AAA616: Program Analysis

Lecture 6 — Abstract Interpretation Framework

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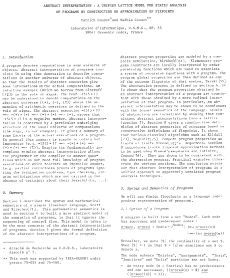
Abstract Interpretation Framework

A powerful framework for designing correct static analysis

- "framework": correct static analysis comes out, reusable
- "powerful": all static analyses are understood in this framework
- "simple": prescription is simple
- "eye-opening": any static analysis is an abstract interpretation



ABSTRACT INTERPRETATION : A UNIFIED LATTICE MODEL FOR STATIC ANALYSIS OF PROGRAMS BY CONSTRUCTION OR APPROXIMATION OF FIXPOINTS

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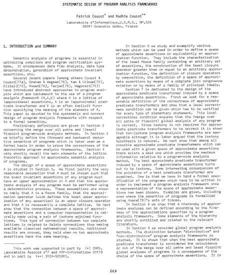

SYSTEMATIC DESIGN OF PROGRAM ANALYSIS FRAMEWORKS

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Step 1: Define Concrete Semantics

The concrete semantics describes the real executions of the program.
Described by semantic domain and function.

- A *semantic domain* D, which is a CPO:
  - D is a partially ordered set with a least element $\bot$.
  - Any increasing chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D has a least upper bound $\bigsqcup_{n \geq 0} d_n$ in D.
- A *semantic function* $F : D \to D$, which is continuous: for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$,

$$F(\bigsqcup_{n \geq 0} d_i) = \bigsqcup_{n \geq 0} F(d_n).$$

Then, the concrete semantics (or collecting semantics) is defined as the least fixed point of *semantic function* $F : D \to D$:

$$\mathit{fix} F = \bigsqcup_{i \in N} F^i(\bot).$$

Step 2: Define Abstract Semantics

Define the abstract semantics of the input program.

- Define an *abstract semantic domain* CPO $\hat{D}$.
  - Intuition: $\hat{D}$ is an abstraction of D
- Define an *abstract semantic function* $\hat{F} : \hat{D} \to \hat{D}$.
  - Intuition: $\hat{F}$ is an abstraction of F.
  - $\hat{F}$ must be monotone:

$$\forall \hat{x}, \hat{y} \in \hat{D}.\ \hat{x} \sqsubseteq \hat{y} \implies \hat{F}(\hat{x}) \sqsubseteq \hat{F}(\hat{y})$$

(or extensive: $\forall x \in \hat{D}.\ x \sqsubseteq \hat{F}(x)$)

Then, static analysis is to compute an upper bound of:

$$\bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\bot)$$

How can we ensure that the result soundly approximate the concrete semantics?

Requirement 1: Galois Connection

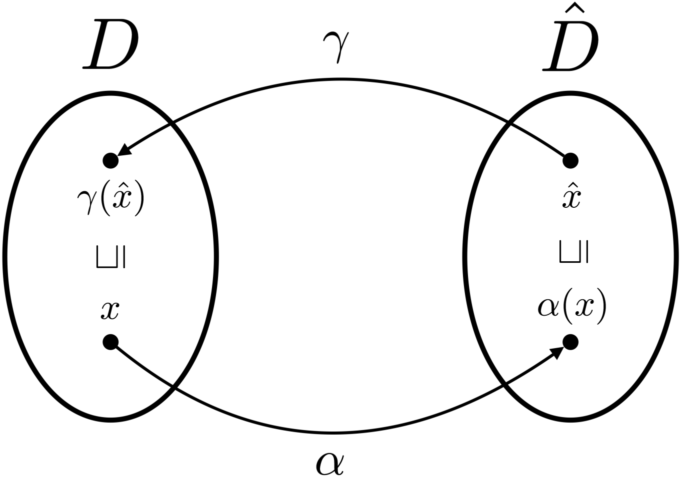
D and $\hat{D}$ must be related with Galois-connection:

