

# Lecture Notes

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# 1 Lecture 1

## 1.1 Introduction to Dynamical Systems

Models of real-world phenomena can often be classified as either *static* or *dynamic*. Furthermore, these systems can either be *discrete* ( $x_1, x_2, \dots$  where  $x_i \in \mathbb{R}$  for  $i \geq 1$ ) or *continuous* ( $x = x(t)$  where  $t \geq 0$  and  $x \in \mathbb{R}$ , and  $\dot{x} = f(x)$ ).

### 1.1.1 Where Do “Dynamical Systems” Come From?

1. Observed phenomena
2. Mathematical model
3. “Solve” the model
4. Make predictions

## 1.2 Autonomous ODEs

**Definition.** *Autonomous ODEs*

We say that an ordinary differential equation is *autonomous* if the right-hand side does not depend on  $t$ .

- The SIR (susceptible, infected, recovered) model is an example of a *first order* system of *autonomous* ODEs.

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \dots, x_n) \end{cases}$$

- We will refer to  $n$  as the *dimension* of the system.

## 2 Lecture 2

### 2.1 Reducing ODEs to First Order Autonomous Systems

Consider the set of differential equations given by

$$\begin{cases} \dot{x} = -\kappa(t)xy, \\ \dot{y} = \kappa(t)xy - \delta y, \\ \dot{z} = \delta y. \end{cases}$$

Introduce a new variable, i.e.  $\tau = \tau(t) = t$ . Then we may rewrite the above as

$$\begin{cases} \dot{x} = -\kappa(\tau)xy, \\ \dot{y} = \kappa(\tau)xy - \delta y, \\ \dot{z} = \delta y, \\ \dot{\tau} = 1. \end{cases}$$

Note that the above system is now autonomous.

**Example. The Pendulum**

We can model the angle  $\theta$  of a pendulum of length  $L > 0$  by

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0.$$

Applying Newton's Second Law, we can get the equations

$$\begin{aligned} mL\ddot{\theta} &= -mg \sin \theta \\ \theta &= \theta(t). \end{aligned}$$

Observe that if we let  $x = \theta$  and  $y = \dot{\theta}$ , then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L} \sin x. \end{cases}$$

**Example. Pendulum with an external force**

If we add an external force to our pendulum, then we get

$$\ddot{\theta} + \frac{g}{L} \sin \theta = \frac{1}{m} F(t).$$

Thus if we let  $x = \theta$ ,  $y = \dot{\theta}$ , and  $z = t$ , then we get

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L} \sin x + \frac{1}{m} F(z) \\ \dot{z} = 1. \end{cases}$$

**Note.** In general, higher order ODEs of the form

$$\frac{d^k x}{dt^k} = f\left(x, \frac{dx}{dt}, \frac{d^2 x}{dt^2}, \dots, \frac{d^{k-1} x}{dt^{k-1}}\right)$$

can be converted into a first order system by taking

$$z_1 = x, z_2 = \frac{dx}{dt}, \dots, z_k = \frac{d^{k-1}x}{dt^{k-1}}.$$

We get the system

$$\begin{cases} \dot{z}_1 = \frac{dx}{dt} = z_2 \\ \dot{z}_2 = \frac{d^2x}{dt^2} = z_3 \\ \vdots \\ \dot{z}_k = f(z_1, z_2, \dots, z_k) \end{cases}$$

### 2.1.1 Flows on the Line

We will now consider systems of the form

$$\dot{x} = f(x)$$

where  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function.

**Example.** Consider the ODE given by

$$\dot{x} = x(x+1)(x-1)^2.$$

We could use separation of variables to solve this.

**Note.** Solutions to ODEs usually come in three different flavors:

- Analytic methods (separation of variables)
- Geometric methods (direction fields)
- Numerical methods (Euler's method)

**Definition.** *Phase Space*

To help us analyze these differential equations, we can plot  $\dot{x}$  against  $x$  on a graph, and see the behavior around zeroes. This is called a *phase space*. If some neighborhood of points around a zero  $x$  tend towards  $x$ , then  $x$  is called a *stable point*. If they tend to move away from  $x$ , then  $x$  is an *unstable point*. On a phase space graph, we denote stable points with  $\bullet$ , unstable points with  $\circ$ , and other points with a half-filled circle.

### 2.1.2 Fixed Points

**Definition.** *Fixed Point*

We say that  $x^*$  is a *fixed point* of the system

$$\dot{x} = f(x)$$

if  $f(x^*) = 0$ . If  $x^*$  is a fixed point then the system has a constant solution given by  $x(t) = x^*$ . These points are also known as equilibrium points, stationary points, rest points, critical points, and steady states.

