Machine Learning & Data Mining
Homework 6 - **1 late hour used**

Caltech CS/CNS/EE 155
February 2022

Policies

- **Due 9 PM, March** 2^{nd}, via Gradescope.
- You are free to collaborate on all of the problems, subject to the collaboration policy stated in the syllabus.
- In this course, we will be using Google Colab for code submissions. You will need a Google account.
- This set uses PyTorch, a Python package for neural networks. We recommend using Google Colab, which comes with PyTorch already installed.

Submission Instructions

- Submit your report as a single .pdf file to Gradescope, under "Set 6 Report".
- In the report, **include any images generated by your code** along with your answers to the questions.
- Submit your code by **sharing a link in your report** to your Google Colab notebook for each problem (see naming instructions below). Make sure to set sharing permissions to at least "Anyone with the link can view". **Links that can not be run by TAs will not be counted as turned in.** Check your links in an incognito window before submitting to be sure.
- For instructions specifically pertaining to the Gradescope submission process, see https://www.gradescope.com/get_started#student-submission.

Google Colab Instructions

For each notebook, you need to save a copy to your drive.

1. Open the github preview of the notebook, and click the icon to open the colab preview.
2. On the colab preview, go to File → Save a copy in Drive.
3. Edit your file name to "lastname_firstname_set_problem", e.g."yue_yisong_set6_prob2.ipynb"

1 Class-Conditional Densities for Binary Data [25 Points, 8 EC Points]

This problem will test your understanding of probabilistic models, especially Naive Bayes. Consider a generative classifier for C classes, with class conditional density $p(x|y)$ and a uniform class prior $p(y)$. Suppose all the D features are binary, $x_j \in \{0, 1\}$. If we assume all of the features are conditionally independent, as in Naive Bayes, we can write:

$$p(x \mid y = c) = \prod_{j=1}^{D} p(x_j \mid y = c)$$

This requires storing DC parameters.

Now consider a different model, which we will call the 'full' model, in which all the features are fully *dependent*.

Problem A [9 points]: Use the chain rule of probability to factorize $p(x \mid y)$, and let $\theta_{xjc} = p(x_j|x_{1,...,j-1}, y = c)$. Assuming we store each θ_{xjc}, how many parameters are needed to represent this factorization? Use big-O notation.

Solution A:

$$\begin{aligned}
p(x \mid y = c) &= p(x_1, x_2, ..., x_j \mid y = c) && (1)\\
&= p(x_j, ..., x_1 \mid y = c) && (2)\\
&= p(x_j \mid x_{j-1}, ..., x_1, y = c)p(x_{j-1}, ..., x_1 \mid y = c) && (3)\\
&= \prod_{j=1}^{D} p(x_j \mid x_{1,...,j-1}, y = c) && (4)\\
&= \prod_{j=1}^{D} \theta_{xjc} && (5)
\end{aligned}$$

All of the θ_{xjc}'s are stored, so since we have D features and need all permutations of previous features (largest of 2^{D-1}), we get $O(2^D)$ parameters for each class for a total of $O(C * 2^D)$ parameters.

Problem B [8 points]: Assume we did no such factorization, and just used the joint probability $p(x \mid y = c)$. How many parameters would we need to estimate in order be able to compute $p(x|y = c)$ for arbitrary x and c? How does this compare to your answer from the previous part? Again, use big-O notation.

Solution B: *With no factorization, we just have all combinations of the D binary features for the C classes for a total of $O(C * 2^D)$. This is the same as part A.*

Problem C [4 points]: Assume the number of features D is fixed. Let there be N training cases. If the sample size N is very small, which model (Naive Bayes or full) is likely to give lower test set error, and why?

Solution C: *The Naive Bayes model is more likely to give lower test set error when the sample size is very small because a full set of parameters would likely overfit the data.*

Problem D [4 points]: If the sample size N is very large, which model (Naive Bayes or full) is likely to give lower test set error, and why?

Solution D: *The full model is more likely to give lower test set error when the sample size is very large because a full set of parameters would likely capture the underlying model of the data better.*

Problem E [8 EC points]: Assume all the parameter estimates have been computed. What is the computational complexity of making a prediction, i.e. computing $p(y \mid x)$, using Naive Bayes for a single test case? What is the computation complexity of making a prediction with the full model? In justifying your answer for the full model, choose either the implementation in 1A or 1B and state your choice. For the full-model case, assume that converting a D-bit vector to an array index is an $O(D)$ operation. Also, recall that we have assumed a uniform class prior.

