Lecture Notes

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1 Lecture 1

1.1 Introduction to Dynamical Systems

Models of real-world phenomena can often be classified as either *static* or *dynamic*. Furthermore, these systems can either be *discrete* ($x_1, x_2, \dots$ where $x_i \in \mathbb{R}$ for $i \geq 1$) or *continuous* ($x = x(t)$ where $t \geq 0$ and $x \in \mathbb{R}$, and $\dot{x} = f(x)$).

1.1.1 Where Do "Dynamical Systems" Come From?

1. Observed phenomena
2. Mathematical model
3. "Solve" the model
4. Make predictions

1.2 Autonomous ODEs

Definition. *Autonomous ODEs*
We say that an ordinary differential equation is *autonomous* if the right-hand side does not depend on t.

- The SIR (susceptible, infected, recovered) model is an example of a *first order* system of *autonomous* ODEs.

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_1(x_1, \dots, x_n) \end{cases}$$

- We will refer to n as the *dimension* of the system.

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2.1 Reducing ODEs to First Order Autonomous Systems

Consider the set of differential equations given by

$$\begin{cases} \dot{x} = -\kappa(t)xy, \\ \dot{y} = \kappa(t)xy - \delta y, \\ \dot{z} = \delta y. \end{cases}$$

Introduce a new variable, i.e. $\tau = \tau(t) = t$. Then we may rewrite the above as

$$\begin{cases} \dot{x} = -\kappa(\tau)xy, \\ \dot{y} = \kappa(\tau)xy - \delta y, \\ \dot{z} = \delta y, \\ \dot{\tau} = 1. \end{cases}$$

Note that the above system is now autonomous.

Example. *The Pendulum*
We can model the angle θ of a pendulum of length $L > 0$ by

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0.$$

Applying Newton's Second Law, we can get the equations

$$\begin{aligned} mL\ddot{\theta} &= -mg\sin\theta \\ \theta &= \theta(t). \end{aligned}$$

Observe that if we let $x = \theta$ and $y = \dot{\theta}$, then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L}\sin\theta. \end{cases}$$

Example. *Pendulum with an external force*
If we add an external force to our pendulum, then we get

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{1}{m}F(t).$$

Thus if we let $x = \theta$, $y = \dot{\theta}$, and $z = t$, then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L}\sin x + \frac{1}{m}F(z) \\ \dot{z} = 1. \end{cases}$$

Note. In general, higher order ODEs of the form

$$\frac{\mathrm{d}^k x}{\mathrm{d}t^k} = f(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \dots, \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}})$$

can be converted into a first order system by taking

$$z_1 = x, z_2 = \frac{\mathrm{d}x}{\mathrm{d}t}, \ldots, z_k = \frac{\mathrm{d}^{k-1}x}{\mathrm{d}t^{k-1}}.$$

We get the system

$$\begin{cases} \dot{z}_1 = \frac{\mathrm{d}x}{\mathrm{d}t} = z_2 \\ \dot{z}_2 = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = z_3 \\ \vdots \\ \dot{z}_k = f(z_1, z_2, \ldots, z_k) \end{cases}$$

2.1.1 Flows on the Line

We will now consider systems of the form

$$\dot{x} = f(x)$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function.

Example. Consider the ODE given by

$$\dot{x} = x(x+1)(x-1)^2.$$

We could use separation of variables to solve this.

Note. Solutions to ODEs usually come in three different flavors:

- Analytic methods (separation of variables)
- Geometric methods (direction fields)
- Numerical methods (Euler's method)

Definition. *Phase Space*
To help us analyze these differential equations, we can plot $\dot{x}$ against x on a graph, and see the behavior around zeroes. This is called a *phase space.* If some neighborhood of points around a zero x tend towards x, then x is called a *stable point.* If they tend to move away from x, then x is an *unstable point.* On a phase space graph, we denote stable points with $\bullet$, unstable points with $\circ$, and other points with a half-filled circle.