$$D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \hat{D}$$

That is, we have

- *abstraction function*: $\alpha \in D \to \hat{D}$
  - represents elements in D as elements of $\hat{D}$
- *concretization function*: $\gamma \in \hat{D} \to D$
  - gives the meaning of elements of $\hat{D}$ in terms of D
- $\forall x \in D, \hat{x} \in \hat{D}.\ \alpha(x) \sqsubseteq \hat{x} \iff x \sqsubseteq \gamma(\hat{x})$
  - α and γ respect the orderings of D and $\hat{D}$
  - If an element $x \in D$ is safely described by $\hat{x} \in \hat{D}$, i.e., $\alpha(d) \sqsubseteq \hat{d}$, then the element described by $\hat{x}$ is also safe w.r.t. x, i.e., $x \sqsubseteq \gamma(\hat{x})$

Galois-Connection



Example: Sign Abstraction

$$\wp(\mathbb{Z}) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\{\bot, +, 0, -, \top\}, \sqsubseteq)$$

$$\begin{array}{rcl}
\alpha(Z) & = & \begin{cases} \bot & Z = \emptyset \\ + & \forall z \in Z.\ z > 0 \\ 0 & Z = \{0\} \\ - & \forall z \in Z.\ z < 0 \\ \top & \text{otherwise} \end{cases} \\
\gamma(\bot) & = & \emptyset \\
\gamma(\top) & = & \mathbb{Z} \\
\gamma(+) & = & \{z \in \mathbb{Z} \mid z > 0\} \\
\gamma(0) & = & \{0\} \\
\gamma(-) & = & \{z \in \mathbb{Z} \mid z < 0\}
\end{array}$$

Example: Interval Abstraction

$$\wp(\mathbb{Z}) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \{\bot\} \cup \{[a, b] \mid a \in \mathbb{Z} \cup \{-\infty\}, b \in \mathbb{Z} \cup \{+\infty\}\}$$

$$\begin{array}{rcl} \gamma(\bot) & = & \emptyset \\ \gamma([a, b]) & = & \{z \in \mathbb{Z} \mid a \leq z \leq b\} \end{array}$$

$$\begin{array}{rcl} \alpha(\emptyset) & = & \bot \\ \alpha(X) & = & [\min X, \max X] \end{array}$$

cf) Alternate Formulation

D and $\hat{D}$ are related with Galois-connection:

$$D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \hat{D}$$

iff (α, γ) satisfies the following conditions:

- α and γ are monotone functions
- $\gamma \circ \alpha$ is extensive, i.e., $\gamma \circ \alpha \sqsupseteq \lambda x.x$
  - abstraction typically loses precision
  - $(\gamma \circ \alpha)(\{1, 3\}) = \{1, 2, 3\}$
- $\alpha \circ \gamma$ is reductive: i.e., $\alpha \circ \gamma \sqsubseteq \lambda x.x$
  - If $\alpha \circ \gamma = \lambda x.x$, Galois-insertion.
  - With Galois-insertion, no two abstract elements describe the same concrete element, which may be true with Galois-connection.

Proof ($\Rightarrow$)

If we have a Galois-connection:

$$\forall x \in D, \hat{x} \in \hat{D}.\ \alpha(x) \sqsubseteq \hat{x} \iff x \sqsubseteq \gamma(\hat{x})$$

then

- $\lambda x.x \sqsubseteq \gamma \circ \alpha$: $\alpha(x) \sqsubseteq \alpha(x)$ and hence $x \sqsubseteq \gamma(\alpha(x))$ by Galois-connection.
- $\alpha \circ \gamma \sqsubseteq \lambda x.x$: $\gamma(\hat{x}) \sqsubseteq \gamma(\hat{x})$ and hence $\alpha(\gamma(\hat{x})) \sqsubseteq \hat{x}$ by Galois-connection.
- γ is monotone: if $\hat{x} \sqsubseteq \hat{y}$, then $\alpha(\gamma(\hat{x})) \sqsubseteq \hat{y}$. Hence $\gamma(\hat{x}) \sqsubseteq \gamma(\hat{y})$ by Galois-connection.
- α is monotone: if $x \sqsubseteq y$, then $x \sqsubseteq \gamma(\alpha(y))$. Hence $\alpha(x) \sqsubseteq \alpha(y)$ by Galois-connection.

