Lecture Notes

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2022-01-04

Contents

1 Lecture 1

1.1 Overview of This Course

"Ancient" history of mathematics

- Egyptian mathematics (c. 3200 BCE – 300 BCE): Number system in base 10, fractions.
- Babylonian mathematics (c. 3300 BCE – 500 BCE): Place-value sexagesimal number system, roots of algebra.
- Greek mathematics (c. 700 BCE – 400 CE): Deduction method, geometry, roots of number theory.
- Mathematics in Asia (c. 1100 BCE – 1200 CE): Roots of number theory and algebra.
- Polynomial equations.
- Development of calculus, infinite series (16^{th}–17^{th} centuries CE).

1.2 The Beginnings

What is "Mathematics"?

1. Logical-deductive mathematics (initiated by the Greeks). This is not the generally accepted view of "mathematics" anymore.
2. Abstract counting problems: "mathematical" exercises without a direct practical use.
3. Practical counting problems: Bookkeeping for bureaucracy, inventories of goods and harvest, calculations involving lengths and areas for farming purposes, etc.
4. Development of a number system.

Note. We will see later that 2–4 all occur almost simultaneously.

1.3 Egyptian Mathematics

Most writing was done on papyrus, which doesn't preserve well, so we don't have much evidence from this period.

1.3.1 Egyptian Number System

The Egyptians had a base 10 number system, with a new Hieroglyphic symbol for each power of 10. This number system is *not* place valued. One simply adds all the symbols to get the number. For instance $\cap\cap\cap|||| = 34$. By comparison, our modern number system is place-valued, which means that the position of each digit matters, i.e. $254 \neq 452$.

1.3.2 Egyptian Arithmetic

Summation is extremely easy, you just write the numbers next to each other to get their sum. Multiplication was done via consecutive doubling and adding. We do this by decomposing one number into powers of 2, and then distributing the other number across this sum. Because of this method, Egyptians had tablets with powers of 2, and tablets with consecutive doublings of many numbers.

We also have sources of:

- Fractions—The Egyptians used almost exclusively fractions of the form $\frac{1}{n}$ (aside from $\frac{2}{3}$ and $\frac{3}{4}$) and used sums of these to express more general fractions.

- Notation—Since any fraction can be written as the sum of fractions of the form $\frac{1}{n}$, any general fraction could be written by just writing $\frac{1}{n}$ fractions next to each other.
- Algebra—We have evidence that they solved some linear and second order (quadratic) equations.
- Geometry—Though limited, we have evidence that the Egyptians could calculate areas and volumes of triangles, circles, pyramids, etc.

However we see no evidence for a general theory for solving these questions, and all of the problems that we know of are elementary and mostly practical. The earliest “advanced” mathematical resources from the Egyptians are from around 1900 BCE, and are predated by Babylonian mathematics.

2 Lecture 2

2.1 Babylonian Mathematics

The Babylonians, in comparison to the Egyptians:

- had a more advanced number system,
- solved more difficult problems,
- solved more abstract ("useless") problems.

Hence people often view the Babylonians in Mesopotamia as being the starting point of "serious" mathematics.

Mesopotamia was located between the Tigris and Euphrates rivers, and its name means "the land between the rivers". Historical texts from Mesopotamia are unusually well-preserved, because texts were made by making impressions on clay tablets, which hardened and preserve well.

2.1.1 Babylonian Number System

The Babylonian number system was a *sexagesimal* (base 60), place-valued system.

Note.

- The number zero didn't exist just yet! So instead of writing "20", they would just write "2", and figure out what number it took from context.
- They also had fractions, but didn't have a symbol for a decimal point, so they again guessed based on context.
- Furthermore, since there is no zero, there was no way of distinguishing between $1 \cdot 60^2 + 30 \cdot 1$, or $1 \cdot 60 + 30 \cdot \frac{1}{60}$, etc.

Later on in Mesopotamia, we see the first usage of the number zero (300 BCE), but only as a place holder to solve the aforementioned issues. This was also done by the Mayans around 400 CE. It was not until 600 CE that the number zero was invented in India, and carried its full weight as a "number".