2.1.2 Fixed Points

Definition. *Fixed Point*
We say that x^* is a *fixed point* of the system

$$\dot{x} = f(x)$$

if $f(x^*) = 0$. If x^* is a fixed point then the system has a constant solution given by $x(t) = x^*$. These points are also known as equilibrium points, stationary points, rest points, critical points, and steady states.

2.1.3 Stability

Definition. *Stability*
Let x^* be a fixed point of the system

$$\dot{x} = f(x).$$

For now, we say that x^* is:

- *Stable* if solutions starting close to x^* approach x^* as $t \to \infty$.
- *Unstable* if solutions starting close to x^* diverge from x^* as $t \to \infty$.
- *Half-stable* if solutions starting close to x^* approach x^* from one side, but diverge from the other side.

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Question. Can we say more about what happens close to fixed points?

Example. Consider the equation given by

$$\dot{x} = x(x+1)(x-1)^2,$$

which has stable points at -1, 0, and 1.
We will use something called the *local method*. We define a new function $\eta(t) = x(t) + 1$, so $x(t) = \eta(t) - 1$. Hence $\dot{x}(t) = \dot{\eta}(t)$. Furthermore,

$$\begin{aligned} x(x+1)(x-1)^2 &= (\eta - 1)\eta(\eta - 2)^2 \\ &= -4\eta + O(\eta^2), && (\eta \to 0) \end{aligned}$$

so $\dot{\eta} \approx -4\eta$. Near $x = -1, \eta = x + 1$ and it satisfies $\dot{\eta} = -4\eta$, so $\eta(t) \approx Ce^{-4t}$. We can see that this approaches 0 as $t \to \infty$, so points around $x = -1$ will approach -1.

In general, we have the following method:
Assume that x^* is a fixed point of $\dot{x} = f(x)$, i.e. $f(x^*) = 0$. Let $\eta = x - x^*$. Then

$$\begin{aligned} \dot{\eta} &= \dot{x} \\ &= f(x) \\ &= f(x^*) + f'(x^*)\eta + \frac{1}{2}f''(x^*)\eta^2 + \cdots \\ &= f'(x^*)\eta + O(\eta^2). && (\eta \to 0) \end{aligned}$$

Hence the equation

$$\dot{\eta} = f'(x^*)\eta$$

is the *linearization* at $x = x^*$. We know that the solution to such a differential equation is

$$\eta(t) = Ce^{f'(x^*)t} = \begin{cases} 0, & f'(x^*) < 0 \\ \pm\infty, & f'(x^*) > 0 \end{cases},$$

as $t \to \infty$. In the first case, the terms near x^* will tend towards x^*, and in the latter they will diverge from x^*.

Theorem. Suppose that x^* is a fixed point of the system $\dot{x} = f(x)$. Then if

- $f'(x^*) < 0$, the fixed point x^* is stable.
- $f'(x^*) > 0$, the fixed point x^* is unstable.

Question. What happens if $f'(x^*) = 0$? Anything can happen/the test is inconclusive.

- Consider the equation $\dot{x} = x^3$. We see that $x^* = 0$ is a critical point, and that $f'(x) = 3x^2$, so $f'(x^*) = 0$. Using our usual graphical methods, we can see that points to the left of x^* will approach x^*, and so will points to the right, and so $x^* = 0$ is a stable point.
- If we use the equation $\dot{x} = -x^3$, we get the direct opposite, that is $x^* = 0$ is unstable despite having the same critical point.
- If we look at the behavior of $\dot{x} = x^2$, then we get that the critical point at $x^* = 0$ is half-stable.
- If we consider the equation $\dot{x} = 0$, then every number on the real line is a critical point, and so the solutions don't move at all.

