MA441-TEST1-ANSWER Instructor : KWANG KIM

Circle your final answers and show your work to receive full credit.

$$(\sec(x))' = \sec(x)\tan(x),\ (\csc(x))' = -\csc(x)\cot(x),\ (\cot(x))' = -csc^2(x)$$

1.[20pt] Find the limit. **Do not use L'hospital rule.**

- $\lim_{x\to -1} \frac{x^2+1}{x-1}$

$$\lim_{x\to -1} \frac{x^2+1}{x-1} = \frac{(-1)^2+1}{(-1)-1} = \frac{2}{-2} = -1$$

- $\lim\limits_{x\to -2} \dfrac{x^2-4}{2x^2+5x+2}$

$$\lim_{x\to -3} \frac{x^2-4}{2x^2+5x+2} = \lim_{x\to -2} \frac{(x+2)(x-2)}{(x+2)(2x+1)} = \lim_{x\to -2} \frac{(x-2)}{(2x+1)} = \frac{(-2)-2}{2(-2)+1} = \frac{-4}{-3} = \frac{4}{3}$$

- $\lim_{t\to 2} \frac{\sqrt{4t+1}-3}{t-2}$

$$= \lim_{t\to 3} \frac{(\sqrt{8t+1}-5)(\sqrt{8t+1}+5)}{(t-3)(\sqrt{8t+1}+5)} = \lim_{t\to 3} \frac{8t+1-25}{(t-3)(\sqrt{8t+1}+5)} = \lim_{t\to 3} \frac{8(t-3)}{(t-3)(\sqrt{8t+1}+5)}$$

$$= \lim_{t\to 3} \frac{8}{\sqrt{8t+1}+5} = \frac{8}{\sqrt{8(3)+1}+5} = \frac{8}{10} = \frac{4}{5}$$

- $\lim_{x\to 0} \frac{\sin(3x)\cdot\sin(2x)}{4x^2}$ Hint:Use $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

$$\lim_{x\to 0} \frac{\sin 5x)\cdot\sin(2x)}{4x^2} = \lim_{x\to 0} \frac{\sin(5x)}{5x}\cdot\frac{\sin(2x)}{2x}\cdot\frac{5\cdot 2}{4} = 1\cdot 1\cdot\frac{5}{2} = \frac{5}{2}$$

2.[30pt] Find $f'(x)$ using the formulas and chain rule. Simplify your answer.

- (7pt) $f(x) = \frac{4x^2-2x}{\sqrt{x}}$

$$f'(x) = (\frac{4x^2}{\sqrt{x}} - \frac{2x}{\sqrt{x}})' = (4x^{\frac{3}{2}} - 2x^{\frac{1}{2}})' = 4(\frac{3}{2}x^{\frac{1}{2}}) - 2(\frac{1}{2}x^{-\frac{1}{2}}) = 6\sqrt{x} - \frac{1}{\sqrt{x}}$$

- (8pt) $f(x) = \frac{2x}{x+1}$

$$f'(x) = \frac{(2x)'(x+1)-(2x)(x+1)'}{(x+1)^2} = \frac{2(x+1)-2x(1)}{(x+1)^2} = \frac{2x+2-2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

- (7pt) $f(x) = \sqrt{x^2+1}$
  Let $y = \sqrt{u},\ u = x^2+1,\ \frac{du}{dx} = 2x$.

$$f'(x) = \frac{dy}{du}\cdot\frac{du}{dx} = (\frac{1}{2\sqrt{u}})\cdot(2x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

- (8pt) $f(x) = (x^2+1)^5\cdot(2x-1)^3$

$$f'(x) = [(x^2+1)^5]'(2x-1)^3 + (x^2+1)^5[(2x-1)^3]' = [5(x^2+1)^4\cdot(2x)](2x-1)^3 + (x^2+1)^5[3(2x-1)^2\cdot(2)]$$

$$= 10x(x^2+1)^4(2x-1)^3 + 6(x^2+1)^5(2x-1)^2 = 2(x^2+1)^4(2x-1)^2[5x(2x-1)+3(x^2+1)]$$

$$= 2(x^2+1)^4(2x-1)^2[13x^2-5x+3]$$

3.[10pt] Find an equation $x^3 + y^3 = 4xy + 1$ of the tangent line to the curve at the given point $(2,1)$.
Let $y' = \frac{dy}{dx}, x' = \frac{dx}{dx} = 1$. Then

$$(xy)' = x'y + xy' = y + xy',\ (y^3) = 3y^2y'$$

Now differentiate the equation w.r.t x.

