Math 131A Lecture Notes

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Contents

1 Lecture 1

1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

1.3 Logical Connections

We usually use the letters P and Q to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: "P and Q", $P \land Q$
2. Disjunctions: "P or Q", $P \lor Q$
3. Implications: "If P, then Q", $P \implies Q$
   (a) If the proposition is false (i.e. if P is false) then the whole statement is true.

   > **Definition.**
   > We say that the statement is *vacuously true.*

4. Negations: "Not P", $\neg P$

1.3.1 Truth Tables

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Prove that if n is an integer, then $n(n+1)$ is even.

Proof. Suppose that n is an integer. Then we have two cases, where either n is even or n is odd. Let n be an even integer such that $n = 2k$ where $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= 2k(2k+1) \\ &= 2(2k^2+k). \end{aligned}$$

Thus we see that $n(n+1)$ is even when n is even. Now let n be odd such that $n = 2m+1$ where $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1). \end{aligned}$$

Thus $n(n+1)$ is also even when n is odd, and so is even for all integers n. $\square$

2 Lecture 2

2.1 Continuation of Logic

2.1.1 De Morgan's Laws

$$\neg(P \lor Q) = \neg P \land \neg Q$$
$$\neg(P \land Q) = \neg P \lor \neg Q$$

Note. Negations turn "and" into "or" and vice versa.

Example. Suppose we have the following statement:

$$P\colon\ x \text{ is even and } x > 0.$$

Then the negation of P would be:

$$\neg P\colon\ x \text{ is odd or } x \leq 0.$$

2.1.2 Converse

Definition. *Converse*
The *converse* of a statement $P \implies Q$ is the statement $Q \implies P$. In general, the converse of a statement says nothing about the original statement.

Example. Consider the statement

$$\text{If } x > 0, \text{ then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0, \text{ then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write $P \iff Q$ instead of $(P \implies Q) \land (Q \implies P)$. In this case, we call P and Q *logically equivalent*. In writing, we say "P if and only if Q".

2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

Lemma 1. Let a be an integer. If a^2 is even, then a is even.

Proof. Suppose a is odd, so $a = 2k + 1$ for some integer k. Then

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus a^2 is odd and this completes the proof. $\square$

2.1.4 Variables and Quantifiers

We have a value x that varies over some values, so we use $P(x)$ to denote a statement that depends on the value of x.

Example. Consider the statement
$$P(x) : x + 2 = 3.$$
The statement is true if and only if $x = 1$.

We have two quantifiers—$\forall$ = "for all", and $\exists$ = "there exists".

- $\forall x : P(x)$ is true if $P(x)$ is true for all x.
- $\exists x : P(x)$ is true if there exists at least one x such that $P(x)$ is true.

We can also have nested quantifiers,
$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

Note. The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of n depends on m.

2.1.5 Proof by Counterexample

After "simplifying" the statement $\neg(\forall x : P(x))$, we get $\exists x : \neg P(x)$. We simply need to find a single counterexample to show that a statement is false for all x.

Example. Consider the statement $\forall x \in \mathbb{R} : x + 2 = 3$. All we need to do is show that there exists some $x \in \mathbb{R}$ such that $x + 2 \neq 3$. This occurs when $x = 0$, so the statement is false.

2.1.6 Proof by Contradiction

Key Idea. We want to show that $P \implies Q$ indirectly.

Lemma 2. We can show that
$$P \implies Q = (\neg P) \lor Q.$$
Then $P \implies Q$ is true if and only if $\neg(P \implies Q)$ is false, and so by Lemma 2 and De Morgan's Laws, $P \land \neg Q$ is false.

For proof by contradiction, we assume P is true and $\neg Q$ is true, and try to show that $P \land \neg Q$ is false (a contradiction).

3 Lecture 3

3.1 More Logic

3.1.1 Proof by Contradiction

To do a proof by contradiction for a statement $P \implies Q$, we assume $P \wedge \neg Q$. We aim to show that $P \wedge \neg Q$ is false (a contradiction).

Theorem — *Irrationality of* $\sqrt{2}$
There is no rational number x such that $x^2 = 2$. In other words, if $x \in \mathbb{Q}$, then $x^2 \neq 2$.

Proof. Suppose towards a contradiction that there exists some $x \in \mathbb{Q}$ such that $x^2 = 2$. Since x is rational, there exist integers p, q such that $q \neq 0$, $\frac{p}{q} = x$, and p and q have no common divisors (other than 1). Then

$$\begin{aligned} x^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since p^2 is even, there exists some integer k such that $p = 2k$. Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

By the same logic as before, we know that q must also be even (they share a common factor of 2). However, this contradicts our original assumption that p and q share no common factors, and this completes the proof. □

3.2 Set Theory

We write $x \in A$ when we want to say that "x is an element of A", and $x \notin A$ when we want to say that "x is not an element of A".

3.2.1 Set Combinations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$.
- Subset (Inclusion): $A \subseteq B$ if and only if $x \in A \implies x \in B$.

  Definition. *Proper Subset*
  A set A is a *proper subset* of a set B if $A \subseteq B$ and there exists some $x \in B$ such that $x \notin A$. We denote this as $A \subset B$.

- Equality: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

  Note (Showing Equality of Sets). If you want to show $A = B$, you need to show both $A \subseteq B$ and $B \subseteq A$. In other words, you must show that for all $x \in A$, we have $x \in B$, and vice versa.

Example. We have $\mathbb{N} = \{1, 2, 3, \dots\}$.

- Let E be the set of even natural numbers. Note that $E \subseteq \mathbb{N}$.
- Let $S = \left\{p \in \mathbb{Q} \mid p^2 < 2\right\} \subseteq \mathbb{Q}$.

Then we have:

- $\mathbb{N} \cup E = \mathbb{N}$.
- $\mathbb{N} \cap E = E$.
- $\mathbb{N} \cap S = \left\{n \in \mathbb{N} \mid n^2 < 2\right\} = \{1\}$.
- $E \cap S = \varnothing$.

  Definition. *Disjoint Sets*
  If $A \cap B = \varnothing$, we call A and B *disjoint* sets.

  Proof. Suppose towards a contradiction that there exists some $x \in E \cap S$, which is to say $x \in E$ and $x \in S$. Since $x \in E$, we know that x is even, and so there exists some integer k such that $x = 2k$. Then
  $$x^2 = (2k)^2 = 4k^2,$$
  so $4 \mid x^2$. Therefore $x \geq 4$, which contradicts the condition for $x \in S$, namely $x^2 < 2$. □

- Given some $n \in \mathbb{N}$, we define $A_n = \mathbb{N} \setminus \{1, \dots, n-1\}$.
  - $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$.
  - $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$.

Definition. *Set Complement*
If $A \subseteq B$, then we define the *complement* of A in B to be $A^c = B \setminus A$.