### 2.1.3 Stability

**Definition.** *Stability*

Let  $x^*$  be a fixed point of the system

$$\dot{x} = f(x).$$

For now, we say that  $x^*$  is:

- *Stable* if solutions starting close to  $x^*$  approach  $x^*$  as  $t \rightarrow \infty$ .
- *Unstable* if solutions starting close to  $x^*$  diverge from  $x^*$  as  $t \rightarrow \infty$ .
- *Half-stable* if solutions starting close to  $x^*$  approach  $x^*$  from one side, but diverge from the other side.

### 3 Lecture 3

**Question.** Can we say more about what happens close to fixed points?

**Example.** Consider the equation given by

$$\dot{x} = x(x+1)(x-1)^2,$$

which has stable points at  $-1$ ,  $0$ , and  $1$ .

We will use something called the *local method*. We define a new function  $\eta(t) = x(t) + 1$ , so  $x(t) = \eta(t) - 1$ . Hence  $\dot{x}(t) = \dot{\eta}(t)$ . Furthermore,

$$\begin{aligned} x(x+1)(x-1)^2 &= (\eta-1)\eta(\eta-2)^2 \\ &= -4\eta + O(\eta^2), \end{aligned} \quad (\eta \rightarrow 0)$$

so  $\dot{\eta} \approx -4\eta$ . Near  $x = -1$ ,  $\eta = x + 1$  and it satisfies  $\dot{\eta} = -4\eta$ , so  $\eta(t) \approx Ce^{-4t}$ . We can see that this approaches 0 as  $t \rightarrow \infty$ , so points around  $x = -1$  will approach  $-1$ .

In general, we have the following method:

Assume that  $x^*$  is a fixed point of  $\dot{x} = f(x)$ , i.e.  $f(x^*) = 0$ . Let  $\eta = x - x^*$ . Then

$$\begin{aligned} \dot{\eta} &= \dot{x} \\ &= f(x) \\ &= f(x^*) + f'(x^*)\eta + \frac{1}{2}f''(x^*)\eta^2 + \dots \\ &= f'(x^*)\eta + O(\eta^2). \end{aligned} \quad (\eta \rightarrow 0)$$

Hence the equation

$$\dot{\eta} = f'(x^*)\eta$$

is the *linearization* at  $x = x^*$ . We know that the solution to such a differential equation is

$$\eta(t) = Ce^{f'(x^*)t} = \begin{cases} 0, & f'(x^*) < 0 \\ \pm\infty, & f'(x^*) > 0 \end{cases},$$

as  $t \rightarrow \infty$ . In the first case, the terms near  $x^*$  will tend towards  $x^*$ , and in the latter they will diverge from  $x^*$ .

**Theorem.** Suppose that  $x^*$  is a fixed point of the system  $\dot{x} = f(x)$ . Then if

- $f'(x^*) < 0$ , the fixed point  $x^*$  is stable.
- $f'(x^*) > 0$ , the fixed point  $x^*$  is unstable.

**Question.** What happens if  $f'(x^*) = 0$ ? Anything can happen/the test is inconclusive.

- Consider the equation  $\dot{x} = x^3$ . We see that  $x^* = 0$  is a critical point, and that  $f'(x) = 3x^2$ , so  $f'(x^*) = 0$ . Using our usual graphical methods, we can see that points to the left of  $x^*$  will approach  $x^*$ , and so will points to the right, and so  $x^* = 0$  is a stable point.
- If we use the equation  $\dot{x} = -x^3$ , we get the direct opposite, that is  $x^* = 0$  is unstable despite having the same critical point.
- If we look at the behavior of  $\dot{x} = x^2$ , then we get that the critical point at  $x^* = 0$  is half-stable.
- If we consider the equation  $\dot{x} = 0$ , then every number on the real line is a critical point, and so the solutions don't move at all.



### 3.1 Potentials

- Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be smooth and consider the system

$$\dot{x} = f(x).$$

- A function  $V: \mathbb{R} \rightarrow \mathbb{R}$  so that

$$f(x) = -V'(x)$$

is called a *potential* for  $f$ .

- Our system can be written as a *gradient flow*

$$\dot{x} = -V'(x).$$

**Note.** Potential functions are *not* unique, since you can always add a constant.

## 4 Lecture 4

**Example.** Consider the differential equation  $\dot{x} = x - x^3$ . Since we have  $\dot{x} = -V'(x)$ , we have

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 = \frac{1}{4}(x^2 - 1)^2 - \frac{1}{4}.$$

When we look at the graph of  $V'(x)$ , we may pretend that there is a ball rolling down a hill from every point, which tells us how to find the stability of points. Each point will settle in the first “well” that it meets.

**Theorem.** Let  $V: \mathbb{R} \rightarrow \mathbb{R}$  be smooth and consider the system

$$\dot{x} = -V'(x).$$

Then the *potential energy*  $V(x(t))$  is non-increasing (as a function of time). Furthermore, if  $x(t)$  is not a fixed point for all  $t \in (T_1, T_2)$ , then the potential energy is strictly decreasing on  $(T_1, T_2)$ .

