The Sally SERIES

Pure and Applied
UNDERGRADUATE TEXTS

15

Partial Differential Equations and Boundary-Value Problems with Applications

Third Edition

Mark A. Pinsky

American Mathematical Society

The Sally SERIES

Pure and Applied
UNDERGRADUATE TEXTS · 15

Partial Differential Equations and Boundary-Value Problems with Applications

Third Edition

Mark A. Pinsky

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification.* Primary 35–01.

For additional information and updates on this book, visit
www.ams.org/bookpages/amstext-15

Library of Congress Cataloging-in-Publication Data

Pinsky, Mark A., 1940–
Partial differential equations and boundary-value problems with applications / Mark A. Pinsky.
p. cm. — (Pure and applied undergraduate texts ; v. 15)
Originally published: 3rd ed. Prospect Heights, Ill. : Waveland Press, 2003.
Includes index.
ISBN 978-0-8218-6889-8 (alk. paper)
1. Differential equations, Partial. 2. Boundary value problems. I. Title.

QA374.P55 2011
515′.353—dc22

2011012736

---

10 9 8 7 6 5 4 3 2 1 16 15 14 13 12 11

Preface

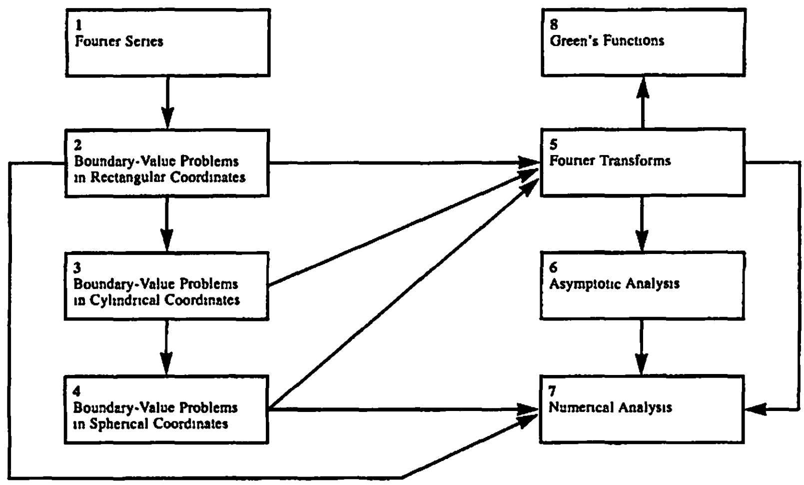
This third edition is an introduction to partial differential equations for students who have finished calculus through ordinary differential equations. The book provides physical motivation, mathematical method, and physical application. Although the first and last are the raison d'être for the mathematics, I have chosen to stress the systematic solution algorithms, based on the methods of separation of variables and Fourier series and integrals. My goal is to achieve a lucid and mathematically correct approach without becoming excessively involved in analysis per se. For example, I have stressed the interpretation of various solutions in terms of asymptotic behavior (for the heat equation) and geometry (for the wave equation).

This new edition builds upon the solid strengths of the previous editions and provides a more patient development of the core concepts. Chapters 0 and 1 have been reorganized and refined to provide more complete examples that will help students master the content. For example, the Sturm-Liouville theory has been rewritten and placed at the end of Chapter 1 just before it is used in Chapter 2. The coverage of infinite series and ordinary differential equations, formerly in Chapter 0, has been moved to appendixes. In addition, we have integrated the applications of Mathematica into the text because computer-assisted methods have become increasingly important in recent years. The previous edition of this text made Mathematica applications available for the first time in a book at this level, and this edition continues this coverage. Each section of the book contains numerous worked examples and a set of exercises. These exercises have been kept to a uniform level of difficulty, and solutions to nearly 450 of the 700 exercises in the text have been provided.

Chapter 0 is a brief introduction to the entire subject of partial differential equations and some technical material that is used frequently throughout the book. Chapters 1 to 4 contain the basic material on Fourier series and boundary-value problems in rectangular, cylindrical, and spherical coordinates. Bessel and Legendre functions are developed in Chapters 3 and 4 for those instructors who want a self-contained development of this material. Instructors who do not wish to use the material on boundary-value problems should cover only Secs. 3.1 and 4.1 in Chapters 3 and 4. These sections contain several interesting boundary-value problems that can be solved without the use of Bessel or Legendre functions.

Chapter 5 develops Fourier transforms and applies them to solve problems in unbounded regions. This material, which may be treated immediately following Chapter 2 if desired, uses real-variable methods. The student is referred to a subsequent course for complex-variable methods.

The student who has finished all the material through Chapter 5 will have a good working knowledge of the classical methods of solution. To complement these basic techniques, I have added chapters on asymptotic analysis (Chapter 6), numerical analysis (Chapter 7), and Green's functions (Chapter 8) for instructors who may have additional time or wish to omit some of the earlier material. The accompanying flowchart plots various paths through the book.



Logical Dependence of Chapters

Chapters 1 and 2 form the heart of the book. They begin with the theory of Fourier series, including a complete discussion of convergence, Parseval's theorem, and the Gibbs phenomenon. We work with the class of piecewise smooth functions, which are infinitely differentiable except at a finite number of points, where all derivatives have left and right limits. Despite the generous dose of theory, it is expected that the student will learn to compute Fourier coefficients and to use Parseval's theorem to estimate the mean square error in approximating a function by the partial sum of its Fourier series. Chapter 1 concludes with Sturm-Liouville theory, which will be used in Chapter 2 and repeatedly throughout the book.

Chapter 2 takes up the systematic study of the wave equation and the heat equation. It begins with steady-state and time-periodic solutions of the heat equation in Sec. 2.1, including applications to heat transfer and to geophysics, and follows with the study of initial-value problems in Secs. 2.2 and 2.3, which are treated by a five-stage method. This systematic breakdown allows the student to separate the steady-state solution from the transient solution (found by the separation-of-variables algorithm) and to verify the uniqueness and asymptotic behavior of the solution as well as to compute the relaxation time. I have found that students can easily appreciate and understand this method, which combines mathematical precision and clear physical interpretation. The five-stage method is used throughout the book, in Secs. 2.5, 3.4, and 4.1. Chapter 2 also includes the wave equation for the vibrating string (Sec. 2.4), solved both by the Fourier series and by the d'Alembert formula. Both methods have advantages and disadvantages, which are discussed in detail. My derivations of both the wave equation and the heat equation are from a three-dimensional viewpoint, which I feel is less artificial and more elegant than many treatments that begin with a one-dimensional formulation.

Following Chapter 2, there is a wide choice in the direction of the course. Those instructors who wish to give a complete treatment of boundary-value problems in cylindrical and spherical coordinates, including Bessel and Legendre functions, will want to cover all of Chapters 3 and 4. Other instructors may ignore this material completely and proceed directly to Chapter 5, on Fourier transforms. An intermediate path might be to cover Secs. 3.1 and/or 4.1, which treat (respectively) Laplace's equation in polar coordinates and spherically symmetric solutions of the heat equation in three dimensions. Neither topic requires any special functions beyond those encountered in trigonometric Fourier series.

Chapter 5 treats Fourier transforms using the complex exponential notation. This is a natural extension of the complex form of the Fourier series, which is covered in Sec. 1.5. Using the Fourier transform, I reduce the heat, Laplace, wave, and telegraph equations to ordinary differential equations with constant coefficients, which can be solved by elementary methods. In many cases, these Fourier representations of the solutions can be rewritten as explicit representations (by what is usually known as the Green function method). The method of images for solving problems on a semi-infinite axis is naturally developed here. The Green functions methods are developed more systematically in Chapter 8. After preparing the one-dimensional case, I give a self-contained treatment of the explicit representation of the solution of Poisson's equation in two and three dimensions. In addition to the traditional physical applications, the Black-Scholes model of option pricing from financial mathematics is included.

Throughout the book I emphasize the asymptotic analysis of *series* solutions of boundary-value problems. Chapter 6 gives an elementary account of asymptotic analysis of *integrals,* in particular the Fourier integral representations of the solutions obtained in Chapter 5. The methods include integration by parts, Laplace's method, and the method of stationary phase. These culminate in an asymptotic analysis of the telegraph equation, which illustrates the group velocity of a wave packet.

No introduction to partial differential equations would be complete without some discussion of approximate solutions and numerical methods. Chapter 7 gives the student some working knowledge of the finite difference solution of the heat equation and Laplace's equation in one and two space dimensions. The material on variational methods first relates differential equations to variational problems and then outlines some direct methods that may be used to arrive at approximate solutions, including the finite element method.

This book was developed from course notes for Mathematics C91-1 in the Integrated Science Program at Northwestern University. The course has been taught to college juniors since 1977; Chapters 1 to 5 are covered in a 10-week quarter. I am indebted to my colleagues Leonard Evens, Robert Speed, Paul Auvil, Gene Birchfield, and Mark Ratner for providing valuable suggestions on the mathematics and its applications. The first draft was written in collaboration with Michael Hopkins. The typing was done by Vicki Davis and Julie Mendelson. The solutions were compiled with the assistance of Mark Scherer. Valuable technical advice was further provided by Edward Reiss and Stuart Antman.

In preparation of this new edition, I received valuable comments and suggestions from Andrew Bernoff, Joseph B. Keller, Thaddeus Ladd, Jeff Miller, Carl Prather, Robert Seeley, and Marshall Slemrod. I also acknowledge the reviewing services of the following individuals: David Bao, University of Houston; William O. Bray, University of Maine; Peter Colwell, Iowa State University; Kenneth A. Heimes, Iowa State University; Yinxi Huang, University of Memphis; Mohammad Kozemi, University of North Carolina–Charlotte; and William Mays, Gloucester Community College (NJ).

In preparation of the past edition, I received valuable comments and suggestions from James W. Brown, Charles Holland, Robert Pego, Mei-Chang Shen, Clark Robinson, Nancy Stanton, Athanassios Tzavaras, David Kapov, and Dennis Kosterman. For the second edition, I also acknowledge the reviewing services of the following individuals: William O. Bray, University of Maine; William E. Fitzgibbon, University of Houston; Peter J. Gingo, University of Akron; Mohammad Kozemi, University of North Carolina–Charlotte; Gilbert N. Lewis, Michigan Technical University; Geoffrey Martin, University of Toledo; Norman Meyers, University of Minnesota–Minneapolis; Allen C. Pipkin, Brown University; R. E. Showalter, University of Texas–Austin; and Grant V. Welland, University of Missouri–St. Louis.

In the preparation of the first edition I was encouraged by John Corrigan of the McGraw-Hill College Division. Preparing the second edition of this text, I benefited from the editorial services of Karen M. Hughes and Richard Wallis. Most recently, for this new edition with Waveland Press, it has been a pleasure to work with Jan Fisher and the staff of Publication Services. The current printing was completed with the editorial assistance of Miron Bekker, Harry R. Hughes, Monica Sharpnack, Nancy Stanton, and Alphonse Sterling.

Mark A. Pinsky

Contents

CHAPTER 0

PRELIMINARIES

INTRODUCTION

This chapter serves as an overview, with some motivation of the origins of partial differential equations and some of the mathematical methods that will be used repeatedly throughout the book. In particular, the technique of *separation of variables* is introduced in Sec. 0.2, and the concept of *orthogonal functions* is introduced in Sec. 0.3 and illustrated through relevant examples. Previous work in vector calculus, infinite series, and ordinary differential equations is reviewed in the appendixes.

0.1. Partial Differential Equations

In this section we introduce the notion of a partial differential equation and illustrate it with various examples.

0.1.1. What is a partial differential equation? From the purely mathematical point of view, a partial differential equation (PDE) is an equation that relates a function u of several variables $x_1, \ldots, x_n$ and its partial derivatives. This is distinguished from an *ordinary differential equation*, which pertains to functions of *one variable*. For example, if a function of two variables is denoted $u(x, y)$, then one may consider the following as examples of partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's equation)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(the wave equation)}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad \text{(the heat equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y) \qquad \text{(Poisson's equation)}$$

In order to simplify the notation, we will often use subscripts to denote the various partial derivatives, so that $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, and so forth. In this notation, the above four examples are written, respectively,

$$u_{xx} + u_{yy} = 0, \qquad u_{xx} - u_{yy} = 0, \qquad u_{xx} - u_y = 0, \qquad u_{xx} + u_{yy} = g$$

The *order* of a PDE is indicated by the highest-order derivative that appears. All of the above four examples are PDEs of second order.