3.2.2 De Morgan's Laws

If I is an index set and $\{A_j\}_{j \in I}$ are subsets of B, then

$$\left(\bigcup_{j \in I} A_j\right)^c = \bigcap_{j \in I} A_j^c, \quad \text{and} \quad \left(\bigcap_{j \in I} A_j\right)^c = \bigcup_{j \in I} A_j^c$$

4 Lecture 4

4.1 Cartesian Product

If I have two sets A and B, then we may form their *Cartesian Product*, which is

$$A \times B = \{(x, y) \mid x \in A \land y \in B\}.$$

Definition. *Binary Relation*
A *binary relation* is a subset $R \subseteq A \times B$. We say $x \in A$ is in relation to $y \in B$ if $(x, y) \in R$. We denote this by

$$xRy \iff (x, y) \in R.$$

Example. Consider the relation

$$\mathbb{R} \times \mathbb{R} \supseteq R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Then the relation is *reflexive*, because xRx. It is also antisymmetric, because $xRy \land yRx \implies x = y$. Finally, this relation is transitive, because $xRy \land yRz \implies xRz$.

These properties only make sense if $A = B$, i.e. $R \subseteq A \times A$, and we say that "R is a relation on A".

Definition. *Partial Order*
If a relation is reflexive, antisymmetric, and transitive on A, then it is a *partial order* on A.

The notion of "less than or equal to" is a partial order for $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, but there exists no partial order for $\mathbb{C}$.

Definition. *Power Set*
For a set A, we may define its *power set* by

$$\mathscr{P}(A) = \{C \mid C \subseteq A\}.$$

Note that set inclusion is a partial order on $\mathscr{P}(A)$.

Definition. *Equivalence Relation*
An *equivalence relation* R over A is a relation that is reflexive, symmetric, and transitive.

Note. Just like a partial order behaves much like $\leq$, an equivalence relation behaves much like $=$.

Definition. *Equivalence Class*
Given an equivalence relation on A.0, we define a new set

$$[x] \coloneqq \{y \in A \mid x \sim y\}.$$

We call $[x]$ the *equivalence class* of x. Any $z \in [x]$ is called a *representative* of the equivalence class $[x]$. In particular, x is a representative of its own equivalence class.

Let A be a set with equivalence relation $\sim$. Then for any $x, y \in A$,

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \varnothing.$$

Proof. Let $x, y \in A$. We know that x is either equivalent to y or it is not. Suppose the former is true and let $z \in [x]$. Thus we know that $z \sim x$ and $x \sim y$, and by transitivity we have $z \sim y$. Thus $z \in [y]$ and $[x] \subseteq [y]$. The reverse argument is the same.

If x is not equivalent to y, then suppose towards a contradiction that $[x] \cap [y] \neq \varnothing$. Let $x \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$. By symmetry we know that $x \sim z$ and by transitivity we have $x \sim y$. We have arrived at the contradiction that x is both equivalent and not equivalent to y. $\square$

Definition. *Function*
A relation $R \subseteq A \times B$ is a *function* if for all $x \in A$ and all $y, z \in B$, we have the following:

- $xRy \wedge xRz \implies y = z$.

In other words, every input x has only one output.

Definition. *Injective Functions*
A function f is *injective* if $f(x_1) = f(x_2) \implies x_1 = x_2$.

Definition. *Surjective Functions*
A function f is *surjective* if for every $y \in B$, there exists some $x \in A$ such that $f(x) = y$.

5 Lecture 5

5.1 The Natural Numbers

We know that the natural numbers are defined via:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

5.1.1 Properties of the Natural Numbers

(P1) $1 \in \mathbb{N}$

(P2) If $n \in \mathbb{N}$, then it has a *successor*, $n + 1 \in \mathbb{N}$

(P3) 1 is *not* the successor of any element of $\mathbb{N}$

(P4) If m, n have the same successor, then $m = n$

Note. The properties above can be abstracted to become:

(P1) $1 \in \mathbb{N}$

(P2) There exists some $S \colon \mathbb{N} \to \mathbb{N}$ where $S(n)$ is the successor n

(P3) $1 \notin \operatorname{range} S$

(P4) S is injective

(P5) Suppose $A \subseteq \mathbb{N}$ with the properties:

(i) $1 \in A$

(ii) If $n \in A$, then $S(n) \in A$

Then $A = \mathbb{N}$.

Theorem — *1.1 (Induction)*
Let $\{P(n) \mid n \in \mathbb{N}\}$ be a set of logical propositions. Suppose that

(i) $P(1)$ is true.

(ii) If $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $A = \{n \mid P(n) \text{ is true}\} \subseteq \mathbb{N}$. By our first assumption, $1 \in A$. By (ii), if $n \in A$, then $n + 1 \in A$. So by (P5) of the natural numbers, we know that $A = \mathbb{N}$. $\square$

Definition. *1.2 (Peano Axioms)*
A triplet $(\mathbb{N}, 1, S)$ is said to be a *system of the naturals* if it satisfies:

1) $\mathbb{N}$ is a set and $1 \in \mathbb{N}$

2) $S \colon \mathbb{N} \to \mathbb{N}$ is a function

3) $1 \notin \operatorname{range} S$

4) S is injective

5) $\forall A \subseteq N$ such that $1 \in A$ and $S(A) \subseteq A$, then $A = \mathbb{N}$

Definition. *Addition*
We define the binary relation $+$ over $\mathbb{N}$:

(i) $\forall n \in \mathbb{N}$, $n + 1 := S(n)$

(ii) $\forall m, n \in \mathbb{N}$, we have $m + S(n) = S(m + n)$

The following properties can be proven from the above definition of addition:

(a) Associativity: $\forall x, y, z \in \mathbb{N}$, we have $(x + y) + z = x + (y + z)$

(b) Commutativity: $\forall x, y \in \mathbb{N}$, we have $x + y = y + x$

(c) Cancellative Law: $\forall x, y, z \in \mathbb{N}$, we have $x + y = y + z \implies x = z$

Theorem — *1.3 (Existence of the Naturals)*
There exists a unique system of the naturals. In other words, there is a bijection between different systems of the naturals and so we may write:

$$(\mathbb{N}, 1, S) \iff (\mathbb{N}', 1', S').$$

5.2 Fields

Definition. *Field*
A *field* is a set with two binary operations,

- $+$, or 'addition'
- $\cdot$, or 'multiplication'

5.2.1 Axioms for Addition

(A1) $x \in \mathbb{F} \land y \in \mathbb{F} \implies x + y \in \mathbb{F}$

(A2) $\forall x, y \in \mathbb{F}$, we have $x + y = y + x$

(A3) $\forall x, y, z \in \mathbb{F}$, we have $(x + y) + z = x + (y + z)$

(A4) There exists some $0 \in \mathbb{F}$ such that $0 + x = x$ for all $x \in \mathbb{F}$

(A5) $\forall x \in \mathbb{F}$, there exists $-x \in \mathbb{F}$ such that $x + (-x) = 0$

