Math 131A Lecture Notes

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2021-09-24

1 Lecture 1

1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

1.3 Logical Connections

We usually use the letters P and Q to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: "P and Q", $P \land Q$
2. Disjunctions: "P or Q", $P \lor Q$
3. Implications: "If P, then Q", $P \implies Q$
   (a) If the proposition is false (i.e. if P is false) then the whole statement is true.

   > **Definition.**
   > We say that the statement is *vacuously true*.

4. Negations: "Not P", $\neg P$

1.3.1 Truth Tables

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Prove that if n is an integer, then $n(n+1)$ is even.

Proof. Suppose that n is an integer. Then we have two cases, where either n is even or n is odd. Let n be an even integer such that $n = 2k$ where $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= 2k(2k+1) \\ &= 2(2k^2+k). \end{aligned}$$

Thus we see that $n(n+1)$ is even when n is even. Now let n be odd such that $n = 2m+1$ where $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1). \end{aligned}$$

Thus $n(n+1)$ is also even when n is odd, and so is even for all integers n. $\square$

2 Lecture 2

2.1 Continuation of Logic

2.1.1 De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

Note. Negations turn "and" into "or" and vice versa.

Example. Suppose we have the following statement:

$$P\colon x \text{ is even and } x > 0.$$

Then the negation of P would be:

$$\neg P\colon x \text{ is odd or } x \leq 0.$$

2.1.2 Converse

Definition. *Converse*
The *converse* of a statement $P \implies Q$ is the statement $Q \implies P$. In general, the converse of a statement says nothing about the original statement.

Example. Consider the statement

$$\text{If } x > 0, \text{ then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0, \text{ then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write $P \iff Q$ instead of $(P \implies Q) \wedge (Q \implies P)$. In this case, we call P and Q *logically equivalent*. In writing, we say "P if and only if Q".

2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

Lemma 1. Let a be an integer. If a^2 is even, then a is even.

Proof. Suppose a is odd, so $a = 2k + 1$ for some integer k. Then

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus a^2 is odd and this completes the proof. $\square$

2.1.4 Variables and Quantifiers

We have a value x that varies over some values, so we use $P(x)$ to denote a statement that depends on the value of x.

Example. Consider the statement
$$P(x) : x + 2 = 3.$$
The statement is true if and only if $x = 1$.

We have two quantifiers—$\forall$ = "for all", and $\exists$ = "there exists".

- $\forall x : P(x)$ is true if $P(x)$ is true for all x.
- $\exists x : P(x)$ is true if there exists at least one x such that $P(x)$ is true.

We can also have nested quantifiers,
$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

Note. The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of n depends on m.

2.1.5 Proof by Counterexample

After "simplifying" the statement $\neg(\forall x : P(x))$, we get $\exists x : \neg P(x)$. We simply need to find a single counterexample to show that a statement is false for all x.

Example. Consider the statement $\forall x \in \mathbb{R} : x + 2 = 3$. All we need to do is show that there exists some $x \in \mathbb{R}$ such that $x + 2 \neq 3$. This occurs when $x = 0$, so the statement is false.

2.1.6 Proof by Contradiction

Key Idea. We want to show that $P \implies Q$ indirectly.

Lemma 2. We can show that
$$P \implies Q = (\neg P) \vee Q.$$
Then $P \implies Q$ is true if and only if $\neg(P \implies Q)$ is false, and so by Lemma 2 and De Morgan's Laws, $P \wedge \neg Q$ is false.

For proof by contradiction, we assume P is true and $\neg Q$ is true, and try to show that $P \wedge \neg Q$ is false (a contradiction).

3 Lecture 3

3.1 More Logic

3.1.1 Proof by Contradiction

To do a proof by contradiction for a statement $P \implies Q$, we assume $P \wedge \neg Q$. We aim to show that $P \wedge \neg Q$ is false (a contradiction).

Theorem — *Irrationality of $\sqrt{2}$*
There is no rational number x such that $x^2 = 2$. In other words, if $x \in \mathbb{Q}$, then $x^2 \neq 2$.

Proof. Suppose towards a contradiction that there exists some $x \in \mathbb{Q}$ such that $x^2 = 2$. Since x is rational, there exist integers p, q such that $q \neq 0$, $\frac{p}{q} = x$, and p and q have no common divisors (other than 1). Then

$$\begin{aligned} x^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since p^2 is even, there exists some integer k such that $p = 2k$. Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

By the same logic as before, we know that q must also be even (they share a common factor of 2). However, this contradicts our original assumption that p and q share no common factors, and this completes the proof. $\square$

3.2 Set Theory

We write $x \in A$ when we want to say that "x is an element of A", and $x \notin A$ when we want to say that "x is not an element of A".