In the case of a function of several variables $u(x_1, \ldots, x_n)$, the most general second-order partial differential equation can be written

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots, u_{x_nx_n}) = 0$$

where the dots imply the other partial derivatives that may occur. In case $n = 1$ we obtain the second-order ordinary differential equation $F(x, u, u', u'') = 0$. The necessary information on ordinary differential equations is reviewed in Appendix A.1.

Another important concept pertaining to a PDE is that of *linearity*. This is most easily described in the context of a differential operator $\mathcal{L}$ applied to a function u. Examples of differential operators are $\mathcal{L}u = \partial u/\partial x$, $\mathcal{L}u = 3u + \sin y\, \partial u/\partial x$, and $\mathcal{L}u = u\, \partial^2 u/\partial x^2$. The operator is said to be *linear* if for any two functions u, v and any constant c,

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \qquad \mathcal{L}(cu) = c\mathcal{L}u$$

A PDE is said to be *linear* if it can be written in the form

$$\mathcal{L}u = g \tag{0.1.1}$$

where $\mathcal{L}$ is a linear differential operator and g is a given function. In case $g = 0$, (0.1.1) is said to be *homogeneous*. For example, three of the above examples (Laplace's equation, the wave equation, and the heat equation) are linear homogeneous PDEs. The most general linear second-order PDE in two variables is written

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

where the functions a, b, c, d, e, f, g are given.

EXERCISES 0.1.1

1. Write down the most general linear first-order PDE in two variables. How many given functions are necessary to specify the PDE?
2. Write down the most general linear first-order PDE in three variables. How many given functions are necessary to specify the PDE?
3. Write down the most general linear first-order homogeneous PDE in two variables. How many given functions are necessary to specify the PDE?
4. Write down the most general linear first-order homogeneous PDE in three variables. How many given functions are necessary to specify the PDE?

5. Define the operator $\mathcal{L}$ by the formula $\mathcal{L}u(x,y) = d(x,y)u_x + e(x,y)u_y + f(x,y)u$. Show that $\mathcal{L}$ is a linear differential operator.
6. Define the operator $\mathcal{L}$ by the formula $\mathcal{L}u(x,y) = a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy}$. Show that $\mathcal{L}$ is a linear differential operator.
7. Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear differential operators. Show that $\mathcal{L}_1+\mathcal{L}_2$ is also a linear differential operator.

0.1.2. Superposition principle and subtraction principle. In the study of ordinary differential equations, it is often possible to write the general solution in a closed form, in terms of arbitrary constants and a set of particular solutions. This is not possible in the case of partial differential equations. To see this in more detail, we cite the example of the second-order equation $u_{xx} = 0$ for the unknown function $u(x,y)$. Integrating once reveals that $u_x(x,y) = C(y)$, while a second integration reveals that $u(x,y) = xC(y) + D(y)$, where C and D are *arbitrary functions.* Clearly, there are infinitely many different choices for each of C and D, so that this solution cannot be specified in terms of a finite number of arbitrary constants. Stated otherwise, the space of solutions is infinite-dimensional.

In order to work effectively with a linear PDE, we must develop rules for combining known solutions. The following principle is basic to all of our future work.

PROPOSITION 0.1.1. **(Superposition principle for homogeneous equations).** *If $u_1, \ldots, u_N$ are solutions of the same linear homogeneous PDE $\mathcal{L}u = 0$, and $c_1, \ldots, c_N$ are constants (real or complex), then $c_1u_1 + \cdots + c_Nu_N$ is also a solution of the PDE.*

Proof. The proof of this depends on the property of linearity. Indeed, we have $\mathcal{L}(u_i) = 0$ for $i = 1, \ldots, n$. Hence

$$\mathcal{L}(c_1u_1 + \cdots + c_Nu_N) = c_1\mathcal{L}(u_1) + \cdots + c_N\mathcal{L}(u_N) = 0 \quad \bullet$$

For example, one may verify that for any constant k, the function $u(x,y) = e^{kx}\cos ky$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$. Therefore, by the superposition principle, the function $u(x,y) = e^{-x}\cos y + 2e^{-3x}\cos 3y - 5e^{-\pi x}\cos \pi y$ is also a solution of Laplace's equation.

The superposition principle does not apply to nonhomogeneous equations. For example, if u_1 and u_2 are solutions of the Poisson equation $u_{xx} + u_{yy} = 1$, then the function $u_1 + u_2$ is the solution of a different equation, namely, $u_{xx} + u_{yy} = 2$. Nevertheless, we have the following important general principle that allows one to relate nonhomogeneous equations to homogeneous equations.

PROPOSITION 0.1.2. **(Subtraction principle for nonhomogeneous equations).** *If u_1 and u_2 are solutions of the same linear nonhomogeneous equation $\mathcal{L}u = g$, then the function $u_1 - u_2$ is a solution of the associated homogeneous equation $\mathcal{L}u = 0$.*

Proof. We have

$$\mathcal{L}(u_1 - u_2) = \mathcal{L}u_1 - \mathcal{L}u_2 = 0 \quad \bullet$$

For example, if u_1 and u_2 are both solutions of the Poisson equation $u_{xx}+u_{yy} = 1$, then $u_1 - u_2$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

The subtraction principle allows us to find the general solution of a nonhomogeneous equation $\mathcal{L}u = g$ once we know a particular solution of the equation and the general solution of the related homogeneous equation $\mathcal{L}u = 0$. The result is expressed as follows.

Corollary. *The general solution of the linear partial differential equation $\mathcal{L}u = g$ can be written in the form*

$$u = U + v$$

where U is a particular solution of the equation $\mathcal{L}U = g$ and v is the general solution of the related homogeneous equation $\mathcal{L}v = 0$.

We illustrate with an example.

EXAMPLE 0.1.1. *Find the general solution $u(x, y)$ of the equation $u_{xx} = 2$.*

Solution. It is immediately verified that the function $u = x^2$ is a solution of the given equation. The general solution of the associated homogeneous equation $u_{xx} = 0$ is $u(x, y) = xg(y) + h(y)$. Therefore the general solution of the nonhomogeneous equation is $u(x, y) = x^2 + xg(y) + h(y)$. $\bullet$

EXERCISES 0.1.2

1. Show that for any constant k, the function $u(x, y) = e^{kx} \cos ky$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.
2. Show that for any constant k, the function $u(x, y) = e^{kx}e^{k^2y}$ is a solution of the heat equation $u_{xx} - u_y = 0$.
3. Show that for any constant k, the function $u(x, y) = e^{kx}e^{-ky}$ is a solution of the wave equation $u_{xx} - u_{yy} = 0$.
4. Show that for any constant k, the function $u(x, y) = (k/2)x^2 + (1 - k)y^2/2$ is a solution of Poisson's equation $u_{xx} + u_{yy} = 1$.

0.1.3. Sources of PDEs in classical physics. Many laws of physics are expressed mathematically as differential equations. The student of elementary mechanics is familiar with Newton's second law of motion, which expresses the acceleration of a system in terms of the forces on the system. In the case of one or more point particles, this translates into a system of ordinary differential equations when the force law is known.

For example, a single spring with Hooke's law of elastic restoration and no frictional forces gives rise to the linear equation of the harmonic oscillator, which is well studied in elementary courses. A system of particles that interact through several springs gives rise to a second-order system of differential equations, which may be resolved into its normal modes— each of which undergoes simple harmonic motion. Newton's law of gravitational attraction gives rise to a more complicated system of nonlinear ordinary differential equations. Generally speaking, whenever we have a finite number of point particles, the mathematical model is a system of ordinary differential equations, where *time* plays the role of independent variable and the positions/velocities of the particles are the dependent variables. In Chapter 2, we will give the complete derivation of the wave equation, which governs the motion of a tightly streched vibrating string.

For time-dependent systems in one spatial dimension, we will use the notation $u(x;t)$ to denote the unknown function that is a solution of the PDE. In the case of two or three spatial dimensions we will use the repective notations $u(x,y;t)$ and $u(x,y,z;t)$ to denote the solution of the PDE.

In the following subsection we will give a simplified derivation of the one-dimensional heat equation. The complete derivation of the heat equation as it applies to three-dimensional systems is found in Chapter 2.

0.1.4. The one-dimensional heat equation. Consider a one-dimensional rod that is capable of conducting heat, and for which we can measure the temperature $u(x;t)$ at the position x at time instant t. We assume that this function has continuous partial derivatives of orders 1 and 2. In order to motivate the discussion, we first consider a finite system of equally spaced points $x_1 < x_2 < \cdots < x_N$. We expect that the temperature will remain constant as a function of time if there is a *local equilibrium*, meaning that the temperature $u(x_i;t)$ is equal to the average of its neighbors; in symbols,

$$\frac{\partial u}{\partial t}(x_i;t) = 0 \qquad \text{if} \qquad u(x_i;t) = \frac{1}{2}u(x_{i-1};t) + \frac{1}{2}u(x_{i+1};t)$$

For example, if the point x_i is at 50 degrees and the neighbor to the left is at 40 degrees while the neighbor to the right is at 60 degrees, then we expect no change in temperature.

On the other hand, if this condition of local equilibrium is not satisfied, then we may expect that the temperature will change, in relation to the amount of disequilibrium. Certainly one expects the temperature to increase if both neighbors are warmer, but also if the average is warmer; for example, if the point x_i is at 50 degrees while the left neighbor is at 45 degrees and the right neighbor is at 65 degrees, then the average is 55 degrees—5 degrees warmer than the home temperature.

In order to quantify this, we postulate the following dynamical law.

The time rate of change of temperature at the point x_i is proportional to the difference between the temperature at x_i and the average of the temperatures at the two neighboring points x_{i-1}, x_{i+1}

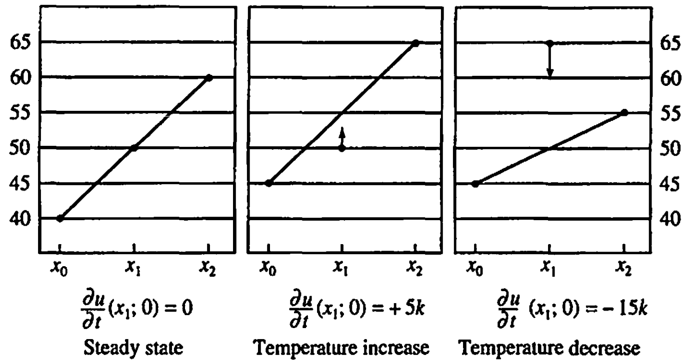


FIGURE 0.1.1 Three different configurations of heat flow dynamics

To translate this into a mathematical statement, we must introduce a constant of proportionality k, which will depend on the properties of the medium. If we have a "good conductor," then k will be large, whereas if we have a "bad conductor," then k will be small. The desired mathematical statement then reads

(0.1.2)

$$\boxed{\frac{\partial u}{\partial t}(x_i; t) = k\left(\frac{1}{2}[u(x_{i+1}; t) + u(x_{i-1}; t)] - u(x_i; t)\right), \qquad i = 2, \ldots, N-1}$$

Figure 0.1.1 presents three different configurations of heat flow dynamics, corresponding to local equilibrium (also called *steady state*), temperature increase, and temperature decrease.