5.2.2 Axioms for Multiplication

(M1) $x \in \mathbb{F} \land y \in \mathbb{F} \implies x \cdot y \in \mathbb{F}$

(M2) $\forall x, y \in \mathbb{F}$, we have $x \cdot y = y \cdot x$

(M3) $\forall x, y, z \in \mathbb{F}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M4) There exists some $1 \in \mathbb{F}$ such that $1 \neq 0$ and $1 \cdot x = x$ for all $x \in \mathbb{F}$

(M5) $\forall x \in \mathbb{F}$, there exists some $\frac{1}{x} \in \mathbb{F}$ such that $x \cdot \frac{1}{x} = 1$

5.2.3 Distributive Law

(D1) $\forall x, y, z \in \mathbb{F}$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$

6 Lecture 6

6.1 The Rationals

On the natural numbers, we have a notion of addition, multiplication, and comparison ($\leq$). We constructed the integers and so we now have:

- A notion of an additive identity, $0 \in \mathbb{Z}$
- Additive inverses

However, we *don't* have:

- Multiplicative inverses

When we think of the rationals, we consider the set

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Note. When dealing with $\mathbb{Q}$, we now have multiplicative inverses.

In particular, $(\mathbb{Q}, +, \cdot, \leq)$ is an *ordered field*.

Definition. *Ordered Field*
An *ordered field* is a field $\mathbb{F}$ which is also an ordered set ($\leq$) such that:

(i) If $x, y, z \in \mathbb{F}$ and $y < z$, then $x + y < x + z$

(ii) If $x, y \in \mathbb{F}$, $x > 0$ and $y > 0$, then $x \cdot y > 0$

Unfortunately, the rational numbers still don't allow us to solve polynomial equations (i.e. $x^2 = 2$).

Definition. *Algebraic Numbers*
A number is called *algebraic* if it solves

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$, $n \in \mathbb{N}$.

Example. Every rational number is algebraic, because it solves the equation

$$qx - p = 0.$$

Definition. *Dividing*
We say $k \in \mathbb{Z}$ *divides* $m \in \mathbb{Z}$ if $\frac{m}{k} \in \mathbb{Z}$.

Theorem — *Rational Zeros Theorem*
Suppose $c_0, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$ and $r \in \mathbb{Q}$ satisfies

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then writing $r = \frac{c}{d}$ with c, d having no common factors, $d \neq 0$, we have:

$$c \text{ divides } c_0$$
$$d \text{ divides } c_n$$

Proof. Since r solves the equation, we have

$$c_n\left(\frac{c}{d}\right)^n+c_{n-1}\left(\frac{c}{d}\right)^{n-1}+\cdots+c_1\left(\frac{c}{d}\right)+c_0=0.$$

Multiplying both sides by d^n, we get

$$c_nc^n+c_{n-1}c^{n-1}d+\cdots+c_1cd^{n-1}+c_0d^n=0.$$

Rearranging, we have

$$c_nc^n=-d(c_{n-1}c^{n-1}-\cdots+c_1cd^{n-2}-c_0d^{n-1}).$$

We know that $d\mid c_nc^n$. Since c and d have no common factors, we know $d\mid c_n$. Rearranging terms again,

$$-c(c_nc^{n-1}+c_{n-1}c^{n-2}d+\cdots+c_1d^{n-1})=c_0d^n.$$

By the same reasoning as before, we have that $c\mid c_0$. □

Corollary. Suppose $r\in\mathbb{Q}$ solves

$$r^n+c_{n-1}r^{n-1}+\cdots+c_1r+c_0=0.$$

Then $r\in\mathbb{Z}$ and $r\mid c_0$.

Proof. Since $r\in\mathbb{Q}$, we may express $r=\frac{c}{d}$, $c\mid c_0$ and $d\mid 1$. From this we know that $d=1$, so $r=c$ and $c\mid c_0$. Therefore $r\in\mathbb{Z}$ and $r\mid c_0$. □

We have some deficiencies for $\mathbb{Q}$:

- There seem to be some "gaps" in $\mathbb{Q}$.

Proposition. We consider the sets $A=\{p\in\mathbb{Q}\mid p\geq 0\text{ and }p^2<2\}$ and $B=\{p\in\mathbb{Q}\mid p\geq 0\text{ and }p^2>2\}$. Notice that A has no largest element and B has no smallest element.

Proof. Given $p\in\mathbb{Q}$, let

$$q=p-\frac{p^2-2}{p^2+2}=\frac{2(p+1)}{p+2}.$$

We also have $q^2-2=\frac{2(p^2-2)}{(p+2)^2}$. If $p\in A$, $p^2-2<0$ so $q>p$ and $q^2-2>0$. If $p\in B$, $p^2-2>0$ so $q<p$ and $q^2-2>0$. □

7 Lecture 7

Definition. *2.11—Upper and Lower Bounds*
Let E be an ordered set and $A \subset E$.

(a) If there exists $x \in E$ such that for all $a \in A$, $a \leq x$, we say A is *bounded above* by x and call x an *upper bound* for A.

(b) Suppose $A \subset E$ is non-empty and bounded above and there exists some $x^* \in E$ such that

   i) x^* is an upper bound for A

   ii) If y is any upper bound for A^*, then $x^* \leq y$

Then we call x^* the *least upper bound for* A, and we write

$$x^* = \sup A. \qquad \text{(sup meaning supremum)}$$

The *greatest lower bound* or *infimum* of a set B, which is bounded below and non-empty, satisfies

i) $\inf B$ is a lower bound for B

ii) If y is any lower bound for B, then $y \leq \inf B$

Example. Suppose

$$A = \left\{p \in \mathbb{Q} \mid p \geq 0, p^2 < 2\right\},$$
$$B = \left\{p \in \mathbb{Q} \mid p \geq 0, p^2 > 2\right\}.$$

Then A is bounded above (say by 2) and B is bounded below (say by 0). In the last lecture we proved that neither $\sup A$ nor $\inf B$ exist in $\mathbb{Q}$ (because the values would have been $\sqrt{2}$).

Example. Let

$$C = \{p \in \mathbb{Q} \mid p < 0\},$$
$$D = \{p \in \mathbb{Q} \mid p \leq 0\}.$$

Then

$$\sup C = \sup D = 0.$$

However, notice that $\sup C \notin C$ and $\sup D \in D$.

Definition. *Maximum*
We define $\max A$ to be the largest element of A, which satisfies:

i) $\max A \in A$

ii) For all $a \in A$, $a \leq \max A$

The definition for minimum is similar.

Definition. *2.12—Least Upper Bound Property (LUBP)*
An *ordered set* E has the *least upper bound property* if the following is true:

i) If $A \subseteq E$, $A \neq \varnothing$, A is bounded above, then $\sup A$ exists and $\sup A \in E$.