3.1 Potentials

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be smooth and consider the system
$$\dot{x} = f(x).$$
- A function $V\colon \mathbb{R} \to \mathbb{R}$ so that
$$f(x) = -V'(x)$$
is called a *potential* for f.
- Our system can be written as a *gradient flow*
$$\dot{x} = -V'(x).$$

Note. Potential functions are *not* unique, since you can always add a constant.

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Example. Consider the differential equation $\dot{x} = x - x^3$. Since we have $\dot{x} = -V'(x)$, we have

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 = \frac{1}{4}(x^2 - 1)^2 - \frac{1}{4}.$$

When we look at the graph of $V'(x)$, we may pretend that there is a ball rolling down a hill from every point, which tells us how to find the stability of points. Each point will settle in the first "well" that it meets.

Theorem. Let $V\colon \mathbb{R} \to \mathbb{R}$ be smooth and consider the system

$$\dot{x} = -V'(x).$$

Then the *potential energy* $V(x(t))$ is non-increasing (as a function of time). Furthermore, if $x(t)$ is not a fixed point for all $t \in (T_1, T_2)$, then the potential energy is strictly decreasing on (T_1, T_2).

Proof. Observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) &= V'(x(t)) \cdot \dot{x}(t) \\ &= -V'(x(t))^2.\end{aligned}$$

Hence the potential energy is non-increasing, as its derivative is always non-positive. Thus if $V'(x_1) = 0$, then x_1 is a critical point! □

Corollary. Let $V\colon \mathbb{R} \to \mathbb{R}$ be smooth and consider the system

$$\dot{x} = -V'(x).$$

If x^* is an isolated critical point of V then

- If it is a local minima of V, it is a stable fixed point.
- If it is a local maxima of V, it is a unstable fixed point.
- If it is an inflection point of V, it is a half-stable fixed point.

Proof. If we imagine the ball analogy again, we can see that if x^* is a local minima then points around it will tend towards x^*, and so it is stable. The opposite happens for divergence near a local maxima, and the analogy still holds for the half-stableness when x^* is an inflection point. □

Note. Every one dimensional system is a gradient flow, because if f is smooth, then we can take the integral and define

$$V(x) := -\int_0^x f(s)\,\mathrm{d}s.$$

4.1 Impossibility of Oscillations

Definition. *Periodic Functions*
If there exists a constant $p > 0$ so that for all t we have

$$x(t + p) = x(t),$$

then we say that p is *periodic*.

Note. All constant functions are periodic.

Theorem. There are no non-constant periodic solutions of the system
$$\dot{x} = f(x).$$

Proof. Suppose that x is a periodic solution, with period $p > 0$. If $0 \leq t \leq p$ then, as the potential energy is non-increasing,
$$V[x(p)] \leq V[x(t)] \leq V(x(0)).$$
Since $x(p) = x(0)$, we have $V[x(t)]$ is constant. Hence $x(t)$ is constant. □

4.2 Numerical Methods

4.2.1 Integral Equations

We want to find a solution of the equation
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

Observe that
$$\begin{aligned} \dot{x} &= f(x) \\ \int_0^t \frac{\mathrm{d}x(s)}{\mathrm{d}s}\,\mathrm{d}s &= \int_0^t f(x(s))\,\mathrm{d}s \\ x(t) - x(0) &= \int_0^t f(x(s))\,\mathrm{d}s \\ x(t) &= x_0 + \int_0^t f(x(s))\,\mathrm{d}s. \end{aligned}$$

We call this an *integral equation*, because the unknown now appears in the integral.

