MA441-TEST3 ANSWER!

$$(\sec(x))' = \sec(x)\tan(x),\ (\csc(x))' = -\csc(x)\cot(x),\ (\cot(x))' = -csc^2(x)$$

**Question 1 (30pt)** *Find the most general anti-derivative of each of the following function.*

*(7pt)* $f(x) = (2x+1)^2$
*Let* $u = 2x+1$. $du = 2dx$ *and* $\frac{du}{2} = dx$.

$$\int u^2\frac{du}{2} = \frac{1}{2}\cdot\frac{u^3}{3} + C = \frac{(x+1)^3}{6} + C$$

*(8pt)* $f(x) = \frac{1-x^2}{\sqrt{x}}$

$$\int\left(x^{-\frac{1}{2}} - x^{\frac{3}{2}}\right)dx = 2x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

*(7pt)* $\int x^3\sqrt{x^2+1}dx$
*Let* $u = x^2+1$. *Then* $x^2 = u-1$, $du = 2xdx$ *and* $\frac{du}{2} = xdx$.

$$\int x^3\sqrt{x^2+1}dx = \int x^2\sqrt{x^2+1}xdx = \int(u-1)\sqrt{u}\frac{du}{2} = \frac{1}{2}\int(u\sqrt{u}-\sqrt{u})du = \frac{1}{2}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + C$$

*(8pt)* $\int\sec^3(x)\tan(x)dx$
*Let* $u = \sec(x)$, $du = \sec(x)\cdot\tan(x)dx$

$$\int\sec^3(x)\tan(x)dx = \int\sec^2(x)\cdot(\sec(x)\cdot\tan(x))\,dx = \int u^2du = \frac{u^3}{3} + C = \frac{\sec^3(x)}{3} + C$$

**Question 2 (30pt)** *Evaluate.(Hint: Use FTC.)*

*(10pt)*

$$\int_{-2}^{2}(x^{11} - \sqrt{2}x^9 - 2x + 1)dx$$

*Since* $y = x^{11} - \sqrt{2}x^9 - 2x$ *is odd and* $y = 1$ *is even,*

$$= \int_{-2}^{2}(1)dx = 2\int_0^2(1)dx = 2[x]_0^2 = 4$$

*(10pt)* $\int_{-2}^{2} f(x)dx$ *where* $f(x) = \begin{cases} 1 & if\ -2\le x\le 0 \\ 3x^2 & if\ 0<x\le 2 \end{cases}$

$$\int_{-2}^{2}f(x)dx = \int_{-2}^{0}f(x)dx + \int_0^2 f(x)dx = \int_{-2}^{0}1dx + \int_0^2 3x^2dx = [x]_{-2}^{0} + \left[3\cdot\frac{x^3}{3}\right]_0^2$$

$$= [(0)-(-2)] + [(8)-(0)] = 10$$

*(10pt)* $\int_0^2 9x^2\sqrt{x^3+1}dx =$ *Let* $u = x^3+1$. $du = 3x^2dx$ *and* $x \to u : 0\to 1,\ 2\to 9$.

$$\int_0^2 9x^2\sqrt{x^3+1}dx = 3\int_0^2\sqrt{x^3+1}(3x^2dx) = 3\int_1^9\sqrt{u}du = 3\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9 = \left[(18) - (\frac{2}{3})\right] = 52$$

**Question 3 (10pt)** *The velocity function (in meters per second) is given for a particle moving along a line.*

$$v(t) = 4t-8, 0\le t\le 3$$

*(5pt) Find the displacement*

$$\int_0^3(v(t))\,dt = [s(t)]_0^3 = \left[2t^2 - 8t\right]_0^3 = s(3) - s(0) = (-6) - (0) = -6$$

*(5pt) Find the distance traveled. From* $|v(t)| = \begin{cases} 4t-8 & ,t\ge 2 \\ -(4t-8) & ,t<2 \end{cases}$

$$\int_0^3(|v(t)|)\,dt = -\int_0^2(v(t))\,dt + \int_2^3(v(t))\,dt = -[s(t)]_0^2 + [s(t)]_2^3 = -\left[2t^2-8t\right]_0^2 + \left[2t^2-8t\right]_2^3$$

$$= -[s(2)-s(0)] + [s(3)-s(2)] = -[(-8)-(0)] + [(-6)-(-8)] = 10$$

**Question 4 (10pt)** *Use the* ***definition*** *of definite integral.*
*Do not use the Fundamental theorem of calculus.*