3.2.1 Set Combinations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$.
- Subset (Inclusion): $A \subseteq B$ if and only if $x \in A \implies x \in B$.

> **Definition.** *Proper Subset*
> A set A is a *proper subset* of a set B if $A \subseteq B$ and there exists some $x \in B$ such that $x \notin A$. We denote this as $A \subset B$.

- Equality: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

> **Note (Showing Equality of Sets).** If you want to show $A = B$, you need to show both $A \subseteq B$ and $B \subseteq A$. In other words, you must show that for all $x \in A$, we have $x \in B$, and vice versa.

Example. We have $\mathbb{N} = \{1, 2, 3, \dots\}$.

- Let E be the set of even natural numbers. Note that $E \subseteq \mathbb{N}$.
- Let $S = \left\{p \in \mathbb{Q} \mid p^2 < 2\right\} \subseteq \mathbb{Q}$.

Then we have:

- $\mathbb{N} \cup E = \mathbb{N}$.
- $\mathbb{N} \cap E = E$.
- $\mathbb{N} \cap S = \left\{n \in \mathbb{N} \mid n^2 < 2\right\} = \{1\}$.
- $E \cap S = \varnothing$.

  Definition. *Disjoint Sets*
  If $A \cap B = \varnothing$, we call A and B *disjoint* sets.

  Proof. Suppose towards a contradiction that there exists some $x \in E \cap S$, which is to say $x \in E$ and $x \in S$. Since $x \in E$, we know that x is even, and so there exists some integer k such that $x = 2k$. Then
  $$x^2 = (2k)^2 = 4k^2,$$
  so $4 \mid x^2$. Therefore $x \geq 4$, which contradicts the condition for $x \in S$, namely $x^2 < 2$. $\square$

- Given some $n \in \mathbb{N}$, we define $A_n = \mathbb{N} \setminus \{1, \dots, n-1\}$.
  - $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$.
  - $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$.

Definition. *Set Complement*
If $A \subseteq B$, then we define the *complement* of A in B to be $A^c = B \setminus A$.

3.2.2 De Morgan's Laws

If I is an index set and $\{A_j\}_{j \in I}$ are subsets of B, then

$$\left(\bigcup_{j \in I} A_j\right)^c = \bigcap_{j \in I} A_j^c, \quad \text{and} \quad \left(\bigcap_{j \in I} A_j\right)^c = \bigcup_{j \in I} A_j^c$$

4 Lecture 4

4.1 Cartesian Product

If I have two sets A and B, then we may form their *Cartesian Product*, which is

$$A \times B = \{(x, y) \mid x_0 in?a \land y \in B\}.$$

Definition. *Binary Relation*
A *binary relation* is a subset $R \subseteq A \times B$. We say $x \in A$ is in relation to $y \in B$ if $(x, y) \in R$. We denote this by

$$xRy \iff (x, y) \in R.$$

Example. Consider the relation

$$\mathbb{R} \times \mathbb{R} \supseteq R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Then the relation is *reflexive*, because xRx. It is also antisymmetric, because $xRy \land yRx \implies x = y$. Finally, this relation is transitive, because $xRy \land yRz \implies xRz$.

These properties only make sense if $A = B$, i.e. $R \subseteq A \times A$, and we say that "R is a relation on A".

Definition. *Partial Order*
If a relation is reflexive, antisymmetric, and transitive on A, then it is a *partial order* on A.

The notion of "less than or equal to" is a partial order for $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, but there exists no partial order for $\mathbb{C}$.

Definition. *Power Set*
For a set A, we may define its *power set* by

$$\mathscr{P}(A) = \{C \mid C \subseteq A\}.$$

Note that set inclusion is a partial order on $\mathscr{P}(A)$.

Definition. *Equivalence Relation*
An *equivalence relation* R over A is a relation that is reflexive, symmetric, and transitive.

Note. Just like a partial order behaves much like $\leq$, an equivalence relation behaves much like $=$.

Definition. *Equivalence Class*
Given an equivalence relation on $A.0$, we define a new set

$$[x] \coloneqq \{y \in A \mid x \sim y\}.$$

We call $[x]$ the *equivalence class* of x. Any $z \in [x]$ is called a *representative* of the equivalence class $[x]$. In particular, x is a representative of its own equivalence class.

