

Math 131A Lecture Notes

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1 Lecture 1

1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

1.3 Logical Connections

We usually use the letters P and Q to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: “ P and Q ”, $P \wedge Q$
2. Disjunctions: “ P or Q ”, $P \vee Q$
3. Implications: “If P , then Q ”, $P \implies Q$
 - (a) If the proposition is false (i.e. if P is false) then the whole statement is true.

Definition.

We say that the statement is *vacuously true*.

4. Negations: “Not P ”, $\neg P$

1.3.1 Truth Tables

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Prove that if n is an integer, then $n(n+1)$ is even.

Proof. Suppose that n is an integer. Then we have two cases, where either n is even or n is odd. Let n be an even integer such that $n = 2k$ where $k \in \mathbb{Z}$. Then we have

$$\begin{aligned}n(n+1) &= 2k(2k+1) \\ &= 2(2k^2 + k).\end{aligned}$$

Thus we see that $n(n+1)$ is even when n is even. Now let n be odd such that $n = 2m+1$ where $m \in \mathbb{Z}$. Then we have

$$\begin{aligned}n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1).\end{aligned}$$

Thus $n(n+1)$ is also even when n is odd, and so is even for all integers n . □

2 Lecture 2

2.1 Continuation of Logic

2.1.1 De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

Note. Negations turn “and” into “or” and vice versa.

Example. Suppose we have the following statement:

$$P: x \text{ is even and } x > 0.$$

Then the negation of P would be:

$$\neg P: x \text{ is odd or } x \leq 0.$$

2.1.2 Converse

Definition. *Converse*

The *converse* of a statement $P \implies Q$ is the statement $Q \implies P$. In general, the converse of a statement says nothing about the original statement.

Example. Consider the statement

$$\text{If } x > 0, \text{ then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0, \text{ then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write $P \iff Q$ instead of $(P \implies Q) \wedge (Q \implies P)$. In this case, we call P and Q *logically equivalent*. In writing, we say “ P if and only if Q ”.

2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

Lemma 1. Let a be an integer. If a^2 is even, then a is even.

Proof. Suppose a is odd, so $a = 2k + 1$ for some integer k . Then

$$\begin{aligned} a^2 &= (2k + 1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus a^2 is odd and this completes the proof. \square

2.1.4 Variables and Quantifiers

We have a value x that varies over some values, so we use $P(x)$ to denote a statement that depends on the value of x .

Example. Consider the statement

$$P(x) : x + 2 = 3.$$

The statement is true if and only if $x = 1$.

We have two quantifiers— \forall = “for all”, and \exists = “there exists”.

- $\forall x : P(x)$ is true if $P(x)$ is true for all x .
- $\exists x : P(x)$ is true if there exists at least one x such that $P(x)$ is true.

We can also have nested quantifiers,

$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

Note. The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of n depends on m .

2.1.5 Proof by Counterexample

After “simplifying” the statement $\neg(\forall x : P(x))$, we get $\exists x : \neg P(x)$. We simply need to find a single counterexample to show that a statement is false for all x .

Example. Consider the statement $\forall x \in \mathbb{R} : x + 2 = 3$. All we need to do is show that there exists some $x \in \mathbb{R}$ such that $x + 2 \neq 3$. This occurs when $x = 0$, so the statement is false.

2.1.6 Proof by Contradiction

Key Idea. We want to show that $P \implies Q$ indirectly.

Lemma 2. We can show that

$$P \implies Q = (\neg P) \vee Q.$$

Then $P \implies Q$ is true if and only if $\neg(P \implies Q)$ is false, and so by Lemma 2 and De Morgan’s Laws, $P \wedge \neg Q$ is false.

For proof by contradiction, we assume P is true and $\neg Q$ is true, and try to show that $P \wedge \neg Q$ is false (a contradiction).

3 Lecture 3

3.1 More Logic

3.1.1 Proof by Contradiction

To do a proof by contradiction for a statement $P \implies Q$, we assume $P \wedge \neg Q$. We aim to show that $P \wedge \neg Q$ is false (a contradiction).

Theorem — *Irrationality of $\sqrt{2}$*

There is no rational number x such that $x^2 = 2$. In other words, if $x \in \mathbb{Q}$, then $x^2 \neq 2$.

Proof. Suppose towards a contradiction that there exists some $x \in \mathbb{Q}$ such that $x^2 = 2$. Since x is rational, there exist integers p, q such that $q \neq 0$, $\frac{p}{q} = x$, and p and q have no common divisors (other than 1). Then

$$\begin{aligned} x^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since p^2 is even, there exists some integer k such that $p = 2k$. Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

By the same logic as before, we know that q must also be even (they share a common factor of 2). However, this contradicts our original assumption that p and q share no common factors, and this completes the proof. \square

3.2 Set Theory

We write $x \in A$ when we want to say that “ x is an element of A ”, and $x \notin A$ when we want to say that “ x is not an element of A ”.

3.2.1 Set Combinations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$.
- Subset (Inclusion): $A \subseteq B$ if and only if $x \in A \implies x \in B$.

