

# Math 131A Lecture Notes

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# 1 Lecture 1

## 1.1 Goals of This Class

The end goal of this class is to go over the Fundamental Theorem of Calculus, and to do that we must first cover:

- What is a real number and what makes them special?
- How do we define the convergence of a sequence of real numbers?
- What is a limit?
- What is continuity?
- What is a derivative?
- What is an integral?

## 1.2 Mathematical Arguments and Logic

We have the following components to reasoning about solutions to a mathematical problem:

- Assumptions
- Logical steps
- Conclusions

## 1.3 Logical Connections

We usually use the letters  $P$  and  $Q$  to denote logical statements (either true or false). We can also use conjunctions to connect logical statements to one another.

1. Conjunctions: “ $P$  and  $Q$ ”,  $P \wedge Q$
2. Disjunctions: “ $P$  or  $Q$ ”,  $P \vee Q$
3. Implications: “If  $P$ , then  $Q$ ”,  $P \implies Q$ 
  - (a) If the proposition is false (i.e. if  $P$  is false) then the whole statement is true.

**Definition.**

We say that the statement is *vacuously true*.

4. Negations: “Not  $P$ ”,  $\neg P$

### 1.3.1 Truth Tables

$P$	$Q$	$P \implies Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

**Example.** Prove that if  $n$  is an integer, then  $n(n+1)$  is even.

*Proof.* Suppose that  $n$  is an integer. Then we have two cases, where either  $n$  is even or  $n$  is odd. Let  $n$  be an even integer such that  $n = 2k$  where  $k \in \mathbb{Z}$ . Then we have

$$\begin{aligned} n(n+1) &= 2k(2k+1) \\ &= 2(2k^2 + k). \end{aligned}$$

Thus we see that  $n(n+1)$  is even when  $n$  is even. Now let  $n$  be odd such that  $n = 2m+1$  where  $m \in \mathbb{Z}$ . Then we have

$$\begin{aligned} n(n+1) &= (2m+1)(2m+1+1) \\ &= (2m+1)(2m+2) \\ &= 2(m+1)(2m+1). \end{aligned}$$

Thus  $n(n+1)$  is also even when  $n$  is odd, and so is even for all integers  $n$ . □

## 2 Lecture 2

### 2.1 Continuation of Logic

#### 2.1.1 De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

**Note.** Negations turn “and” into “or” and vice versa.

**Example.** Suppose we have the following statement:

$$P: x \text{ is even and } x > 0.$$

Then the negation of  $P$  would be:

$$\neg P: x \text{ is odd or } x \leq 0.$$

#### 2.1.2 Converse

**Definition.** *Converse*

The *converse* of a statement  $P \implies Q$  is the statement  $Q \implies P$ . In general, the converse of a statement says nothing about the original statement.

**Example.** Consider the statement

$$\text{If } x > 0, \text{ then } x^3 \neq 0.$$

The converse is then

$$\text{If } x^3 \neq 0, \text{ then } x > 0.$$

Note that the converse is false even though the original statement is true.

If we know both the original statement and the converse are true, then we may write  $P \iff Q$  instead of  $(P \implies Q) \wedge (Q \implies P)$ . In this case, we call  $P$  and  $Q$  *logically equivalent*. In writing, we say “ $P$  if and only if  $Q$ ”.

#### 2.1.3 Proof by Contrapositive

We can show that

$$(\neg Q) \implies (\neg P) \iff P \implies Q.$$

This gives us another way of proving the original statement, since the contrapositive has the same truth values.

**Lemma 1.** Let  $a$  be an integer. If  $a^2$  is even, then  $a$  is even.

*Proof.* Suppose  $a$  is odd, so  $a = 2k + 1$  for some integer  $k$ . Then

$$\begin{aligned} a^2 &= (2k + 1)^2 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Thus  $a^2$  is odd and this completes the proof.  $\square$

### 2.1.4 Variables and Quantifiers

We have a value  $x$  that varies over some values, so we use  $P(x)$  to denote a statement that depends on the value of  $x$ .

**Example.** Consider the statement

$$P(x) : x + 2 = 3.$$

The statement is true if and only if  $x = 1$ .

We have two quantifiers— $\forall$  = “for all”, and  $\exists$  = “there exists”.

- $\forall x : P(x)$  is true if  $P(x)$  is true for all  $x$ .
- $\exists x : P(x)$  is true if there exists at least one  $x$  such that  $P(x)$  is true.

We can also have nested quantifiers,

$$\forall m \in \mathbb{N}, \exists n \in \mathbb{N} : m < n.$$

**Note.** The order of quantifiers matters. In the above example, the quantifiers tell us that our choice of  $n$  depends on  $m$ .

### 2.1.5 Proof by Counterexample

After “simplifying” the statement  $\neg(\forall x : P(x))$ , we get  $\exists x : \neg P(x)$ . We simply need to find a single counterexample to show that a statement is false for all  $x$ .

**Example.** Consider the statement  $\forall x \in \mathbb{R} : x + 2 = 3$ . All we need to do is show that there exists some  $x \in \mathbb{R}$  such that  $x + 2 \neq 3$ . This occurs when  $x = 0$ , so the statement is false.

### 2.1.6 Proof by Contradiction

**Key Idea.** We want to show that  $P \implies Q$  indirectly.

**Lemma 2.** We can show that

$$P \implies Q = (\neg P) \vee Q.$$

Then  $P \implies Q$  is true if and only if  $\neg(P \implies Q)$  is false, and so by Lemma 2 and De Morgan’s Laws,  $P \wedge \neg Q$  is false.

For proof by contradiction, we assume  $P$  is true and  $\neg Q$  is true, and try to show that  $P \wedge \neg Q$  is false (a contradiction).

## 3 Lecture 3

### 3.1 More Logic

#### 3.1.1 Proof by Contradiction

To do a proof by contradiction for a statement  $P \implies Q$ , we assume  $P \wedge \neg Q$ . We aim to show that  $P \wedge \neg Q$  is false (a contradiction).

**Theorem** — *Irrationality of  $\sqrt{2}$*

There is no rational number  $x$  such that  $x^2 = 2$ . In other words, if  $x \in \mathbb{Q}$ , then  $x^2 \neq 2$ .

*Proof.* Suppose towards a contradiction that there exists some  $x \in \mathbb{Q}$  such that  $x^2 = 2$ . Since  $x$  is rational, there exist integers  $p, q$  such that  $q \neq 0$ ,  $\frac{p}{q} = x$ , and  $p$  and  $q$  have no common divisors (other than 1). Then

$$\begin{aligned} x^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ p^2 &= 2q^2. \end{aligned}$$

Since  $p^2$  is even, there exists some integer  $k$  such that  $p = 2k$ . Thus

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2. \end{aligned}$$

By the same logic as before, we know that  $q$  must also be even (they share a common factor of 2). However, this contradicts our original assumption that  $p$  and  $q$  share no common factors, and this completes the proof.  $\square$

### 3.2 Set Theory

We write  $x \in A$  when we want to say that “ $x$  is an element of  $A$ ”, and  $x \notin A$  when we want to say that “ $x$  is not an element of  $A$ ”.

#### 3.2.1 Set Combinations

- Union:  $A \cup B = \{x \mid x \in A \vee x \in B\}$ .
- Intersection:  $A \cap B = \{x \mid x \in A \wedge x \in B\}$ .
- Difference:  $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$ .
- Subset (Inclusion):  $A \subseteq B$  if and only if  $x \in A \implies x \in B$ .

**Definition.** *Proper Subset*

A set  $A$  is a *proper subset* of a set  $B$  if  $A \subseteq B$  and there exists some  $x \in B$  such that  $x \notin A$ . We denote this as  $A \subset B$ .

- Equality:  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$ .

**Note (Showing Equality of Sets).** If you want to show  $A = B$ , you need to show both  $A \subseteq B$  and  $B \subseteq A$ . In other words, you must show that for all  $x \in A$ , we have  $x \in B$ , and vice versa.



**Example.** We have  $\mathbb{N} = \{1, 2, 3, \dots\}$ .

- Let  $E$  be the set of even natural numbers. Note that  $E \subseteq \mathbb{N}$ .
- Let  $S = \{p \in \mathbb{Q} \mid p^2 < 2\} \subseteq \mathbb{Q}$ .

Then we have:

- $\mathbb{N} \cup E = \mathbb{N}$ .
- $\mathbb{N} \cap E = E$ .
- $\mathbb{N} \cap S = \{n \in \mathbb{N} \mid n^2 < 2\} = \{1\}$ .
- $E \cap S = \emptyset$ .

**Definition.** *Disjoint Sets*

If  $A \cap B = \emptyset$ , we call  $A$  and  $B$  *disjoint* sets.

*Proof.* Suppose towards a contradiction that there exists some  $x \in E \cap S$ , which is to say  $x \in E$  and  $x \in S$ . Since  $x \in E$ , we know that  $x$  is even, and so there exists some integer  $k$  such that  $x = 2k$ . Then

$$x^2 = (2k)^2 = 4k^2,$$

so  $4 \mid x^2$ . Therefore  $x \geq 4$ , which contradicts the condition for  $x \in S$ , namely  $x^2 < 2$ .  $\square$

- Given some  $n \in \mathbb{N}$ , we define  $A_n = \mathbb{N} \setminus \{1, \dots, n-1\}$ .
  - $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$ .
  - $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ .

**Definition.** *Set Complement*

If  $A \subseteq B$ , then we define the *complement* of  $A$  in  $B$  to be  $A^c = B \setminus A$ .

### 3.2.2 De Morgan's Laws

If  $I$  is an index set and  $\{A_j\}_{j \in I}$  are subsets of  $B$ , then

$$\left( \bigcup_{j \in I} A_j \right)^c = \bigcap_{j \in I} A_j^c, \quad \text{and} \quad \left( \bigcap_{j \in I} A_j \right)^c = \bigcup_{j \in I} A_j^c$$

## 4 Lecture 4

### 4.1 Cartesian Product

If I have two sets  $A$  and  $B$ , then we may form their *Cartesian Product*, which is

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}.$$

**Definition.** *Binary Relation*

A *binary relation* is a subset  $R \subseteq A \times B$ . We say  $x \in A$  is in relation to  $y \in B$  if  $(x, y) \in R$ . We denote this by

$$xRy \iff (x, y) \in R.$$

**Example.** Consider the relation

$$\mathbb{R} \times \mathbb{R} \supseteq R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}.$$

Then the relation is *reflexive*, because  $xRx$ . It is also *antisymmetric*, because  $xRy \wedge yRx \implies x = y$ . Finally, this relation is *transitive*, because  $xRy \wedge yRz \implies xRz$ .

These properties only make sense if  $A = B$ , i.e.  $R \subseteq A \times A$ , and we say that “ $R$  is a relation on  $A$ ”.

**Definition.** *Partial Order*

If a relation is reflexive, antisymmetric, and transitive on  $A$ , then it is a *partial order* on  $A$ .

The notion of “less than or equal to” is a partial order for  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ , but there exists no partial order for  $\mathbb{C}$ .

**Definition.** *Power Set*

For a set  $A$ , we may define its *power set* by

$$\mathcal{P}(A) = \{C \mid C \subseteq A\}.$$

Note that set inclusion is a partial order on  $\mathcal{P}(A)$ .

**Definition.** *Equivalence Relation*

An *equivalence relation*  $R$  over  $A$  is a relation that is reflexive, symmetric, and transitive.

**Note.** Just like a partial order behaves much like  $\leq$ , an equivalence relation behaves much like  $=$ .

**Definition.** *Equivalence Class*

Given an equivalence relation on  $A$ , we define a new set

$$[x] := \{y \in A \mid x \sim y\}.$$

We call  $[x]$  the *equivalence class* of  $x$ . Any  $z \in [x]$  is called a *representative* of the equivalence class  $[x]$ . In particular,  $x$  is a representative of its own equivalence class.

Let  $A$  be a set with equivalence relation  $\sim$ . Then for any  $x, y \in A$ ,

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \emptyset.$$

*Proof.* Let  $x, y \in A$ . We know that  $x$  is either equivalent to  $y$  or it is not. Suppose the former is true and let  $z \in [x]$ . Thus we know that  $z \sim x$  and  $x \sim y$ , and by transitivity we have  $z \sim y$ . Thus  $z \in [y]$  and  $[x] \subseteq [y]$ . The reverse argument is the same.