*Proof.* Observe that

$$\begin{aligned} \frac{d}{dt}V(x(t)) &= V'(x(t)) \cdot \dot{x}(t) \\ &= -V'(x(t))^2. \end{aligned}$$

Hence the potential energy is non-increasing, as its derivative is always non-positive. Thus if  $V'(x_1) = 0$ , then  $x_1$  is a critical point!  $\square$

**Corollary.** Let  $V: \mathbb{R} \rightarrow \mathbb{R}$  be smooth and consider the system

$$\dot{x} = -V'(x).$$

If  $x^*$  is an isolated critical point of  $V$  then

- If it is a local minima of  $V$ , it is a stable fixed point.
- If it is a local maxima of  $V$ , it is an unstable fixed point.
- If it is an inflection point of  $V$ , it is a half-stable fixed point.

*Proof.* If we imagine the ball analogy again, we can see that if  $x^*$  is a local minima then points around it will tend towards  $x^*$ , and so it is stable. The opposite happens for divergence near a local maxima, and the analogy still holds for the half-stableness when  $x^*$  is an inflection point.  $\square$

**Note.** Every one dimensional system is a gradient flow, because if  $f$  is smooth, then we can take the integral and define

$$V(x) := - \int_0^x f(s) \, ds.$$

### 4.1 Impossibility of Oscillations

**Definition.** *Periodic Functions*

If there exists a constant  $p > 0$  so that for all  $t$  we have

$$x(t + p) = x(t),$$

then we say that  $p$  is *periodic*.

**Note.** All constant functions are periodic.

**Theorem.** There are no non-constant periodic solutions of the system

$$\dot{x} = f(x).$$

*Proof.* Suppose that  $x$  is a periodic solution, with period  $p > 0$ . If  $0 \leq t \leq p$  then, as the potential energy is non-increasing,

$$V[x(p)] \leq V[x(t)] \leq V[x(0)].$$

Since  $x(p) = x(0)$ , we have  $V[x(t)]$  is constant. Hence  $x(t)$  is constant.  $\square$

## 4.2 Numerical Methods

### 4.2.1 Integral Equations

We want to find a solution of the equation

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

Observe that

$$\begin{aligned} \dot{x} &= f(x) \\ \int_0^t \frac{dx(s)}{ds} ds &= \int_0^t f(x(s)) ds \\ x(t) - x(0) &= \int_0^t f(x(s)) ds \\ x(t) &= x_0 + \int_0^t f(x(s)) ds. \end{aligned}$$

We call this an *integral equation*, because the unknown now appears in the integral.

**Example.** Write the equation

$$\begin{cases} \dot{x} = \sin x \\ x(0) = 1 \end{cases}$$

as an integral equation.

We have that

$$x(t) = 1 + \int_0^t \sin(x(s)) ds.$$

### 4.2.2 Numerical Approximation

Suppose we have the equation

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

Let's take  $\Delta t > 0$  small. Then we have

$$x(\Delta t) = x_0 + \int_0^{\Delta t} f(x(s)) ds.$$

We will approximate  $x(s) \approx x_0$  on the interval  $(0, \Delta t)$ . Thus we have

$$\begin{aligned} x(\Delta t) &= x_0 + \int_0^{\Delta t} f(x(s)) \, ds \\ &= x_0 + \int_0^{\Delta t} f(x_0) \, ds \\ &= x_0 + f(x_0)\Delta t. \end{aligned} \qquad (x_1 := x_0 + f(x_0)\Delta t)$$

Euler's method is to repeat the above to get  $x_2 \approx x(2\Delta t)$ . We have

$$\begin{aligned} x_2 &= x_1 + f(x_1)\Delta t, \\ x_3 &= x_2 + f(x_2)\Delta t, \\ &\dots \\ x_{n+1} &= x_n + f(x_n)\Delta t. \end{aligned}$$

## 5 Lecture 5

### 5.1 Euler's Method

- We want to approximate the solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

- Given a time step  $\Delta t$ , for  $n \geq 0$  define

$$x_{n+1} = x_n + f(x_n)\Delta t$$

- Thus we take  $x_n$  to be our approximation to  $x(n\Delta t)$ .

