SAMPLE ANSWER
Circle your final answers and show your work to get a full credit.

$$(\sec(x))' = \sec(x)\tan(x),\ (\csc(x))' = -\csc(x)\cot(x),\ (\cot(x))' = -csc^2(x)$$

Question 1 *(30pt) Find the limit including* $\infty, -\infty$.

(6pt) $\lim\limits_{x\to\infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$ ***Answer.*** *Since for* $r > 0$, $\lim_{x\to\infty} \frac{1}{x^r} = 0$.

$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8} = \lim_{x\to\infty} \frac{3\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{4}{x^2}}{2\frac{x^2}{x^2} + 5\frac{x}{x^2} - \frac{8}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} + \frac{4}{x^2}}{2 + 5\frac{1}{x} - \frac{8}{x^2}} = \frac{3}{2}$$

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(6pt) $\lim\limits_{x\to-\infty} \dfrac{x^2 + x}{3 - x}$ ***Answer.***

$$\lim_{x\to-\infty} \frac{x^2 + x}{3 - x} = \lim_{x\to-\infty} \frac{\frac{x^2}{x} + \frac{x}{x}}{\frac{3}{x} - \frac{x}{x}} = \lim_{x\to-\infty} \frac{x + 1}{\frac{3}{x} - 1} = \infty$$

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(6pt) $\lim\limits_{x\to\infty} \left(\sqrt{9x^2 + x} - 3x\right)$. ***Answer.***

$$\lim_{x\to\infty} \left(\sqrt{9x^2 + 1} - 3x\right) = \lim_{x\to\infty} \left(\sqrt{9x^2 + 1} - 3x\right) \cdot \frac{\left(\sqrt{9x^2 + 1} + 3x\right)}{\left(\sqrt{9x^2 + 1} + 3x\right)} = \lim_{x\to\infty} \frac{9x^2 + 1 - 9x^2}{\sqrt{9x^2 + x} + 3x} = \lim_{x\to\infty} \frac{1}{\sqrt{9x^2 + x} + 3x} = 0$$

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(6pt) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2 + 1}}{x - 1}$ ***Answer.*** *Since* $x > 0 \Rightarrow \sqrt{x^2} = x$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{x}{x} - \frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{1 - \frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{2\frac{x^2}{x^2} + \frac{1}{x^2}}}{1 - \frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{1 - \frac{1}{x}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

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(6pt) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{2x^2 + 1}}{x - 1}$ ***Answer.*** *Since* $x < 0 \Rightarrow \sqrt{x^2} = -x$,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{x}{x} - \frac{1}{x}} = \lim_{x\to-\infty} \frac{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}{1 - \frac{1}{x}} = \lim_{x\to-\infty} -\frac{\sqrt{2\frac{x^2}{x^2} + \frac{1}{x^2}}}{1 - \frac{1}{x}} = \lim_{x\to-\infty} -\frac{\sqrt{2 + \frac{1}{x^2}}}{1 - \frac{1}{x}} = -\frac{\sqrt{2}}{1} = -\sqrt{2}$$

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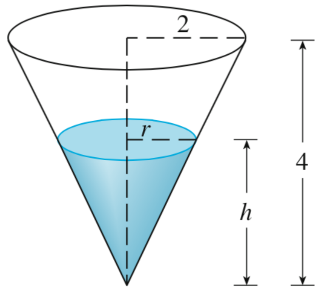
Question 2 *(10pt) A water tank has the shape of an inverted circular cone with base radius* 2m *and height* 4m*. If water is being pumped into the tank at a rate of* $2m^3/min$*, find* ***the rate at which the water level is rising*** *when the water is 3m deep. Hint:* $V = \frac{1}{3}\pi r^2 h$*. Write the exact answer, not a decimal.*

Answer.



Let $h' = \frac{dh}{dt}$, $r' = \frac{dr}{dt}$, $V' = \frac{dV}{dt}$. From the conditions, $V' = 2(m^3/min)$, $h = 3(m)$. From the similarity of two right triangles,

$$4 : 2 = h : r \Rightarrow 2h = 4r \Rightarrow r = \frac{h}{2}$$

Therefore

$$V = \frac{\pi}{3}(\frac{h}{2})^2 h = \frac{\pi}{12}h^3$$

Differentiate with respect to t.

$$V' = \frac{\pi}{12}3h^2h' = \frac{\pi}{4}h^2h'$$

$$2 = \frac{\pi}{4}(3)^2h' \Rightarrow h' = 2 \cdot \frac{4}{9\pi} = \frac{8}{9\pi}(m/min)$$

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Question 3 *(10pt) Find the linearization of the function of the function* $f(x) = \sqrt{x+3}$ *at* $a = 1$ *and use it to approximate the numbers* $\sqrt{3.98}$*.*

Answer.