If  $x$  is not equivalent to  $y$ , then suppose towards a contradiction that  $[x] \cap [y] \neq \emptyset$ . Let  $x \in [x] \cap [y]$ . Then  $z \sim x$  and  $z \sim y$ . By symmetry we know that  $x \sim z$  and by transitivity we have  $x \sim y$ . We have arrived at the contradiction that  $x$  is both equivalent and not equivalent to  $y$ .  $\square$

**Definition.** *Function*

A relation  $R \subseteq A \times B$  is a *function* if for all  $x \in A$  and all  $y, z \in B$ , we have the following:

- $xRy \wedge xRz \implies y = z$ .

In other words, every input  $x$  has only one output.

**Definition.** *Injective Functions*

A function  $f$  is *injective* if  $f(x_1) = f(x_2) \implies x_1 = x_2$ .

**Definition.** *Surjective Functions*

A function  $f$  is *surjective* if for every  $y \in B$ , there exists some  $x \in A$  such that  $f(x) = y$ .

## 5 Lecture 5

### 5.1 The Natural Numbers

We know that the natural numbers are defined via:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

#### 5.1.1 Properties of the Natural Numbers

- (P1)  $1 \in \mathbb{N}$
- (P2) If  $n \in \mathbb{N}$ , then it has a *successor*,  $n + 1 \in \mathbb{N}$
- (P3) 1 is *not* the successor of any element of  $\mathbb{N}$
- (P4) If  $m, n$  have the same successor, then  $m = n$

**Note.** The properties above can be abstracted to become:

- (P1)  $1 \in \mathbb{N}$
- (P2) There exists some  $S: \mathbb{N} \rightarrow \mathbb{N}$  where  $S(n)$  is the successor  $n$
- (P3)  $1 \notin \text{range } S$
- (P4)  $S$  is injective
- (P5) Suppose  $A \subseteq \mathbb{N}$  with the properties:
  - (i)  $1 \in A$
  - (ii) If  $n \in A$ , then  $S(n) \in A$
 Then  $A = \mathbb{N}$ .

#### Theorem — 1.1 (Induction)

Let  $\{P(n) \mid n \in \mathbb{N}\}$  be a set of logical propositions. Suppose that

- (i)  $P(1)$  is true.
- (ii) If  $P(n)$  is true, then  $P(n + 1)$  is true.

Then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

*Proof.* Let  $A = \{n \mid P(n) \text{ is true}\} \subseteq \mathbb{N}$ . By our first assumption,  $1 \in A$ . By (ii), if  $n \in A$ , then  $n + 1 \in A$ . So by (P5) of the natural numbers, we know that  $A = \mathbb{N}$ .  $\square$

#### Definition. 1.2 (Peano Axioms)

A triplet  $(\mathbb{N}, 1, S)$  is said to be a *system of the naturals* if it satisfies:

- 1)  $\mathbb{N}$  is a set and  $1 \in \mathbb{N}$
- 2)  $S: \mathbb{N} \rightarrow \mathbb{N}$  is a function
- 3)  $1 \notin \text{range } S$
- 4)  $S$  is injective
- 5)  $\forall A \subseteq \mathbb{N}$  such that  $1 \in A$  and  $S(A) \subseteq A$ , then  $A = \mathbb{N}$

**Definition.** *Addition*

We define the binary relation  $+$  over  $\mathbb{N}$ :

- (i)  $\forall n \in \mathbb{N}, n + 1 := S(n)$
- (ii)  $\forall m, n \in \mathbb{N}, \text{ we have } m + S(n) = S(m + n)$

The following properties can be proven from the above definition of addition:

- (a) Associativity:  $\forall x, y, z \in \mathbb{N}, \text{ we have } (x + y) + z = x + (y + z)$
- (b) Commutativity:  $\forall x, y \in \mathbb{N}, \text{ we have } x + y = y + x$
- (c) Cancellative Law:  $\forall x, y, z \in \mathbb{N}, \text{ we have } x + y = y + z \implies x = z$

**Theorem — 1.3** (*Existence of the Naturals*)

There exists a unique system of the naturals. In other words, there is a bijection between different systems of the naturals and so we may write:

$$(\mathbb{N}, 1, S) \iff (\mathbb{N}', 1', S').$$

## 5.2 Fields

**Definition.** *Field*

A *field* is a set with two binary operations,

- $+$ , or ‘addition’
- $\cdot$ , or ‘multiplication’

### 5.2.1 Axioms for Addition

- (A1)  $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x + y \in \mathbb{F}$
- (A2)  $\forall x, y \in \mathbb{F}, \text{ we have } x + y = y + x$
- (A3)  $\forall x, y, z \in \mathbb{F}, \text{ we have } (x + y) + z = x + (y + z)$
- (A4) There exists some  $0 \in \mathbb{F}$  such that  $0 + x = x$  for all  $x \in \mathbb{F}$
- (A5)  $\forall x \in \mathbb{F}, \text{ there exists } -x \in \mathbb{F} \text{ such that } x + (-x) = 0$

### 5.2.2 Axioms for Multiplication

- (M1)  $x \in \mathbb{F} \wedge y \in \mathbb{F} \implies x \cdot y \in \mathbb{F}$
- (M2)  $\forall x, y \in \mathbb{F}, \text{ we have } x \cdot y = y \cdot x$
- (M3)  $\forall x, y, z \in \mathbb{F}, \text{ we have } (x \cdot y) \cdot z = x \cdot (y \cdot z)$
- (M4) There exists some  $1 \in \mathbb{F}$  such that  $1 \neq 0$  and  $1 \cdot x = x$  for all  $x \in \mathbb{F}$
- (M5)  $\forall x \in \mathbb{F}, \text{ there exists some } \frac{1}{x} \in \mathbb{F} \text{ such that } x \cdot \frac{1}{x} = 1$

### 5.2.3 Distributive Law

- (D1)  $\forall x, y, z \in \mathbb{F}, \text{ we have } x \cdot (y + z) = x \cdot y + x \cdot z$

## 6 Lecture 6

### 6.1 The Rationals

On the natural numbers, we have a notion of addition, multiplication, and comparison ( $\leq$ ). We constructed the integers and so we now have:

- A notion of an additive identity,  $0 \in \mathbb{Z}$
- Additive inverses

However, we *don't* have:

- Multiplicative inverses

When we think of the rationals, we consider the set

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

**Note.** When dealing with  $\mathbb{Q}$ , we now have multiplicative inverses.

In particular,  $(\mathbb{Q}, +, \cdot, \leq)$  is an *ordered field*.

**Definition.** *Ordered Field*

An *ordered field* is a field  $\mathbb{F}$  which is also an ordered set ( $\leq$ ) such that:

- (i) If  $x, y, z \in \mathbb{F}$  and  $y < z$ , then  $x + y < x + z$
- (ii) If  $x, y \in \mathbb{F}$ ,  $x > 0$  and  $y > 0$ , then  $x \cdot y > 0$

Unfortunately, the rational numbers still don't allow us to solve polynomial equations (i.e.  $x^2 = 2$ ).

**Definition.** *Algebraic Numbers*

A number is called *algebraic* if it solves

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where  $c_0, \dots, c_n \in \mathbb{Z}$ ,  $c_n \neq 0$ ,  $n \in \mathbb{N}$ .

**Example.** Every rational number is algebraic, because it solves the equation

$$qx - p = 0.$$

**Definition.** *Dividing*

We say  $k \in \mathbb{Z}$  *divides*  $m \in \mathbb{Z}$  if  $\frac{m}{k} \in \mathbb{Z}$ .

**Theorem — Rational Zeros Theorem**

Suppose  $c_0, \dots, c_n \in \mathbb{Z}$ ,  $c_n \neq 0$  and  $r \in \mathbb{Q}$  satisfies

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then writing  $r = \frac{c}{d}$  with  $c, d$  having no common factors,  $d \neq 0$ , we have:

$$\begin{aligned} c &\text{ divides } c_0 \\ d &\text{ divides } c_n \end{aligned}$$

*Proof.* Since  $r$  solves the equation, we have

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0.$$

Multiplying both sides by  $d^n$ , we get

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Rearranging, we have

$$c_n c^n = -d(c_{n-1} c^{n-1} + \cdots + c_1 c d^{n-2} + c_0 d^{n-1}).$$

We know that  $d \mid c_n c^n$ . Since  $c$  and  $d$  have no common factors, we know  $d \mid c_n$ . Rearranging terms again,

$$-c(c_n c^{n-1} + c_{n-1} c^{n-2} d + \cdots + c_1 d^{n-1}) = c_0 d^n.$$

By the same reasoning as before, we have that  $c \mid c_0$ . □

**Corollary.** Suppose  $r \in \mathbb{Q}$  solves

$$r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

Then  $r \in \mathbb{Z}$  and  $r \mid c_0$ .

*Proof.* Since  $r \in \mathbb{Q}$ , we may express  $r = \frac{c}{d}$ ,  $c \mid c_0$  and  $d \mid 1$ . From this we know that  $d = 1$ , so  $r = c$  and  $c \mid c_0$ . Therefore  $r \in \mathbb{Z}$  and  $r \mid c_0$ . □

We have some deficiencies for  $\mathbb{Q}$ :

- There seem to be some “gaps” in  $\mathbb{Q}$ .

**Proposition.** We consider the sets  $A = \{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 < 2\}$  and  $B = \{p \in \mathbb{Q} \mid p \geq 0 \text{ and } p^2 > 2\}$ . Notice that  $A$  has no largest element and  $B$  has no smallest element.

*Proof.* Given  $p \in \mathbb{Q}$ , let

$$q = p - \frac{p^2 - 2}{p^2 + 2} = \frac{2(p+1)}{p+2}.$$

We also have  $q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2}$ . If  $p \in A$ ,  $p^2 - 2 < 0$  so  $q > p$  and  $q^2 - 2 > 0$ . If  $p \in B$ ,  $p^2 - 2 > 0$  so  $q < p$  and  $q^2 - 2 < 0$ . □

## 7 Lecture 7

### Definition. 2.11—Upper and Lower Bounds

Let  $E$  be an ordered set and  $A \subset E$ .

- (a) If there exists  $x \in E$  such that for all  $a \in A$ ,  $a \leq x$ , we say  $A$  is *bounded above* by  $x$  and call  $x$  an *upper bound* for  $A$ .
- (b) Suppose  $A \subset E$  is non-empty and bounded above and there exists some  $x^* \in E$  such that
  - i)  $x^*$  is an upper bound for  $A$
  - ii) If  $y$  is any upper bound for  $A^*$ , then  $x^* \leq y$

Then we call  $x^*$  the *least upper bound* for  $A$ , and we write

$$x^* = \sup A. \quad (\text{sup meaning supremum})$$

The *greatest lower bound* or *infimum* of a set  $B$ , which is bounded below and non-empty, satisfies

- i)  $\inf B$  is a lower bound for  $B$
- ii) If  $y$  is any lower bound for  $B$ , then  $y \leq \inf B$

**Example.** Suppose

$$A = \{p \in \mathbb{Q} \mid p \geq 0, p^2 < 2\},$$

$$B = \{p \in \mathbb{Q} \mid p \geq 0, p^2 > 2\}.$$

Then  $A$  is bounded above (say by 2) and  $B$  is bounded below (say by 0). In the last lecture we proved that neither  $\sup A$  nor  $\inf B$  exist in  $\mathbb{Q}$  (because the values would have been  $\sqrt{2}$ ).

**Example.** Let

$$C = \{p \in \mathbb{Q} \mid p < 0\},$$

$$D = \{p \in \mathbb{Q} \mid p \leq 0\}.$$

Then

$$\sup C = \sup D = 0.$$

However, notice that  $\sup C \notin C$  and  $\sup D \in D$ .

### Definition. Maximum

We define  $\max A$  to be the largest element of  $A$ , which satisfies:

- i)  $\max A \in A$
- ii) For all  $a \in A$ ,  $a \leq \max A$

The definition for minimum is similar.

### Definition. 2.12—Least Upper Bound Property (LUBP)

An ordered set  $E$  has the *least upper bound property* if the following is true:

- i) If  $A \subseteq E$ ,  $A \neq \emptyset$ ,  $A$  is bounded above, then  $\sup A$  exists and  $\sup A \in E$ .