#### 5.1.1 Truncation Error of a Numerical Method

- Let  $x_n \approx x(n\Delta t)$ .
- We define the *local truncation error* to be

$$e_1 = x(\Delta t) - x_1.$$

We will use Taylor's theorem with Lagrange residue to approximate the error:

$$\begin{aligned} x(\Delta t) &= x(0) + x'(0)\Delta t + \frac{x''(\xi)}{2}(\Delta t)^2 & (\xi \in (0, \Delta t)) \\ &= x_0 + f(x_0)\Delta t + \frac{f'(\xi)f(\xi)}{2}(\Delta t)^2. & (\dot{x} = f, \text{ Chain rule}) \end{aligned}$$

Thus if  $f$  and  $f'$  are bounded and continuous, then  $|\text{Remainder}| \leq C(\Delta t)^2$ . Substituting into our local truncation error definition, we have

$$\begin{aligned} e_1 &= x(\Delta t) - x_1 \\ &= x(\Delta t) - (x_0 + f(x_0)\Delta t) \\ &= \frac{x''(\xi)}{2}(\Delta t)^2. \end{aligned}$$

Hence  $e_1$  is bounded above by  $C(\Delta t)^2$ .

**Note.** If we decrease  $\Delta t$ , then we decrease our error as well.

Euler's method is of first order because  $|e_1| \leq C(\Delta t)^2$ . If we apply Euler's method  $n$  times over some interval, then we will get  $n$  errors:

$$|e_1| + |e_2| + \cdots + |e_n| \leq Cn(\Delta t)^2 = Cn \left( \frac{T}{n} \right) \Delta t = CT\Delta t.$$

#### How can we improve our results?

- Take smaller time steps  $\Delta t$ .
  - Unfortunately, this means that we need to perform more computations.
  - There will be more “round-off errors”.
- Improve the approximation (see next section)

### 5.1.2 Improved Euler's Method

- We still want to approximate the solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

- For  $n \geq 0$ :
  - We make our first approximation

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t.$$

- Use this to make a better approximation

$$x_{n+1} = x_n + \frac{1}{2}(f(x_n) + f(\tilde{x}_{n+1}))\Delta t.$$

- Take  $x_n$  to be our approximation for  $x(n\Delta t)$ .

The local truncation error for the improved Euler's method is of the form  $C(\Delta t)^3$ , and the global error is  $CT(\Delta t)^2$ .

### 5.1.3 Runge-Kutta 4th Order Method

- We still want to approximate the solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

- For  $n \geq 0$ , take:
  - $k_n^{(1)} = f(x_n)\Delta t$
  - $k_n^{(2)} = f(x_n + \frac{1}{2}k_n^{(1)})\Delta t$
  - $k_n^{(3)} = f(x_n + \frac{1}{2}k_n^{(2)})\Delta t$
  - $k_n^{(4)} = f(x_n + \frac{1}{2}k_n^{(3)})\Delta t$

- Then we may set

$$x_{n+1} = x_n + \frac{1}{6}(k_n^{(1)} + 2k_n^{(2)} + 2k_n^{(3)} + k_n^{(4)}).$$

- The local truncation error satisfies

$$|e_1| \leq C(\Delta t)^5,$$

and the global error is 4<sup>th</sup> order, satisfying  $CT(\Delta t)^4$ .

## 5.2 Existence and Uniqueness of Solutions

### **Theorem** — *Cauchy-Peano Existence Theorem*

Let  $f: (a, b) \rightarrow \mathbb{R}$  be continuous and  $x_0 \in (a, b)$ . Then there exists some  $\delta > 0$  and a solution  $x: [-\delta, \delta] \rightarrow \mathbb{R}$  of the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

**Theorem** — *Picard–Lindelöf Existence and Uniqueness Theorem*

Let  $f: (a, b) \rightarrow \mathbb{R}$  be continuous and  $x_0 \in (a, b)$ . If  $f$  is locally Lipschitz continuous, then there exists a *unique local solution*  $\bar{x} \in C^1(I, \mathbb{R})$  of the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

where  $I$  is some interval around 0.

## 6 Lecture 6

### 6.1 Existence and Uniqueness

**Definition.** *Locally Lipschitz Continuity*

Let  $f: (a, b) \rightarrow \mathbb{R}$ . We call  $f$  *locally Lipschitz continuous* if for every  $[d, c] \subseteq (a, b)$  there exists  $k > 0$  such that

$$|f(x) - f(y)| \leq k|x - y|. \quad \forall x, y \in [d, c]$$

**Definition.** *Global Lipschitz Continuity*

A function  $f: (a, b) \rightarrow \mathbb{R}$  is said to be *global Lipschitz continuous* if there exists  $L > 0$  such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in (a, b)$$

**Fact.** Continuously differentiable functions are Lipschitz continuous, and so are continuous.

**Example.**

- The function  $f(x) = \sqrt{|x|}$  is continuous but it is not Lipschitz continuous.
- The function  $g(x) = |x|$  is not differentiable but it is Lipschitz continuous on  $[-1, 1]$ .

**Note.** In general, if it has a cusp, then it is not Lipschitz continuous.

#### 6.1.1 Finite Time Blowup

**Example.** Does the solution of

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$$

blow up in finite time?

If we solve the differential equation, then we get  $x(t) = \frac{1}{1-t}$  (which is not well defined for all  $t > 0$ ).

