Math 131A Lecture Notes

Kyle Chui

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Contents

1 Lecture 1

1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

1.3 Logical Connections

We usually use the letters P and Q to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: "P and Q", $P \wedge Q$
2. Disjunctions: "P or Q", $P \vee Q$
3. Implications: "If P, then Q", $P \implies Q$
   (a) If the proposition is false (i.e. if P is false) then the whole statement is true.

   > **Definition.**
   > We say that the statement is *vacuously true*.

4. Negations: "Not P", $\neg P$

1.3.1 Truth Tables

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Prove that if n is an integer, then $n(n+1)$ is even.

Proof. Suppose that n is an integer. Then we have two cases, where either n is even or n is odd. Let n be an even integer such that $n = 2k$ where $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= 2k(2k+1) \\ &= 2(2k^2+k). \end{aligned}$$

Thus we see that $n(n+1)$ is even when n is even. Now let n be odd such that $n = 2m+1$ where $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1). \end{aligned}$$

Thus $n(n+1)$ is also even when n is odd, and so is even for all integers n. $\square$

2 Lecture 2

2.1 Continuation of Logic

2.1.1 De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

Note. Negations turn "and" into "or" and vice versa.

Example. Suppose we have the following statement:

$$P\colon\ x \text{ is even and } x > 0.$$

Then the negation of P would be:

$$\neg P\colon\ x \text{ is odd or } x \leq 0.$$

2.1.2 Converse

Definition. *Converse*
The *converse* of a statement $P \implies Q$ is the statement $Q \implies P$. In general, the converse of a statement says nothing about the original statement.

Example. Consider the statement

$$\text{If } x > 0\text{, then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0\text{, then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write $P \iff Q$ instead of $(P \implies Q) \wedge (Q \implies P)$. In this case, we call P and Q *logically equivalent.* In writing, we say "P if and only if Q".

2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

Lemma 1. Let a be an integer. If a^2 is even, then a is even.

Proof. Suppose a is odd, so $a = 2k + 1$ for some integer k. Then

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus a^2 is odd and this completes the proof. $\square$

2.1.4 Variables and Quantifiers

We have a value x that varies over some values, so we use $P(x)$ to denote a statement that depends on the value of x.

Example. Consider the statement
$$P(x) : x + 2 = 3.$$
The statement is true if and only if $x = 1$.

We have two quantifiers—$\forall$ = "for all", and $\exists$ = "there exists".

- $\forall x : P(x)$ is true if $P(x)$ is true for all x.
- $\exists x : P(x)$ is true if there exists at least one x such that $P(x)$ is true.

We can also have nested quantifiers,
$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

Note. The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of n depends on m.

2.1.5 Proof by Counterexample

After "simplifying" the statement $\neg(\forall x : P(x))$, we get $\exists x : \neg P(x)$. We simply need to find a single counterexample to show that a statement is false for all x.

Example. Consider the statement $\forall x \in \mathbb{R} : x + 2 = 3$. All we need to do is show that there exists some $x \in \mathbb{R}$ such that $x + 2 \neq 3$. This occurs when $x = 0$, so the statement is false.

2.1.6 Proof by Contradiction

Key Idea. We want to show that $P \implies Q$ indirectly.

Lemma 2. We can show that
$$P \implies Q = (\neg P) \vee Q.$$
Then $P \implies Q$ is true if and only if $\neg(P \implies Q)$ is false, and so by Lemma 2 and De Morgan's Laws, $P \wedge \neg Q$ is false.

For proof by contradiction, we assume P is true and $\neg Q$ is true, and try to show that $P \wedge \neg Q$ is false (a contradiction).

3 Lecture 3

3.1 More Logic

3.1.1 Proof by Contradiction

To do a proof by contradiction for a statement $P \implies Q$, we assume $P \wedge \neg Q$. We aim to show that $P \wedge \neg Q$ is false (a contradiction).

Theorem — *Irrationality of $\sqrt{2}$*
There is no rational number x such that $x^2 = 2$. In other words, if $x \in \mathbb{Q}$, then $x^2 \neq 2$.

Proof. Suppose towards a contradiction that there exists some $x \in \mathbb{Q}$ such that $x^2 = 2$. Since x is rational, there exist integers p, q such that $q \neq 0$, $\frac{p}{q} = x$, and p and q have no common divisors (other than 1). Then

$$\begin{aligned} x^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since p^2 is even, there exists some integer k such that $p = 2k$. Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

By the same logic as before, we know that q must also be even (they share a common factor of 2). However, this contradicts our original assumption that p and q share no common factors, and this completes the proof. □

3.2 Set Theory

We write $x \in A$ when we want to say that "x is an element of A", and $x \notin A$ when we want to say that "x is not an element of A".

3.2.1 Set Combinations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$.
- Subset (Inclusion): $A \subseteq B$ if and only if $x \in A \implies x \in B$.

> **Definition.** *Proper Subset*
> A set A is a *proper subset* of a set B if $A \subseteq B$ and there exists some $x \in B$ such that $x \notin A$. We denote this as $A \subset B$.

- Equality: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

> **Note (Showing Equality of Sets).** If you want to show $A = B$, you need to show both $A \subseteq B$ and $B \subseteq A$. In other words, you must show that for all $x \in A$, we have $x \in B$, and vice versa.

Example. We have $\mathbb{N} = \{1, 2, 3, \dots\}$.

- Let E be the set of even natural numbers. Note that $E \subseteq \mathbb{N}$.
- Let $S = \left\{p \in \mathbb{Q} \mid p^2 < 2\right\} \subseteq \mathbb{Q}$.

Then we have:

- $\mathbb{N} \cup E = \mathbb{N}$.
- $\mathbb{N} \cap E = E$.
- $\mathbb{N} \cap S = \left\{n \in \mathbb{N} \mid n^2 < 2\right\} = \{1\}$.
- $E \cap S = \varnothing$.

  Definition. *Disjoint Sets*
  If $A \cap B = \varnothing$, we call A and B *disjoint* sets.

  Proof. Suppose towards a contradiction that there exists some $x \in E \cap S$, which is to say $x \in E$ and $x \in S$. Since $x \in E$, we know that x is even, and so there exists some integer k such that $x = 2k$. Then
  $$x^2 = (2k)^2 = 4k^2,$$
  so $4 \mid x^2$. Therefore $x \geq 4$, which contradicts the condition for $x \in S$, namely $x^2 < 2$. $\square$

- Given some $n \in \mathbb{N}$, we define $A_n = \mathbb{N} \setminus \{1, \dots, n-1\}$.
  - $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}$.
  - $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$.

Definition. *Set Complement*
If $A \subseteq B$, then we define the *complement* of A in B to be $A^c = B \setminus A$.

3.2.2 De Morgan's Laws

If I is an index set and $\{A_j\}_{j\in I}$ are subsets of B, then
$$\left(\bigcup_{j\in I} A_j\right)^c = \bigcap_{j\in I} A_j^c, \quad \text{and} \quad \left(\bigcap_{j\in I} A_j\right)^c = \bigcup_{j\in I} A_j^c$$

4 Lecture 4

4.1 Cartesian Product

If I have two sets A and B, then we may form their *Cartesian Product*, which is

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}.$$

Definition. *Binary Relation*
A *binary relation* is a subset $R \subseteq A \times B$. We say $x \in A$ is in relation to $y \in B$ if $(x, y) \in R$. We denote this by

$$xRy \iff (x, y) \in R.$$

Example. Consider the relation

$$\mathbb{R} \times \mathbb{R} \supseteq R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Then the relation is *reflexive*, because xRx. It is also antisymmetric, because $xRy \wedge yRx \implies x = y$. Finally, this relation is transitive, because $xRy \wedge yRz \implies xRz$.

These properties only make sense if $A = B$, i.e. $R \subseteq A \times A$, and we say that "R is a relation on A".

Definition. *Partial Order*
If a relation is reflexive, antisymmetric, and transitive on A, then it is a *partial order* on A.

The notion of "less than or equal to" is a partial order for $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, but there exists no partial order for $\mathbb{C}$.

Definition. *Power Set*
For a set A, we may define its *power set* by

$$\mathscr{P}(A) = \{C \mid C \subseteq A\}.$$

Note that set inclusion is a partial order on $\mathscr{P}(A)$.

Definition. *Equivalence Relation*
An *equivalence relation* R over A is a relation that is reflexive, symmetric, and transitive.

Note. Just like a partial order behaves much like $\leq$, an equivalence relation behaves much like $=$.

Definition. *Equivalence Class*
Given an equivalence relation on $A.0$, we define a new set

$$[x] \coloneqq \{y \in A \mid x \sim y\}.$$

We call $[x]$ the *equivalence class* of x. Any $z \in [x]$ is called a *representative* of the equivalence class $[x]$. In particular, x is a representative of its own equivalence class.

Let A be a set with equivalence relation $\sim$. Then for any $x, y \in A$,

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \varnothing.$$

Proof. Let $x, y \in A$. We know that x is either equivalent to y or it is not. Suppose the former is true and let $z \in [x]$. Thus we know that $z \sim x$ and $x \sim y$, and by transitivity we have $z \sim y$. Thus $z \in [y]$ and $[x] \subseteq [y]$. The reverse argument is the same.

If x is not equivalent to y, then suppose towards a contradiction that $[x] \cap [y] \neq \varnothing$. Let $x \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$. By symmetry we know that $x \sim z$ and by transitivity we have $x \sim y$. We have arrived at the contradiction that x is both equivalent and not equivalent to y. $\square$

Definition. *Function*
A relation $R \subseteq A \times B$ is a *function* if for all $x \in A$ and all $y, z \in B$, we have the following:

- $xRy \wedge xRz \implies y = z$.

In other words, every input x has only one output.

Definition. *Injective Functions*
A function f is *injective* if $f(x_1) = f(x_2) \implies x_1 = x_2$.

Definition. *Surjective Functions*
A function f is *surjective* if for every $y \in B$, there exists some $x \in A$ such that $f(x) = y$.

5 Lecture 5

5.1 The Natural Numbers

We know that the natural numbers are defined via:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

5.1.1 Properties of the Natural Numbers

(P1) $1 \in \mathbb{N}$

(P2) If $n \in \mathbb{N}$, then it has a *successor*, $n + 1 \in \mathbb{N}$

(P3) 1 is *not* the successor of any element of $\mathbb{N}$

(P4) If m, n have the same successor, then $m = n$

Note. The properties above can be abstracted to become:

(P1) $1 \in \mathbb{N}$

(P2) There exists some $S\colon \mathbb{N} \to \mathbb{N}$ where $S(n)$ is the successor n

(P3) $1 \notin \operatorname{Ran} S$

(P4) S is injective

(P5) Suppose $A \subseteq \mathbb{N}$ with the properties:

  (i) $1 \in A$

  (ii) If $n \in A$, then $S(n) \in A$

  Then $A = \mathbb{N}$.

Theorem — *1.1 (Induction)*
Let $\{P(n) \mid n \in \mathbb{N}\}$ be a set of logical propositions. Suppose that

(i) $P(1)$ is true.

(ii) If $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $A = \{n \mid P(n) \text{ is true}\} \subseteq \mathbb{N}$. By our first assumption, $1 \in A$. By (ii), if $n \in A$, then $n + 1 \in A$. So by (P5) of the natural numbers, we know that $A = \mathbb{N}$. $\square$

Definition. *1.2 (Peano Axioms)*
A triplet $(\mathbb{N}, 1, S)$ is said to be a *system of the naturals* if it satisfies:

1) $\mathbb{N}$ is a set and $1 \in \mathbb{N}$

2) $S\colon \mathbb{N} \to \mathbb{N}$ is a function

3) $1 \notin \operatorname{Ran} S$

4) S is injective

5) $\forall A \subseteq N$ such that $1 \in A$ and $S(A) \subseteq A$, then $A = \mathbb{N}$

Definition. *Addition*
We define the binary relation $+$ over $\mathbb{N}$:

(i) $\forall n \in \mathbb{N}$, $n + 1 := S(n)$

(ii) $\forall m, n \in \mathbb{N}$, we have $m + S(n) = S(m + n)$

The following properties can be proven from the above definition of addition:

(a) Associativity: $\forall x, y, z \in \mathbb{N}$, we have $(x + y) + z = x + (y + z)$

(b) Commutativity: $\forall x, y \in \mathbb{N}$, we have $x + y = y + x$

(c) Cancellative Law: $\forall x, y, z \in \mathbb{N}$, we have $x + y = y + z \implies x = z$

Theorem — *1.3 (Existence of the Naturals)*
There exists a unique system of the naturals. In other words, there is a bijection between different systems of the naturals and so we may write:

$$(\mathbb{N}, 1, S) \iff (\mathbb{N}', 1', S').$$

5.2 Fields

Definition. *Field*
A *field* is a set with two binary operations,

- $+$, or 'addition'
- $\cdot$, or 'multiplication'

5.2.1 Axioms for Addition

(A1) $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x + y \in \mathbb{F}$

(A2) $\forall x, y \in \mathbb{F}$, we have $x + y = y + x$

(A3) $\forall x, y, z \in \mathbb{F}$, we have $(x + y) + z = x + (y + z)$

(A4) There exists some $0 \in \mathbb{F}$ such that $0 + x = x$ for all $x \in \mathbb{F}$

(A5) $\forall x \in \mathbb{F}$, there exists $-x \in \mathbb{F}$ such that $x + (-x) = 0$

5.2.2 Axioms for Multiplication

(M1) $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x \cdot y \in \mathbb{F}$

(M2) $\forall x, y \in \mathbb{F}$, we have $x \cdot y = y \cdot x$

(M3) $\forall x, y, z \in \mathbb{F}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M4) There exists some $1 \in \mathbb{F}$ such that $1 \neq 0$ and $1 \cdot x = x$ for all $x \in \mathbb{F}$

(M5) $\forall x \in \mathbb{F}$, there exists some $\frac{1}{x} \in \mathbb{F}$ such that $x \cdot \frac{1}{x} = 1$

5.2.3 Distributive Law

(D1) $\forall x, y, z \in \mathbb{F}$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$

6 Lecture 6

6.1 The Rationals

On the natural numbers, we have a notion of addition, multiplication, and comparison ($\leq$). We constructed the integers and so we now have:

- A notion of an additive identity, $0 \in \mathbb{Z}$
- Additive inverses

However, we *don't* have:

- Multiplicative inverses

When we think of the rationals, we consider the set

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Note. When dealing with $\mathbb{Q}$, we now have multiplicative inverses.

