$$

for all $n \geq N$. Consider the restriction map

$$\pi_{n,N} : X^I_{n,\infty} \twoheadrightarrow X^I_{N,\infty}.$$

Since characteristic ideals are multiplicative in short exact sequences, the kernel of the above surjective map must be finite. However, a theorem of R. Greenberg (see [Gre78, Theorem §1]) ensures that there are no non-trivial finite submodules inside $X^I_{n,\infty}$. This forces the kernel to be trivial, i.e.,

$$X^I_{n,\infty} = X^I_{N,\infty}.$$

Via the main conjecture of Iwasawa theory for imaginary quadratic fields [Rub91, Theorem 4.1(i)], it is enough to show that

$$U^I_{n,\infty}/\overline{\mathcal{C}}^I_{n,\infty} = U^I_{N,\infty}/\overline{\mathcal{C}}^I_{N,\infty}.$$

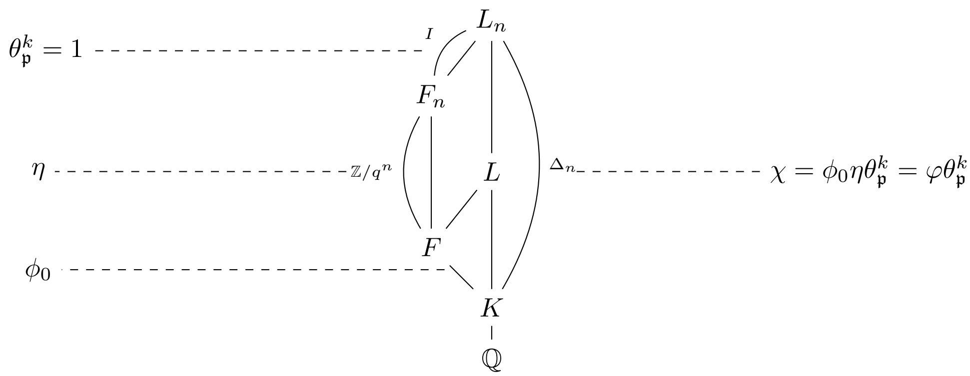
This in turn would follow from

$$U^\chi_{n,\infty} = \overline{\mathcal{C}}^\chi_{n,\infty}$$

for all characters χ of Δ_n that do not factor through Δ_N with $\chi|_I = 1$. Taking $\mathfrak{h}_n$ and applying Theorem 3.1 with K_0 taken to be F_n, it is now enough to show that for all such χ, there exists $\lambda \in \mathcal{O}_K \setminus \mu_K$ such that $\lambda \equiv 1 \mod \mathfrak{f}$ and $(\lambda, 6q\mathfrak{p}) = 1$ satisfying

$$\mathrm{ord}_\pi\left(\left(N(\lambda) - \lambda^k \varphi(\sigma_{(\lambda)})\right) \cdot L^{(\mathrm{alg})}_{\mathfrak{h}}(\overline{\varphi\psi^k})\right) = 0, \tag{2}$$

where $\chi = \varphi\theta_{\mathfrak{p}}^k$, with φ being a character of H_n and $k = p-1$ (so that its restriction to I is trivial). In particular, the condition on E/K_0 being anomalous is irrelevant. We draw another field diagram for convenience of the reader



Note that (2) is equivalent to saying that

$$\left(\lambda\overline{\lambda} - \lambda^k\varphi(\sigma_{(\lambda)})\right) \cdot L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\varphi\psi^k}) \not\equiv 0 \pmod{\pi\mathcal{O}}.$$

Now, observe that

$$\begin{aligned}\left(\lambda\overline{\lambda} - \lambda^k\varphi(\sigma_{(\lambda)})\right) \equiv 0 \pmod{\pi\mathcal{O}} &\Leftrightarrow \varphi(\sigma_{(\lambda)}) \equiv \overline{\lambda}\lambda^{1-k} \pmod{\pi\mathcal{O}}\\ &\Leftrightarrow \eta\phi_0(\sigma_{(\lambda)}) \equiv \overline{\lambda}\lambda^{1-k} \pmod{\pi\mathcal{O}}\\ &\Leftrightarrow \eta(\sigma_{(\lambda)}) \equiv \overline{\lambda}\lambda^{1-k}\phi_0^{-1}(\sigma_{(\lambda)}) \pmod{\pi\mathcal{O}}.\end{aligned}$$