Let A be a set with equivalence relation $\sim$. Then for any $x, y \in A$,

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \varnothing.$$

Proof. Let $x, y \in A$. We know that x is either equivalent to y or it is not. Suppose the former is true and let $z \in [x]$. Thus we know that $z \sim x$ and $x \sim y$, and by transitivity we have $z \sim y$. Thus $z \in [y]$ and $[x] \subseteq [y]$. The reverse argument is the same.

If x is not equivalent to y, then suppose towards a contradiction that $[x] \cap [y] \neq \varnothing$. Let $x \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$. By symmetry we know that $x \sim z$ and by transitivity we have $x \sim y$. We have arrived at the contradiction that x is both equivalent and not equivalent to y. □

Definition. *Function*
A relation $R \subseteq A \times B$ is a *function* if for all $x \in A$ and all $y, z \in B$, we have the following:

- $xRy \wedge xRz \implies y = z$.

In other words, every input x has only one output.

Definition. *Injective Functions*
A function f is *injective* if $f(x_1) = f(x_2) \implies x_1 = x_2$.

Definition. *Surjective Functions*
A function f is *surjective* if for every $y \in B$, there exists some $x \in A$ such that $f(x) = y$.

5 Lecture 5

5.1 The Natural Numbers

We know that the natural numbers are defined via:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

5.1.1 Properties of the Natural Numbers

(P1) $1 \in \mathbb{N}$

(P2) If $n \in \mathbb{N}$, then it has a *successor*, $n + 1 \in \mathbb{N}$

(P3) 1 is *not* the successor of any element of $\mathbb{N}$

(P4) If m, n have the same successor, then $m = n$

Note. The properties above can be abstracted to become:

(P1) $1 \in \mathbb{N}$

(P2) There exists some $S \colon \mathbb{N} \to \mathbb{N}$ where $S(n)$ is the successor n

(P3) $1 \notin \operatorname{range} S$

(P4) S is injective

(P5) Suppose $A \subseteq \mathbb{N}$ with the properties:

(i) $1 \in A$

(ii) If $n \in A$, then $S(n) \in A$

Then $A = \mathbb{N}$.

Theorem — *1.1 (Induction)*
Let $\{P(n) \mid n \in \mathbb{N}\}$ be a set of logical propositions. Suppose that

(i) $P(1)$ is true.

(ii) If $P(n)$ is true, then $P(n + 1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $A = \{n \mid P(n) \text{ is true}\} \subseteq \mathbb{N}$. By our first assumption, $1 \in A$. By (ii), if $n \in A$, then $n + 1 \in A$. So by (P5) of the natural numbers, we know that $A = \mathbb{N}$. $\square$

Definition. *1.2 (Peano Axioms)*
A triplet $(\mathbb{N}, 1, S)$ is said to be a *system of the naturals* if it satisfies:

1) $\mathbb{N}$ is a set and $1 \in \mathbb{N}$

2) $S \colon \mathbb{N} \to \mathbb{N}$ is a function

3) $1 \notin \operatorname{range} S$

4) S is injective

5) $\forall A \subseteq N$ such that $1 \in A$ and $S(A) \subseteq A$, then $A = \mathbb{N}$

Definition. *Addition*
We define the binary relation $+$ over $\mathbb{N}$:

(i) $\forall n \in \mathbb{N}$, $n + 1 := S(n)$

(ii) $\forall m, n \in \mathbb{N}$, we have $m + S(n) = S(m + n)$

The following properties can be proven from the above definition of addition:

(a) Associativity: $\forall x, y, z \in \mathbb{N}$, we have $(x + y) + z = x + (y + z)$

(b) Commutativity: $\forall x, y \in \mathbb{N}$, we have $x + y = y + x$

(c) Cancellative Law: $\forall x, y, z \in \mathbb{N}$, we have $x + y = y + z \implies x = z$

Theorem — *1.3 (Existence of the Naturals)*
There exists a unique system of the naturals. In other words, there is a bijection between different systems of the naturals and so we may write:

$$(\mathbb{N}, 1, S) \iff (\mathbb{N}', 1', S').$$

5.2 Fields

Definition. *Field*
A *field* is a set with two binary operations,

- $+$, or 'addition'
- $\cdot$, or 'multiplication'

5.2.1 Axioms for Addition

(A1) $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x + y \in \mathbb{F}$

(A2) $\forall x, y \in \mathbb{F}$, we have $x + y = y + x$

(A3) $\forall x, y, z \in \mathbb{F}$, we have $(x + y) + z = x + (y + z)$

(A4) There exists some $0 \in \mathbb{F}$ such that $0 + x = x$ for all $x \in \mathbb{F}$