Solution E:

2 Sequence Prediction [75 Points]

In this problem, we will explore some of the various algorithms associated with Hidden Markov Models (HMMs), as discussed in lecture. We have also uploaded a note on HMMs to the github that might be helpful.

Sequence Prediction

These next few problems will require extensive coding, so be sure to start early!

- You will write an implementation for the hidden Markov model in the cell for `HMM Code` in the notebook given to you, within the appropriate functions where indicated. There should be no need to write additional functions or use NumPy in your implementation, but feel free to do so if you would like.
- You can (and should!) use the helper cells for each of the subproblems in the notebook, namely `2A`, `2Bi`, `2Bii`, `2C`, `2D`, and `2F`. These can be used to run and check your implementations for each of the corresponding problems. The cells provide useful output in an easy-to-read format. There is no need to modify these cells.
- Lastly, the cell for `Utility` contains some functions used for loading data directly from the class github repository. There is no need to modify this cell.

The supplementary data folder of the class github repository contains 6 files titled `sequence_data0.txt`, `sequence_data1.txt`, ... , `sequence_data5.txt`. Each file specifies a **trained** HMM. The first row contains two tab-delimited numbers: the number of states Y and the number of types of observations X (i.e. the observations are $0, 1, ..., X-1$). The next Y rows of Y tab-delimited floating-point numbers describe the state transition matrix. Each row represents the current state, each column represents a state to transition to, and each entry represents the probability of that transition occurring. The next Y rows of X tab-delimited floating-point numbers describe the output emission matrix, encoded analogously to the state transition matrix. The file ends with 5 possible emissions from that HMM.

The supplementary data folder also contains one additional file titled `ron.txt`. This is used in problems 2C and 2D and is explained in greater detail there.

Problem A [10 points]: For each of the six trained HMMs, find the max-probability state sequence for each of the five input sequences at the end of the corresponding file. To complete this problem, you will have to implement the Viterbi algorithm (in `viterbi()` of the `HiddenMarkovModel` object). Write your implementation well, as we will be reusing it in a later problem. See the end of problem 2B for a big hint! Note that you do not need to worry about underflow in this part.

In your report, show your results on the 6 files. (Copy-pasting the results of the cell for `2A` suffices.)

Solution A:

https://colab.research.google.com/drive/14sCWTj1aG6N_udpz1On_RODx6OrBAUbA?usp=sharing

File #0:

Emission Sequence	Max Probability State Sequence
25421	31033
01232367534	22222100310
5452674261527433	1031003103222222
72262131645122672 55	1310331000033100310
02471206023520510102552 41	22222222222222222222103

File #1:

Emission Sequence	Max Probability State Sequence
77550	22222
7224523677	2222221000
505767442426747	222100003310031
72134131645536112267	10310310000310333100
47336677714500510602530 41	22210000032222231032222 23

File #2:

Emission Sequence	Max Probability State Sequence
60622	11111
4687981156	2100202111
815833657775062	021011111111111
21310222515963505015	020201111111111111021
65031994525712740063200 25	11102021111111020211102 11

File #3:

Emission Sequence	Max Probability State Sequence
13661	00021
2102213421	3131310213
166066262165133	133333133133100
531646621121626341 56	200000213131310021 33
152354100512323022630 6256	131002133331331333131333 133

File #4:

Emission Sequence	Max Probability State Sequence
23664	01124
3630535602	0111201112
350201162150142	011244012441112
00214005402015146362	11201112412444011112
21112665246651435625344 50	20120124241240111124111 24

File #5:

Emission Sequence	Max Probability State Sequence
68535	10111
4546566636	1111111111
638436858181213	110111010000011
132403383084445146 88	00010000000111111100
01116644344413825336326 26	21111111111111001111101 01

Problem B [17 points]: For each of the six trained HMMs, find the probabilities of emitting the five input sequences at the end of the corresponding file. To complete this problem, you will have to implement the Forward algorithm and the Backward algorithm. You may assume that the initial state is randomly selected along a uniform distribution (the starting state transition probabilities are defined in `self.A_start` in `HiddenMarkovModel`). Again, write your implementation well, as we will be reusing it in a later problem.

Note that the probability of emitting an input sequence can be found by using either the α vectors from the Forward algorithm or the β vectors from the Backward algorithm. You don't need to worry about this, as it is done for you in `probability_alphas()` and `probability_betas()`.

