

Math 131A Lecture Notes

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1 Lecture 1

1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

1.3 Logical Connections

We usually use the letters P and Q to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: “ P and Q ”, $P \wedge Q$
2. Disjunctions: “ P or Q ”, $P \vee Q$
3. Implications: “If P , then Q ”, $P \implies Q$

- (a) If the proposition is false (i.e. if P is false) then the whole statement is true.

Definition.

We say that the statement is *vacuously true*.

4. Negations: “Not P ”, $\neg P$

1.3.1 Truth Tables

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Prove that if n is an integer, then $n(n+1)$ is even.

Proof. Suppose that n is an integer. Then we have two cases, where either n is even or n is odd. Let n be an even integer such that $n = 2k$ where $k \in \mathbb{Z}$. Then we have

$$\begin{aligned}n(n+1) &= 2k(2k+1) \\ &= 2(2k^2 + k).\end{aligned}$$

Thus we see that $n(n+1)$ is even when n is even. Now let n be odd such that $n = 2m+1$ where $m \in \mathbb{Z}$. Then we have

$$\begin{aligned}n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1).\end{aligned}$$

Thus $n(n+1)$ is also even when n is odd, and so is even for all integers n . □

2 Lecture 2

2.1 Continuation of Logic

2.1.1 De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

Note. Negations turn “and” into “or” and vice versa.

Example. Suppose we have the following statement:

$$P: x \text{ is even and } x > 0.$$

Then the negation of P would be:

$$\neg P: x \text{ is odd or } x \leq 0.$$

2.1.2 Converse

Definition. *Converse*

The *converse* of a statement $P \implies Q$ is the statement $Q \implies P$. In general, the converse of a statement says nothing about the original statement.

Example. Consider the statement

$$\text{If } x > 0, \text{ then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0, \text{ then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write $P \iff Q$ instead of $(P \implies Q) \wedge (Q \implies P)$. In this case, we call P and Q *logically equivalent*. In writing, we say “ P if and only if Q ”.

2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

Lemma 1. Let a be an integer. If a^2 is even, then a is even.

Proof. Suppose a is odd, so $a = 2k + 1$ for some integer k . Then

$$\begin{aligned} a^2 &= (2k + 1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus a^2 is odd and this completes the proof. \square

2.1.4 Variables and Quantifiers

We have a value x that varies over some values, so we use $P(x)$ to denote a statement that depends on the value of x .

Example. Consider the statement

$$P(x) : x + 2 = 3.$$

The statement is true if and only if $x = 1$.

We have two quantifiers— \forall = “for all”, and \exists = “there exists”.

- $\forall x : P(x)$ is true if $P(x)$ is true for all x .
- $\exists x : P(x)$ is true if there exists at least one x such that $P(x)$ is true.

We can also have nested quantifiers,

$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

Note. The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of n depends on m .

2.1.5 Proof by Counterexample

After “simplifying” the statement $\neg(\forall x : P(x))$, we get $\exists x : \neg P(x)$. We simply need to find a single counterexample to show that a statement is false for all x .

Example. Consider the statement $\forall x \in \mathbb{R} : x + 2 = 3$. All we need to do is show that there exists some $x \in \mathbb{R}$ such that $x + 2 \neq 3$. This occurs when $x = 0$, so the statement is false.

2.1.6 Proof by Contradiction

Key Idea. We want to show that $P \implies Q$ indirectly.

Lemma 2. We can show that

$$P \implies Q = (\neg P) \vee Q.$$

Then $P \implies Q$ is true if and only if $\neg(P \implies Q)$ is false, and so by Lemma 2 and De Morgan’s Laws, $P \wedge \neg Q$ is false.

For proof by contradiction, we assume P is true and $\neg Q$ is true, and try to show that $P \wedge \neg Q$ is false (a contradiction).

3 Lecture 3

3.1 More Logic

3.1.1 Proof by Contradiction

To do a proof by contradiction for a statement $P \implies Q$, we assume $P \wedge \neg Q$. We aim to show that $P \wedge \neg Q$ is false (a contradiction).

Theorem — *Irrationality of $\sqrt{2}$*

There is no rational number x such that $x^2 = 2$. In other words, if $x \in \mathbb{Q}$, then $x^2 \neq 2$.

Proof. Suppose towards a contradiction that there exists some $x \in \mathbb{Q}$ such that $x^2 = 2$. Since x is rational, there exist integers p, q such that $q \neq 0$, $\frac{p}{q} = x$, and p and q have no common divisors (other than 1). Then

$$\begin{aligned} x^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since p^2 is even, there exists some integer k such that $p = 2k$. Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

By the same logic as before, we know that q must also be even (they share a common factor of 2). However, this contradicts our original assumption that p and q share no common factors, and this completes the proof. \square

3.2 Set Theory

We write $x \in A$ when we want to say that “ x is an element of A ”, and $x \notin A$ when we want to say that “ x is not an element of A ”.