(A5) $\forall x \in \mathbb{F}$, there exists $-x \in \mathbb{F}$ such that $x + (-x) = 0$

5.2.2 Axioms for Multiplication

(M1) $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x \cdot y \in \mathbb{F}$

(M2) $\forall x, y \in \mathbb{F}$, we have $x \cdot y = y \cdot x$

(M3) $\forall x, y, z \in \mathbb{F}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(M4) There exists some $1 \in \mathbb{F}$ such that $1 \neq 0$ and $1 \cdot x = x$ for all $x \in \mathbb{F}$

(M5) $\forall x \in \mathbb{F}$, there exists some $\frac{1}{x} \in \mathbb{F}$ such that $x \cdot \frac{1}{x} = 1$

5.2.3 Distributive Law

(D1) $\forall x, y, z \in \mathbb{F}$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$

6 Lecture 6

6.1 The Rationals

On the natural numbers, we have a notion of addition, multiplication, and comparison ($\leq$). We constructed the integers and so we now have:

- A notion of an additive identity, $0 \in \mathbb{Z}$
- Additive inverses

However, we *don't* have:

- Multiplicative inverses

When we think of the rationals, we consider the set

$$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}.$$

Note. When dealing with $\mathbb{Q}$, we now have multiplicative inverses.

In particular, $(\mathbb{Q}, +, \cdot, \leq)$ is an *ordered field*.

Definition. *Ordered Field*
An *ordered field* is a field $\mathbb{F}$ which is also an ordered set ($\leq$) such that:

(i) If $x, y, z \in \mathbb{F}$ and $y < z$, then $x + y < x + z$

(ii) If $x, y \in \mathbb{F}$, $x > 0$ and $y > 0$, then $x \cdot y > 0$

Unfortunately, the rational numbers still don't allow us to solve polynomial equations (i.e. $x^2 = 2$).

Definition. *Algebraic Numbers*
A number is called *algebraic* if it solves

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$, $n \in \mathbb{N}$.

Example. Every rational number is algebraic, because it solves the equation

$$qx - p = 0.$$

Definition. *Dividing*
We say $k \in \mathbb{Z}$ *divides* $m \in \mathbb{Z}$ if $\frac{m}{k} \in \mathbb{Z}$.

Theorem — *Rational Zeros Theorem*
Suppose $c_0, \ldots, c_n \in \mathbb{Z}$, $c_n \neq 0$ and $r \in \mathbb{Q}$ satisfies

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then writing $r = \frac{c}{d}$ with c, d having no common factors, $d \neq 0$, we have:

$$\begin{gathered} c \text{ divides } c_0 \\ d \text{ divides } c_n \end{gathered}$$

Proof. Since r solves the equation, we have

$$c_n\left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0.$$

Multiplying both sides by d^n, we get

$$c_nc^n + c_{n-1}c^{n-1}d + \cdots + c_1cd^{n-1} + c_0d^n = 0.$$

Rearranging, we have

$$c_nc^n = -d(c_{n-1}c^{n-1} - \cdots + c_1cd^{n-2} - c_0d^{n-1}).$$

We know that $d \mid c_nc^n$. Since c and d have no common factors, we know $d \mid c_n$. Rearranging terms again,

$$-c(c_nc^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_1d^{n-1}) = c_0d^n.$$

By the same reasoning as before, we have that $c \mid c_0$. $\square$

Corollary. Suppose $r \in \mathbb{Q}$ solves

$$r^n + c_{n-1}r^{n-1} + \cdots + c_1r + c_0 = 0.$$

Then $r \in \mathbb{Z}$ and $r \mid c_0$.

Proof. Since $r \in \mathbb{Q}$, we may express $r = \frac{c}{d}$, $c \mid c_0$ and $d \mid 1$. From this we know that $d = 1$, so $r = c$ and $c \mid c_0$. Therefore $r \in \mathbb{Z}$ and $r \mid c_0$. $\square$

We have some deficiencies for $\mathbb{Q}$:

- There seem to be some "gaps" in $\mathbb{Q}$.

Proposition. We consider the sets $A = \left\{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 < 2\right\}$ and $B = \left\{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 > 2\right\}$. Notice that A has no largest element and B has no smallest element.

Proof. Given $p \in \mathbb{Q}$, let

$$q = p - \frac{p^2 - 2}{p^2 + 2} = \frac{2(p+1)}{p+2}.$$

We also have $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. If $p \in A$, $p^2 - 2 < 0$ so $q > p$ and $q^2 - 2 > 0$. If $p \in B$, $p^2 - 2 > 0$ so $q < p$ and $q^2 - 2 > 0$. $\square$

7 Lecture 7

Definition. *2.11—Upper and Lower Bounds*
Let E be an ordered set and $A \subset E$.

