삼차원 벡터 $a=(a_1,a_2,a_3)^T, b=(b_1,b_2,b_3)^T$ 에 대하여 Cross product 'X'를 다음과 같이 정의할 때, 다음 성질들을 증명하시오. ($e_1=(1,0,0)^T, e_2=(0,1,0)^T, e_3=(0,0,1)^T$로 놓자.)

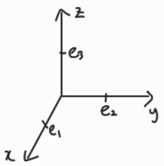
$$a \times b \ := \ \|a\| \ \|b\| \sin\theta \ \mathrm{n}$$ $\Rightarrow$ Cross product 의 결과는 벡터!

(θ는 a와 b가 이루는 내각이고, n은 오른손 법칙에 따른 a, b에 수직인 단위벡터)

(A-1) $e_i \times e_i = (0,0,0)^T \quad i=1,2,3$

$$e_i \times e_i = \|e_i\|\|e_i\| \cdot \sin 0^\circ \cdot n = 0 \quad (\because \sin 0^\circ = 0)$$

(A-2) $e_1 \times e_2 = e_3, \ \ e_2 \times e_3 = e_1, \ \ e_3 \times e_1 = e_2$



$$e_1 \times e_2 = \|e_1\|\|e_2\| \sin 90^\circ \cdot e_3 = 1\cdot1\cdot1\cdot e_3 = e_3$$
$$e_2 \times e_3 = \|e_2\|\|e_3\| \sin 90^\circ \cdot e_1 = 1\cdot1\cdot1\cdot e_1 = e_1$$
$$e_3 \times e_1 = \|e_3\|\|e_1\| \sin 90^\circ \cdot e_2 = 1\cdot1\cdot1\cdot e_2 = e_2$$

cf) $e_2 \times e_1 = -e_3$ ($\because$ e_2에서 e_1으로 오른손 법칙 쓰면 e_3와 반대방향)
$e_3 \times e_2 = -e_1$
$e_1 \times e_3 = -e_2$

(B-1) $a=a_1e_1+a_2e_2+a_3e_3, \ b=b_1e_1+b_2e_2+b_3e_3$ 와 (A)를 이용:

$$a \times b = C_{11}e_1 + C_{12}e_2 + C_{13}e_3 \quad (\text{단, } C_{ij} \text{ 는 } \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{의 Cofactor})$$

$$a \times b = (a_1e_1+a_2e_2+a_3e_3) \times (b_1e_1+b_2e_2+b_3e_3)$$
$$= a_1b_1\underbrace{e_1\times e_1}_{=0} + a_1b_2\underbrace{e_1\times e_2}_{=e_3} + a_1b_3\underbrace{e_1\times e_3}_{=-e_2} + a_2b_1\underbrace{e_2\times e_1}_{=-e_3} + a_2b_2\underbrace{e_2\times e_2}_{=0} + a_2b_3\underbrace{e_2\times e_3}_{=e_1} + a_3b_1\underbrace{e_3\times e_1}_{=e_2} + a_3b_2\underbrace{e_3\times e_2}_{=-e_1} + a_3b_3\underbrace{e_3\times e_3}_{=0}$$
$$= a_1b_2e_3 - a_1b_3e_2 - a_2b_1e_3 + a_2b_3e_1 + a_3b_1e_2 - a_3b_2e_1$$
$$= (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2 \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = a_3b_1 - a_1b_3 \qquad C_{13} = (-1)^{1+3}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2b_3 - a_3b_2$$

$$C_{11}e_1 + C_{12}e_2 + C_{13}e_3 = (a_2b_3 - a_3b_2)e_1 + (a_3b_1 + a_1b_3)e_2 + (a_2b_3 - a_3b_2)e_3$$

(B-2) $a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (\text{1행 선택하면}) \ e_1 \cdot C_{11} + e_2 \cdot C_{12} + e_3 \cdot C_{13}$$

