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Detecting discontinuities in two-dimensional signals sampled on a grid

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Abstract: In this paper we consider the problem of detecting, from a finite discrete set of points, the curves across which a two-dimensional function is discontinuous. We propose a strategy based on wavelets which allows to discriminate the edge points from points in which the function has steep gradients or extrema.

Keywords: Edge detection, discontinuities, wavelets, polyharmonic splines.

Mathematics Subject Classification: 65D15, 65R10, 65R30.

1 Introduction

Detection of edges (i.e jumps in a function) is of importance in many scientific applications including signal and image processing, geology, geophysics, economics, medicine.
In two dimensions, given a finite and discrete set of data, the problem is to detect the curves across which the function is discontinuous. The accurate detection of discontinuity curves, often referred as fault lines or edges, is of basic importance to analyze and recover a certain phenomenon correctly. In fact, usually, the most important information is carried by irregular structures. We can think for instance to depth or subsoil faults which represent discontinuities caused by severe movements of the earth crust. Their localization provides useful information on the occurrence of oil reservoirs. Another important example is the analysis of medical images as the magnetic resonance (MRI) where these lines may indicate the presence of some pathology.
The importance of this topic, is also evident from the literature where we find methods both for the univariate case (see for instance [3],[11], [17], [25],[26]) and for the two dimensions (see [1], [2], [6], [9],[12], [13], [18], [23], [24]).
Some of them are based on the theory of wavelets which are the ideal tool to study localized changes as jumps (edges) and sharp variations (steep gradients) of the underlying function in one dimension as well as several dimensions. Moreover, it is well-known that it is possible to characterize the global and local regularity of a signal by the decay of its wavelets coefficients across the scales.

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The main problem of this approach and also of the other methods in the literature, is to be able to distinguish between discontinuities and steep gradients that could be identified as jumps.
For this reason, we address here a detection strategy which, exploiting the properties of wavelets, allows to detect the edges and to discriminate the edge points from those describing other smooth features of the underlying function.
The method has low computational cost and leads to an efficient algorithm which is completely automatic.
In particular, we will use the polyharmonic wavelets which have the attractive property of being a non-separable basis and, even if they are not compactly supported, have the right localization properties for this kind of problem.

Before starting, we formalize the problem. We suppose to have a finite set of functional data given on a grid of a subset Ω of $\mathbb{R}^2$,

$$z_i = f(2^{-N}i), \quad i \in \mathcal{A} := \left\{\mathbb{Z}^2 \cap 2^N\Omega\right\}, \tag{1}$$

where $f(x)$ is discontinuous across an unknown curve g of Ω and smooth in any neighborhood of Ω which does not intersect g. We call D_0, the given data set

$$D_0 = \{(2^{-N}i, z_i), \quad i \in \mathcal{A} := \left\{\mathbb{Z}^2 \cap 2^N\Omega\right\}\}, \tag{2}$$

and we assume $\Omega = [0,1]^2$.
Our aim is to detect the points of the sample belonging or near the discontinuity curve g.

The paper is organized as follows. In Section 2 we give some basic notations, we recall the multiresolution setup and we give a characterization of discontinuities. In Section 3, we present the detection method and we discuss the properties needed by the wavelets when used in edge detection and we suggest a possible choice: the polyharmonic ones. We recall their definition and we state some useful results for our purposes. In Section 4, we show some numerical results which validate the effectiveness of the method here presented and the good performances of polyharmonic wavelets in this kind of problem.

2 Preliminaries

2.1 Basic notations and definitions

Throughout this paper, d is the dimension of the space.
$D^\alpha f$ denotes the derivative of f of total order α in $\mathbb{R}^d$. $\|\bullet\|$ denotes the euclidean norm on $\mathbb{R}^d$. $*$ denotes the convolution product: for all functions f and g in $L^1(\mathbb{R}^d)$ and all vectors u in $\ell^1(\mathbb{Z}^d)$

$$f * g := \int_{\mathbb{R}^d} f(x) g(\bullet - x) dx, \quad , \quad u * f := \sum_{j \in \mathbb{Z}^d} u_j \, f(\bullet - j).$$

We use standard notation for the inner product on $L^2(\mathbb{R}^d)$, i.e.

$$(f, g) := \int_{\mathbb{R}^d} f(x) \, g(x) dx.$$

$\hat{}$ is the Fourier transform, for any function f in $L^1(\mathbb{R}^d)$,

$$\widehat{f}(\omega) := \int_{\mathbb{R}^d} f(x) e^{-i\omega \, x} dx.$$

We call $\mathcal{RB}(s)$, $s \in \mathbb{R}$, the class of functions $\varphi(x)$ satisfying

$$|\varphi(x)| \leq \frac{C}{(1+\|x\|)^s},\ x \in \mathbb{R}^d$$

for some positive constant C, and $\mathcal{B}(s)$ the class of vectors $v = \{v_k\}_{k\in\mathbb{Z}^d}$ such that

$$|v_k| \leq \frac{C}{(1+\|k\|)^s},\ k \in \mathbb{Z}^d.$$

For $0 < \alpha < 1$, we say that f is Hölder continuous of order α, or α-regular at x_0 if $|f(x) - f(x_0)| \leq C\|x - x_0\|^\alpha$.

