1. S={[1, 1; 0, 0], [0, 0; 1, 1], [1, 0; 0, 1], [0, 1; 1, 1]} 는 $M_{2\times 2}$의 Basis임을 증명하시오.

$\dim(M) = |S| = 4$

(a) S는 Linearly Indepedent하다.

$$a\begin{pmatrix}1&1\\0&0\end{pmatrix} + b\begin{pmatrix}0&0\\1&1\end{pmatrix} + c\begin{pmatrix}1&0\\0&1\end{pmatrix} + d\begin{pmatrix}0&1\\1&1\end{pmatrix} = \begin{pmatrix}a+c & a+d\\ b+d & b+c+d\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$a+c=0$
$a+d=0$ } $\Rightarrow c=d$
$b+d=0$ } $\Rightarrow a=b=-d$
$b+c+d=0$ } $\Rightarrow d=0$

$\therefore a=b=c=d=0$ $\therefore$ Linearly Independent

(b) S는 $M_{2\times 2}$를 span한다.

$$x_1\begin{pmatrix}1&1\\0&0\end{pmatrix} + x_2\begin{pmatrix}0&0\\1&1\end{pmatrix} + x_3\begin{pmatrix}1&0\\0&1\end{pmatrix} + x_4\begin{pmatrix}0&1\\1&1\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$\begin{pmatrix}x_1+x_3 & x_1+x_4\\ x_2+x_4 & x_2+x_3+x_4\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$x_1+x_3=a$
$x_1+x_4=b$
$x_2+x_4=c$
$x_2+x_3+x_4=d$] $\Rightarrow x_3=d-c$

$x_1=a+c-d$ $x_4=-a+b-c+d$ $x_2=a-b+2c-d$

$\therefore$ Span한다

2. B = $\{1+x,\ -1+2x,\ x^2\}$ 는 P_2의 Basis임을 증명하시오.

2차 이하의 다항식

$\dim(P_2) = |B| = 3$

(a) B는 Linearly Indepedent하다.

$a(1+x)+b(-1+2x)+cx^2 = (a-b)+(a+2b)x+cx^2 = 0$

$\therefore a-b=0,\ a+2b=0,\ c=0$ 즉 $a=b=c=0$ $\therefore$ Linearly Independent

(b) B는 P_2를 span한다.

$\alpha(1+x)+\beta(-1+2x)+\gamma x^2 = (\alpha-\beta)+(\alpha+2\beta)x+\gamma x^2 = a+bx+cx^2$

$\therefore \alpha-\beta=a,\ \alpha+2\beta=b,\ \gamma=c$ $\therefore \alpha=\frac{2a+b}{3},\ \beta=\frac{b-a}{3},\ c=\gamma$ $\therefore$ Span한다

$\longrightarrow$

3. A=LU=(1, 0, 0; 1, 1, 0; 7, -1, 2)(1, 0, 1, 4, 5; 0, 1, 2, 2, 1; 0, 0, 0, 1, 1) 이다.

(a) Basis of $\mathbb{N}(A)$

Basis of $N(A)$ = Basis of $N(U)$.

$$\overset{U}{\begin{pmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}} \begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

x	y	z	s	t
-1	-2	1	0	0
-1	1	0	-1	1

$$N(U) = \left\langle \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle \qquad \dim(N(A)) = 2 \subseteq R^5$$

(b) Basis of $\mathbb{C}(A)$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 4 & 5 \\ 1 & 1 & 3 & 6 & 6 \\ 7 & -1 & 5 & 28 & 36 \end{pmatrix}$$

pivot column은 1열, 2열, 4열. $\therefore$ $C(A)$의 Basis $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 28 \end{pmatrix} \right\}$

(c) Basis of $\mathbb{R}(A)$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 4 & 5 \\ 1 & 1 & 3 & 6 & 6 \\ 7 & -1 & 5 & 28 & 36 \end{pmatrix}$$

pivot row는 1행, 2행, 3행. $\therefore$ $R(A)$의 Basis $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 5 \\ 28 \\ 36 \end{pmatrix} \right\}$