**Note.**  $\mathbb{Q}$  does not have the least upper bound property.

**Theorem — 2.13 (Existence of  $\mathbb{R}$ )**

There exists an ordered field  $\mathbb{R}$  which has

- i)  $\mathbb{Q}$  as a sub-field
- ii) The least upper bound property

## 7.1 Fundamental Properties of the Real Numbers (because of LUBP)

**Theorem — 2.14 (Archimedean Property of  $\mathbb{R}$ )**

If  $x, y \in \mathbb{R}$ , and  $x > 0$ , then  $\exists n \in \mathbb{N}$  such that  $n \cdot x > y$ .

*Proof.* Let  $A = \{nx \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ . Suppose towards a contradiction that there is no such  $n$  that satisfies the statement above. In other words, for all  $n \in \mathbb{N}$ ,  $nx \leq y$ . Thus  $A$  is bounded above by  $y$ . Since  $A$  is nonempty and is a subset of  $\mathbb{R}$ , we know that  $\sup A$  exists. Consider the value given by  $\sup A - x$ , which is not an upper bound for  $A$ . Then we know there exists some  $z \in A$  such that  $\sup A - x < z$ . Since  $z = mx$  (because  $z \in A$ ), we have

$$\begin{aligned}\sup A - x &< z \\ \sup A - x &< mx \\ \sup A &< (m+1)x.\end{aligned}$$

We know that  $(m+1)x \in A$ , which contradicts the definition of  $\sup A$ . □

Some remarks:

1. Let  $x = 1$ . Then  $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$  such that  $n > y$ .
2. Let  $y = 1$ . Then  $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$  such that  $x > \frac{1}{n} > 0$ .

**Theorem — 2.15 (Density of  $\mathbb{Q}$  in  $\mathbb{R}$ )**

For all  $x, y \in \mathbb{R}$ ,  $x < y$ , there exists some  $p \in \mathbb{Q}$  such that  $x < p < y$ .

*Proof.* Fix  $x < y$ . Then by the Archimedean property we have some  $n \in \mathbb{N}$  such that  $n(y - x) > 1$ , or  $y - x > \frac{1}{n}$ . We may suppose  $x > 0$ , because otherwise we either have  $x < 0 < y$  or  $x < y < 0$  (multiply all sides by  $-1$ ).

We want to show that

$$nx < m < ny.$$

Since  $nx + 1 < ny$ , we have  $nx < m < nx + 1$ , or  $m - 1 < nx < m$ . If  $nx \in \mathbb{Z}$ , we can take  $m = nx + 1$ . Thus

$$x < x + \frac{1}{n} = \frac{nx + 1}{n} = \frac{m}{n} < \frac{ny}{n} < y.$$

Otherwise we have  $nx \notin \mathbb{Z}$ . We then apply the following lemma:

**Lemma.** If  $x \in \mathbb{R}$ , there exists a  $k \in \mathbb{Z}$  such that  $k - 1 \leq x \leq k$ .

Then  $m - 1 < nx < m$ , as desired. □

## 8 Lecture 8

### 8.1 A Construction of the Real Numbers

We want to show that there exists some field  $\mathbb{R}$  such that  $\mathbb{Q}$  is a sub-field and  $\mathbb{R}$  has the least upper bound property. This construction will be via Dedekind cuts.

**Definition.** 2.17—*Cut*

A *cut* in  $\mathbb{Q}$  is a pair of subsets  $A, B \subset \mathbb{Q}$  such that:

- (i)  $A \cap B = \emptyset$ ,  $A \cup B = \mathbb{Q}$ ,  $A, B \neq \emptyset$  (They partition  $\mathbb{Q}$ )
- (ii) If  $a \in A$  and  $b \in B$ , then  $a < b$
- (iii)  $A$  contains no largest element

**Example.** An example of such a cut could be  $A = \{p \in \mathbb{Q} \mid p < 1\}$  and  $B = \{p \in \mathbb{Q} \mid p \geq 1\}$ . We say that  $A \mid B$  is a cut. Another such example is  $A = \{p \in \mathbb{Q} \mid p \leq 0 \text{ or } p^2 < 2\}$  and  $B = \{p \in \mathbb{Q} \mid p > 0 \text{ and } p^2 \geq 2\}$

**Definition.** 2.18—*The Reals*

We may define

$$\mathbb{R} = \{X \mid X \text{ is a cut in } \mathbb{Q}\}.$$

In order to show that the above is a valid definition for  $\mathbb{R}$ , we must show:

- (i)  $\mathbb{Q}$  is contained in  $\mathbb{R}$  in some “natural way”
- (ii)  $\mathbb{R}$  is an ordered field
- (iii)  $\mathbb{R}$  has the least upper bound property

**Definition.** 2.19—*Partial Order for the Reals*

We define a *partial order* on  $\mathbb{R}$  as follows: If  $X = A \mid B$  and  $Y = C \mid D$ , then we say  $X < Y$  if and only if  $A \subset C$  and  $X \leq Y$  if and only if  $A \subseteq C$ .

We will show that the reals contain the rationals.

*Proof.* We will begin by showing that  $\mathbb{Q} \subseteq \mathbb{R}$ . We say that  $A \mid B$  is a *rational cut* if for some  $c \in \mathbb{Q}$ , we have  $A = \{p \in \mathbb{Q} \mid p < c\}$  and  $B = \mathbb{Q} \setminus A$ . We will use  $c^*$  to denote the rational cut at  $c$ . Then we may associate every  $c \in \mathbb{Q}$  with a corresponding rational cut  $c^* \in \mathbb{R}$ .  $\square$

**Theorem — 2.20 (The Reals have the LUBP)**

With respect to the partial order  $\leq$  defined earlier, we may show that  $\mathbb{R}$  has the least upper bound property.

*Proof.* Let  $\mathcal{C}$  be any non-empty collection of cuts which are bounded above, say by the cut  $X$ . We want to show that  $\sup \mathcal{C}$  exists in  $\mathbb{R}$ , so  $\sup \mathcal{C}$  is itself a cut. A candidate for  $\sup \mathcal{C}$  is  $C \mid D$ , where

$$C = \{a \in \mathbb{Q} \mid \text{There exists a cut } A \mid B \in \mathcal{C} \text{ such that } a \in A\}.$$

Now,  $Z = C \mid D$  is a cut in  $\mathbb{Q}$ . We claim that  $C$  has no largest element. If  $a \in C$ , then there exists a cut  $A \mid B \in \mathcal{C}$  such that  $a \in A$ . Since  $A$  is a part of a cut, it has no largest element, so there exists some  $a' \in A$  such that  $a < a'$ . Thus  $a' \in C$  so  $C$  has no largest element.

We must now show that  $Z$  is the least upper bound of  $\mathcal{C}$ . For any  $A \mid B \in \mathcal{C}$ ,  $A \subseteq C$ . Therefore  $A \mid B \leq C \mid D = Z$  and  $Z$  is an upper bound for  $\mathcal{C}$ . We must now show that it is the *least* upper

bound. Let  $Z' = C' \mid D'$  be an upper bound for  $\mathcal{C}$ . Then  $A \mid B \leq C' \mid D'$  and  $A \subseteq C'$  for any  $A \mid B \in \mathcal{C}$ . Now by definition of  $C$ , we have  $C \subseteq C'$ . Therefore  $C \mid D \leq C' \mid D'$ , so  $Z \leq Z'$ . We have  $Z = \sup \mathcal{C} \in \mathbb{R}$ , so  $\mathbb{R}$  has the least upper bound property.  $\square$

We now show that  $\mathbb{R}$  is a field, namely an ordered field. We define the binary relations  $+$  and  $\cdot$  as follows: Given two cuts  $A \mid B$  and  $C \mid D$ , we may define

$$E = A + C = \{p \in \mathbb{Q} \mid p = a + c \text{ for some } a \in A, c \in C\}.$$

**Note.** This set summation is known as the Minkowski sum.

We must check that  $E \mid F$  is a cut. We must also show that the additive identity for  $\mathbb{R}$  is 0, i.e. we must show that  $0 + x = x + 0 = x$  for all  $x \in \mathbb{R}$ . To show the existence of additive inverses, show that for any cut  $A \mid B$ , there exists some  $C \mid D \in \mathbb{R}$  such that  $A \mid B + C \mid D = 0$ .

Similarly, we define multiplication by

$$E = \{p \in \mathbb{Q} \mid p = ac \text{ for some } a \in A, c \in C\}.$$

## 8.2 Interesting Questions

- (a) Can we cut  $\mathbb{R}$  to get something larger? No, because every possible cut in  $\mathbb{R}$  is an element of  $\mathbb{R}$ . This is because  $\mathbb{R}$  has the least upper bound property and every cut in  $\mathbb{R}$  would just be a “real cut” at  $\sup A$ .
- (b) Is  $\mathbb{R}$  unique in some natural way? Yes. If you take any other ordered field  $\mathbb{F}$  such that  $\mathbb{Q} \subseteq \mathbb{F}$  and  $\mathbb{F}$  has the least upper bound property, then there exists some bijection between  $\mathbb{F}$  and  $\mathbb{R}$ .
- (c) What about  $+\infty$  and  $-\infty$ ? We can’t treat these as real numbers using Dedekind cuts, as either  $A$  or  $B$  would have to be empty.

## 9 Lecture 9

### 9.1 Sequences

**Example.** *Approximating  $\pi$*

We know that the area of the unit circle should be  $\pi$ . We can approximate the area of a unit circle by inscribing various shapes in the circle and finding their areas (giving us a sequence of lower bounds). If we inscribe an equilateral triangle in the circle, we find that its side length is  $\sqrt{3}$ , so the area of the triangle is  $a_1 = \frac{3\sqrt{3}}{4} \approx 1.299$ .

If we use a square instead, its side length is  $\sqrt{2}$ , so we have a new lower bound of  $a_2 = 2 < \pi$ .

Using a regular pentagon, we have a new approximation as given by  $a_3 = \frac{5}{2} \sin\left(\frac{2\pi}{5}\right) < \pi$ .

Continuing this pattern, we get

$$a_n = \frac{n+2}{2} \sin\left(\frac{2\pi}{n+2}\right) < \pi.$$

Notice that for all  $n \in \mathbb{N}$ , we have  $a_n < a_{n+1} < \pi$ . Computing this value for larger  $n$ , we have

$$a_{100} \approx 3.1396$$

$$a_{1000} \approx 3.141572$$

$$a_{10000} \approx 3.14159245.$$

Similarly, we may obtain upper bounds for  $\pi$  by circumscribing regular polygons around the unit circle. By circumscribing a square, we know that  $\pi < 4$ , so we know that there is some upper bound for what  $\pi$  is equal to. Thus we know that there exists some  $a \in \mathbb{R}$  such that  $a_n \approx a$  for some sufficiently large  $n$ .

In other words, there exists some  $a \in \mathbb{R}$  such that  $a_n$  converges to  $a$  as  $n$  approaches  $\infty$ .

One issue to note is that we are using  $\pi$  in our formula for  $a_n$  to approximate the value of  $\pi$ . Using some trigonometric identities, we may circumvent this by evaluating

$$a_{2n} = \frac{n}{2} \left( n - \sqrt{n - 4a_n^2} \right).$$

**Definition.** *3.1—Sequences*

A *sequence* is a function  $f: \mathbb{N} \rightarrow \mathbb{R}$ . Instead of writing  $f(1), f(2), \dots, f(n)$ , we tend to write a sequence as  $f_1, f_2, \dots, f_n$ .

**Note (Notations for Sequences).** There are many different notations for expressing sequences, a few popular notations being used below:

$$\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}.$$

#### 9.1.1 Behaviors of Sequences

- (a) “Convergence”—Getting closer and closer to a given point.
- (b) “Divergence”—Getting closer and closer to  $\pm\infty$ .
- (c) “Oscillation”—The sequence does not approach any value in particular.

### 9.1.2 Facts From the Homework

We define the absolute value function

$$|\cdot| : \mathbb{R} \rightarrow [0, \infty) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We have the following properties of the absolute value function:

- (i)  $|xy| = |x| |y|$  for all  $x, y \in \mathbb{R}$ .
- (ii)  $|x - y| \leq \ell$  if and only if  $y - \ell \leq x \leq y + \ell$ , where  $\ell \geq 0$  and  $x, y \in \mathbb{R}$ .
- (iii) **Triangle Inequality:**  $|x + y| \leq |x| + |y|$ .