2.2 Multiresolution and wavelet decomposition

We recall the multidimensional multiresolution setup (see for instance [20]).
Let $\phi \in L^2(\mathbb{R}^d)$ be a function which satisfies the stability condition

$$A \sum_{k\in\mathbb{Z}^d} |\lambda_k|^2 \leq \left\| \sum_{k\in\mathbb{Z}^d} \lambda_k \phi(\bullet - k) \right\|_2^2 \leq B \sum_{k\in\mathbb{Z}^d} |\lambda_k|^2, \tag{3}$$

for any $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$. The constants A, B are such that $0 < A \leq B$.
We associate with ϕ an infinite sequence of closed subspaces $\{V_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ defined as $V_n := \left\{\sum_{k\in\mathbb{Z}^d} \lambda_k \phi_{n,k}(x) : \lambda \in l^2(\mathbb{Z}^d)\right\}$, where $\phi_{n,k}(x) := 2^{nd/2}\phi(2^n x - k)$.
We say that ϕ admits multiresolution, or that the sequence $\{V_n\}_{n\in\mathbb{Z}}$ forms a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$, if, in addition to (3), we have

$$V_n \subseteq V_{n+1},\ n \in \mathbb{Z}, \quad \overline{(\cup_{n\in\mathbb{Z}} V_n)} = L^2(\mathbb{R}^d), \quad \cap_{n\in\mathbb{Z}} V_n = \varnothing.$$

We also say that n is the scale or the resolution level and that ϕ is the scaling function.
Let W_n be the orthogonal complement of V_n in V_{n+1}, that is $V_{n+1} = V_n \oplus W_n$, $E = [0,1]^d \cap \mathbb{Z}^d$ and $E' = E \setminus \{0\}^d$.
It is known that there exist $2^d - 1$ wavelets ψ^e, $e \in E'$ that span W_n. The sequence $\{W_n\}_{n\in\mathbb{Z}}$ provides an orthogonal decomposition of $L^2(\mathbb{R}^d)$

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

This means that any $f \in L^2(\mathbb{R}^d)$ can be decomposed into an orthogonal series

$$f = \sum_{j\in\mathbb{Z}} D_j, \quad D_j \in W_j,$$

with

$$D_j = \sum_{k\in\mathbb{Z}^d} \sum_{e\in E'} c^e_{j,k} \psi^e_{j,k}(x), \quad \psi^e_{j,k} := 2^{jd/2}\psi^e(2^j x - k), \quad c^e_{j,k} = (f, \widetilde{\psi}^e_{j,k}),$$

where $\widetilde{\psi}^e$ is the dual function of ψ^e. Equivalently, f can be decomposed in two parts

$$f = P_{V_n} f + \sum_{j\geq n} D_j.$$

The smooth part $P_{V_n} f$ is the projection of f on the resolution space V_n, and the remaining part is formed by the details we need to complete it. In the detail coefficients $c^e_{j,k}$ are contained the information on the main features of the function (discontinuities, steep gradients, extrema).

2.3 Characterization of jumps

It is known that it is possible to characterize the local regularity of order α of a signal by the decay of its wavelets coefficients across the scales n (see [7], [10], [19]). These known results can be resumed in the following proposition.

Proposition 1 *Let $\psi^e(x) \in C^1(\mathbb{R}^d)$. Assume that $\psi^e(x)$ and $D^1\psi^e(x)$ belong to $\mathcal{RB}(d+2)$. If a function f is Hölder continuous of order α at x_0, $0<\alpha<1$, then*

$$|c^e_{n,k}| = |\int_{\mathbb{R}^d} f(x)\widetilde{\psi}^e_{n,k}(x)dx| < C_1(1+\|x_0-2^{-n}k\|^\alpha)2^{-n(d/2+\alpha)} \tag{4}$$

where $n>0$, $k\in\mathbb{Z}^d$, and $e\in E'$. Conversely, if for some $\epsilon>0$, and some $0<\alpha<1$

$$\max_{k\in I}|c^e_{n,k}| = O(2^{-n(d/2+\alpha)}), \qquad I=\{\mathbb{Z}^d\cap 2^{-n}[x_0-\epsilon, x_0+\epsilon]^d\}, \tag{5}$$

then f is Hölder continuous of order α at x_0.

This proposition says us that the coefficient moduli in the neighborhoods of the points in which the function is α-regular, $0<\alpha<1$, goes to zero as the scale n increases. Then, in the points in which the function is discontinuous, we expect to have coefficients with modulus that does not got to zero as $n\to+\infty$.
In fact we have proved the following result. From now on we consider $d=2$.

Proposition 2 *Let the assumptions of Proposition 1 hold, and consider a function $f(x)\in L^2(\mathbb{R}^2)$ which is discontinuous across a curve g of $\mathbb{R}^2$ and smooth in any neighborhood which does not intersect g. For any $k\in\mathbb{Z}^2$ such that $k/2^n$ belongs to a neighborhood of a discontinuity point, let $x_0\in g$ be its nearest point. Then, the corresponding wavelet coefficient is such that*

$$c^e_{n,k} = K_n + O(2^{-n}), \tag{6}$$

where K_n is a non zero constant tending to the jump size at x_0 as $n\mapsto\infty$.