**Theorem.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be Lipschitz continuous. Then there exists a unique *global* solution  $x: \mathbb{R} \rightarrow \mathbb{R}$  of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

**Theorem.** Let  $f \leq g$  be smooth and let  $x_0 \leq y_0$ . Suppose that  $x$  and  $y$  are solutions of the ODES

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y) \\ y(0) = y_0 \end{cases}$$

on an interval  $[0, T]$ . Then  $x(t) \leq y(t)$  for all  $t \in [0, T]$ .



## 6.2 Bifurcations

### 6.2.1 External Parameters

- Consider the ODE

$$\dot{x} = f(x, r)$$

where  $r$  is a parameter of the model.

- **Question.** How do the dynamics vary as we vary  $r$ ?

**Example.** Consider the differential equation  $\dot{x} = r + x^2$ . Depending on the value of  $r$ , we could have:

- 2 critical points, one of which is stable (when  $r < 0$ )
- 1 critical point, which is half-stable (when  $r = 0$ )
- 0 critical points, (when  $r > 0$ )

This change in behavior as  $r$  goes from negative to positive is called a *bifurcation*. We say that a bifurcation occurs at  $(x^*, r^*) = (0, 0)$ .

## 7 Lecture 7

### 7.1 Saddle-Node Bifurcations

**Definition.** *Bifurcation*

Consider the following autonomous system

$$\dot{x} = f(x, \lambda)$$

where  $x \in \mathbb{R}$  and  $\lambda \in \mathbb{R}$ . A *bifurcation* occurs at parameter  $\lambda = \lambda_0$  if there are parameter values  $\lambda_1$  arbitrarily close to  $\lambda_0$  with dynamics topologically inequivalent from those at  $\lambda_0$ .

### 7.2 Identifying Bifurcations

If the system

$$\dot{x} = f(x, r)$$

has a bifurcation at  $(x, r) = (x^*, r^*)$ , then

$$f(x^*, r^*) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(x^*, r^*) = 0.$$

**Note.** The converse is *not* necessarily true! This helps us find *possible* bifurcation points, but we still need to perform further analysis in order to check whether these points are *actually* bifurcation points.

*Proof.* We define  $\eta = x - x^*$ , which implies that  $\dot{\eta} = \frac{\partial f}{\partial x}(x^*, r^*)\eta$ , which is stable if  $\frac{\partial f}{\partial x}(x^*, r^*) < 0$  and unstable if  $\frac{\partial f}{\partial x}(x^*, r^*) > 0$ . Hence there is a change in stability when  $\frac{\partial f}{\partial x}(x^*, r^*) = 0$ .  $\square$

## 8 Lecture 8

### 8.1 Transcritical Bifurcations

We are analyzing differential equations of the form

$$\dot{x} = rx - x^2 = x(r - x).$$

We see that there are fixed points at  $x = 0, r$ . To find the bifurcations, we solve for when

$$\begin{aligned} f(x, r) &= rx - x^2 = 0, \\ \frac{\partial f}{\partial x}(x, r) &= r - 2x = 0. \end{aligned}$$

Hence we have  $2x^2 - x^2 = x^2 = 0$ , so the only possible bifurcation occurs at  $x = r = 0$ .

## 9 Lecture 9

### 9.1 Subcritical Pitchfork

Look at the bifurcation diagram for

$$\dot{x} = rx + x^3.$$

### 9.2 Supercritical Pitchfork

Look at the bifurcation diagram for

$$\dot{x} = rx - x^3.$$

### 9.3 Symmetry

Note that the Saddle-node bifurcation ( $\dot{x} = r + x^2$ ) and the Transcritical bifurcation ( $\dot{x} = rx - x^2$ ) are *not* symmetric with respect to  $y = -x$ , i.e.  $x \rightarrow -x$ . For the Saddle-node bifurcation, we have

$$\begin{aligned}\dot{x} &= r + x^2 \\ -\dot{y} &= r + x^2 \\ \dot{y} &= -r - y^2,\end{aligned}$$

and the Transcritical bifurcation yields

$$\begin{aligned}\dot{x} &= rx - x^2 \\ -\dot{y} &= -ry - y^2 \\ \dot{y} &= ry + y^2.\end{aligned}$$

However, for the subcritical pitchfork bifurcation, we have

$$\begin{aligned}\dot{x} &= rx + x^3 \\ -\dot{y} &= -ry - y^3 \\ \dot{y} &= ry + y^3,\end{aligned}$$

so it *is* symmetric.

### 9.4 Hysteresis

Consider the ODE given by

$$\dot{x} = rx + x^3 - x^5.$$

## 10 Lecture 10

**Note.** Hysteresis is a concept that appears due to the non-reversibility as the parameter  $r$  varies. If we look at the bifurcation diagram's stable branches, we take one path as we increase  $r$  past 0, but take a different path as we decrease  $r$  back below 0.