In particular, $(\mathbb{Q}, +, \cdot, \leq)$ is an *ordered field*.

Definition. *Ordered Field*
An *ordered field* is a field $\mathbb{F}$ which is also an ordered set ($\leq$) such that:

(i) If $x, y, z \in \mathbb{F}$ and $y < z$, then $x + y < x + z$

(ii) If $x, y \in \mathbb{F}$, $x > 0$ and $y > 0$, then $x \cdot y > 0$

Unfortunately, the rational numbers still don't allow us to solve polynomial equations (i.e. $x^2 = 2$).

Definition. *Algebraic Numbers*
A number is called *algebraic* if it solves

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$, $n \in \mathbb{N}$.

Example. Every rational number is algebraic, because it solves the equation

$$qx - p = 0.$$

Definition. *Dividing*
We say $k \in \mathbb{Z}$ *divides* $m \in \mathbb{Z}$ if $\frac{m}{k} \in \mathbb{Z}$.

Theorem — *Rational Zeros Theorem*
Suppose $c_0, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$ and $r \in \mathbb{Q}$ satisfies

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then writing $r = \frac{c}{d}$ with c, d having no common factors, $d \neq 0$, we have:

$$\begin{gathered} c \text{ divides } c_0 \\ d \text{ divides } c_n \end{gathered}$$

Proof. Since r solves the equation, we have

$$c_n\left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0.$$

Multiplying both sides by d^n, we get

$$c_nc^n + c_{n-1}c^{n-1}d + \cdots + c_1cd^{n-1} + c_0d^n = 0.$$

Rearranging, we have

$$c_nc^n = -d(c_{n-1}c^{n-1} - \cdots + c_1cd^{n-2} - c_0d^{n-1}).$$

We know that $d \mid c_nc^n$. Since c and d have no common factors, we know $d \mid c_n$. Rearranging terms again,

$$-c(c_nc^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_1d^{n-1}) = c_0d^n.$$

By the same reasoning as before, we have that $c \mid c_0$. $\square$

Corollary. Suppose $r \in \mathbb{Q}$ solves

$$r^n + c_{n-1}r^{n-1} + \cdots + c_1r + c_0 = 0.$$

Then $r \in \mathbb{Z}$ and $r \mid c_0$.

Proof. Since $r \in \mathbb{Q}$, we may express $r = \frac{c}{d}$, $c \mid c_0$ and $d \mid 1$. From this we know that $d = 1$, so $r = c$ and $c \mid c_0$. Therefore $r \in \mathbb{Z}$ and $r \mid c_0$. $\square$

We have some deficiencies for $\mathbb{Q}$:

- There seem to be some "gaps" in $\mathbb{Q}$.

Proposition. We consider the sets $A = \left\{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 < 2\right\}$ and $B = \left\{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 > 2\right\}$. Notice that A has no largest element and B has no smallest element.

Proof. Given $p \in \mathbb{Q}$, let

$$q = p - \frac{p^2 - 2}{p^2 + 2} = \frac{2(p+1)}{p+2}.$$

We also have $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. If $p \in A$, $p^2 - 2 < 0$ so $q > p$ and $q^2 - 2 > 0$. If $p \in B$, $p^2 - 2 > 0$ so $q < p$ and $q^2 - 2 > 0$. $\square$

7 Lecture 7

Definition. *2.11—Upper and Lower Bounds*
Let E be an ordered set and $A \subset E$.

(a) If there exists $x \in E$ such that for all $a \in A$, $a \leq x$, we say A is *bounded above* by x and call x an *upper bound* for A.

(b) Suppose $A \subset E$ is non-empty and bounded above and there exists some $x^* \in E$ such that

  i) x^* is an upper bound for A

  ii) If y is any upper bound for A^*, then $x^* \leq y$

Then we call x^* the *least upper bound for* A, and we write

$$x^* = \sup A. \qquad \text{(sup meaning supremum)}$$

The *greatest lower bound* or *infimum* of a set B, which is bounded below and non-empty, satisfies

i) $\inf B$ is a lower bound for B

ii) If y is any lower bound for B, then $y \leq \inf B$

Example. Suppose

$$A = \left\{p \in \mathbb{Q} \mid p \geq 0, p^2 < 2\right\},$$
$$B = \left\{p \in \mathbb{Q} \mid p \geq 0, p^2 > 2\right\}.$$

Then A is bounded above (say by 2) and B is bounded below (say by 0). In the last lecture we proved that neither $\sup A$ nor $\inf B$ exist in $\mathbb{Q}$ (because the values would have been $\sqrt{2}$).

Example. Let

$$C = \{p \in \mathbb{Q} \mid p < 0\},$$
$$D = \{p \in \mathbb{Q} \mid p \leq 0\}.$$

Then

$$\sup C = \sup D = 0.$$

However, notice that $\sup C \notin C$ and $\sup D \in D$.

Definition. *Maximum*
We define $\max A$ to be the largest element of A, which satisfies:

i) $\max A \in A$

ii) For all $a \in A$, $a \leq \max A$

The definition for minimum is similar.

Definition. *2.12—Least Upper Bound Property (LUBP)*
An *ordered set* E has the *least upper bound property* if the following is true:

i) If $A \subseteq E$, $A \neq \varnothing$, A is bounded above, then $\sup A$ exists and $\sup A \in E$.

Note. $\mathbb{Q}$ does not have the least upper bound property.

Theorem — *2.13 (Existence of $\mathbb{R}$)*
There exists an ordered field $\mathbb{R}$ which has

i) $\mathbb{Q}$ as a sub-field

ii) The least upper bound property

7.1 Fundamental Properties of the Real Numbers (because of LUBP)

Theorem — *2.14 (Archimedean Property of $\mathbb{R}$)*
If $x, y \in \mathbb{R}$, and $x > 0$, then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Proof. Let $A = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Suppose towards a contradiction that there is no such n that satisfies the statement above. In other words, for all $n \in \mathbb{N}$, $nx \leq y$. Thus A is bounded above by y. Since A is nonempty and is a subset of $\mathbb{R}$, we know that $\sup A$ exists. Consider the value given by $\sup A - x$, which is not an upper bound for A. Then we know there exists some $z \in A$ such that $\sup A - x < z$. Since $z = mx$ (because $z \in A$), we have

$$\begin{aligned}\sup A - x &< z\\ \sup A - x &< mx\\ \sup A &< (m+1)x.\end{aligned}$$

We know that $(m+1)x \in A$, which contradicts the definition of $\sup A$. $\square$

Some remarks:

1. Let $x = 1$. Then $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > y$.
2. Let $y = 1$. Then $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x > \frac{1}{n} > 0$.

Theorem — *2.15 (Density of $\mathbb{Q}$ in $\mathbb{R}$)*
For all $x, y \in \mathbb{R}$, $x < y$, there exists some $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Fix $x < y$. Then by the Archimedean property we have some $n \in \mathbb{N}$ such that $n(y - x) > 1$, or $y - x > \frac{1}{n}$. We may suppose $x > 0$, because otherwise we either have $x < 0 < y$ or $x < y < 0$ (multiply all sides by -1).
We want to show that

$$nx < m < ny.$$

Since $nx + 1 < ny$, we have $nx < m < nx + 1$, or $m - 1 < nx < m$. If $nx \in \mathbb{Z}$, we can take $m = nx + 1$. Thus

$$x < x + \frac{1}{n} = \frac{nx+1}{n} = \frac{m}{n} < \frac{ny}{n} < y.$$

Otherwise we have $nx \notin \mathbb{Z}$. We then apply the following lemma:

Lemma. If $x \in \mathbb{R}$, there exists a $k \in \mathbb{Z}$ such that $k - 1 \leq x \leq k$.

Then $m - 1 < nx < m$, as desired. $\square$

8 Lecture 8

8.1 A Construction of the Real Numbers

We want to show that there exists some field $\mathbb{R}$ such that $\mathbb{Q}$ is a sub-field and $\mathbb{R}$ has the least upper bound property. This construction will be via Dedekind cuts.

Definition. *2.17—Cut*
A *cut* in $\mathbb{Q}$ is a pair of subsets $A, B \subset \mathbb{Q}$ such that:

(i) $A \cap B = \varnothing$, $A \cup B = \mathbb{Q}$, $A, B \neq \varnothing$ (They partition $\mathbb{Q}$)

(ii) If $a \in A$ and $b \in B$, then $a < b$

(iii) A contains no largest element

Example. An example of such a cut could be $A = \{p \in \mathbb{Q} \mid p < 1\}$ and $B = \{p \in \mathbb{Q} \mid p \geq 1\}$. We say that $A \mid B$ is a cut. Another such example is $A = \left\{p \in \mathbb{Q} \mid p \leq 0 \text{ or } p^2 < 2\right\}$ and $B = \left\{p \in \mathbb{Q} \mid p > 0 \text{ and } p^2 \geq 2\right\}$

Definition. *2.18—The Reals*
We may define
$$\mathbb{R} = \{X \mid X \text{ is a cut in } \mathbb{Q}\}.$$

In order to show that the above is a valid definition for $\mathbb{R}$, we must show:

(i) $\mathbb{Q}$ is contained in $\mathbb{R}$ in some "natural way"

(ii) $\mathbb{R}$ is an ordered field

(iii) $\mathbb{R}$ has the least upper bound property

Definition. *2.19—Partial Order for the Reals*
We define a *partial order* on $\mathbb{R}$ as follows: If $X = A \mid B$ and $Y = C \mid D$, then we say $X < Y$ if and only if $A \subset C$ and $X \leq Y$ if and only if $A \subseteq C$.

We will show that the reals contain the rationals.

Proof. We will begin by showing that $\mathbb{Q} \subseteq \mathbb{R}$. We say that $A \mid B$ is a *rational cut* if for some $c \in \mathbb{Q}$, we have $A = \{p \in \mathbb{Q} \mid p < c\}$ and $B = \mathbb{Q} \setminus A$. We will use c^* to denote the rational cut at c. Then we may associate every $c \in \mathbb{Q}$ with a corresponding rational cut $c^* \in \mathbb{R}$. □

Theorem — *2.20 (The Reals have the LUBP)*
With respect to the partial order $\leq$ defined earlier, we may show that $\mathbb{R}$ has the least upper bound property.

Proof. Let $\mathscr{C}$ be any non-empty collection of cuts which are bounded above, say by the cut X. We want to show that $\sup \mathscr{C}$ exists in $\mathbb{R}$, so $\sup \mathscr{C}$ is itself a cut. A candidate for $\sup \mathscr{C}$ is $C \mid D$, where
$$C = \{a \in \mathbb{Q} \mid \text{There exists a cut } A \mid B \in \mathscr{C} \text{ such that } a \in A\}.$$
Now, $Z = C \mid D$ is a cut in $\mathbb{Q}$. We claim that C has no largest element. If $a \in C$, then there exists a cut $A \mid B \in \mathscr{C}$ such that $a \in A$. Since A is a part of a cut, it has no largest element, so there exists some $a' \in A$ such that $a < a'$. Thus $a' \in C$ so C has no largest element.
We must now show that Z is the least upper bound of $\mathscr{C}$. For any $A \mid B \in \mathscr{C}, A \subseteq C$. Therefore $A \mid B \leq C \mid D = Z$ and Z is an upper bound for $\mathscr{C}$. We must now show that it is the *least* upper

bound. Let $Z' = C' \mid D'$ be an upper bound for $\mathscr{C}$. Then $A \mid B \leq C' \mid D'$ and $A \subseteq C'$ for any $A \mid B \in \mathscr{C}$. Now by definition of C, we have $C \subseteq C'$. Therefore $C \mid D \leq C' \mid D'$, so $Z \leq Z'$. We have $Z = \sup \mathscr{C} \in \mathbb{R}$, so $\mathbb{R}$ has the least upper bound property. $\square$

We now show that $\mathbb{R}$ is a field, namely an ordered field. We define the binary relations $+$ and $\cdot$ as follows: Given two cuts $A \mid B$ and $C \mid D$, we may define

$$E = A + C = \{p \in \mathbb{Q} \mid p = a + c \text{ for some } a \in A, c \in C\}.$$

Note. This set summation is known as the Minkowski sum.

We must check that $E \mid F$ is a cut. We must also show that the additive identity for $\mathbb{R}$ is 0, i.e. we must show that $0 + x = x + 0 = x$ for all $x \in \mathbb{R}$. To show the existence of additive inverses, show that for any cut $A \mid B$, there exists some $C \mid D \in \mathbb{R}$ such that $A \mid B + C \mid D = 0$.

Similarly, we define multiplication by

$$E = \{p \in \mathbb{Q} \mid p = ac \text{ for some } a \in A, c \in C\}.$$

8.2 Interesting Questions

(a) Can we cut $\mathbb{R}$ to get something larger? No, because every possible cut in $\mathbb{R}$ is an element of $\mathbb{R}$. This is because $\mathbb{R}$ has the least upper bound property and every cut in $\mathbb{R}$ would just be a "real cut" at $\sup A$.

(b) Is $\mathbb{R}$ unique in some natural way? Yes. If you take any other ordered field $\mathbb{F}$ such that $\mathbb{Q} \subseteq \mathbb{F}$ and $\mathbb{F}$ has the least upper bound property, then there exists some bijection between $\mathbb{F}$ and $\mathbb{R}$.

(c) What about $+\infty$ and $-\infty$? We can't treat these as real numbers using Dedekind cuts, as either A or B would have to be empty.