Definition. *Proper Subset*

A set A is a *proper subset* of a set B if $A \subseteq B$ and there exists some $x \in B$ such that $x \notin A$. We denote this as $A \subset B$.

- Equality: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Note (Showing Equality of Sets). If you want to show $A = B$, you need to show both $A \subseteq B$ and $B \subseteq A$. In other words, you must show that for all $x \in A$, we have $x \in B$, and vice versa.

Example. We have $\mathbb{N} = \{1, 2, 3, \dots\}$.

- Let E be the set of even natural numbers. Note that $E \subseteq \mathbb{N}$.
- Let $S = \{p \in \mathbb{Q} \mid p^2 < 2\} \subseteq \mathbb{Q}$.

Then we have:

- $\mathbb{N} \cup E = \mathbb{N}$.
- $\mathbb{N} \cap E = E$.
- $\mathbb{N} \cap S = \{n \in \mathbb{N} \mid n^2 < 2\} = \{1\}$.
- $E \cap S = \emptyset$.

Definition. *Disjoint Sets*

If $A \cap B = \emptyset$, we call A and B *disjoint* sets.

Proof. Suppose towards a contradiction that there exists some $x \in E \cap S$, which is to say $x \in E$ and $x \in S$. Since $x \in E$, we know that x is even, and so there exists some integer k such that $x = 2k$. Then

$$x^2 = (2k)^2 = 4k^2,$$

so $4 \mid x^2$. Therefore $x \geq 4$, which contradicts the condition for $x \in S$, namely $x^2 < 2$. \square

- Given some $n \in \mathbb{N}$, we define $A_n = \mathbb{N} \setminus \{1, \dots, n-1\}$.
 - $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$.
 - $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Definition. *Set Complement*

If $A \subseteq B$, then we define the *complement* of A in B to be $A^c = B \setminus A$.

3.2.2 De Morgan's Laws

If I is an index set and $\{A_j\}_{j \in I}$ are subsets of B , then

$$\left(\bigcup_{j \in I} A_j \right)^c = \bigcap_{j \in I} A_j^c, \quad \text{and} \quad \left(\bigcap_{j \in I} A_j \right)^c = \bigcup_{j \in I} A_j^c$$

4 Lecture 4

4.1 Cartesian Product

If I have two sets A and B , then we may form their *Cartesian Product*, which is

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}.$$

Definition. *Binary Relation*

A *binary relation* is a subset $R \subseteq A \times B$. We say $x \in A$ is in relation to $y \in B$ if $(x, y) \in R$. We denote this by

$$xRy \iff (x, y) \in R.$$

Example. Consider the relation

$$\mathbb{R} \times \mathbb{R} \supseteq R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Then the relation is *reflexive*, because xRx . It is also *antisymmetric*, because $xRy \wedge yRx \implies x = y$. Finally, this relation is *transitive*, because $xRy \wedge yRz \implies xRz$.

These properties only make sense if $A = B$, i.e. $R \subseteq A \times A$, and we say that “ R is a relation on A ”.

Definition. *Partial Order*

If a relation is reflexive, antisymmetric, and transitive on A , then it is a *partial order* on A .

The notion of “less than or equal to” is a partial order for \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} , but there exists no partial order for \mathbb{C} .

Definition. *Power Set*

For a set A , we may define its *power set* by

$$\mathcal{P}(A) = \{C \mid C \subseteq A\}.$$

Note that set inclusion is a partial order on $\mathcal{P}(A)$.

Definition. *Equivalence Relation*

An *equivalence relation* R over A is a relation that is reflexive, symmetric, and transitive.

Note. Just like a partial order behaves much like \leq , an equivalence relation behaves much like $=$.

Definition. *Equivalence Class*

Given an equivalence relation on A , we define a new set

$$[x] := \{y \in A \mid x \sim y\}.$$

We call $[x]$ the *equivalence class* of x . Any $z \in [x]$ is called a *representative* of the equivalence class $[x]$. In particular, x is a representative of its own equivalence class.

Let A be a set with equivalence relation \sim . Then for any $x, y \in A$,

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \emptyset.$$

Proof. Let $x, y \in A$. We know that x is either equivalent to y or it is not. Suppose the former is true and let $z \in [x]$. Thus we know that $z \sim x$ and $x \sim y$, and by transitivity we have $z \sim y$. Thus $z \in [y]$ and $[x] \subseteq [y]$. The reverse argument is the same.

If x is not equivalent to y , then suppose towards a contradiction that $[x] \cap [y] \neq \emptyset$. Let $x \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$. By symmetry we know that $x \sim z$ and by transitivity we have $x \sim y$. We have arrived at the contradiction that x is both equivalent and not equivalent to y . \square

Definition. *Function*

A relation $R \subseteq A \times B$ is a *function* if for all $x \in A$ and all $y, z \in B$, we have the following:

- $xRy \wedge xRz \implies y = z$.

In other words, every input x has only one output.

Definition. *Injective Functions*

A function f is *injective* if $f(x_1) = f(x_2) \implies x_1 = x_2$.

Definition. *Surjective Functions*

A function f is *surjective* if for every $y \in B$, there exists some $x \in A$ such that $f(x) = y$.