The above mathematical model of heat flow can be expected to be rigorously valid for a finite system of equally spaced points $x_1 < x_2 < \cdots < x_N$. Equation (0.1.2) is a system of ordinary differential equations that can be solved by algebraic methods, if necessary. If we now consider these points as an approximation to a continuum of points, then we can expect this model to be valid as a first approximation when the spacing tends to zero. In order to obtain a partial differential equation we apply Taylor's theorem with remainder:

$$u(x_{i+1}; t) - u(x_i; t) = (x_{i+1} - x_i)u_x(x_i; t) + \frac{1}{2}(x_{i+1} - x_i)^2\, u_{xx}(x_i'; t)$$
$$u(x_{i-1}; t) - u(x_i; t) = (x_{i-1} - x_i)u_x(x_i; t) + \frac{1}{2}(x_{i-1} - x_i)^2\, u_{xx}(x_i''; t)$$

where the points x_i', x_i'' satisfy $x_{i-1} \le x_i'' \le x_i \le x_i' \le x_{i+1}$. Recalling that the points are equally spaced, let $\Delta x = x_{i+1} - x_i$ be the common spacing, and substitute into (0.1.2) to obtain

$$\frac{\partial u}{\partial t}(x_i;t) = \frac{k(\Delta x)^2}{4}\left(u_{xx}(x_i';t) + u_{xx}(x_i'';t)\right) \tag{0.1.3}$$

The final simplification is to assert that, if the spacing is very small, then the values of the second partial derivative will vary very little from the nearby points x_i, x_i', x_i'', and thus we can replace the two values of the second partial derivatives by the value at the point x_i. Defining $K = k(\Delta x)^2/2$, we obtain the heat equation

$$\boxed{\frac{\partial u}{\partial t} = K u_{xx}} \tag{0.1.4}$$

The constant K is called the *diffusivity.*

With no further information, the heat equation (0.1.4) will have infinitely many solutions. In order to specify a solution of the heat equation, we consider various boundary conditions and initial conditions. Assuming that the rod occupies the interval $0 < x < L$ of the x-axis, we consider three types of boundary conditions at the endpoint $x = 0$:

$$\begin{aligned} \mathrm{I} &: u(0;t) = T_0 \\ \mathrm{II} &: u_x(0;t) = 0 \\ \mathrm{III} &: -u_x(0;t) = h(T_e - u(0;t)) \qquad \text{where } h > 0 \end{aligned}$$

Boundary condition I signifies that the temperature at the end $x = 0$ is held constant. In practice this could occur as the result of heating the end by means of some device. Boundary condition II signifies that there is no heat flow at the end $x = 0$. In practice this could occur by means of *insulation*, which prohibits the flow of heat at this end. Boundary condition III is sometimes called *Newton's law of cooling*: the negative of the partial derivative is interpreted as the *heat flux*, i.e., the rate of heat flow *out* of the end $x = 0$, and is required to be proportional to the difference between the outside temperature T_e and the endpoint temperature $u(0;t)$. If this difference is large, then we may expect heat to flow out of the rod at a rapid rate. If T_e is less than the endpoint temperature, then $u(0;t) > T_e$ and the rate will be negative, so that we may expect heat to flow *into* the rod from the exterior. The concept of flux will be discussed in more detail in Chapter 2, when we derive the three-dimensional heat equation.

Similarly, we can have each of the three boundary conditions present at the end $x = L$; in detail,

$$\begin{aligned} \mathrm{I} &: u(L;t) = T_0 \\ \mathrm{II} &: u_x(L;t) = 0 \\ \mathrm{III} &: u_x(L;t) = h(T_e - u(L;t)) \qquad \text{where } h > 0 \end{aligned}$$

The constants T_0, h, and T_e may be the same as for the endpoint $x = 0$ or may have different values. The interpretations are exactly the same as for the endpoint

$x = 0$, with one small exception: in the third boundary condition (III), the heat flux at the end $x = L$ is written without the minus sign, since this measures the rate of heat flow *out of* the end $x = L$. As before, we expect that if the external temperature T_e is much greater than the endpoint temperature $u(L;t)$, then the rate of heat flow out of the end will be large, whereas if the external temperature is less than the endpoint temperature, then the heat flow out will be negative.

A typical boundary-value problem for the heat equation will have one boundary condition for each end $x = 0$ and $x = L$. Considering all possible cases, we have nine different combinations, of which we list the first three below:

$$\begin{aligned} u(0;t) &= T_0, \quad u(L;t) = T_L \\ u(0;t) &= T_0, \quad u_x(L;t) = 0 \\ u(0;t) &= T_0, \quad u_x(L;t) = h(T_e - u(L;t)) \end{aligned}$$

The final piece of information used to specify the solution is the *inital data.* This is simply written

$$u(x;0) = f(x), \qquad 0 < x < L$$

This signifies that the temperature is known at time $t = 0$ and is given by the function $f(x), 0 < x < L$. Note that we do not insist that this agree with the values of the solution at the endpoints $x = 0$, $x = L$. Specification of boundary conditions and initial conditions is known as the *initial-boundary-value problem.* In Chapter 2 we will make a detailed study of this for the one-dimensional heat equation.

In the remainder of this subsection we will determine the *steady-state* solutions of the heat equation corresponding to the various boundary conditions. u is said to be a steady-state solution if $\partial u/\partial t = 0$. Referring to the heat equation (0.1.4), this is equivalent to the statement that $u_{xx} = 0$.

EXAMPLE 0.1.2. *Find the steady-state solution of the heat equation with the boundary conditions* $u(0;t) = T_1$, $u(L;t) = T_2$.

Solution. Since the solution is independent of time, we can write $u = U(x)$, with $U''(x) = 0$. The general solution of this is a linear function: $U(x) = Ax + B$. The boundary condition at $x = 0$ gives $B = T_1$, whereas the boundary condition at $x = L$ gives $AL + B = T_2$, $A = (T_2 - T_1)/L$. The steady-state solution is

$$U(x) = T_1 + \frac{T_2 - T_1}{L}x = T_2\frac{x}{L} + T_1\left(1 - \frac{x}{L}\right) \quad \bullet$$

EXAMPLE 0.1.3. *Find the steady-state solution of the heat equation with the boundary conditions* $u(0;t) = T_1$, $u_x(L;t) = h(T_e - u(L;t))$.

Solution. Since the solution is independent of time, we can write $u = U(x)$, with $U''(x) = 0$. The general solution of this is a linear function: $U(x) = Ax + B$. The boundary condition at $x = 0$ gives $B = T_1$, whereas the boundary condition at $x = L$ gives $A = h(T_e - AL - B)$, $A = -h(T_1 - T_e)/(1 + hL)$ and the steady-state solution

$$U(x) = T_1 - \frac{hx}{1+hL}(T_1 - T_e) = T_e\frac{hx}{1+hL} + T_1\frac{1+h(L-x)}{1+hL} \quad \bullet$$

EXERCISES 0.1.4

1. Find the steady-state solution of the heat equation with the boundary conditions $u(0;t) = T_1$, $u_x(L;t) = 0$.
2. Find the steady-state solution of the heat equation with the boundary conditions $u_x(0;t) = h(T_0 - u(0;t)), u_x(L;t) = \Phi$, where h, T_0, Φ are positive constants.
3. Find the steady-state solution of the heat equation with the boundary conditions $-u_x(0;t) = h(T_0 - u(0;t)), u_x(L;t) = h(T_1 - u(L;t))$ where T_0, T_1, h are constants with $h > 0$.

0.1.5. Classification of second-order PDEs. It is impossible to formulate a general existence theorem that applies to all linear partial differential equations, even if we restrict attention to the important case of second-order equations. Instead, it is more natural to specify a solution through a set of boundary conditions or initial conditions related to the equation. For example, the solution of the heat equation $u_t = Ku_{xx}$ in the region $0 < x < L$, $0 < t < \infty$ may be specified uniquely in terms of the initial conditions at $t = 0$ and the boundary conditions at $x = 0$ and $x = L$. On the other hand, the solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ in the region $0 < x < L$, $0 < t < \infty$ is uniquely obtained in terms of the boundary conditions at $x = 0$, $x = L$ and *two* initial conditions, pertaining to the solution $u(x;0)$ and its time derivative $\partial u/\partial t(x;0)$. In order to put this in a more general context, one may classify the second-order linear partial differential equation as follows:

(0.1.5)

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

If $4ac - b^2 > 0$, the PDE (0.1.5) is called *elliptic.*

If $4ac - b^2 = 0$, the PDE (0.1.5) is called *parabolic.*

If $4ac - b^2 < 0$, the PDE (0.1.5) is called *hyperbolic.*

For example, Laplace's equation and Poisson's equation are both elliptic, while the wave equation is hyperbolic. The heat equation is parabolic. General theorems about these classes of equations are stated and proved in more advanced texts and reference books. Here we indicate the types of boundary conditions that are natural for each of the three types of equations.

If the equation is elliptic, we may solve the *Dirichlet problem*, namely, in a region D to find a solution of $\mathcal{L}u = g$ that further satisfies the boundary condition that $u = \phi(x,y)$ on the boundary of D. For example, the physical problem of determining the electrostatic potential function $u(x,y)$ in the interior of the cylindrical region $x^2+y^2 < R^2$ when the charge density $\rho(x,y)$ is specified and the boundary is required to be an equipotential surface leads to the elliptic boundary-value problem

$$\begin{aligned} u_{xx} + u_{yy} &= -\rho(x,y) \qquad && x^2+y^2<R^2 \\ u(x,y) &= C && x^2+y^2=R^2 \end{aligned}$$

If the equation is parabolic or hyperbolic, it is natural to solve the *Cauchy problem*, which amounts to specifying the solution and its time derivative on the line $t=0$ as well as specifying the relevant boundary conditions. Here we indicate the Cauchy problem for the equation of the vibrating string, which will be derived in complete detail in Chapter 2:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad && t>0,\ 0<x<L \\ u(x;0) &= f_1(x) && 0<x<L \\ u_t(x;0) &= f_2(x) && 0<x<L \\ u(0;t) &= 0,\ u(L;t)=0 && t>0 \end{aligned}$$

The initial conditions f_1, f_2 represent the initial position and velocity of the vibrating string. The boundary conditions at $x=0$ and $x=L$ signify that ends of the string are fixed for all time at the position $u=0$.

EXERCISES 0.1.5

Classify each of the following second-order equations as elliptic, parabolic, or hyperbolic.

1. $u_{xx} + 3u_{xy} + u_{yy} + 2u_x - u_y = 0$
2. $u_{xx} + 3u_{xy} + 8u_{yy} + 2u_x - u_y = 0$
3. $u_{xx} - 2u_{xy} + u_{yy} + 2u_x - u_y = 0$
4. $u_{xx} + xu_{yy} = 0$

0.2. Separation of Variables

0.2.1. What is a separated solution? A fundamental technique for obtaining solutions of linear partial differential equations is the method of *separation of variables.* This means that we look for particular solutions in the form $u(x,y) = X(x)Y(y)$ and try to obtain ordinary differential equations for $X(x)$ and $Y(y)$. These equations will contain a parameter called the *separation constant.* The function $u(x,y)$ is called a *separated solution.* Then we can use the superposition principle to obtain more general solutions of a linear homogeneous PDE as sums of separated solutions.

0.2.2. Separated solutions of Laplace's equation. It is especially simple to obtain separated solutions for Laplace's equation, $u_{xx} + u_{yy} = 0$.

If we let $u(x,y) = X(x)Y(y)$ and substitute in Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assumed to be nonzero), we obtain

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

The first term depends only on x, while the second term depends only on y. The sum can equal zero only if both terms are constants that sum to zero. In order to express this in terms of a single parameter, we introduce the constant λ and obtain the system of two ordinary differential equations

$$\frac{X''(x)}{X(x)} = \lambda, \qquad \frac{Y''(y)}{Y(y)} = -\lambda$$

λ is the *separation constant*. These equations may be written in the more standard form

$$X''(x) - \lambda X(x) = 0 \tag{0.2.1}$$

$$Y''(y) + \lambda Y(y) = 0 \tag{0.2.2}$$

Both of these are second-order homogeneous linear ordinary differential equations, which may be solved in terms of exponential functions, trigonometric functions, or linear functions, depending on the sign of λ.[1] To proceed further, we consider separately the three cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

Case 1. If $\lambda > 0$, we write $\lambda = k^2$, where $k > 0$. The general solutions to (0.2.1) and (0.2.2) are

$$\begin{aligned} X(x) &= A_1 e^{kx} + A_2 e^{-kx} \\ Y(y) &= A_3 \cos ky + A_4 \sin ky \end{aligned}$$

where A_1, A_2, A_3, A_4 are arbitrary constants. These cannot be determined until we have imposed further conditions, which will be done later.

Case 2. If $\lambda = 0$, we have the equations $X'' = 0$, $Y'' = 0$, for which the general solutions to (0.2.1) and (0.2.2) are linear functions:

$$\begin{aligned} X(x) &= A_1 x + A_2 \\ Y(y) &= A_3 y + A_4 \end{aligned}$$

where A_1, A_2, A_3, A_4 are arbitrary constants.

[1] For a review of ordinary differential equations, consult Appendix A.1.