3.2.1 Set Combinations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$.
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
- Difference: $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$.
- Subset (Inclusion): $A \subseteq B$ if and only if $x \in A \implies x \in B$.

Definition. *Proper Subset*

A set A is a *proper subset* of a set B if $A \subseteq B$ and there exists some $x \in B$ such that $x \notin A$. We denote this as $A \subset B$.

- Equality: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Note (Showing Equality of Sets). If you want to show $A = B$, you need to show both $A \subseteq B$ and $B \subseteq A$. In other words, you must show that for all $x \in A$, we have $x \in B$, and vice versa.

Example. We have $\mathbb{N} = \{1, 2, 3, \dots\}$.

- Let E be the set of even natural numbers. Note that $E \subseteq \mathbb{N}$.
- Let $S = \{p \in \mathbb{Q} \mid p^2 < 2\} \subseteq \mathbb{Q}$.

Then we have:

- $\mathbb{N} \cup E = \mathbb{N}$.
- $\mathbb{N} \cap E = E$.
- $\mathbb{N} \cap S = \{n \in \mathbb{N} \mid n^2 < 2\} = \{1\}$.
- $E \cap S = \emptyset$.

Definition. *Disjoint Sets*

If $A \cap B = \emptyset$, we call A and B *disjoint* sets.

Proof. Suppose towards a contradiction that there exists some $x \in E \cap S$, which is to say $x \in E$ and $x \in S$. Since $x \in E$, we know that x is even, and so there exists some integer k such that $x = 2k$. Then

$$x^2 = (2k)^2 = 4k^2,$$

so $4 \mid x^2$. Therefore $x \geq 4$, which contradicts the condition for $x \in S$, namely $x^2 < 2$. \square

- Given some $n \in \mathbb{N}$, we define $A_n = \mathbb{N} \setminus \{1, \dots, n-1\}$.
 - $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$.
 - $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Definition. *Set Complement*

If $A \subseteq B$, then we define the *complement* of A in B to be $A^c = B \setminus A$.

3.2.2 De Morgan's Laws

If I is an index set and $\{A_j\}_{j \in I}$ are subsets of B , then

$$\left(\bigcup_{j \in I} A_j \right)^c = \bigcap_{j \in I} A_j^c, \quad \text{and} \quad \left(\bigcap_{j \in I} A_j \right)^c = \bigcup_{j \in I} A_j^c$$

4 Lecture 4

4.1 Cartesian Product

If I have two sets A and B , then we may form their *Cartesian Product*, which is

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}.$$

Definition. *Binary Relation*

A *binary relation* is a subset $R \subseteq A \times B$. We say $x \in A$ is in relation to $y \in B$ if $(x, y) \in R$. We denote this by

$$xRy \iff (x, y) \in R.$$

Example. Consider the relation

$$\mathbb{R} \times \mathbb{R} \supseteq R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Then the relation is *reflexive*, because xRx . It is also *antisymmetric*, because $xRy \wedge yRx \implies x = y$. Finally, this relation is *transitive*, because $xRy \wedge yRz \implies xRz$.

These properties only make sense if $A = B$, i.e. $R \subseteq A \times A$, and we say that “ R is a relation on A ”.

Definition. *Partial Order*

If a relation is reflexive, antisymmetric, and transitive on A , then it is a *partial order* on A .

The notion of “less than or equal to” is a partial order for \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} , but there exists no partial order for \mathbb{C} .

Definition. *Power Set*

For a set A , we may define its *power set* by

$$\mathcal{P}(A) = \{C \mid C \subseteq A\}.$$

Note that set inclusion is a partial order on $\mathcal{P}(A)$.

Definition. *Equivalence Relation*

An *equivalence relation* R over A is a relation that is reflexive, symmetric, and transitive.

Note. Just like a partial order behaves much like \leq , an equivalence relation behaves much like $=$.

Definition. *Equivalence Class*

Given an equivalence relation on A , we define a new set

$$[x] := \{y \in A \mid x \sim y\}.$$

We call $[x]$ the *equivalence class* of x . Any $z \in [x]$ is called a *representative* of the equivalence class $[x]$. In particular, x is a representative of its own equivalence class.

Let A be a set with equivalence relation \sim . Then for any $x, y \in A$,

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \emptyset.$$

Proof. Let $x, y \in A$. We know that x is either equivalent to y or it is not. Suppose the former is true and let $z \in [x]$. Thus we know that $z \sim x$ and $x \sim y$, and by transitivity we have $z \sim y$. Thus $z \in [y]$ and $[x] \subseteq [y]$. The reverse argument is the same.

If x is not equivalent to y , then suppose towards a contradiction that $[x] \cap [y] \neq \emptyset$. Let $x \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$. By symmetry we know that $x \sim z$ and by transitivity we have $x \sim y$. We have arrived at the contradiction that x is both equivalent and not equivalent to y . \square

Definition. *Function*

A relation $R \subseteq A \times B$ is a *function* if for all $x \in A$ and all $y, z \in B$, we have the following:

- $xRy \wedge xRz \implies y = z$.