Example. Write the equation
$$\begin{cases} \dot{x} = \sin x \\ x(0) = 1 \end{cases}$$
as an integral equation.
We have that
$$x(t) = 1 + \int_0^t \sin(x(s))\,\mathrm{d}s.$$

4.2.2 Numerical Approximation

Suppose we have the equation
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

Let's take $\Delta t > 0$ small. Then we have
$$x(\Delta t) = x_0 + \int_0^{\Delta t} f(x(s))\,\mathrm{d}s.$$

We will approximate $x(s) \approx x_0$ on the interval $(0, \Delta t)$. Thus we have

$$\begin{aligned} x(\Delta t) &= x_0 + \int_0^{\Delta t} f(x(s))\,\mathrm{d}s \\ &= x_0 + \int_0^{\Delta t} f(x_0)\,\mathrm{d}s \\ &= x_0 + f(x_0)\Delta t. && (x_1 \coloneqq x_0 + f(x_0)\Delta t) \end{aligned}$$

Euler's method is to repeat the above to get $x_2 \approx x(2\Delta t)$. We have

$$\begin{aligned} x_2 &= x_1 + f(x_1)\Delta t, \\ x_3 &= x_2 + f(x_2)\Delta t, \\ &\cdots \\ x_{n+1} &= x_n + f(x_n)\Delta t. \end{aligned}$$

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5.1 Euler's Method

- We want to approximate the solution of
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$
- Given a time step Δt, for $n \geq 0$ define
$$x_{n+1} = x_n + f(x_n)\Delta t$$
- Thus we take x_n to be our approximation to $x(n\Delta t)$.

5.1.1 Truncation Error of a Numerical Method

- Let $x_n \approx x(n\Delta t)$.
- We define the *local truncation error* to be
$$e_1 = x(\Delta t) - x_1.$$

We will use Taylor's theorem with Lagrange residue to approximate the error:
$$\begin{aligned} x(\Delta t) &= x(0) + x'(0)\Delta t + \frac{x''(\xi)}{2}(\Delta t)^2 && (\xi \in (0, \Delta t)) \\ &= x_0 + f(x_0)\Delta t + \frac{f'(\xi)f(\xi)}{2}(\Delta t)^2. && (\dot{x} = f\text{, Chain rule}) \end{aligned}$$
Thus if f and f' are bounded and continuous, then $|\text{Remainder}| \leq C(\Delta t)^2$. Substituting into our local truncation error definition, we have
$$\begin{aligned} e_1 &= x(\Delta t) - x_1 \\ &= x(\Delta t) - (x_0 + f(x_0)\Delta t) \\ &= \frac{x''(\xi)}{2}(\Delta t)^2. \end{aligned}$$
Hence e_1 is bounded above by $C(\Delta t)^2$.

Note. If we decrease Δt, then we decrease our error as well.

Euler's method is of first order because $|e_1| \leq C(\Delta t)^2$. If we apply Euler's method n times over some interval, then we will get n errors:
$$|e_1| + |e_2| + \cdots + |e_n| \leq Cn(\Delta t)^2 = Cn\left(\frac{T}{n}\right)\Delta t = CT\Delta t.$$

How can we improve our results?

- Take smaller time steps Δt.
  - Unfortunately, this means that we need to perform more computations.
  - There will be more "round-off errors".
- Improve the approximation (see next section)

5.1.2 Improved Euler's Method

- We still want to approximate the solution of
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$
- For $n \geq 0$:
  - We make our first approximation
  $$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t.$$
  - Use this to make a better approximation
  $$x_{n+1} = x_n + \frac{1}{2}\left(f(x_n) + f(\tilde{x}_{n+1})\right)\Delta t.$$
- Take x_n to be our approximation for $x(n\Delta t)$.

The local truncation error for the improved Euler's method is of the form $C(\Delta t)^3$, and the global error is $CT(\Delta t)^2$.