Proof ($\Leftarrow$)

- Assume $\alpha(x) \sqsubseteq \hat{x}$. Since γ is monotone, $\gamma(\alpha(x)) \sqsubseteq \gamma(\hat{x})$. Because $\gamma \circ \alpha$ is extensive, we have $x \sqsubseteq \gamma(\hat{x})$.
- Assume $x \sqsubseteq \gamma(\hat{x})$. Since α is monotone, $\alpha(x) \sqsubseteq \alpha(\gamma(\hat{x}))$. Because $\alpha \circ \gamma$ is reductive, we have $\alpha(x) \sqsubseteq \hat{x}$.

Properties of Galois-Connection (1)

Given $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \hat{D}$, we have:

- $\gamma \circ \alpha \circ \gamma = \gamma$
  - From $\alpha \circ \gamma \sqsubseteq \lambda x.x$ and monotonicity of γ, we have $\gamma \circ \alpha \circ \gamma \sqsubseteq \gamma$. We have $\gamma \circ \alpha \circ \gamma \sqsupseteq \gamma$ from $\gamma \circ \alpha \sqsupseteq \lambda x.x$.
- $\alpha \circ \gamma \circ \alpha = \alpha$
- $\alpha \circ \gamma$ and $\gamma \circ \alpha$ are idempotent:

$$(\alpha \circ \gamma)^2 = \alpha \circ \gamma, (\gamma \circ \alpha)^2 = \gamma \circ \alpha$$

- γ uniquely determines $\alpha (D, \hat{D}$ complete lattices):

$$\alpha(d) = \bigsqcap \{\hat{d} \mid d \sqsubseteq \gamma(\hat{d})\}$$

  which implies that $\alpha(d)$ is the best abstraction of d.
- α uniquely determines γ:

$$\gamma(\hat{d}) = \bigsqcup \{d \mid \alpha(d) \sqsubseteq \hat{d}\}$$

Properties of Galois-Connection (2)

- α is strict, i.e., $\alpha(\bot) = \hat{\bot}$. Proof. From $\bot \sqsubseteq \gamma(\hat{\bot})$, we have $\alpha(\bot) \sqsubseteq \hat{\bot}$ by Galois-connection.
- α is continuous: for any chain S in D,

$$\alpha(\bigsqcup_{x \in S} x) = \bigsqcup_{x \in S} \alpha(x).$$

Proof. Since α is monotonic,

$$\bigsqcup_{x \in S} \alpha(x) \sqsubseteq \alpha(\bigsqcup_{x \in S} x).$$

Since $\lambda x.x \sqsubseteq \gamma \circ \alpha$ and γ is monotonic,

$$\bigsqcup_{x \in S} x \sqsubseteq \bigsqcup_{x \in S} \gamma(\alpha(x)) \sqsubseteq \gamma(\bigsqcup_{x \in S} \alpha(x))$$

By Galois-connection, we have

$$\alpha(\bigsqcup_{x \in S} x) \sqsubseteq \bigsqcup_{x \in S} \alpha(x)$$

Deriving Galois-Connections

- Pointwise lifting: Given $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \hat{D}$ and a set S, then

$$S \to D \underset{\alpha'}{\overset{\gamma'}{\leftrightarrows}} S \to \hat{D}$$

  with $\alpha'(f) = \lambda s \in S. \alpha(f(s))$ and $\gamma(f) = \lambda s \in S. \gamma(f(s))$.
- Composition: Given $X_1 \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} X_2 \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} X_3$, we have

$$X_1 \underset{\alpha_2 \circ \alpha_1}{\overset{\gamma_1 \circ \gamma_2}{\leftrightarrows}} X_3$$

Requirement 2: $\hat{F}$ and F

- $\hat{F}$ is a sound abstraction of F:

$$F \circ \gamma \sqsubseteq \gamma \circ \hat{F} \quad (\alpha \circ F \sqsubseteq \hat{F} \circ \alpha)$$