**Note.** One consequence of this is

$$\begin{aligned} |x - y| &= |x - z + z - y| \\ &\leq |x - z| + |z - y|. \end{aligned}$$

In other words, the distance between two points  $x$  and  $y$  is always less than or equal to the sum of the distances of  $x$  and  $y$  to a third point,  $z$ .

We say that  $a_n$  approaches  $a$  if “ $|a_n - a|$  gets arbitrarily small as  $n$  gets arbitrarily large”.

### 9.1.3 Convergence

**Definition.** 3.2—*Convergence*

A sequence  $(x_n)$  of real numbers is said to *converge* to an  $x \in \mathbb{R}$  if for all  $\varepsilon > 0$ , there exists  $N > 0$  such that

$$|x_n - x| < \varepsilon$$

for all  $n > N$ . If  $(x_n)$  converges to  $x$ , we also write:

$$x_n \rightarrow x \text{ as } n \rightarrow \infty \quad \text{or} \quad \lim_{n \rightarrow \infty} x_n = x.$$

We call  $x$  the *limit* of the sequence  $(x_n)$ . We say that a sequence *diverges* if it does not converge.

**Note.** By the Archimedean property, we can always take  $N \in \mathbb{N}$ . In general,  $N$  is a function of  $\varepsilon$ . We fix  $\varepsilon > 0$ , and use that to find some sufficient  $N$  so that the sequence converges.

**Example.** Consider the sequence  $x_n = \frac{1}{n^2}$ . We suspect that  $x_n \rightarrow 0 \in \mathbb{R}$  as  $n \rightarrow \infty$ . Fix  $\varepsilon > 0$ . If we let  $N = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$  and  $n > N$ , we have

$$\begin{aligned} |x_n - 0| &= \left| \frac{1}{n^2} \right| \\ &= \frac{1}{n^2} \\ &< \frac{1}{N^2} \\ &< \varepsilon. \end{aligned}$$

Thus,  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ , so

$$\lim_{n \rightarrow \infty} x_n = 0.$$

## 10 Lecture 10

### 10.1 Convergence of Sequences Using the Definition of Convergence

From last lecture, we have a basic definition of convergence. However, this is not very helpful for determining whether a sequence converges or not because we have to actually know what the sequence converges to.

**Example.** Consider the sequence  $x_n = \frac{4n^3+n}{n^3-6}$ . We have an intuition that this converges to 4 by dividing the leading coefficients of the numerator and denominator.

We have

$$\begin{aligned} |x_n - 4| &= \left| \frac{4n^3 + n}{n^3 - 6} - 4 \right| \\ &= \left| \frac{(4n^3 + n) - 4(n^3 - 6)}{n^3 - 6} \right| \\ &= \left| \frac{n + 24}{n^3 - 6} \right| \\ &= \frac{|n + 24|}{|n^3 - 6|} \\ &= \frac{n + 24}{n^3 - 6} \end{aligned}$$

We want to find some  $N$  such that  $\frac{n+24}{|n^3-6|} < \varepsilon$  for all  $n > N$ . Thus we want to show

$$\frac{n + 24}{|n^3 - 6|} \leq \frac{C}{n^2} \leq \varepsilon.$$

If  $n \geq 24$ , we have  $n + 24 \leq 2n$ . Suppose we wish for  $|n^3 - 6| > 0$ , so  $n \geq 2$ . Furthermore, note that  $n^3 - 6 \geq \frac{1}{2}n^3$  when  $n \geq 12^{\frac{1}{3}}$ . Thus for  $n \geq 24$ , we have

$$\begin{aligned} \frac{n + 24}{|n^3 - 6|} &\leq \frac{2n}{\frac{1}{2}n^3} \\ &= \frac{4}{n^2}. \end{aligned}$$

Thus we take

$$N = \max \left( 24, \left\lceil \sqrt{\frac{4}{\varepsilon}} \right\rceil \right) = \max \left( 24, \left\lceil \frac{2}{\sqrt{\varepsilon}} \right\rceil \right).$$

Thus for  $n > N$ , we have

$$\frac{n + 24}{|n^3 - 6|} < \varepsilon.$$

*Proof.* Fix  $\varepsilon > 0$  and choose  $N = \max \left( 24, \left\lceil \frac{2}{\sqrt{\varepsilon}} \right\rceil \right)$ . Then

$$\begin{aligned} |x_n - 4| &= \frac{n + 24}{|n^3 - 6|} \\ &\leq \frac{4}{n^2} \\ &< \varepsilon \end{aligned}$$

for all  $n > N$ . Thus  $\lim_{n \rightarrow \infty} x_n = 4$ . □

**Example.** Show that  $x_n = (-1)^n$  diverges.

Note that  $x_{2k} = 1$ , and  $x_{2k+1} = -1$  for some integer  $k$ .

*Proof.* Suppose towards a contradiction that  $x_n$  converges to a point  $x$ . Since we know that the sequence only takes on the values 1 and  $-1$ , we know that

$$\begin{aligned} 2 &= |(1 - x) + (x + 1)| \\ &\leq |1 - x| + |x + 1| \\ &2 \leq |x_{2n} - x| + |x - x_{2n+1}|. \end{aligned}$$

Since  $x_n \rightarrow x$ , given  $\varepsilon > 0$ , there exists  $N$  such that  $|x_n - x| < \varepsilon$  for all  $n > N$ . For  $n > N$ , we have  $|x_{2n} - x| < \varepsilon$  and  $|x - x_{2n+1}| < \varepsilon$ . Thus we have  $2 \leq 2\varepsilon$ , so  $1 \leq \varepsilon$ , which contradicts that  $\varepsilon$  may be arbitrarily small. Hence the sequence diverges.  $\square$

## 10.2 Limit Laws

**Proposition 3.4** (Limits are Unique)—If  $x_n \rightarrow x$  and  $x_n \rightarrow y$ , then  $x = y$ .

Idea: The points in the sequence have to be arbitrarily close to  $x$  and  $y$  simultaneously.

We see that

$$\begin{aligned} |x - y| &\leq |x - x_n| + |x_n - y| \\ &= \varepsilon, \end{aligned}$$

for any  $\varepsilon > 0$ . By a previous theorem, we have  $x = y$ .

*Proof.* Fix  $\varepsilon > 0$ . Since  $x_n \rightarrow x$ , there exists  $N_1 \in \mathbb{N}$  such that  $|x_n - x| < \frac{\varepsilon}{2}$  for all  $n > N_1$ . Since  $x_n \rightarrow y$ , there exists  $N_2 \in \mathbb{N}$  such that  $|x_n - y| < \frac{\varepsilon}{2}$  for all  $n > N_2$ . Then for  $n > \max(N_1, N_2)$ , the triangle inequality implies that

$$\begin{aligned} |x - y| &\leq |x_n - x| + |y - x_n| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since we have  $|x - y| < \varepsilon$  for any  $\varepsilon > 0$ , we have  $x = y$ .  $\square$

**Definition.** 3.5—*Boundedness for Sequences*

A sequence  $(x_n)$  is *bounded* if there exists some real number  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Note.** In the above, our choice of  $M$  does not depend on  $n$ . Additionally, all convergent sequences are bounded (since we may just choose an arbitrary  $\varepsilon$  and we know that  $x_n$  will at some point rest between  $x - \varepsilon$  and  $x + \varepsilon$ ).

**Theorem — 3.6** (*Convergent Sequences are Bounded*)

Suppose  $x_n \rightarrow x$ . Then for  $\varepsilon = 1$ , we may find a  $N \in \mathbb{N}$  such that  $|x_n - x| < 1$  for all  $n > N$ . Then when  $n > N$ , we have

$$\begin{aligned} |x_n| &\leq |x_n - x| + |x| \\ &\leq 1 + |x|. \end{aligned}$$

Hence, for  $M = \max(|x_1|, |x_2|, \dots, |x_N|, 1 + |x|)$ , we see that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

**Theorem — 3.7 (Algebraic Limit Theorem)**

Let  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} y_n = y$ . Then

- (i)  $\lim_{n \rightarrow \infty} (ax_n + by_n) = ax + by$  for all  $a, b \in \mathbb{R}$ .
- (ii)  $\lim_{n \rightarrow \infty} (x_n \cdot y_n) = xy$ .
- (iii)  $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$ , provided  $y \neq 0$ .



## 11 Lecture 11

### 11.1 Proof for Algebraic Limit Theorem

We want to show that

$$|x_n y_n - xy| < \varepsilon$$

for all  $\varepsilon > 0$ .

*Proof.* Observe that

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &= |x_n(y_n - y) + (x_n - x)y| \\ &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n| |y_n - y| + |y| |x_n - x| \end{aligned}$$

Since convergent sequences are bounded, we know that there exist some  $M > 0$  such that  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ .

$$\leq M |y_n - y| + |y| |x_n - x|$$

Fix  $\varepsilon > 0$ . Then there exists  $N_1$  such that  $|x_n - x| < \frac{\varepsilon}{2(1+|y|)}$  for all  $n > N_1$ . Furthermore, we know there exists some  $N_2$  such that  $|y_n - y| < \frac{\varepsilon}{2(1+M)}$  for all  $n > N_2$ .

$$\begin{aligned} &< \frac{M\varepsilon}{2(1+M)} + \frac{|y|\varepsilon}{2(1+|y|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

for all  $\varepsilon > 0$ . Therefore the limit exists and

$$\lim_{n \rightarrow \infty} x_n y_n = xy.$$

□

We now want to show that

$$\lim_{n \rightarrow \infty} \left( \frac{x_n}{y_n} \right) = \frac{x}{y}.$$

*Proof.* Observe that

$$\begin{aligned} \left| \frac{x_n}{y_n} - \frac{x}{y} \right| &= \left| \frac{x_n y - x y_n}{y_n y} \right| \\ &= \frac{1}{|y_n| |y|} |x_n y - x y_n| \\ &= \frac{1}{|y_n| |y|} |x_n y - xy + xy - x y_n| \\ &\leq \frac{1}{|y_n| |y|} |y(x_n - x) + x(y - y_n)| \\ &= \frac{1}{|y_n| |y|} (|y| |x_n - x| + |x| |y - y_n|) \end{aligned}$$

We now need to show that  $\frac{1}{|y_n|}$  is upper bounded by some value. Since  $(y_n)$  converges, we know that for all  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that  $|y_n - y| < \varepsilon$ . Suppose we choose  $\varepsilon = \frac{y}{2}$ , so we have  $\frac{1}{y_n} \leq \frac{2}{y}$ . In a fashion similar to the previous part, we may manipulate the above expression to complete the proof.

□

**Theorem — 3.8 (Order Limit Theorem)**

Assume  $x_n \rightarrow x$  and  $y_n \rightarrow y$ .

- (i) If  $x_n \geq 0$  for all  $n \in \mathbb{N}$ , then  $x \geq 0$ .
- (ii) If  $x_n \leq y_n$  for all  $n \in \mathbb{N}$ , then  $x \leq y$ .
- (iii) If there exists  $a \in \mathbb{R}$  such that  $a \leq x_n$  for all  $n \in \mathbb{N}$ , then  $a \leq x$ . If there exists  $b \in \mathbb{R}$  such that  $b \geq x_n$  for all  $n \in \mathbb{N}$ , then  $b \geq x$ .

**Note.** Strict inequalities are not necessarily respected! If  $x_n \rightarrow x$  and  $x_n > 0$ , then the most we can say is that  $x_n \geq 0$ . For example, consider the case where  $x_n = \frac{1}{n} > 0$ , and  $x = 0$ .

## 11.2 Monotone Sequences

**Definition. 3.9—Monotonicity**

A sequence  $(x_n)$  is *increasing* if  $x_n \leq x_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence  $(x_n)$  is *decreasing* if  $x_n \geq x_{n+1}$  for all  $n \in \mathbb{N}$ . We call both increasing and decreasing sequences *monotone* or *monotonic*.

**Note.** If you iterate the definition for an increasing sequence, you get  $x_n \leq x_m$  for  $n \leq m$ . A similar result comes from iterating the definition for a decreasing sequence.