In other words, every input x has only one output.

Definition. *Injective Functions*

A function f is *injective* if $f(x_1) = f(x_2) \implies x_1 = x_2$.

Definition. *Surjective Functions*

A function f is *surjective* if for every $y \in B$, there exists some $x \in A$ such that $f(x) = y$.

5 Lecture 5

5.1 The Natural Numbers

We know that the natural numbers are defined via:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

5.1.1 Properties of the Natural Numbers

- (P1) $1 \in \mathbb{N}$
- (P2) If $n \in \mathbb{N}$, then it has a *successor*, $n + 1 \in \mathbb{N}$
- (P3) 1 is *not* the successor of any element of \mathbb{N}
- (P4) If m, n have the same successor, then $m = n$

Note. The properties above can be abstracted to become:

- (P1) $1 \in \mathbb{N}$
- (P2) There exists some $S: \mathbb{N} \rightarrow \mathbb{N}$ where $S(n)$ is the successor n
- (P3) $1 \notin \text{range } S$
- (P4) S is injective
- (P5) Suppose $A \subseteq \mathbb{N}$ with the properties:
 - (i) $1 \in A$
 - (ii) If $n \in A$, then $S(n) \in A$Then $A = \mathbb{N}$.

Theorem — 1.1 (*Induction*)

Let $\{P(n) \mid n \in \mathbb{N}\}$ be a set of logical propositions. Suppose that

- (i) $P(1)$ is true.
- (ii) If $P(n)$ is true, then $P(n + 1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $A = \{n \mid P(n) \text{ is true}\} \subseteq \mathbb{N}$. By our first assumption, $1 \in A$. By (ii), if $n \in A$, then $n + 1 \in A$. So by (P5) of the natural numbers, we know that $A = \mathbb{N}$. \square

Definition. 1.2 (*Peano Axioms*)

A triplet $(\mathbb{N}, 1, S)$ is said to be a *system of the naturals* if it satisfies:

- 1) \mathbb{N} is a set and $1 \in \mathbb{N}$
- 2) $S: \mathbb{N} \rightarrow \mathbb{N}$ is a function
- 3) $1 \notin \text{range } S$
- 4) S is injective
- 5) $\forall A \subseteq \mathbb{N}$ such that $1 \in A$ and $S(A) \subseteq A$, then $A = \mathbb{N}$

Definition. *Addition*

We define the binary relation $+$ over \mathbb{N} :

- (i) $\forall n \in \mathbb{N}, n + 1 := S(n)$
- (ii) $\forall m, n \in \mathbb{N}, \text{ we have } m + S(n) = S(m + n)$

The following properties can be proven from the above definition of addition:

- (a) Associativity: $\forall x, y, z \in \mathbb{N}, \text{ we have } (x + y) + z = x + (y + z)$
- (b) Commutativity: $\forall x, y \in \mathbb{N}, \text{ we have } x + y = y + x$
- (c) Cancellative Law: $\forall x, y, z \in \mathbb{N}, \text{ we have } x + y = y + z \implies x = z$

Theorem — 1.3 (*Existence of the Naturals*)

There exists a unique system of the naturals. In other words, there is a bijection between different systems of the naturals and so we may write:

$$(\mathbb{N}, 1, S) \iff (\mathbb{N}', 1', S').$$

5.2 Fields

Definition. *Field*

A *field* is a set with two binary operations,

- $+$, or ‘addition’
- \cdot , or ‘multiplication’

5.2.1 Axioms for Addition

- (A1) $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x + y \in \mathbb{F}$
- (A2) $\forall x, y \in \mathbb{F}, \text{ we have } x + y = y + x$
- (A3) $\forall x, y, z \in \mathbb{F}, \text{ we have } (x + y) + z = x + (y + z)$
- (A4) There exists some $0 \in \mathbb{F}$ such that $0 + x = x$ for all $x \in \mathbb{F}$
- (A5) $\forall x \in \mathbb{F}, \text{ there exists } -x \in \mathbb{F} \text{ such that } x + (-x) = 0$

5.2.2 Axioms for Multiplication

- (M1) $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x \cdot y \in \mathbb{F}$
- (M2) $\forall x, y \in \mathbb{F}, \text{ we have } x \cdot y = y \cdot x$
- (M3) $\forall x, y, z \in \mathbb{F}, \text{ we have } (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- (M4) There exists some $1 \in \mathbb{F}$ such that $1 \neq 0$ and $1 \cdot x = x$ for all $x \in \mathbb{F}$
- (M5) $\forall x \in \mathbb{F}, \text{ there exists some } \frac{1}{x} \in \mathbb{F} \text{ such that } x \cdot \frac{1}{x} = 1$

5.2.3 Distributive Law

- (D1) $\forall x, y, z \in \mathbb{F}, \text{ we have } x \cdot (y + z) = x \cdot y + x \cdot z$

6 Lecture 6

6.1 The Rationals

On the natural numbers, we have a notion of addition, multiplication, and comparison (\leq). We constructed the integers and so we now have:

- A notion of an additive identity, $0 \in \mathbb{Z}$
- Additive inverses

However, we *don't* have:

- Multiplicative inverses

When we think of the rationals, we consider the set

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Note. When dealing with \mathbb{Q} , we now have multiplicative inverses.