Proof: Consider the kth-wavelet coefficient $c^e_{n,k}$, $e\in E'$ and let S_n a circle centered at $k/2^n$ with radius 2^{-n}. It is easy to show that

$$c^e_{n,k} = (f,\widetilde{\psi}^e_{n,k}) = \int_{S_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx + O(2^{-n}). \tag{7}$$

Now, we assume that S_n intersects the discontinuity curve which divides S_n in two sets S^1_n containing $k/2^n$ and S^2_n. Then

$$c^e_{n,k} = \int_{S^1_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx + \int_{S^2_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx + O(2^{-n}). \tag{8}$$

Since $f(x)$ is smooth in S^1_n and in S^2_n, we can write

$$\begin{aligned} f(x) &= f(k/2^n)+O(2^{-n}), \quad x\in S^1_n \\ f(x) &= f(\bar{k}/2^n)+O(2^{-n}), \quad x\in S^2_n, \end{aligned} \tag{9}$$

where $\bar{k}/2^n$ is the nearest point of $2^{-n}\mathbb{Z}^2\cap S^2_n$ to $k/2^n$. Now

$$\int_{S^2_n} f(x)\widetilde{\psi}^e_{n,k}(x) = \int_{S_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx - \int_{S^1_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx$$

$$
\begin{aligned}
&= (f, \widetilde{\psi}^e_{n,k}) - \int_{\mathbb{R}^2 \setminus S_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx - \int_{S_n^1} f(x)\widetilde{\psi}^e_{n,k}(x)dx \\
&= -\int_{\mathbb{R}^2 \setminus S_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx - \int_{S_n^1} f(x)\widetilde{\psi}^e_{n,k}(x)dx. \qquad (10)
\end{aligned}
$$

From (7), we have that $\int_{\mathbb{R}^2 \setminus S_n} f(x)\widetilde{\psi}^e_{n,k}(x)dx = O(2^{-n})$, and then

$$
\int_{S_n^2} f(x)\widetilde{\psi}^e_{n,k}(x)dx = -\int_{S_n^1} f(x)\widetilde{\psi}^e_{n,k}(x)dx + O(2^{-n}). \qquad (11)
$$

Using (9) and (11), (8) becomes

$$
\begin{aligned}
c^e_{n,k} &= \left(f(k/2^n) - f(\bar{k}/2^n)\right) \int_{S_n^1} \widetilde{\psi}^e_{n,k}(x)dx + O(2^{-n}) \\
&= \bar{C}\left(f(k/2^n) - f(\bar{k}/2^n)\right) + O(2^{-n}), \qquad (12)
\end{aligned}
$$

where $\bar{C}$ is a constant not depending on n and we get the proof. □

3 The Detection Method

These theoretical results suggest a characterization of jumps. It is clear that the coefficients with bigger modulus are relevant to the important features in the signal (as the discontinuities). So the idea is to look for the "maxima" of the coefficient moduli. For doing this we need to define what these maxima are and how to deal with the three sets $\{c^e_{n,k}\}_{k\in\mathbb{Z}^2}$, $e \in E' = \{(1,0),(0,1),(1,1)\}$. Each of them analyzes the function in a particular direction: the horizontal, the vertical and the positive diagonal.

We may interpret them as follows. At each point $k/2^n$, $k \in \mathbb{Z}^2$ we associate a vector $\mathbf{v}^e_{n,k}$ whose direction is given by $e \in E'$ and whose modulus is $|c^e_{n,k}|$.

We put together the information given by them, associating to each grid point only one vector $\mathbf{V}_{n,k}$ defined as $\sum_{e\in E'} \mathbf{v}^e_{n,k}$.

We consider its modulus

$$
|\mathbf{V}_{n,k}| = \sqrt{(c^{(0,1)}_{n,k} + \frac{\sqrt{2}}{2}c^{(1,1)}_{n,k})^2 + (c^{(1,0)}_{n,k} + \frac{\sqrt{2}}{2}c^{(1,1)}_{n,k})^2}, \qquad (13)
$$

and the angle it forms with the horizontal

$$
ang(\mathbf{V}_{n,k}) = atan[\frac{c^{(0,1)}_{n,k} + \frac{\sqrt{2}}{2}c^{(1,1)}_{n,k}}{c^{(1,0)}_{n,k} + \frac{\sqrt{2}}{2}c^{(1,1)}_{n,k}}]. \qquad (14)
$$

We define *wavelet modulus maxima*, the points $\bar{k}/2^n$ in which (13) is maximum in the direction given by (14).

It is trivial to see that also $|\mathbf{V}_{n,k}|$ satisfies Proposition 1 and Proposition 2.

From Proposition 1, we know that in a neighborhood of an α-regular point of f, $|\mathbf{V}_{n,k}|$ goes to zero as $2^{-n(\alpha+1)}$ when $n \to +\infty$. Conversely, if in a neighborhood of x_0 all the $|\mathbf{V}_{n,k}|$ converge to zero as $2^{-n(\alpha+1)}$ for $n \to +\infty$, then f is α-regular at x_0.

On the other side, Proposition 2, says that in a neighborhood of a point x_0 belonging to the discontinuity curve g, there is at least a point $k/2^n$ such that $|\mathbf{V}_{n,k}|$ does not converge to zero but to a constant proportional to the jump size at x_0. Moreover, if in a neighborhood of a point x_0

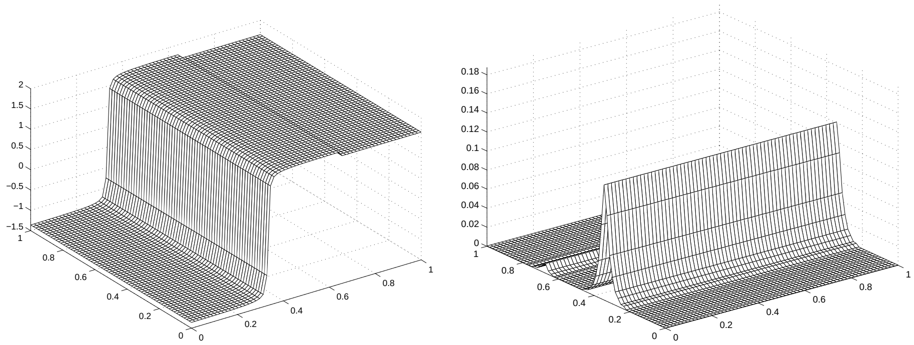

Figure 1: On the left the signal: it is discontinuous across the line $y = 0.7$. On the right the values of (13) computed starting from $2^7 \times 2^7$ gridded data

some $|\mathbf{V}_{n,k}|$ do not converge to zero but to a constant $K \neq 0$, by Proposition 1, follows that the function can not be α-regular for some α and then we have a discontinuity at x_0.
Then, as already said, we can conclude that the more significant information is given by the wavelet modulus maxima and their asymptotic behavior tell us if they correspond or not to the discontinuities of the function.
In addition, as direct consequence of Proposition 1 and 2, we can prove also the following

