1. 행렬 $A=(1,1,-1,1;-3,-3,3,1;2,1,-1,3;-1,-1,3,-3)$에 대하여 다음에 답하시오. (강의자료 p5-7)

(a) 행렬 A에 가우스 소거법을 적용하면서 $PA=L'U'$를 만족하는 permutation matrix P, Lower/Upper 삼각행렬 L', U'을 구하시오.

$$A=\begin{pmatrix}1&1&-1&1\\-3&-3&3&1\\2&1&-1&3\\-1&-1&3&-3\end{pmatrix}\quad P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}\quad L'=\begin{pmatrix}1&0&0&0\\2&1&0&0\\-1&0&1&0\\-3&0&0&1\end{pmatrix}\rightarrow$$

$$\downarrow$$

$$\begin{pmatrix}1&1&-1&1\\0&0&0&4\\0&-1&1&1\\0&0&2&-2\end{pmatrix}\rightarrow\begin{pmatrix}1&1&-1&1\\0&-1&1&1\\0&0&0&4\\0&0&2&-2\end{pmatrix}\rightarrow\begin{pmatrix}1&1&-1&1\\0&-1&1&1\\0&0&2&-2\\0&0&0&4\end{pmatrix}$$

$$\therefore P=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}\quad L'=\begin{pmatrix}1&0&0&0\\2&1&0&0\\-1&0&1&0\\-3&0&0&1\end{pmatrix}\quad U'=\begin{pmatrix}1&1&-1&1\\0&-1&1&1\\0&0&2&-2\\0&0&0&4\end{pmatrix}$$

<검산>

$$\underset{P}{\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}}\underset{A}{\begin{pmatrix}1&1&-1&1\\-3&-3&3&1\\2&1&-1&3\\-1&-1&3&-3\end{pmatrix}}=\underset{PA}{\begin{pmatrix}1&1&-1&1\\2&1&-1&3\\-1&-1&3&-3\\-3&-3&3&1\end{pmatrix}}\quad \underset{L'}{\begin{pmatrix}1&0&0&0\\2&1&0&0\\-1&0&1&0\\-3&0&0&1\end{pmatrix}}\underset{U'}{\begin{pmatrix}1&1&-1&1\\0&-1&1&1\\0&0&2&-2\\0&0&0&4\end{pmatrix}}=\begin{pmatrix}1&1&-1&1\\2&1&-1&3\\-1&-1&3&3\\-3&-3&3&1\end{pmatrix}$$

(b) (a)에서 구한 행렬들을 이용해, $B_1=(-5,11,-6,13)^T$, $B_2=(-2,10,1,4)^T$ 에 대하여 $AX_1=B_1$, $AX_2=B_2$를 만족하는 해 X_1, X_2 를 구하시오.

$$\underset{L'}{\begin{pmatrix}1&0&0&0\\2&1&0&0\\-1&0&1&0\\-3&0&0&1\end{pmatrix}}\underbrace{\underset{U'}{\begin{pmatrix}1&1&-1&1\\0&-1&1&1\\0&0&2&-2\\0&0&0&4\end{pmatrix}}\underset{X_1}{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}}}_{\begin{pmatrix}u\\v\\s\\t\end{pmatrix}}=\underset{P}{\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}}\underset{B_1}{\begin{pmatrix}-5\\11\\-6\\13\end{pmatrix}}=\underset{B_1'}{\begin{pmatrix}-5\\-6\\13\\11\end{pmatrix}}$$

<검산> $AX_1=B_1$

$$\text{Forward}\Rightarrow\begin{pmatrix}u\\v\\s\\t\end{pmatrix}=\begin{pmatrix}-5\\4\\8\\-4\end{pmatrix}\qquad \text{Backward}\Rightarrow\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}1\\-2\\3\\-1\end{pmatrix}=X_1$$

$$\underset{L'}{\begin{pmatrix}1&0&0&0\\2&1&0&0\\-1&0&1&0\\-3&0&0&1\end{pmatrix}}\underbrace{\underset{U'}{\begin{pmatrix}1&1&-1&1\\0&-1&1&1\\0&0&2&-2\\0&0&0&4\end{pmatrix}}\underset{X_1}{\begin{pmatrix}x\\y\\z\\w\end{pmatrix}}}_{\begin{pmatrix}u\\v\\s\\t\end{pmatrix}}=\underset{P}{\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}}\underset{B_2}{\begin{pmatrix}-2\\10\\1\\4\end{pmatrix}}=\underset{B_2'}{\begin{pmatrix}-2\\1\\4\\10\end{pmatrix}}$$

<검산> $AX_2=B_2$

$$\text{Forward}\Rightarrow\begin{pmatrix}u\\v\\s\\t\end{pmatrix}=\begin{pmatrix}-2\\5\\2\\4\end{pmatrix}\qquad \text{Backward}\Rightarrow\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}1\\-2\\2\\1\end{pmatrix}=X_2$$

2. 임의의 permutation matrix (단위행렬의 행들의 순서를 바꾼 행렬)을 P라 할 때, 다음을 증명해 보시오. $PP^T = I$, 즉 P^T가 P의 역행렬이다.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \leftarrow e_1 \\ \leftarrow e_2 \\ \leftarrow e_3 \end{matrix} \qquad P = \begin{pmatrix} e_{\sigma(1)} \\ e_{\sigma(2)} \\ e_{\sigma(3)} \end{pmatrix}$$

$$ex)\ P = \begin{pmatrix} e_3 \\ e_1 \\ e_2 \end{pmatrix} \Rightarrow PP^T = \begin{pmatrix} e_3 \\ \hline e_1 \\ \hline e_2 \end{pmatrix} \left(e_3^T \,\middle|\, e_1^T \,\middle|\, e_2^T \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

3. 행렬 A와 단위행렬 I를 이용하여 역행렬 A^{-1}을 구하는 과정은 다음과 같다. (강의자료 p9)

$[\ A\ |\ I\ ]$ --> $[\ U\ |\ V\ ]$ --> $[\ I\ |\ A^{-1}\ ]$ (U: upper triangular matrix)

다음 행렬 A에 대하여 행렬 U, V와 역행렬 A^{-1}을 구하시오.

(a) $A = (-1, 1, 2; 2, 0, -1; -2, 1, 3)$

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 2 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \end{array}\right)$$

$$\boxed{\begin{matrix} AA^{-1} = I \\ A \mid I \\ \hookrightarrow I \mid A^{-1} \end{matrix}}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 4 & -1 & -3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 2 & 0 & 8 & -2 & -6 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 5 & -2 & -4 \\ 0 & 2 & 0 & 8 & -2 & -6 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 1 \end{array}\right)$$

(b) $A = (-1, 2, 2; 2, -1, 0; 3, 0, -1)$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 3 & 4 & 2 & 1 & 0 \\ 0 & 6 & 5 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 3 & 4 & 2 & 1 & 0 \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & -\frac{5}{9} & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -\frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \\ 0 & 3 & 0 & \frac{2}{3} & -\frac{5}{3} & \frac{4}{3} \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} -1 & 2 & 0 & \frac{1}{3} & -\frac{4}{3} & \frac{2}{3} \\ 0 & 3 & 0 & \frac{2}{3} & -\frac{5}{3} & \frac{4}{3} \\ 0 & 0 & -3 & -1 & -2 & 1 \end{array}\right)$$