Implement the Forward algorithm. In your report, show your results on the 6 files.
Implement the Backward algorithm. In your report, show your results on the 6 files.

After you complete problems 2A and 2B, you can compare your results (the probabilities from the forward and backward algorithms should be the same) for the file titled `sequence_data0.txt` with the values given in the table below:

Dataset	Emission Sequence	Max-probability State Sequence	Probability of Sequence
0	25421	31033	4.537e-05
0	01232367534	22222100310	1.620e-11
0	5452674261527433	1031003103222222	4.348e-15
0	7226213164512267255	1310331000033100310	4.739e-18
0	024712060235205101025524 1	22222222222222222222222103	9.365e-24

Solution B:

File #0:

Emission Sequence	Probability of Emitting Sequence
25421	4.537e-05
01232367534	1.620e-11
5452674261527433	4.348e-15
7226213164512267255	4.739e-18
024712060235205101025524 1	9.365e-24

File #1:

Emission Sequence	Probability of Emitting Sequence
77550	1.181e-04
7224523677	2.033e-09
505767442426747	2.477e-13
72134131645536112267	8.871e-20
473366777145005106025304 1	3.740e-24

File #2:

Emission Sequence	Probability of Emitting Sequence
60622	2.088e-05
4687981156	5.181e-11
815833657775062	3.315e-15
21310222515963505015	5.126e-20
6503199452571274006320025	1.297e-25

File #3:

Emission Sequence	Probability of Emitting Sequence
13661	1.732e-04
2102213421	8.285e-09
166066262165133	1.642e-12
531646621121626341 56	1.063e-16
152354100512323022630 6256	4.535e-22

File #4:

Emission Sequence	Probability of Emitting Sequence
23664	1.141e-04
3630535602	4.326e-09
350201162150142	9.793e-14
00214005402015146362	4.740e-18
21112665246651435625344 50	5.618e-22

File #5:

Emission Sequence	Probability of Emitting Sequence
68535	1.322e-05
4546566636	2.867e-09
638436858181213	4.323e-14
132403383084445146 88	4.629e-18
01116644344413825336 3262 6	1.440e-22

File #0:

Emission Sequence	Probability of Emitting Sequence
25421	4.537e-05
01232367534	1.620e-11
5452674261527433	4.348e-15
7226213164512267255	4.739e-18
024712060235205101025524 1	9.365e-24

File #1:

Emission Sequence	Probability of Emitting Sequence
77550	1.181e-04
7224523677	2.033e-09
505767442426747	2.477e-13
72134131645536112267	8.871e-20
473366777145005106025304 1	3.740e-24

File #2:

Emission Sequence	Probability of Emitting Sequence
60622	2.088e-05
4687981156	5.181e-11
815833657775062	3.315e-15
21310222515963505015	5.126e-20
6503199452571274006320025	1.297e-25

File #3:

Emission Sequence	Probability of Emitting Sequence
13661	1.732e-04
2102213421	8.285e-09
166066262165133	1.642e-12
531646621121626341 56	1.063e-16
152354100512323022630 6256	4.535e-22

File #4:

Emission Sequence	Probability of Emitting Sequence
23664	1.141e-04
3630535602	4.326e-09
350201162150142	9.793e-14
00214005402015146362	4.740e-18
21112665246651435625344 50	5.618e-22

File #5:

Emission Sequence	Probability of Emitting Sequence
68535	1.322e-05
4546566636	2.867e-09
638436858181213	4.323e-14
132403383084445146 88	4.629e-18
01116644344413825336 3262 6	1.440e-22

HMM Training

Ron is an avid music listener, and his genre preferences at any given time depend on his mood. Ron's possible moods are happy, mellow, sad, and angry. Ron experiences one mood per day (as humans are known to do) and chooses one of ten genres of music to listen to that day depending on his mood.

Ron's roommate, who is known to take to odd hobbies, is interested in how Ron's mood affects his music selection, and thus collects data on Ron's mood and music selection for six years (2190 data points). This data is contained in the supplementary file `ron.txt`. Each row contains two tab-delimited strings: Ron's mood and Ron's genre preference that day. The data is split into 12 sequences, each corresponding to half a year's worth of observations. The sequences are separated by a row containing only the character `-`.

