

# COSE212: Programming Languages

## Lecture 1 — Inductive Definitions (1)

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# Inductive Definitions

Inductive definition (induction) is widely used in the study of programming languages and computer science in general: e.g.,

- The syntax and semantics of programming languages
- Data structures (e.g., lists, trees, graphs)

Induction is a technique for formally defining a set:

- The set is defined in terms of itself.
- The only way of defining an infinite set by a finite means.

Three styles to inductive definition:

- Top-down
- Bottom-up
- Rules of inference

## Example (Top-Down)

Let us define a certain subset  $S$  of natural numbers ( $\mathbb{N}$ ) as follows:

### Definition ( $S$ )

A natural number  $n$  is in  $S$  if and only if

- ①  $n = 0$ , or
- ②  $n - 3 \in S$ .

The definition is *inductive*, because the set is defined in terms of itself.  
What is the set  $S$ ?

## Example (Continued)

Let us see what natural numbers are in  $S$ .

- $0$  is in  $S$  because of the first condition of the definition.
- $3$  is in  $S$  because  $3 - 3 = 0$  and  $0$  is in  $S$ .
- $6$  is in  $S$  because  $6 - 3 = 3$  and  $3$  is in  $S$ .
- $\dots$

We can conjecture that  $\{0, 3, 6, 9, \dots\} \subseteq S$ .

### Proof by mathematical induction .

We show that  $3k \in S$  for all  $k \in \mathbb{N}$ .

- 1 Base case:  $3k \in S$  when  $k = 0$ .
- 2 Inductive case: Assume  $3k \in S$  (Induction Hypothesis, I.H.).  
Then show  $3 \cdot (k + 1) \in S$ , which holds because  
 $3 \cdot (k + 1) - 3 = 3k \in S$  by the induction hypothesis.



## Example (Continued)

What about other numbers? Does  $S$  contain only the multiples of 3?

- For instance,  $1 \in S$ ? No. Because the first condition is not true, the second condition must be true for 1 to be in  $S$ . However, it is not true because  $1 - 3 = -2$  is not a natural number. Similarly, we can show that  $2 \notin S$ .
- What about 4? Because  $4 - 3 = 1 \notin S$ ,  $4 \notin S$ .

By similar reasoning, we can conjecture that if  $n$  is not a multiple of 3 then  $n$  is not in  $S$ . In other words,  $S$  contains multiples of 3 only: i.e.,

$$\{0, 3, 6, 9, \dots\} \supseteq S.$$

### Proof by contradiction.

Let  $n = 3k + q$  ( $q = 1$  or  $2$ ) and assume  $n \in S$ . By the definition of  $S$ ,  $n - 3, n - 6, \dots, n - 3k \in S$ . Thus,  $S$  must include 1 or 2, a contradiction. □

# A Bottom-up Definition

An alternative inductive definition of  $S$ :

## Definition ( $S$ )

$S$  is the *smallest* set such that  $S \subseteq \mathbb{N}$  and  $S$  satisfies the following two conditions:

- ①  $0 \in S$ , and
- ② if  $n \in S$ , then  $n + 3 \in S$ .

- The two conditions imply  $\{0, 3, 6, 9, \dots\} \subseteq S$ .
- The two conditions do not imply  $\{0, 3, 6, 9, \dots\} \supseteq S$ . E.g.,
  - ▶  $\mathbb{N}$  satisfies the conditions:  $0 \in \mathbb{N}$  and if  $n \in \mathbb{N}$  then  $n + 3 \in \mathbb{N}$ .
  - ▶  $\{0, 3, 6, 9, \dots\} \cup \{1, 4, 7, 10, \dots\}$  satisfies the conditions.
- This is why the definition requires  $S$  to be the **smallest** such a set.
- The smallest set that satisfies the two conditions is unique:

$$S = \{0, 3, 6, 9, \dots\}.$$

# Rules of Inference

The third way is to define the set with inference rules. An inference rule is of the form:

$$\frac{A}{B}$$

- $A$ : hypothesis (antecedent)
- $B$ : conclusion (consequent)
- “if  $A$  is true then  $B$  is also true”.
- $\overline{B}$ : axiom (inference rule without hypothesis)

The hypothesis may contain multiple statements:

$$\frac{A \quad B}{C}$$

“If both  $A$  and  $B$  are true then so is  $C$ ”.

# Rules of Inferences

The set  $S$  is defined as inference rules as follows:

## Definition ( $S$ )

$$\overline{0 \in S} \qquad \frac{n \in S}{(n + 3) \in S}$$

Interpret the rules as follows:

“A natural number  $n$  is in  $S$  iff  $n \in S$  can be derived from the axiom by applying the inference rules finitely many times”

For example,  $3 \in S$  because we can find a “proof/derivation tree”:

$$\frac{\overline{0 \in S}}{3 \in S} \begin{array}{l} \text{the axiom} \\ \text{the second rule} \end{array}$$

but  $1, 2, 4, \dots \notin S$  because we cannot find proofs. Note that this interpretation enforces that  $S$  is the smallest set closed under the inference rules.



# Exercises

- ① What set is defined by the following inductive rules?

$$\frac{}{3} \quad \frac{x \quad y}{x + y}$$

- ② What set is defined by the following inductive rules?

$$\frac{}{()} \quad \frac{x}{(x)} \quad \frac{x \quad y}{xy}$$

- ③ Define the following set as rules of inference:

$$S = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \dots\}$$

- ④ Define the following set as rules of inference:

$$S = \{x^n y^{n+1} \mid n \in \mathbb{N}\}$$

# Summary

In inductive definitions, a set is defined in terms of itself. Three styles:

- Top-down
- Bottom-up
- Rules of inference

In PL, we mainly use the rules-of-inference method.