Note. There are still remnants of the Babylonian sexagesimal number system in the modern day.

- Time: Hours and minutes are divided into 60 pieces
- Angle measurements are still done in multiples/factors of 60, i.e. $360°$.

2.1.2 Babylonian Algebra

This notation was also used to solve basic algebra questions, but we don't know how exactly these problems were solved (probably the same method as the Egyptians, guess and check). It is believed that math problems were given to students in order to train them in literacy, numeracy, and applications to administration. The Babylonians also knew about the quadratic formula, although it was described in words as opposed to being given as an equation. Furthermore, although they knew of the quadratic formula, they did not have negative numbers, and so found only one solution to quadratic equations. This is the first instance we know of where a civilization has an "algorithmic" solution to a problem.

2.1.3 Babylonian Geometry

We have some sources of Babylonian geometry that indicate that they knew of the Pythagorean Theorem. One example of this is the "IM 67118" tablet, which gives the sides of a rectangle given its area and diagonal length. There is a tablet called “Plimpton 322” that contains Pythagorean triples, or integer triplets a, b, c satisfying $a^2 + b^2 = c^2$.

3 Lecture 3

Another tablet titled "YBC7289" contains approximations for $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$ that has an error of less than 1 in 2 million.

3.1 Greek Mathematics

Note. A lot of "Greek" mathematicians did not come from what is nowadays Greece, but rather the old Hellenistic empire (which was larger).

3.1.1 The Theorem of Pythagoras

Probably the oldest "mathematical theorem", known (in some form) to:

- Egyptians
- Babylonians
- Chinese
- Indians
- Greeks

before Pythagoras.

What we know about Pythagoras with (quasi-)certainty

- He existed (~580 BCE–500 BCE)
- He was born in Samos (modern day Turkey)
- He left no writings himself
- He founded a school in Croton (c. 540 BCE)
  - We say "school" but it was more like a sect, with strict rules and secrecy
  - Their central belief was "All is number", and that every number has a meaning
  - The number 10 was a "holy number", often drawn as a triangle
  - Theory of music based on whole fractions
  - Largely responsible for introducing "rigorous" mathematics
  - They viewed "pure" mathematics as the opposite of "applied" mathematics

Theorem — *Pythagorean Theorem*
Given a right triangle with legs a and b and hypotenuse c, we have

$$a^2 + b^2 = c^2.$$

Definition. *Pythagorean Triples*
Triples (a, b, c) that satisfy $a^2 + b^2 = c^2$ are called *Pythagorean triples.*

General Formula: For positive integers p, q, r such that $p \geq q$, we can use the following equations:

$$\begin{aligned} a &= (p^2 - q^2)r \\ b &= 2pqr \\ c &= (p^2 + q^2)r \end{aligned}$$

"Facts":

- Less general formulas were known to Pythagoras' school, Plato, Babylonians...
- The first statement *and* proof of this formula is from Euclid's "Elements", book 10

3.1.2 Rational points on the circle

Finding rational points on the unit circle with rational coordinates is actually very related to finding Pythagorean triples.

4 Lecture 4

4.1 Rational Points on the Circle

Observation. If $a^2 + b^2 = c^2$, then

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Hence rational points on the unit circle correspond to Pythagorean triples.

4.2 Chord-Tangent Construction

- The idea is most likely due to Diophantos, circa 250 CE.
- Worked out in detail in the 17$^{\text{th}}$ century (Lagrange, Euler, etc.).
- Take a line through $(-1, 0)$ to (x, y) with slope t. The equation of this line is thus $y = t(x + 1)$.
- We claim that R has rational coordinates if and only if t is rational.

Proof. ($\Rightarrow$) Suppose $R = (x_0, y_0)$, with $x_0, y_0 \in \mathbb{Q}$. The slope of the line can be computed to be $\frac{y_0}{1+x_0} = t$. Thus t is rational.
($\Leftarrow$) Suppose t is rational. Since the line intersects the unit circle, we have

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + t(x+1)^2 &= 1 \\ x^2 + tx^2 + 2tx + t &= 1 \\ (1+t)x^2 + (2t)x + (t-1) &= 0. \end{aligned}$$

Observation: If x_0, x_1 are solutions of $ax^2 + bx + c = 0$, then we know that we can factor into $a(x - x_0)(x - x_1) = 0$. Thus by equating coefficients, we see that x_0 is rational if x_1 is rational.