Proposition 3 *Under the assumptions of Proposition 1 and 2, we have that a function $f(x)$ is discontinuous at x_0 if and only if in a neighborhood of x_0 we have, for some k,*

$$\frac{|V_{n,k}|}{|V_{n+l,k}|} \to 1 \quad as \quad n \to +\infty, \quad l \in \mathbb{N} \tag{15}$$

We have now to see how to use the above results in practice, that is when we have only a gridded sample D_0 with size $2^N \times 2^N$ (see (1)).
First, we need to map the discrete signal D_0 onto the resolution space V_N by some operator which approximates the projection $P_N f$ of the function f on V_N. One effective method is to construct an element interpolating the data, but, often, because of computational efforts in computing the interpolating coefficients, people use the quasi interpolant operator.
When we have approximated the signal in V_N, we decompose it, we compute the wavelet coefficients and the values (13). Then, we find the wavelet modulus maxima at the present resolution level.
We know that in this set there are points corresponding to discontinuities. The problem, as shown in fig. 1, is that in this set we find also maxima corresponding to "smooth behavior" of the underlying function, as steep gradients or extrema, and at the moment, we are not able to say which are the maxima actually associated with the discontinuities.
For this reason we need to study a strategy to discriminate the edge maxima from the others.
Keeping in mind the theoretical results stated above, one possibility could be: to start from the given level N, to decompose the signal up to a prefixed level $s < N$ and to study the evolution of the maxima across the levels j, $j = s, \ldots, N$.
The disadvantage of this approach is that we smooth more and more the original signal and this means to loose the important features of the underlying function as the discontinuities we are looking for, especially if we have jumps of small sizes.

For this reason we follow a different approach.
We fix an integer l such that $1 \leq l \leq [N/2]$. From the given data set D_0, we extract l coarser subsets of gridded data D_j with step size $2^{-(N-j)}$, $j = 1, \ldots, l$.

For each set $D_l, \dots, D_1, D_0$, we perform only one wavelet decomposition. In this way, we construct the wavelet coefficients starting from sets containing the original information on the signal irregularities.
Then, for $j = l, \dots, 0$ we compute

1. the discrete functions

$$W_j = \{\frac{k}{2^{N-j-1}}, |\mathbf{V}_{N-j-1,k}|, \, k \in \mathbb{Z}^2 \cap 2^{N-j-1}[0,1]^2\},$$

2. the wavelet modulus maxima sets
$WMM_j = \left\{ \left(\kappa/2^{N-j-1}, |\mathbf{V}_{N-j-1,k}| \right), \kappa \in K_j \subset \mathbb{Z}^2 \cap 2^{N-j-1}[0,1]^2 \right\},$

and we look how they behave as j goes from l to 0: For the maxima corresponding to regularity points, we have a strong reduction of the moduli $|\mathbf{V}_{N-j-1,k}|$ while, according to Proposition 3, for the maxima corresponding to discontinuity points, the moduli $|\mathbf{V}_{N-j-1,\kappa}|$ remain of the same order for $j = l, \dots, 0$.
So we have the following procedure for the detection.
We consider the set WMM_0. For each $\kappa \in K_0$, we find in each set WMM_j, $j = 1, \dots, l$, the nearest point to $\kappa/2^{N-1}$

$$\left(\bar{\kappa}/2^{N-j-1}, |\mathbf{V}_{N-j-1,\bar{\kappa}}| \right) \in WMM_j, \; j = 1, \dots, l$$

and we compute the ratios

$$R_j = \frac{|\mathbf{V}_{N-1,\kappa}|}{|\mathbf{V}_{N-j-1,\bar{\kappa}}|}, \; j = 1, \dots, l$$

If, for $j = 1, \dots, l$

$$0.5 \leq R_j \leq 1.5,$$

(that is to say the moduli remain of the same order) we take $\kappa/2^{N-1}$ as discontinuity point, otherwise we reject it.

Remark: Usually it suffices to fix $l = 2$ or $l = 3$ and once fixed l and chosen the wavelet family, the algorithm is completely automatic.

We want now to discuss which wavelet is convenient to choose.
According to the assumptions of Proposition 1, we have to choose a wavelet with a good decay that means that its support can be considered numerically compact and then it is not so important to have compactly supported basis. Moreover it is very important that the wavelet has the right localization properties. Too localized wavelets may need larger sample to discriminate the edges from other smooth behaviors, and this, of course, causes a growth of the computational cost. On the other hand, wavelets with the right localization, are able to detect the discontinuities also from samples with lower dimension an then with a saving of the cost.
Another desirable property is to have non separable wavelets. This means that each set of coefficients analyzes the function in a particular direction: the horizontal, the vertical and the positive diagonal, and so there aren't only two favorite directions, as in the tensor product case in which we have the horizontal, the vertical directions, and we create a diagonal cross term which hasn't a straightforward interpretation. In our case each wavelet is actually associated to a privileged direction.
A class of wavelets satisfying to all the aforesaid requirements is the polyharmonic one. Then in the next section we recall their definition, their main properties and we prove some results which validate this choice from a theoretical point of view. Finally, we show some numerical results which confirm their goodness also in practice.