- or, alternatively,

$$\alpha(x) \sqsubseteq \hat{x} \implies \alpha(F(x)) \sqsubseteq \hat{F}(\hat{x})$$

Best Abstract Semantics

From $D \xleftrightarrows[\alpha]{\gamma} \hat{D}$ and $F \circ \gamma \sqsubseteq \gamma \circ \hat{F}$, we have

$$
\begin{aligned}
\alpha \circ F \circ \gamma &\sqsubseteq \alpha \circ \gamma \circ \hat{F} && \alpha \text{ is monotone} \\
&\sqsubseteq \hat{F} && \alpha \circ \gamma \sqsubseteq \lambda x.x
\end{aligned}
$$

The result means that $\alpha \circ F \circ \gamma$ is the best abstraction of F and any sound abstraction $\hat{F}$ of F is greater than $\alpha \circ F \circ \gamma$.

Composition

When $\boldsymbol{F}, \boldsymbol{F'}$ are concrete operators and $\hat{\boldsymbol{F}}, \hat{\boldsymbol{F}}'$ are abstract operators, if $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{F}}'$ are sound abstractions of $\boldsymbol{F}$ and $\boldsymbol{F'}$, respectively, then $\hat{\boldsymbol{F}} \circ \hat{\boldsymbol{F}}'$ is a sound abstraction of $\boldsymbol{F} \circ \boldsymbol{F'}$.

Fixpoint Transfer Theorems

Theorem (Fixpoint Transfer)

Let D *and* $\hat{D}$ *be related by Galois-connection* $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \hat{D}$. *Let* $F : D \to D$ *be a continuous function and* $\hat{F} : \hat{D} \to \hat{D}$ *be a monotone function such that* $\alpha \circ F \sqsubseteq \hat{F} \circ \alpha$. *Then,*

$$\alpha(\mathit{fix} F) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot}).$$

Theorem (Fixpoint Transfer2)

Let D *and* $\hat{D}$ *be related by Galois-connection* $D \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \hat{D}$. *Let* $F : D \to D$ *be a continuous function and* $\hat{F} : \hat{D} \to \hat{D}$ *be a monotone function such that* $\alpha(x) \sqsubseteq \hat{x} \implies \alpha(F(x)) \sqsubseteq \hat{F}(\hat{x})$. *Then,*

$$\alpha(\mathit{fix} F) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot}).$$

Proof of Fixpoint Transfer

- From $\alpha \circ F \sqsubseteq \hat{F} \circ \alpha$, we can derive
$$\forall n \in \mathbb{N}.\ \alpha \circ F^n \sqsubseteq \hat{F}^n \circ \alpha \quad (\forall n \in \mathbb{N}.\ \alpha(F^n(\bot)) \sqsubseteq \hat{F}^n(\hat{\bot}))$$
by induction as follows:
$$\begin{array}{lcll} \alpha \circ F^{n+1} & = & \alpha \circ F \circ F^n & \\ & \sqsubseteq & \alpha \circ F \circ \gamma \circ \alpha \circ F^n & \cdots \alpha \circ F \text{ is mono. and } \lambda x.x \sqsubseteq \gamma \circ \alpha \\ & \sqsubseteq & \alpha \circ F \circ \gamma \circ \hat{F}^n \circ \alpha & \cdots \alpha \circ F \circ \gamma \text{ is mono. and by I.H.} \\ & \sqsubseteq & \hat{F} \circ \hat{F}^n \circ \alpha & \cdots \alpha \circ F \circ \gamma \sqsubseteq \hat{F} \end{array}$$
- Since $\alpha, F, \hat{F}$ are monotone, $\{\alpha(F^i(\bot))\}_i$ and $\{\hat{F}^i(\hat{\bot})\}_i$ are chains, and
$$\bigsqcup_{i \in \mathbb{N}} \alpha(F^i(\bot)) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot}) \tag{1}$$
- Since α and F are continuous,
$$\bigsqcup_{i \in \mathbb{N}} \alpha(F^i(\bot)) = \alpha(\bigsqcup_{i \in \mathbb{N}} (F^i(\bot))) = \alpha(\mathit{fix} F)$$
By replacing the left-hand side of (1), we have
$$\alpha(\mathit{fix} F) \sqsubseteq \bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot})$$