In particular, $(\mathbb{Q}, +, \cdot, \leq)$ is an *ordered field*.

Definition. *Ordered Field*

An *ordered field* is a field \mathbb{F} which is also an ordered set (\leq) such that:

- (i) If $x, y, z \in \mathbb{F}$ and $y < z$, then $x + y < x + z$
- (ii) If $x, y \in \mathbb{F}$, $x > 0$ and $y > 0$, then $x \cdot y > 0$

Unfortunately, the rational numbers still don't allow us to solve polynomial equations (i.e. $x^2 = 2$).

Definition. *Algebraic Numbers*

A number is called *algebraic* if it solves

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, \dots, c_n \in \mathbb{Z}$, $c_n \neq 0$, $n \in \mathbb{N}$.

Example. Every rational number is algebraic, because it solves the equation

$$qx - p = 0.$$

Definition. *Dividing*

We say $k \in \mathbb{Z}$ *divides* $m \in \mathbb{Z}$ if $\frac{m}{k} \in \mathbb{Z}$.

Theorem — Rational Zeros Theorem

Suppose $c_0, \dots, c_n \in \mathbb{Z}$, $c_n \neq 0$ and $r \in \mathbb{Q}$ satisfies

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then writing $r = \frac{c}{d}$ with c, d having no common factors, $d \neq 0$, we have:

$$\begin{aligned} c &\text{ divides } c_0 \\ d &\text{ divides } c_n \end{aligned}$$

Proof. Since r solves the equation, we have

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0.$$

Multiplying both sides by d^n , we get

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Rearranging, we have

$$c_n c^n = -d(c_{n-1} c^{n-1} + \cdots + c_1 c d^{n-2} + c_0 d^{n-1}).$$

We know that $d \mid c_n c^n$. Since c and d have no common factors, we know $d \mid c_n$. Rearranging terms again,

$$-c(c_n c^{n-1} + c_{n-1} c^{n-2} d + \cdots + c_1 d^{n-1}) = c_0 d^n.$$

By the same reasoning as before, we have that $c \mid c_0$. □

Corollary. Suppose $r \in \mathbb{Q}$ solves

$$r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then $r \in \mathbb{Z}$ and $r \mid c_0$.

Proof. Since $r \in \mathbb{Q}$, we may express $r = \frac{c}{d}$, $c \mid c_0$ and $d \mid 1$. From this we know that $d = 1$, so $r = c$ and $c \mid c_0$. Therefore $r \in \mathbb{Z}$ and $r \mid c_0$. □

We have some deficiencies for \mathbb{Q} :

- There seem to be some “gaps” in \mathbb{Q} .

Proposition. We consider the sets $A = \{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 > 2\}$. Notice that A has no largest element and B has no smallest element.

Proof. Given $p \in \mathbb{Q}$, let

$$q = p - \frac{p^2 - 2}{p^2 + 2} = \frac{2(p+1)}{p+2}.$$

We also have $q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2}$. If $p \in A$, $p^2 - 2 < 0$ so $q > p$ and $q^2 - 2 > 0$. If $p \in B$, $p^2 - 2 > 0$ so $q < p$ and $q^2 - 2 < 0$. □

7 Lecture 7

Definition. 2.11—Upper and Lower Bounds

Let E be an ordered set and $A \subset E$.

- (a) If there exists $x \in E$ such that for all $a \in A$, $a \leq x$, we say A is *bounded above* by x and call x an *upper bound* for A .
- (b) Suppose $A \subset E$ is non-empty and bounded above and there exists some $x^* \in E$ such that
 - i) x^* is an upper bound for A
 - ii) If y is any upper bound for A^* , then $x^* \leq y$

Then we call x^* the *least upper bound* for A , and we write

$$x^* = \sup A. \quad (\text{sup meaning supremum})$$

The *greatest lower bound* or *infimum* of a set B , which is bounded below and non-empty, satisfies

- i) $\inf B$ is a lower bound for B
- ii) If y is any lower bound for B , then $y \leq \inf B$

Example. Suppose

$$A = \{p \in \mathbb{Q} \mid p \geq 0, p^2 < 2\},$$

$$B = \{p \in \mathbb{Q} \mid p \geq 0, p^2 > 2\}.$$

Then A is bounded above (say by 2) and B is bounded below (say by 0). In the last lecture we proved that neither $\sup A$ nor $\inf B$ exist in \mathbb{Q} (because the values would have been $\sqrt{2}$).