3.1 The Polyharmonic Wavelets

Polyharmonic splines can be defined in any dimension d through an interpolation problem (see [8]). The m-harmonic spline interpolating f on a set X is the unique solution of minimizing the semi-norm

$$\left(\int_{\mathbb{R}^d} \sum_{\alpha \in \mathbb{N}^d, |\alpha|=m} \frac{m!}{\alpha!} \left(\frac{\partial^\alpha u}{\partial x^\alpha}(x) \right)^2 dx \right)^{1/2}$$

among all the functions u in the Sobolev space $H^m(\mathbb{R}^d)$ interpolating f on X.
Let Δ_1 be the discrete version of the Laplace operator Δ, defined by $\Delta_1 f = \sum_{j=1}^d (f(\bullet - e_j) - 2f + f(\bullet + e_j))$, $(e_j)_k := \delta_{j,k}$, $1 \le j,k \le d$, and $\Delta^m := m^{th}$-iterate of Δ.
In the case of cardinal mesh, $X = \mathbb{Z}^d$, Rabut [22] has proved that the m-harmonic B-spline, $m \in \mathbb{N}$, $m > d/2$, with knots in $\mathbb{Z}^d$, is $\phi_m := \Delta_1^m v_m$, where v_m is a fundamental solution of Δ^m,

$$v_m = \begin{cases} \dfrac{1}{2^{2m}\pi^{d/2}} \dfrac{(-1)^{1-d/2}}{(m-d/2)!\ \Gamma(m)} \|\bullet\|^{2m-d} \ln \|\bullet\|^2 & , d \text{ even} \\ \dfrac{1}{2^{2m}\pi^{d/2}} \dfrac{(-1)^m \Gamma(\frac{d}{2}-m)}{\Gamma(m)} \|\bullet\|^{2m-d} & , d \text{ odd.} \end{cases} \qquad 8/3$$

The polyharmonic B-spline ϕ_m is a valid scaling function for constructing a MRA $\{V_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ (see [16], [21]) and the (pre-)wavelets are defined by means of the Lagrangian polyharmonic spline. Namely, let L_{2m} denote the Lagrangean $2m$-harmonic spline, i.e. the polyharmonic spline in $H^{2m}(\mathbb{R}^d)$ interpolating the data $(j, \delta_j)_{j\in\mathbb{Z}^d}$, we define

$$\psi_m := (\Delta^m L_{2m})(2\bullet) \quad , \quad \psi_m^e := \psi_m \left(\bullet + \frac{e}{2} \right) \ , \ e \in E'. \tag{16}$$

The functions ψ_m and ψ_m^e, $e \in E'$ are m-harmonic splines with knots in $\mathbb{Z}^d/2$. For any $n \in \mathbb{Z}$ and $e \in E'$, let W_n^e denote the space spanned by the family $\{\psi_m^e(2^n x - j)\}_{j\in\mathbb{Z}^d}$ and let W_n be the one spanned by $\{\psi_m^e(2^n \bullet - j)\}_{j\in\mathbb{Z}^d, e\in E'}$. Hence, $W_n = \uplus_{e'\in E'} W_n^{e'}$, $n \in \mathbb{Z}$, and the sequence $\{W_n\}_{n\in\mathbb{Z}}$ provides an orthogonal decomposition of $L^2(\mathbb{R}^d)$ (see [4] for more details). Then the functions of the family $\{\psi_m^e(2^n x - k)\}_{e\in E', k\in\mathbb{Z}^d}$ are orthogonal on different scales, while the orthogonality fails if we take functions of W_n, i.e. on the same level n.
The very important fact is that we can use these pre-wavelets in the applications, in fact we find in [4] an explicit construction of the MRA, and in [5] an efficient algorithm for performing the wavelet decomposition and recomposition of a discrete signal.

3.1.1 Main Properties

The polyharmonic splines have many interesting properties. For our purposes, we recall here that ϕ_m and ψ_m^e belong to $C^{2m-d-1}(\mathbb{R}^d)$ and that their decay is [4]

$$|\phi_m(x)| \le \frac{C}{(1+\|x\|)^{d+2}}, \qquad |\psi_m^e(x)| \le \frac{C}{(1+\|x\|)^{d+2}}, \quad e \in E. \tag{17}$$

Moreover, in the following proposition, we have proved that also $D^l \phi_m(x)$ and $D^l \psi_m^e(x)$, $e \in E$, $|l| \le d+1$, decay as (17).

Proposition 4 *Let $l \in \mathbb{N}^d$ such that $|l| \le d+1$. Then, $D^l\phi_m(x) \in \mathcal{RC}(d+2)$ and $D^l\psi_m^e(x) \in \mathcal{RC}(d+2)$, $e \in E$, provided that $2m-d-|l| > 0$.*