**Example. Monotone Sequences**

- $x_n = 1 - \frac{1}{n}$  is increasing (and converges to 1).
- $x_n = \frac{1}{2^n}$  is decreasing (and converges to 0).
- $x_n = n$  is increasing (but is neither convergent nor bounded).
- $x_n = (-1)^n$  is not monotonic (and does not converge).

**Note.** In the above examples, if a sequence is monotonic, then if it is bounded it converges, otherwise diverges. Can we prove this?

**Theorem — 3.10 (Monotone Convergence Theorem)**

Every monotonic and bounded sequence in  $\mathbb{R}$  necessarily converges.

*Proof.* Let  $(x_n)$  be a sequence in  $\mathbb{R}$  which is both increasing and bounded (same argument works for decreasing). If we look at the set

$$S = \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R},$$

we see that this set is non-empty and bounded above (because  $(x_n)$  is bounded). Thus by the least upper bound property of  $\mathbb{R}$  we know that  $\sup S$  exists. We claim that  $x = \sup S$  is the limit for  $(x_n)$ . Let  $\varepsilon > 0$ . Then there exists some  $x_N \in S$  such that  $x_N > x - \varepsilon$ . We know that  $(x_n)$  is increasing, then  $x - \varepsilon < x_N < x_n$  for all  $n > N$ . Furthermore, since  $x$  is the supremum of  $S$ , we know that  $x_n < x < x + \varepsilon$ . Thus we have

$$x - \varepsilon < x_n < x + \varepsilon,$$

so  $|x_n - x| < \varepsilon$  for all  $\varepsilon > 0$  and all  $n > N$ . □

## 12 Lecture 12

**Example.** Consider the sequence defined by  $x_1 = 2$ , and for  $n \geq 2$

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right).$$

We don't have a nice function for  $x_n$  in terms of  $n$ , so proving convergence by normal means is non-ideal. To find the limit of  $(x_n)$ , we:

- Show that  $(x_n)$  is monotonic (decreasing in this case) and bounded
- Apply the Monotonic Convergence Theorem, and say that there exists some  $x \in \mathbb{R}$  such that  $x_n \rightarrow x$
- Apply limit laws to actually find the value of  $x$

**Note.** If  $x_n \rightarrow x$ , we know that  $x_{n+1} \rightarrow x$ .

Taking the first few terms, we see

$$x_1 = 2, x_2 = \frac{3}{2}, \dots$$

We guess that the sequence is always bounded, namely  $1 \leq x_n \leq 2$  for all  $n \in \mathbb{N}$ . We can prove this by induction. To show that the sequence is decreasing, we just need to show that for all  $n \in \mathbb{N}$ , we have  $x_{n+1} - x_n \leq 0$ . We can then use induction again, along with the fact that the previous inequality can be expressed as a function of  $x_n$ , to show that the sequence is decreasing.

We can see that the sequence  $(x_n)$  is strictly greater than zero, so we may use the Algebraic Limit Theorem. Thus we have

$$\begin{aligned} x &= \frac{1}{2} \left( x + \frac{2}{x} \right) \\ 2x &= x + \frac{2}{x} \\ x &= \frac{2}{x} \\ x^2 &= 2 \\ x &= \sqrt{2}. \end{aligned}$$

### 12.1 Subsequences

Consider the diverging sequence given by  $x_n = (-1)^n$ . If we take the even terms, we see that  $x_{2n} = 1$ , and if we take the odd terms, we have  $x_{2n+1} = -1$ . Thus we see that “parts” of our diverging sequence are actually convergent sequences.

**Definition.** *Subsequences*

Let  $(x_n)$  be a sequence and

$$n_1 < n_2 < n_3 < \dots$$

be a strictly increasing sequence of natural numbers. Then for  $k \in \mathbb{N}$ , the sequence  $(x_{n_k})$  is a *subsequence* of the original sequence  $(x_n)$ .

**Example.** *Subsequences*

1. Consider  $x_n = (-1)^n$ . Then  $x_{2k} = 1$  is a subsequence. Similarly,  $x_{2k+1} = -1$  is also a subsequence.
2. Consider  $x_n = \frac{1}{n} = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ . A valid subsequence could be  $x_{n_k} = \frac{1}{2k}$  or  $x_{n_k} = \frac{1}{10^k}$ . However, the sequence given by

$$n_1 = 10, n_2 = 50, n_3 = 20, n_4 = 5, n_5 = 1000$$

is not a subsequence because the indices are not strictly increasing.

**Observation.** Suppose we have  $(n_k)$  is strictly increasing. Then we know that  $n_k \geq k$ , because it is strictly increasing and starts at 1. Then if  $x_{n_k} = \frac{1}{n_k}$ , then we see

$$x_{n_k} \leq \frac{1}{k} \rightarrow 0.$$

Thus we see that for *any* subsequence of our convergent sequence  $x_n$ , it converges to the same limit as  $(x_n)$ .

**Proposition 3.12** Every subsequence of a converging sequence converges and does so to the same limit as the original sequence.

**Lemma.** We will show that  $n_k \geq k$  for all  $k \in \mathbb{N}$ , assuming that  $n_k$  is a strictly increasing sequence.

*Proof.* We proceed via induction. Observe that for  $n_1 \in \mathbb{N}$ , we have  $n_1 \geq 1$ . Suppose  $n_k \geq k$ . Then since  $(n_k)$  is strictly increasing, we have  $n_{k+1} > n_k \geq k$ , so  $n_{k+1} \geq k+1$ .  $\square$

*Proof.* Suppose  $(x_n)$  converges to  $x$ . Let  $(x_{n_k})$  be a subsequence of  $(x_n)$ . We claim that  $x_{n_k} \rightarrow x$ . Let  $\varepsilon > 0$ . Then there exists some  $N \in \mathbb{N}$  such that  $|x_n - x| < \varepsilon$  for all  $n > N$ . Since  $n_k \geq k$  for  $k > N$ , we have  $|x_{n_k} - x| < \varepsilon$  for all  $n_k > N$ . Thus  $x_{n_k} \rightarrow x$  as  $k \rightarrow \infty$ .  $\square$

**Note.** If  $x_n \rightarrow x$ , then  $x_{n+1} \rightarrow x$  because  $(x_{n+1})$  is a subsequence of  $(x_n)$ .

**Note.** Proposition 3.12 can be used to prove divergence, by showing that two subsequences of  $(x_n)$  converge to different values. For example, if  $(x_n) = (-1)^n$ , then

$$\begin{aligned} x_{2n} &= (-1)^{2n} = 1 \\ x_{2n+1} &= (-1)^{2n+1} = -1, \end{aligned}$$

so  $(x_n)$  diverges.

**Proposition 3.13** Every sequence has a monotonic subsequence.

*Proof.* We need to carefully select our subsequence. Let  $(x_n)$  be a sequence and

$$D = \{n \in \mathbb{N} \mid x_n > x_m \text{ for all } m > n\} \subseteq \mathbb{N}.$$

If  $n \in D$ , then  $x_n > x_m$  for all  $m > n$ . We say that if  $n \in D$ , that  $x_n$  is *dominant*. We now consider when  $D$  is finite and when  $D$  is infinite.

1. If  $D$  is infinite, then there exists  $\{n_k\} \subseteq D$  such that  $x_{n_k}$  is dominant. Then  $x_{n_{k+1}} < x_{n_k}$  and so  $(x_{n_k})$  is a subsequence which is decreasing, and so monotonic.

2. If  $D$  is finite, then there exists some  $N$  such that  $\max D = N$ . Thus the last dominant term of our sequence is  $x_N$ . Hence there exists some  $n_1 > N$  such that  $x_{n_1} > x_{N+1}$ . However, since  $n_1 > N$ , we have  $n_1 \notin D$ , so there must exist some  $x_{n_2} > x_{n_1+1}$ . By induction on  $k \in \mathbb{N}$ , we get  $(n_k)$ , so there exist

$$n_1 < n_2 < n_3 < \cdots$$

such that

$$x_{n_1} < x_{n_2} < \cdots < x_{n_k} < x_N.$$

Therefore  $(x_{n_k})$  is an increasing subsequence, and so is monotonic.

□

**Theorem — 3.14 (Bolzano–Weierstrass)**

Every bounded sequence in  $\mathbb{R}$  has a converging subsequence.

*Proof.* Suppose we have a bounded sequence  $(x_n)$  in  $\mathbb{R}$ . Then by Proposition 3.13 we know that there exists some subsequence  $(x_{n_k})$  that is monotonic, which is also bounded. Thus by the Monotone Convergence Theorem we have that  $(x_{n_k})$  converges in  $\mathbb{R}$ . □

**Note.** Bolzano–Weierstrass doesn't tell us anything about the original sequence, just that there are converging subsequences.

## 13 Lecture 13

### 13.1 Cauchy Sequences

The notion of a Cauchy sequence is a sequence that doesn't necessarily converge, but whose terms get "arbitrarily close" to each other.

**Definition.** *Cauchy Sequence*

A sequence  $(x_n)$  is *Cauchy* if for all  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } n, m > N.$$

**Note.** We don't need to know any limit  $x$  in order to determine if a sequence is Cauchy.

**Example.** *Cauchy Sequences*

- 1)  $x_n = \frac{1}{n}$  is Cauchy.
- 2)  $x_n = (-1)^n$  is not Cauchy.
- 3)  $x_n = n$  is not Cauchy.

**Proposition 3.16** Every convergent sequence is Cauchy.

*Proof.* Let  $(x_n)$  be a convergent sequence. Then for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ , we have  $|x_n - x| < \frac{\varepsilon}{2}$ . Let  $m > N$ . Thus

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x| + |x - x_m| \\ &= |x_n - x| + |x_m - x| \\ &< \varepsilon. \end{aligned}$$

Therefore  $(x_n)$  is Cauchy. □

**Proposition 3.17** Every Cauchy sequence is bounded.

*Proof.* Let  $(x_n)$  be Cauchy. For  $\varepsilon = 1$ , we know there exists  $N \in \mathbb{N}$  such that

$$|x_n - x_m| < 1 \quad \text{for all } n, m > N.$$

Thus  $|x_n| \leq 1 + |x_{N+1}|$  for all  $n > N$ . Suppose we choose

$$M = \max(|x_1|, |x_2|, \dots, |x_N|, 1 + |x_{N+1}|).$$

Then  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ , and so  $(x_n)$  is bounded. □

### 13.2 Completeness

**Theorem — 3.18 (Completeness of  $\mathbb{R}$ )**

$\mathbb{R}$  is *complete*, in other words every Cauchy sequence in  $\mathbb{R}$  converges.

*Proof.* Let  $(x_n)$  be Cauchy in  $\mathbb{R}$ . We have two steps:

- (1) Identify some candidate for the limit.
- (2) Show that the sequence converges to the candidate.

(1) By Proposition 3.17, we know that  $(x_n)$  is bounded. By Bolzano–Weierstrass, there exists some subsequence  $x_{n_k} \rightarrow x \in \mathbb{R}$  as  $k \rightarrow \infty$ . This is our candidate for our limit.

(2) We now show that our sequence converges to  $x$ . Bolzano–Weierstrass tells us that for a subsequence of  $(x_n)$ , we have the terms getting arbitrarily close to  $x$ . Because  $(x_n)$  is Cauchy, we “control” the remaining terms by “proximity” to  $(x_{n_k})$ .

Fix  $\varepsilon > 0$ ,  $x_{n_k} \rightarrow x$ , there exists  $N_1 \in \mathbb{N}$  such that

$$x_{\frac{n}{2}} < \frac{\varepsilon}{2} \quad \text{for all } n_k > N_1.$$

Since  $(x_n)$  is Cauchy, there exists some  $N_2 \in \mathbb{N}$  such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \text{for all } n, m > N_2.$$

So for any  $n > \max(N_1, N_2)$ , and  $n_k > \max(N_1, N_2)$ ,

$$\begin{aligned} |x_n - x| &\leq |x_n - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore  $x_n \rightarrow x$ . □

**Note.** The above proof makes use of Bolzano–Weierstrass, which applies to the reals, not rationals. Thus we think that the rationals are not complete.

### Theorem — 3.19 (Incompleteness of $\mathbb{Q}$ )

The rationals are *not* complete. More precisely, there exists a Cauchy sequence of rational numbers that *does not* converge.