Case 3. If $\lambda < 0$, we write $\lambda = -l^2$, where $l > 0$; the general solutions of (0.2.1) and (0.2.2) are

$$\begin{aligned} X(x) &= A_1 \cos lx + A_2 \sin lx \\ Y(y) &= A_3 e^{ly} + A_4 e^{-ly} \end{aligned}$$

To summarize, we have found the following separated solutions of Laplace's equation:

$$u(x,y) = \begin{cases} (A_1 e^{kx} + A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky) & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) \\ (A_1 \cos lx + A_2 \sin lx)(A_3 e^{ly} + A_4 e^{-ly}) & l > 0 \end{cases}$$

We can also write the separated solutions of Laplace's equation in terms of *hyperbolic functions*. These are defined by the formulas

$$\sinh a = \frac{1}{2}(e^a - e^{-a}), \qquad \cosh a = \frac{1}{2}(e^a + e^{-a})$$

From this it follows immediately that

$$e^a = \cosh a + \sinh a, \qquad e^{-a} = \cosh a - \sinh a$$

Using this notation, we can write the separated solutions of Laplace's equation in the equivalent form

$$u(x,y) = \begin{cases} (A_1 \sinh kx + A_2 \cosh kx)(A_3 \cos ky + A_4 \sin ky) & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) \\ (A_1 \cos lx + A_2 \sin lx)(A_3 \sinh ly + A_4 \cosh ly) & l > 0 \end{cases}$$

We emphasize that the constants A_1, A_2, A_3, A_4 will change when we make this change of notation. But the *form* of the solution remains unchanged; put otherwise, the classes of separated solutions defined by the two sets of notations are identical.

To derive these, we assumed that $u(x,y) \neq 0$. Having now obtained the explicit forms, we can verify independently that in each case $u(x,y)$ satisfies Laplace's equation.

EXAMPLE 0.2.1. *Verify that the preceding separated solutions satisfy Laplace's equation.*

Solution. In case $\lambda > 0$, we have

$$u(x,y) = (A_1 e^{kx} + A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky)$$

so that

$$\begin{aligned} u_x &= (kA_1 e^{kx} - kA_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky) \\ u_{xx} &= (k^2 A_1 e^{kx} + k^2 A_2 e^{-kx})(A_3 \cos ky + A_4 \sin ky) \\ u_y &= (A_1 e^{kx} + A_2 e^{-kx})(-kA_3 \sin ky + kA_4 \cos ky) \\ u_{yy} &= (A_1 e^{kx} + A_2 e^{-kx})(-k^2 A_3 \cos ky - k^2 A_4 \sin ky) \end{aligned}$$

The second and fourth terms are negatives of one another. Therefore $u_{xx} + u_{yy} = 0$, and we have verified Laplace's equation in case $\lambda > 0$.

In case $\lambda = 0$ we have

$$\begin{aligned} u_x &= A_1(A_3y + A_4), \qquad & u_{xx} = 0 \\ u_y &= (A_1x + A_2)A_3, \qquad & u_{yy} = 0 \end{aligned}$$

so that both of the partial derivatives u_{xx} and u_{yy} are zero and Laplace's equation is immediate in this case. The verification for $\lambda < 0$ is left to the exercises. •

EXERCISES 0.2.2

1. Verify that $u(x, y) = (A_1 \cos lx + A_2 \sin lx)(A_3e^{ly} + A_4e^{-ly})$ satisfies Laplace's equation, for any $l > 0$.
2. Suppose that $u(x, y)$ is a solution of Laplace's equation. If θ is a fixed real number, define the function $v(x, y) = u(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. Show that $v(x, y)$ is a solution of Laplace's equation.
3. Apply the result of the previous exercise to the separated solutions of Laplace's equation of the form $u(x, y) = (A_1e^{kx} + A_2e^{-kx})(A_3 \cos ky + A_4 \sin ky)$, to obtain additional solutions of Laplace's equation. Are these new solutions separated?
4. From the definitions of the hyperbolic functions, prove the following properties:
   (a) $\sinh 0 = 0$, $\cosh 0 = 1$
   (b) $(d/dx)(\sinh x) = \cosh x$, $(d/dx)(\cosh x) = \sinh x$
   (c) $\cosh x \geq 1$ for all x
   (d) $\cosh x \geq \sinh x$ for all x
   (e) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
   (f) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

0.2.3. Real and complex separated solutions. In the previous subsection we found all of the separated solutions of Laplace's equation, in terms of trigonometric functions, exponential functions, and linear functions using a real separation constant.

In looking for separated solutions of a PDE, it is often convenient to allow the functions $X(x)$ and $Y(y)$ to be complex-valued, corresponding to a complex separation constant. The following proposition shows that the real and imaginary parts of *any* complex-valued solution will again satisfy the PDE.

PROPOSITION 0.2.1. *Let $u(x, y) = v_1(x, y) + iv_2(x, y)$ be a complex-valued solution of the linear PDE*

$$\mathcal{L}u = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$$

where a, b, c, d, e, f, g are real-valued functions of (x, y). Then $v_1(x, y) = \operatorname{Re} u(x, y)$ satisfies the PDE $\mathcal{L}u = g$, and $v_2(x, y) = \operatorname{Im} u(x, y)$ satisfies the associated homogeneous PDE $\mathcal{L}u = 0$.

Proof. The operation of partial differentiation is linear; thus

$$\begin{aligned} u_x &= (v_1)_x + i(v_2)_x \\ u_{xx} &= (v_1)_{xx} + i(v_2)_{xx} \end{aligned}$$

with similar expressions for u_y, u_{yy}, and u_{xy}. Substituting these into the partial differential equation and separating the real and imaginary parts yields the result. •

We illustrate this technique with the example of Laplace's equation. Letting $u(x,y) = X(x)Y(y)$, consider a purely imaginary separation constant in the form $\lambda = 2ik^2$, where $k > 0$. This leads to the two ordinary differential equations

$$X''(x) - 2ik^2 X(x) = 0 \tag{0.2.3}$$

$$Y''(y) + 2ik^2 Y(y) = 0 \tag{0.2.4}$$

These can be solved in terms of the complex exponential function, using the observation that $[k(1+i)]^2 = 2ik^2$, $[k(1-i)]^2 = -2ik^2$. Thus

$$X(x) = A_1 e^{k(1+i)x} + A_2 e^{-k(1+i)x}, \qquad Y(y) = A_3 e^{k(1-i)y} + A_4 e^{-k(1-i)y}$$

Multiplying these, we obtain the complex separated solutions

$$u(x,y) = \begin{cases} e^{k(x+y)} e^{ik(x-y)} \\ e^{k(x-y)} e^{ik(x+y)} \\ e^{k(y-x)} e^{-ik(x+y)} \\ e^{-k(x+y)} e^{ik(y-x)} \end{cases}$$

When we take the real and imaginary parts, we obtain the following real-valued solutions of Laplace's equation:

$$u(x,y) = \begin{cases} e^{k(x+y)} \cos k(x-y), & e^{k(x+y)} \sin k(x-y) \\ e^{k(x-y)} \cos k(x+y), & e^{k(x-y)} \sin k(x+y) \\ e^{k(y-x)} \cos k(x+y), & e^{k(y-x)} \sin k(x+y) \\ e^{-k(x+y)} \cos k(y-x), & e^{-k(x+y)} \sin k(y-x) \end{cases}$$

When we consider more general linear PDEs, complex-valued separated solutions may always be found if the functions a, b, c, d, e, f that occur in the equation are independent of (x,y); in this case we speak of a *PDE with constant coefficients*, whose solutions may be written as exponential functions.

PROPOSITION 0.2.2. *Consider the linear homogeneous PDE*

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

Suppose that a, b, c, d, e, f are real constants. Then there exist complex separated solutions of the form

$$u(x,y) = e^{\alpha x} e^{\beta y}$$

for appropriate choices of the complex numbers α, β.

Proof. We first note that the ordinary rules for differentiating $e^{\alpha x}$ are valid for complex-valued functions. For example, if $\alpha = a + ib$,

$$\begin{aligned}
\frac{d}{dx}(e^{\alpha x}) &= \frac{d}{dx}[e^{ax}(\cos bx + i\sin bx)] \\
&= ae^{ax}\cos bx - be^{ax}\sin bx \\
&\quad + i(ae^{ax}\sin bx + be^{ax}\cos bx) \\
&= e^{ax}(a+ib)(\cos bx + i\sin bx) \\
&= (a+ib)e^{(a+ib)x} \\
&= \alpha e^{\alpha x}
\end{aligned}$$

Similarly, $(d^2/dx^2)(e^{\alpha x}) = \alpha^2 e^{\alpha x}$, with similar expressions for (d/dy) and (d^2/dy^2). Applying this to $u(x,y) = e^{\alpha x}e^{\beta y}$, we have $u_x = \alpha u$, $u_{xx} = \alpha^2 u$, $u_y = \beta u$, $u_{yy} = \beta^2 u$, $u_{xy} = \alpha\beta u$. Substituting these into the PDE, we must have

$$(a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + f)e^{\alpha x}e^{\beta y} = 0$$

But $e^{\alpha x}e^{\beta y} \neq 0$; therefore we obtain a solution if and only if α, β satisfy the quadratic equation

$$\boxed{a\alpha^2 + b\alpha\beta + c\beta^2 + d\alpha + e\beta + f = 0} \tag{0.2.5}$$

For a given value of β, we may solve this equation for α to obtain in general two roots α_1, α_2. Alternatively, we may fix α and solve for β to obtain in general two roots β_1, β_2. This proves the proposition. •

In the case of Laplace's equation, the quadratic equation (0.2.5) is $\alpha^2+\beta^2 = 0$. If α is real, then β must be purely imaginary; conversely if β is real, then α is purely imaginary. These two cases correspond to the separated solutions found in the previous subsection by solving (0.2.1) and (0.2.2). The solutions originating from (0.2.3) correspond to values of α for which α^2 is purely imaginary.

We now turn to some examples involving the heat equation, where complex separated solutions are useful.

EXAMPLE 0.2.2. *Find separated solutions of the PDE $u_{xx} - u_t = 0$ in the form $u(x,t) = e^{i\mu x}e^{\beta t}$, with μ real.*

Solution. Substituting $u(x,t) = e^{i\mu x}e^{\beta t}$ in the PDE yields the quadratic equation $-\mu^2 - \beta = 0$. Thus $\beta = -\mu^2$, and we have the separated solutions

$$\begin{aligned}
u(x,t) &= e^{i\mu x}e^{-\mu^2 t} \\
&= \cos\mu x\, e^{-\mu^2 t} + i(\sin\mu x\, e^{-\mu^2 t})
\end{aligned}$$

Taking the real and imaginary parts, we obtain the real-valued separated solutions

$$u(x;t) = \sin\mu x\, e^{-\mu^2 t}, \qquad u(x;t) = \cos\mu x\, e^{-\mu^2 t}$$

By taking linear combinations, we may write the general real-valued separated solution as

$$u(x;t) = (A_1 \sin \mu x + A_2 \cos \mu x)e^{-\mu^2 t}$$

where A_1, A_2 are arbitrary constants. •

In the above example the solutions tend to zero when the time t tends to infinity. In some problems we may wish to obtain a solution that oscillates in time, to represent a periodic disturbance.

EXAMPLE 0.2.3. *Find separated solutions of the PDE* $u_{xx} - u_t = 0$ *in the form* $u(x,t) = e^{\alpha x}e^{i\omega t}$, *where* ω *is real and positive.*

Solution. Substituting $u(x,t) = e^{\alpha x}e^{i\omega t}$ in the PDE $u_t - u_{xx} = 0$ yields the quadratic equation $\alpha^2 - i\omega = 0$. This equation has two solutions, which may be obtained as follows. Writing the complex number i in the polar form $i = e^{i\pi/2}$, we have the two square roots $i^{1/2} = \pm e^{i\pi/4} = \pm(1+i)/\sqrt{2}$. Therefore the solutions of the quadratic equation are $\alpha = \pm(1+i)\sqrt{\omega/2}$. The separated solutions are

$$u(x,t) = \begin{cases} \exp[x(1+i)\sqrt{\omega/2}]\exp(i\omega t) \\ \qquad\qquad = \exp(x\sqrt{\omega/2})\exp[i(\omega t + x\sqrt{\omega/2})] \\ \exp[-x(1+i)\sqrt{\omega/2}]\exp(i\omega t) \\ \qquad\qquad = \exp[-x\sqrt{\omega/2})\exp[i(\omega t - x\sqrt{\omega/2})] \end{cases}$$

Taking the real and imaginary parts, we have the real-valued solutions

$$u(x,t) = \begin{cases} e^{x\sqrt{\omega/2}}\cos(\omega t + x\sqrt{\omega/2}) \\ e^{x\sqrt{\omega/2}}\sin(\omega t + x\sqrt{\omega/2}) \\ e^{-x\sqrt{\omega/2}}\cos(\omega t - x\sqrt{\omega/2}) \\ e^{-x\sqrt{\omega/2}}\sin(\omega t - x\sqrt{\omega/2}) \end{cases}$$

These real-valued solutions are no longer in the separated form $X(x)T(t)$. But because they arise as the real and imaginary parts of complex separated solutions, we refer to them as *quasi-separated solutions.* •

If some of the coefficients a, b, c, d, e, f are not constant, we will no longer have separated solutions in the form of exponential functions. Even worse, the equation may not admit *any* nonconstant separated solutions, for example, $u_x + (x+y)u_y = 0$ (see the exercises). Nevertheless, various classes of equations can still be solved by the separation of variables. For example, for any equation of the form

$$a(x)u_{xx} + c(y)u_{yy} + d(x)u_x + e(y)u_y = 0$$

if we divide by $X(x)Y(y)$, we have

$$\left[a(x)\frac{X''(x)}{X(x)} + d(x)\frac{X'(x)}{X(x)}\right] + \left[c(y)\frac{Y''(y)}{Y(y)} + e(y)\frac{Y'(y)}{Y(y)}\right] = 0$$

The term in the first set of brackets depends only on x, while the term in the second set depends only on y; therefore both are constant and we have reduced the problem to ordinary differential equations. Introducing the separation constant λ, we have in detail

$$\begin{aligned} a(x)X''(x) + d(x)X'(x) + \lambda X(x) &= 0 \\ c(y)Y''(y) + e(y)Y'(y) - \lambda Y(y) &= 0 \end{aligned}$$

The following example gives a concrete illustration.