Note. $\mathbb{Q}$ does not have the least upper bound property.

Theorem — *2.13 (Existence of $\mathbb{R}$)*
There exists an ordered field $\mathbb{R}$ which has

i) $\mathbb{Q}$ as a sub-field

ii) The least upper bound property

7.1 Fundamental Properties of the Real Numbers (because of LUBP)

Theorem — *2.14 (Archimedean Property of $\mathbb{R}$)*
If $x, y \in \mathbb{R}$, and $x > 0$, then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Proof. Let $A = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Suppose towards a contradiction that there is no such n that satisfies the statement above. In other words, for all $n \in \mathbb{N}$, $nx \leq y$. Thus A is bounded above by y. Since A is nonempty and is a subset of $\mathbb{R}$, we know that $\sup A$ exists. Consider the value given by $\sup A - x$, which is not an upper bound for A. Then we know there exists some $z \in A$ such that $\sup A - x < z$. Since $z = mx$ (because $z \in A$), we have

$$\begin{aligned} \sup A - x &< z \\ \sup A - x &< mx \\ \sup A &< (m+1)x. \end{aligned}$$

We know that $(m+1)x \in A$, which contradicts the definition of $\sup A$. $\square$

Some remarks:

1. Let $x = 1$. Then $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > y$.

2. Let $y = 1$. Then $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x > \frac{1}{n} > 0$.

Theorem — *2.15 (Density of $\mathbb{Q}$ in $\mathbb{R}$)*
For all $x, y \in \mathbb{R}$, $x < y$, there exists some $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Fix $x < y$. Then by the Archimedean property we have some $n \in \mathbb{N}$ such that $n(y - x) > 1$, or $y - x > \frac{1}{n}$. We may suppose $x > 0$, because otherwise we either have $x < 0 < y$ or $x < y < 0$ (multiply all sides by -1).
We want to show that

$$nx < m < ny.$$

Since $nx + 1 < ny$, we have $nx < m < nx + 1$, or $m - 1 < nx < m$. If $nx \in \mathbb{Z}$, we can take $m = nx + 1$. Thus

$$x < x + \frac{1}{n} = \frac{nx+1}{n} = \frac{m}{n} < \frac{ny}{n} < y.$$

Otherwise we have $nx \notin \mathbb{Z}$. We then apply the following lemma:

Lemma. If $x \in \mathbb{R}$, there exists a $k \in \mathbb{Z}$ such that $k - 1 \leq x \leq k$.

Then $m - 1 < nx < m$, as desired. $\square$

8 Lecture 8

8.1 A Construction of the Real Numbers

We want to show that there exists some field $\mathbb{R}$ such that $\mathbb{Q}$ is a sub-field and $\mathbb{R}$ has the least upper bound property. This construction will be via Dedekind cuts.

Definition. *2.17—Cut*
A *cut* in $\mathbb{Q}$ is a pair of subsets $A, B \subset \mathbb{Q}$ such that:

(i) $A \cap B = \varnothing$, $A \cup B = \mathbb{Q}$, $A, B \neq \varnothing$ (They partition $\mathbb{Q}$)

(ii) If $a \in A$ and $b \in B$, then $a < b$

(iii) A contains no largest element

Example. An example of such a cut could be $A = \{p \in \mathbb{Q} \mid p < 1\}$ and $B = \{p \in \mathbb{Q} \mid p \geq 1\}$. We say that $A \mid B$ is a cut. Another such example is $A = \left\{p \in \mathbb{Q} \mid p \leq 0 \text{ or } p^2 < 2\right\}$ and $B = \left\{p \in \mathbb{Q} \mid p > 0 \text{ and } p^2 \geq 2\right\}$

Definition. *2.18—The Reals*
We may define
$$\mathbb{R} = \{X \mid X \text{ is a cut in } \mathbb{Q}\}.$$

In order to show that the above is a valid definition for $\mathbb{R}$, we must show:

(i) $\mathbb{Q}$ is contained in $\mathbb{R}$ in some “natural way”

(ii) $\mathbb{R}$ is an ordered field

(iii) $\mathbb{R}$ has the least upper bound property

Definition. *2.19—Partial Order for the Reals*
We define a *partial order* on $\mathbb{R}$ as follows: If $X = A \mid B$ and $Y = C \mid D$, then we say $X < Y$ if and only if $A \subset C$ and $X \leq Y$ if and only if $A \subseteq C$.

We will show that the reals contain the rationals.

Proof. We will begin by showing that $\mathbb{Q} \subseteq \mathbb{R}$. We say that $A \mid B$ is a *rational cut* if for some $c \in \mathbb{Q}$, we have $A = \{p \in \mathbb{Q} \mid p < c\}$ and $B = \mathbb{Q} \setminus A$. We will use c^* to denote the rational cut at c. Then we may associate every $c \in \mathbb{Q}$ with a corresponding rational cut $c^* \in \mathbb{R}$. □

Theorem — *2.20 (The Reals have the LUBP)*
With respect to the partial order $\leq$ defined earlier, we may show that $\mathbb{R}$ has the least upper bound property.

Proof. Let $\mathscr{C}$ be any non-empty collection of cuts which are bounded above, say by the cut X. We want to show that $\sup \mathscr{C}$ exists in $\mathbb{R}$, so $\sup \mathscr{C}$ is itself a cut. A candidate for $\sup \mathscr{C}$ is $C \mid D$, where
$$C = \{a \in \mathbb{Q} \mid \text{There exists a cut } A \mid B \in \mathscr{C} \text{ such that } a \in A\}.$$
Now, $Z = C \mid D$ is a cut in $\mathbb{Q}$. We claim that C has no largest element. If $a \in C$, then there exists a cut $A \mid B \in \mathscr{C}$ such that $a \in A$. Since A is a part of a cut, it has no largest element, so there exists some $a' \in A$ such that $a < a'$. Thus $a' \in C$ so C has no largest element.
We must now show that Z is the least upper bound of $\mathscr{C}$. For any $A \mid B \in \mathscr{C}, A \subseteq C$. Therefore $A \mid B \leq C \mid D = Z$ and Z is an upper bound for $\mathscr{C}$. We must now show that it is the *least* upper

bound. Let $Z' = C' \mid D'$ be an upper bound for $\mathscr{C}$. Then $A \mid B \leq C' \mid D'$ and $A \subseteq C'$ for any $A \mid B \in \mathscr{C}$. Now by definition of C, we have $C \subseteq C'$. Therefore $C \mid D \leq C' \mid D'$, so $Z \leq Z'$. We have $Z = \sup \mathscr{C} \in \mathbb{R}$, so $\mathbb{R}$ has the least upper bound property. $\square$

We now show that $\mathbb{R}$ is a field, namely an ordered field. We define the binary relations $+$ and $\cdot$ as follows: Given two cuts $A \mid B$ and $C \mid D$, we may define

$$E = A + C = \{p \in \mathbb{Q} \mid p = a + c \text{ for some } a \in A, c \in C\}.$$

Note. This set summation is known as the Minkowski sum.