Problem C [10 points]: Use a single M-step to train a supervised Hidden Markov Model on the data in `ron.txt`. What are the learned state transition and output emission matrices?

Tip: the $(1,1)$ entry of your transition matrix should be `2.833e-01`, and the $(1,1)$ entry of your observation matrix should be `1.486e-01`.

Solution C:

```
Transition Matrix:
########################################################################
2.833e-01   4.714e-01   1.310e-01   1.143e-01
2.321e-01   3.810e-01   2.940e-01   9.284e-02
1.040e-01   9.760e-02   3.696e-01   4.288e-01
1.883e-01   9.903e-02   3.052e-01   4.075e-01


Observation Matrix:
########################################################################
1.486e-01   2.288e-01   1.533e-01   1.179e-01   4.717e-02   5.189e-02   2.830e-02   1.297e-01   9.198e-02   2.358e-03
1.062e-01   9.653e-03   1.931e-02   3.089e-02   1.699e-01   4.633e-02   1.489e-01   2.394e-01   1.371e-01   1.004e-01
1.194e-01   4.299e-02   6.529e-02   9.076e-02   1.768e-01   2.022e-01   4.618e-02   5.096e-02   7.803e-02   1.274e-01
1.694e-01   3.871e-02   1.468e-01   1.823e-01   4.839e-02   6.290e-02   9.032e-02   2.581e-02   2.161e-01   1.935e-02
```

Problem D [15 points]: Now suppose that Ron has a third roommate who is also interested in how Ron's mood affects his music selection. This roommate is lazier than the other one, so he simply steals the first roommate's data. Unfortunately, he only manages to grab half the data, namely, Ron's choice of music for each of the 2190 days.

In this problem, we will train an unsupervised Hidden Markov Model on this data. Recall that unsupervised HMM training is done using the Baum-Welch algorithm and will require repeated EM steps. For this problem, we will use 4 hidden states and run the algorithm for 1000 iterations. The transition and observation matrices are initialized for you in the helper functions `supervised_learning()` and `unsupervised_learning()` such that they are random and normalized.

What are the learned state transition and output emission matrices? Please report the result using random seed state 1, as is done by default in the notebook.

Tips for debugging:

- The rows of the state transition and output emitting matrices should sum to 1.
- Your matrices should not change drastically every iteration.
- After many iterations, your matrices should converge.
- If you used random seed 1 for this computation (as is done by default in the notebook), the $(1, 1)$ entry of the state transition matrix should be `5.075e-01`, and the $(1, 1)$ entry of the output emission matrix should be `1.117e-01`.

Solution D:

```
Transition Matrix:
##########################################################################
5.075e-01   4.596e-01   6.533e-09   3.292e-02
3.127e-03   2.107e-04   9.964e-01   2.733e-04
1.195e-09   6.886e-02   9.686e-16   9.311e-01
6.203e-01   3.796e-01   1.555e-05   1.579e-04


Observation Matrix:
##########################################################################
1.117e-01   1.525e-01   7.740e-02   1.975e-02   1.594e-01   4.574e-13   3.556e-16   2.475e-01   1.139e-01   1.180e-01
1.205e-01   2.548e-15   1.103e-01   1.751e-01   3.656e-04   2.190e-01   1.002e-01   6.178e-02   1.323e-01   8.053e-02
1.276e-01   2.665e-02   5.788e-02   1.682e-01   1.700e-01   6.969e-02   1.254e-01   3.940e-02   1.627e-01   5.244e-02
1.918e-01   8.206e-02   1.376e-01   8.725e-02   1.152e-01   1.209e-01   1.033e-01   3.101e-02   1.308e-01   5.847e-38
```

Problem E [5 points]: How do the transition and emission matrices from 2C and 2D compare? Which do you think provides a more accurate representation of Ron's moods and how they affect his music choices? Justify your answer. Suggest one way that we may be able to improve the method (supervised or unsupervised) that you believe produces the less accurate representation.

Solution E: *The matrices from 2C have no really small values ($< 10^{-3}$), while the matrices from 2D have values that are nearly 0 ($< 10^{-15}$). I would likely think that 2C is a more accurate representation because it doesn't seem to make sense that there is a nearly 0 chance to have a certain emission or a certain transition, especially the self transitions of staying in the same mood. One way we could improve the unsupervised method would be to increase the data we feed to it.*

Sequence Generation

Hidden Markov Models fall under the umbrella of generative models and therefore can be used to not only predict sequential data, but also to generate it.