EXAMPLE 0.2.4. *Find all of the real-valued separated solutions of the PDE* $u_{xx} + y^2 u_{yy} + y u_y = 0$ *valid for* $y > 0$.

Solution. We let $u(x,y) = X(x)Y(y)$ and obtain the separated equations

$$X''(x) + \lambda X(x) = 0 \tag{0.2.6}$$

$$y^2 Y''(y) + y Y'(y) - \lambda Y(y) = 0 \tag{0.2.7}$$

Equation (0.2.6) has constant coefficients and was solved previously; equation (0.2.7) is a form of Euler's equidimensional equation, which can also be solved explicitly. We consider separately the cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

If $\lambda = k^2 > 0$, then the general solution of (0.2.6) is $X(x) = A_1 \cos kx + A_2 \sin kx$. Meanwhile (0.2.7) can be solved by a power $Y(y) = y^r$, where $r(r-1) + r - k^2 = 0$; thus $r = \pm k$ and the general solution $Y(y) = A_3 y^k + A_4 y^{-k}$.

If $\lambda = 0$, then the general solution of (0.2.6) is $X(x) = A_1 + A_2 x$, while (0.2.7) becomes $y^2 Y'' + y Y' = 0$, which has the general solution $Y(y) = A_3 + A_4 \log y$ valid for $y > 0$.

If $\lambda = -l^2 < 0$, then the general solution of (0.2.6) is $X(x) = A_1 e^{lx} + A_2 e^{-lx}$, while (0.2.7) becomes $y^2 Y'' + y Y' + l^2 Y = 0$, which has the general solution $Y(y) = A_3 \cos(l \log y) + A_4 \sin(l \log y)$.

Putting these together, we have the most general real-valued separated solution:

$$u(x,y) = \begin{cases} (A_1 \cos kx + A_2 \sin kx)(A_3 y^k + A_4 y^{-k}) & k > 0 \\ (A_1 + A_2 x)(A_3 + A_4 \log y) & \\ (A_1 e^{lx} + A_2 e^{-lx})(A_3 \cos(l \log y) + A_4 \sin(l \log y)) & l > 0 \end{cases} \qquad \bullet$$

EXERCISES 0.2.3

1. Find the separated equations satisfied by $X(x)$, $Y(y)$ for the following partial differential equations:
   (a) $u_{xx} - 2u_{yy} = 0$ (b) $u_{xx} + u_{yy} + 2u_x = 0$
   (c) $x^2 u_{xx} - 2y u_y = 0$ (d) $u_{xx} + u_x + u_y - u = 0$
2. Which of the following are solutions of Laplace's equation?
   (a) $u(x,y) = e^x \cos 2y$ (b) $u(x,y) = e^x \cos y + e^y \cos x$
   (c) $u(x,y) = e^x e^y$ (d) $u(x,y) = (3x+2)e^y$

In Exercises 3–7, find the separated solutions of the indicated equations.

3. $u_{xx} + 2u_x + u_{yy} = 0$
4. $u_{xx} + u_{yy} + 3u = 0$
5. $x^2 u_{xx} + xu_x + u_{yy} = 0$
6. $u_{xx} - u_{yy} + u = 0$
7. $u_{xx} + yu_y + u = 0$
8. This exercise provides an example of a homogeneous linear partial differential equation with no separated solutions other than $u(x,y) \equiv$ constant. Suppose that $u(x,y) = X(x)Y(y)$ is a solution of the equation $u_x + (x+y)u_y = 0$. Show that $X(x)$ and $Y(y)$ are both constant. [*Hint:* Show first that $X'(x)/X(x) + (x+y)(Y'(y)/Y(y)) = 0$ and deduce that $X'(x)/X(x) = cx+d$, $Y'(y)/Y(y) = -c$ for suitable constants c, d. By solving these ordinary differential equations, show that the PDE is satisfied if and only if $c = 0$, $d = 0$.]

0.2.4. Separated solutions with boundary conditions. In many problems we need separated solutions that satisfy certain additional conditions, which are suggested by the physics of the problem. They may be in the form of *boundary conditions* or *conditions of boundedness.* We shall now illustrate these by means of examples.

EXAMPLE 0.2.5. *Find the separated solutions of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L$, $y > 0$ that satisfy the boundary conditions $u(0,y) = 0$, $u(L,y) = 0$, $u(x,0) = 0$.*

Solution. From the discussion in subsection 0.2.2 we have the separated solutions of three types, depending on the separation constant.

$$u(x,y) = \begin{cases} (A_1 \sinh kx + A_2 \cosh kx)(A_3 \cos ky + A_4 \sin ky) & k > 0 \\ (A_1 x + A_2)(A_3 y + A_4) & \\ (A_1 \cos lx + A_2 \sin lx)(A_3 \sinh ly + A_4 \cosh ly) & l > 0 \end{cases}$$

In the first case, we must have $0 = u(0,y) = A_2(A_3 \cos ky + A_4 \sin ky)$, so $A_2 = 0$, while $0 = u(L,y) = A_1 \sinh kL(A_3 \cos ky + A_4 \sin ky)$ implies that $A_1 = 0$, so this case does not produce any separated solutions that satisfy the boundary conditions.

In the second case, we must have $0 = u(0,y) = A_2(A_3 y + A_4)$, so $A_2 = 0$, and $0 = u(L,y) = A_1 L(A_3 y + A_4)$, so $A_1 = 0$. Therefore this case does not produce any separated solutions that satisfy the boundary conditions.

In the third case, we must have $0 = u(0,y) = A_1(A_3 \sinh ly + A_4 \cosh ly)$, so that $A_1 = 0$; and $0 = u(L,y) = A_2 \sin Ll(A_3 \sinh ly + A_4 \cosh ly)$ has a nonzero solution if and only if $\sin Ll = 0$, which is satisfied if and only if $Ll = n\pi$ for some $n = 1, 2, 3, \ldots$. To satisfy the boundary condition $u(x,0) = 0$, we must have $A_4 = 0$. Writing $A = A_2 A_3$, we have obtained the following separated solutions of Laplace's equation satisfying the boundary conditions:

$$u(x,y) = A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \qquad n = 1, 2, \ldots \quad \bullet$$

The following example occurs repeatedly in the solution of the heat equation in Chapter 2.

EXAMPLE 0.2.6. *Find the separated solutions $u(x;t)$ of the heat equation $u_{xx} - u_t = 0$ in the region $0 < x < L$, $t > 0$ that satisfy the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.*

Solution. In Example 0.2.2 we found the real-valued separated solutions

$$u(x;t) = (A_1 \sin \mu x + A_2 \cos \mu x) e^{-\mu^2 t}$$

In order to satisfy the boundary condition at $x = 0$ we must have $0 = u(0;t) = A_2 e^{-\mu^2 t}$, which is satisfied if and only if $A_2 = 0$. In order to satisfy the boundary condition at $x = L$, we must have $0 = u(L;t) = A_1 (\sin \mu L)\, e^{-\mu^2 t}$. This is satisfied if and only if $\mu L = n\pi$ for some $n = 1, 2, \ldots$. Therefore the separated solutions satisfying the boundary conditions are of the form

$$u(x;t) = A_1 \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 t}, \qquad n = 1, 2, \ldots \quad \bullet$$

The next example occurs repeatedly in the discussion of the vibrating string in Chapter 2, Sec. 2.4.

EXAMPLE 0.2.7. *Find the separated solutions of the wave equation $u_{tt} - c^2 u_{xx} = 0$ that satisfy the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.*

Solution. Assuming the separated form $u(x;t) = X(x)T(t)$, it follows that $X(x)T''(t) - c^2 X''(x)T(t) = 0$. Thus $X''(x) + \lambda X(x) = 0$, $T''(t) + \lambda c^2 T(t) = 0$. The boundary conditions require $X(0) = 0$, $X(L) = 0$; thus $X(x) = A_3 \sin(n\pi x/L)$, $T(t) = A_1 \cos(n\pi ct/L) + A_2 \sin(n\pi ct/L)$ for constants A_1, A_2, A_3. The required separated solutions are

$$u(x;t) = (A_1 \cos(n\pi ct/L) + A_2 \sin(n\pi ct/L)) \sin(n\pi x/L) \qquad n = 1, 2, \ldots \quad \bullet$$

In all of the preceding examples we used one or more boundary conditions to pick out certain values of the separation constant that satisfy the boundary conditions. This can also be carried out through conditions of *boundedness* as indicated in the following examples. Physically these represent a *stationary solution*, corresponding to a system that has been in existence over a very long period of time.

EXAMPLE 0.2.8. *Find the complex separated solutions $u(x;t)$ of the wave equation $u_{tt} - c^2 u_{xx} = 0$, which are bounded in the form $|u(x;t)| \le M$ for some constant M and all t, $-\infty < t < \infty$.*

Solution. Taking $u(x;t) = e^{ax+bt}$ and substituting in the wave equation, we have $b^2 - c^2a^2 = 0$; thus $b = \pm ca$. The separated solutions are of the form $u(x;t) = e^{ax}e^{cat}$, $e^{ax}e^{-cat}$. This solution is bounded for all t if and only if a is pure imaginary, $a = ik$ for k real. Thus the solutions are $u(x;t) = e^{ik(x+ct)}$, $e^{ik(x-ct)}$. The real (quasi-separated) solutions are $\cos k(x+ct)$, $\cos k(x-ct)$, $\sin k(x+ct)$, $\sin k(x-ct)$. •

The final example, concerning stationary solutions of the heat equation, will be developed in more detail in Chapter 2, Sec. 2.1, in connection with heat flow in the earth.

EXAMPLE 0.2.9. *Find the complex separated solutions $u(x;t)$ of the heat equation $u_t - u_{xx} = 0$, which are bounded in the form $|u(x;t)| \le M$ for some constant M and all t, $-\infty < t < \infty$.*

Solution. Taking $u(x;t) = e^{ax+bt}$ and substituting in the heat equation, we have $b - a^2 = 0$. In order that this solution be bounded for all $t, -\infty < t < \infty$, it is necessary that the constant b be purely imaginary; otherwise the solution would tend to $+\infty$ for large $|t|$ if b had a nonzero real part. Hence we set $b = i\omega$, where ω is real. Assuming $\omega > 0$, the equation $a^2 = i\omega$ has two solutions,

$$a = \sqrt{\frac{\omega}{2}}(1+i), \qquad a = -\sqrt{\frac{\omega}{2}}(1+i)$$

leading to the separated solution

$$u(x;t) = e^{i\omega t}\left(A_1 e^{\sqrt{\omega/2}(1+i)x} + A_2 e^{-\sqrt{\omega/2}(1+i)x}\right)$$

If $\omega < 0$, then the equation $a^2 = i\omega$ has two solutions,

$$a = \sqrt{\frac{|\omega|}{2}}(1-i), \qquad a = -\sqrt{\frac{|\omega|}{2}}(1-i)$$

leading to the separated solution

$$u(x;t) = e^{i\omega t}\left(A_1 e^{\sqrt{|\omega|/2}(1-i)x} + A_2 e^{-\sqrt{|\omega|/2}(1-i)x}\right) \quad \bullet$$

The alert reader will note that these separated solutions are closely related to those found in Example 0.2.3, where we stipulated in advance that ω be real and positive. Now we have shown that the reality of ω can be deduced from the qualitative condition of boundedness of the solution for all time.