Here, we have written $\varphi = \eta\phi_0$, where η is a character of $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/q^n$. Note that η has exact order q^m for some $m \geq 1$. Therefore, $\eta(\sigma_{(\lambda)})$ is a primitive q^m-th root of unity. But, modulo π, the q-power roots of unity are distinct. Therefore, for almost all η,

$$\mathrm{ord}_\pi\left(N(\lambda) - \lambda^k\varphi(\sigma_{(\lambda)})\right) = 0.$$

By Theorem 2.2 and Remark 2.3, one can choose $N \gg 0$ such that $\mathrm{ord}_\pi\left(L_{\mathfrak{h}}^{(\mathrm{alg})}(\overline{\varphi\psi^k})\right) = 0$ holds for all characters χ of Δ_n with $n \geq N$, which do not factor through Δ_N. This establishes (2) and the proof of the theorem is now complete. □

We are now in a position to prove Theorem A from the Introduction. We repeat the statement below for the convenience of the reader.

Theorem. *Let K be an imaginary quadratic field of class number 1. Let p and q be distinct primes (≥ 5) which split in K. Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ coprime to pq such that $\mathfrak{g}$ is a product of split primes and that $w_{\mathfrak{g}} = 1$. Let $F = \mathscr{R}(\mathfrak{g}q)$ be a prime-to-p extension of K and $\mathscr{R}(\mathfrak{g}q^\infty)^{\mathrm{ac}}/F$ be the anticyclotomic $\mathbb{Z}_q$-extension. Then, there exists an integer N such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)).$$

Proof. Let the p-class group of F_n (resp. F_N) be denoted by $A(F_n)$ (resp. $A(F_N)$). Since p does not divide $[F_n : F_N]$, we have an injection

$$A(F_N) \hookrightarrow A(F_n). \tag{3}$$

It follows from global class field theory that for all $n \geq 0$, we have the identification

$$A_{n,\infty} \simeq \mathrm{Gal}(M_{n,\infty}/L_{n,\infty}),$$

where $M_{n,\infty}$ is the maximal abelian unramified p-extension of $L_{n,\infty}$. Recall the four-term exact sequence introduced in (1). It follows from Theorem 3.2 that $A_{n,\infty} = A_{N,\infty}$. Now, consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A_{N,\infty} \arrow[r] \arrow[d] & A_{n,\infty} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A(F_N) \arrow[r] & A(F_n) & \end{tikzcd}$$

where the vertical maps are given by restriction and are surjective because the extension $L_{n,\infty}/F_n$ and $L_{N,\infty}/F_N$ are totally ramified at primes above $\mathfrak{p}$. As explained above, the top horizontal row is an isomorphism. Therefore, the bottom row is a surjective map as well. When combined with (3), we see that the bottom row is in fact an isomorphism. This completes the proof of the theorem. □

4. Asymptotic growth of fine Selmer groups of abelian varieties

4.1. **Definition of fine Selmer groups.** Suppose F is a number field. Throughout, $A_{/F}$ is a fixed abelian variety. We fix a finite set S of primes of F containing p, the primes dividing the conductor of A, as well as the archimedean primes. Denote by F_S, the maximal algebraic extension of F unramified outside S. For every (possibly infinite) extension L of F contained in F_S, write $G_S(L) = \mathrm{Gal}(F_S/L)$. Write $S(L)$ for the set of primes of L above S. If L is a finite extension of F and w is a place of L, we write L_w for its completion at w; when L/F is infinite, it is the union of completions of all finite sub-extensions of L.