Example. Let

$$C = \{p \in \mathbb{Q} \mid p < 0\},$$

$$D = \{p \in \mathbb{Q} \mid p \leq 0\}.$$

Then

$$\sup C = \sup D = 0.$$

However, notice that $\sup C \notin C$ and $\sup D \in D$.

Definition. Maximum

We define $\max A$ to be the largest element of A , which satisfies:

- i) $\max A \in A$
- ii) For all $a \in A$, $a \leq \max A$

The definition for minimum is similar.

Definition. 2.12—Least Upper Bound Property (LUBP)

An ordered set E has the *least upper bound property* if the following is true:

- i) If $A \subseteq E$, $A \neq \emptyset$, A is bounded above, then $\sup A$ exists and $\sup A \in E$.

Note. \mathbb{Q} does not have the least upper bound property.

Theorem — 2.13 (Existence of \mathbb{R})

There exists an ordered field \mathbb{R} which has

- i) \mathbb{Q} as a sub-field
- ii) The least upper bound property

7.1 Fundamental Properties of the Real Numbers (because of LUBP)

Theorem — 2.14 (Archimedean Property of \mathbb{R})

If $x, y \in \mathbb{R}$, and $x > 0$, then $\exists n \in \mathbb{N}$ such that $n \cdot x > y$.

Proof. Let $A = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Suppose towards a contradiction that there is no such n that satisfies the statement above. In other words, for all $n \in \mathbb{N}$, $nx \leq y$. Thus A is bounded above by y . Since A is nonempty and is a subset of \mathbb{R} , we know that $\sup A$ exists. Consider the value given by $\sup A - x$, which is not an upper bound for A . Then we know there exists some $z \in A$ such that $\sup A - x < z$. Since $z = mx$ (because $z \in A$), we have

$$\begin{aligned}\sup A - x &< z \\ \sup A - x &< mx \\ \sup A &< (m+1)x.\end{aligned}$$

We know that $(m+1)x \in A$, which contradicts the definition of $\sup A$. □

Some remarks:

1. Let $x = 1$. Then $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > y$.
2. Let $y = 1$. Then $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x > \frac{1}{n} > 0$.

Theorem — 2.15 (Density of \mathbb{Q} in \mathbb{R})

For all $x, y \in \mathbb{R}$, $x < y$, there exists some $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Fix $x < y$. Then by the Archimedean property we have some $n \in \mathbb{N}$ such that $n(y - x) > 1$, or $y - x > \frac{1}{n}$. We may suppose $x > 0$, because otherwise we either have $x < 0 < y$ or $x < y < 0$ (multiply all sides by -1).

We want to show that

$$nx < m < ny.$$

Since $nx + 1 < ny$, we have $nx < m < nx + 1$, or $m - 1 < nx < m$. If $nx \in \mathbb{Z}$, we can take $m = nx + 1$. Thus

$$x < x + \frac{1}{n} = \frac{nx + 1}{n} = \frac{m}{n} < \frac{ny}{n} < y.$$

Otherwise we have $nx \notin \mathbb{Z}$. We then apply the following lemma:

Lemma. If $x \in \mathbb{R}$, there exists a $k \in \mathbb{Z}$ such that $k - 1 \leq x \leq k$.

Then $m - 1 < nx < m$, as desired. □

8 Lecture 8

8.1 A Construction of the Real Numbers

We want to show that there exists some field \mathbb{R} such that \mathbb{Q} is a sub-field and \mathbb{R} has the least upper bound property. This construction will be via Dedekind cuts.

Definition. 2.17—*Cut*

A *cut* in \mathbb{Q} is a pair of subsets $A, B \subset \mathbb{Q}$ such that:

- (i) $A \cap B = \emptyset$, $A \cup B = \mathbb{Q}$, $A, B \neq \emptyset$ (They partition \mathbb{Q})
- (ii) If $a \in A$ and $b \in B$, then $a < b$
- (iii) A contains no largest element

Example. An example of such a cut could be $A = \{p \in \mathbb{Q} \mid p < 1\}$ and $B = \{p \in \mathbb{Q} \mid p \geq 1\}$. We say that $A \mid B$ is a cut. Another such example is $A = \{p \in \mathbb{Q} \mid p \leq 0 \text{ or } p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \mid p > 0 \text{ and } p^2 \geq 2\}$

Definition. 2.18—*The Reals*

We may define

$$\mathbb{R} = \{X \mid X \text{ is a cut in } \mathbb{Q}\}.$$

In order to show that the above is a valid definition for \mathbb{R} , we must show:

- (i) \mathbb{Q} is contained in \mathbb{R} in some “natural way”
- (ii) \mathbb{R} is an ordered field
- (iii) \mathbb{R} has the least upper bound property

Definition. 2.19—*Partial Order for the Reals*

We define a *partial order* on \mathbb{R} as follows: If $X = A \mid B$ and $Y = C \mid D$, then we say $X < Y$ if and only if $A \subset C$ and $X \leq Y$ if and only if $A \subseteq C$.