We must check that $E \mid F$ is a cut. We must also show that the additive identity for $\mathbb{R}$ is 0, i.e. we must show that $0 + x = x + 0 = x$ for all $x \in \mathbb{R}$. To show the existence of additive inverses, show that for any cut $A \mid B$, there exists some $C \mid D \in \mathbb{R}$ such that $A \mid B + C \mid D = 0$.

Similarly, we define multiplication by

$$E = \{p \in \mathbb{Q} \mid p = ac \text{ for some } a \in A, c \in C\}.$$

8.2 Interesting Questions

(a) Can we cut $\mathbb{R}$ to get something larger? No, because every possible cut in $\mathbb{R}$ is an element of $\mathbb{R}$. This is because $\mathbb{R}$ has the least upper bound property and every cut in $\mathbb{R}$ would just be a "real cut" at $\sup A$.

(b) Is $\mathbb{R}$ unique in some natural way? Yes. If you take any other ordered field $\mathbb{F}$ such that $\mathbb{Q} \subseteq \mathbb{F}$ and $\mathbb{F}$ has the least upper bound property, then there exists some bijection between $\mathbb{F}$ and $\mathbb{R}$.

(c) What about $+\infty$ and $-\infty$? We can't treat these as real numbers using Dedekind cuts, as either A or B would have to be empty.

9 Lecture 9

9.1 Sequences

Example. *Approximating π*
We know that the area of the unit circle should be π. We can approximate the area of a unit circle by inscribing various shapes in the circle and finding their areas (giving us a sequence of lower bounds).
If we inscribe an equilateral triangle in the circle, we find that its side length is $\sqrt{3}$, so the area of the triangle is $a_1 = \frac{3\sqrt{3}}{4} \approx 1.299$.
If we use a square instead, its side length is $\sqrt{2}$, so we have a new lower bound of $a_2 = 2 < \pi$.
Using a regular pentagon, we have a new approximation as given by $a_3 = \frac{5}{2}\sin\left(\frac{2\pi}{5}\right) < \pi$.
Continuing this pattern, we get

$$a_n = \frac{n+2}{2}\sin\left(\frac{2\pi}{n+2}\right) < \pi.$$

Notice that for all $n \in \mathbb{N}$, we have $a_n < a_{n+1} < \pi$. Computing this value for larger n, we have

$$\begin{aligned} a_{100} &\approx 3.1396 \\ a_{1000} &\approx 3.141572 \\ a_{10000} &\approx 3.14159245. \end{aligned}$$

Similarly, we may obtain upper bounds for π by circumscribing regular polygons around the unit circle. By circumscribing a square, we know that $\pi < 4$, so we know that there is some upper bound for what π is equal to. Thus we know that there exists some $a \in \mathbb{R}$ such that $a_n \approx a$ for some sufficiently large n.
In other words, there exists some $a \in \mathbb{R}$ such that a_n converges to a as n approaches ∞.
One issue to note is that we are using π in our formula for a_n to approximate the value of π. Using some trigonometric identities, we may circumvent this by evaluating

$$a_{2n} = \frac{n}{2}\left(n - \sqrt{n - 4a_n^2}\right).$$

Definition. *3.1—Sequences*
A *sequence* is a function $f\colon \mathbb{N} \to \mathbb{R}$. Instead of writing $f(1), f(2), \dots, f(n)$, we tend to write a sequence as $f_1, f_2, \dots, f_n$.

Note (Notations for Sequences). There are many different notations for expressing sequences, a few popular notations being used below:

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}.$$

9.1.1 Behaviors of Sequences

(a) "Convergence"—Getting closer and closer to a given point.

(b) "Divergence"—Getting closer and closer to $\pm\infty$.

(c) "Oscillation"—The sequence does not approach any value in particular.

9.1.2 Facts From the Homework

We define the absolute value function

$$|\cdot| : \mathbb{R} \to [0, \infty) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We have the following properties of the absolute value function:

(i) $|xy| = |x|\,|y|$ for all $x, y \in \mathbb{R}$.

(ii) $|x - y| \leq \ell$ if and only if $y - \ell \leq x \leq y + \ell$, where $\ell \geq 0$ and $x, y \in \mathbb{R}$.

(iii) **Triangle Inequality:** $|x + y| \leq |x| + |y|$.

> **Note.** One consequence of this is
>
> $$\begin{aligned} |x - y| &= |x - z + z - y| \\ &\leq |x - z| + |z - y|. \end{aligned}$$
>
> In other words, the distance between two points x and y is always less than or equal to the sum of the distances of x and y to a third point, z.

We say that a_n approaches a if "$|a_n - a|$ gets arbitrarily small as n gets arbitrarily large".

9.1.3 Convergence

> **Definition.** *3.2—Convergence*
> A sequence (x_n) of real numbers is said to *converge* to an $x \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N > 0$ such that
>
> $$|x_n - x| < \varepsilon$$
>
> for all $n > N$. If (x_n) converges to x, we also write:
>
> $$x_n \to x \text{ as } n \to \infty \quad \text{or} \quad \lim_{n\to\infty} x_n = x.$$
>
> We call x the *limit* of the sequence (x_n). *W*e say that a sequence *diverges* if it does not converge.

> **Note.** By the Archimedean property, we can always take $N \in \mathbb{N}$. In general, N is a function of ε. We fix $\varepsilon > 0$, and use that to find some sufficient N so that the sequence converges.

> **Example.** Consider the sequence $x_n = \frac{1}{n^2}$. We suspect that $x_n \to 0 \in \mathbb{R}$ as $n \to \infty$. Fix $\varepsilon > 0$. If we let $N = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$ and $n > N$, we have
>
> $$\begin{aligned} |x_n - 0| &= \left| \frac{1}{n^2} \right| \\ &= \frac{1}{n^2} \\ &< \frac{1}{N^2} \\ &< \varepsilon. \end{aligned}$$
>
> Thus, $x_n \to 0$ as $n \to \infty$, so
>
> $$\lim_{n\to\infty} x_n = 0.$$

10 Lecture 10

10.1 Convergence of Sequences Using the Definition of Convergence

From last lecture, we have a basic definition of convergence. However, this is not very helpful for determining whether a sequence converges or not because we have to actually know what the sequence converges to.

Example. Consider the sequence $x_n = \frac{4n^3+n}{n^3-6}$. We have an intuition that this converges to 4 by dividing the leading coefficients of the numerator and denominator.