Proof. First, we consider $D^l\phi_m(x)$ with $|l|=1$. In this case, for the existence of the derivatives, it is necessary that $2m-d-1>0$.
For simplicity, we set $g_j(x)=\partial\phi_m(x)/\partial x_j$ and we consider its fourier transform $\hat{g}_j(\omega)=-i\omega_j\hat{\phi}_m(\omega)$. It is known that $D^\alpha\hat{\phi}_m(\omega)$, $|\alpha|\le d+1$, is locally summable. Since $D^\alpha\hat{\phi}_m(\omega)=O(\|\omega\|^{-2m})$, as $\|\omega\|\to\infty$, we have that $|D^\alpha\hat{\phi}_m(\omega)|\in L^1(\mathbb{R}^d)$ (see [4]).
Now, let us consider $D^\alpha\hat{g}_j(\omega)$, $|\alpha|\le d+1$. Its expression is a linear combination of the terms $\hat{\phi}_m(\omega)$ and $\omega_j D^\alpha\hat{\phi}_m(\omega)$. Then $D^\alpha\hat{g}_j(\omega)$ is locally summable and when $\|\omega\|\to\infty$, we have that $D^\alpha\hat{g}_j(\omega)=O(\|\omega\|^{-2m+1})$. Remembering that $2m-1>d$, we conclude that also $|D^\alpha\hat{g}_j(\omega)|\in L^1(\mathbb{R}^d)$, $|\alpha|\le d+1$.
Then, it follows that $g_j(x)=o(\|x\|^{-d-1})$ as $\|x\|\to\infty$. Since $g_j(x)$ admits a series expansion out of a certain neighborhood of the origin, we get $g_j(x)=O(\|x\|^{-d-2})$ as $\|x\|\to\infty$.
We may iterate this procedure to the successive derivatives $D^l\phi_m(x)$ with $|l|>1$. In this case we need $2m-d-|l|>0$ and going on as in the previous case, we get the proof.
We turn now to $D^l\psi_m^e(x)$, $e\in E$, $|l|\le d+1$. Since the wavelet $\psi_m\in W_0$ is in V_1, and since V_1 is generated by $\{\phi_m(2\bullet-j)\}_{k\in\mathbb{Z}^d}$, there exists a unique vector $p=(p_j)_{j\in\mathbb{Z}}\in l^2(\mathbb{Z}^d)$ such that $\psi_m=(p*\phi_m)(2\bullet)$, and it has been proved in [4], that $p\in\mathcal{B}(d+2)$.
By (16) it follows that $\psi_m^e=(p*\phi_m)(2\bullet+e)$, $e\in E$. Then $D^l\psi_m^e(x)=2(p*D^l\phi_m)(2\bullet+e)$, $e\in E$, and since the convolution between a vector of $\mathcal{B}(s)$ and a function of $\mathcal{RB}(s)$ is again a function of $\mathcal{RB}(s)$ (see [4]), we get the proof. □

In addition we have proved that

Proposition 5 *ψ_m^e, $e\in E$ and their duals $\tilde{\psi}_m^e$ have $2m-1$ vanishing moments, that is*

$$\int_{\mathbb{R}^d} x_1^{l_1}\dots x_d^{l_d}\psi_m^e(x)dx=0,\quad \int_{\mathbb{R}^d} x_1^{l_1}\dots x_d^{l_d}\tilde{\psi}_m^e(x)dx=0,\quad e\in E, \tag{18}$$

for all $l\in\mathbb{N}^d$ such that $|l|=l_1+\cdots+l_d\le 2m-1$.

Proof. Let us consider the Fourier transform of $\psi_m^e(x)$, $e\in E'$. By the wavelet definition (16), we have

$$\hat{\psi}_m^e(\omega)=\exp(-ie\omega)\hat{\psi}_m(\omega), \tag{19}$$

and

$$\hat{\psi}_m(\omega)=2^{-d}\|\omega\|^{2m}\hat{L}_{2m}(\omega/2). \tag{20}$$

The polyharmonic cardinal spline $L_m(x)$ decay exponentially (see [15]) and then its Fourier transform

$$\hat{L}_m(\omega)=\frac{\|\omega\|^{-2m}}{\sum_{k\in\mathbb{Z}^d}\|\omega-2k\pi\|^{-2m}} \tag{21}$$

is C^∞. So $\hat{\psi}_m(\omega)$ is C^∞ too.
Moreover it is easy to verify that $\hat{\psi}_m(\omega)=O(\|\omega\|^{2m})$ as $\omega\to0$. In fact, by definition

$$\hat{\psi}_m(\omega)=2^{-d}\|\omega\|^{2m}\frac{\|\omega/2\|^{-4m}}{\sum_{k\in\mathbb{Z}^d}\|\omega/2-2k\pi\|^{-4m}}.$$

Since we have

$$\sum_{k\in\mathbb{Z}^d}\|\omega/2-2k\pi\|^{-4m}>\|w/2\|^{-4m},$$

we get

$$\hat{\psi}_m(\omega)\le K\|\omega\|^{2m}.$$

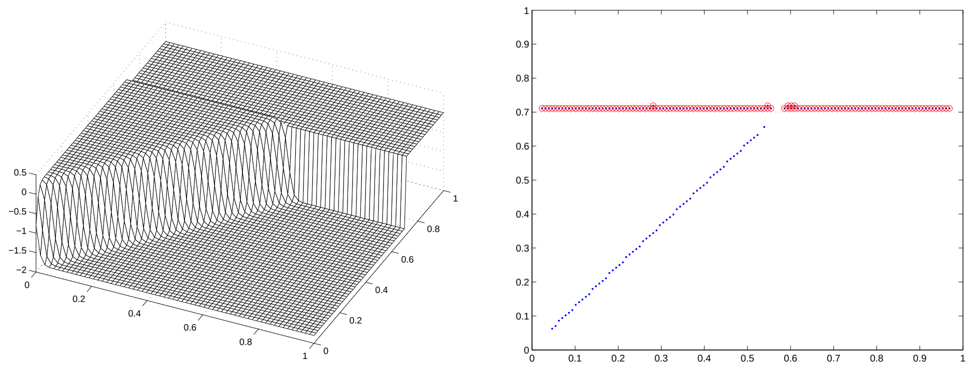

Figure 2: Left: Test function $f1$. Right: the results of the detection, the points are the wavelet modulus maxima and the circle the wavelet modulus maxima detected as discontinuity points