Definition 4.1. Let L/F be an algebraic extension. The *p-primary fine Selmer group of* A over L is defined as

$$\mathrm{Sel}_0(A/L) = \ker\left(H^1\left(G_S(L), A[p^\infty]\right) \to \bigoplus_{v \in S} H^1\left(L_v, A[p^\infty]\right)\right).$$

Similarly, the *p-fine Selmer group of* A over L is defined as

$$\mathrm{Sel}_0(A[p]/L) = \ker\left(H^1\left(G_S(L), A[p]\right) \to \bigoplus_{v \in S} H^1\left(L_v, A[p]\right)\right).$$

It is easy to observe that if F_∞/F is an infinite extension,

$$\mathrm{Sel}_0\left(A/F_\infty\right) = \varinjlim_L \mathrm{Sel}_0\left(A/L\right), \quad \mathrm{Sel}_0\left(A[p]/F_\infty\right) = \varinjlim_L \mathrm{Sel}_0\left(A[p]/L\right),$$

where the inductive limits are taken with respect to the restriction maps and L runs over all finite extensions of F contained in F_∞. Next, we define the notion of p-rank of an abelian group G.

Definition 4.2. Let G be an abelian group. Define the *p-rank* of G as

$$r_p(G) = r_p(G[p]) := \dim_{\mathbb{F}_p}\left(G[p]\right).$$

4.2. **Growth of fine Selmer groups in $\mathbb{Z}_q$-extensions.** In this section, we prove the following theorem which essentially says that the p-part of the class group and the p-primary fine Selmer group have similar growth behaviour in $\mathbb{Z}_q$-extensions.

Theorem 4.3. *Let A be a d-dimensional abelian variety defined over any number field F. Let $S(F)$ be a finite set of primes in F consisting precisely of the primes above q, the primes of bad reduction of A, and the Archimedean primes. Let F_∞/F be a fixed $\mathbb{Z}_q$ extension such that primes in $S(F)$ are finitely decomposed in F_∞/F and suppose $[F_n : F] = q^n$. Further suppose that $A[p] \subseteq A(F)$. Then*

$$\left| r_p \left(\mathrm{Sel}_0 \left(A/F_n \right) \right) - 2d r_p \left(\mathrm{Cl}(F_n) \right) \right| = O(1).$$

If $A[p] \subseteq A(F)$, then the action of G_F on $A[p]$ is trivial. Let $A^\vee$ be the dual abelian variety. The action on the dual representation, $A^\vee[p]$ is also trivial. This tells us that $A^\vee[p] \subseteq A^\vee(F)$. Therefore, Theorem 4.3 allows us to deduce the following result.

Corollary 4.4. *With the same hypothesis as in Theorem 4.3*

$$\left| r_p \left(\mathrm{Sel}_0 \left(A/F_n \right) \right) - r_p \left(\mathrm{Sel}_0 \left(A^\vee/F_n \right) \right) \right| = O(1).$$

To prove Theorem 4.3, we need a few lemmas.

Lemma 4.5. *Consider the following short exact sequence of of co-finitely generated abelian groups*

$$P \to Q \to R \to S.$$

Then,

$$\left| r_p(Q) - r_p(R) \right| \leq 2 r_p(P) + r_p(S).$$

Proof. See [LM16, Lemma 3.2]. □

Lemma 4.6. *Let F_∞ be any $\mathbb{Z}_q$-extension of F such that all the primes in $S(F)$ are finitely decomposed. Let F_n be the subfield of F_∞ such that $[F_n : F] = q^n$. Then*

$$\left| r_p \left(\mathrm{Cl}(F_n) \right) - r_p \left(\mathrm{Cl}_S(F_n) \right) \right| = O(1).$$

Proof. For each F_n, we write $S_f(F_n)$ for the set of finite primes of F_n above S_f. For each n, we have the following exact sequence

$$\mathbb{Z}^{\left| S_f(F_n) \right|} \longrightarrow \mathrm{Cl}(F_n) \xrightarrow{\alpha_n} \mathrm{Cl}_S(F_n) \longrightarrow 0$$

