Lecture Notes

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2022-01-04

1 Lecture 1

1.1 Introduction to Dynamical Systems

Models of real-world phenomena can often be classified as either *static* or *dynamic*. Furthermore, these systems can either be *discrete* ($x_1, x_2, \dots$ where $x_i \in \mathbb{R}$ for $i \geqslant 1$) or *continuous* ($x = x(t)$ where $t \geqslant 0$ and $x \in \mathbb{R}$, and $\dot{x} = f(x)$).

1.1.1 Where Do "Dynamical Systems" Come From?

1. Observed phenomena
2. Mathematical model
3. "Solve" the model
4. Make predictions

1.2 Autonomous ODEs

Definition. *Autonomous ODEs*
We say that an ordinary differential equation is *autonomous* if the right-hand side does not depend on t.

- The SIR (susceptible, infected, recovered) model is an example of a *first order* system of *autonomous* ODEs.

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_1(x_1, \dots, x_n) \end{cases}$$

- We will refer to n as the *dimension* of the system.

2 Lecture 2

2.1 Reducing ODEs to First Order Autonomous Systems

Consider the set of differential equations given by

$$\begin{cases} \dot{x} = -\kappa(t)xy, \\ \dot{y} = \kappa(t)xy - \delta y, \\ \dot{z} = \delta y. \end{cases}$$

Introduce a new variable, i.e. $\tau = \tau(t) = t$. Then we may rewrite the above as

$$\begin{cases} \dot{x} = -\kappa(\tau)xy, \\ \dot{y} = \kappa(\tau)xy - \delta y, \\ \dot{z} = \delta y, \\ \dot{\tau} = 1. \end{cases}$$

Note that the above system is now autonomous.

Example. *The Pendulum*
We can model the angle θ of a pendulum of length $L > 0$ by

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0.$$

Applying Newton's Second Law, we can get the equations

$$\begin{aligned} mL\ddot{\theta} &= -mg\sin\theta \\ \theta &= \theta(t). \end{aligned}$$

Observe that if we let $x = \theta$ and $y = \dot{\theta}$, then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L}\sin\theta. \end{cases}$$

Example. *Pendulum with an external force*
If we add an external force to our pendulum, then we get

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{1}{m}F(t).$$

Thus if we let $x = \theta$, $y = \dot{\theta}$, and $z = t$, then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L}\sin x + \frac{1}{m}F(z) \\ \dot{z} = 1. \end{cases}$$

Note. In general, higher order ODEs of the form

$$\frac{\mathrm{d}^k x}{\mathrm{d}t^k} = f(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \dots, \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}})$$

can be converted into a first order system by taking

$$z_1 = x, z_2 = \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, z_k = \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}}.$$

We get the system

$$\begin{cases} \dot{z}_1 = \frac{\mathrm{d}x}{\mathrm{d}t} = z_2 \\ \dot{z}_2 = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = z_3 \\ \vdots \\ \dot{z}_k = f(z_1, z_2, \ldots, z_k) \end{cases}$$

2.1.1 Flows on the Line

We will now consider systems of the form

$$\dot{x} = f(x)$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function.

Example. Consider the ODE given by

$$\dot{x} = x(x+1)(x-1)^2.$$

We could use separation of variables to solve this.

Note. Solutions to ODEs usually come in three different flavors:

- Analytic methods (separation of variables)
- Geometric methods (direction fields)
- Numerical methods (Euler's method)

Definition. *Phase Space*
To help us analyze these differential equations, we can plot $\dot{x}$ against x on a graph, and see the behavior around zeroes. This is called a *phase space*. If some neighborhood of points around a zero x tend towards x, then x is called a *stable point*. If they tend to move away from x, then x is an *unstable point*. On a phase space graph, we denote stable points with $\bullet$, unstable points with $\circ$, and other points with a half-filled circle.