Computing $\bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot})$

- If the abstract domain $\hat{D}$ has finite height (i.e., all chains are finite), we can directly calculate

$$\bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot}).$$

- If the domain $\hat{D}$ has infinite height, the computation may not terminate. In this case, we find a finite chain $\hat{X}_0 \sqsubseteq \hat{X}_1 \sqsubseteq \hat{X}_2 \sqsubseteq \ldots$ such that

$$\bigsqcup_{i \in \mathbb{N}} \hat{F}^i(\hat{\bot}) \sqsubseteq \lim_{i \in \mathbb{N}} \hat{X}_i$$

Fixpoint Accerlation with Widening

Define finite chain $\hat{X}_i$ by an widening operator $\nabla : \hat{D} \times \hat{D} \to \hat{D}$:

$$\begin{array}{rcll} \hat{X}_0 & = & \hat{\bot} & \\ \hat{X}_i & = & \hat{X}_{i-1} & \text{if } \hat{F}(\hat{X}_{i-1}) \sqsubseteq \hat{X}_{i-1} \\ & = & \hat{X}_{i-1} \nabla \hat{F}(\hat{X}_{i-1}) & \text{otherwise} \end{array} \tag{2}$$

Conditions on ∇:

- $\forall a, b \in \hat{D}.\ (a \sqsubseteq a \nabla b) \ \wedge \ (b \sqsubseteq a \nabla b)$
- For all increasing chains $(x_i)_i$, the increasing chain $(y_i)_i$ defined as

$$y_i = \begin{cases} x_0 & \text{if } i = 0 \\ y_{i-1} \nabla x_i & \text{if } i > 0 \end{cases}$$

  eventually stabilizes (i.e., the chain is finite).

Decreasing Iterations with Narrowing

- We can refine the widening result $\mathbf{lim}_{i\in\mathbb{N}}\, \hat{X}_i$ by a narrowing operator $\triangle : \hat{D} \times \hat{D} \to \hat{D}$.
- Compute chain $(\hat{Y}_i)_i$

$$\hat{Y}_i = \begin{cases} \mathbf{lim}_{i\in\mathbb{N}}\, \hat{X}_i & \text{if } i = 0 \\ \hat{Y}_{i-1} \triangle \hat{F}(\hat{Y}_{i-1}) & \text{if } i > 0 \end{cases} \tag{3}$$

- Conditions on $\triangle$
  - $\forall a, b \in \hat{D}.\ a \sqsubseteq b \implies a \sqsubseteq a \triangle b \sqsubseteq b$
  - For all decreasing chain $(x_i)_i$, the decreasing chain $(y_i)_i$ defined as

$$y_i = \begin{cases} x_i & \text{if i} = 0 \\ y_{i-1} \triangle x_i & \text{if } i > 0 \end{cases}$$

eventually stabilizes.

Safety of Widening and Narrowing

Theorem (Widening's Safety)

Let $\hat{\boldsymbol{D}}$ be a CPO, $\hat{\boldsymbol{F}} : \hat{\boldsymbol{D}} \to \hat{\boldsymbol{D}}$ a monotone function, $\bigtriangledown : \hat{\boldsymbol{D}} \times \hat{\boldsymbol{D}} \to \hat{\boldsymbol{D}}$ a widening operator. Then, chain $(\hat{\boldsymbol{X}}_i)_i$ defined as (2) eventually stabilizes and

$$\bigsqcup_{i \in \mathbb{N}} \hat{\boldsymbol{F}}^i(\hat{\bot}) \sqsubseteq \lim_{i \in \mathbb{N}} \hat{\boldsymbol{X}}_i.$$

Theorem (Narrowing's Safety)