### 10.1 Taylor's Theorem

#### 10.1.1 Single Variable

Let  $F(t)$  be continuous and  $\frac{d^n F}{dt^n}$  is continuous for  $1 \leq n \leq N + 1$ . Then we can write

$$F(t) = \sum_{n=0}^N \frac{1}{n!} \frac{d^n F}{dt^n}(0) t^n + R_N(t),$$

where  $R_N(t) = \underbrace{\frac{1}{(N+1)!} \frac{d^{N+1} F}{dt^{N+1}}(\tilde{t}) t^{N+1}}_{\text{Lagrange form residue}}$  for some  $\tilde{t} \in (0, t)$ .

#### 10.1.2 Multi Variable

Let  $f(x, r)$  be smooth (so  $\frac{\partial^n}{\partial x^n} \frac{\partial^m}{\partial r^m} f$  is continuous for all  $m, n \geq 1$ ). Then let  $F(t) = f(tx, tr)$  and apply Taylor's Theorem. We have

$$\frac{d^n}{dt^n} F(t) = \sum_{j=0}^n \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(tx, tr) x^{n-j} r^j.$$

When we take  $t = 1$ , we have

$$\begin{aligned} f(x, t) &= F(1) \\ &= \sum_{n=0}^N \frac{1}{n!} \frac{d^n F}{dt^n}(0) + R_N(0) \\ &= \sum_{n=0}^N \sum_{j=0}^n \frac{1}{(n-j)!j!} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(0, 0) x^{n-j} r^j + R_N(0) \end{aligned}$$

#### **Theorem** — *Taylor's Theorem*

Suppose all partial derivatives of  $f(x, r)$  up to order  $N + 1$  are continuous. Then,

$$f(x, r) = \sum_{n=0}^N \sum_{j=0}^n \frac{1}{(n-j)!j!} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(0, 0) x^{n-j} r^j + R_N(x, r),$$

where the remainder term can be written as

$$R_N(x, r) = \sum_{j=0}^{N+1} \frac{1}{(N+1-j)!j!} \frac{\partial^{N+1} f}{\partial x^{N+1-j} \partial r^j}(tx, tr) x^{N+1-j} r^j,$$

for some  $0 < t < 1$ .

**Example. Special Case**

Consider the case where  $N = 2$  for the Taylor expansion. Plugging that into the formula, we have that the quadratic expansion of  $f(x, r)$  at  $(0, 0)$  is

$$\begin{aligned} f(x, r) &= f(0, 0) + \frac{\partial f}{\partial x}(0, 0) \cdot x + \frac{\partial f}{\partial r}(0, 0) \cdot r \\ &\quad + \frac{\partial^2 f}{\partial x^2}(0, 0) \cdot \frac{x^2}{2} + \frac{\partial^2 f}{\partial x \partial r}(0, 0) \cdot xr + \frac{\partial^2 f}{\partial r^2}(0, 0) \cdot \frac{r^2}{2}. \end{aligned}$$

## 10.2 Normal Forms

Consider the Taylor expansion for a saddle-node bifurcation about the point  $(x^*, r^*)$ :

$$\begin{aligned} \dot{x} &= f(x, r) \\ &= f(x^*, r^*) + \underbrace{\frac{\partial f}{\partial x}(x^*, r^*)(x - x^*)}_{q_1} + \underbrace{\frac{\partial f}{\partial r}(x^*, r^*)(r - r^*)}_{p_1} \\ &\quad + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x^*, r^*)(x - x^*)^2}_{q_2} + \underbrace{\frac{\partial^2 f}{\partial x \partial r}(x^*, r^*)(x - x^*)(r - r^*)}_{p_2} \\ &\quad + \underbrace{\frac{1}{2} \frac{\partial^2 f}{\partial r^2}(x^*, r^*)(r - r^*)^2}_{Q} + \text{higher order terms.} \end{aligned}$$

**Theorem.** Suppose that  $f(x^*, r^*) = 0$ ,  $q_1 = 0$ ,  $p_1 \neq 0$ ,  $q_2 \neq 0$ . Then  $\dot{x} = f(x, r)$  undergoes a saddle-node bifurcation at  $(x^*, r^*)$  and

$$\dot{x} = \frac{\partial f}{\partial r}(x^*, r^*)(r - r^*) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x^*, r^*)(x - x^*)^2 + \mathcal{O}(\varepsilon^3)$$

for  $|r - r^*| < \varepsilon^2$  and  $|x - x^*| < \varepsilon$ . Moreover, there exists a change of variables

$$(t, x, r) \mapsto (s, y, R)$$

such that

$$\dot{x} = p_1(r - r^*) + q_2(x - x^*)^2 + \text{higher order terms}$$

takes the form

$$\frac{dy}{ds} = R + y^2 \quad (\text{Saddle-node bifurcation})$$

near  $(0, 0) = (y(x^*), R(r^*))$ .