*Proof.* Consider the sequence given by

$$x_1 = 2, x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right).$$

We see that  $(x_n)$  is a sequence of rationals. We know that it does not converge in the rationals, because if it did it would have to converge to  $\sqrt{2} \notin \mathbb{Q}$ . It remains to show that it is Cauchy. From one of the homeworks we know that  $|x_n - |x_{n+1}|| < \frac{1}{2^n}$  for all  $n \in \mathbb{N}$ . We can take telescoping sums by writing (for  $n > m$ ):

$$\begin{aligned} |x_n - x_m| &= \left| \sum_{k=m}^{n-1} (x_{k+1} - x_k) \right| \\ &\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \\ &< \sum_{k=m}^{n-1} \frac{1}{2^k}. \end{aligned}$$

This is a geometric series that sums to

$$\begin{aligned} &= \frac{\left(\frac{1}{2}\right)^m - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\ &= \frac{4}{2^m}, \end{aligned}$$

which converges to 0 as  $m \rightarrow \infty$ . Thus  $(x_n)$  is Cauchy in  $\mathbb{Q}$ . □

### 13.3 Bonus: Diverging to Infinity

**Definition.** 3.20—*Diverging to  $\pm\infty$*

Let  $(x_n)$  be a sequence in  $\mathbb{R}$ . We say that  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$  if: for all  $M > 0$ , there exists  $N \in \mathbb{N}$  such that  $x_n > M$  for all  $n > N$ .

Similarly, we say  $x_n \rightarrow -\infty$  if for all  $M > 0$ , there exists  $N \in \mathbb{N}$  such that  $x_n < -M$  for all  $n > N$ .



## 14 Lecture 14

### 14.1 Diverging Limits and Multiplication

**Proposition 3.21** Let  $x_n \rightarrow +\infty$  and  $y_n \rightarrow y > 0$ . Then

$$\lim_{n \rightarrow \infty} (x_n y_n) = +\infty.$$

The intuition via the Algebraic Limit Theorem tells us that this limit should approach “ $x \cdot y = +\infty \cdot y = +\infty$ ”.

**Note.** We can’t always allow the case when  $y = 0$ , i.e. consider the case where  $y_n = 0$ , so

$$\lim_{n \rightarrow \infty} x_n \cdot 0 = \lim_{n \rightarrow \infty} 0 = 0.$$

**Proposition 3.22** Let  $(x_n)$ ,  $x_n \geq 0$ . Then  $x_n \rightarrow +\infty$  if and only if  $\frac{1}{x_n} \rightarrow 0$ .

**Example.** Consider sequences of the form  $x_n = n^p$ , where  $x_n \geq 0$  and  $p > 0$ . Then  $x_n \rightarrow +\infty$  and  $\frac{1}{x_n} = \frac{1}{n^p} \rightarrow 0$ .

### 14.2 Diverging Limits and Addition

**Note.** Limits diverging to infinity don’t behave nearly as well when subjected to addition and subtraction, in comparison to multiplication and division. For example, if  $x_n \rightarrow +\infty$  and  $y_n \rightarrow \infty$ , we can’t say for sure what  $x_n + y_n$  converges to, if anything at all.

### 14.3 Notion of Infinity

We introduce symbols  $+\infty$  and  $-\infty$ , and define the *extended real numbers*:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the notion of  $\leq$  to  $\overline{\mathbb{R}}$  as follows: For all  $x \in \overline{\mathbb{R}}$ , we have

$$-\infty < x < +\infty.$$

We aren’t going to extend arithmetic (addition, subtraction, multiplication, division, etc.) to our new symbols. We also introduce the notation:

$$\begin{aligned} [a, +\infty) &= \{x \in \mathbb{R} \mid a \leq x\}. \\ (-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\}. \\ (-\infty, +\infty) &= \mathbb{R}. \end{aligned}$$

**Note.** If  $A \neq \emptyset$  and  $A \subseteq \mathbb{R}$  which is unbounded above, then  $\sup A = +\infty$ . We should always write  $+\infty$  and  $-\infty$ , not just  $\infty$ .

### 14.4 More on Sequences

Let  $(x_n)$  be a sequence in  $\mathbb{R}$ . Define

$$L = \{x_0 \in \overline{\mathbb{R}} \mid \text{There exists a subsequence } (x_{n_k}) \text{ such that } x_{n_k} \rightarrow x_0\} \subseteq \overline{\mathbb{R}}.$$

We may think of the above by “the set of all possible sub sequential limits of  $(x_n)$ ”. We continue to study the three possible behaviors:

- Case 1:  $x_n \rightarrow x$  (Converging)

Then we know that  $L = \{x\} \in \mathbb{R}$  because if the sequence converges to a point  $x$ , then all sub sequences must converge to the same  $x$ .

- Case 2:  $x_n \rightarrow +\infty$ . Then  $L = \{+\infty\}$ .

In other words, any subsequence of a sequence diverging to  $+\infty$  must also diverge to  $+\infty$ .

*Proof.* Suppose  $(x_{n_k})$  is a subsequence of  $(x_n)$ , which diverges to  $+\infty$ . Thus for all  $M > 0$ , there exists some  $N$  such that  $x_n > M$  for all  $n > N$ . We know that  $n_k \geq k$ , and so fixing  $M > 0$ , there exists  $N$  such that

$$n_k \geq k > N,$$

so  $x_{n_k} > M$ . Thus  $x_{n_k} \rightarrow +\infty$  as  $k \rightarrow +\infty$ . □

A similar result can be shown for if  $x_n \rightarrow -\infty$ .

- Case 3: “Oscillating Sequences”

Characterized by  $L$  having more than one element in the extended reals.

### Theorem — 3.22

A sequence of real numbers which does not converge in  $\overline{\mathbb{R}}$  has at least two distinct limit points, i.e. there exists two subsequences converging to different things.

*Proof.* Case 1:  $(x_n)$  is unbounded from above.

We can extract a subsequence  $(x_{n_k})$  such that  $x_{n_k} \rightarrow +\infty$ . Since  $x_n$  is unbounded above, for all  $M > 0$ , there exists  $n \in \mathbb{N}$  such that  $x_n > M$ . We consider the subsequence

$$(x_n)_{n=n_1+1}^\infty,$$

which is also unbounded above. Let  $M = 1$ , so there exists  $n_1 \in \mathbb{N}$  such that  $x_{n_1} > 1$ . We then choose  $M = 2$ , and get  $n_2 > n_1$  such that

$$(x_n)_{n=n_2+1}^\infty,$$

and continue this pattern to construct a subsequence  $(x_{n_k})$  such that  $x_{n_k} \geq k \rightarrow +\infty$ , so  $+\infty \in L$ .

Case 1.1:  $(x_n)$  is also unbounded from below. Same arguments give you a subsequence  $(x_{n_\ell})$  such that  $x_{n_\ell} \rightarrow -\infty$ , and so  $-\infty \in L$ .

Case 1.2:  $(x_n)$  is bounded from below.

Since  $(x_n)$  does not converge in the extended real numbers, it does *not* diverge to  $+\infty$ . Then there exists  $M > 0$  for all  $N \in \mathbb{N}$ , such that  $x_n \leq M$  for some  $n > N$ . Thus we may extract a bounded subsequence. If  $N = 1$ , then there exists  $n_1 > 1$  such that  $x_{n_1} \leq M$ . If  $N = n_1$ , there exists  $n_2 > n_1$  such that  $x_{n_2} \leq M$ . We continue this process until we have constructed our subsequence, i.e.

$$x_{n_k} \leq M \quad \text{for all } k \in \mathbb{N}.$$

Bolzano–Weierstrass tells us that there exists a further subsequence  $(x_{n_{k_\ell}})$  such that

$$x_{n_{k_\ell}} \rightarrow x \in \mathbb{R}.$$

We know that  $(x_{n_{k_\ell}})$  is a subsequence of our subsequence  $(x_{n_k})$ , and so it must be a subsequence of our original sequence  $(x_n)$ . Case 3:  $(x_n)$  is bounded. From the homework we know that

$$\liminf_{n \rightarrow \infty} x_n, \limsup_{x \rightarrow \infty} x_n \in L.$$

Since  $(x_n)$  does not converge in  $\overline{\mathbb{R}}$ , we have that the limits are distinct and so  $L$  has more than 1 element. □

## 15 Lecture 15

### 15.1 Infinite Series

Infinite series are *extremely* useful in applications, i.e. Taylor and Fourier series.

**Definition.** 3.24—*Infinite Series*

Let  $(x_n)_{n=m}^{\infty}$  be a sequence in  $\mathbb{R}$  ( $m \in \mathbb{Z}$ ). We define the sequence  $(S_n)_{n=m}^{\infty}$  to be defined by

$$S_n = \sum_{k=m}^n x_k.$$

We call  $(S_n)$  the sequence of *partial sums* of  $(x_n)$ . The *infinite series* given by

$$\sum_{n=m}^{\infty} x_n$$

is said to converge if the sequence of partial sums converges, and we write

$$\sum_{n=m}^{\infty} x_n := \lim_{n \rightarrow \infty} S_n.$$

A series that *doesn't converge* is said to *diverge*. If  $S_n \rightarrow \pm\infty$ , we say that

$$\sum_{n=m}^{\infty} x_n = \pm\infty.$$

### 15.2 Important Examples of Infinite Series

1) Geometric Series—Series of the form

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots,$$

where  $x \in \mathbb{R}$ . We know that

$$\begin{aligned} S_n &= \sum_{k=0}^n x^k \\ &= 1 + x + x^2 + \cdots + x^n. \end{aligned}$$

With some algebra we find that for all  $n \in \mathbb{N}$ ,  $x \neq 1$ ,

$$S_n = \frac{1 - x^{n+1}}{1 - x}.$$

*Proof.* Observe that

$$\begin{aligned} (1 - x)S_n &= (1 - x)(1 + x + x^2 + \cdots + x^n) \\ &= 1 + x + \cdots + x^n - x - x^2 - \cdots - x^{n+1} \\ &= 1 - x^{n+1}. \end{aligned}$$

□

In the edge case where  $x = 1$ , we have that

$$\sum_{n=0}^{\infty} 1 = 1 + 1 + \cdots \rightarrow +\infty.$$

If  $|x| < 1$ , then  $x^{n+1} \rightarrow 0$ , and so

$$S_n \rightarrow \frac{1}{1-x}.$$

If  $|x| > 1$ , then  $(S_n)$  diverges. If  $x = 1$  we have  $(S_n)$  diverges. When  $x = -1$  we have

$$S_n = \frac{1 - (-1)^{n+1}}{2},$$

which does not converge. Thus we have that  $(S_n)$  converges if and only if  $|x| < 1$ , and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

2) Inverse Squares—Series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

Then we have

$$S_n = \sum_{k=1}^n \frac{1}{k^2}.$$

Since this sequence is increasing, it suffices to find an upper bound to show that it converges. Observe that for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} S_n &= 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \cdots \\ &\leq 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{n(n-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} + \frac{1}{n}\right) \\ &\leq 2 - \frac{1}{n} \\ &\leq 2. \end{aligned}$$

Thus  $(S_n)$  is bounded above and so by Monotone Convergence Theorem, we know that

$$\lim_{n \rightarrow \infty} S_n$$

exists.

3) The Harmonic Series—The infinite series given by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

**Theorem** — *Divergence of the Harmonic Series*

The Harmonic Series diverges!

*Proof.* Let us write the harmonic series by

$$S_n = \sum_{k=1}^n \frac{1}{k}.$$

Since  $S_n$  is increasing, it suffices to show that the series is unbounded from above. The idea here is that we will group the terms by powers of 2. Observe that for all  $n \in \mathbb{N}$ , there exists some  $m \in \mathbb{N}$  such that  $2^m \leq n$ , and so

$$\begin{aligned} S_n &\geq S_{2^m} \\ &= \sum_{k=1}^{2^m} \frac{1}{k} \\ &= 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \cdots + \left( \frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m} \right) \\ &\geq 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \cdots + \frac{1}{8} \right) + \cdots + \left( \frac{1}{2^m} + \cdots + \frac{1}{2^m} \right) \\ &= 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{m-1}}{2^m} \\ &= 1 + \underbrace{\frac{1}{2} + \cdots + \frac{1}{2}}_{m \text{ times}} \\ &= 1 + \frac{m}{2} \rightarrow +\infty. \end{aligned}$$

□

### 15.3 Infinite Series Tests

**Theorem — 3.23 (Cauchy condensation test)**

Suppose  $(x_n)$  is *decreasing* and  $x_n \geq 0$  for all  $n \in \mathbb{N}$ . Then, the series  $\sum_{n=1}^{\infty} x_n$  converges if and only if the series

$$\sum_{n=0}^{\infty} \underbrace{2^n x_{2^n}}_{y_n}$$

converges.