We will show that the reals contain the rationals.

Proof. We will begin by showing that $\mathbb{Q} \subseteq \mathbb{R}$. We say that $A \mid B$ is a *rational cut* if for some $c \in \mathbb{Q}$, we have $A = \{p \in \mathbb{Q} \mid p < c\}$ and $B = \mathbb{Q} \setminus A$. We will use c^* to denote the rational cut at c . Then we may associate every $c \in \mathbb{Q}$ with a corresponding rational cut $c^* \in \mathbb{R}$. \square

Theorem — 2.20 (The Reals have the LUBP)

With respect to the partial order \leq defined earlier, we may show that \mathbb{R} has the least upper bound property.

Proof. Let \mathcal{C} be any non-empty collection of cuts which are bounded above, say by the cut X . We want to show that $\sup \mathcal{C}$ exists in \mathbb{R} , so $\sup \mathcal{C}$ is itself a cut. A candidate for $\sup \mathcal{C}$ is $C \mid D$, where

$$C = \{a \in \mathbb{Q} \mid \text{There exists a cut } A \mid B \in \mathcal{C} \text{ such that } a \in A\}.$$

Now, $Z = C \mid D$ is a cut in \mathbb{Q} . We claim that C has no largest element. If $a \in C$, then there exists a cut $A \mid B \in \mathcal{C}$ such that $a \in A$. Since A is a part of a cut, it has no largest element, so there exists some $a' \in A$ such that $a < a'$. Thus $a' \in C$ so C has no largest element.

We must now show that Z is the least upper bound of \mathcal{C} . For any $A \mid B \in \mathcal{C}$, $A \subseteq C$. Therefore $A \mid B \leq C \mid D = Z$ and Z is an upper bound for \mathcal{C} . We must now show that it is the *least* upper

bound. Let $Z' = C' \mid D'$ be an upper bound for \mathcal{C} . Then $A \mid B \leq C' \mid D'$ and $A \subseteq C'$ for any $A \mid B \in \mathcal{C}$. Now by definition of C , we have $C \subseteq C'$. Therefore $C \mid D \leq C' \mid D'$, so $Z \leq Z'$. We have $Z = \sup \mathcal{C} \in \mathbb{R}$, so \mathbb{R} has the least upper bound property. \square

We now show that \mathbb{R} is a field, namely an ordered field. We define the binary relations $+$ and \cdot as follows: Given two cuts $A \mid B$ and $C \mid D$, we may define

$$E = A + C = \{p \in \mathbb{Q} \mid p = a + c \text{ for some } a \in A, c \in C\}.$$

Note. This set summation is known as the Minkowski sum.

We must check that $E \mid F$ is a cut. We must also show that the additive identity for \mathbb{R} is 0, i.e. we must show that $0 + x = x + 0 = x$ for all $x \in \mathbb{R}$. To show the existence of additive inverses, show that for any cut $A \mid B$, there exists some $C \mid D \in \mathbb{R}$ such that $A \mid B + C \mid D = 0$.

Similarly, we define multiplication by

$$E = \{p \in \mathbb{Q} \mid p = ac \text{ for some } a \in A, c \in C\}.$$

8.2 Interesting Questions

- (a) Can we cut \mathbb{R} to get something larger? No, because every possible cut in \mathbb{R} is an element of \mathbb{R} . This is because \mathbb{R} has the least upper bound property and every cut in \mathbb{R} would just be a “real cut” at $\sup A$.
- (b) Is \mathbb{R} unique in some natural way? Yes. If you take any other ordered field \mathbb{F} such that $\mathbb{Q} \subseteq \mathbb{F}$ and \mathbb{F} has the least upper bound property, then there exists some bijection between \mathbb{F} and \mathbb{R} .
- (c) What about $+\infty$ and $-\infty$? We can’t treat these as real numbers using Dedekind cuts, as either A or B would have to be empty.

9 Lecture 9

9.1 Sequences

Example. *Approximating π*

We know that the area of the unit circle should be π . We can approximate the area of a unit circle by inscribing various shapes in the circle and finding their areas (giving us a sequence of lower bounds). If we inscribe an equilateral triangle in the circle, we find that its side length is $\sqrt{3}$, so the area of the triangle is $a_1 = \frac{3\sqrt{3}}{4} \approx 1.299$.

If we use a square instead, its side length is $\sqrt{2}$, so we have a new lower bound of $a_2 = 2 < \pi$.

Using a regular pentagon, we have a new approximation as given by $a_3 = \frac{5}{2} \sin\left(\frac{2\pi}{5}\right) < \pi$.