EXERCISES 0.2.4

1. Find the separated solutions $u(x,y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L$, $y > 0$ that satisfy the boundary conditions $u_x(0,y) = 0$, $u_x(L,y) = 0$, $u(x,0) = 0$.

2. Find the separated solutions $u(x,y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L$, $y > 0$, that satisfy the boundary conditions $u(0,y) = 0$, $u(L,y) = 0$ and the boundedness condition $|u(x,y)| \le M$ for $y > 0$, where M is a constant independent of (x,y).
3. Find the separated solutions $u(x;t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.
4. Find the separated solutions $u(x;t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u_x(0;t) = 0$, $u_x(L;t) = 0$.
5. Find the separated solutions $u(x;t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$ that satisfy the boundary conditions $u(0;t) = 0$, $u_x(L;t) = 0$.

0.3. Orthogonal Functions

Separated solutions of linear partial differential equations with suitable boundary conditions lead to systems of *orthogonal functions*, which are introduced in this section. The most important system of orthogonal functions gives rise to the trigonometric Fourier series, which will be discussed in Chapter 1, including the more general *Sturm-Liouville eigenvalue problem.* In order to formulate the property of orthogonality, we first introduce the general notion of *inner product.*

0.3.1. Inner product space of functions. The notions of dot product, distance, orthogonality, and projection, which are familiar for vectors in three dimensions, can also be formulated for real-valued functions on an interval $a \le x \le b$. The basic notion is the *inner product* of two functions $\varphi(x)$, $\psi(x)$ on the interval $a \le x \le b$. This is defined by the integral

$$\boxed{\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\psi(x)\,dx} \tag{0.3.1}$$

For example, on the interval $0 \le x \le 1$, we have $\langle x, e^{x^2} \rangle = \int_0^1 x e^{x^2}\,dx = \frac{1}{2}(e-1) = 0.86$, to two decimal places.

The inner product defined by (0.3.1) has many properties in common with the ordinary dot product of two vectors in three-dimensional space, defined by $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$. The analogy between the inner product and the three-dimensional dot product is intuitive if we think of the integral as a "continuous sum" of the pointwise products $\varphi(x)\psi(x)$, a generalization of the three-dimensional dot product formula.

The inner product is *linear* and *homogeneous* in both arguments. This means that, for any functions φ_1, φ_2, ψ_1, ψ_2 and any real number a,

$$\begin{aligned}
\langle \varphi_1, \psi_1 + \psi_2 \rangle &= \langle \varphi_1, \psi_1 \rangle + \langle \varphi_1, \psi_2 \rangle \\
\langle \varphi_1 + \varphi_2, \psi_1 \rangle &= \langle \varphi_1, \psi_1 \rangle + \langle \varphi_2, \psi_1 \rangle \\
\langle a\varphi_1, \psi_1 \rangle &= a\langle \varphi_1, \psi_1 \rangle \\
\langle \varphi_1, a\psi_1 \rangle &= a\langle \varphi_1, \psi_1 \rangle
\end{aligned}$$

The proofs of these properties are left as exercises.

Definition Two functions φ, ψ are *orthogonal* on the interval $a \le x \le b$ if and only if $\langle \varphi, \psi \rangle = 0$.

This definition requires some comment. It is formulated as a generalization of the notion of perpendicularity for vectors in three-dimensional space, which is expressed as the equation $\mathbf{v} \cdot \mathbf{w} = 0$. In working with functions, it is difficult to visualize the notion of orthogonality, as we are accustomed to for vectors in two- and three-dimensional space. In particular, there is no suggestion that the graphs of the two orthogonal functions intersect at 90 degrees.

A few examples may help to illustrate these concepts.

EXAMPLE 0.3.1. *Show that the functions $\phi(x) = \sin x$, $\psi(x) = \cos x$ are orthogonal on the interval $0 \le x \le \pi$ but are not orthogonal on the interval $0 \le x \le \pi/2$.*

Solution. The inner product on the interval $0 \le x \le \pi$ is computed as the integral

$$\int_0^\pi \sin x \, \cos x \, dx = \frac{1}{2}(\sin x)^2 \Big|_0^\pi = 0$$

If we do the same computation on the interval $0 \le x \le \pi/2$, we obtain

$$\int_0^{\pi/2} \sin x \, \cos x \, dx = \frac{1}{2}(\sin x)^2 \Big|_0^{\pi/2} = \frac{1}{2}$$

Therefore we have orthogonality in the first case but not in the second case. •

For more than two functions, we say that $(\varphi_1, \ldots, \varphi_N)$ are orthogonal if $\langle \varphi_i, \varphi_j \rangle = 0$ for $i \ne j$. This is illustrated by the next example.

EXAMPLE 0.3.2. *Show that the set of functions* $\sin x, \sin 2x, \ldots, \sin Nx$ *is orthogonal on the interval* $0 \le x \le \pi$ *for any* $N \ge 2$.

Solution. The inner product on the interval $0 \le x \le \pi$ is computed as the integral

$$\int_0^\pi \sin mx \, \sin nx \, dx$$

We use the trigonometric identity

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$$

If $m \neq n$, the integral of each cosine function is a sine function, which vanishes at the endpoints $x = 0$, $x = \pi$. Therefore each of the integrals is zero, and we have proved orthogonality. •

The *norm* of a function is the nonnegative number $||\varphi||$ that satisfies

$$||\varphi||^2 = \langle \varphi, \varphi \rangle$$

For example, on the interval $0 \le x \le \pi$,

$$||\sin x||^2 = \int_0^\pi \sin^2 x \, dx = \int_0^\pi \frac{1}{2}(1 - \cos 2x) dx = \frac{1}{2}\pi$$

The *distance* between φ and ψ is defined by $d(\varphi, \psi) = ||\varphi - \psi||$. For example, the distance between $\sin x$ and $\cos x$ on the interval $0 \le x \le \pi$ is obtained from

$$[d(\sin x, \cos x)]^2 = \int_0^\pi (\sin x - \cos x)^2 \, dx = \int_0^\pi (\sin^2 x + \cos^2 x) \, dx = \pi$$

so that the distance is given by $d = \sqrt{\pi} \sim 1.77$ to two decimals. Since these two functions are orthogonal, one may think of a "right triangle" in the space of functions, for which we have computed the hypotenuse.

In order to formulate the notion of *angle* for functions on an interval, we recall that for vectors in three-dimensional space we have the dot product formula $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta$, where θ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ and $||\mathbf{v}||$, $||\mathbf{w}||$ are the lengths of the respective vectors. Hence the cosine of the angle between the two vectors may be computed as the ratio of the dot product to the product of the lengths. In order to extend this to functions on an interval, we need to know that the corresponding ratio is not greater than 1 in absolute value. This is known as the *Schwarz inequality*.

PROPOSITION 0.3.1. *Suppose that $\varphi(x)$, $\psi(x)$ are nonzero functions defined on an interval $a \le x \le b$. Then*

$$\boxed{\langle \varphi, \psi \rangle^2 \le ||\varphi||^2 ||\psi||^2} \tag{0.3.2}$$

Proof. By the linearity and homogeneity of the inner product, we have, for any real number t,

$$\begin{aligned} D(t) := \|\varphi - t\psi\|^2 &= \|\varphi\|^2 - 2t\langle\varphi,\psi\rangle + t^2\|\psi\|^2 \\ &= \|\psi\|^2\left(t^2 - 2t\frac{\langle\varphi,\psi\rangle}{\|\psi\|^2} + \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^4}\right) \\ &\quad + \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \\ &= \|\psi\|^2\left(t - \frac{\langle\varphi,\psi\rangle}{\|\psi\|^2}\right)^2 + \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \end{aligned}$$

From these transformations we see that this quadratic function of t is nonnegative and has a global minimum at $t = t_0$, where $t_0 = \langle\varphi,\psi\rangle/\|\psi\|^2$; at this point the value of the function is nonnegative and given explicitly by

$$D(t_0) = \left(\|\varphi\|^2 - \frac{\langle\varphi,\psi\rangle^2}{\|\psi\|^2}\right) \geq 0$$

which completes the proof of the Schwarz inequality. •

In case the equality sign holds in equation (0.3.2), we expect that the functions $\varphi(x)$, $\psi(x)$ will be proportional to one another, analogous to the case of three-dimensional vectors that are colinear. This is rigorously true if both functions $\varphi(x)$, $\psi(x)$ are *continuous*: from the above computations, the integral of the nonnegative continuous function $|\varphi(x) - t_0\psi(x)|^2$ is equal to zero. But this means that the function must be identically zero, so that we conclude $\varphi(x) - t_0\psi(x) = 0$ for all x, $a \leq x \leq b$; thus we have established the desired proportionality, with the proportionality constant t_0. If one of the functions fails to be continuous, we cannot conclude that the integrand is zero everywhere, but only *almost everywhere*[2] (for example, a finite set).

0.3.2. Projection of a function onto an orthogonal set. We now discuss minimizing properties of orthogonal functions. This will motivate the definition of *Fourier coefficients* in a general setting. Let $(\varphi_1, \ldots, \varphi_N)$ be a set of orthogonal functions with $\|\varphi_i\| \neq 0$ for $1 \leq i \leq N$. If f is an arbitrary function, we compute the minimum of

$$D(c_1, \ldots, c_N) = \|f - (c_1\varphi_1 + \cdots + c_N\varphi_N)\|^2$$

where $(c_1, \ldots, c_N)$ range over all real values. In other words, we are trying to find the best "mean square approximation" of the given function $f(x), a \leq x \leq b$, by means of linear combinations of the members of the orthogonal set.

[2]This means that the set of exceptional values can be included in a union of intervals whose total length is arbitrarily small.

PROPOSITION 0.3.2. *The minimization problem has the following properties:*

- *The minimum is attained uniquely when*

$$c_i = \hat{c}_i := \frac{\langle f, \varphi_i \rangle}{||\varphi_i||^2}, \qquad 1 \le i \le N$$

- *The minimum distance is given by*

$$d^2_{\min} = ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2}$$

- *The Fourier coefficients* $\hat{c}_1, \ldots, \hat{c}_N$ *satisfy Bessel's inequality*

$$\hat{c}_1^2 ||\varphi_1||^2 + \cdots + \hat{c}_N^2 ||\varphi_N||^2 \le ||f||^2$$

The function $\hat{c}_1\varphi_1 + \cdots + \hat{c}_N\varphi_N$ *is called the* **projection** *of* f *onto the orthogonal set* $(\varphi_1, \ldots, \varphi_N)$*;* $\hat{c}_i$ *is called the* ith **Fourier coefficient** *of* f.

Proof. The proof of these facts can be done by rewriting the formula for D. We use the linearity and homogeneity of the inner product to write

$$\begin{aligned} D(c_1, \ldots, c_N) &= ||f||^2 - 2\sum_{i=1}^{N} c_i \langle f, \varphi_i \rangle + \sum_{i=1}^{N} c_i^2 ||\varphi_i||^2 \\ &= \sum_{i=1}^{N} ||\varphi_i||^2 \left(c_i^2 - 2c_i \frac{\langle f, \varphi_i \rangle}{||\varphi_i||^2} + \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^4} \right) \\ &\quad + ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} \\ &= \sum_{i=1}^{N} ||\varphi_i||^2 \left(c_i - \frac{\langle f, \varphi_i \rangle}{||\varphi_i||^2} \right)^2 + ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} \\ &\ge ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} \end{aligned}$$

Clearly, the minimum is achieved when $c_i = \hat{c}_i := \langle f, \varphi_i \rangle / ||\varphi_i||^2$, as required. The value of the minimum is

$$D(\hat{c}_1, \ldots, \hat{c}_N) = ||f||^2 - \sum_{i=1}^{N} \frac{\langle f, \varphi_i \rangle^2}{||\varphi_i||^2} = ||f||^2 - \sum_{i=1}^{N} \hat{c}_i^2 ||\varphi_i||^2$$

as required. Since this is nonnegative, Bessel's inequality is merely the statement that $D(\hat{c}_1, \ldots, \hat{c}_N) \ge 0$. •

As a first example, we consider the orthogonal set consisting of the three functions $\{\sin x, \sin 2x, \sin 3x\}$ on the interval $0 \le x \le \pi$.

