Math 131A Lecture Notes

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2021-09-24

1 Lecture 1

1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

1.3 Logical Connections

We usually use the letters P and Q to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: "P and Q", $P \land Q$
2. Disjunctions: "P or Q", $P \lor Q$
3. Implications: "If P, then Q", $P \implies Q$
   (a) If the proposition is false (i.e. if P is false) then the whole statement is true.

   > **Definition.**
   > We say that the statement is *vacuously true*.

4. Negations: "Not P", $\neg P$

1.3.1 Truth Tables

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example. Prove that if n is an integer, then $n(n+1)$ is even.

Proof. Suppose that n is an integer. Then we have two cases, where either n is even or n is odd. Let n be an even integer such that $n = 2k$ where $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= 2k(2k+1) \\ &= 2(2k^2 + k). \end{aligned}$$

Thus we see that $n(n+1)$ is even when n is even. Now let n be odd such that $n = 2m+1$ where $m \in \mathbb{Z}$. Then we have

$$\begin{aligned} n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1). \end{aligned}$$

Thus $n(n+1)$ is also even when n is odd, and so is even for all integers n. $\square$

2 Lecture 2

2.1 Continuation of Logic

2.1.1 De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

Note. Negations turn "and" into "or" and vice versa.

Example. Suppose we have the following statement:

$$P\colon x \text{ is even and } x > 0.$$

Then the negation of P would be:

$$\neg P\colon x \text{ is odd or } x \leq 0.$$

2.1.2 Converse

Definition. *Converse*
The *converse* of a statement $P \implies Q$ is the statement $Q \implies P$. In general, the converse of a statement says nothing about the original statement.

Example. Consider the statement

$$\text{If } x > 0 \text{, then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0 \text{, then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write $P \iff Q$ instead of $(P \implies Q) \wedge (Q \implies P)$. In this case, we call P and Q *logically equivalent*. In writing, we say "P if and only if Q".

2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

Lemma 1. Let a be an integer. If a^2 is even, then a is even.

Proof. Suppose a is odd, so $a = 2k + 1$ for some integer k. Then

$$\begin{aligned} a^2 &= (2k+1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus a^2 is odd and this completes the proof. $\square$

2.1.4 Variables and Quantifiers

We have a value x that varies over some values, so we use $P(x)$ to denote a statement that depends on the value of x.

Example. Consider the statement
$$P(x) : x + 2 = 3.$$
The statement is true if and only if $x = 1$.

We have two quantifiers—$\forall$ = "for all", and $\exists$ = "there exists".

- $\forall x : P(x)$ is true if $P(x)$ is true for all x.
- $\exists x : P(x)$ is true if there exists at least one x such that $P(x)$ is true.

We can also have nested quantifiers,
$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

Note. The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of n depends on m.

2.1.5 Proof by Counterexample

After "simplifying" the statement $\neg(\forall x : P(x))$, we get $\exists x : \neg P(x)$. We simply need to find a single counterexample to show that a statement is false for all x.

Example. Consider the statement $\forall x \in \mathbb{R} : x + 2 = 3$. All we need to do is show that there exists some $x \in \mathbb{R}$ such that $x + 2 \neq 3$. This occurs when $x = 0$, so the statement is false.

2.1.6 Proof by Contradiction

Key Idea. We want to show that $P \implies Q$ indirectly.

Lemma 2. We can show that
$$P \implies Q = (\neg P) \vee Q.$$
Then $P \implies Q$ is true if and only if $\neg(P \implies Q)$ is false, and so by Lemma 2 and De Morgan's Laws, $P \wedge \neg Q$ is false.

For proof by contradiction, we assume P is true and $\neg Q$ is true, and try to show that $P \wedge \neg Q$ is false (a contradiction).