Continuing this pattern, we get

$$a_n = \frac{n+2}{2} \sin\left(\frac{2\pi}{n+2}\right) < \pi.$$

Notice that for all $n \in \mathbb{N}$, we have $a_n < a_{n+1} < \pi$. Computing this value for larger n , we have

$$a_{100} \approx 3.1396$$

$$a_{1000} \approx 3.141572$$

$$a_{10000} \approx 3.14159245.$$

Similarly, we may obtain upper bounds for π by circumscribing regular polygons around the unit circle. By circumscribing a square, we know that $\pi < 4$, so we know that there is some upper bound for what π is equal to. Thus we know that there exists some $a \in \mathbb{R}$ such that $a_n \approx a$ for some sufficiently large n .

In other words, there exists some $a \in \mathbb{R}$ such that a_n converges to a as n approaches ∞ .

One issue to note is that we are using π in our formula for a_n to approximate the value of π . Using some trigonometric identities, we may circumvent this by evaluating

$$a_{2n} = \frac{n}{2} \left(n - \sqrt{n - 4a_n^2} \right).$$

Definition. *3.1—Sequences*

A *sequence* is a function $f: \mathbb{N} \rightarrow \mathbb{R}$. Instead of writing $f(1), f(2), \dots, f(n)$, we tend to write a sequence as f_1, f_2, \dots, f_n .

Note (Notations for Sequences). There are many different notations for expressing sequences, a few popular notations being used below:

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}.$$

9.1.1 Behaviors of Sequences

- (a) “Convergence”—Getting closer and closer to a given point.
- (b) “Divergence”—Getting closer and closer to $\pm\infty$.
- (c) “Oscillation”—The sequence does not approach any value in particular.

9.1.2 Facts From the Homework

We define the absolute value function

$$|\cdot| : \mathbb{R} \rightarrow [0, \infty) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We have the following properties of the absolute value function:

- (i) $|xy| = |x| |y|$ for all $x, y \in \mathbb{R}$.
- (ii) $|x - y| \leq \ell$ if and only if $y - \ell \leq x \leq y + \ell$, where $\ell \geq 0$ and $x, y \in \mathbb{R}$.
- (iii) **Triangle Inequality:** $|x + y| \leq |x| + |y|$.

Note. One consequence of this is

$$\begin{aligned} |x - y| &= |x - z + z - y| \\ &\leq |x - z| + |z - y|. \end{aligned}$$

In other words, the distance between two points x and y is always less than or equal to the sum of the distances of x and y to a third point, z .

We say that a_n approaches a if “ $|a_n - a|$ gets arbitrarily small as n gets arbitrarily large”.

9.1.3 Convergence

Definition. 3.2—*Convergence*

A sequence (x_n) of real numbers is said to *converge* to an $x \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists $N > 0$ such that

$$|x_n - x| < \varepsilon$$

for all $n > N$. If (x_n) converges to x , we also write:

$$x_n \rightarrow x \text{ as } n \rightarrow \infty \quad \text{or} \quad \lim_{n \rightarrow \infty} x_n = x.$$

We call x the *limit* of the sequence (x_n) . We say that a sequence *diverges* if it does not converge.

Note. By the Archimedean property, we can always take $N \in \mathbb{N}$. In general, N is a function of ε . We fix $\varepsilon > 0$, and use that to find some sufficient N so that the sequence converges.

Example. Consider the sequence $x_n = \frac{1}{n^2}$. We suspect that $x_n \rightarrow 0 \in \mathbb{R}$ as $n \rightarrow \infty$. Fix $\varepsilon > 0$. If we let $N = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$ and $n > N$, we have

$$\begin{aligned} |x_n - 0| &= \left| \frac{1}{n^2} \right| \\ &= \frac{1}{n^2} \\ &< \frac{1}{N^2} \\ &< \varepsilon. \end{aligned}$$

Thus, $x_n \rightarrow 0$ as $n \rightarrow \infty$, so

$$\lim_{n \rightarrow \infty} x_n = 0.$$

10 Lecture 10

10.1 Convergence of Sequences Using the Definition of Convergence

From last lecture, we have a basic definition of convergence. However, this is not very helpful for determining whether a sequence converges or not because we have to actually know what the sequence converges to.

Example. Consider the sequence $x_n = \frac{4n^3+n}{n^3-6}$. We have an intuition that this converges to 4 by dividing the leading coefficients of the numerator and denominator.