Then $D^l\hat{\psi}_m(\omega)|_{\omega=0} = 0$ for any l such that $|l| \leq 2m-1$. Being $D^l\hat{\psi}_m(\omega)|_{\omega=0} = (-i)^{|l|}\int_{\mathbb{R}^d} x_1^{l_1}\dots x_d^{l_d}\psi_m(x)dx$, we conclude that $\psi_m(x)$ and $\psi_m^e(x)$ have $2m-1$ vanishing moments.
Let now consider the dual wavelets $\tilde{\psi}_m^e(x)$. It is known that

$$\widehat{\tilde{\psi}}_m^e(\omega) = \frac{\hat{\psi}_m^e(\omega)}{\sum_{k\in\mathbb{Z}^d}|\hat{\psi}_m^e(\omega+2k\pi)|^2}. \tag{22}$$

Then $\widehat{\tilde{\psi}}_m^e(\omega)$ is C^∞, and since for $e \in E$, $\psi_m^e(x)$ is a Riesz basis with constants $0 < A^e \leq B^e$, we have

$$\frac{\hat{\psi}_m^e(\omega)}{B^e} \leq \widehat{\tilde{\psi}}_m^e(\omega) \leq \frac{\hat{\psi}_m^e(\omega)}{A^e}. \tag{23}$$

Then also $\widehat{\tilde{\psi}}_m^e(\omega)) = O(\|\omega\|^{2m})$ as $\omega \to 0$ and with the same arguments used for $\psi_m^e(x)$ we get the proof. $\square$

4 Numerical Results

In this section we present some numerical experiments that validate the efficacy of the method presented in the paper and the good performance of polyharmonic wavelets in edge detection.
We have chosen the polyharmonic splines of order $m = 2$ and used the algorithm proposed in [5] for performing the decomposition of the given signal.

Example 1. We start from the function

$$f1(x,y) = \begin{cases} \tanh(50y-60x) - 0.8, & 0 \leq y < 0.7,\ x \in [0,1], \\ 0, & y \geq 0.7,\ x \in [0,1], \end{cases}$$

which has steep gradients along the line $y = 6/5x$ and is discontinuous across the line $y = 0.7$. We have considered a gridded sample of size $2^8 \times 2^8$. In Fig. 2, we can see on the left the function $f1$ and on the right the results of the detection algorithm: the points are the wavelet modulus maxima and the circle the wavelet modulus maxima detected as discontinuity points.

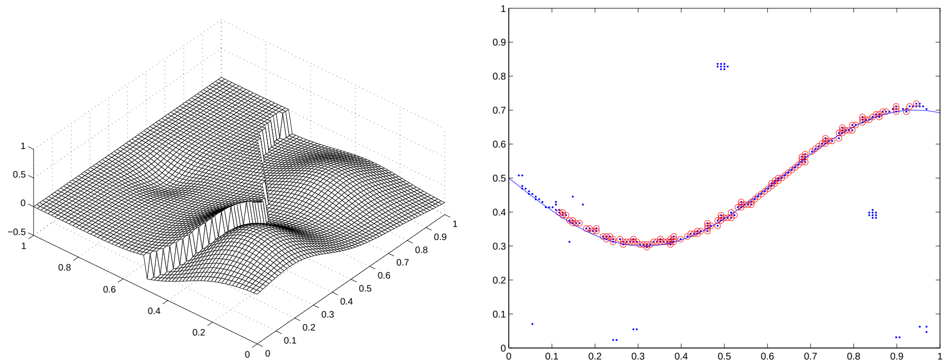

Figure 3: Left: Test function $f2$. Right: the results of the detection, the points are the wavelet modulus maxima and the circle the wavelet modulus maxima detected as discontinuity points

Example 2. Here we have considered the well known Franke's function $fr(x,y)$ and the function $f2(x,y)$ defined as

$$f2(x,y) = \begin{cases} fr(x,y), & -0.2\sin(5x) + 0.5 < y \leq 1,\ x \in [0,1], \\ fr(x,y) - 0.4, & 0 \leq y \leq -0.2\sin(5x) + 0.5,\ x \in [0,1], \end{cases}$$

which has three extrema and is discontinuous across the curve

$$y = -0.2\sin(5x) + 0.5.$$

We have considered a gridded sample of size $2^8 \times 2^8$. We can see in Fig. 3 that in the wavelet modulus maxima set we find also the extrema of the underlying function which are rejected by the detection algorithm.

Example 3. Finally, we want to show a practical example, the phantom function [2], used as benchmark test in magnetic resonance imaging. The larger ellipse represent the brain and it contains several small ellipses representing features in the brain. The domain of this function is $\Omega = [-1,1]^2$ and we have considered a grid of size $2^9 \times 2^9$. In Fig. 4, we can see the phantom function and on the right the results of the detection algorithm.

5 Concluding Remarks

In this paper we have considered the problem of detecting, from a finite discrete set of points, the curves across which a two-dimensional function is discontinuous. We have proposed a strategy based on wavelets which allows to discriminate the edge points from points in which the function has steep gradients or extrema. The method leads to an efficient algorithm which is completely automatic.

We have also addressed which wavelet is convenient to choose for this kind of problem. In particular we have considered the polyharmonic wavelets which have the attractive property of being a non-separable basis and, even if they are not compactly supported, have the right localization properties for this kind of problem. The numerical experiments have shown that this is a good choice also in practice.