*Proof.* ( $\Leftarrow$ ) Suppose  $\sum 2^n x_{2^n}$  converges. Then

$$t_n = \sum_{k=1}^n 2^k x_{2^k}$$

converges, and so is bounded. Let

$$S_n = \sum_{k=0}^n x_k.$$

We want to show that  $S_n < t_m$  (bounded above). The idea here is to partition  $[1, n]$  into dyadic intervals. Fix  $n \in \mathbb{N}$  and let  $m$  be large enough so that  $n \leq 2^{m+1} - 1$ . Then we have

$$\begin{aligned} S_n &\leq S_{2^{m+1}-1} \\ &= \sum_{k=1}^{2^{m+1}-1} x_k \\ &= \sum_{\ell=0}^m \sum_{k=2^\ell}^{2^{\ell+1}-1} x_k \\ &\leq \sum_{\ell=0}^m \max_{2^\ell \leq k \leq 2^{\ell+1}-1} x_k \cdot \sum_{k=2^\ell}^{2^{\ell+1}-1} 1 \\ &\leq \sum_{\ell=0}^m x_{2^\ell} (2^{\ell+1} - 1 - 2^\ell + 1) \\ &\leq \sum_{\ell=0}^m 2^\ell x_{2^\ell} \\ &= t_m. \end{aligned}$$

□

**Corollary 3.24** The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if  $p > 1$ .

*Proof.*

$$\begin{aligned}\sum_{n=0}^N 2^n x_{2^n} &= \sum_{n=0}^N 2^n \left(\frac{1}{2^n}\right)^p \\ &= \sum_{n=0}^N \frac{2^n}{2^{np}} \\ &= \sum_{n=0}^N \left(\frac{1}{2^{p-1}}\right)^N,\end{aligned}$$

which converges by Geometric series.

□

## 16 Lecture 16

### Definition. 3.25—Cauchy Criterion for Series

We say a series  $\sum x_n$  satisfies the *Cauchy Criterion* if the sequence of partial sums  $(S_n)$  is a Cauchy sequence. In other words, for all  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that

$$|S_n - S_m| < \varepsilon,$$

for all  $m, n > N$ . Written another way, we have that for all  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that

$$\left| \sum_{k=m+1}^n x_k \right| < \varepsilon$$

for all  $n \geq m > N$ .

### Theorem — 3.26

A series  $\sum x_n$  satisfies the Cauchy criterion if and only if the series converges.

*Proof.* ( $\Rightarrow$ ) We know that the sequence of partial sums is Cauchy. Since the reals are complete, we have that the partial sums converge, and so the series converges.

( $\Leftarrow$ ) If the series converges, the sequence of partial sums is converges, and so is Cauchy. Therefore the series satisfies the Cauchy criterion.  $\square$

**Corollary 3.27** If  $\sum x_n$  converges, then  $\lim_{n \rightarrow \infty} x_n = 0$ .

*Proof.* Let  $(S_n)$  denote the sequence of partial sums of  $(x_n)$ . Since  $(S_n)$  converges, we know that it satisfies the Cauchy criterion. Since  $(S_n)$  is Cauchy, we have that for all  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that

$$\left| \sum_{k=1}^n x_k - \sum_{k=1}^m x_k \right| = \left| \sum_{k=m+1}^n x_k \right| < \varepsilon,$$

for all  $m, n > N$ . Letting  $n = m + 1$ , we have

$$\begin{aligned} \left| \sum_{k=m+1}^n x_k \right| &< \varepsilon \\ \left| \sum_{k=m+1}^{m+1} x_k \right| &< \varepsilon \\ |x_{m+1}| &< \varepsilon. \end{aligned}$$

Thus for all  $n > N$ , we have  $|x_n| < \varepsilon$ , and so

$$\lim_{n \rightarrow \infty} x_n = 0.$$

$\square$

### 16.1 Various Convergence Tests

**Note.** This gives us the “divergence test”. If the limit of the summand of a series does not converge to 0, then the series diverges.



**Example. Divergence Test**

Consider the series given by

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + 1}.$$

The above series diverges because the summand converges to 1 (by Algebraic Limit Theorem). However, the sequence converging is a necessary and non-sufficient condition to show that the series converges, i.e. consider  $\sum x_n = \frac{1}{n}$ .

**Theorem — 3.28 (Comparison Test)**

Assume  $(x_n), (y_n)$  are sequences in  $\mathbb{R}$ ,  $y_n \geq 0$  for all  $n \in \mathbb{N}$ . Then

- (i) If the series  $\sum y_n$  converges and  $|x_n| \leq y_n$  for all  $n \in \mathbb{N}$ , then  $\sum x_n$  converges.
- (ii) If the series  $\sum y_n$  diverges and  $|x_n| \geq y_n$  for all  $n \in \mathbb{N}$ , then  $\sum x_n$  diverges.

*Proof.* (i) Suppose  $n > m$ . Then

$$\begin{aligned} \left| \sum_{k=m+1}^n x_k \right| &\leq \sum_{k=m+1}^n |x_k| \\ &\leq \sum_{k=m+1}^n y_k. \end{aligned}$$

Since  $\sum y_n$  converges, it satisfies the Cauchy criterion. Thus we fix some  $\varepsilon > 0$ , and there exists some  $N \in \mathbb{N}$  such that

$$\sum_{k=m+1}^n y_k < \varepsilon,$$

for all  $n > m > N$ . Continuing the inequality from before we get

$$\left| \sum_{k=m+1}^n x_k \right| < \varepsilon,$$

so  $\sum x_k$  is Cauchy, and so converges.

- (ii) Let  $(X_n)$  be partial sums for  $\sum x_n$ , and  $(Y_n)$  be partial sums for  $\sum y_n$ . By domination, we know that  $X_n \geq Y_n$  for all  $n \in \mathbb{N}$ . Since  $\sum y_n$  diverges, we have  $Y_n \rightarrow +\infty$ . This forces  $X_n$  to diverge to  $+\infty$  as well.

□

**Note.** The domination conditions in the previous theorem need only apply for all but finitely many  $n$ , rather than for all  $n \in \mathbb{N}$ . In other words, we only need those conditions to hold “from some point onwards”. This is because a series’ convergence is dictated by the convergence/divergence of its “tail”:

$$\sum_{n=1}^{\infty} x_n = \underbrace{\sum_{n=1}^N x_n}_{\text{Finite}} + \underbrace{\sum_{n=N+1}^{\infty} x_n}_{\text{“Tail”}}.$$

## 16.2 Convergent Series and the Harmonics

**Question.** How close can we get to the Harmonic Series and still converge?

By the  $p$ -series test, we know that  $\sum \frac{1}{n^p} < +\infty$  if and only if  $p > 1$ . Thus for any  $\varepsilon > 0$ , we have  $\sum \frac{1}{n} \cdot \frac{1}{n^\varepsilon} = \sum \frac{1}{n^{1+\varepsilon}}$ , which converges. The question remains whether we can replace  $\frac{1}{n^\varepsilon}$  by something that decays *slower*.

We know that for any  $a > 0$ , there exists  $N \in \mathbb{N}$  such that  $\log n \leq n^a$  for all  $n > N$ . In other words, logarithms grow slower than any exponential. We know that the series

$$\sum \frac{1}{n^p \log(1+n)}$$

converges when  $p > 1$ , by comparison with  $\sum \frac{1}{n^p}$ . Here we try replacing the  $n^\varepsilon$  from earlier with  $\log(1+n)$ . Then

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^\beta}$$

converges if and only if  $\beta > 1$ . We can continue to get closer to the  $\beta = 1$  case by adding more logs, i.e.

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)(\log \log n)^\alpha}$$

converges if and only if  $\alpha > 1$ .

### 16.3 Absolute and Conditional Convergence

**Theorem — 3.29** (*Absolute Convergence Test*)

If the series  $\sum |x_n|$  converges, then  $\sum x_n$  converges.

*Proof.* Comparison test—we have  $|x_n| \leq |x_n|$ . □

**Note.** The converse of the above is generally false, as  $\sum \frac{1}{n}$  diverges but  $\sum \frac{(-1)^n}{n}$  converges.

**Definition. 3.30—Absolute/Conditional Convergence**

A series  $\sum x_n$  converges *absolutely* if  $\sum |x_n|$  converges. If the series  $\sum x_n$  converges and  $\sum |x_n|$  diverges, we say  $\sum x_n$  converges *conditionally*.

## 17 Lecture 17

### Theorem — 3.31 (Alternating Series Test)

Let  $(x_n)$  be a sequence such that

- (i)  $(x_n)$  is decreasing
- (ii)  $\lim_{n \rightarrow \infty} x_n = 0$

Then

$$\sum_{n=1}^{\infty} (-1)^{n+1} x_n$$

converges.

*Proof.* Let

$$S_n = \sum_{k=1}^n (-1)^{k+1} x_k.$$

Then we have

$$\begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 - x_2 \\ S_3 &= x_1 - x_2 + x_3 \\ S_4 &= x_1 - x_2 + x_3 - x_4. \end{aligned}$$

Since  $(x_n)$  is decreasing, we have that  $(S_{2n})$  is increasing (as  $x_{2n-1} - x_{2n-2}$  is non-negative). Furthermore, we have  $(S_{2n+1})$  is decreasing, as  $x_{2n} - x_{2n-1}$  is non-positive. Thus we have the inequality

$$S_2 \leq S_{2n} \leq S_{2n+1} \leq S_1.$$

By the Monotone Convergence Theorem, we can say that both partial sum subsequences converge, say  $S_{2n} \rightarrow a$  and  $S_{2n+1} \rightarrow b$ , for some  $a, b \in \mathbb{R}$ . By the Ordered Limit Theorem we know that  $a \leq b$ . It remains to show that  $a = b$  and that any subsequence of  $(S_n)$  converges to  $a$ . Notice that

$$S_{2n+1} - S_{2n} = (-1)^{2n+2} x_{2n+1},$$

and so by Algebraic Limit Theorem we have  $b - a = 0$  (since  $(x_n)$  converges to 0). Thus  $a = b$ . Fixing  $m \in \mathbb{N}$ , we know there exists some  $n \in \mathbb{N}$  such that

$$S_{2n} \leq S_n \leq S_{2n+1}.$$

By Squeeze Theorem, we have  $S_n \rightarrow a$  as  $n \rightarrow \infty$ . □

**Note.** This theorem is quite helpful for helping us test for conditional convergence.

**Example.** Consider the series given by  $\frac{1}{n^p}$ , where  $p > 0$ . Observe that this sequence is both decreasing and converges to zero. Hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

converges. Furthermore, even something that decays much slower like

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log \log \log \log(1+n)}$$

converges.

**Theorem — 3.32 (Ratio Test)**

Suppose  $x_n \neq 0$  for all  $n \in \mathbb{N}$  and

$$L := \lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right|.$$

Then the series  $\sum x_n$ :

- (i) If  $L < 1$ , the series converges absolutely.
- (ii) If  $L > 1$ , the series diverges.
- (iii) If  $L = 1$ , the test is inconclusive.

*Proof.* (i) For all  $\varepsilon > 0$ , there exists some  $N$  such that for all  $n > N$ ,

$$\left| \left| \frac{x_{n+1}}{x_n} \right| - L \right| < \varepsilon$$

$$|x_{n+1}| < (L + \varepsilon) |x_n|.$$

If  $L < 1$ , we may choose  $\varepsilon > 0$  sufficiently small so that  $L + \varepsilon = L' < 1$ . Thus we have

$$|x_2| < L' |x_1|$$

$$|x_3| < L' |x_2| < (L')^2 |x_1|$$

$$|x_4| < L' |x_3| < (L')^3 |x_1|.$$

Iterating inductively, we see  $|x_{n+k}| \leq (L')^k |x_n|$  for all  $n > N$ . We then may define

$$|x_n| \leq \begin{cases} \max(|x_1|, \dots, |x_N|) & \text{for } 1 \leq n \leq N, \\ (L')^{n-N} |x_N| & \text{for } n > N. \end{cases}$$

Summing this up, we get

$$\sum_{n=1}^{\infty} |x_n| = \underbrace{\sum_{n=1}^N \max(|x_1|, \dots, |x_N|)}_{\text{finite}} + \underbrace{\sum_{n=N+1}^{\infty} (L')^{n-N} |x_N|}_{\text{converges geometrically}}$$

converges.