9 Lecture 9

9.1 Sequences

Example. *Approximating π*
We know that the area of the unit circle should be π. We can approximate the area of a unit circle by inscribing various shapes in the circle and finding their areas (giving us a sequence of lower bounds).
If we inscribe an equilateral triangle in the circle, we find that its side length is $\sqrt{3}$, so the area of the triangle is $a_1 = \frac{3\sqrt{3}}{4} \approx 1.299$.
If we use a square instead, its side length is $\sqrt{2}$, so we have a new lower bound of $a_2 = 2 < \pi$.
Using a regular pentagon, we have a new approximation as given by $a_3 = \frac{5}{2}\sin\left(\frac{2\pi}{5}\right) < \pi$.
Continuing this pattern, we get

$$a_n = \frac{n+2}{2}\sin\left(\frac{2\pi}{n+2}\right) < \pi.$$

Notice that for all $n \in \mathbb{N}$, we have $a_n < a_{n+1} < \pi$. Computing this value for larger n, we have

$$\begin{aligned} a_{100} &\approx 3.1396 \\ a_{1000} &\approx 3.141572 \\ a_{10000} &\approx 3.14159245. \end{aligned}$$

Similarly, we may obtain upper bounds for π by circumscribing regular polygons around the unit circle. By circumscribing a square, we know that $\pi < 4$, so we know that there is some upper bound for what π is equal to. Thus we know that there exists some $a \in \mathbb{R}$ such that $a_n \approx a$ for some sufficiently large n.
In other words, there exists some $a \in \mathbb{R}$ such that a_n converges to a as n approaches ∞.
One issue to note is that we are using π in our formula for a_n to approximate the value of π. Using some trigonometric identities, we may circumvent this by evaluating

$$a_{2n} = \frac{n}{2}\left(n - \sqrt{n - 4a_n^2}\right).$$

Definition. *3.1—Sequences*
A *sequence* is a function $f\colon \mathbb{N} \to \mathbb{R}$. Instead of writing $f(1), f(2), \dots, f(n)$, we tend to write a sequence as $f_1, f_2, \dots, f_n$.

Note (Notations for Sequences). There are many different notations for expressing sequences, a few popular notations being used below:

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}.$$

9.1.1 Behaviors of Sequences

(a) "Convergence"—Getting closer and closer to a given point.

(b) "Divergence"—Getting closer and closer to $\pm\infty$.

(c) "Oscillation"—The sequence does not approach any value in particular.

9.1.2 Facts From the Homework

We define the absolute value function

$$|\cdot| : \mathbb{R} \to [0, \infty) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We have the following properties of the absolute value function:

(i) $|xy| = |x|\,|y|$ for all $x, y \in \mathbb{R}$.

(ii) $|x - y| \leq \ell$ if and only if $y - \ell \leq x \leq y + \ell$, where $\ell \geq 0$ and $x, y \in \mathbb{R}$.

(iii) **Triangle Inequality:** $|x + y| \leq |x| + |y|$.

> **Note.** One consequence of this is
>
> $$\begin{aligned} |x - y| &= |x - z + z - y| \\ &\leq |x - z| + |z - y|\,. \end{aligned}$$
>
> In other words, the distance between two points x and y is always less than or equal to the sum of the distances of x and y to a third point, z.

We say that a_n approaches a if "$|a_n - a|$ gets arbitrarily small as n gets arbitrarily large".

9.1.3 Convergence

Definition. *3.2—Convergence*
A sequence (x_n) of real numbers is said to *converge* to an $x \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|x_n - x| < \varepsilon$$

for all $n > N$. If (x_n) converges to x, we also write:

$$x_n \to x \text{ as } n \to \infty \quad \text{or} \quad \lim_{n \to \infty} x_n = x.$$

We call x the *limit* of the sequence (x_n). We say that a sequence *diverges* if it does not converge.

Note. By the Archimedean property, we can always take $N \in \mathbb{N}$. In general, N is a function of ε. We fix $\varepsilon > 0$, and use that to find some sufficient N so that the sequence converges.

Example. Consider the sequence $x_n = \frac{1}{n^2}$. We suspect that $x_n \to 0 \in \mathbb{R}$ as $n \to \infty$. Fix $\varepsilon > 0$. If we let $N = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$ and $n > N$, we have

$$\begin{aligned} |x_n - 0| &= \left| \frac{1}{n^2} \right| \\ &= \frac{1}{n^2} \\ &< \frac{1}{N^2} \\ &< \varepsilon. \end{aligned}$$

Thus, $x_n \to 0$ as $n \to \infty$, so

$$\lim_{n \to \infty} x_n = 0.$$

10 Lecture 10

10.1 Convergence of Sequences Using the Definition of Convergence

From last lecture, we have a basic definition of convergence. However, this is not very helpful for determining whether a sequence converges or not because we have to actually know what the sequence converges to.

Example. Consider the sequence $x_n = \frac{4n^3+n}{n^3-6}$. We have an intuition that this converges to 4 by dividing the leading coefficients of the numerator and denominator.

We have

$$\begin{aligned}|x_n - 4| &= \left|\frac{4n^3+n}{n^3-6} - 4\right| \\ &= \left|\frac{(4n^3+n) - 4(n^3-6)}{n^3-6}\right| \\ &= \left|\frac{n+24}{n^3-6}\right| \\ &= \frac{|n+24|}{|n^3-6|} \\ &= \frac{n+24}{n^3-6}\end{aligned}$$

We want to find some N such that $\frac{n+24}{|n^3-6|} < \varepsilon$ for all $n > N$. Thus we want to show

$$\frac{n+24}{|n^3-6|} \leq \frac{C}{n^2} \leq \varepsilon.$$

If $n \geq 24$, we have $n + 24 \leq 2n$. Suppose we wish for $\left|n^3 - 6\right| > 0$, so $n \geq 2$. Furthermore, note that $n^3 - 6 \geq \frac{1}{2}n^3$ when $n \geq 12^{\frac{1}{3}}$. Thus for $n \geq 24$, we have

$$\begin{aligned}\frac{n+24}{|n^3-6|} &\leq \frac{2n}{\frac{1}{2}n^3} \\ &= \frac{4}{n^2}.\end{aligned}$$

Thus we take

$$N = \max\left(24, \left\lceil\sqrt{\frac{4}{\varepsilon}}\right\rceil\right) = \max\left(24, \left\lceil\frac{2}{\sqrt{\varepsilon}}\right\rceil\right).$$

Thus for $n > N$, we have

$$\frac{n+24}{|n^3-6|} < \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and choose $N = \max\left(24, \left\lceil\frac{2}{\sqrt{\varepsilon}}\right\rceil\right)$. Then

$$\begin{aligned}|x_n - 4| &= \frac{n+24}{|n^3-6|} \\ &\leq \frac{4}{n^2} \\ &< \varepsilon\end{aligned}$$

for all $n > N$. Thus $\lim_{n\to\infty} x_n = 4$. $\square$

Example. Show that $x_n = (-1)^n$ diverges.

Note that $x_{2k} = 1$, and $x_{2k+1} = -1$ for some integer k.

Proof. Suppose towards a contradiction that x_n converges to a point x. Since we know that the sequence only takes on the values 1 and -1, we know that

$$\begin{aligned} 2 &= \lvert (1-x) + (x+1) \rvert \\ &\leq \lvert 1-x \rvert + \lvert x+1 \rvert \\ 2 &\leq \lvert x_{2n} - x \rvert + \lvert x - x_{2n+1} \rvert . \end{aligned}$$

Since $x_n \to x$, given $\varepsilon > 0$, there exists N such that $\lvert x_n - 1 \rvert < \varepsilon$ for all $n > N$. For $n > N$, we have $\lvert x_{2n} - x \rvert < \varepsilon$ and $\lvert x - x_{2n+1} \rvert < \varepsilon$. Thus we have $2 \leq 2\varepsilon$, so $1 \leq \varepsilon$, which contradicts that ε may be arbitrarily small. Hence the sequence diverges. □

10.2 Limit Laws

Proposition 3.4 (Limits are Unique)—If $x_n \to x$ and $x_n \to y$, then $x = y$.
Idea: The points in the sequence have to be arbitrarily close to x and y simultaneously.
We see that

$$\begin{aligned} \lvert x - y \rvert &\leq \lvert x - x_n \rvert + \lvert x_n - y \rvert \\ &= \varepsilon, \end{aligned}$$

for any $\varepsilon > 0$. By a previous theorem, we have $x = y$.

Proof. Fix $\varepsilon > 0$. Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that $\lvert x_n - x \rvert < \frac{\varepsilon}{2}$ for all $n > N_1$. Since $x_n \to y$, there exists $N_2 \in \mathbb{N}$ such that $\lvert x_n - y \rvert < \frac{\varepsilon}{2}$ for all $n > N_2$. Then for $n > \max(N_1, N_2)$, the triangle inequality implies that

$$\begin{aligned} \lvert x - y \rvert &\leq \lvert x_n - x \rvert + \lvert y - x_n \rvert \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since we have $\lvert x - y \rvert < \varepsilon$ for any $\varepsilon > 0$, we have $x = y$. □

Definition. *3.5—Boundedness for Sequences*
A sequence (x_n) is *bounded* if there exists some real number $M > 0$ such that $\lvert x_n \rvert \leq M$ for all $n \in \mathbb{N}$.

Note. In the above, our choice of M does not depend on n. Additionally, all convergent sequences are bounded (since we may just choose an arbitrary ε and we know that x_n will at some point rest between $x - \varepsilon$ and $x + \varepsilon$).

Theorem — *3.6 (Convergent Sequences are Bounded)*
Suppose $x_n \to x$. Then for $\varepsilon = 1$, we may find a $N \in \mathbb{N}$ such that $\lvert x_n - x \rvert < 1$ for all $n > N$. Then when $n > N$, we have

$$\begin{aligned} \lvert x_n \rvert &\leq \lvert x_n - x \rvert + \lvert x \rvert \\ &\leq 1 + \lvert x \rvert . \end{aligned}$$

Hence, for $M = \max(\lvert x_1 \rvert, \lvert x_2 \rvert, \ldots, \lvert x_N \rvert, 1 + \lvert x \rvert)$, we see that $\lvert x_n \rvert \leq M$ for all $n \in \mathbb{N}$.

Theorem — *3.7 (Algebraic Limit Theorem)*
Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

(i) $\lim_{n\to\infty}(ax_n + by_n) = ax + by$ for all $a, b \in \mathbb{R}$.

(ii) $\lim_{n\to\infty}(x_n \cdot y_n) = xy$.

(iii) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$, provided $y \neq 0$.

11 Lecture 11

11.1 Proof for Algebraic Limit Theorem

We want to show that

$$|x_n y_n - xy| < \varepsilon$$

for all $\varepsilon > 0$.

Proof. Observe that

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) + (x_n - x)y| \\
&\leq |x_n(y_n - y)| + |y(x_n - x)| \\
&= |x_n|\,|y_n - y| + |y|\,|x_n - x|
\end{aligned}$$

Since convergent sequences are bounded, we know that there exist some $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

$$\leq M\,|y_n - y| + |y|\,|x_n - x|$$

Fix $\varepsilon > 0$. Then there exists N_1 such that $|x_n - x| < \frac{\varepsilon}{2(1+|y|)}$ for all $n > N_1$. Furthermore, we know there exists some N_2 such that $|y_n - y| < \frac{\varepsilon}{2(1+M)}$ for all $n > N_2$.

$$\begin{aligned}
&< \frac{M\varepsilon}{2(1+M)} + \frac{|y|\,\varepsilon}{2(1+|y|)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon,
\end{aligned}$$

for all $\varepsilon > 0$. Therefore the limit exists and

$$\lim_{n\to\infty} x_n y_n = xy.$$

□

We now want to show that

$$\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}.$$

Proof. Observe that

$$\begin{aligned}
\left|\frac{x_n}{y_n} - \frac{x}{y}\right| &= \left|\frac{x_n y - x y_n}{y_n y}\right| \\
&= \frac{1}{|y_n|\,|y|}\,|x_n y - x y_n| \\
&= \frac{1}{|y_n|\,|y|}\,|x_n y - xy + xy - x y_n| \\
&\leq \frac{1}{|y_n|\,|y|}\,|y(x_n - x) + x(y - y_n)| \\
&= \frac{1}{|y_n|\,|y|}\,(|y|\,|x_n - x| + |x|\,|y - y_n|)
\end{aligned}$$

We now need to show that $\frac{1}{|y_n|}$ is upper bounded by some value. Since (y_n) converges, we know that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$. Suppose we choose $\varepsilon = \frac{y}{2}$, so we have $\frac{1}{y_n} \leq \frac{2}{y}$. In a fashion similar to the previous part, we may manipulate the above expression to complete the proof.

□

Theorem — *3.8 (Order Limit Theorem)*
Assume $x_n \to x$ and $y_n \to y$.

(i) If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x \geq 0$.

(ii) If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

(iii) If there exists $a \in \mathbb{R}$ such that $a \leq x_n$ for all $n \in \mathbb{N}$, then $a \leq x$. If there exists $b \in \mathbb{R}$ such that $b \geq x_n$ for all $n \in \mathbb{N}$, then $b \geq x$.

Note. Strict inequalities are not necessarily respected! If $x_n \to x$ and $x_n > 0$, then the most we can say is that $x_n \geq 0$. For example, consider the case where $x_n = \frac{1}{n} > 0$, and $x = 0$.

11.2 Monotone Sequences

Definition. *3.9—Monotonicity*
A sequence (x_n) is *increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence (x_n) is *decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. We call both increasing and decreasing sequences *monotone* or *monotonic*.

Note. If you iterate the definition for an increasing sequence, you get $x_n \leq x_m$ for $n \leq m$. A similar result comes from iterating the definition for a decreasing sequence.

Example. *Monotone Sequences*

- $x_n = 1 - \frac{1}{n}$ is increasing (and converges to 1).
- $x_n = \frac{1}{2^n}$ is decreasing (and converges to 0).
- $x_n = n$ is increasing (but is neither convergent nor bounded).
- $x_n = (-1)^n$ is not monotonic (and does not converge).

Note. In the above examples, if a sequence is monotonic, then if it is bounded it converges, otherwise diverges. Can we prove this?

Theorem — *3.10 (Monotone Convergence Theorem)*
Every monotonic and bounded sequence in $\mathbb{R}$ necessarily converges.

Proof. Let (x_n) be a sequence in $\mathbb{R}$ which is both increasing and bounded (same argument works for decreasing). If we look at the set

$$S = \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R},$$

we see that this set is non-empty and bounded above (because (x_n) is bounded). Thus by the least upper bound property of $\mathbb{R}$ we know that $\sup S$ exists. We claim that $x = \sup S$ is the limit for (x_n). Let $\varepsilon > 0$. Then there exists some $x_N \in S$ such that $x_N > x - \varepsilon$. We know that (x_n) is increasing, then $x - \varepsilon < x_N < x_n$ for all $n > N$. Furthermore, since x is the supremum of S, we know that $x_n < x < x + \varepsilon$. Thus we have

$$x - \varepsilon < x_n < x + \varepsilon,$$

so $|x_n - x| < \varepsilon$ for all $\varepsilon > 0$ and all $n > N$. $\square$

12 Lecture 12

Example. Consider the sequence defined by $x_1 = 2$, and for $n \geq 2$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

We don't have a nice function for x_n in terms of n, so proving convergence by normal means is non-ideal. To find the limit of (x_n), we:

- Show that (x_n) is monotonic (decreasing in this case) and bounded
- Apply the Monotonic Convergence Theorem, and say that there exists some $x \in \mathbb{R}$ such that $x_n \to x$
- Apply limit laws to actually find the value of x

> **Note.** If $x_n \to x$, we know that $x_{n+1} \to x$.