5.1.3 Runge–Kutta 4th Order Method

- We still want to approximate the solution of
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$
- For $n \geq 0$, take:
  - $k_n^{(1)} = f(x_n)\Delta t$
  - $k_n^{(2)} = f(x_n + \frac{1}{2}k_n^{(1)})\Delta t$
  - $k_n^{(3)} = f(x_n + \frac{1}{2}k_n^{(2)})\Delta t$
  - $k_n^{(4)} = f(x_n + \frac{1}{2}k_n^{(3)})\Delta t$
- Then we may set
$$x_{n+1} = x_n + \frac{1}{6}(k_n^{(1)} + 2k_n^{(2)} + 2k_n^{(3)} + k_n^{(4)}).$$
- The local truncation error satisfies
$$|e_1| \leq C(\Delta t)^5,$$
and the global error is 4^{th} order, satisfying $CT(\Delta t)^4$.

5.2 Existence and Uniqueness of Solutions

Theorem — *Cauchy–Peano Existence Theorem*
Let $f\colon (a,b) \to \mathbb{R}$ be continuous and $x_0 \in (a,b)$. Then there exists some $\delta > 0$ and a solution $x\colon [-\delta, \delta] \to \mathbb{R}$ of the IVP
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

Theorem — *Picard–Lindelöf Existence and Uniqueness Theorem*
Let $f\colon (a,b) \to \mathbb{R}$ be continuous and $x_0 \in (a,b)$. If f is locally Lipschitz continuous, then there exists a *unique local solution* $\bar{x} \in C^1(I, \mathbb{R})$ of the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

where I is some interval around 0.

6 Lecture 6

6.1 Existence and Uniqueness

Definition. *Locally Lipschitz Continuity*
Let $f\colon (a,b) \to \mathbb{R}$. We call f *locally Lipschitz continuous* if for every $[d,c] \subseteq (a,b)$ there exists $k > 0$ such that
$$|f(x) - f(y)| \leq k\,|x - y|\,. \qquad \forall x, y \in [d,c]$$

Definition. *Global Lipschitz Continuity*
A function $f\colon (a,b) \to \mathbb{R}$ is said to be *global Lipschitz continuous* if there exists $L > 0$ such that
$$|f(x) - f(y)| \leq L\,|x - y| \qquad \forall x, y \in (a,b)$$

Fact. Continuously differentiable functions are Lipschitz continuous, and so are continuous.

Example.

- The function $f(x) = \sqrt{|x|}$ is continuous but it is not Lipschitz continuous.
- The function $g(x) = |x|$ is not differentiable but it is Lipschitz continuous on $[-1,1]$.

Note. In general, if it has a cusp, then it is not Lipschitz continuous.

6.1.1 Finite Time Blowup

Example. Does the solution of
$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$$
blow up in finite time?
If we solve the differential equation, then we get $x(t) = \frac{1}{1-t}$ (which is not well defined for all $t > 0$).

Theorem. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then there exists a unique *global* solution $x\colon \mathbb{R} \to \mathbb{R}$ of
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Theorem. Let $f \leq g$ be smooth and let $x_0 \leq y_0$. Suppose that x and y are solutions of the ODES
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y) \\ y(0) = y_0 \end{cases}$$
on an interval $[0,T]$. Then $x(t) \leq y(t)$ for all $t \in [0,T]$.

6.2 Bifurcations

6.2.1 External Parameters

- Consider the ODE
$$\dot{x} = f(x, r)$$
where r is a parameter of the model.
- **Question.** How do the dynamics vary as we vary r?

Example. Consider the differential equation $\dot{x} = r + x^2$. Depending on the value of r, we could have:

- 2 critical points, one of which is stable (when $r < 0$)
- 1 critical point, which is half-stable (when $r = 0$)
- 0 critical points, (when $r > 0$)

This change in behavior as r goes from negative to positive is called a *bifurcation.* We say that a bifurcation occurs at $(x^*, r^*) = (0, 0)$.

7 Lecture 7

7.1 Saddle-Node Bifurcations

Definition. *Bifurcation*
Consider the following autonomous system

$$\dot{x} = f(x, \lambda)$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. A *bifurcation* occurs at parameter $\lambda = \lambda_0$ if there are parameter values λ_1 arbitrarily close to λ_0 with dynamics topologically inequivalent from those at λ_0.