- $a = 1$. So $f(a) = \sqrt{1+3} = 2$.
- $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{x+3}}(x+3)' = \frac{1}{2\sqrt{x+3}}$$

- $L(x) = f(a) + f'(a)(x-a)$
  Since $f(1) = 2$, $f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$,

$$L(x) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4}$$

- From $f(x) \approx L(x)$

$$\sqrt{3.98} = \sqrt{(0.98)+3} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

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Question 4 *(10pt) Let* $f(x) = x^3 - x$, $a = 0$, $b = 2$*. Find all* $c \in (a,b) = (0,2)$ *satisfying Mean Value theorem.*

Answer.

$$f(b) = f(a) + f'(c)(b-a), \; c \in (a,b)$$

From $f'(x) = 3x^2 - 1$, $f(2) = (2)^3 - (2) = 6$, $f(0) = 0$

$$6 = 0 + (3c^2-1)(2-0) \Rightarrow 6 = 2(3c^2-1) \Rightarrow 3 = 3c^2 - 1 \Rightarrow 4 = 3c^2 \Rightarrow c = \pm\frac{2}{\sqrt{3}}$$

But $c \in (0,2)$. Therefore $c = \frac{2}{\sqrt{3}}$. ■

Question 5 *(10pt) Find the absolute max/min value of the function*

$$f(x) = \frac{x}{x^2+4}, \quad -1 \le x \le 3$$

- *Find the critical number(s).* ***Answer.***

$$f'(x) = \frac{(x)'(x^2+4)-(x)(x^2+4)'}{(x^2+4)^2} = \frac{x^2+4-x(2x)}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+4)^2} = -\frac{x^2-4}{(x^2+4)^2} = -\frac{(x+2)(x-2)}{(x^2+4)^2}$$

  - $f'(c) = 0 \Rightarrow c = -2, 2$. *But* $c \in [-1, 3]$. *Therefore* $c = 2$.
  - $f'(c)$ *is undefined. No such* c.

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- *Find the absolute max and absolute min value of the function* f.
  Answer. *From* $f(-1) = -\frac{1}{5}$, $f(2) = \frac{1}{4}$, $f(3) = \frac{3}{13}$,

$$absolute\ Max : \frac{1}{4}, \ absolute\ Min : -\frac{1}{5}$$

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Question 6 *(20pt)* $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. *Find* f' *and* f''*.Hint: To factor* f' *use the grouping Method.*
Answer.

$$f'(x) = 12\,x^3 - 12\,x^2 - 12\,x + 12 = 12x^2(x-1) - 12x(x-1) = 12(x-1)(x^2-1) = 12\,(x+1)(x-1)^2$$

x	$-\infty$		-1		1		∞
$f'(x)$		$-$	0	$+$	0	$+$	
$f(x)$	∞	$\searrow$	*LMin*	$\nearrow$	$\nearrow$	$\nearrow$	∞

$$f''(x) = 36\,x^2 - 24\,x - 12 = 12\,(3\,x+1)(x-1)$$

x	$-\infty$		$-\frac{1}{3}$		1		∞
$f''(x)$		$+$	0	$-$	0	$+$	
$f(x)$	∞	$\smile$	*inflection*	$\frown$	*inflection*	$\smile$	∞

■

- *Find the intervals on which* f *is increasing.* $\mathbf{(-1, \infty)}$
- *Find the intervals on which* f *is decreasing.* $\mathbf{(-\infty, -1)}$
- *Find the local minimum value(s).* $\boldsymbol{f(-1) = -10}$.
- *Find the local maximum value(s).* ***No***.
- *Find the intervals on which* f *is concave upward.* $\mathbf{(-\infty, -\frac{1}{3}) \cup (1, \infty)}$
- *Find the intervals on which* f *is concave downward.* $\mathbf{(-\frac{1}{3}, 1)}$
- *Find (an) inflection point(s) if exists.* $\mathbf{(-\frac{1}{3}, -\frac{94}{27}),\ (1, 6)}$

Question 7 *(20pt)* $y = \frac{x^2}{x^2+3}$ *Find* y', y''. *Determine the signs of* y' *and* y''.

$$f(x) = \frac{x^2}{x^2+3} = \frac{x^2+3-3}{x^2+3} = 1 - \frac{3}{x^2+3}$$
$$f'(x) = -\frac{(0)(x^2+3)-(3)(2x)}{(x^2+3)^2} = -\frac{-6x}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

Therefore critical number(s) $c = 0$.

x	$-\infty$		0		∞
$f'(x)$		-		$+$	
$f(x)$		$\searrow$	0	$\nearrow$	

From $[(x^2+3)^2]' = 2(x^2+3)(2x) = 4x(x^2+3)$

$$f''(x) = \frac{(6)(x^2+3)^2-(6x)(4x)(x^2+3)}{(x^2+3)^4} = \frac{(6)(x^2+3)[(x^2+3)-(4x)(x)]}{(x^2+3)^4}$$
$$= \frac{6[x^2+3-4x^2]}{(x^2+3)^3} = \frac{-6[3x^2-3]}{(x^2+3)^3} = \frac{-18(x^2-1)}{(x^2+3)^3} = \frac{-18(x+1)(x-1)}{(x^2+3)^3}$$

Therefore

$$f''(x) = 0 \text{ or } f''(x) : DNE \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

x	$-\infty$		-1		1		∞
$f''(x)$		-	0	$+$	0	-	
$f(x)$		$\frown$	$\frac{1}{4}$	$\smile$	$\frac{1}{4}$	$\frown$	

- *Find the intervals on which f is increasing. (Exclude endpoint(s) if exist(s).)*

$$(0, \infty)$$

- *Find the intervals on which f is decreasing. (Exclude endpoint(s) if exist(s).)*

$$(-\infty, 0)$$

- *Find the local minimum value(s).*

$$f(0) = 0$$

- *Find the local maximum value(s). No.*
- *Find the intervals on which f is concave upward.*

$$(-1, 1)$$

- *Find the intervals on which f is concave downward.*

$$(-\infty, -1) \cup (1, \infty)$$

- *Find (an) inflection point(s) if exists.*

$$\left(-1, \frac{1}{4}\right), \left(1, \frac{1}{4}\right)$$