EXAMPLE 0.3.3. *Find the projection of the function $f(x) = 1$ onto the orthogonal set $\{\sin x, \sin 2x, \sin 3x\}$ on the interval $0 \le x \le \pi$ and compute $d_{\min}$.*

Solution. We first note that the norms are given by

$$\|\varphi_m\|^2 = \int_0^\pi \sin^2 mx\, dx = \frac{1}{2}\int_0^\pi (1 - \cos 2mx)\, dx = \frac{\pi}{2}$$

From Proposition 0.3.2, the Fourier coefficients are

$$\begin{aligned}
\hat{c}_1 &= \frac{\int_0^\pi \sin x\, dx}{\int_0^\pi \sin^2 x\, dx} = \frac{\cos x|_{x=\pi}^{x=0}}{\pi/2} = \frac{4}{\pi} \\
\hat{c}_2 &= \frac{\int_0^\pi \sin 2x\, dx}{\int_0^\pi \sin^2 2x\, dx} = \frac{1}{2}\frac{\cos 2x|_{x=\pi}^{x=0}}{\pi/2} = 0 \\
\hat{c}_3 &= \frac{\int_0^\pi \sin 3x\, dx}{\int_0^\pi \sin^2 3x\, dx} = \frac{1}{3}\frac{\cos 3x|_{x=\pi}^{x=0}}{\pi/2} = \frac{4}{3\pi}
\end{aligned}$$

The projection is the function

$$s(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$$

The minimum distance is obtained from

$$\begin{aligned}
d_{\min}^2 &= \|f\|^2 - \sum_{i=1}^{3} \hat{c}_i^2 \|\varphi_i\|^2 \\
&= \pi - \left(\frac{4}{\pi}\right)^2 \left(\frac{\pi}{2}\right) - \left(\frac{4}{3\pi}\right)^2 \left(\frac{\pi}{2}\right) \\
&= 3.14 - (1.27)^2(1.57) - (0.42)^2(1.57) \\
&= 0.33
\end{aligned}$$

to two decimal places. •

In the next example we consider an orthogonal set of three polynomial functions on the interval $-1 \le x \le 1$. This is closely related to the *Legendre polynomial expansion,* which will be considered in Chapter 4.

EXAMPLE 0.3.4. *Find the projection of the function $f(x) = \cos(\pi x/2)$ on the orthogonal set $(1, x, x^2 - \frac{1}{3})$ on the interval $-1 \le x \le 1$, and compute $d_{\min}$.*

Solution. The solution may be written in the form

$$s(x) = \hat{c}_0 + \hat{c}_1 x + \hat{c}_2\left(x^2 - \frac{1}{3}\right)$$

where the Fourier coefficients $\hat{c}_0$, $\hat{c}_1$, $\hat{c}_2$ are computed from the equations

$$\begin{aligned}
\hat{c}_0 \int_{-1}^{1} dx &= \int_{-1}^{1} \cos\frac{\pi x}{2} dx \\
\hat{c}_1 \int_{-1}^{1} x^2\, dx &= \int_{-1}^{1} x\cos\frac{\pi x}{2}\, dx \\
\hat{c}_2 \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\cos\frac{\pi x}{2}\, dx
\end{aligned}$$

The first of these is straightforward since

$$\int_{-1}^{1} \cos\frac{\pi x}{2}\, dx = \frac{2}{\pi}\sin\frac{\pi x}{2}\bigg|_{x=-1}^{x=1} = \frac{4}{\pi}; \quad \text{thus } \hat{c}_0 = \frac{2}{\pi}$$

The next is also easy since the function $x\cos\pi x/2$ is odd; thus $\hat{c}_1 = 0$. To perform the final integral, we write

$$\begin{aligned}
\int_{-1}^{1} x^2\cos\frac{\pi x}{2}\, dx &= \frac{2}{\pi}\int_{-1}^{1} x^2 d\left(\sin\frac{\pi x}{2}\right) \\
&= \frac{2x^2}{\pi}\sin\frac{\pi x}{2}\bigg|_{-1}^{1} - \frac{4}{\pi}\int_{-1}^{1} x\sin\frac{\pi x}{2}\, dx \\
&= \frac{4}{\pi} + \frac{8}{\pi^2}\int_{-1}^{1} x\, d\left(\cos\frac{\pi x}{2}\right) \\
&= \frac{4}{\pi} + \frac{8}{\pi^2}\left(x\cos\frac{\pi x}{2}\bigg|_{-1}^{1} - \int_{-1}^{1}\cos\frac{\pi x}{2}\, dx\right) \\
&= \frac{4}{\pi} - \frac{32}{\pi^3}
\end{aligned}$$

Combining this with the previous integral, we have

$$\begin{aligned}
\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)\cos\frac{\pi x}{2}\, dx &= \frac{4}{\pi} - \frac{32}{\pi^3} - \frac{4}{3\pi} \\
&= \frac{8\pi^2 - 96}{3\pi^3}
\end{aligned}$$

But $\int_{-1}^{1}(x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1}(x^4 - \frac{2}{3}x^2 + \frac{1}{9})dx = \frac{2}{5} - \left(\frac{2}{3}\right)^2 + \frac{2}{9} = \frac{8}{45}$. Therefore $\hat{c}_2 = \frac{45}{8}(8\pi^2 - 96)/3\pi^3 = 15(\pi^2 - 12)/\pi^3$. Thus the required orthogonal projection is

$$s(x) = \frac{2}{\pi} + \frac{15(\pi^2 - 12)}{\pi^3}\left(x^2 - \frac{1}{3}\right)$$

To compute $d_{\min}$, we have, to four decimals,

$$\begin{aligned}
\hat{c}_0^2 &= \left(\frac{2}{\pi}\right)^2 = 0.4053 \\
\|1\|^2 &= \int_{-1}^{1} dx = 2 \\
\hat{c}_2^2 &= \left[\frac{15(\pi^2 - 12)}{\pi^3}\right]^2 = 1.0622 \\
\left\|x^2 - \frac{1}{3}\right\|^2 &= \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = 0.1778 \\
\left\|\cos\frac{\pi x}{2}\right\|^2 &= \int_{-1}^{1} \cos^2\frac{\pi x}{2} dx = \frac{1}{2}\int_{-1}^{1}(1 + \cos\pi x)dx = 1
\end{aligned}$$

Thus, to four decimals,

$$d_{\min}^2 = 1 - (0.4053)(2) - (1.0622)(0.1778) = 0.0004$$

and, to two decimals, $d_{\min} = 0.02$. •

It is instructive to compare the orthogonal projection with the corresponding values of $\cos(\pi x/2)$ at some representative points. For example, to four decimal places of accuracy, we have

$$\begin{aligned}
s(0) &= (0.6366) + \tfrac{1}{3}(1.0306) = 0.9801 \\
s(1) &= (0.6366) - \tfrac{2}{3}(1.0306) = 0.0505 \\
s\left(\frac{1}{2}\right) &= (0.6366) + \tfrac{1}{12}(1.0306) = 0.7225 \\
s\left(\frac{1}{3}\right) &= (0.6366) + \tfrac{2}{9}(1.0306) = 0.8656 \\
s\left(\frac{2}{3}\right) &= (0.6366) - \tfrac{1}{9}(1.0306) = 0.5221
\end{aligned}$$

On the other hand, the corresponding values of $\cos(\pi x/2)$ are 1, 0, 0.7071, 0.8667, 0.5000.

0.3.3. Orthonormal sets of functions. The formulas for the Fourier coefficients and the minimum distance become especially simple when the functions $(\varphi_1, \ldots, \varphi_N)$ are *orthonormal*. This means that $\langle\varphi_i, \varphi_j\rangle = 0$ for $i \neq j$ and $\langle\varphi_i, \varphi_i\rangle = 1$, $1 \le i \le N$. Thus we have for orthonormal functions

$$\hat{c}_i = \langle f, \varphi_i\rangle \qquad 1 \le i \le N \tag{0.3.3}$$

$$d_{\min}^2 = D(\hat{c}_1, \ldots, \hat{c}_N) = \|f\|^2 - (\hat{c}_1^2 + \cdots + \hat{c}_N^2) \tag{0.3.4}$$

If $(\varphi_1, \ldots, \varphi_N)$ is an orthogonal set of functions, we obtain an orthonormal set by replacing φ_i by $\varphi_i/\|\varphi_i\|$, $1 \le i \le N$.

EXAMPLE 0.3.5. *Let $\varphi_1 = 1$, $\varphi_2 = \sin x$, $\varphi_3 = \cos x$ for $-\pi < x < \pi$. Verify that this is an orthogonal set and find the corresponding orthonormal set.*

Solution. Direct computation reveals that each of the inner products $\langle \varphi_i, \varphi_j \rangle$ is zero for $i \neq j$. To find the orthonormal set, we compute

$$\begin{aligned}
\|\varphi_1\|^2 &= \int_{-\pi}^{\pi} dx = 2\pi \\
\|\varphi_2\|^2 &= \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi \\
\|\varphi_3\|^2 &= \int_{-\pi}^{\pi} \cos^2 x \, dx = \pi
\end{aligned}$$

The orthonormal set is $(1/\sqrt{2\pi}, (\sin x)/\sqrt{\pi}, (\cos x)/\sqrt{\pi})$. •

In many problems we are given an *infinite* orthonormal set

$$(\varphi_n)_{n \geq 1} = (\varphi_1, \varphi_2, \dots)$$

To study such a set, we apply the above procedure to the finite orthonormal set $(\varphi_1, \dots, \varphi_n)$. The Fourier coefficients are

$$\hat{c}_i = \langle f, \varphi_i \rangle, \qquad 1 \leq i \leq N$$

which don't depend on N. Furthermore we have Bessel's inequality: for each N

$$\sum_{i=1}^{N} \hat{c}_i^2 \leq \|f\|^2 \qquad N = 1, 2, \dots$$

This is valid for every $N = 1, 2, \dots$; hence the infinite series $\sum_{i=1}^{\infty} \hat{c}_i^2$ converges and we have

$$\boxed{\sum_{i=1}^{\infty} \hat{c}_i^2 \leq \|f\|^2} \tag{0.3.5}$$

This is formulated as follows.

PROPOSITION 0.3.3. *Suppose that $(\varphi_n)_{n \geq 1} = (\varphi_1, \varphi_2, \dots)$ is an infinite orthonormal set of functions and that f is a function for which $\int_a^b |f(x)|^2 \, dx < \infty$. Then the series of sums of squares of Fourier coefficients converges and satisfies the Bessel inequality (0.3.5).*

0.3.4. Parseval's equality, completeness, and mean square convergence. If we have an infinite orthonormal set, it may happen that Bessel's inequality (0.3.5) is an equality, namely

$$\boxed{\sum_{i=1}^{\infty} \hat{c}_i^2 = \|f\|^2} \tag{0.3.6}$$

This is called *Parseval's equality.* We will show that Parseval's equality is equivalent to the *mean square convergence* of the series $\sum_{i=1}^{\infty} \hat{c}_i\varphi_i$, which is defined by the limiting statement

$$\lim_{N\to\infty} \left\| f - \sum_{i=1}^{N} \hat{c}_i\varphi_i \right\|^2 = 0 \tag{0.3.7}$$

The formal statement of equivalence follows.

PROPOSITION 0.3.4. *Let $(\varphi_n)_{n\geq 1}$ be an orthonormal set and f a function with $\int_a^b f(x)^2\,dx < \infty$. Parseval's equality is true if and only if we have mean square convergence of the series $\sum_{i=1}^{\infty} \hat{c}_i\varphi_i$.*

Proof. Let $\hat{c}_i = \langle f, \varphi_i\rangle$ be the ith Fourier coefficient of f. Then by expanding the inner product and using orthonormality on the left side, we have

$$\begin{aligned}\left\| f - \sum_{i=1}^{N} \hat{c}_i\varphi_i \right\|^2 &= \|f\|^2 - 2\sum_{i=1}^{N} \hat{c}_i\langle f, \varphi_i\rangle + \sum_{i=1}^{N} \hat{c}_i^2 \\ &= \|f\|^2 - \sum_{i=1}^{N} \hat{c}_i^2\end{aligned}$$

Letting $N \to \infty$, we see that the right side tends to zero if and only if Parseval's equality is valid. The left side tends to zero (by definition) if and only if we have mean square convergence. Therefore the proposition is proved. •

One may note that Parseval's equality is not true for an arbitrary function. For example, the set of functions $\pi^{-1/2}(\sin nx, \cos nx)_{n\geq 1}$ is an orthonormal set for $-\pi \leq x \leq \pi$. The function $f(x) = 1$ has all Fourier coefficients zero; indeed, $\int_{-\pi}^{\pi} \sin nx\,dx = 0 = \int_{-\pi}^{\pi} \cos nx\,dx, n \geq 1$. Yet $\|f\|^2 = \int_{-\pi}^{\pi} 1\,dx = 2\pi$. In this case Bessel's inequality is the statement that $0 = \sum_{i=1}^{\infty} \hat{c}_i^2 < \|f\|^2 = 2\pi$.