7.2 Identifying Bifurcations

If the system

$$\dot{x} = f(x, r)$$

has a bifurcation at $(x, r) = (x^*, r^*)$, then

$$f(x^*, r^*) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(x^*, r^*) = 0.$$

Note. The converse is *not* necessarily true! This helps us find *possible* bifurcation points, but we still need to perform further analysis in order to check whether these points are *actually* bifurcation points.

Proof. We define $\eta = x - x^*$, which implies that $\dot{\eta} = \frac{\partial f}{\partial x}(x^*, r^*)\eta$, which is stable if $\frac{\partial f}{\partial x}(x^*, r^*) < 0$ and unstable if $\frac{\partial f}{\partial x}(x^*, r^*) > 0$. Hence there is a change in stability when $\frac{\partial f}{\partial x}(x^*, r^*) = 0$. $\square$

8 Lecture 8

8.1 Transcritical Bifurcations

We are analyzing differential equations of the form

$$\dot{x} = rx - x^2 = x(r - x).$$

We see that there are fixed points at $x = 0, r$. To find the bifurcations, we solve for when

$$\begin{aligned} f(x, r) &= rx - x^2 = 0, \\ \frac{\partial f}{\partial x}(x, r) &= r - 2x = 0. \end{aligned}$$

Hence we have $2x^2 - x^2 = x^2 = 0$, so the only possible bifurcation occurs at $x = r = 0$.

9 Lecture 9

9.1 Subcritical Pitchfork

Look at the bifurcation diagram for
$$\dot{x} = rx + x^3.$$

9.2 Supercritical Pitchfork

Look at the bifurcation diagram for
$$\dot{x} = rx - x^3.$$

9.3 Symmetry

Note that the Saddle-node bifurcation ($\dot{x} = r + x^2$) and the Transcritical bifurcation ($\dot{x} = rx - x^2$) are *not* symmetric with respect to $y = -x$, i.e. $x \to -x$. For the Saddle-node bifurcation, we have

$$\begin{aligned}
\dot{x} &= r + x^2 \\
-\dot{y} &= r + x^2 \\
\dot{y} &= -r - y^2,
\end{aligned}$$

and the Transcritical bifurcation yields

$$\begin{aligned}
\dot{x} &= rx - x^2 \\
-\dot{y} &= -ry - y^2 \\
\dot{y} &= ry + y^2.
\end{aligned}$$

However, for the subcritical pitchfork bifurcation, we have

$$\begin{aligned}
\dot{x} &= rx + x^3 \\
-\dot{y} &= -ry - y^3 \\
\dot{y} &= ry + y^3,
\end{aligned}$$

so it *is* symmetric.

9.4 Hysteresis

Consider the ODE given by
$$\dot{x} = rx + x^3 - x^5.$$

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Note. Hysteresis is a concept that appears due to the non-reversibility as the parameter r varies. If we look at the bifurcation diagram's stable branches, we take one path as we increase r past 0, but take a different path as we decrease r back below 0.

10.1 Taylor's Theorem

10.1.1 Single Variable

Let $F(t)$ be continuous and $\frac{\mathrm{d}^n F}{\mathrm{d}t^n}$ is continuous for $1 \leq n \leq N+1$. Then we can write

$$F(t) = \sum_{n=0}^{N} \frac{1}{n!} \frac{\mathrm{d}^n F}{\mathrm{d}t^n}(0) t^n + R_N(t),$$

where $R_N(t) = \underbrace{\frac{1}{(N+1)!} \frac{\mathrm{d}^{N+1} F}{\mathrm{d}t^{N+1}} \left(\tilde{t}\right) t^{n+1}}_{\text{Lagrange form residue}}$ for some $\tilde{t} \in (0, t)$.

