1. A=[4, -1; -1, 4]일 때 다음 행렬들의 고유치 및 고유벡터를 직접 계산해 구해보시오.

(a) A, A^2

$A=\begin{pmatrix}4 & -1\\ -1 & 4\end{pmatrix}$, $A-\lambda I=\begin{pmatrix}4-\lambda & -1\\ -1 & 4-\lambda\end{pmatrix}$ $|A-\lambda I| = \lambda^2-8\lambda+15=0$. $\lambda=5,3$

$\lambda=5$ 일 때 $\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}-1 & -1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}$

$\lambda=3$ 일 때 $\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\therefore A: \left(5, \begin{pmatrix}-1\\ 1\end{pmatrix}\right), \left(3, \begin{pmatrix}1\\ 1\end{pmatrix}\right)$

$A^2: \left(25, \begin{pmatrix}-1\\ 1\end{pmatrix}\right), \left(9, \begin{pmatrix}1\\ 1\end{pmatrix}\right)$ $Ax=\lambda x \Rightarrow A^2x=A\lambda x=\lambda Ax=\lambda\lambda x=\lambda^2 x$

(b) A^{-1}

$A^{-1}: \left(\frac{1}{5}, \begin{pmatrix}-1\\ 1\end{pmatrix}\right), \left(\frac{1}{3}, \begin{pmatrix}1\\ 1\end{pmatrix}\right)$ $Ax=\lambda x \Rightarrow A^{-1}Ax=A^{-1}\lambda x \Rightarrow x=\lambda A^{-1}x \Rightarrow \frac{1}{\lambda}x=A^{-1}x$

(c) $A+4I, A^2+2A$

$A+4I: \left(9, \begin{pmatrix}-1\\ 1\end{pmatrix}\right), \left(7, \begin{pmatrix}1\\ 1\end{pmatrix}\right)$ $(A+4I)x = Ax+4Ix = \lambda x+4x=(\lambda+4)x$

$A^2+2A: \left(15, \begin{pmatrix}-1\\ 1\end{pmatrix}\right), \left(35, \begin{pmatrix}1\\ 1\end{pmatrix}\right)$ $(A^2+2A)x = A^2x+2Ax = \lambda^2x+2\lambda x=(\lambda^2+2\lambda)x$

2. A=[1,-1,0; -1,2,-1; 0,-1,1]에 대하여 답하시오. — A가 symmetric이면 Diagonalizable 임

(a) A의 고유치 및 고유벡터를 모두 구하시오.

$$|A-\lambda I| = \begin{vmatrix}1-\lambda & -1 & 0\\ -1 & 2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{vmatrix} = (1-\lambda)\begin{vmatrix}2-\lambda & -1\\ -1 & 1-\lambda\end{vmatrix} - (-1)\begin{vmatrix}-1 & -1\\ 0 & 1-\lambda\end{vmatrix} + 0\begin{vmatrix}-1 & 2-\lambda\\ 0 & -1\end{vmatrix}$$

$$= (1-\lambda)\left((2-\lambda)(1-\lambda)-1\right)+1\cdot(\lambda-1) = (1-\lambda)(\lambda^2-3\lambda+1)+(\lambda-1)$$

$$= (1-\lambda)(\lambda^2-3\lambda) = \lambda(1-\lambda)(\lambda-3)=0 \quad \lambda=3,1,0$$

$\lambda=3$ 일 때 $\begin{pmatrix}-2 & -1 & 0\\ -1 & -1 & -1\\ 0 & -1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}-2 & -1 & 0\\ 0 & -\frac{1}{2} & -1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$. $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ -2\\ 1\end{pmatrix}$

$\lambda=1$ 일 때 $\begin{pmatrix}0 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}-1 & 1 & -1\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$. $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$

$\lambda=0$ 일 때 $\begin{pmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$

$\therefore \left(3, \begin{pmatrix}1\\ -2\\ 1\end{pmatrix}\right), \left(1, \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}\right), \left(0, \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\right)$

(b) $\mathrm{A}=\mathrm{S}\Lambda\mathrm{S}^{-1}$을 만족하는 3x3 행렬 S와 Λ를 구하시오.

$S=\begin{pmatrix}1 & -1 & 1\\ -2 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}$, $\Lambda=\begin{pmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$

3. 다음을 증명해 보시오.

(a) nxn 행렬 B가 Invertible이면 A와 $B^{-1}AB$는 같은 eigenvalue를 갖는다.

$B^{-1}AB \to (\lambda, X)$ 가정. $B^{-1}ABX = \lambda X \Rightarrow ABX = \lambda BX$ $\therefore A \to (\lambda, BX)$

(b) nxn 행렬 B가 Invertible이면 AB와 BA는 같은 eigenvalue를 갖는다.

BA 와 $B^{-1}(BA)B$ 의 eigenvalue 같음 $B^{-1}BAB = AB$ $(\because (a))$

4. A=[3,2; 2,6]과 x=[3;1]에 대하여 $A^{100}x$를 구하시오. A가 Diagonalizable ⇒ A^kX 계산가능

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 14 = 0. \quad \lambda = 7, 2$$

$\lambda = 7$일 때 $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ ... 편하게 $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = 2$일 때 $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$X = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = aX_1 + bX_2$$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A^{100}X = A^{100}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = A^{100}(aX_1 + bX_2) = aA^{100}X_1 + bA^{100}X_2 = a\lambda^{100}X_1 + b\lambda^{100}X_2$$

$$= \begin{pmatrix} a \cdot 7^{100} - 2b \cdot 2^{100} \\ 2a \cdot 7^{100} + b \cdot 2^{100} \end{pmatrix} = \begin{pmatrix} 7^{100} + 2^{101} \\ 2 \cdot 7^{100} - 2^{100} \end{pmatrix}$$