Taking the first few terms, we see

$$x_1 = 2, x_2 = \frac{3}{2}, \ldots$$

We guess that the sequence is always bounded, namely $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. We can prove this by induction. To show that the sequence is decreasing, we just need to show that for all $n \in \mathbb{N}$, we have $x_{n+1} - x_n \leq 0$. We can then use induction again, along with the fact that the previous inequality can be expressed as a function of x_n, to show that the sequence is decreasing.
We can see that the sequence (x_n) is strictly greater than zero, so we may use the Algebraic Limit Theorem. Thus we have

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\ 2x &= x + \frac{2}{x} \\ x &= \frac{2}{x} \\ x^2 &= 2 \\ x &= \sqrt{2}. \end{aligned}$$

12.1 Subsequences

Consider the diverging sequence given by $x_n = (-1)^n$. If we take the even terms, we see that $x_{2n} = 1$, and if we take the odd terms, we have $x_{2n+1} = -1$. Thus we see that "parts" of our diverging sequence are actually convergent sequences.

> **Definition.** *Subsequences*
> Let (x_n) be a sequence and
> $$n_1 < n_2 < n_3 < \cdots$$
> be a strictly increasing sequence of natural numbers. Then for $k \in \mathbb{N}$, the sequence (x_{n_k}) is a *subsequence* of the original sequence (x_n).

Example. *Subsequences*

1. Consider $x_n = (-1)^n$. Then $x_{2k} = 1$ is a subsequence. Similarly, $x_{2k+1} = -1$ is also a subsequence.

2. Consider $x_n = \frac{1}{n} = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$. A valid subsequence could be $x_{n_k} = \frac{1}{2k}$ or $x_{n_k} = \frac{1}{10^k}$. However, the sequence given by

$$n_1 = 10, n_2 = 50, n_3 = 20, n_4 = 5, n_5 = 1000$$

is not a subsequence because the indices are not strictly increasing.

Observation. Suppose we have (n_k) is strictly increasing. Then we know that $n_k \geq k$, because it is strictly increasing and starts at 1. Then if $x_{n_k} = \frac{1}{n_k}$, then we see

$$x_{n_k} \leq \frac{1}{k} \to 0.$$

Thus we see that for *any* subsequence of our convergent sequence x_n, it converges to the same limit as (x_n).

Proposition 3.12 Every subsequence of a converging sequence converges and does so to the same limit as the original sequence.

Lemma. We will show that $n_k \geq k$ for all $k \in \mathbb{N}$, assuming that n_k is a strictly increasing sequence.

Proof. We proceed via induction. Observe that for $n_1 \in \mathbb{N}$, we have $n_1 \geq 1$. Suppose $n_k \geq k$ Then since (n_k) is striclty increasing, we have $n_{k+1} > n_k \geq k$, so $n_{k+1} \geq k + 1$. □

Proof. Suppose (x_n) converges to x. Let (x_{n_k}) be a subsequence of (x_n). We claim that $x_{n_k} \to x$. Let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n > N$. Since $n_k \geq k$ for $k > N$, we have $|x_{n_k} - x| < \varepsilon$ for all $n_k > N$. Thus $x_{n_k} \to x$ as $k \to \infty$. □

Note. If $x_n \to x$, then $x_{n+1} \to x$ because (x_{n+1}) is a subsequence of (x_n).

Note. Proposition 3.12 can be used to prove divergence, by showing that two subsequences of (x_n) converge to different values. For example, if $(x_n) = (-1)^n$, then

$$\begin{aligned} x_{2n} &= (-1)^{2n} = 1 \\ x_{2n+1} &= (-1)^{2n+1} = -1, \end{aligned}$$

so (x_n) diverges.

Proposition 3.13 Every sequence has a monotonic subsequence.

Proof. We need to carefully select our subsequence. Let (x_n) be a sequence and

$$D = \{n \in \mathbb{N} \mid x_n > x_m \text{ for all } m > n\} \subseteq \mathbb{N}.$$

If $n \in D$, then $x_n > x_m$ for all $m > n$. We say that if $n \in D$, that x_n is *dominant*. We now consider when D is finite and when D is infinite.

1. If D is infinite, then there exists $\{n_k\} \subseteq D$ such that x_{n_k} is dominant. Then $x_{n_{k+1}} < x_{n_k}$ and so (x_{n_k}) is a subsequence which is decreasing, and so monotonic.

2. If D is finite, then there exists some N such that $\max D = N$. Thus the last dominant term of our sequence is x_N. Hence there exists some $n_1 > N$ such that $x_{n_1} > x_{N+1}$. However, since $n_1 > N$, we have $n_1 \notin D$, so there must exist some $x_{n_2} > x_{n_1+1}$. By induction on $k \in \mathbb{N}$, we get (n_k), so there exist

$$n_1 < n_2 < n_3 < \cdots$$

such that

$$x_{n_1} < x_{n_2} < \cdots < x_{n_k} < x_N.$$

Therefore (x_{n_k}) is an increasing subsequence, and so is monotonic.

□

Theorem — *3.14 (Bolzano–Weierstrass)*
Every bounded sequence in $\mathbb{R}$ has a converging subsequence.

Proof. Suppose we have a bounded sequence (x_n) in $\mathbb{R}$. Then by Proposition 3.13 we know that there exists some subsequence (x_{n_k}) that is monotonic, which is also bounded. Thus by the Monotone Convergence Theorem we have that (x_{n_k}) converges in $\mathbb{R}$. □

Note. Bolzano–Weierstrass doesn't tell us anything about the original sequence, just that there are converging subsequences.

13 Lecture 13

13.1 Cauchy Sequences

The notion of a Cauchy sequence is a sequence that doesn't necessarily converge, but whose terms get "arbitrarily close" to each other.

Definition. *Cauchy Sequence*
A sequence (x_n) is *Cauchy* if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } n, m > N.$$

Note. We don't need to know any limit x in order to determine if a sequence is Cauchy.

Example. *Cauchy Sequences*

1) $x_n = \frac{1}{n}$ is Cauchy.
2) $x_n = (-1)^n$ is not Cauchy.
3) $x_n = n$ is not Cauchy.

Proposition 3.16 Every convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n - x| < \frac{\varepsilon}{2}$. Let $m > N$. Thus

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x| + |x - x_m| \\ &= |x_n - x| + |x_m - x| \\ &< \varepsilon. \end{aligned}$$

Therefore (x_n) is Cauchy. □

Proposition 3.17 Every Cauchy sequence is bounded.

Proof. Let (x_n) be Cauchy. For $\varepsilon = 1$, we know there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < 1 \quad \text{for all } n, m > N.$$

Thus $|x_n| \leq 1 + |x_{N+1}|$ for all $n > N$. Suppose we choose

$$M = \max(|x_1|, |x_2|, \ldots, |x_N|, 1 + |x_{N+1}|).$$

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$, and so (x_n) is bounded. □

13.2 Completeness

Theorem — *3.18 (Completeness of $\mathbb{R}$)*
$\mathbb{R}$ is *complete*, in other words every Cauchy sequence in $\mathbb{R}$ converges.

Proof. Let (x_n) be Cauchy in $\mathbb{R}$. We have two steps:

(1) Identify some candidate for the limit.

(2) Show that the sequence converges to the candidate.

(1) By Proposition 3.17, we know that (x_n) is bounded. By Bolzano–Weierstrass, there exists some subsequence $x_{n_k} \to x \in \mathbb{R}$ as $k \to \infty$. This is our candidate for our limit.
(2) We now show that our sequence converges to x. Bolzano–Weierstrass tells us that for a subsequence of (x_n), we have the terms getting arbitrarily close to x. Because (x_n) is Cauchy, we "control" the remaining terms by "proximity" to (x_{n_k}).
Fix $\varepsilon > 0$, $x_{n_k} \to x$, there exists $N_1 \in \mathbb{N}$ such that

$$x_{\frac{n}{2}} < \frac{\varepsilon}{2} \quad \text{for all } n_k > N_1.$$

Since (x_n) is Cauchy, there exists some $N_2 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \text{for all } n, m > N_2.$$

So for any $n > \max(N_1, N_2)$, and $n_k > \max(N_1, N_2)$,

$$\begin{aligned} |x_n - x| &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore $x_n \to x$. □

Note. The above proof makes use of Bolzano–Weierstrass, which applies to the reals, not rationals. Thus we think that the rationals are not complete.

Theorem — *3.19 (Incompleteness of $\mathbb{Q}$)*
The rationals are *not* complete. More precisely, there exists a Cauchy sequence of rational numbers that *does not* converge.

Proof. Consider the sequence given by

$$x_1 = 2, x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

We see that (x_n) is a sequence of rationals. We know that it does not converge in the rationals, because if it did it would have to converge to $\sqrt{2} \notin \mathbb{Q}$. It remains to show that it is Cauchy. From one of the homeworks we know that $|x_n - |x_{n+1}|| < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. We can take telescoping sums by writing (for $n > m$):

$$\begin{aligned} |x_n - x_m| &= \left|\sum_{k=m}^{n-1} (x_{k+1} - x_k)\right| \\ &\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \\ &< \left|\sum_{k=m}^{n-1} \frac{1}{2^k}\right|. \end{aligned}$$

This is a geometric series that sums to

$$= \frac{\left(\frac{1}{2}\right)^m - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$
$$= \frac{4}{2^m},$$

which converges to 0 as $m \to \infty$. Thus (x_n) is Cauchy in $\mathbb{Q}$. $\square$

13.3 Bonus: Diverging to Infinity

Definition. *3.20—Diverging to* $\pm\infty$
Let (x_n) be a sequence in $\mathbb{R}$. We say that $x_n \to \infty$ as $n \to \infty$ if: for all $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n > N$.
Similarly, we say $x_n \to -\infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n > N$.

14 Lecture 14

14.1 Diverging Limits and Multiplication

Proposition 3.21 Let $x_n \to +\infty$ and $y_n \to y > 0$. Then

$$\lim_{n\to\infty} (x_n y_n) = +\infty.$$

The intuition via the Algebraic Limit Theorem tells us that this limit should approach "$x \cdot y = +\infty \cdot y = +\infty$".

Note. We can't always allow the case when $y = 0$, i.e. consider the case where $y_n = 0$, so

$$\lim_{n\to\infty} x_n \cdot 0 = \lim_{n\to\infty} 0 = 0.$$

Proposition 3.22 Let (x_n), $x_n \geq 0$. Then $x_n \to +\infty$ if and only if $\frac{1}{x_n} \to 0$.

Example. Consider sequences of the form $x_n = n^p$, where $x_n \geq 0$ and $p > 0$. Then $x_n \to +\infty$ and $\frac{1}{x_n} = \frac{1}{n^p} \to 0$.

14.2 Diverging Limits and Addition

Note. Limits diverging to infinity don't behave nearly as well when subjected to addition and subtraction, in comparison to multiplication and division. For example, if $x_n \to +\infty$ and $y_n \to \infty$, we can't say for sure what $x_n + y_n$ converges to, if anything at all.

14.3 Notion of Infinity

We introduce symbols $+\infty$ and $-\infty$, and define the *extended real numbers*:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the notion of $\leq$ to $\overline{\mathbb{R}}$ as follows: For all $x \in \overline{\mathbb{R}}$, we have

$$-\infty < x < +\infty.$$

We aren't going to extend arithmetic (addition, subtraction, multiplication, division, etc.) to our new symbols. We also introduce the notation:

$$\begin{aligned}[a, +\infty) &= \{x \in \mathbb{R} \mid a \leq x\}.\\ (-\infty, b) &= \{x \in \mathbb{R} \mid x \leq b\}.\\ (-\infty, +\infty) &= \mathbb{R}.\end{aligned}$$

Note. If $A \neq \varnothing$ and $A \subseteq \mathbb{R}$ which is unbounded above, then $\sup A = +\infty$. We should always write $+\infty$ and $-\infty$, not just ∞.

14.4 More on Sequences

Let (x_n) be a sequence in $\mathbb{R}$. Define

$$L = \left\{x_0 \in \overline{\mathbb{R}} \mid \text{There exists a subsequence } (x_{n_k}) \text{ such that } x_{n_k} \to x_0\right\} \subseteq \overline{\mathbb{R}}.$$

We may think of the above by "the set of all possible sub sequential limits of (x_n)". We continue to study the three possible behaviors:

- Case 1: $x_n \to x$ (Converging)

  Then we know that $L = \{x\} \in \mathbb{R}$ because if the sequence converges to a point x, then all sub sequences must converge to the same x.

- Case 2: $x_n \to +\infty$. Then $L = \{+\infty\}$.

  In other words, any subsequence of a sequence diverging to $+\infty$ must also diverge to $+\infty$.

  Proof. Suppose (x_{n_k}) is a subsequence of (x_n), which diverges to $+\infty$. Thus for all $M > 0$, there exists some N such that $x_n > M$ for all $n > N$. We know that $n_k \geq k$, and so fixing $M > 0$, there exists N such that
  $$n_k \geq k > N,$$
  so $x_{n_k} > M$. Thus $x_{n_k} \to +\infty$ as $k \to +\infty$. □

  A similar result can be shown for if $x_n \to -\infty$.

- Case 3: "Oscillating Sequences"

  Characterized by L having more than one element in the extended reals.

Theorem — *3.22*
A sequence of real numbers which does not converge in $\overline{\mathbb{R}}$ has at least two distinct limit points, i.e. there exists two subsequences converging to different things.