10.1.2 Multi Variable

Let $f(x, r)$ be smooth (so $\frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial r^m} f$ is continuous for all $m, n \geq 1$). Then let $F(t) = f(tx, tr)$ and apply Taylor's Theorem. We have

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} F(t) = \sum_{j=0}^{n} \binom{n}{j} \frac{\partial^n f}{\partial_x^{n-j} \partial_r^j}(tx, tr) x^{n-j} r^j.$$

When we take $t = 1$, we have

$$\begin{aligned} f(x, t) &= F(1) \\ &= \sum_{n=0}^{N} \frac{1}{n!} \frac{\mathrm{d}^n F}{\mathrm{d}t^n}(0) + R_N(0) \\ &= \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{1}{(n-j)! j!} \frac{\partial^n f}{\partial_x^{n-j} \partial r^j}(0, 0) x^{n-j} r^j + R_N(0) \end{aligned}$$

Theorem — *Taylor's Theorem*
Suppose all partial derivatives of $f(x, r)$ up to order $N+1$ are continuous. Then,

$$f(x, r) = \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{1}{(n-j)! j!} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(0, 0) x^{n-j} r^j + R_N(x, r),$$

where the remainder term can be written as

$$R_N(x, r) = \sum_{j=0}^{N+1} \frac{1}{(N+1-j)! j!} \frac{\partial^{N+1} f}{\partial x^{N+1-j} \partial r^j}(tx, tr) x^{N+1-j} r^j,$$

for some $0 < t < 1$.

Example. *Special Case*
Consider the case where $N = 2$ for the Taylor expansion. Plugging that into the formula, we have that the quadratic expansion of $f(x,r)$ at $(0,0)$ is

$$\begin{aligned}f(x,r) &= f(0,0) + \frac{\partial f}{\partial x}(0,0)\cdot x + \frac{\partial f}{\partial r}(0,0)\cdot r\\ &\quad + \frac{\partial f}{\partial x \partial r}(0,0)\cdot xr + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(0,0)\cdot x^2 + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial r^2}(0,0)\cdot r^2.\end{aligned}$$

10.2 Normal Forms

Consider the Taylor expansion for a saddle-node bifurcation about the point (x^*, r^*):

$$\begin{aligned}\dot{x} &= f(x,r)\\ &= f(x^*,r^*) + \underbrace{\partial x f(x^*,r^*)}_{q_1}(x-x^*) + \underbrace{\partial r f(x^*,r^*)}_{p_1}(r-r^*)\\ &\quad + \underbrace{\frac{1}{2}\partial_{xx} f(x^*,r^*)}_{q_2}(x-x^*)^2 + \underbrace{\partial_{xr} f(x^*,r^*)}_{p_2}(x-x^*)(r-r^*)\\ &\quad + \underbrace{\frac{1}{2}\partial_{rr} f(x^*,r^*)}_{Q}(r-r^*)^2 + \text{higher order terms.}\end{aligned}$$

Theorem. Suppose that $f(x^*,r^*) = 0$, $q_1 = 0$, $p_1 \neq 0$, $q_2 \neq 0$. Then $\dot{x} = f(x,r)$ undergoes a saddle-node bifurcation at (x^*,r^*) and

$$\dot{x} = \frac{\partial f}{\partial r}(x^*,r^*)(r-r^*) + \frac{1}{2}\partial_{xx} f(x^*,r^*)(x-x^*)^2 + \mathcal{O}(\varepsilon^3)$$

for $|r-r^*| < \varepsilon^2$ and $|x-x^*| < \varepsilon$. Moreover, there exists a change of variables

$$(t,x,r) \mapsto (s,y,R)$$

such that

$$\dot{x} = p_1(r-r^*) + q_2(x-x^*)^2 + \text{higher order terms}$$

takes the form

$$\frac{\mathrm{d}y}{\mathrm{d}s} = R + y^2 \qquad \text{(Saddle-node bifurcation)}$$

near $(0,0) = (y(x^*), R(r^*))$.