2.1.2 Fixed Points

Definition. *Fixed Point*
We say that x^* is a *fixed point* of the system

$$\dot{x} = f(x)$$

if $f(x^*) = 0$. If x^* is a fixed point then the system has a constant solution given by $x(t) = x^*$. These points are also known as equilibrium points, stationary points, rest points, critical points, and steady states.

2.1.3 Stability

Definition. *Stability*
Let x^* be a fixed point of the system
$$\dot{x} = f(x).$$
For now, we say that x^* is:

- *Stable* if solutions starting close to x^* approach x^* as $t \to \infty$.
- *Unstable* if solutions starting close to x^* diverge from x^* as $t \to \infty$.
- *Half-stable* if solutions starting close to x^* approach x^* from one side, but diverge from the other side.

3 Lecture 3

Question. Can we say more about what happens close to fixed points?

Example. Consider the equation given by

$$\dot{x} = x(x+1)(x-1)^2,$$

which has stable points at -1, 0, and 1.
We will use something called the *local method*. We define a new function $\eta(t) = x(t) + 1$, so $x(t) = \eta(t) - 1$. Hence $\dot{x}(t) = \dot{\eta}(t)$. Furthermore,

$$\begin{aligned} x(x+1)(x-1)^2 &= (\eta - 1)\eta(\eta - 2)^2 \\ &= -4\eta + O(\eta^2), && (\eta \to 0) \end{aligned}$$

so $\dot{\eta} \approx -4\eta$. Near $x = -1, \eta = x + 1$ and it satisfies $\dot{\eta} = -4\eta$, so $\eta(t) \approx Ce^{-4t}$. We can see that this approaches 0 as $t \to \infty$, so points around $x = -1$ will approach -1.

In general, we have the following method:
Assume that x^* is a fixed point of $\dot{x} = f(x)$, i.e. $f(x^*) = 0$. Let $\eta = x - x^*$. Then

$$\begin{aligned} \dot{\eta} &= \dot{x} \\ &= f(x) \\ &= f(x^*) + f'(x^*)\eta + \frac{1}{2}f''(x^*)\eta^2 + \cdots \\ &= f'(x^*)\eta + O(\eta^2). && (\eta \to 0) \end{aligned}$$

Hence the equation

$$\dot{\eta} = f'(x^*)\eta$$

is the *linearization* at $x = x^*$. We know that the solution to such a differential equation is

$$\eta(t) = Ce^{f'(x^*)t} = \begin{cases} 0, & f'(x^*) < 0 \\ \pm\infty, & f'(x^*) > 0 \end{cases},$$

as $t \to \infty$. In the first case, the terms near x^* will tend towards x^*, and in the latter they will diverge from x^*.

Theorem. Suppose that x^* is a fixed point of the system $\dot{x} = f(x)$. Then if

- $f'(x^*) < 0$, the fixed point x^* is stable.
- $f'(x^*) > 0$, the fixed point x^* is unstable.

Question. What happens if $f'(x^*) = 0$? Anything can happen/the test is inconclusive.

- Consider the equation $\dot{x} = x^3$. We see that $x^* = 0$ is a critical point, and that $f'(x) = 3x^2$, so $f'(x^*) = 0$. Using our usual graphical methods, we can see that points to the left of x^* will approach x^*, and so will points to the right, and so $x^* = 0$ is a stable point.
- If we use the equation $\dot{x} = -x^3$, we get the direct opposite, that is $x^* = 0$ is unstable despite having the same critical point.
- If we look at the behavior of $\dot{x} = x^2$, then we get that the critical point at $x^* = 0$ is half-stable.
- If we consider the equation $\dot{x} = 0$, then every number on the real line is a critical point, and so the solutions don't move at all.

3.1 Potentials

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be smooth and consider the system

$$\dot{x} = f(x).$$

- A function $V\colon \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = -V'(x)$$

  is called a *potential* for f.

- Our system can be written as a *gradient flow*

$$\dot{x} = -V'(x).$$

Note. Potential functions are *not* unique, since you can always add a constant.