Proof. Case 1: (x_n) is unbounded from above.
We can extract a subsequence (x_{n_k}) such that $x_{n_k} \to +\infty$. Since x_n is unbounded above, for all $M > 0$, there exists $n \in \mathbb{N}$ such that $x_n > M$. We consider the subsequence
$$(x_n)_{n=n_1+1}^{\infty},$$
which is also unbounded above. Let $M = 1$, so there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} > 1$. We then choose $M = 2$, and get $n_2 > n_1$ such that
$$(x_n)_{n=n_2+1}^{\infty},$$
and continue this pattern to construct a subsequence (x_{n_k}) such that $x_{n_k} \geq k \to +\infty$, so $+\infty \in L$.
Case 1.1: (x_n) is also unbounded from below. Same arguments give you a subsequence (x_{n_ℓ}) such that $x_{n_\ell} \to -\infty$, and so $-\infty \in L$.
Case 1.2: (x_n) is bounded from below.
Since (x_n) does not converge in the extended real numbers, it does *not* diverge to $+\infty$. Then there exists $M > 0$ for all $N \in \mathbb{N}$, such that $x_n \leq M$ for some $n > N$. Thus we may extract a bounded subsequence. If $N = 1$, then there exists $n_1 > 1$ such that $x_{n_1} \leq M$. If $N = n_1$, there exists $n_2 > n_1$ such that $x_{n_2} \leq M$. We continue this process until we have constructed our subsequence, i.e.
$$x_{n_k} \leq M \quad \text{for all } k \in \mathbb{N}.$$
Bolzano–Weierstrass tells us that there exists a further subsequence $(x_{n_{k_\ell}})$ such that
$$x_{n_{k_\ell}} \to x \in \mathbb{R}.$$
We know that $(x_{n_{k_\ell}})$ is a subsequence of our subsequence (x_{n_k}), and so it must be a subsequence of our original sequence (x_n). Case 3: (x_n) is bounded. From the homework we know that
$$\liminf_{n\to\infty} x_n, \limsup_{x\to\infty} x_n \in L.$$
Since (x_n) does not converge in $\overline{\mathbb{R}}$, we have that the limits are distinct and so L has more than 1 element. □

15 Lecture 15

15.1 Infinite Series

Infinite series are *extremely* useful in applications, i.e. Taylor and Fourier series.

Definition. *3.24—Infinite Series*
Let $(x_n)_{n=m}^\infty$ be a sequence in $\mathbb{R}$ ($m \in \mathbb{Z}$). We define the sequence $(S_n)_{n=m}^\infty$ to be defined by

$$S_n = \sum_{k=m}^n x_k.$$

We call (S_n) the sequence of *partial sums* of (x_n). The *infinite series* given by

$$\sum_{n=m}^\infty x_n$$

is said to converge if the sequence of partial sums converges, and we write

$$\sum_{n=m}^\infty x_n \coloneqq \lim_{n\to\infty} S_n.$$

A series that *doesn't converge* is said to *diverge*. If $S_n \to \pm\infty$, we say that

$$\sum_{n=m}^\infty x_n = \pm\infty.$$

15.2 Important Examples of Infinite Series

1) Geometric Series—Series of the form

$$\sum_{n=0}^\infty x^n = 1 + x + x^2 + \cdots,$$

where $x \in \mathbb{R}$. We know that

$$\begin{aligned} S_n &= \sum_{k=0}^n x^k \\ &= 1 + x + x^2 + \cdots + x^k. \end{aligned}$$

With some algebra we find that for all $n \in \mathbb{N}$, $x \neq 1$,

$$S_n = \frac{1 - x^{n+1}}{1 - x}.$$

Proof. Observe that

$$\begin{aligned} (1-x)S_n &= (1-x)(1 + x + x^2 + \cdots + x^n) \\ &= 1 + x + \cdots + x^n - x - x^2 - \cdots - x^{n+1} \\ &= 1 - x^{n+1}. \end{aligned}$$

$\square$

In the edge case where $x = 1$, we have that

$$\sum_{n=0}^{\infty} 1 = 1 + 1 + \cdots \to +\infty.$$

If $|x| < 1$, then $x^{n+1} \to 0$, and so

$$S_n \to \frac{1}{1-x}.$$

If $|x| > 1$, then (S_n) diverges. If $x = 1$ we have (S_n) diverges. When $x = -1$ we have

$$S_n = \frac{1 - (-1)^{n+1}}{2},$$

which does not converge. Thus we have that (S_n) converges if and only if $|x| < 1$, and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

2) Inverse Squares—Series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Then we have

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Since this sequence is increasing, it suffices to find an upper bound to show that it converges. Observe that for all $n \in \mathbb{N}$,

$$\begin{aligned}
S_n &= 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \cdots \\
&\leq 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n(n-1)} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} + \frac{1}{n}\right) \\
&\leq 2 - \frac{1}{n} \\
&\leq 2.
\end{aligned}$$

Thus (S_n) is bounded above and so by Monotone Convergence Theorem, we know that

$$\lim_{n\to\infty} S_n$$

exists.

3) The Harmonic Series—The infinite series given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Theorem — *Divergence of the Harmonic Series*
The Harmonic Series diverges!

Proof. Let us write the harmonic series by

$$S_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Since S_n is increasing, it suffices to show that the series is unbounded from above. The idea here is that we will group the terms by powers of 2. Observe that for all $n \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ such that $2^m \leq n$, and so

$$\begin{aligned}
S_n &\geq S_{2^m} \\
&= \sum_{k=1}^{2^m} \frac{1}{k} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right) \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^m} + \cdots + \frac{1}{2^m}\right) \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{m-1}}{2^m} \\
&= 1 + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{m \text{ times}} \\
&= 1 + \frac{m}{2} \to +\infty.
\end{aligned}$$

$\square$

15.3 Infinite Series Tests

Theorem — *3.23 (Cauchy condensation test)*
Suppose (x_n) is *decreasing* and $x_n \geq 0$ for all $n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} \underbrace{2^n x_{2^n}}_{y_n}$$

converges.

Proof. ($\Leftarrow$) Suppose $\sum 2^n x_{2^n}$ converges. Then

$$t_n = \sum_{k=1}^{n} 2^k x_{2^k}$$

converges, and so is bounded. Let

$$S_n = \sum_{k=0}^{n} x_k.$$

We want to show that $S_n < t_m$ (bounded above). The idea here is to partition $[1, n]$ into dyadic intervals. Fix $n \in \mathbb{N}$ and let m be large enough so that $n \leq 2^{m+1} - 1$. Then we have

$$\begin{aligned} S_n &\leq S_{2^{m+1}-1} \\ &= \sum_{k=1}^{2^{m+1}-1} x_k \\ &= \sum_{\ell=0}^{m} \sum_{k=2^\ell}^{2^{\ell+1}-1} x_k \\ &\leq \sum_{\ell=0}^{m} \max_{2^\ell \leq k \leq 2^{\ell+1}-1} x_k \cdot \sum_{k=2^\ell}^{2^{\ell+1}-1} 1 \\ &\leq \sum_{\ell=0}^{m} x_{2^\ell}(2^{\ell+1} - 1 - 2^\ell + 1) \\ &\leq \sum_{\ell=0}^{m} 2^\ell x_{2^\ell} \\ &= t_m. \end{aligned}$$

□

Corollary 3.24 The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proof.

$$\begin{aligned}\sum_{n=0}^{N} 2^n x_{2^n} &= \sum_{n=0}^{N} 2^n \left(\frac{1}{2^n}\right)^p \\ &= \sum_{n=0}^{N} \frac{2^n}{2^{np}} \\ &= \sum_{n=0}^{N} \left(\frac{1}{2^{p-1}}\right)^N,\end{aligned}$$

which converges by Geometric series. □

16 Lecture 16

Definition. *3.25—Cauchy Criterion for Series*
We say a series $\sum x_n$ satisfies the *Cauchy Criterion* if the sequence of partial sums (S_n) is a Cauchy sequence. In other words, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|S_n - S_m| < \varepsilon,$$

for all $m, n > N$. Written another way, we have that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\left|\sum_{k=m+1}^{n} x_k\right| < \varepsilon$$

for all $n \geq m > N$.

Theorem — *3.26*
A series $\sum x_n$ satisfies the Cauchy criterion if and only if the series converges.

Proof. ($\Rightarrow$) We know that the sequence of partial sums is Cauchy. Since the reals are complete, we have that the partial sums converge, and so the series converges.
($\Leftarrow$) If the series converges, the sequence of partial sums is converges, and so is Cauchy. Therefore the series satisfies the Cauchy criterion. $\square$

Corollary 3.27 If $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let (S_n) denote the sequence of partial sums of (x_n). Since (S_n) converges, we know that it satisfies the Cauchy criterion. Since (S_n) is Cauchy, we have that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right| = \sum_{k=m+1}^{n} x_k < \varepsilon,$$

for all $m, n > N$. Letting $n = m + 1$, we have

$$\begin{aligned}\left|\sum_{k=m+1}^{n} x_k\right| &< \varepsilon \\ \left|\sum_{k=m+1}^{m+1} x_k\right| &< \varepsilon \\ |x_{m+1}| &< \varepsilon.\end{aligned}$$

Thus for all $n > N$, we have $|x_n| < \varepsilon$, and so

$$\lim_{n\to\infty} x_n = 0.$$

$\square$

16.1 Various Convergence Tests

Note. This gives us the "divergence test". If the limit of the summand of a series does not converge to 0, then the series diverges.

Example. *Divergence Test*
Consider the series given by
$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}.$$
The above series diverges because the summand converges to 1 (by Algebraic Limit Theorem). However, the sequence converging a necessary and non-sufficient condition to show that the series converges, i.e. consider $\sum x_n = \frac{1}{n}$.

Theorem — *3.28 (Comparison Test)*
Assume (x_n), (y_n) are sequences in $\mathbb{R}$, $y_n \geq 0$ for all $n \in \mathbb{N}$. Then

(i) If the series $\sum y_n$ converges and $|x_n| \leq y_n$ for all $n \in \mathbb{N}$, then $\sum x_n$ converges.

(ii) If the series $\sum y_n$ diverges and $|x_n| \geq y_n$ for all $n \in \mathbb{N}$, then $\sum x_n$ diverges.

Proof. (i) Suppose $n > m$. Then
$$\begin{aligned}\left|\sum_{k=m+1}^{n} x_k\right| &\leq \sum_{k=m+1}^{n} |x_k| \\ &\leq \sum_{k=m+1}^{n} y_k.\end{aligned}$$
Since $\sum y_n$ converges, it satisfies the Cauchy criterion. Thus we fix some $\varepsilon > 0$, and there exists some $N \in \mathbb{N}$ such that
$$\sum_{k=m+1}^{n} y_k < \varepsilon,$$
for all $n > m > N$. Continuing the inequality from before we get
$$\left|\sum_{k=m+1}^{n} x_k\right| < \varepsilon,$$
so $\sum x_k$ is Cauchy, and so converges.

(ii) Let (X_n) be partial sums for $\sum x_n$, and (Y_n) be partial sums for $\sum y_n$. By domination, we know that $X_n \geq Y_n$ for all $n \in \mathbb{N}$. Since $\sum y_n$ diverges, we have $Y_n \to +\infty$. This forces X_n to diverge to $+\infty$ as well.

□

Note. The domination conditions in the previous theorem need only apply for all but finitely many n, rather than for all $n \in \mathbb{N}$. In other words, we only need those conditions to hold "from some point onwards". This is because a series' convergence is dictated by the convergence/divergence of its "tail":
$$\sum_{n=1}^{\infty} x_n = \underbrace{\sum_{n=1}^{N} x_n}_{\text{Finite}} + \underbrace{\sum_{n=N+1}^{\infty} x_n}_{\text{"Tail"}}.$$

16.2 Convergent Series and the Harmonics

Question. How close can we get to the Harmonic Series and still converge?

By the p-series test, we know that $\sum \frac{1}{n^p} < +\infty$ if and only if $p > 1$. Thus for any $\varepsilon > 0$, we have $\sum \frac{1}{n} \cdot \frac{1}{n^\varepsilon} = \sum \frac{1}{n^{1+\varepsilon}}$, which converges. The question remains whether we can replace $\frac{1}{n^\varepsilon}$ by something that decays *slower*.

We know that for any $a > 0$, there exists $N \in \mathbb{N}$ such that $\log n \leq n^a$ for all $n > N$. In other words, logarithms grow slower than any exponential. We know that the series

$$\sum \frac{1}{n^p \log(1+n)}$$

converges when $p > 1$, by comparison with $\sum \frac{1}{n^p}$. Here we try replacing the n^ε from earlier with $\log(1+n)$. Then

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\beta}$$

converges if and only if $\beta > 1$. We can continue to get closer to the $\beta = 1$ case by adding more logs, i.e.

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)(\log\log n)^\alpha}$$

converges if and only if $\alpha > 1$.

16.3 Absolute and Conditional Convergence

Theorem — *3.29 (Absolute Convergence Test)*
If the series $\sum |x_n|$ converges, then $\sum x_n$ converges.

Proof. Comparison test—we have $|x_n| \leq |x_n|$. □

Note. The converse of the above is generally false, as $\sum \frac{1}{n}$ diverges but $\sum \frac{(-1)^n}{n}$ converges.

Definition. *3.30—Absolute/Conditional Convergence*
A series $\sum x_n$ converges *absolutely* if $\sum |x_n|$ converges. If the series $\sum x_n$ converges and $\sum |x_n|$ diverges, we say $\sum x_n$ converges *conditionally.*

17 Lecture 17

Theorem — *3.31 (Alternating Series Test)*
Let (x_n) be a sequence such that

(i) (x_n) is decreasing

(ii) $\lim_{n\to\infty} x_n = 0$

Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} x_n$$

converges.

Proof. Let

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} x_k.$$

Then we have

$$\begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 - x_2 \\ S_3 &= x_1 - x_2 + x_3 \\ S_4 &= x_1 - x_2 + x_3 - x_4. \end{aligned}$$

Since (x_n) is decreasing, we have that (S_{2n}) is increasing (as $x_{2n-1} - x_{2n-2}$ is non-negative). Furthermore, we have (S_{2n+1}) is decreasing, as $x_{2n} - x_{2n-1}$ is non-positive. Thus we have the inequality

$$S_2 \le S_{2n} \le S_{2n+1} \le S_1.$$

By the Monotone Convergence Theorem, we can say that both partial sum subsequences converge, say $S_{2n} \to a$ and $S_{2n+1} \to b$, for some $a, b \in \mathbb{R}$. By the Ordered Limit Theorem we know that $a \le b$. It remains to show that $a = b$ and that any subsequence of (S_n) converges to a. Notice that

$$S_{2n+1} - S_{2n} = (-1)^{2n+2} x_{2n+1},$$

and so by Algebraic Limit Theorem we have $b - a = 0$ (since (x_n) converges to 0). Thus $a = b$. Fixing $m \in \mathbb{N}$, we know there exists some $n \in \mathbb{N}$ such that

$$S_{2n} \le S_n \le S_{2n+1}.$$

By Squeeze Theorem, we have $S_n \to a$ as $n \to \infty$. □

Note. This theorem is quite helpful for helping us test for conditional convergence.

