

Lecture Notes

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1 Lecture 1

1.1 Introduction to Dynamical Systems

Models of real-world phenomena can often be classified as either *static* or *dynamic*. Furthermore, these systems can either be *discrete* (x_1, x_2, \dots where $x_i \in \mathbb{R}$ for $i \geq 1$) or *continuous* ($x = x(t)$ where $t \geq 0$ and $x \in \mathbb{R}$, and $\dot{x} = f(x)$).

1.1.1 Where Do “Dynamical Systems” Come From?

1. Observed phenomena
2. Mathematical model
3. “Solve” the model
4. Make predictions

1.2 Autonomous ODEs

Definition. *Autonomous ODEs*

We say that an ordinary differential equation is *autonomous* if the right-hand side does not depend on t .

- The SIR (susceptible, infected, recovered) model is an example of a *first order* system of *autonomous* ODEs.

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \dots, x_n) \end{cases}$$

- We will refer to n as the *dimension* of the system.

2 Lecture 2

2.1 Reducing ODEs to First Order Autonomous Systems

Consider the set of differential equations given by

$$\begin{cases} \dot{x} = -\kappa(t)xy, \\ \dot{y} = \kappa(t)xy - \delta y, \\ \dot{z} = \delta y. \end{cases}$$

Introduce a new variable, i.e. $\tau = \tau(t) = t$. Then we may rewrite the above as

$$\begin{cases} \dot{x} = -\kappa(\tau)xy, \\ \dot{y} = \kappa(\tau)xy - \delta y, \\ \dot{z} = \delta y, \\ \dot{\tau} = 1. \end{cases}$$

Note that the above system is now autonomous.

Example. The Pendulum

We can model the angle θ of a pendulum of length $L > 0$ by

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0.$$

Applying Newton's Second Law, we can get the equations

$$\begin{aligned} mL\ddot{\theta} &= -mg \sin \theta \\ \theta &= \theta(t). \end{aligned}$$

Observe that if we let $x = \theta$ and $y = \dot{\theta}$, then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L} \sin x. \end{cases}$$

Example. Pendulum with an external force

If we add an external force to our pendulum, then we get

$$\ddot{\theta} + \frac{g}{L} \sin \theta = \frac{1}{m} F(t).$$

Thus if we let $x = \theta$, $y = \dot{\theta}$, and $z = t$, then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L} \sin x + \frac{1}{m} F(z) \\ \dot{z} = 1. \end{cases}$$

Note. In general, higher order ODEs of the form

$$\frac{d^k x}{dt^k} = f\left(x, \frac{dx}{dt}, \frac{d^2 x}{dt^2}, \dots, \frac{d^{k-1} x}{dt^{k-1}}\right)$$

can be converted into a first order system by taking

$$z_1 = x, z_2 = \frac{dx}{dt}, \dots, z_k = \frac{d^{k-1}x}{dt^{k-1}}.$$

We get the system

$$\begin{cases} \dot{z}_1 = \frac{dx}{dt} = z_2 \\ \dot{z}_2 = \frac{d^2x}{dt^2} = z_3 \\ \vdots \\ \dot{z}_k = f(z_1, z_2, \dots, z_k) \end{cases}$$

2.1.1 Flows on the Line

We will now consider systems of the form

$$\dot{x} = f(x)$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function.

Example. Consider the ODE given by

$$\dot{x} = x(x+1)(x-1)^2.$$

We could use separation of variables to solve this.

Note. Solutions to ODEs usually come in three different flavors:

- Analytic methods (separation of variables)
- Geometric methods (direction fields)
- Numerical methods (Euler's method)

Definition. *Phase Space*

To help us analyze these differential equations, we can plot \dot{x} against x on a graph, and see the behavior around zeroes. This is called a *phase space*. If some neighborhood of points around a zero x tend towards x , then x is called a *stable point*. If they tend to move away from x , then x is an *unstable point*. On a phase space graph, we denote stable points with \bullet , unstable points with \circ , and other points with a half-filled circle.

2.1.2 Fixed Points

Definition. *Fixed Point*

We say that x^* is a *fixed point* of the system

$$\dot{x} = f(x)$$

if $f(x^*) = 0$. If x^* is a fixed point then the system has a constant solution given by $x(t) = x^*$. These points are also known as equilibrium points, stationary points, rest points, critical points, and steady states.

2.1.3 Stability

Definition. *Stability*

Let x^* be a fixed point of the system

$$\dot{x} = f(x).$$

For now, we say that x^* is:

- *Stable* if solutions starting close to x^* approach x^* as $t \rightarrow \infty$.
- *Unstable* if solutions starting close to x^* diverge from x^* as $t \rightarrow \infty$.
- *Half-stable* if solutions starting close to x^* approach x^* from one side, but diverge from the other side.