- (ii) We want to show that for all  $\varepsilon > 0$ , there exists  $N$  such that  $|x_{n+1}| > (L - \varepsilon) |x_n|$  for all  $n > N$ . Since  $L > 1$ , we may choose some  $\varepsilon > 0$  small such that  $L' = L - \varepsilon > 1$ . Iterating again, we get

$$|x_n| > (L')^{n-N} |x_N|$$

for all  $n > N$ . Since  $L' > 1$ , we have that the series diverges by Geometric Series. □

**Note.** The test fails when  $L = 1$  because there's no way to choose an  $\varepsilon > 0$  such that  $L' < 1$  or  $L' > 1$ , so we can't claim that the series converges/diverges by Geometric Series.

**Note.** Alternatively, we may actually replace the  $L$  in parts (i) and (ii) with  $\limsup$  and  $\liminf$ , respectively. This makes a stronger statement since we know that  $\limsup$  and  $\liminf$  always exist.

**Theorem — 3.33 (Root Test)**

Let  $(x_n)$  be a sequence such that

$$\lim_{n \rightarrow \infty} |x_n|^{\frac{1}{n}} = L.$$

Then the series  $\sum x_n$ :

- (i) If  $L < 1$  then  $\sum x_n$  converges absolutely.
- (ii) If  $L > 1$  then  $\sum x_n$  diverges.
- (iii) If  $L = 1$  then the test is inconclusive.

*Proof.* Since we have that

$$\lim_{n \rightarrow \infty} |x_n|^{\frac{1}{n}} = L,$$

we have that for all  $\varepsilon > 0$ , there exists  $N$  such that  $\left| |x_n|^{\frac{1}{n}} - L \right| < \varepsilon$  for all  $n > N$ . Thus we have

$$L - \varepsilon < |x_n|^{\frac{1}{n}} < L + \varepsilon.$$

If  $L < 1$ , we may choose  $\varepsilon > 0$  such that  $L' = L + \varepsilon < 1$ , so

$$\begin{aligned} |x_n|^{\frac{1}{n}} &< L' \\ |x_n| &< (L')^n. \end{aligned}$$

Thus by the comparison test with a geometric series we see that  $\sum x_n$  converges absolutely. If  $L > 1$ , we may choose  $\varepsilon > 0$  such that  $L' = L - \varepsilon > 1$ , so

$$\begin{aligned} |x_n|^{\frac{1}{n}} &> L' \\ |x_n| &> (L')^n. \end{aligned}$$

Thus by the comparison test with a geometric series we see that  $\sum x_n$  diverges.  $\square$

**Note.** In a fashion similar to the Ratio Test, if  $L = 1$ , we can't compare  $\sum x_n$  to a geometric series, and so the test is inconclusive.

**Note.** If we consider the Ratio and Root Tests in the more general setting (with  $\limsup$  and  $\liminf$  instead of  $\lim$ ), then the Root Test is “better” than the Ratio Test. However, if we just look at regular limits, then the tests yield the *same information*. From Ross:

$$\liminf \left| \frac{x_{n+1}}{x_n} \right| \leq \limsup |x_n|^{\frac{1}{n}} \leq \limsup \left| \frac{x_{n+1}}{x_n} \right|.$$

For the above, we “lose a little bit” when we use the Ratio Test, and have fewer cases where we can be bounded by a geometric series.

**Example.** An example where the Root Test works but the Ratio Test fails is for the series

$$\sum_{n=1}^{\infty} 2^{(-1)^n} 2^{-n}.$$

## 18 Lecture 18

### 18.1 Introduction to Continuity

We now want to discuss functions on the reals,

$$f: A \subseteq \mathbb{R} \rightarrow \mathbb{R}, \quad \text{dom}(A) = f.$$

We have a basic understanding of continuity—“if  $x \in A$  is *close* to  $c \in A$ , then  $f(x)$  is *close* to  $f(c)$ ”. Written a bit more rigorously, we can say that “ $f$  is continuous at a point  $c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$ ”. However, we haven’t defined the limit of a function yet. We’ll call the limit above a “functional limit”.

**Example.** *Dirichlet’s Function*

Let us define

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

**Question.** What number should we associate with

$$\lim_{x \rightarrow \frac{1}{2}} g(x)?$$

We take a sequence of points in  $\mathbb{R}$ , apply  $g$  to them, and see what that approaches as those points approach  $\frac{1}{2}$ . There exists some  $(x_n) \in \mathbb{Q}$  such that  $x_n \rightarrow \frac{1}{2}$ , so  $g(x_n) = 1$ . Thus

$$\lim_{n \rightarrow \infty} g(x_n) = 1.$$

However, by the density of the irrationals, we know there exists some  $(y_n) \in \mathbb{R} \setminus \mathbb{Q}$  such that  $y_n \rightarrow \frac{1}{2}$ , so  $g(y_n) = 0$ . Thus

$$\lim_{n \rightarrow \infty} g(y_n) = 0.$$

Since these two values disagree with each other, we say that the limit does not exist. In particular for Dirichlet’s function, we say that “ $g$  is discontinuous *everywhere*”.

**Example.** Let us consider the function given by

$$h(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Consider what the value of

$$\lim_{x \rightarrow 0} h(x)$$

should be. Let  $(x_n) \subseteq \mathbb{R}$  such that  $x_n \rightarrow 0$ . Thus we have

$$|h(x_n)| \leq |x_n| \rightarrow 0.$$

Thus by Squeeze Theorem we have that  $h(x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore “ $\lim_{x \rightarrow 0} h(x) = 0$ ” and “ $h$  is continuous at  $x = 0$  and discontinuous elsewhere”.

### 18.2 Functional Limits

**Definition.** 4.1—*Limit Points*

Let  $A \subseteq \mathbb{R}$ . We say  $c \in \mathbb{R}$  is a *limit point* of  $A$  if there exists a sequence such that

- $(x_n) \subseteq A$

- $x_n \neq c$  for all  $n \in \mathbb{N}$
- $\lim_{n \rightarrow \infty} x_n = c$

In other words,  $c$  is a limit point if a sequence of points in  $A$  (not equal to  $c$ ) converges to  $c$ .

**Note.** A limit point  $c$  does not necessarily have to belong to the set  $A$ .

**Example.**

- Let  $A = (0, 1)$ . If  $x \in A$ , then  $x_n = x - \frac{1}{n}$  converges to  $x$  when  $n$  is sufficiently large, so all points in  $A$  are limit points. Note that if we let  $c = 1$ , we can still define  $x_n = 1 - \frac{1}{n} \rightarrow 1$  to see that 1 is also a limit point of  $A$  despite not belonging to  $A$ .
- If  $A = [0, 1]$ , then  $1 \in A$  and every point in  $A$  is a limit point of  $A$ .
- Every interval for the form  $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$ ,  $a < b$  contains all of its limit points. We write  $L_A \subseteq \mathbb{R}$  to be the set of all limit points.

**Note.** The set of limit points  $L_A$  of a set  $A$  in general can look very different from  $A$  itself, and we can't draw any conclusions about one being the subset of the other.

**Definition. Isolated Point**

A point in  $A$  that is not a limit point of  $A$  is called a *isolated point*.

**Definition. 4.2—Closed Set**

A set containing its limit points is called a *closed set*.

**Definition. 4.3—Functional Limit**

Let  $A \subseteq \mathbb{R}$ ,  $c \in \mathbb{R}$  be a limit point of  $A$ , and  $f: A \rightarrow \mathbb{R}$  ( $\text{dom}(f) = A$ ). Then we define

$$\lim_{x \rightarrow c} f(x) = L$$

for some  $L \in \mathbb{R}$ , if for *every* sequence  $(x_n)$  in  $A$  which converges to  $c$ , we have

$$\lim_{n \rightarrow \infty} f(x_n) = L.$$

**Note.** We extend the definition to allow  $c \in \mathbb{R} \cup \{+\infty, -\infty\}$  so we may have limits of the form

$$\lim_{x \rightarrow +\infty} f(x) \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x). \quad (L_A \subseteq \overline{\mathbb{R}})$$

Furthermore, we extend  $L \in \mathbb{R} \cup \{+\infty, -\infty\}$  so we may have limits of the form

$$\lim_{x \rightarrow c} f(x) = \pm\infty.$$



**Example.**

- 1) Let  $f(x) = x^2$ , where  $A = \mathbb{R}$ . Furthermore, let  $c \in \mathbb{R}$  and  $x_n \rightarrow c$ . Then  $f(x_n) = x_n^2 \rightarrow c^2$  by Algebraic Limit Theorem. Hence,

$$\lim_{x \rightarrow c} f(x) = c^2$$

- 2) Let  $f(x) = \frac{x^2 - 4}{x - 2}$ . We have that  $A = (-\infty, 2) \cup (2, \infty)$ . We can see that 2 is a limit point of  $A$ . Let  $(x_n) \subseteq A$  such that  $x_n \rightarrow 2$ , and  $x_n \neq 2$  for all  $n$ . Then

$$\begin{aligned} f(x_n) &= \frac{x_n^2 - 4}{x_n - 2} \\ &= \frac{(x_n + 2)(x_n - 2)}{x_n - 2} \\ &= x_n + 2, \end{aligned}$$

which converges to 4. Hence,  $\lim_{x \rightarrow 2} f(x) = 4$ .

- 3) Let  $f(x) = \frac{1}{(x-2)^3}$ , so  $A = \mathbb{R} \setminus \{2\}$ . We see that  $\pm\infty$  are limit points of  $A$ , and want to show that

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0.$$

Let  $x_n \rightarrow +\infty$ , and  $(x_n) \subseteq A$ . Then  $f(x_n) = \frac{1}{(x_n - 2)^3} \rightarrow 0$ .

What about  $\lim_{x \rightarrow 2} f(x)$ ? Looking at the graph, we suspect that the limit does not exist. If  $(x_n) \subseteq A$  such that  $x_n > 2$  and  $x_n \rightarrow 2$ , then

$$f(x_n) = \frac{1}{(x_n - 2)^3} \rightarrow +\infty.$$

If  $(y_n) \subseteq A$  such that  $y_n < 2$  and  $y_n \rightarrow 2$ , then

$$f(y_n) = \frac{1}{(y_n - 2)^3} \rightarrow -\infty.$$

We found two sequences such that when  $f$  is applied, they converge to different values, and so  $\lim_{x \rightarrow 2} f(x)$  does not exist. However, we take the notion of “one-sided” limits:

- $\lim_{x \rightarrow 2^+} f(x) = +\infty$ .
- $\lim_{x \rightarrow 2^-} f(x) = -\infty$ .

**Definition.** 4.4—*Left and Right Hand Limits*

Let  $c$  be a limit point of  $A \subseteq \mathbb{R}$ . We define the *right-hand limit* of  $f: A \rightarrow \mathbb{R}$  by

$$\lim_{x \rightarrow c^+} f(x) = c$$

if for every  $(x_n) \subseteq A$ ,  $x_n > c$  and  $x_n \rightarrow c$  we have

$$\lim_{n \rightarrow \infty} f(x_n) = L.$$

Similarly, we define the *left-hand limit* of  $f: A \rightarrow \mathbb{R}$  by

$$\lim_{x \rightarrow c^-} f(x) = c$$

if for every  $(x_n) \subseteq A$ ,  $x_n < c$  and  $x_n \rightarrow c$  we have

$$\lim_{n \rightarrow \infty} f(x_n) = L.$$

**Theorem — 4.5**

If  $f: A \rightarrow \mathbb{R}$ ,  $c$  a limit point of  $A$ , Then

$$\lim_{x \rightarrow c} f(x) = L \quad (\text{exists})$$

if and only if

$$\lim_{x \rightarrow c^+} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^-} f(x) = L.$$