We have

$$\begin{aligned} |x_n - 4| &= \left| \frac{4n^3 + n}{n^3 - 6} - 4 \right| \\ &= \left| \frac{(4n^3 + n) - 4(n^3 - 6)}{n^3 - 6} \right| \\ &= \left| \frac{n + 24}{n^3 - 6} \right| \\ &= \frac{|n + 24|}{|n^3 - 6|} \\ &= \frac{n + 24}{n^3 - 6} \end{aligned}$$

We want to find some N such that $\frac{n+24}{|n^3-6|} < \varepsilon$ for all $n > N$. Thus we want to show

$$\frac{n + 24}{|n^3 - 6|} \leq \frac{C}{n^2} \leq \varepsilon.$$

If $n \geq 24$, we have $n + 24 \leq 2n$. Suppose we wish for $|n^3 - 6| > 0$, so $n \geq 2$. Furthermore, note that $n^3 - 6 \geq \frac{1}{2}n^3$ when $n \geq 12^{\frac{1}{3}}$. Thus for $n \geq 24$, we have

$$\begin{aligned} \frac{n + 24}{|n^3 - 6|} &\leq \frac{2n}{\frac{1}{2}n^3} \\ &= \frac{4}{n^2}. \end{aligned}$$

Thus we take

$$N = \max \left(24, \left\lceil \sqrt{\frac{4}{\varepsilon}} \right\rceil \right) = \max \left(24, \left\lceil \frac{2}{\sqrt{\varepsilon}} \right\rceil \right).$$

Thus for $n > N$, we have

$$\frac{n + 24}{|n^3 - 6|} < \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and choose $N = \max \left(24, \left\lceil \frac{2}{\sqrt{\varepsilon}} \right\rceil \right)$. Then

$$\begin{aligned} |x_n - 4| &= \frac{n + 24}{|n^3 - 6|} \\ &\leq \frac{4}{n^2} \\ &< \varepsilon \end{aligned}$$

for all $n > N$. Thus $\lim_{n \rightarrow \infty} x_n = 4$. □

Example. Show that $x_n = (-1)^n$ diverges.

Note that $x_{2k} = 1$, and $x_{2k+1} = -1$ for some integer k .

Proof. Suppose towards a contradiction that x_n converges to a point x . Since we know that the sequence only takes on the values 1 and -1 , we know that

$$\begin{aligned} 2 &= |(1 - x) + (x + 1)| \\ &\leq |1 - x| + |x + 1| \\ 2 &\leq |x_{2n} - x| + |x - x_{2n+1}|. \end{aligned}$$

Since $x_n \rightarrow x$, given $\varepsilon > 0$, there exists N such that $|x_n - x| < \varepsilon$ for all $n > N$. For $n > N$, we have $|x_{2n} - x| < \varepsilon$ and $|x - x_{2n+1}| < \varepsilon$. Thus we have $2 \leq 2\varepsilon$, so $1 \leq \varepsilon$, which contradicts that ε may be arbitrarily small. Hence the sequence diverges. \square

10.2 Limit Laws

Proposition 3.4 (Limits are Unique)—If $x_n \rightarrow x$ and $x_n \rightarrow y$, then $x = y$.

Idea: The points in the sequence have to be arbitrarily close to x and y simultaneously.

We see that

$$\begin{aligned} |x - y| &\leq |x - x_n| + |x_n - y| \\ &= \varepsilon, \end{aligned}$$

for any $\varepsilon > 0$. By a previous theorem, we have $x = y$.

Proof. Fix $\varepsilon > 0$. Since $x_n \rightarrow x$, there exists $N_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n > N_1$. Since $x_n \rightarrow y$, there exists $N_2 \in \mathbb{N}$ such that $|x_n - y| < \frac{\varepsilon}{2}$ for all $n > N_2$. Then for $n > \max(N_1, N_2)$, the triangle inequality implies that

$$\begin{aligned} |x - y| &\leq |x_n - x| + |y - x_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since we have $|x - y| < \varepsilon$ for any $\varepsilon > 0$, we have $x = y$. \square

Definition. 3.5—*Boundedness for Sequences*

A sequence (x_n) is *bounded* if there exists some real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Note. In the above, our choice of M does not depend on n . Additionally, all convergent sequences are bounded (since we may just choose an arbitrary ε and we know that x_n will at some point rest between $x - \varepsilon$ and $x + \varepsilon$).

Theorem — 3.6 (*Convergent Sequences are Bounded*)

Suppose $x_n \rightarrow x$. Then for $\varepsilon = 1$, we may find a $N \in \mathbb{N}$ such that $|x_n - x| < 1$ for all $n > N$. Then when $n > N$, we have

$$\begin{aligned} |x_n| &\leq |x_n - x| + |x| \\ &\leq 1 + |x|. \end{aligned}$$

Hence, for $M = \max(|x_1|, |x_2|, \dots, |x_N|, 1 + |x|)$, we see that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem — 3.7 (Algebraic Limit Theorem)

Let $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. Then

- (i) $\lim_{n \rightarrow \infty} (ax_n + by_n) = ax + by$ for all $a, b \in \mathbb{R}$.
- (ii) $\lim_{n \rightarrow \infty} (x_n \cdot y_n) = xy$.
- (iii) $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$, provided $y \neq 0$.