(C-1) Triple Product Formula for $a=(a_1,a_2,a_3)^T, b=(b_1,b_2,b_3)^T, c=(c_1,c_2,c_3)^T$

$$(a\times b)\cdot c=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

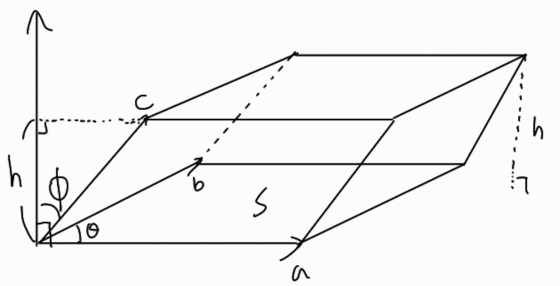
$$(a\times b)\cdot c=(C_{11}e_1+C_{12}e_2+C_{13}e_3)\cdot(c_1e_1+c_2e_2+c_3e_3)$$
$$=C_{11}c_1+C_{12}c_2+C_{13}c_3=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=-\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(C-2) $(a\times b)\cdot c=(b\times c)\cdot a=(c\times a)\cdot b$

$$(a\times b)\cdot c=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}=(b\times c)\cdot a$$

$$(a\times b)\cdot c=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=(c\times a)\cdot b$$

(C-3) $(a\times b)\cdot c$ = ± (a, b, c를 세 변으로 하는 평행육면체(parallelepiped)의 부피)

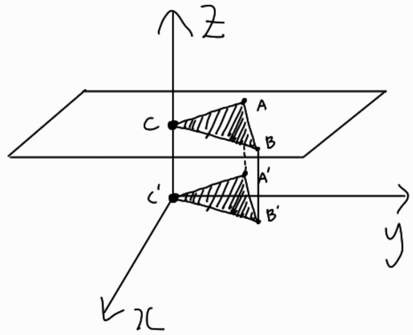


$$(a\times b)\cdot c$$
$$=\|a\times b\|\|c\|\cos\phi$$
$$=\|a\|\|b\|\sin\theta\,\|c\|\cos\phi$$
$$V=S\cdot h=\underbrace{\|a\|\|b\|\sin\theta}_{\text{평행사변형 넓이}}\cdot\underbrace{\|c\|\cos\phi}_{\text{높이}}$$

이때 $h=c\cdot\cos\phi$

(D-1) (C) 이용: $A=(x_1, y_1, 1)$, $B=(x_2, y_2, 1)$, $C=(0,0,1)$로 이루어진 삼각형의 넓이 S

$$S = \frac{1}{2}\begin{vmatrix} x_1 & y_2 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$



$$\triangle ABC = S = V = \frac{1}{2}\text{평행육면체 부피} = \frac{1}{2}(A\times B)\cdot C = \frac{1}{2}\begin{vmatrix} A-C \\ B-C \\ C \end{vmatrix}$$

$$V = S\cdot \underbrace{1}_{\text{높이}=1}$$

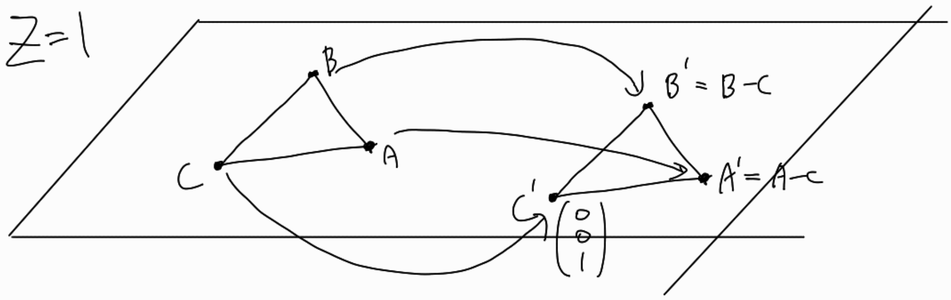
$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times 1$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} \times 1$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

(D-2) $A=(x_1, y_1, 1)$, $B=(x_2, y_2, 1)$, $C=(x_3, y_3, 1)$로 이루어진 삼각형의 넓이 S

$$S = \frac{1}{2}\begin{vmatrix} x_1 & y_2 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



$$S = \frac{1}{2}\begin{vmatrix} A-C, & 1 \\ B-C, & 1 \\ 0 \quad 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1-x_3 & y_1-y_3 & 1 \\ x_2-x_3 & y_2-y_3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$\times x_3$ $\times y_3$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$