Let $\hat{\boldsymbol{D}}$ be a CPO, $\hat{\boldsymbol{F}} : \hat{\boldsymbol{D}} \to \hat{\boldsymbol{D}}$ a monotone function, $\bigtriangleup : \hat{\boldsymbol{D}} \times \hat{\boldsymbol{D}} \to \hat{\boldsymbol{D}}$ a narrowing operator. Then, chain $(\hat{\boldsymbol{Y}}_i)_i$ defined as (3) eventually stabilizes and

$$\bigsqcup_{i \in \mathbb{N}} \hat{\boldsymbol{F}}^i(\hat{\bot}) \sqsubseteq \lim_{i \in \mathbb{N}} \hat{\boldsymbol{Y}}_i.$$

Proof of Widening's Safety

- We first show that $\{\hat{F}(\hat{X}_i)\}_i$ is an increasing chain (if so, by the second condition of widening, the widening sequence $\{\hat{X}_i\}_i$ eventually stabilizes). Note that, by (2), $\hat{F}(\hat{X}_{i+1})$ is either $\hat{F}(\hat{X}_i)$ or $\hat{F}(\hat{X}_i \mathbin{\nabla} \hat{F}(\hat{X}_i))$. Since $\hat{X}_i \sqsubseteq \hat{X}_i \mathbin{\nabla} \hat{F}(\hat{X}_i)$ and $\hat{F}$ is monotone, for all i we have

$$\hat{F}(\hat{X}_i) \sqsubseteq \hat{F}(\hat{X}_{i+1})$$

- We next show that $\forall i \in \mathbb{N}.\ \hat{F}^i(\hat{\bot}) \sqsubseteq \hat{X}_i$.
  - Base case. $\hat{F}^0(\hat{\bot}) = \hat{\bot} \sqsubseteq \hat{X}_0$.
  - Inductive case. From the induction hypothesis (I.H.), i.e., $\hat{F}^i(\hat{\bot}) \sqsubseteq \hat{X}_i$, and the monotonicity of $\hat{F}$, we have

$$\hat{F}^{i+1}(\hat{\bot}) \sqsubseteq \hat{F}(\hat{X}_i) \tag{4}$$

    There are two cases to consider:
    1. When $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_i$ and $\hat{X}_{i+1} = \hat{X}_i$: we have $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_{i+1}$ and therefore $\hat{F}^{i+1}(\hat{\bot}) \sqsubseteq \hat{X}_{i+1}$.
    2. When $\hat{F}(\hat{X}_i) \not\sqsubseteq \hat{X}_i$ and $\hat{X}_{i+1} = \hat{X}_i \mathbin{\nabla} \hat{F}_i(\hat{X}_i)$: by the condition of ∇, $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_i \mathbin{\nabla} \hat{F}_i(\hat{X}_i) = \hat{X}_{i+1}$. Thus, $\hat{F}(\hat{X}_i) \sqsubseteq \hat{X}_{i+1}$ for all $i \in \mathbb{N}$. By (4), we have $\hat{F}^{i+1}(\hat{\bot}) \sqsubseteq \hat{X}_{i+1}$.

Proof of Narrowing's Safety

- We first show that $\{\hat{F}(\hat{Y}_i)\}_i$ is a decreasing chain (if so, by the second condition of narrowing, the narrowing sequence $\{\hat{Y}_i\}_i$ eventually stabilizes). We can show that $\{\hat{F}(\hat{Y}_i)\}_i$ is a decreasing chain if the following holds:

$$\forall i \in \mathbb{N}.\ \hat{Y}_i \sqsupseteq \hat{F}(\hat{Y}_i). \tag{5}$$

  This is because, from (5), we have $\hat{Y}_i \sqsupseteq \hat{Y}_i \triangle \hat{F}(\hat{Y}_i) \sqsupseteq \hat{F}(\hat{Y}_i)$ and by the mono. of $\hat{F}$, we have

$$\hat{F}(\hat{Y}_i) \sqsupseteq \hat{F}(\hat{Y}_i \bigtriangleup \hat{F}(\hat{Y}_i)) = \hat{F}(\hat{Y}_{i+1})$$

  Proof of (5): exercise.

- We next show that $\forall i \in \mathbb{N}.\ \hat{F}^i(\hat{\bot}) \sqsubseteq \hat{Y}_i$ by induction (exercise).