Example. Consider the series given by $\frac{1}{n^p}$, where $p > 0$. Observe that this sequence is both decreasing and converges to zero. Hence
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$
converges. Furthermore, even something that decays much slower like
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log\log\log\log(1+n)}$$
converges.

Theorem — *3.32 (Ratio Test)*
Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and
$$L \coloneqq \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|.$$
Then the series $\sum x_n$:

(i) If $L < 1$, the series converges absolutely.

(ii) If $L > 1$, the series diverges.

(iii) If $L = 1$, the test is inconclusive.

Proof. (i) For all $\varepsilon > 0$, there exists some N such that for all $n > N$,
$$\left|\left|\frac{x_{n+1}}{x_n}\right| - L\right| < \varepsilon$$
$$|x_{n+1}| < (L+\varepsilon)\,|x_n|\,.$$
If $L < 1$, we may choose $\varepsilon > 0$ sufficiently small so that $L + \varepsilon = L' < 1$. Thus we have
$$\begin{aligned} |x_2| &< L'x_1 \\ |x_3| &< L'x_2 < (L')^2 x_1 \\ |x_4| &< L'x_3 < (L')^3 x_1. \end{aligned}$$
Iterating inductively, we see $|x_{n+k}| \le (L')^k\,|x_n|$ for all $n > N$. We then may define
$$|x_n| \le \begin{cases} \max(|x_1|, \cdots, |x_N|) & \text{for } 1 \le n \le N, \\ (L')^{n-N}\,|x_N| & \text{for } n > N. \end{cases}$$
Summing this up, we get
$$\sum_{n=1}^{\infty} |x_n| = \underbrace{\sum_{n=1}^{N} \max(|x_1|, \cdots, |x_N|)}_{\text{finite}} + \underbrace{\sum_{n=N+1}^{\infty} (L')^{n-N}\,|x_N|}_{\text{converges geometrically}}$$
converges.

(ii) We want to show that for all $\varepsilon > 0$, there exists N such that $|x_{n+1}| > (L-\varepsilon)\,|x_n|$ for all $n > N$. Since $L > 1$, we may choose some $\varepsilon > 0$ small such that $L' = L - \varepsilon > 1$. Iterating again, we get
$$|x_n| > (L')^{n-N}\,|x_N|$$
for all $n > N$. Since $L' > 1$, we have that the series diverges by Geometric Series.

$\square$

Note. The test fails when $L = 1$ because there's no way to choose an $\varepsilon > 0$ such that $L' < 1$ or $L' > 1$, so we can't claim that the series converges/diverges by Geometric Series.

Note. Alternatively, we may actually replace the L in parts (i) and (ii) with $\limsup$ and $\liminf$, respectively. This makes a stronger statement since we know that $\limsup$ and $\liminf$ always exist.

Theorem — *3.33 (Root Test)*
Let (x_n) be a sequence such that
$$\lim_{n\to\infty} |x_n|^{\frac{1}{n}} = L.$$
Then the series $\sum x_n$:

(i) If $L < 1$ then $\sum x_n$ converges absolutely.

(ii) If $L > 1$ then $\sum x_n$ diverges.

(iii) If $L = 1$ then the test is inconclusive.

Proof. Since we have that
$$\lim_{n\to\infty} |x_n|^{\frac{1}{n}} = L,$$
we have that for all $\varepsilon > 0$, there exists N such that $\left||x_n|^{\frac{1}{n}} - L\right| < \varepsilon$ for all $n > N$. Thus we have
$$L - \varepsilon < |x_n|^{\frac{1}{n}} < L + \varepsilon.$$
If $L < 1$, we may choose $\varepsilon > 0$ such that $L' = L + \varepsilon < 1$, so
$$\begin{aligned} |x_n|^{\frac{1}{n}} &< L' \\ |x_n| &< (L')^n. \end{aligned}$$
Thus by the comparison test with a geometric series we see that $\sum x_n$ converges absolutely. If $L > 1$, we may choose $\varepsilon > 0$ such that $L' = L - \varepsilon > 1$, so
$$\begin{aligned} |x_n|^{\frac{1}{n}} &> L' \\ |x_n| &> (L')^n. \end{aligned}$$
Thus by the comparison test with a geometric series we see that $\sum x_n$ diverges. $\square$

Note. In a fashion similar to the Ratio Test, if $L = 1$, we can't compare $\sum x_n$ to a geometric series, and so the test is inconclusive.

Note. If we consider the Ratio and Root Tests in the more general setting (with $\limsup$ and $\liminf$ instead of $\lim$), then the Root Test is "better" than the Ratio Test. However, if we just look at regular limits, then the tests yield the *same information.* From Ross:
$$\liminf \left|\frac{x_{n+1}}{x_n}\right| \leq \limsup |x_n|^{\frac{1}{n}} \leq \limsup \left|\frac{x_{n+1}}{x_n}\right|.$$
For the above, we "lose a little bit" when we use the Ratio Test, and have fewer cases where we can be bounded by a geometric series.

Example. An example where the Root Test works but the Ratio Test fails is for the series
$$\sum_{n=1}^{\infty} 2^{(-1)^n} 2^{-n}.$$

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18.1 Introduction to Continuity

We now want to discuss functions on the reals,

$$f\colon A \subseteq \mathbb{R} \to \mathbb{R}, \quad \operatorname{dom}(A) = f.$$

We have a basic understanding of continuity—"if $x \in A$ is *close* to $c \in A$, then $f(x)$ is *close* to $f(c)$". Written a bit more rigorously, we can say that "f is continuous at a point c if $\lim_{x\to c} f(x) = f(c)$". However, we haven't defined the limit of a function yet. We'll call the limit above a "functional limit".

Example. *Dirichlet's Function*
Let us define

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Question. What number should we associate with

$$\lim_{x\to\frac{1}{2}} g(x)?$$

We take a sequence of points in $\mathbb{R}$, apply g to them, and see what that approaches as those points approach $\frac{1}{2}$. There exists some $(x_n) \in \mathbb{Q}$ such that $x_n \to \frac{1}{2}$, so $g(x_n) = 1$. Thus

$$\lim_{n\to\infty} g(x_n) = 1.$$

However, by the density of the irrationals, we know there exists some $(y_n) \in \mathbb{R} \setminus \mathbb{Q}$ such that $y_n \to \frac{1}{2}$, so $g(y_n) = 0$. Thus

$$\lim_{n\to\infty} g(y_n) = 0.$$

Since these two values disagree with each other, we say that the limit does not exist. In particular for Dirichlet's function, we say that "g is discontinuous *everywhere*".

Example. Let us consider the function given by

$$h(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Consider what the value of

$$\lim_{x\to 0} h(x)$$

should be. Let $(x_n) \subseteq \mathbb{R}$ such that $x_n \to 0$. Thus we have

$$|h(x_n)| \leq |x_n| \to 0.$$

Thus by Squeeze Theorem we have that $h(x_n) \to 0$ as $n \to \infty$. Therefore "$\lim_{x\to 0} h(x) = 0$" and "h is continuous at $x = 0$ and discontinuous elsewhere".

18.2 Functional Limits

Definition. *4.1—Limit Points*
Let $A \subseteq \mathbb{R}$. We say $c \in \mathbb{R}$ is a *limit point* of A if there exists a sequence such that

- $(x_n) \subseteq A$

- $x_n \neq c$ for all $n \in \mathbb{N}$
- $\lim\limits_{n\to\infty} x_n = c$

In other words, c is a limit point if a sequence of points in A (not equal to c) converges to c.

Note. A limit point c does not necessarily have to belong to the set A.

Example.

- Let $A = (0, 1)$. If $x \in A$, then $x_n = x - \frac{1}{n}$ converges to x when n is sufficiently large, so all points in A are limit points. Note that if we let $c = 1$, we can still define $x_n = 1 - \frac{1}{n} \to 1$ to see that 1 is also a limit point of A despite not belonging to A.
- If $A = [0, 1]$, then $1 \in A$ and every point in A is a limit point of A.
- Every interval for the form $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$, $a < b$ contains all of its limit points. We write $L_A \subseteq \mathbb{R}$ to be the set of all limit points.

Note. The set of limit points L_A of a set A in general can look very different from A itself, and we can't draw any conclusions about one being the subset of the other.

Definition. *Isolated Point*
A point in A that is not a limit point of A is called a *isolated point*.

Definition. *4.2—Closed Set*
A set containing its limit points is called a *closed set*.

Definition. *4.3—Functional Limit*
Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ be a limit point of A, and $f\colon A \to \mathbb{R}$ ($\operatorname{dom}(f) = A$). Then we define

$$\lim_{x\to c} f(x) = L$$

for some $L \in \mathbb{R}$, if for *every* sequence (x_n) in A which converges to c, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Note. We extend the definition to allow $c \in \mathbb{R} \cup \{+\infty, -\infty\}$ so we may have limits of the form

$$\lim_{x\to+\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x). \qquad (L_A \subseteq \overline{\mathbb{R}})$$

Furthermore, we extend $L \in \mathbb{R} \cup \{+\infty, -\infty\}$ so we may have limits of the form

$$\lim_{x\to c} f(x) = \pm\infty.$$

Example.

1) Let $f(x) = x^2$, where $A = \mathbb{R}$. Furthermore, let $c \in \mathbb{R}$ and $x_n \to c$. Then $f(x_n) = x_n^2 \to c^2$ by Algebraic Limit Theorem. Hence,
$$\lim_{x \to c} f(x) = c^2$$

2) Let $f(x) = \frac{x^2-4}{x-2}$. We have that $A = (-\infty, 2) \cup (2, \infty)$. We can see that 2 is a limit point of A. Let $(x_n) \subseteq A$ such that $x_n \to 2$, and $x_n \neq 2$ for all n. Then
$$\begin{aligned} f(x_n) &= \frac{x_n^2 - 4}{x_n - 2} \\ &= \frac{(x_n + 2)(x_n - 2)}{x_n - 2} \\ &= x_n + 2, \end{aligned}$$
which converges to 4. Hence, $\lim\limits_{x \to 2} f(x) = 4$.

3) Let $f(x) = \frac{1}{(x-2)^3}$, so $A = \mathbb{R} \setminus \{2\}$. We see that $\pm\infty$ are limit points of A, and want to show that
$$\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$
Let $x_n \to +\infty$, and $(x_n) \subseteq A$. Then $f(x_n) = \frac{1}{(x_n-2)^3} \to 0$.

What about $\lim\limits_{x \to 2} f(x)$? Looking at the graph, we suspect that the limit does not exist. If $(x_n) \subseteq A$ such that $x_n > 2$ and $x_n \to 2$, then
$$f(x_n) = \frac{1}{(x_n - 2)^3} \to +\infty.$$
If $(y_n) \subseteq A$ such that $y_n < 2$ and $y_n \to 2$, then
$$f(y_n) = \frac{1}{(y_n - 2)^3} \to -\infty.$$
We found two sequences such that when f is applied, they converge to different values, and so $\lim\limits_{x \to 2} f(x)$ does not exist. However, we take the notion of "one-sided" limits:

- $\lim\limits_{x \to 2^+} f(x) = +\infty$.
- $\lim\limits_{x \to 2^-} f(x) = -\infty$.

Definition. *4.4—Left and Right Hand Limits*
Let c be a limit point of $A \subseteq \mathbb{R}$. We define the *right-hand limit* of $f\colon A \to \mathbb{R}$ by
$$\lim_{x \to c^+} f(x) = c$$
if for every $(x_n) \subseteq A$, $x_n > c$ and $x_n \to c$ we have
$$\lim_{n \to \infty} f(x_n) = L.$$
Similarly, we define the *left-hand limit* of $f\colon A \to \mathbb{R}$ by
$$\lim_{x \to c^-} f(x) = c$$

if for every $(x_n) \subseteq A$, $x_n < c$ and $x_n \to c$ we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Theorem — *4.5*
If $f\colon A \to \mathbb{R}$, c a limit point of A, Then

$$\lim_{x\to c} f(x) = L \tag{exists}$$

if and only if

$$\lim_{x\to c^+} f(x) = L \quad \text{and} \quad \lim_{x\to c^-} f(x) = L.$$

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Theorem — *4.6 (Algebraic Limit Theorem for Functional Limits)*
Let f, g be defined on A, c a limit point of A, and

$$\lim_{x\to c} f(x) = L, \lim_{x\to c} g(x) = M.$$

Then

(i) $\lim_{x\to c}[af(x) + bg(x)] = aL + bM$ for all $a, b \in \mathbb{R}$.

(ii) $\lim_{x\to c}[f(x)\cdot g(x)] = L \cdot M$.

(iii) $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, provided $M \neq 0$.

Proposition 4.7 (Divergence Criterion for Functional Limits) Let $f\colon A \to \mathbb{R}$, c a limit point of A. If $(x_n), (y_n) \subseteq A$ such that $x_n \neq c, y_n \neq c$ and $x_n \to c, y_n \to c$, and

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n),$$

then $\lim_{x\to c} f(x)$ does not exist.

Example. Consider the function $f(x) = \sin(\frac{1}{x})$, where $A = \mathbb{R} \setminus \{0\}$. We claim that $\lim_{x\to 0} f(x)$ does not exist.
If we choose $x_n = \frac{1}{2n\pi}$, we have $f(x_n) = \sin(2\pi n) \to 0$. If we choose $y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, then we get $f(y_n) = \sin(2\pi n + \frac{\pi}{2}) \to 1$. Therefore the functional limit does not exist.