*Hint:* $\sum_{i=1}^{n} 1 = n,\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2},\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},\ \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

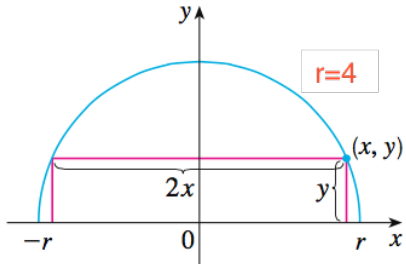
$$\int_0^1 x^3dx$$

*Let* $b = 1, a = 0$. *Then*

$$\Delta x = \frac{b-a}{n} = \frac{1}{n},\ x_i = a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}, f(x_i) = 2\left(\frac{i}{n}\right)^3$$

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}i^3 = \lim_{n\to\infty}\frac{2}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty}\frac{n^4+\ldots}{4n^4} = \frac{1}{4}$$

**Question 5 (10pt)** *Find the area of the largest rectangle that can be inscribed in a semicircle of radius $r = 4$.*




*Hint: Describe $y$ in terms of $x$.* $0 \le x \le 4,\ 0 \le y \le 4$.
*From* $x^2 + y^2 = 4^2$, $y = \sqrt{4^2 - x^2} \ge 0$. *The area function of the reactangle is*

$$a(x) = 2xy = 2x \cdot \sqrt{4^2 - x^2}$$

*From* $(\sqrt{4^2 - x^2})' = \frac{-2x}{2\sqrt{4^2 - x^2}}$

$$a'(x) = (2x)'(\sqrt{4^2 - x^2}) + 2x(\sqrt{4^2 - x^2})' = 2\sqrt{4^2 - x^2} + \frac{2x(-2x)}{2\sqrt{4^2 - x^2}}$$

$$a'(x) = 0 \Rightarrow 2\sqrt{4^2 - x^2} = \frac{2x^2}{\sqrt{4^2 - x^2}} \Rightarrow 4^2 - x^2 = x^2 \Rightarrow 4^2 = 2x^2 \Rightarrow x = 2\sqrt{2} \ge 0$$

$$a'(x) : DNE \Rightarrow x = \pm 4 \ge 0 \therefore x = 4$$

*Critical numbers:* $2\sqrt{2},\ 4$. *End points:* $0,\ 4$.

$$a(0) = 0,\ a(2\sqrt{2}) = 16,\ a(4) = 0$$

$$MAX : 16$$

**Question 6 (10pt)** *Find* $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \cos^2(\theta) d\theta$.
*Let* $F(x) = \int \cos^2(x) dx$. *Then* $F'(x) = \cos^2(x)$. *From FTC 2,*

$$\frac{d}{dx}\int_1^{x^2} \sec(t) dt = \frac{d}{dx}\left(F(x^2) - F(1)\right) = F'(x^2)(x^2)' - 0 = \cos^2(x^2) \cdot (2x) = 2x \cdot \cos^2(x^2)$$

**Question 7 (Extra:10pt)** *Find an anti-derivative.*

$$\int x^3 \sqrt{x^2 + 4} dx$$

*Let* $u = x^2 + 4$. *Then* $x^2 = u - 4$, $du = 2xdx$ *and* $\frac{du}{2} = xdx$.

$$\int x^3 \sqrt{x^2 + 4} dx = \int x^2 \sqrt{x^2 + 4} x dx = \int (u - 4)\sqrt{u}\frac{du}{2} = \frac{1}{2}\int (u\sqrt{u} - 4\sqrt{u}) du = \frac{1}{2}\int \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{2}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - 4\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{(x^2 + 4)^{\frac{5}{2}}}{5} - \frac{4(x^2 + 4)^{\frac{3}{2}}}{3} + C$$