(see [NSW08, Lemma 10.3.12]). Since the class group is always finite, it follows that $\ker(\alpha_n)$ is finite. Also, $r_p \left(\ker(\alpha_n) \right) \leq \left| S_f(F_n) \right|$ and $r_p \left(\ker(\alpha_n)/p \right) \leq \left| S_f(F_n) \right|$. By Lemma 4.5,

$$\left| r_p \left(\mathrm{Cl}(F_n) \right) - r_p \left(\mathrm{Cl}_S(F_n) \right) \right| \leq 2 \left| S_f(F_n) \right| = O(1).$$

□

Lemma 4.7. *Let F_∞/F be a $\mathbb{Z}_q$-extension and let F_n be the subfield of F_∞ such that $[F_n : F] = q^n$. Let A be an abelian variety defined over F. Suppose that all primes of $S(F)$ are finitely decomposed in F_∞/F. Then*

$$\left| r_p \left(\mathrm{Sel}_0(A[p]/F_n) \right) - r_p \left(\mathrm{Sel}_0(A/F_n) \right) \right| = O(1).$$

Proof. Consider the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_0(A[p]/F_n) \arrow[r] \arrow[d, "s_n"] & H^1(G_S(F_n), A[p]) \arrow[r] \arrow[d, "f_n"] & \bigoplus_{v\in S(F_n)} H^1(F_{n,v_n}, A[p]) \arrow[d, "\gamma_n"] \\
0 \arrow[r] & \mathrm{Sel}_0(A/F_n)[p] \arrow[r] & H^1(G_S(F_n),\ A[p^\infty])[p] \arrow[r] & \bigoplus_{v_n\in S(F_n)} H^1(F_{n,v_n},\ A[p^\infty])[p]
\end{tikzcd}$$

Both f_n and γ_n are surjective. The kernel of these maps are given by

$$\ker(f_n) = A(F_n)[p^\infty]/p,$$
$$\ker(\gamma_n) = \bigoplus_{v_n\in S(F_n)} A(F_{n,v_n})[p^\infty]/p.$$

Observe that $r_p\left(\ker(s_n)\right) \leq r_p\left(\ker(f_n)\right) \leq 2d$ and that $r_p\left(\ker(\gamma_n)\right) \leq 2d\left|S_f(F_n)\right|$. By hypothesis, $S_f(F_n)$ is bounded as n varies. It follows from the snake lemma that both $r_p\left(\ker(s_n)\right)$ and $r_p\left(\mathrm{coker}(s_n)\right)$ are finite and bounded. Applying Lemma 4.5 to the following exact sequence

$$0 \to \ker(s_n) \to \mathrm{Sel}_0(A[p]/F_n) \to \mathrm{Sel}_0(A/F_n)[p] \to \mathrm{coker}(s_n) \to 0$$

completes the proof. □

Proof of Theorem 4.3. By hypothesis, $A[p] \subseteq A(F)$. Therefore, $A[p] \simeq (\mathbb{Z}/p)^{2d}$. We have

$$H^1\left(G_S(F_n),\ A[p]\right) = \mathrm{Hom}\left(G_S(F_n),\ A[p]\right).$$

There are similar identifications for the local cohomology groups. Thus,

$$\mathrm{Sel}_0\left(A[p]/F_n\right) \simeq \mathrm{Hom}\left(\mathrm{Cl}_S(F_n), A[p]\right) \simeq \mathrm{Cl}_S(F_n)[p]^{2d}$$

as abelian groups. Therefore,

$$r_p\left(\mathrm{Sel}_0\left(A[p]/F_n\right)\right) = 2d r_p\left(\mathrm{Cl}_S(F_n)\right).$$

The theorem now follows from Lemmas 4.6 and 4.7. □

Let p^{e_n} be the largest power of p that divides the class number of F_n. If e_n is bounded then it follows (trivially) that the p-rank is bounded. Thus, the following corollary is immediate.

