Lecture Notes

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1 Lecture 1

The goal of this class is to *quantify randomness*. The main topics for the term are:

1. The fundamentals of probability theory, including conditional probability and enumeration arguments.
2. Discrete and continuous random variables.
3. Sequences of i.i.d. random variables, including the Weak Law of Large Numbers and the Central Limit Theorem.

1.1 Properties of Probability

Probability theory takes place inside a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition. *Probability Space*
A *probability space* is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying:

1. A non-empty set Ω, called the *sample space*.
2. A set $\mathcal{F}$ of subsets of Ω satisfying certain properties:
   - Elements of $\mathcal{F}$ are called *events*.
   - Events $A_1, A_2, \ldots, A_k$ are called *mutually exclusive* if they are *pairwise disjoint*, i.e. if $i \neq j$ then $A_i \cap A_j = \varnothing$.
   - Events $A_1, A_2, \ldots, A_k$ are called *exhaustive* if their union is the sample space, i.e.

$$\bigcup_{j=1}^{k} A_j = \Omega.$$

   - For this class you may ignore $\mathcal{F}$ and assume that all subsets of Ω are events.
3. A function $\mathbb{P}\colon \mathcal{F} \to [0,1]$, called a *probability measure*, which satisfies:
   - $\mathbb{P}[\Omega] = 1$, or "the probability that something happens is 1".
   - If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] = \sum_{j=1}^{n} \mathbb{P}[A_j].$$

   - If $A_1, A_2, \ldots$ are mutually exclusive events, then

$$\mathbb{P}\left[\bigcup_{j=1}^{\infty} A_j\right] = \sum_{j=1}^{\infty} \mathbb{P}[A_j].$$

Example. Suppose I flip two fair coins. Then the sample space can be written as $\Omega = \{HH, HT, TH, TT\}$. The probability measure should be defined as

$$\begin{aligned} P[HH] &= \frac{1}{4} \\ P[HT] &= \frac{1}{4} \\ P[TH] &= \frac{1}{4} \\ P[TT] &= \frac{1}{4}. \end{aligned}$$

The probability of getting exactly one head is hence $\mathbb{P}[\{HT, TH\}] = \mathbb{P}[HT] + \mathbb{P}[TH] = \frac{1}{2}$.

Theorem. $\mathbb{P}[\varnothing] = 0$.

Proof. We know that Ω and $\varnothing$ are mutually exclusive, since, $\Omega \cap \varnothing = \varnothing$. Thus

$$\begin{aligned} \mathbb{P}[\Omega] &= \mathbb{P}[\Omega \cup \varnothing] \\ &= \mathbb{P}[\Omega] + \mathbb{P}[\varnothing], \end{aligned}$$

and so $\mathbb{P}[\varnothing] = 0$. $\square$

Theorem. If $A \subseteq \Omega$ is an event and $A' = \Omega \setminus A$ then

$$\mathbb{P}[A] = 1 - \mathbb{P}[A'].$$

Proof. Since we have that $A' = \Omega \setminus A$, we know that $A' \cap A = \varnothing$, so they are mutually exclusive. Thus we have

$$\begin{aligned} \mathbb{P}[\Omega] &= \mathbb{P}[A \cup A'] \\ 1 &= \mathbb{P}[A] + \mathbb{P}[A'] \\ \mathbb{P}[A] &= 1 - \mathbb{P}[A']. \end{aligned}$$

$\square$

Theorem. If $A \subseteq B$ then

$$\mathbb{P}[B \setminus A] = \mathbb{P}[B] - \mathbb{P}[A].$$

Proof. We know that $B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \varnothing$. Hence

$$\mathbb{P}[B] = \mathbb{P}[A \cup (B \setminus A)] = \mathbb{P}[A] + \mathbb{P}[B \setminus A],$$

and the result follows. $\square$

Theorem. If $A \subseteq B$ then $\mathbb{P}[A] \leq \mathbb{P}[B]$.

Proof. From the previous theorem we have

$$\mathbb{P}[A] \leq \mathbb{P}[A] + \mathbb{P}[B \setminus A] = \mathbb{P}[B].$$

$\square$

2 Lecture 2

2.1 Inclusion-Exclusion Principle

Theorem — *Inclusion-Exclusion Principle*
If $A, B \subseteq \Omega$ are events, then

$$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B].$$

Proof. Observe that we may write

$$A \cup B = [A \setminus (A \cap B)] \cup [A \cap B] \cup [B \setminus (A \cap B)],$$

where $A \cap B$, $B \setminus (A \cap B)$, and $A \setminus (A \cap B)$ are mutually exclusive. Hence

$$\begin{aligned}\mathbb{P}[A \cup B] &= \mathbb{P}[A \setminus (A \cap B)] + \mathbb{P}[B \setminus (A \cap B)] + \mathbb{P}[A \cap B] \\ &= (\mathbb{P}[A] - \mathbb{P}[A \cap B]) + (\mathbb{P}[B] - \mathbb{P}[A \cap B]) + \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B].\end{aligned}$$

□

Theorem — *Union Bound*
If $A_1, A_2, \dots, A_n \subseteq \Omega$ are events, then

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] \leq \sum_{j=1}^{n} \mathbb{P}[A_j].$$

Proof. We proceed via proof by induction. Observe that for $n = 1$, we have $\mathbb{P}[A_1] \leq \mathbb{P}[A_1]$, which is obviously true. Suppose that this statements holds for some $k \geq 1$. Then

$$\begin{aligned}\mathbb{P}\left[\bigcup_{j=1}^{k+1} A_j\right] &= \mathbb{P}\left[\left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}\right] \\ &= \mathbb{P}\left[\bigcup_{j=1}^{k} A_j\right] + \mathbb{P}[A_{k+1}] - \mathbb{P}\left[\left(\bigcup_{j=1}^{k} A_j\right) \cap A_{k+1}\right] \\ &\leq \mathbb{P}\left[\bigcup_{j=1}^{k} A_j\right] + \mathbb{P}[A_{k+1}] \\ &\leq \sum_{j=1}^{k} \mathbb{P}[A_j] + \mathbb{P}[A_{k+1}] \\ &= \sum_{j=1}^{k+1} \mathbb{P}[A_j].\end{aligned}$$

Hence the statement holds for $k + 1$, and so holds for all natural numbers n. □

2.2 Mutual Independence

Definition. *Independence*
We say that two events $A, B \subseteq \Omega$ are *independent* if

$$\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B].$$

If two events are not independent, then we say that they are *dependent*.

Definition. *Mutual Independence*
We say that events $A_1, \dots, A_n \subseteq \Omega$ are *mutually independent* if, given any $1 \leq k \leq n$ and $1 \leq j_1 < j_2 \cdots < j_k \leq n$ we have

$$\mathbb{P}\left[\bigcap_{\ell=1}^{k} A_{j_\ell}\right] = \prod_{\ell=1}^{k} \mathbb{P}[A_{j_\ell}].$$