We have

$$\begin{aligned}|x_n - 4| &= \left|\frac{4n^3+n}{n^3-6} - 4\right| \\ &= \left|\frac{(4n^3+n)-4(n^3-6)}{n^3-6}\right| \\ &= \left|\frac{n+24}{n^3-6}\right| \\ &= \frac{|n+24|}{|n^3-6|} \\ &= \frac{n+24}{n^3-6}\end{aligned}$$

We want to find some N such that $\frac{n+24}{|n^3-6|} < \varepsilon$ for all $n > N$. Thus we want to show

$$\frac{n+24}{|n^3-6|} \leq \frac{C}{n^2} \leq \varepsilon.$$

If $n \geq 24$, we have $n + 24 \leq 2n$. Suppose we wish for $\left|n^3 - 6\right| > 0$, so $n \geq 2$. Furthermore, note that $n^3 - 6 \geq \frac{1}{2}n^3$ when $n \geq 12^{\frac{1}{3}}$. Thus for $n \geq 24$, we have

$$\begin{aligned}\frac{n+24}{|n^3-6|} &\leq \frac{2n}{\frac{1}{2}n^3} \\ &= \frac{4}{n^2}.\end{aligned}$$

Thus we take

$$N = \max\left(24, \left\lceil\sqrt{\frac{4}{\varepsilon}}\right\rceil\right) = \max\left(24, \left\lceil\frac{2}{\sqrt{\varepsilon}}\right\rceil\right).$$

Thus for $n > N$, we have

$$\frac{n+24}{|n^3-6|} < \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and choose $N = \max\left(24, \left\lceil\frac{2}{\sqrt{\varepsilon}}\right\rceil\right)$. Then

$$\begin{aligned}|x_n - 4| &= \frac{n+24}{|n^3-6|} \\ &\leq \frac{4}{n^2} \\ &< \varepsilon\end{aligned}$$

for all $n > N$. Thus $\lim_{n\to\infty} x_n = 4$. ∎

Example. Show that $x_n = (-1)^n$ diverges.

Note that $x_{2k} = 1$, and $x_{2k+1} = -1$ for some integer k.

Proof. Suppose towards a contradiction that x_n converges to a point x. Since we know that the sequence only takes on the values 1 and -1, we know that

$$\begin{aligned} 2 &= \lvert(1-x)+(x+1)\rvert \\ &\leq \lvert 1-x\rvert + \lvert x+1\rvert \\ 2 &\leq \lvert x_{2n}-x\rvert + \lvert x-x_{2n+1}\rvert . \end{aligned}$$

Since $x_n \to x$, given $\varepsilon > 0$, there exists N such that $\lvert x_n - 1\rvert < \varepsilon$ for all $n > N$. For $n > N$, we have $\lvert x_{2n} - x\rvert < \varepsilon$ and $\lvert x - x_{2n+1}\rvert < \varepsilon$. Thus we have $2 \leq 2\varepsilon$, so $1 \leq \varepsilon$, which contradicts that ε may be arbitrarily small. Hence the sequence diverges. □

10.2 Limit Laws

Proposition 3.4 (Limits are Unique)—If $x_n \to x$ and $x_n \to y$, then $x = y$.

Idea: The points in the sequence have to be arbitrarily close to x and y simultaneously.

We see that

$$\begin{aligned} \lvert x-y\rvert &\leq \lvert x-x_n\rvert + \lvert x_n-y\rvert \\ &= \varepsilon, \end{aligned}$$

for any $\varepsilon > 0$. By a previous theorem, we have $x = y$.

Proof. Fix $\varepsilon > 0$. Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that $\lvert x_n - x\rvert < \frac{\varepsilon}{2}$ for all $n > N_1$. Since $x_n \to y$, there exists $N_2 \in \mathbb{N}$ such that $\lvert x_n - y\rvert < \frac{\varepsilon}{2}$ for all $n > N_2$. Then for $n > \max(N_1, N_2)$, the triangle inequality implies that

$$\begin{aligned} \lvert x-y\rvert &\leq \lvert x_n-x\rvert + \lvert y-x_n\rvert \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since we have $\lvert x-y\rvert < \varepsilon$ for any $\varepsilon > 0$, we have $x = y$. □

Definition. *3.5—Boundedness for Sequences*
A sequence (x_n) is *bounded* if there exists some real number $M > 0$ such that $\lvert x_n\rvert \leq M$ for all $n \in \mathbb{N}$.

Note. In the above, our choice of M does not depend on n. Additionally, all convergent sequences are bounded (since we may just choose an arbitrary ε and we know that x_n will at some point rest between $x - \varepsilon$ and $x + \varepsilon$).

Theorem — *3.6 (Convergent Sequences are Bounded)*
Suppose $x_n \to x$. Then for $\varepsilon = 1$, we may find a $N \in \mathbb{N}$ such that $\lvert x_n - x\rvert < 1$ for all $n > N$. Then when $n > N$, we have

$$\begin{aligned} \lvert x_n\rvert &\leq \lvert x_n - x\rvert + \lvert x\rvert \\ &\leq 1 + \lvert x\rvert . \end{aligned}$$

Hence, for $M = \max(\lvert x_1\rvert, \lvert x_2\rvert, \ldots, \lvert x_N\rvert, 1 + \lvert x\rvert)$, we see that $\lvert x_n\rvert \leq M$ for all $n \in \mathbb{N}$.

Theorem — *3.7 (Algebraic Limit Theorem)*
Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then

(i) $\lim_{n\to\infty}(ax_n + by_n) = ax + by$ for all $a, b \in \mathbb{R}$.

(ii) $\lim_{n\to\infty}(x_n \cdot y_n) = xy$.

(iii) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$, provided $y \neq 0$.

11 Lecture 11

11.1 Proof for Algebraic Limit Theorem

We want to show that

$$|x_n y_n - xy| < \varepsilon$$

for all $\varepsilon > 0$.

Proof. Observe that

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + (x_n - x)y| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n|\,|y_n - y| + |y|\,|x_n - x| \end{aligned}$$

Since convergent sequences are bounded, we know that there exist some $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

$$\leq M\,|y_n - y| + |y|\,|x_n - x|$$

Fix $\varepsilon > 0$. Then there exists N_1 such that $|x_n - x| < \frac{\varepsilon}{2(1+|y|)}$ for all $n > N_1$. Furthermore, we know there exists some N_2 such that $|y_n - y| < \frac{\varepsilon}{2(1+M)}$ for all $n > N_2$.

$$\begin{aligned} &< \frac{M\varepsilon}{2(1+M)} + \frac{|y|\,\varepsilon}{2(1+|y|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

for all $\varepsilon > 0$. Therefore the limit exists and

$$\lim_{n\to\infty} x_n y_n = xy.$$

□

We now want to show that

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{x}{y}.$$

Proof. Observe that

$$\begin{aligned} \left|\frac{x_n}{y_n} - \frac{x}{y}\right| &= \left|\frac{x_n y - x y_n}{y_n y}\right| \\ &= \frac{1}{|y_n|\,|y|}\,|x_n y - x y_n| \\ &= \frac{1}{|y_n|\,|y|}\,|x_n y - xy + xy - x y_n| \\ &\leq \frac{1}{|y_n|\,|y|}\,|y(x_n - x) + x(y - y_n)| \\ &= \frac{1}{|y_n|\,|y|}\,(|y|\,|x_n - x| + |x|\,|y - y_n|) \end{aligned}$$

We now need to show that $\frac{1}{|y_n|}$ is upper bounded by some value. Since (y_n) converges, we know that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$. Suppose we choose $\varepsilon = \frac{y}{2}$, so we have $\frac{1}{y_n} \leq \frac{2}{y}$. In a fashion similar to the previous part, we may manipulate the above expression to complete the proof.