## 11 Lecture 11

### 11.1 Normal Forms for Saddle-Node Bifurcations

**Note.** The mapping in the previous theorem basically just maps  $(x^*, r^*)$  to the origin in the  $yR$ -plane.

**Example.** Consider the ODE given by

$$\dot{x} = r + x - 1 - e^{x-1}.$$

To find possible bifurcation points, we find where

$$\begin{aligned} r + x - 1 - e^{x-1} &= 0, \\ 1 - e^{x-1} &= 0. \end{aligned}$$

Hence the only possible bifurcation point occurs at  $(1, 1)$ . We now Taylor expand  $f$  about  $(1, 1)$  to get

$$\begin{aligned} f(x, r) &= r + x - 1 - \left( 1 + (x - 1) + \frac{1}{2}(x - 1)^2 + \dots \right) \\ &= (r - 1) - \frac{1}{2}(x - 1)^2 + \mathcal{O}(|x - 1|^3). \end{aligned}$$

We can see that the above takes the form

$$\frac{dy}{ds} = R + y^2,$$

so we have a saddle-node bifurcation.

### 11.2 Normal Forms for Transcritical Bifurcations

**Theorem.** Suppose that  $\dot{x} = f(x, r)$  has a bifurcation at  $(x, r) = (x^*, r^*)$ . If

$$\begin{aligned} \frac{\partial^n f}{\partial r^n}(x^*, r^*) &= 0 \quad \text{for all } n \\ \frac{\partial^2 f}{\partial x \partial r}(x^*, r^*) &\neq 0 \\ \frac{\partial^2 f}{\partial x^2}(x^*, r^*) &\neq 0 \end{aligned}$$

Then  $(x^*, r^*)$  is a transcritical bifurcation.

## 12 Lecture 12

### 12.1 Dimensional Analysis

**Note.** For dimensional analysis:

- Express all of your time derivatives in a unit-less form by substituting another variable, i.e.  $\tau := \frac{t}{T}$ .
- Make the equation truly dimensionless by dividing all sides by the common unit, i.e. a unit of force.

Since we are not well-equipped to study second-order differential equations yet, we try to find certain conditions that make the second-order term vanish to zero (but not any of the other terms). Consider the following differential equation:

$$\frac{r}{gT^2} \cdot \frac{d^2\phi}{d\tau^2} = -\frac{b}{mgT} \frac{d\phi}{d\tau} - \sin \phi + \frac{r\Omega^2}{g} \sin \phi \cos \phi.$$

We wish to study a regime where

$$\frac{b}{mgT} = 1,$$

so we have that  $T = \frac{b}{mg}$ . Plugging this  $T$  into the second-order term above, we get

$$\underbrace{\frac{r}{g\left(\frac{b}{mg}\right)^2}}_{\varepsilon} \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \underbrace{\frac{r\Omega^2}{g}}_{\gamma} \sin \phi \cos \phi$$

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi.$$

If  $\varepsilon \ll 1$ , then we get that

$$\frac{r}{g\left(\frac{b}{mg}\right)^2} \ll 1$$

$$m^2gr \ll b^2.$$

Thus we are left with the equation

$$0 = -\frac{d\phi}{d\tau} - \sin \phi + \gamma \sin \phi \cos \phi$$

$$\frac{d\phi}{d\tau} = -\sin \phi + \gamma \sin \phi \cos \phi$$

$$\frac{d\phi}{d\tau} = \sin \phi (\gamma \cos \phi - 1).$$

**Note.** The non-dimensionalized form of our model is

$$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = \sin \phi (\gamma \cos \phi - 1).$$

If  $\varepsilon \ll 1$ , then

$$\frac{d\phi}{d\tau} = \sin \phi (\gamma \cos \phi - 1).$$



### 12.1.1 Bifurcation Analysis of the First-order Equation

We see that we have fixed points when  $\phi = n\pi$  for all  $n \in \mathbb{Z}$  or  $\cos \phi = \frac{1}{\gamma}$ . When  $0 < \gamma < 1$ , then there are no solutions to  $\cos \gamma = \phi$ . For  $\gamma > 1$ , we are going to have two solutions, symmetric about  $\phi = 0$ , given by  $\phi^* = \pm \cos^{-1}(\frac{1}{\gamma})$ . Finally, when  $\gamma = 1$ , then  $\phi = 0$ .