Theorem — *4.8 (Quantitative Version of Functional Limits)*
Let $f\colon A \to \mathbb{R}$, c a limit point of A ($c \in \mathbb{R}, L \in \mathbb{R}$). Then the following are equivalent:

(i) $\lim_{x\to c} f(x) = L$

(ii) For all $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$

Proof. (i) $\Rightarrow$ (ii) We proceed via contrapositive. Suppose that (ii) does not hold. Then there exists $\varepsilon_0 > 0$ for all $\delta > 0$, such that $|f(x) - L| \geq \varepsilon_0$ whenever $0 < |x - c| < \delta$. Letting $\delta = 1$, we know there exists $\varepsilon_0 > 0$ such that whenever $0 < |x - c| < \delta = 1$, we have $|f(x) - L| \geq \varepsilon_0$. Pick x_1 such that $0 < |x_1 - c| < 1$. Inductively choose $\delta = \frac{1}{n}$, to get a sequence $(x_n) \subseteq A$ where $x_n \neq c$ and $x_n \to c$ (since $|x_n - c| < \frac{1}{n}$). Notice that $f(x_n) - L \geq \varepsilon_0$ for all $n \in \mathbb{N}$ because $0 < |x_n - c| < \frac{1}{n} = \delta$. This contradicts $f(x) \to L$, so $f(x) \not\to L$ and

$$\lim_{x\to c} f(x) \neq L.$$

(ii) $\Rightarrow$ (i) Let $(x_n) \subseteq A$, $x_n \neq c$, $x_n \to c$. Fix $\varepsilon > 0$. Then there exists $\delta > 0$, such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. Since $x_n \to c$, there exists $N = N(\delta)$ such that $|x_n - c| < \delta$ for all $n > N$. Thus $|f(x_n) - L| < \varepsilon$ for all $n > N$. Hence

$$\lim_{n\to\infty} f(x_n) = L,$$

and so

$$\lim_{x\to c} f(x) = L.$$

$\square$

Note. The above is also commonly referred to as the (ε, δ) definition for functional limits.

Example. Let $f(x) = x^2$, $A = \mathbb{R}$. We wish to show that $\lim_{x\to c} f(x) = c^2$ for any $c \in \mathbb{R}$.
Scratch Work. We see that

$$\begin{aligned}\left|f(x) - c^2\right| &= \left|x^2 - c^2\right| \\ &= |x+c| \cdot |x-c|\end{aligned}$$

If we choose $\delta < 1$, then we have $|x+c| \leq |x-c| + 2|c| < \delta + 2|c| < 1 + 2|c|$. Thus

$$\begin{aligned}&< |x+c|\,\delta \\ &< (1+2|c|)\delta \\ &< \varepsilon.\end{aligned}$$

Thus we know that we should choose $\delta = \min(1, \frac{\varepsilon}{1+2|c|})$.

Proof. Given $\varepsilon > 0$, pick $\delta = \min(1, \frac{\varepsilon}{1+2|c|})$. Then

$$\begin{aligned}\left|f(x) - c^2\right| &\leq |x+c|\,|x-c| \\ &< (1+2|c|)\delta \\ &< \varepsilon,\end{aligned}$$

whenever $0 < |x-c| < \delta$. Hence

$$\lim_{x\to c} f(x) = c^2.$$

$\square$

19.1 Continuity

Definition. *4.9—Continuity*
We say that $f\colon A \to \mathbb{R}$ is *continuous at* $c \in A$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $0 < |x-c| < \delta$.

Note. How is this different from Theorem 4.8?

- We need f defined at c, i.e. $f(c)$ exists.
- We are allowing $x = c$.
- We don't say that c is a limit point of A.
  - Every point is continuous at isolated points of a set, since it trivially satisfies the conditions.

Theorem — *4.11*
Let $f\colon A \to \mathbb{R}$ be a limit point of A. Then f is continuous at c if and only if

$$\lim_{x\to c} f(x) = f(c).$$

Moreover, for any point $c \in A$, f is continuous at c if and only if for any $(x_n) \subseteq A$ such that $x_n \to c$, $f(x_n) \to f(c)$.

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Theorem — *4.12*
Let $f, g\colon A \to \mathbb{R}$ be continuous at $c \in A$. Then:

(i) $af + bg$ is continuous at c, for any $a, b \in \mathbb{R}$.

(ii) $f \cdot g$ is continuous at c.

(iii) $\frac{f}{g}$ is continuous at c, whenever the quotient is defined.

Example.

1) $g(x) = x$ is continuous on $\mathbb{R}$, which is to say that for all $c \in \mathbb{R}$, g is continuous at c. Additionally, where $a \in \mathbb{R}$, is continuous on $\mathbb{R}$.

2) By Theorem 4.12 and the examples given above, we have that *any polynomial* is continuous on $\mathbb{R}$:
$$p(x) = a_n x^n + \cdots + a_1 x + a_0,$$
for $a_0, \dots, a_n \in \mathbb{R}, n \in \mathbb{N}$.

3) Every rational function is continuous wherever they are defined. We have for polynomials p, q, that
$$r(x) = \frac{p(x)}{q(x)}.$$

4) $f(x) = \sin(\frac{1}{x})$ on $\mathbb{R} \setminus \{0\}$. Since $\lim\limits_{x\to 0} f(x)$ does not exist, f is not continuous at $x = 0$.

5) Consider the function
$$g(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$
This function is continuous at $x = 0$ since the function gets squeezed to 0.

6) $f(x) = \sqrt{x}$, $A = [0, \infty)$. We know that f is continuous on A, but we need to consider the cases where $c = 0$ and $c \neq 0$. For the former, we have
$$\begin{aligned} |f(x) - f(0)| &= \left|\sqrt{x}\right| \\ &= \sqrt{x} \\ &< \varepsilon, \end{aligned}$$
so we choose $\delta = \varepsilon^2$. When $c \neq 0$, we have
$$\begin{aligned} |f(x) - f(c)| &= \left|\sqrt{x} - \sqrt{c}\right| \\ &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ &\leq \frac{1}{\sqrt{c}}\,|x - c| \\ &< \frac{\delta}{\sqrt{c}}, \end{aligned}$$
so we choose $\delta = \varepsilon\sqrt{c}$.

Proposition 4.13 Let $f\colon A \to \mathbb{R}$, $g\colon B \to \mathbb{R}$, where $A, B \subseteq \mathbb{R}$. Assume $\operatorname{Ran}(f) \subseteq B$ so that the composition $(g \circ f)(x)$ is well defined. Assume that f is continuous at $c \in A$ and g is continuous at $f(c) \in B$. Then $g \circ f$ is continuous at $c \in A$.

Proof. We use sequences. Let $(x_n) \subseteq A$, $x_n \to c$. Since f is continuous at c, we know that $f(x_n) \to f(c)$. Since g is continuous at $f(c)$, we know that $g(f(x_n)) \to g(f(c))$, so $(g \circ f)(x_n) \to (g \circ f)(c)$. Therefore $g \circ f$ is continuous at $c \in A$. $\square$

Example. We are going to assume for this class that $f(x) = \cos x, \sin x, e^x, \log x$ are all continuous functions wherever they are defined. Thus $g(x) = \cos(e^x), e^{\sin x}, \sin(\sin(e^x))$ are all continuous on $\mathbb{R}$.

20.1 The Extreme Value Theorem

Definition. *4.14—Compactness*
We say that a set $K \subseteq \mathbb{R}$ is *compact* if for any sequence $(x_n) \subseteq K$ there exists a subsequence (x_{n_k}) which converges to $x \in K$.

Note. This is a generalization of the notion of closed sets, which says that a set is closed if it contains all of its limit points.

Example. *Compact Sets*

1) The set $K = [a, b]$ is compact, where $a \leq b$.

   Let $(x_n) \subseteq K$. We see that K is bounded, so (x_n) is bounded. Thus by Bolzano–Weierstrass, we have there exists some subsequence (x_{n_k}) such that $x_{n_k} \to x \in \mathbb{R}$. Since $a \leq x_{n_k} \leq b$, we know that $a \leq x \leq b$, so $x \in K$.

2) The set (a, b) is not compact. Observe that $x_n = b - \frac{b-a}{2n} \subseteq (a, b)$, but $x_n \to b \notin (a, b)$. Thus it is not compact.

3) $[0, \infty)$ is not compact, because $x_n = n$ diverges to $+\infty$, and so every subsequence (x_{n_k}) must also diverge to $+\infty$.

Theorem — *4.15 (Heine–Borel Theorem)*
A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. ($\Leftarrow$) Suppose $K \subseteq \mathbb{R}$ is closed and bounded. We claim that K is compact. Let $(x_n) \subseteq K$. Since K is bounded, Bolzano–Weierstrass says that there exists some subsequence $(x_{n_k}) \subseteq K$ such that $x_{n_k} \to x \in \mathbb{R}$. Since K is closed, we know that $x \in K$, and so K is compact.
($\Rightarrow$) Suppose K is compact. We first show that K is bounded. Suppose towards a contradiction that K is not bounded, so there exists some sequence $(x_n) \subseteq K$ such that $|x_n| \to +\infty$. However, this means that every subsequence $|x_{n_k}|$ must also diverge to $+\infty$, which contradicts compactness.
We now show that K is closed, which is to say that if $(x_n) \subseteq K$ such that $x_n \to x \in \mathbb{R}$, then $x \in K$. By compactness, there exists a subsequence (x_{n_k}) such that $x_{n_k} \to y \in K$. However, since every subsequence of a convergent sequence must converge to the same value, we have $x = y \in K$ and so K is closed. $\square$

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Note (Why do we like compact sets?).

- In general, continuous functions don't map closed sets to closed sets.
- Continuous functions map compact sets to compact sets.
- Over compact sets, continuous functions have both minima and maxima.

Theorem — *4.16*
Let $f\colon A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is also compact. In particular, f is bounded on K (by Heine–Borel):

$$\exists M > 0 \text{ such that } |f(x)| \leq M,$$

for all $x \in K$.

Proof. Let $(y_n) \subseteq f(K)$, so there exists some sequence $(x_n) \subseteq K$ such that $f(x_n) = y_n$. We know that K is compact, so there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x \in K$. Thus we have $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$. ☐

Theorem — *4.17 (Extreme Value Theorem)*
If $f\colon K \to \mathbb{R}$ is continuous on compact K, then there exists $x_1, x_2 \in K$ such that

$$f(x_1) \leq f(x) \leq f(x_2),$$

for all $x \in K$. In other words, f has a maxima and minima over K.

Lemma. If a non-empty set $K \subseteq \mathbb{R}$ is compact, then it has a maxima and a minima.

Proof. We begin by showing that K has a maxima. By Heine–Borel, if a set is compact it must be closed and bounded, so K is closed and bounded. Since K is bounded and non-empty, it must have a supremum, and so $\sup K$ exists. We know that there exists a sequence $(x_n) \subseteq K$ that converges to $\sup K$. However, K is closed so $\sup K \in K$, and so K has a maxima.
We may apply similar logic to show that K has a minima. Since K is bounded and non-empty, it must have an infimum, so $\inf K$ exists. We know that there exists a sequence $(x_n) \subseteq K$ that converges to $\inf K$. Because K is closed we have $\inf K \in K$, so K has a minima. ☐

Proof. By Theorem 4.16, we have that $f(K)$ is compact. Thus $f(K)$ has maxima and minima. ☐

21.1 The Intermediate Value Theorem

Theorem — *4.18 (Intermediate Value Theorem)*
Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $L \in \mathbb{R}$ such that

$$\begin{aligned} f(a) < L < f(b) \quad &\text{or} \\ f(b) < L < f(a). \end{aligned}$$

Then, there exists some $c \in (a, b)$ such that $f(c) = L$.

Proof. We have some simplifying assumptions that make the proof easier:

- By shifting f, we may assume that $L = 0$. What this means is that we shift the function f up or down until our L becomes 0.
- We suppose that
$$f(a) < 0 < f(b),$$
as the other case is similar.

Consider the set $A = \{x \in [a, b] \mid f(x) \leq 0\}$. We claim that $\sup A$ not only exists, but also is in $[a, b]$. Since $f(a) < 0$, we have $a \in A$ and so $A \neq \varnothing$. Furthermore, A is bounded above by b, and so by the least upper bound property we have $c = \sup A$ exists. Note that $c \in (a, b)$ because $a \leq c \leq b$.
We now show that $f(c) = 0$. Suppose towards a contradiction that $f(c) > 0$. Since $c = \sup A$, there exists some sequence $(x_n) \subseteq A$ such that $x_n \to c$. Since f is continuous we have $f(x_n) \to f(c)$. Furthermore, as $(x_n) \subseteq A$ we have $f(x_n) \leq 0$ for all n, so $f(c) \leq 0$, a contradiction.
Now suppose towards a contradiction that $f(c) < 0$. Let $z_n = \min(b, c + \frac{1}{n})$, so $(z_n) \subseteq [a, b] \setminus A$. We know that $z_n \to c$, and by continuity of f, $f(z_n) \to f(c)$. Since (z_n) is not a subset of A, $f(z_n) > 0$. Thus by Ordered Limit Theorem we have $f(c) \geq 0$, a contradiction.
Therefore $f(c) = 0$. $\square$

Note. The zero found by the Intermediate Value Theorem is *not necessarily unique*.

21.1.1 Consequences of Intermediate Value Theorem

Corollary 4.19 If f is continuous on an interval I, then $f(I)$ is an interval or a single point.

Theorem — *Inverse Function Theorem*
Let I be an interval and $f\colon I \to \mathbb{R}$ be strictly increasing (so $f(x) < f(y)$ if $x < y$) and continuous on I. Then $f^{-1}\colon f(I) \to \mathbb{R}$ exists, and is strictly increasing/decreasing and continuous on $f(I)$.

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22.1 Uniform Continuity

Definition. *Uniform Continuity*
Let $f\colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}$. We say f is *uniformly continuous* over A if:

- For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $|x - y| < \delta$ for all $x, y \in A$.

Note. Our value of δ is independent of our selection of points, i.e. δ does not depend on x or y.

Note. Regular continuity is more concerned with whether or not a function is continuous at a given point, whereas uniform continuity is more concerned with whether or not a function is uniformly continuous *over a given set.*

Proposition 4.21 Every uniformly continuous function is continuous.