□

Theorem — *3.8 (Order Limit Theorem)*
Assume $x_n \to x$ and $y_n \to y$.

(i) If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x \geq 0$.

(ii) If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

(iii) If there exists $a \in \mathbb{R}$ such that $a \leq x_n$ for all $n \in \mathbb{N}$, then $a \leq x$. If there exists $b \in \mathbb{R}$ such that $b \geq x_n$ for all $n \in \mathbb{N}$, then $b \geq x$.

Note. Strict inequalities are not necessarily respected! If $x_n \to x$ and $x_n > 0$, then the most we can say is that $x_n \geq 0$. For example, consider the case where $x_n = \frac{1}{n} > 0$, and $x = 0$.

11.2 Monotone Sequences

Definition. *3.9—Monotonicity*
A sequence (x_n) is *increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence (x_n) is *decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. We call both increasing and decreasing sequences *monotone* or *monotonic.*

Note. If you iterate the definition for an increasing sequence, you get $x_n \leq x_m$ for $n \leq m$. A similar result comes from iterating the definition for a decreasing sequence.

Example. *Monotone Sequences*

- $x_n = 1 - \frac{1}{n}$ is increasing (and converges to 1).
- $x_n = \frac{1}{2^n}$ is decreasing (and converges to 0).
- $x_n = n$ is increasing (but is neither convergent nor bounded).
- $x_n = (-1)^n$ is not monotonic (and does not converge).

Note. In the above examples, if a sequence is monotonic, then if it is bounded it converges, otherwise diverges. Can we prove this?

Theorem — *3.10 (Monotone Convergence Theorem)*
Every monotonic and bounded sequence in $\mathbb{R}$ necessarily converges.

Proof. Let (x_n) be a sequence in $\mathbb{R}$ which is both increasing and bounded (same argument works for decreasing). If we look at the set

$$S = \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R},$$

we see that this set is non-empty and bounded above (because (x_n) is bounded). Thus by the least upper bound property of $\mathbb{R}$ we know that $\sup S$ exists. We claim that $x = \sup S$ is the limit for (x_n). Let $\varepsilon > 0$. Then there exists some $x_N \in S$ such that $x_N > x - \varepsilon$. We know that (x_n) is increasing, then $x - \varepsilon < x_N < x_n$ for all $n > N$. Furthermore, since x is the supremum of S, we know that $x_n < x < x + \varepsilon$. Thus we have

$$x - \varepsilon < x_n < x + \varepsilon,$$

so $|x_n - x| < \varepsilon$ for all $\varepsilon > 0$ and all $n > N$. $\square$

12 Lecture 12

Example. Consider the sequence defined by $x_1 = 2$, and for $n \geq 2$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

We don't have a nice function for x_n in terms of n, so proving convergence by normal means is non-ideal. To find the limit of (x_n), we:

- Show that (x_n) is monotonic (decreasing in this case) and bounded
- Apply the Monotonic Convergence Theorem, and say that there exists some $x \in \mathbb{R}$ such that $x_n \to x$
- Apply limit laws to actually find the value of x

> **Note.** If $x_n \to x$, we know that $x_{n+1} \to x$.

Taking the first few terms, we see

$$x_1 = 2, x_2 = \frac{3}{2}, \dots$$

We guess that the sequence is always bounded, namely $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. We can prove this by induction. To show that the sequence is decreasing, we just need to show that for all $n \in \mathbb{N}$, we have $x_{n+1} - x_n \leq 0$. We can then use induction again, along with the fact that the previous inequality can be expressed as a function of x_n, to show that the sequence is decreasing.
We can see that the sequence (x_n) is strictly greater than zero, so we may use the Algebraic Limit Theorem. Thus we have

$$\begin{aligned} x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\ 2x &= x + \frac{2}{x} \\ x &= \frac{2}{x} \\ x^2 &= 2 \\ x &= \sqrt{2}. \end{aligned}$$

12.1 Subsequences

Consider the diverging sequence given by $x_n = (-1)^n$. If we take the even terms, we see that $x_{2n} = 1$, and if we take the odd terms, we have $x_{2n+1} = -1$. Thus we see that "parts" of our diverging sequence are actually convergent sequences.

> **Definition.** *Subsequences*
> Let (x_n) be a sequence and
> $$n_1 < n_2 < n_3 < \cdots$$
> be a strictly increasing sequence of natural numbers. Then for $k \in \mathbb{N}$, the sequence (x_{n_k}) is a *subsequence* of the original sequence (x_n).

Example. *Subsequences*

1. Consider $x_n = (-1)^n$. Then $x_{2k} = 1$ is a subsequence. Similarly, $x_{2k+1} = -1$ is also a subsequence.
2. Consider $x_n = \frac{1}{n} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$. A valid subsequence could be $x_{n_k} = \frac{1}{2k}$ or $x_{n_k} = \frac{1}{10^k}$. However, the sequence given by
$$n_1 = 10, n_2 = 50, n_3 = 20, n_4 = 5, n_5 = 1000$$
is not a subsequence because the indices are not strictly increasing.

Observation. Suppose we have (n_k) is strictly increasing. Then we know that $n_k \geq k$, because it is strictly increasing and starts at 1. Then if $x_{n_k} = \frac{1}{n_k}$, then we see
$$x_{n_k} \leq \frac{1}{k} \to 0.$$
Thus we see that for *any* subsequence of our convergent sequence x_n, it converges to the same limit as (x_n).

Proposition 3.12 Every subsequence of a converging sequence converges and does so to the same limit as the original sequence.

Lemma. We will show that $n_k \geq k$ for all $k \in \mathbb{N}$, assuming that n_k is a strictly increasing sequence.

Proof. We proceed via induction. Observe that for $n_1 \in \mathbb{N}$, we have $n_1 \geq 1$. Suppose $n_k \geq k$ Then since (n_k) is striclty increasing, we have $n_{k+1} > n_k \geq k$, so $n_{k+1} \geq k + 1$. $\square$

Proof. Suppose (x_n) converges to x. Let (x_{n_k}) be a subsequence of (x_n). We claim that $x_{n_k} \to x$. Let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n > N$. Since $n_k \geq k$ for $k > N$, we have $|x_{n_k} - x| < \varepsilon$ for all $n_k > N$. Thus $x_{n_k} \to x$ as $k \to \infty$. $\square$

Note. If $x_n \to x$, then $x_{n+1} \to x$ because (x_{n+1}) is a subsequence of (x_n).

