Correction! KWANG-KIM

$$f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$$

Hint

$$f(x) = [x^2 \cdot (6-x)]^{\frac{1}{3}}$$

Find $f'(x)$.

Method I

Rule

$$([g(x)]^n)' = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

$$f'(x) = \{[x^2 \cdot (6-x)]^{\frac{1}{3}}\}' = \frac{1}{3}[x^2 \cdot (6-x)]^{-\frac{2}{3}} \cdot [x^2 \cdot (6-x)]'$$
$$= \frac{1}{3}[x^2 \cdot (6-x)]^{-\frac{2}{3}} \cdot [6x^2 - x^3]' = \frac{1}{3}[x^2 \cdot (6-x)]^{-\frac{2}{3}} \cdot [12x - 3x^2]$$
$$= \frac{1}{3}[x^2 \cdot (6-x)]^{-\frac{2}{3}} \cdot [-3x(x-4)] = \frac{-3x(x-4)}{3[x^2 \cdot (6-x)]^{\frac{2}{3}}}$$
$$= \frac{-x(x-4)}{[x^2 \cdot (6-x)]^{\frac{2}{3}}}$$

Since $[x^2 \cdot (6-x)]^{\frac{2}{3}} = x^{\frac{4}{3}} \cdot (6-x)^{\frac{2}{3}}$ and $(6-x)^{\frac{2}{3}} = (x-6)^{\frac{2}{3}}$, by canceling x, we conclude

$$f'(x) = \frac{-(x-4)}{x^{\frac{1}{3}} \cdot (6-x)^{\frac{2}{3}}} = \frac{-(x-4)}{x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}}$$

We also want to mention that cancellation by x does not change \$f'(x) \$ since we still exclude $x = 0$.

Method II - implicit differentiation

Let

$$y = [x^2 \cdot (6-x)]^{\frac{1}{3}}$$

then

$$y^3 = x^2 \cdot (6-x) = 6x^2 - 3x^3$$

For our convenience, let

$$x' = \frac{dx}{dx} = 1, y' = \frac{dy}{dx}$$

Now differentiate above equation w.r.t x.

$$3y^2 \cdot y' = 12x - 3x^2$$

$$y^2 \cdot y' = -x(x-4)$$

$$y' = \frac{-x(x-4)}{y^2} = \frac{-x(x-4)}{[x^2 \cdot (6-x)]^{\frac{2}{3}}}$$

Then you can simplify further using the computation in Method I.

Find $f''(x)$

Method I

$$f'(x) = -\frac{(x-4)}{x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}}$$

From the quotient rule,

$$f''(x) = -\frac{(x-4)'[x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}] - (x-4)[x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}]'}{x^{\frac{2}{3}} \cdot (x-6)^{\frac{4}{3}}}$$

Since

$$[x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}]' = \frac{1}{3}x^{-\frac{2}{3}} \cdot (x-6)^{\frac{2}{3}} + x^{\frac{1}{3}} \cdot \frac{2}{3}(x-6)^{-\frac{1}{3}}$$

$$= \frac{1}{3}[x^{-\frac{2}{3}} \cdot (x-6)^{\frac{2}{3}} + x^{\frac{1}{3}} \cdot 2(x-6)^{-\frac{1}{3}}] \cdot \frac{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}$$

$$= \frac{(x-6)+2x}{3x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}} = \frac{3x-6}{3x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}} = \frac{x-2}{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}$$

we have

$$f'(x) = -\frac{[x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}] - (x-4)\left(\frac{3x-6}{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}\right)}{x^{\frac{2}{3}} \cdot (x-6)^{\frac{4}{3}}}$$

$$= -\frac{[x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}]\left(\frac{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}\right) - (x-4)\left(\frac{3x-6}{x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}}\right)}{x^{\frac{2}{3}} \cdot (x-6)^{\frac{4}{3}}}$$

$$= -\frac{x(x-6)-(x-4)(x-2)}{x^{\frac{2}{3}}(x-6)^{\frac{4}{3}} \cdot x^{\frac{2}{3}}(x-6)^{\frac{1}{3}}} = -\frac{-8}{x^{\frac{4}{3}}(x-6)^{\frac{5}{3}}} = \frac{8}{x^{\frac{4}{3}}(x-6)^{\frac{5}{3}}}$$

Method II - implicit differentiation

$$g = f'(x) = -\frac{(x-4)}{x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}}$$

$$x(x-6)^2 g^3 = -(x-4)^3$$

Now differentiate above equation w.r.t x.

$$[(x-6)^2 + x \cdot 2(x-6)]g^3 + x(x-6)^2 \cdot 3g^2 \cdot g' = -3(x-4)^2$$
$$[(x-6)(x-6+2x)]g^3 + x(x-6)^2 \cdot 3g^2 \cdot g' = -3(x-4)^2$$
$$[(x-6)(3x-6)]g^3 + x(x-6)^2 \cdot 3g^2 \cdot g' = -3(x-4)^2$$
$$[(x-6)(x-2)]g^3 + x(x-6)^2 \cdot g^2 \cdot g' = -(x-4)^2$$
$$x(x-6)^2 \cdot g^2 \cdot g' = -(x-2)(x-6)g^3 - (x-4)^2$$

For simplicity, let us multiply $(x-4)$ everywhere.

$$(x-4)x(x-6)^2 \cdot g^2 \cdot g' = -(x-4)(x-2)(x-6)g^3 - (x-4)^3$$

From $x(x-6)^2 g^3 = -(x-4)^3$,

$$x(x-4)(x-6)^2 g^2 \cdot g' = -(x-4)(x-2)(x-6)g^3 + x(x-6)^2 g^3$$

$$x(x-4)(x-6) \cdot g' = g \cdot [-(x-4)(x-2) + x(x-6)] = g \cdot (-8)$$

Therefore

$$f''(x) = g' = \frac{1}{x(x-4)(x-6)} \cdot \frac{-(x-4)}{x^{\frac{1}{3}} \cdot (x-6)^{\frac{2}{3}}} \cdot (-8) = \frac{8}{x^{\frac{4}{3}} \cdot (x-6)^{\frac{5}{3}}}$$