Corollary 4.8. *Let $p \neq q$. Let $F/\mathbb{Q}$ be any finite extension of $\mathbb{Q}$ and F_∞/F be any $\mathbb{Z}_q$-extension of F. Let p^{e_n} be the exact power of p dividing the class number of the n-th intermediate field F_n. Let $A_{/F}$ be an abelian variety such that $A[p] \subseteq A(F)$. If e_n is bounded as $n \to \infty$, then $r_p\left(\mathrm{Sel}_0\left(A/F_n\right)\right)$ is bounded independently of n.*

In addition to Theorem A, there are some other results in the literature where it is known that the p-part of the class group stabilizes in a $\mathbb{Z}_q$-extension (when p, q are distinct primes). These were discussed briefly in the Introduction and are recorded here more precisely.

(1) ([Was78, Theorem]) Let $F/\mathbb{Q}$ be an abelian extension of $\mathbb{Q}$ and F_∞/F be the cyclotomic $\mathbb{Z}_q$-extension of F. If p^{e_n} be the exact power of p dividing the class number of the n-th intermediate field F_n, then e_n is bounded as $n \to \infty$.

(2) ([Lam15, Theorem 7.10]) Let p, q be fixed odd distinct primes both ≥ 5, K be an imaginary quadratic field of class number 1 where p and q split, and $E_{/K}$ be an elliptic curves with CM by $\mathcal{O}_K$ and good reduction at p, q. Let K_∞ be the $\mathbb{Z}_q$ extensions of K which is unramified outside $\mathfrak{q}$ (resp. $\overline{\mathfrak{q}}$). Let $\mathfrak{g}$ be a fixed ideal of $\mathcal{O}_K$ such that it is coprime to pq and $F = \mathscr{R}(\mathfrak{g}\mathfrak{q})$ is of degree prime-to-p over F. Then, the p-part of the class number

stabilizes in $FK_\infty = \mathscr{R}(\mathfrak{g}\mathfrak{q}^\infty)$. However, since p is assumed to be unramified in F in *loc. cit.*, the hypothesis $A[p] \subseteq A(F)$ in Theorem 4.3 is unlikely to hold. The same can be said regarding the setting studied in Theorem A.

Theorem 4.9. *With notation as above, suppose that the p-rank of the fine Selmer group, $r_p\left(\mathrm{Sel}_0(A/F_n)\right)$ stabilizes in a $\mathbb{Z}_q$-extension of F. Then there exists $n \gg 0$, such that for all $m \geq n$*

$$\mathrm{Sel}_0(A/F_n) = \mathrm{Sel}_0(A/F_m).$$

Proof. The following argument is similar to the one presented in [Lam14, p. 15], where instead of classical Selmer groups, we consider fine Selmer groups. Consider the extension F_m/F_n. Then $[F_m : F_n] = q^{m-n} = t$ (say). The restriction map

$$\mathrm{Gal}\left(\overline{F}/F_n\right) \longrightarrow \mathrm{Gal}\left(\overline{F}/F_m\right)$$

induces the restriction homomorphism

$$\mathrm{res} : \mathrm{Sel}_0(A/F_n) \longrightarrow \mathrm{Sel}_0(A/F_m).$$

Since $\gcd(q,p) = 1$, this maps is an injection. Moreover, we have

$$\mathrm{Sel}_0(A/F_n) \xrightarrow{\mathrm{res}} \mathrm{Sel}_0(A/F_m) \xrightarrow{\mathrm{cores}} \mathrm{Sel}_0(A/F_n) \xrightarrow{t^{-1}} \mathrm{Sel}_0(A/F_n)$$

where $\mathrm{cores} \circ \mathrm{res} = \mathrm{t}$. Since the composition $\mathrm{res} \circ \mathrm{cores} \circ \mathrm{t}^{-1}$ is the identity map, the exact sequence

$$0 \longrightarrow \mathrm{Sel}_0(A/F_n) \longrightarrow \mathrm{Sel}_0(A/F_m) \longrightarrow \mathrm{Sel}_0(A/F_m)\big/ \mathrm{Sel}_0(A/F_n) \longrightarrow 0$$

is split exact.