Note. Proposition 3.12 can be used to prove divergence, by showing that two subsequences of (x_n) converge to different values. For example, if $(x_n) = (-1)^n$, then
$$\begin{aligned} x_{2n} &= (-1)^{2n} = 1 \\ x_{2n+1} &= (-1)^{2n+1} = -1, \end{aligned}$$
so (x_n) diverges.

Proposition 3.13 Every sequence has a monotonic subsequence.

Proof. We need to carefully select our subsequence. Let (x_n) be a sequence and
$$D = \{n \in \mathbb{N} \mid x_n > x_m \text{ for all } m > n\} \subseteq \mathbb{N}.$$
If $n \in D$, then $x_n > x_m$ for all $m > n$. We say that if $n \in D$, that x_n is *dominant*. We now consider when D is finite and when D is infinite.

1. If D is infinite, then there exists $\{n_k\} \subseteq D$ such that x_{n_k} is dominant. Then $x_{n_{k+1}} < x_{n_k}$ and so (x_{n_k}) is a subsequence which is decreasing, and so monotonic.

2. If D is finite, then there exists some N such that $\max D = N$. Thus the last dominant term of our sequence is x_N. Hence there exists some $n_1 > N$ such that $x_{n_1} > x_{N+1}$. However, since $n_1 > N$, we have $n_1 \notin D$, so there must exist some $x_{n_2} > x_{n_1+1}$. By induction on $k \in \mathbb{N}$, we get (n_k), so there exist
$$n_1 < n_2 < n_3 < \cdots$$
such that
$$x_{n_1} < x_{n_2} < \cdots < x_{n_k} < x_N.$$
Therefore (x_{n_k}) is an increasing subsequence, and so is monotonic.

□

Theorem — *3.14 (Bolzano–Weierstrass)*
Every bounded sequence in $\mathbb{R}$ has a converging subsequence.

Proof. Suppose we have a bounded sequence (x_n) in $\mathbb{R}$. Then by Proposition 3.13 we know that there exists some subsequence (x_{n_k}) that is monotonic, which is also bounded. Thus by the Monotone Convergence Theorem we have that (x_{n_k}) converges in $\mathbb{R}$. □

Note. Bolzano–Weierstrass doesn't tell us anything about the original sequence, just that there are converging subsequences.

13 Lecture 13

13.1 Cauchy Sequences

The notion of a Cauchy sequence is a sequence that doesn't necessarily converge, but whose terms get "arbitrarily close" to each other.

Definition. *Cauchy Sequence*
A sequence (x_n) is *Cauchy* if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } n, m > N.$$

Note. We don't need to know any limit x in order to determine if a sequence is Cauchy.

Example. *Cauchy Sequences*

1) $x_n = \frac{1}{n}$ is Cauchy.
2) $x_n = (-1)^n$ is not Cauchy.
3) $x_n = n$ is not Cauchy.

Proposition 3.16 Every convergent sequence is Cauchy.

Proof. Let (x_n) be a convergent sequence. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n - x| < \frac{\varepsilon}{2}$. Let $m > N$. Thus

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x| + |x - x_m| \\ &= |x_n - x| + |x_m - x| \\ &< \varepsilon. \end{aligned}$$

Therefore (x_n) is Cauchy. □

Proposition 3.17 Every Cauchy sequence is bounded.

Proof. Let (x_n) be Cauchy. For $\varepsilon = 1$, we know there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < 1 \quad \text{for all } n, m > N.$$

Thus $|x_n| \leq 1 + |x_{N+1}|$ for all $n > N$. Suppose we choose

$$M = \max(|x_1|, |x_2|, \ldots, |x_N|, 1 + |x_{N+1}|).$$

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$, and so (x_n) is bounded. □

13.2 Completeness

Theorem — *3.18 (Completeness of $\mathbb{R}$)*
$\mathbb{R}$ is *complete*, in other words every Cauchy sequence in $\mathbb{R}$ converges.

Proof. Let (x_n) be Cauchy in $\mathbb{R}$. We have two steps:

(1) Identify some candidate for the limit.

(2) Show that the sequence converges to the candidate.

(1) By Proposition 3.17, we know that (x_n) is bounded. By Bolzano–Weierstrass, there exists some subsequence $x_{n_k} \to x \in \mathbb{R}$ as $k \to \infty$. This is our candidate for our limit.
(2) We now show that our sequence converges to x. Bolzano–Weierstrass tells us that for a subsequence of (x_n), we have the terms getting arbitrarily close to x. Because (x_n) is Cauchy, we "control" the remaining terms by "proximity" to (x_{n_k}).
Fix $\varepsilon > 0$, $x_{n_k} \to x$, there exists $N_1 \in \mathbb{N}$ such that

$$x_{\frac{n}{2}} < \frac{\varepsilon}{2} \quad \text{for all } n_k > N_1.$$

Since (x_n) is Cauchy, there exists some $N_2 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \text{for all } n, m > N_2.$$

So for any $n > \max(N_1, N_2)$, and $n_k > \max(N_1, N_2)$,

$$\begin{aligned}
|x_n - x| &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore $x_n \to x$. □

Note. The above proof makes use of Bolzano–Weierstrass, which applies to the reals, not rationals. Thus we think that the rationals are not complete.

Theorem — *3.19 (Incompleteness of $\mathbb{Q}$)*
The rationals are *not* complete. More precisely, there exists a Cauchy sequence of rational numbers that *does not* converge.

Proof. Consider the sequence given by

$$x_1 = 2, x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

We see that (x_n) is a sequence of rationals. We know that it does not converge in the rationals, because if it did it would have to converge to $\sqrt{2} \notin \mathbb{Q}$. It remains to show that it is Cauchy. From one of the homeworks we know that $|x_n - |x_{n+1}|| < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. We can take telescoping sums by writing (for $n > m$):

$$\begin{aligned}
|x_n - x_m| &= \left|\sum_{k=m}^{n-1} (x_{k+1} - x_k)\right| \\
&\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \\
&< \left|\sum_{k=m}^{n-1} \frac{1}{2^k}\right|.
\end{aligned}$$

This is a geometric series that sums to

$$\begin{aligned} &= \frac{\left(\frac{1}{2}\right)^m - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\ &= \frac{4}{2^m}, \end{aligned}$$

which converges to 0 as $m \to \infty$. Thus (x_n) is Cauchy in $\mathbb{Q}$. □

13.3 Bonus: Diverging to Infinity

Definition. *3.20—Diverging to* $\pm\infty$
Let (x_n) be a sequence in $\mathbb{R}$. We say that $x_n \to \infty$ as $n \to \infty$ if: for all $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n > N$.
Similarly, we say $x_n \to -\infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n > N$.