We wish to find when  $\frac{\partial f}{\partial \phi} = 0$ , so

$$\begin{aligned} \cos \phi(\gamma \cos \phi - 1) + \sin \phi(-\gamma \sin \phi) &= \gamma \cos^2 \phi - \cos \phi - \gamma \sin^2 \phi \\ &= \gamma \cos^2 \phi - \cos \phi - \gamma(1 - \cos^2 \phi) \\ &= 2\gamma \cos^2 \phi - \cos \phi - \gamma. \end{aligned}$$

## 13 Lecture 13

We now perform stability analysis on the differential equation from last lecture. When we are at the fixed point  $\phi = 0$ , we have that

$$\begin{aligned}\left.\frac{\partial f}{\partial \phi}\right|_{\phi=0} &= (2\gamma \cos^2 \phi - \cos \phi - \gamma)|_{\phi=0} \\ &= \gamma - 1.\end{aligned}$$

- If  $\gamma > 1$ , then we have an unstable fixed point at  $\phi = 0$ .
- If  $0 < \gamma < 1$ , then we have a stable fixed point at  $\phi = 0$ .
- If  $\gamma = 1$ , there is a possible bifurcation at  $\phi = 0$ .

We now consider the fixed points  $\phi = \pm\pi$ .

$$\begin{aligned}\left.\frac{\partial f}{\partial \phi}\right|_{\phi=\pi} &= (2\gamma \cos^2 \phi - \cos \phi - \gamma)|_{\phi=\pi} \\ &= \gamma + 1.\end{aligned}$$

- We have that this fixed point is unstable for all  $\gamma > 0$ .

Finally, we consider the fixed points  $\phi = \pm \cos^{-1}(\frac{1}{\gamma})$ .

$$\begin{aligned}\left.\frac{\partial f}{\partial \phi}\right|_{\phi=\pm \cos^{-1}(1/\gamma)} &= (2\gamma \cos^2 \phi - \cos \phi - \gamma)|_{\phi=\pm \cos^{-1}(1/\gamma)} \\ &= \frac{1}{\gamma} - \gamma.\end{aligned}$$

- If  $\gamma > 1$ , then we have stable fixed points.
- If  $0 < \gamma < 1$ , then we have unstable fixed points.
- If  $\gamma = 1$ , there is a possible bifurcation.

### 13.1 Singularly Perturbed Problems

We consider problems of the form

$$\varepsilon \phi'' + \phi' = f(\phi).$$

We wish to see how the solutions of  $\phi' = f(\phi)$  relate to the solutions of the above.

### 13.2 Another Non-dimensionalization Problem

Consider a damped pendulum with length  $L$ , mass  $m$ , friction constant  $k$  and gravitational constant  $g$ . By Newton's Second Law we have

$$mL\ddot{\theta} = -k\dot{\theta} - mg \sin \theta.$$

**Question.** In which region can we approximate it by a first order differential equation?

We again define  $\tau := \frac{t}{T}$ , where  $T > 0$  is a typical time scale. Thus

$$\frac{d\theta}{d\tau} = \frac{d\theta}{dt} \cdot \frac{dt}{d\tau} = \frac{d\theta}{dt} \cdot T.$$

Hence we have

$$\frac{d^2\theta}{d\tau^2} = \frac{d^2\theta}{dt^2} \cdot \frac{1}{T^2}.$$

Therefore the original equation can be rewritten as

$$\frac{mL}{T^2} \cdot \frac{d^2\theta(\tau)}{d\tau^2} = -\frac{k}{T} \cdot \frac{d\theta(\tau)}{d\tau} - mg \sin(\theta(\tau))$$

Since the units of each of these terms is force, we divide everything by  $mg$ , yielding

$$\frac{L}{T^2g} \cdot \frac{d^2\theta(\tau)}{d\tau^2} = -\frac{k}{Tmg} \cdot \frac{d\theta(\tau)}{d\tau} - \sin(\theta(\tau)).$$

We choose  $T = \frac{k}{mg}$ , giving us a coefficient of

$$\varepsilon = \frac{L}{g \left( \frac{k}{mg} \right)^2} = \frac{m^2gL}{k^2},$$

so

$$\varepsilon \ddot{\theta} = -\dot{\theta} - \sin \theta.$$

If we take  $\varepsilon \ll 1$  then  $\dot{\theta} = -\sin \theta$ , which is quite a good approximation to the original problem.

## 14 Lecture 14

### 14.1 Imperfect Bifurcations

We consider differential equations of the form

$$\dot{x} = f(x, r, h).$$

Note that in this case we have *two* parameters. How can we study fixed points/stability/bifurcations?

**Example.** Consider the differential equation given by  $\dot{x} = h + rx - x^3$ . We can rewrite this as  $\dot{x} = x(r - x^2) - (-h)$ , and we let  $y_1 = x(r - x^2)$  and  $y_2 = -h$ . When  $h = 0$ , we have a supercritical pitchfork bifurcation. We call  $h$  the *imperfection parameter*. We first fix  $r$  and vary  $h$ . Critical points occur whenever  $y_1(x) = y_2(x)$ . We see that when  $r \leq 0$  we have 1 stable critical point.