Let us write $\mathrm{Sel}_0(A/F_n) = (\mathbb{Q}_p/\mathbb{Z}_p)^{s_n} \oplus T_n$, where $s_n \geq 0$ and T_n is a finite p-group. Then,

$$r_p\left(\mathrm{Sel}_0(A/F_n)\right) = s_n + r_p(T_n).$$

The injection $\mathrm{Sel}_0(A/F_n) \hookrightarrow \mathrm{Sel}_0(A/F_m)$ tells us that $s_m \geq s_n$. If the p-rank $r_p\left(\mathrm{Sel}_0(A/F_n)\right)$ eventually stabilizes it follows that s_n also stabilizes. In particular, the cokernel of the injection, which we denote by $C_{m,n}$, is finite when $n \gg 0$. By duality, we have the short exact sequence

$$0 \to C_{m,n}^\vee \to \mathbb{Z}_p^{s_m} \oplus T_m \to \mathbb{Z}_p^{s_n} \oplus T_n \to 0.$$

When $s_m = s_n$, $C_{m,n}$ has to be finite. Consequently, the image of $C_{m,n}^\vee$ in $\mathrm{Sel}_0(A/F_n)^\vee$ is contained inside T_m. Furthermore, since the short exact sequence splits, we deduce the isomorphism

$$T_m = T_n \oplus C_{m,n}^\vee.$$

As s_n stabilizes, $r_p(T_n)$ also stabilizes. Therefore, $C_{m,n}^\vee$ has to be 0 eventually. $\square$

Theorem B is now an immediate corollary of Theorems 4.3 and 4.9.

Corollary 4.10. *Let p, q be distinct odd primes. Let F be any number field and $A_{/F}$ be an abelian variety such that $A[p] \subseteq A(F)$. Let F_∞/F be a $\mathbb{Z}_q$-extension where the primes above q, the primes of bad reduction of A, and the archimedean primes are finitely decomposed. If the p-part of the class group stabilizes, i.e., there exists $N \gg 0$ such that for all $n \geq N$,*

$$\mathrm{ord}_p(h(F_n)) = \mathrm{ord}_p(h(F_N)),$$

then the growth of the p-primary fine Selmer group stabilizes in the $\mathbb{Z}_q$-extension as well, i.e., for all $n \geq N$,

$$\mathrm{Sel}_0(A/F_n) = \mathrm{Sel}_0(A/F_N).$$

Appendix A. Existence of CM elliptic curves with only split primes in the conductor

In this appendix, we give an elementary proof that given any imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ of class number 1, there exists a CM elliptic curve (over K) such that it has conductor divisible only by split primes of K. The imaginary quadratic fields of class number 1 are precisely the following

$$\mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-2}),\ \mathbb{Q}(\sqrt{-3}),\ \mathbb{Q}(\sqrt{-7}),\ \mathbb{Q}(\sqrt{-11}),\ \mathbb{Q}(\sqrt{-19}),\ \mathbb{Q}(\sqrt{-43}),\mathbb{Q}(\sqrt{-67}),\ \mathbb{Q}(\sqrt{-167}).$$

When $d = 1, 2,$ and 3, we produce explicit examples. For $d \geq 7$, we can prove a general result.

Let K denote one of the nine imaginary quadratic fields with class number 1. Suppose that $E_{/K}$ is an elliptic curve with CM by an order $\mathcal{O}$ in K. We know that the j-invariant $j(E)$ is an integer in this case, so E must be a twist of the base extension of an elliptic curve defined over $\mathbb{Q}$. For $d > 3$, the base curve is uniquely determined (up to isomorphism over K) by the condition that it has CM by $\mathcal{O}_K$ and bad reduction at the ramified prime $\mathfrak{p} = (\sqrt{-d})$. For $d = 1, 2,$ and 3 there are several choices for the elliptic curve over $\mathbb{Q}$ (see [CP19, Remark 3.1]).