(a) If there exists $x \in E$ such that for all $a \in A$, $a \leq x$, we say A is *bounded above* by x and call x an *upper bound* for A.

(b) Suppose $A \subset E$ is non-empty and bounded above and there exists some $x^* \in E$ such that

   i) x^* is an upper bound for A

   ii) If y is any upper bound for A^*, then $x^* \leq y$

Then we call x^* the *least upper bound for* A, and we write

$$x^* = \sup A. \qquad \text{(sup meaning supremum)}$$

The *greatest lower bound* or *infimum* of a set B, which is bounded below and non-empty, satisfies

i) $\inf B$ is a lower bound for B

ii) If y is any lower bound for B, then $y \leq \inf B$

Example. Suppose

$$A = \left\{p \in \mathbb{Q} \mid p \geq 0, p^2 < 2\right\},$$
$$B = \left\{p \in \mathbb{Q} \mid p \geq 0, p^2 > 2\right\}.$$

Then A is bounded above (say by 2) and B is bounded below (say by 0). In the last lecture we proved that neither $\sup A$ nor $\inf B$ exist in $\mathbb{Q}$ (because the values would have been $\sqrt{2}$).

Example. Let

$$C = \{p \in \mathbb{Q} \mid p < 0\},$$
$$D = \{p \in \mathbb{Q} \mid p \leq 0\}.$$

Then

$$\sup C = \sup D = 0.$$

However, notice that $\sup C \notin C$ and $\sup D \in D$.

Definition. *Maximum*
We define $\max A$ to be the largest element of A, which satisfies:

i) $\max A \in A$

ii) For all $a \in A$, $a \leq \max A$

The definition for minimum is similar.

Definition. *2.12—Least Upper Bound Property (LUBP)*
An *ordered set* E has the *least upper bound property* if the following is true:

i) If $A \subseteq E$, $A \neq \varnothing$, A is bounded above, then $\sup A$ exists and $\sup A \in E$.

Note. $\mathbb{Q}$ does not have the least upper bound property.

Theorem — *2.13 (Existence of $\mathbb{R}$)*
There exists an ordered field $\mathbb{R}$ which has

i) $\mathbb{Q}$ as a sub-field

ii) The least upper bound property

7.1 Fundamental Properties of the Real Numbers (because of LUBP)

Theorem — *2.14 (Archimedean Property of $\mathbb{R}$)*
If $x, y \in \mathbb{R}$, and $x > 0$, then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Proof. Let $A = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Suppose towards a contradiction that there is no such n that satisfies the statement above. In other words, for all $n \in \mathbb{N}$, $nx \leq y$. Thus A is bounded above by y. Since A is nonempty and is a subset of $\mathbb{R}$, we know that $\sup A$ exists. Consider the value given by $\sup A - x$, which is not an upper bound for A. Then we know there exists some $z \in A$ such that $\sup A - x < z$. Since $z = mx$ (because $z \in A$), we have

$$\begin{aligned}\sup A - x &< z\\ \sup A - x &< mx\\ \sup A &< (m+1)x.\end{aligned}$$

We know that $(m+1)x \in A$, which contradicts the definition of $\sup A$. □

Some remarks:

1. Let $x = 1$. Then $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > y$.
2. Let $y = 1$. Then $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x > \frac{1}{n} > 0$.

Theorem — *2.15 (Density of $\mathbb{Q}$ in $\mathbb{R}$)*
For all $x, y \in \mathbb{R}$, $x < y$, there exists some $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Fix $x < y$. Then by the Archimedean property we have some $n \in \mathbb{N}$ such that $n(y - x) > 1$, or $y - x > \frac{1}{n}$. We may suppose $x > 0$, because otherwise we either have $x < 0 < y$ or $x < y < 0$ (multiply all sides by -1).
We want to show that

$$nx < m < ny.$$

Since $nx + 1 < ny$, we have $nx < m < nx + 1$, or $m - 1 < nx < m$. If $nx \in \mathbb{Z}$, we can take $m = nx + 1$. Thus

$$x < x + \frac{1}{n} = \frac{nx+1}{n} = \frac{m}{n} < \frac{ny}{n} < y.$$

Otherwise we have $nx \notin \mathbb{Z}$. We then apply the following lemma:

Lemma. If $x \in \mathbb{R}$, there exists a $k \in \mathbb{Z}$ such that $k - 1 \leq x \leq k$.

Then $m - 1 < nx < m$, as desired. □