We already know that $x_1 = -1$ is a solution to our equation, so we know that x must be rational. Since $y = t(x_0 + 1)$, it must be rational as well, so R has rational coordinates. ☐

If we actually solve the equation, we get the point R has coordinates $(-1, 0)$ or $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. If we write $t = \frac{p}{q}$ for integers p, q, then we have

$$x_0 = \frac{p^2 - q^2}{p^2 + q^2}, y_0 = \frac{2pq}{p^2 + q^2}.$$

Thus we may convert this into a true Pythagorean triple via

$$\begin{aligned} a &= (p^2 - q^2)r \\ b &= 2pqr \\ c &= (p^2 + q^2)r \end{aligned}$$

Note. The general takeaway from this proof is that you can use "known" points to help you deduce properties of certain "unknown" points. In this case, we found an intersection point between a line and a circle, using that we are given the other intersection point.

5 Lecture 5

5.1 Irrational Numbers

Claim. The square root of 2 is irrational.

Lemma. If a^2 is even, then so is a.

Proof. We proceed via proof by contrapositive. Suppose a is odd, so $a = 2k+1$ for some $k \in \mathbb{Z}$. Then

$$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Hence a^2 is odd. $\square$

Proof. Suppose towards a contradiction that $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ where $q \neq 0$. We may also assume that the greatest common factor of p, q is 1, since we otherwise could have pulled the factor out. Hence

$$\begin{aligned} \left(\frac{p}{q}\right)^2 &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since p^2 is even, p must be even, and so $p = 2k$ for some $k \in \mathbb{Z}$. Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

Since q^2 is even, so must q. However, this contradicts our assumption that the greatest common factor of p and q is 1. $\square$

The original Greek conclusion was that "geometric quantities" were different from "numbers". They used rational numbers to "describe" irrational numbers. This later led to the "theory of proportions" and the method of exhaustion, which inspired Dedekind cuts.

Note. The theorem of Pythagoras also gives a notion of *distance* between points in the plane, e.g.

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

5.2 Greek Geometry

The most important resource for Greek geometry is Euclid's "The Elements".

- It was written circa 300 BCE
- It consists of 465 theorems and proofs in 13 books
- Mostly comprised of works from earlier mathematicians
- Main contributions:
  - The deductive method
  - Unified framework for "Euclidean geometry".

Facts about Euclid:

- Lived in Alexandria
- Called the "father of geometry"

6 Lecture 6

6.1 Greek Geometry

6.1.1 The Deductive Method

The "Elements" from Euclid is the first mathematical text where statements are derived step by step, starting from axioms ("postulates") following certain rules ("common notions").

(1) Definitions: It starts with 23 definitions, i.e. of "point", "line", "parallel lines", "circle", etc.

(2) Postulates ("Ruler–Compass Construction"):

   (a) To draw a straight line from any point to any point.
   (b) To produce a finite straight line continuously into a straight line.
   (c) To describe a circle with any center and distance ("radius").
   (d) All right angles are equal to each other.
   (e) If a straight line falling on two straight lines makes the interior angles on the same sides less than two right angles, then the two straight lines, if produced indefinitely, meet on the side which the angles are less than two right angles.

(3) Common Notions:

   (a) Things which are equal to the same thing are also equal to each other ("Transitivity of equality").
   (b) If equals are added to equals, the wholes are equal.
   (c) If equals are subtracted from equals, the remainders are equal.
   (d) Things which coincide with one another are equal to one another.
   (e) The whole is greater than the part.

   > **Note.** The word "equal" means "have the same length/area/volume", whereas the word "coincide" is when two objects are "really the same".

> **Note.** Often, Euclid uses "visually obvious" assumptions that are not mentioned in the postulates.

> **Example.** *Proposition I.35 of the "Elements"*
> "Parallelograms with the same base and the same height are equal".

> **Note.** Some of the proofs are flawed because they make visual assumptions of the layouts of problems, rather than use equations and variables.