Proof. Fix $y = c$. Then for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon$$

whenever $|x - c| < \delta$, for all $x \in \mathbb{R}$. $\square$

Note. We will show later that

$$\{f\colon A \to \mathbb{R} \mid f \text{ is uniformly continuous on } A\} \subsetneq \{f\colon A \to \mathbb{R} \mid f \text{ is continuous on } A\}.$$

22.1.1 Equivalent Formulation for Uniform Continuity

We say a function f is uniformly continuous on A if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{x,y\in A}(|f(x) - f(y)|) < \varepsilon,$$

whenever $|x - y| < \delta$.

Proposition 4.22 Sequential Criterion for Non-uniform Continuity

A function $f\colon A \to \mathbb{R}$ *fails* to be uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and two sequences $(x_n), (y_n)$ in A such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

Proof. ($\Rightarrow$) There exists $\varepsilon_0 > 0$, such that for all $\delta > 0$, we have

$$\sup_{\substack{x,y\in A\\|x-y|<\delta}} |f(x) - f(y)| \geq 2\varepsilon_0.$$

Thus there exists $(x_n), (y_n) \subseteq A$ such that $|x_n - y_n| < \frac{1}{n}$ and

$$\begin{aligned}|f(x_n) - f(y_n)| &\geq \sup_{\substack{x,y\in A\\|x-y|<\delta}} |f(x) - f(y)| - \varepsilon_0\\ &\geq 2\varepsilon_0 - \varepsilon_0\\ &= \varepsilon_0.\end{aligned}$$

($\Leftarrow$) Suppose towards a contradiction that f is uniformly continuous on A. Put $\varepsilon = \varepsilon_0$, then for $\delta = \delta(\varepsilon_0)$, there exists N such that $|x_n - y_n| < \delta$ for all $n > N$. Thus for $n > N$,

$$\begin{aligned}
\varepsilon_0 &\leq |f(x_n) - f(y_n)| \\
&\leq \sup_{\substack{x,y\in A\\|x-y|<\delta}} |f(x) - f(y)| \\
&< \varepsilon_0,
\end{aligned}$$

a contradiction. □

Note. The above proposition does not imply that $x_n \to x$ and $y_n \to x$, both of them can diverge. All it assumes is that $|x_n - y_n| \to 0$.

22.1.2 Enemies of Uniform Continuity

- Rapid growth: Either as $|x| \to +\infty$ or there exists some asymptote in a bounded domain. For example, consider the functions $f(x) = x^2$ on $\mathbb{R}$, and $f(x) = \frac{1}{x}$ on $(0, 1)$.
- Rapid oscillation: Too much variation in small intervals. For example, consider the function $f(x) = \sin(\frac{1}{x})$.

Example. Let $f(x) = x^2$ over $\mathbb{R}$, and $x_n = n$ and $y_n = n + \frac{1}{n}$. Thus we have $|x_n - y_n| = \frac{1}{n} \to 0$. However,

$$\begin{aligned}
|f(x_n) - f(y_n)| &= \left|n^2 - \left(n + \frac{1}{n}\right)^2\right| \\
&= \left|n^2 - \left(n^2 + 2 + \frac{1}{n^2}\right)\right| \\
&= \left|2 + \frac{1}{n^2}\right| \\
&\geq 2.
\end{aligned}$$

Thus f is not uniformly continuous over $\mathbb{R}$.

Theorem — *4.23*
A function that is continuous on a compact set is *uniformly continuous* on that set.

Proof. Suppose towards a contradiction that $f\colon K \to \mathbb{R}$ is continuous but not uniformly continuous, where K is a compact subset of $\mathbb{R}$. By Proposition 4.22, we know there exists $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subseteq K$ such that

$$|x_n - y_n| \to 0 \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Since K is compact, we know there exists some sequence of points (x_{n_k}) such that $x_{n_k} \to x \in K$. We look at the subsequence of (y_{n_k}), and see that $y_{n_k} = y_{n_k} - x_{n_k} + x_{n_k}$. Taking the limit of both sides, we have $y_{n_k} \to x$. Since f is compact, $f(x_{n_k}) - f(y_{n_k}) \to f(x) - f(x) = 0$, a contradiction. □

We are studying uniform continuity because it is useful for integration.

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23.1 Differentiation

The derivative is just the slope of a tangent line to a function at a point. We can get this slope by taking the limit of the slopes of a bunch of secant lines.

Definition. *5.1—Derivative at a Point*
Let $I = (a, b)$ be an interval, $a < b$, $f : I \to \mathbb{R}$ and let $c \in I$. The *derivative of f at c* is defined by:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

provided the limit exists (i.e. it converges to a real number). If $f'(c)$ exists for all $c \in I$, then we say that f is *differentiable* on I.

Note.

1) If we put $h = x - c$, we may rewrite the limit as

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(h)}{h}.$$

2) It is often useful to think of f' as a function in its own right: $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$. We may write $f' \colon \mathrm{dom}(f') \to \mathbb{R}$, where

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(h)}{h}.$$

Example. *Derivatives*

1) Let $f(x) = x^n$, $n \in \mathbb{N}$. We claim that f is differentiable on $\mathbb{R}$ and $f'(x) = nx^{n-1}$.

   Proof. Fix $c \in \mathbb{R}$, so

$$\begin{aligned}\frac{f(x) - f(c)}{x - c} &= \frac{x^n - c^n}{x - c} \\ &= \frac{(x - c)(x^{n-1} + \cdots + c^{n-1})}{x - c} \\ &= \sum_{j=0}^{n-1} x^j c^{n-1-j}.\end{aligned}$$

   Since polynomials are continuous, we may take the limit of this as x goes to c, which is

$$\begin{aligned}f'(c) &= \lim_{x \to c} \left(\sum_{j=0}^{n-1} x^j c^{n-1-j}\right) \\ &= \sum_{j=0}^{n-1} c^j \cdot c^{n-1-j} \\ &= \sum_{j=0}^{n-1} c^{n-1} \\ &= nc^{n-1}.\end{aligned}$$

   □

2) The function $f(x) = |x|$ is differentiable on $\mathbb{R} \setminus \{0\}$, and not differentiable at $x = 0$. At $x \neq 0$, we have $f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$. We see that

$$\lim_{x \to 0^-} \frac{f(x)}{x} = -1 \neq 1 = \lim_{x \to 0^+} \frac{f(x)}{x}.$$

   In particular, this is an example of a function that is continuous everywhere but not necessarily continuous everywhere.

3) Bolzano's function (Weierstrass function): A function that is continuous everywhere but differentiable nowhere.

Proposition 5.2 If a function $f \colon I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

Proof. For $x \neq c$, we may write

$$f(x) = f(c) + \frac{f(x) - f(c)}{x - c} \cdot (x - c).$$

Thus we have

$$\lim_{x \to c} f(x) = f(c) + f'(c) \cdot 0 = f(c).$$

Therefore f is continuous at c. □

Theorem — 5.3
Let $f, g\colon I \to \mathbb{R}$ and f, g are differentiable at $c \in I$. Then:

(i) For all $a, b \in \mathbb{R}$, we have $af + bg$ is differentiable at c, and
$$(af + bg)'(c) = af'(c) + bg'(c).$$

(ii) Leibniz Rule: We know that fg is differentiable at c, in particular
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

Proof. When $x \neq c$, we have
$$\frac{f(x)g(x) - f(c)g(c)}{x - c} = f(x) \cdot \frac{g(x) - g(c)}{x - c} + g(x) \cdot \frac{f(x) - f(c)}{x - c}.$$
Taking the limit, we have what we are looking for. $\square$

(iii) If $g(c) \neq 0$, then $\frac{f}{g}$ is differentiable at c, and
$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Proposition 5.4 (Chain Rule): Let $f\colon I \to \mathbb{R}$, $g\colon J \to \mathbb{R}$ such that $f(I) \subseteq J$, so $g \circ f$ is well-defined on I. Then if f is differentiable at $c \in I$ and g is differentiable at $g(c) \in J$, then $g \circ f\colon I \to \mathbb{R}$ is differentiable at c and
$$(g \circ f)'(c) = f'(c) \cdot g'(f(c)).$$

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Proposition 5.5 Let $I = (a, b)$, $f\colon I \to \mathbb{R}$, $c \in I$. Then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists in } \mathbb{R},$$

if and only if there exists $L \in \mathbb{R}$, $R\colon I \to \mathbb{R}$ such that $\lim_{x \to c} R(x) = 0$ and

$$f(x) = f(c) + (x - c)L + (x - c)R(x).$$

Note. The above more or less states that the derivative of f at a point c exists if and only if when we approximate f with a line, the remainder function goes to 0. Additionally, it says that if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c),$$

then $f'(c) = L$.

Proof. ($\Rightarrow$) Assume $\dfrac{f(x) - f(c)}{x - c} = L \in \mathbb{R}$. Define

$$R(x) \coloneqq \begin{cases} \frac{f(x) - f(c)}{x - c} - L & \text{if } x \neq c, \\ 0 & \text{if } x = \text{c}. \end{cases}$$

Then

$$\lim_{x \to c} R(x) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} - L \right) = L - L = 0.$$

Thus for $x \neq c$, we have $(x - c)(L + R(x)) = f(x) - f(c)$.

($\Leftarrow$) If $x \neq c$, we write

$$\frac{f(x) - f(c)}{x - c} = L + R(x).$$

Thus by Algebraic Limit Theorem,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L + 0 = L.$$

$\square$

Note. We can also rewrite the linear approximation by

$$f(c + h) = f(c) + hL + hR(h),$$

for some other remainder function where

$$\lim_{h \to 0} R(h) = 0.$$

We now have the tools necessary to prove the Chain Rule.

Proof. By Proposition 5.5, we may write

$$\begin{aligned} f(c + h) &= f(c) + f'(c)h + hR_1(h) \\ g(f(c) + \widetilde{h}) &= g(f(c)) + \widetilde{h}g'(f(c)) + \widetilde{h}R_2(\widetilde{h}). \end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{g(f(c+h))-g(f(c))}{h} &= \frac{1}{h}\left(g(f(c)+\underbrace{hf'(c)+hR_1(h)}_{\widetilde{h}})-g(f(c))\right)\\
&= \frac{1}{h}\left(g(f(c)+\widetilde{h})-g(f(c))\right)\\
&= \frac{1}{h}\left(g(f(c))+\widetilde{h}g'(f(c))+\widetilde{h}R_2(\widetilde{h})-g(f(c))\right)\\
&= \frac{\widetilde{h}}{h}\left(g'(f(c))+R_2(\widetilde{h})\right)\\
&= (f'(c)+R_1(h))\cdot(g'(f(c))+R_2(\widetilde{h}))\\
&= (f'(c)+R_1(h))\cdot(g'(f(c))+R_2(hf'(c)+hR_1(h))).
\end{aligned}$$

Note that by taking the limit as $h\to 0$, we may apply the Algebraic Limit Theorem, getting:

$$\begin{aligned}
\lim_{h\to 0}\frac{(g\circ f)(c+h)-(g\circ f)(c)}{h} &= \lim_{h\to 0}(f'(c)+R_1(h))\cdot(g'(f(c))+R_2(hf'(c)+hR_1(h)))\\
&= f'(c)\cdot(g'(f(c))+0)\\
&= f'(c)\cdot g'(f(c)).
\end{aligned}$$

Hence $(g\circ f)'(c)$ exists and is equal to $f'(c)\cdot g'(f(c))$. □

Question. Is the derivative of a function a continuous function?
In general, no. Consider the function defined for all $n\in\mathbb{N}\cup\{0\}$ by

$$f_n(x)=\begin{cases} x^n\sin\left(\frac{1}{x}\right) & x\neq 0,\\ 0 & x=0.\end{cases}$$

When $n=0$, we see that f_0 is not even continuous at $x=0$.

When $n=1$, f_1 is continuous at $x=0$ because of Squeeze Theorem. When $x\neq 0$, $f_1'(x)=\sin(\frac{1}{x})-\frac{1}{x}\cos(\frac{1}{x})$, which doesn't have a limit as x goes to 0. Similarly, if we take a difference quotient, we still get that the limit doesn't exist so f_1 is not differentiable at $x=0$.

When $n=2$, we have that f_2 is continuous at $x=0$ by Squeeze Theorem. Through some algebra, we see that f_2 is differentiable on $\mathbb{R}\setminus\{0\}$, and by taking a difference quotient we are able to see that f_2 *is* differentiable at $x=0$. However, f_2' is not continuous on $\mathbb{R}$.

Note. It is possible to have a function that is differentiable everywhere, but that derivative is *not* necessarily continuous everywhere.

24.1 Properties of Derivatives

We can use derivatives to help us locate extrema.
Proposition 5.6 (Interior Extremum Theorem) Let $I=(a,b)$, $f\colon I\to\mathbb{R}$, $c\in I$. If c is an extremum point for f (i.e. a max or min), and f is differentiable at c, then $f'(c)=0$.

Proof. Since I is an interval, we can find sequences $x_n<c<y_n$ such that $x_n\to c$ and $y_n\to c$. Suppose f has a maximum at c. Then $f(y_n)\leq f(c)$ for all $n\in\mathbb{N}$, so $f(y_n)-f(c)\leq 0$. Thus

$$f'(c)=\lim_{n\to\infty}\frac{f(y_n)-f(c)}{y_n-c}\leq 0.$$

Similarly, we have $f(x_n)\leq f(c)$ for all $n\in\mathbb{N}$, so $f(x_n)-f(c)\leq 0$. Thus

$$f'(c)=\lim_{n\to\infty}\frac{f(x_n)-f(c)}{x_n-c}\geq 0.$$

Hence $f'(c)=0$. □

Note. A local maxima of f at $c \in (a, b)$ exists if there exists $\delta > 0$ such that $f(x) \leq f(c)$ for all $x \in (c - \delta, c + \delta)$.

Note. The converse of Proposition 5.6 is false, i.e. if there exists c such that $f'(c) = 0$, it *is not necessarily true* that $f(c)$ is an extremum. For instance, consider $f(x) = x^3$ at $c = 0$.

Proposition 5.7 (Location of Extrema) Let $f\colon [a, b] \to \mathbb{R}$, continuous on $[a, b]$, differentiable on (a, b). Then f has extrema in $[a, b]$ and they occur:

(i) At a.

(ii) At b.

(iii) At some $c \in (a, b)$ where $f'(c) = 0$.

Example. Consider the function $f(x) = x^2$ over the interval $[-1, 1]$. Then we have maxima at the endpoints, $c = \pm 1$. The minima is at $0 \in (-1, 1)$, and $f'(0) = 0$.