$$3x^2 + 3y^2y' = 4(y + xy')$$

Solve it for y'.

$$3x^2 + 3y^2y' = 4y + 4xy' \Leftrightarrow 3y^2y' - 4xy' = 4y - 3x^2$$

$$(3y^2 - 4x)y' = 4y - 3x^2 \Leftrightarrow y' = \frac{4y - 3x^2}{3y^2 - 4x}$$

The slope of tangent line at $(2,1)$ is

$$m = \frac{4(1) - 3(2)^2}{3(1)^2 - 4(2)} = \frac{-8}{-5} = \frac{8}{5}$$

Or from $3x^2 + 3y^2y' = 4(y + xy')$, replace (x,y) with $(2,1)$.

$$3(2)^2 + 3(1)^2y' = 4(1 + 2y') \Leftrightarrow 12 + 3y' = 4 + 8y' \Leftrightarrow 8 = 5y' \Leftrightarrow y' = \frac{8}{5}$$

The equation of tangent line of $x^3 + y^3 = 4xy + 1$ at $(2,1)$ is

$$y - 1 = \frac{8}{5}(x - 2)$$

4.[15pt] Find $f'(x)$ using the definition of the derivative if $f(x) = 2x^2 - 3x + 1$.

$$f(x+h) = 2(x+h)^2 - 3(x+h) + 1 = 2(x^2 + 2xh + h^2) - 3x - 3h + 1 = 2x^2 + 4xh + 2h^2 - 3x - 3h + 1$$

$$f(x+h) - f(x) = f(x+h) - [2x^2 - 3x + 1] = 2x^2 + 4xh + 2h^2 - 3x - 3h + 1 - 2x^2 + 3x - 1 = 4xh + 2h^2 - 3h = h(4x + 2h - 3)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{h(4x + 2h - 3)}{h} = \lim_{h\to 0}(4x + 2h - 3) = 4x - 3$$

5.[15pt] Let

$$f(x) = \begin{cases} \cos(x) + 1 & if\ x > 0 \\ 2 & if\ x = 0 \\ |x - 5| - 3 & if\ x < 0 \end{cases}$$

- Find $\lim_{x\to 0^-} f(x)$ if exists.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} [|x - 5| - 3] = |(0) - 5| - 3 = 5 - 3 = 2$$

- Find $\lim_{x\to 0^+} f(x)$ if exists.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} [\cos(x) + 1] = 1 + 1 = 2$$

- Find $\lim_{x\to 0} f(x)$ if exists.
  Since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$,

$$\lim_{x\to 0} f(x) = 2$$

- Is f continuous at $x = 0$? Explain it.
  Since $\lim_{x\to 0} f(x) = 2 = f(0)$, f is the continuous at $x = 0$.

6.[10pt] If $h(x) = \sqrt{f(x) - 2g(x)}$ where $f(1) = 13,\ g(1) = 2,\ f'(1) = 5,\ g'(1) = 7$, find $h'(1)$.
Since

$$h'(x) = \frac{1}{2\sqrt{f(x) - 2g(x)}} \cdot (f'(x) - 2g'(x)) = \frac{f'(x) - 2g'(x)}{2\sqrt{f(x) - 2g(x)}}$$

$$h'(1) = \frac{f'(1) - 2g'(1)}{2\sqrt{f(1) - 2g(1)}} = \frac{5 - 2(7)}{2\sqrt{13 - 2(2)}} = \frac{-9}{6} = -\frac{3}{2}$$

Extra.[5pt]

$$f(x) = sin(x^2) \cdot cos(x)$$

$$f'(x) = [sin(x^2)]' \cdot cos(x) + sin(x^2) \cdot [cos(x)]' = [cos(x^2) \cdot (2x)] \cdot cos(x) + sin(x^2)[-sin(x)] = 2x \cdot cos(x^2) \cdot cos(x) - sin(x) \cdot sin(x^2)$$