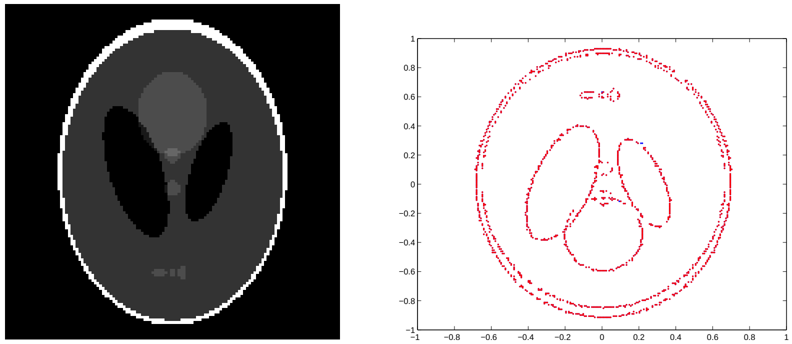

Figure 4: Left: The phantom function. Right: the results of the detection, the points are the wavelet modulus maxima detected as discontinuity points

References

[1] Allasia G., Besenghi R., De Rossi A., A scattered data approximation scheme for the detection of fault lines. In *Mathematical methods for curves and surfaces* (Oslo, 2000), 25–34, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 2001.

[2] Archibald R., Gelb A., Yoon J., Polynomial fitting for edge detection in irregularly sampled signals and images, Accepted on SIAM Journal of Num. Anal.

[3] Bozzini M., F. De Tisi, M. Rossini:Irregularity detection from noisy data with wavelets. In: Wavelets, Images, and Surface Fitting, pp. 59–66, Wellesley MA: A. K. Peters 1994.

[4] B. Bacchelli, M. Bozzini, C. Rabut and M. L. Varas, Decomposition and Reconstruction of Multidimensional Signals using polyharmonic pre-wavelets, Appl. comput. Harmon. anal. **18** (2005) 282-299.

[5] B. Bacchelli, M. Bozzini, C.Rabut, A fast wavelet algorithm for multidimensional signal using polyharmonic splines, in Curves and Surfaces Fitting: Saint-Malo 2002, NashboroPress, Nashville, TN (2003) 21–30.

[6] Canny J., A computational approach to edge detection. IEEE Trans.Pattern. Anal. Machine Intell. **8** (1986) 679-698.

[7] Daubechies I., *Ten lectures on wavelets*, SIAM, Philadelphia (1992).

[8] Duchon J., Interpolation des fonctions de deux variables suivant le principe de la flexion des plaques minces, R.A.I.R.O. Analyse Numerique **10** (1976) no. 12, 345-369.

[9] Gutzmer T., and A. Iske, Detection of discontinuities in scattered data approximation. Numerical Algorithms **16** (1997) 155-170.

[10] Jaffard S., Exposants de Hölder en des points donnes et coefficients d'ondeletts, C. R. Acad. Sci. Paeis, **308**, Series I (1989) 79-81.

[11] Lee D., Coping with Discontinuities in Computer Vision: Their Detection, Classification and Measurement IEEE Transaction on Pattern Analysis and Machine Intelligence, **12** (1990) 321-344.

[12] Parra, M. C., López de Silanes, M. C. Some results obtained in the study of fault detection from scattered data. Seventh Zaragoza-Pau Conference on Applied Mathematics and Statistics (Spanish) (Jaca, 2001), Monogr. Semin. Mat. Garca Galdeano, 27, Univ. Zaragoza, Zaragoza, (2003) 483–490.

[13] López de Silanes, M. C., Parra, M. C., Torrens, J. J., Vertical and oblique fault detection in explicit surfaces, J. Comput. Appl. Math. **140** no. 1-2 (2002) 559–585.

[14] Madych W.R., Polyharmonic splines, multiscale analysis and entire functions. Multivariate approximation and interpolation (Duisburg, 1989), Intern. Er. Numer. math., 94, pp. 205–216 Birkhauser, Basel, 1990.

[15] Madych W. R. and S. A. Nelson, Polyharmonic cardinal splines, Journal of Approximation Theory **60**, 141-156 (1990).

[16] Madych W.R., Some elementary properties of multiresolution analyses of $L^2(\mathbb{R}^n)$. Wavelets, 259–294, Wavelet Anal. Appl., 2, Academic Press, Boston, MA, 1992

[17] Mallat S., W. L. Hwang, Singularity detection and processing with wavelets, IEEE Transactions on Information Theory, *38*, 617–643 (1992).

[18] Mallat S., S. Zhong, Characterization of signals from multiscale edges, IEEE Trans. Pattern Anal. Machine Int. PAMI *14* (1992).

[19] Mallat S., *A wavelet tour of signal processing*. Academic Press (2001).

[20] Meyer Y., *Wavelets and Operators*, Cambridge University Press, Cambridge, 1992.

[21] Micchelli C., C. Rabut, and F. Utreras, Using the refinement equation for the construction of pre-wavelets, III: Elliptic splines, Numerical algorithms I (1991) 331-352.

[22] C. Rabut, B-splines Polyarmoniques Cardinales: Interpolation, Quasi-interpolation, filtrage, Thèse d'Etat, Université de Toulouse (1990).

[23] Rossini M.: Irregularity detection from noisy data in one and two dimension, Numer. Algor. *16*, 283–301 (1997).

[24] Rossini M., 2D- Discontinuity detection from scattered data. Computing **61**, (1998) 215–234.

[25] Wang Y., Yumps and sharp cusp detection by wavelets, Biometrika **82**, no. 2 (1995) 385-397.

[26] Wei M., De Pierro A. R.,Yin J., Iterative methods based on polynomial interpolation filters to detect discontinuities and recover point values from Fourier data, IEE Trans. on Signal Process. **53**, no. 1 (2005) 136-146.