(i) Case $K = \mathbb{Q}(\sqrt{-1})$.
The discriminant of K is $D_K = -4$. Consider the elliptic curve $E =$ 25.1-CMa1. This elliptic curve is a twist of 256.1-CMa1 which is a base change of 64.a4. It has bad reduction only at $2 + \sqrt{-1}$ which is a split prime above 5 since

$$5 = (2 + \sqrt{-1})(2 - \sqrt{-1}).$$

(ii) Case $K = \mathbb{Q}(\sqrt{-2})$.
The discriminant of K is $D_K = -8$. Consider the elliptic curve 256.d1 over $\mathbb{Q}$ and its base change to K is the elliptic curve 1024.1-CMb1. A twist of this latter curve is $E =$ 9.3-CMa1 which has bad reduction only at $1 - \sqrt{-2}$, which is a split prime above 3 since

$$3 = (1 + \sqrt{-2})(1 - \sqrt{-2}).$$

(iii) Case $K = \mathbb{Q}(\sqrt{-3})$.
The discriminant of K is $D_K = -3$. Consider the elliptic curve $E =$ 2401.3-CMa1 which arises as a twist of 81.1-CMa1. This latter curve is obtained by base extension of 27.a4 over $\mathbb{Q}$. The primes of bad reduction of E are both primes in K above 7, which admits the splitting

$$7 = (2 + \sqrt{-3})(2 - \sqrt{-3}).$$

(iv) Other cases.
When K is one of the remaining imaginary quadratic fields, the discriminant $D_K = -d \equiv 1 \pmod 4$. In the following proposition, we show that in this case as well there always exist CM elliptic curves with bad reduction only at split primes.

Proposition A.1. *Let $d > 3$ and $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field with class number 1. There exists an elliptic curve over K with CM by $\mathcal{O}_K$ whose conductor is divisible only by primes that split in K.*

Proof. For $d = 7, 11, 19, 43, 67,$ and 167, fix the following elliptic curves $A_{/\mathbb{Q}}$: 49.a4, 121.b2, 361.a2, 1849.b2, 4489.b2, and 26569.a2, respectively. As explained above, any elliptic curve over K with CM by $\mathcal{O}_K$ is obtained as a quadratic twist of the respective $A_{/K}$.

Henceforth, set $\mathfrak{P} = \sqrt{-d}$. It follows from [CP19, Theorem 3.3] that if we twist $A_{/K}$ by a character corresponding to $K(\sqrt{\alpha})$ where $\alpha = \mathfrak{P}\mathfrak{Q}$ such that $\mathfrak{Q}$ is a prime of K distinct from $\mathfrak{P}$

satisfying $\mathfrak{Q} \equiv u^2\sqrt{-d} \mod 4\mathcal{O}_K$ for some $u \in \mathcal{O}_K$, then the twisted elliptic curve (over K) has good reduction everywhere except at $\mathfrak{Q}$. Therefore, it is enough to show that we may choose $\mathfrak{Q}$ to be a split prime in K.

Let $r \in \mathbb{Z}$ such that $(4r + \sqrt{-d})(4r - \sqrt{-d}) = 16r^2 + d$ is an odd rational prime. Such r exists for all possible values of d. For example, we may take r to be 1 when $d = 7, 43, 67$, $r = 2$ when $d = 19$, $r = 3$ when $d = 167$ and $r = 6$ when $d = 11$. Then $4r + \sqrt{-d}$ is a split prime of K and $4r + \sqrt{-d} \equiv 1^2\sqrt{-d} \mod 4\mathcal{O}_K$. In particular, we may apply [CP19, Theorem 3.3] with $\mathfrak{q} = 4r + \sqrt{-d}$ and $u = 1$. This completes the proof of the proposition. □

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(Kundu) Department of Mathematics, University of British Columbia, Vancouver BC, V6T 1Z2, Canada

Email address: dkundu@math.ubc.ca

(Lei) Département de Mathématiques et de Statistique, Université Laval, Pavillion Alexandre-Vachon, 1045 Avenue de la Médecine, Québec, QC, Canada G1V 0A6

Email address: antonio.lei@mat.ulaval.ca