Problem F [5 points]: Run the cell for this problem. The code in the cell loads the trained HMMs from the files titled `sequence_data0.txt`, ..., `sequence_data5.txt` and uses the six models to probabilistically generate five sequences of emissions from each model, each of length 20. In your report, show your results.

Solution F:

```
File #0:
Generated Emission
######################################################################
27507237522455716641
57515563321515722540
42557003747152141777
27725570547474747752
11072754741251154744

File #1:
Generated Emission
######################################################################
55777446575240622057
11271547133774320102
55465434321572351007
50454047244666615510
53404323555174025374

File #2:
Generated Emission
######################################################################
79752756112715762398
62612627348724929612
07319652265592751626
14812741675914950855
95711670205554371836

File #3:
Generated Emission
######################################################################
30563333230062322142
62325030232614062211
03661061211041224066
61641250020411664226
50455651614055313566

File #4:
Generated Emission
######################################################################
34166203035066231165
51164226356523064460
46341654135233210104
60152355133613250641
24206143116010643133

File #5:
Generated Emission
######################################################################
48253618434461433181
86806115602103852805
33660333158116338414
31366464836406080240
44620803353216536433
```

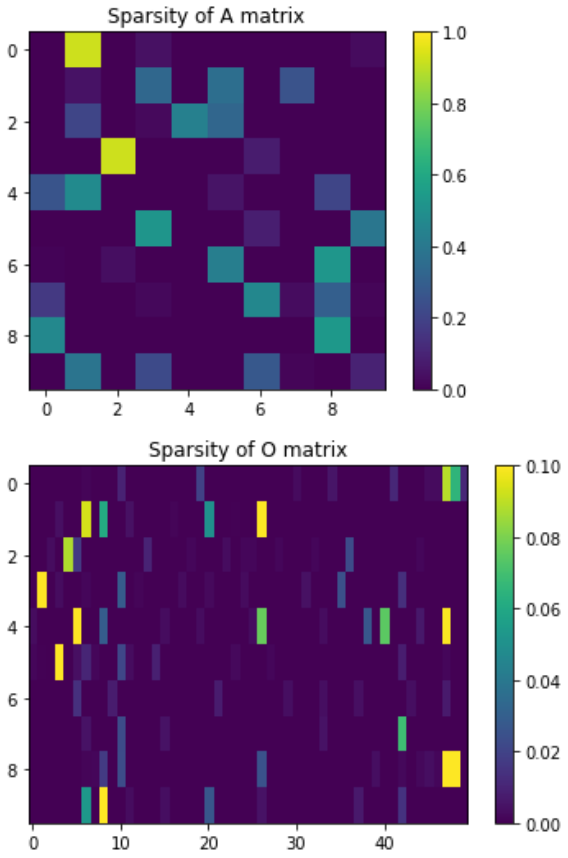
Visualization & Analysis

Once you have implemented the HMM code part of the notebook, load and run the cells for the following subproblems. Here you will apply the HMM you have implemented to the Constitution. There is no coding required for this part, only analysis.

Answer the following problems in the context of the visualizations in the notebook.

Problem G [3 points]: What can you say about the sparsity of the trained A and O matrices? How does this sparsity affect the transition and observation behaviour at each state?

Solution G:



Both the A and O matrices are very sparse. This cause the transition and observation behaviour at each state to be fairly defined with only a single or couple options to go to or observe at each state.

Problem H [5 points]: How do the sample emission sentences from the HMM change as the number of hidden states is increased? What happens in the special case where there is only one hidden state? In general, when the number of hidden states is unknown while training an HMM for a fixed observation set, can we increase the training data likelihood by allowing more hidden states?

Solution H: *As the number of hidden states is increased, the same emission sentences become more like real sentences. When there is only one hidden state, we seem to get completely random words. When the number of hidden states is unknown, we can increase the training data likelihood by allowing more hidden states and to make sparser and stronger associations between states.*

Problem I [5 points]: Pick a state that you find semantically meaningful, and analyze this state and its wordcloud. What does this state represent? How does this state differ from the other states? Back up your claim with a few key words from the wordcloud.

Solution I: *State 2 seems to represent all the major nouns of the constitution. We have words such as president, senate, congress, state, etc. This differs from other states that might have more to do with actions or sections in the constitution rather than the major nouns, with the major words of some of the other states being elected, vacancies, establish, two, three, one, etc.*