If Parseval's equality holds for all functions f with $\int_a^b f(x)^2\,dx < \infty$, then we say that the orthonormal set is *complete* on the interval $a \leq x \leq b$. For example, in Chapter 1 it will be shown that the trigonometric system consisting of $\{1/\sqrt{2\pi}, (\sin nx)/\sqrt{\pi}, (\cos nx)/\sqrt{\pi}\}_{n\geq 1}$ is complete on the interval $-\pi \leq x \leq \pi$.

0.3.5. Weighted inner product. In many problems we are required to deal with a weighted inner product with respect to a positive *weight function* $\rho(x), a \leq x \leq b$. This is defined by the integral

$$\langle \varphi, \psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)\,dx$$

This has the same properties of linearity and homogeneity as the ordinary inner product. We say that two functions φ, ψ are orthogonal with respect to the weight function $\rho(x), a \leq x \leq b$, if $\langle \varphi, \psi\rangle_\rho = 0$.

Weighted orthogonality arises when we make a change of variable by means of an increasing differentiable function $x = h(y)$. The ordinary inner product is transformed as follows:

$$\int_a^b \varphi(x)\psi(x)\,dx = \int_c^d \varphi(h(y))\psi(h(y))h'(y)\,dy$$

Therefore we see that if $\varphi(x), \psi(x)$ are orthogonal on the interval $a \le x \le b$, then the functions $\varphi(h(y)), \psi(h(y))$ are orthogonal with respect to the weight function $h'(y)$ on the interval $c \le y \le d$, where $a = h(c)$, $b = h(d)$.

EXAMPLE 0.3.6. *Given the orthogonal functions $P_1(x) = x$, $P_2(x) = 3x^2 - 1$ on the interval $-1 \le x \le 1$, find the weighted orthogonality relation on the interval $0 \le y \le \pi$ under the transformation $x = -\cos y$.*

Solution. We have the transformed functions $P_1(h(y)) = -\cos y$, $P_2(y) = 3\cos^2 y - 1$, with the weight function $\rho(y) = h'(y) = \sin y$. •

0.3.6. Gram-Schmidt orthogonalization. When we deal with separated solutions of boundary-value problems in PDEs, the property of orthogonality is often immediately verified. This will be discussed in more detail in the following chapters. Nevertheless, it is interesting to know how we may manufacture orthogonal sets of functions from arbitrary sets of functions, by the so-called *Gram-Schmidt procedure.*[3] Suppose that $(\varphi_1, \ldots, \varphi_n)$ is a given set of functions, not necessarily orthogonal. Instead we suppose linear independence, i.e., that there are no relations of the form $c_1\varphi_1 + \cdots + c_n\varphi_n = 0$ among the $(\varphi_1, \ldots, \varphi_n)$, other than the trivial relation where $c_1 = 0, \ldots, c_n = 0$. In particular, $\|\varphi_i\| \ne 0$ for $1 \le i \le n$. Then we define

$$\begin{aligned}
\psi_1 &= \varphi_1 \\
\psi_2 &= \varphi_2 - \frac{\langle \varphi_2, \psi_1\rangle}{\langle \psi_1, \psi_1\rangle}\psi_1 \\
\psi_3 &= \varphi_3 - \frac{\langle \varphi_3, \psi_2\rangle}{\langle \psi_2, \psi_2\rangle}\psi_2 - \frac{\langle \varphi_3, \psi_1\rangle}{\langle \psi_1, \psi_1\rangle}\psi_1 \\
&\vdots \\
\psi_n &= \varphi_n - \sum_{i=1}^{n-1} \frac{\langle \varphi_n, \psi_i\rangle}{\langle \psi_i, \psi_i\rangle}\psi_i
\end{aligned}$$

The functions $(\psi_1, \ldots, \psi_n)$ are orthogonal. These formulas may seem less mysterious if we note that in the ith formula we are subtracting from φ_i its projection onto the orthogonal set $\psi_1, \ldots, \psi_{i-1}$.

[3]This material is not used in the subsequent chapters.

The sets $(\varphi_1, \dots, \varphi_n)$ and $(\psi_1, \dots, \psi_n)$ have the same *linear span*; i.e., any function of the form $f = c_1\varphi_1 + \cdots + c_n\varphi_n$ can be written in the form $d_1\psi_1 + \cdots + d_n\psi_n$ for appropriate $(d_1, \dots, d_n)$, and the converse is also true.

EXAMPLE 0.3.7. *Let $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ for $0 \le x \le 1$. Apply the Gram-Schmidt procedure to find the orthogonal functions ψ_1, ψ_2, ψ_3.*

Solution. We have $\psi_1 = \varphi_1 = 1$, $\langle \varphi_2, \psi_1 \rangle = \int_0^1 x\,dx = \frac{1}{2}$, $\langle \psi_1, \psi_1 \rangle = 1$. Thus $\psi_2 = x - \frac{1}{2}$. The remaining inner products are

$$\begin{aligned}
\langle \varphi_3, \psi_2 \rangle &= \int_0^1 x^2 \left(x - \frac{1}{2}\right) dx = \frac{1}{4} - \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{12} \\
\langle \psi_2, \psi_2 \rangle &= \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12} \\
\langle \varphi_3, \psi_1 \rangle &= \int_0^1 x^2 dx = \frac{1}{3}
\end{aligned}$$

Thus $\psi_3 = x^2 - (x - \frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}$. The orthogonal functions are 1, $x - \frac{1}{2}$, $x^2 - x + \frac{1}{6}$, $0 \le x \le 1$. •

0.3.7. Complex inner product. In dealing with complex-valued functions, it is necessary to modify the definition of inner product and orthogonality. The guiding principle is that the norm of a function should be a nonnegative number. With this in mind, we define the complex inner product and norm on the interval $a < x < b$ as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\bar{\psi}(x)\,dx \tag{0.3.8}$$

$$\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle} \ge 0 \tag{0.3.9}$$

where the bar denotes the complex conjugate of a function, defined by $\bar{\psi}(x) = f(x) - ig(x)$ when $\psi(x) = f(x) + ig(x)$. *Orthogonality* of complex-valued functions is defined by the requirement that the complex inner product be zero: $\langle \varphi, \psi \rangle = 0$.

The properties of linearity and homogeneity of the complex inner product are almost identical to those of the real inner product, with the exception that we have $\langle \varphi, a\psi \rangle = \bar{a}\langle \varphi, \psi \rangle$ for any complex constant. We record here the appropriate statement of Schwarz's inequality.

PROPOSITION 0.3.5. *Suppose that $\varphi(x)$ and $\psi(x)$, $a < x < b$, are complex-valued functions. Then $|\langle \varphi, \psi \rangle| \le \|\varphi\|\,\|\psi\|$. If equality holds and both functions are continuous, then the functions are proportional: $C_1\varphi(x) = C_2\psi(x)$ for some complex constants C_1, C_2.*

The proof is suggested as an optional exercise.

EXERCISES 0.3

1. Let $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ on the interval $0 \le x \le 1$. Find the following inner products:
   (a) $\langle \varphi_1, \varphi_2 \rangle$ (b) $\langle \varphi_1, \varphi_3 \rangle$
   (c) $\|\varphi_1 - \varphi_2\|^2$ (d) $\|\varphi_1 + 3\varphi_2\|^2$
2. Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?
$$\varphi_1 = \sin 2\pi x \qquad \varphi_2 = x \qquad \varphi_3 = \cos 2\pi x \qquad \varphi_4 = 1$$
3. Let $\bar{f} = \hat{c}_1\varphi_1 + \cdots + \hat{c}_n\varphi_n$ be the projection of f on the orthogonal set $(\varphi_1, \ldots, \varphi_n)$. Show that $f - \bar{f}$ is orthogonal to each of the functions $(\varphi_1, \ldots, \varphi_n)$.
4. Find the projection of the function $\sin \pi x$ on the orthogonal set $(1, x - \frac{1}{2})$ on the interval $0 \le x \le 1$ and compute the minimum distance $d_{\min}$.
5. Find the projection of the function $f(x) = \cos^2 x$ on the orthogonal set $(1, \cos x, \cos 2x)$ on the interval $-\pi \le x \le \pi$.
6. Let $\varphi_1(x) = 1$, $\varphi_2(x) = x/|x|$, $\varphi_3(x) = x^2 - \frac{1}{3}$ for $-1 \le x \le 1$.
   (a) Show that $(\varphi_1, \varphi_2, \varphi_3)$ form an orthogonal set.
   (b) Find the projection of $f(x) = x$ on this orthogonal set and compute the minimum distance $d_{\min}$.
7. Let $(\varphi_1, \varphi_2, \varphi_3)$ be an orthonormal set of functions on the interval $-1 \le x \le 1$, and let f be any function of the form $f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)$.
   (a) Show that $\|f\|^2 = a_1^2 + a_2^2 + a_3^2$.
   (b) Show that $\langle f, \varphi_1 \rangle = a_1$, $\langle f, \varphi_2 \rangle = a_2$, $\langle f, \varphi_3 \rangle = a_3$.
8. Let $(\varphi_1, \varphi_2, \varphi_3)$ be an orthonormal set of functions on the interval $-1 \le x \le 1$, and let $f(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x)$, $g(x) = b_1\varphi_1(x) + b_2\varphi_2(x) + b_3\varphi_3(x)$.
   (a) Show that $\langle f, g \rangle = a_1b_1 + a_2b_2 + a_3b_3$.
   (b) Discuss the relation with the three-dimensional dot product formula.
9. Define the angle between two nonzero functions φ, ψ by the formula $\cos\theta = \langle \varphi, \psi \rangle / \|\varphi\|\|\psi\|$, $0 \le \theta \le \pi$.
   (a) If φ and ψ are orthogonal, show that $\theta = \pi/2$.
   (b) If φ and ψ are proportional, show that $\theta = 0$ or $\theta = \pi$.
   (c) If $\theta = 0$ or π, does it follow that φ and ψ are necessarily proportional? (*Hint:* Compute $\|\varphi - c\psi\|^2$ and write it as a perfect square.)
   (d) Compute θ if $\varphi(x) = 1$, $\psi(x) = x$ for $0 \le x \le 1$.
10. (a) Apply the Gram-Schmidt procedure to obtain orthogonal functions beginning with the functions $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ for $-1 \le x \le 1$.
   (b) Find the orthonormal set corresponding to the orthogonal set found in part (a).

11. Prove that the inner product defined by (0.3.1) satisfies $\langle\varphi_1, \psi_1 + \psi_2\rangle = \langle\varphi_1, \psi_1\rangle + \langle\varphi_1, \psi_2\rangle$.
12. Prove that the inner product defined by (0.3.1) satisfies $\langle\varphi_1 + \varphi_2, \psi_1\rangle = \langle\varphi_1, \psi_1\rangle + \langle\varphi_2, \psi_1\rangle$.
13. Prove that the inner product defined by (0.3.1) satisfies $\langle a\varphi_1, \psi_1\rangle = a\langle\varphi_1, \psi_1\rangle$.
14. Prove that the inner product defined by (0.3.1) satisfies $\langle\varphi_1, a\psi_1\rangle = a\langle\varphi_1, \psi_1\rangle$.
15. Prove the complex form of Schwarz's inequality. [*Hint:* Examine the non-negative quadratic polynomial $G(t,s) = \|t\psi - s\varphi e^{-i\theta}\|^2$, where the inner product has the polar form $\langle\varphi, \psi\rangle = Re^{i\theta}$. Check that the discriminant $= R^2 = \|\varphi\|^2\|\psi\|^2 - \langle\varphi, \psi\rangle^2 \geq 0$.]

CHAPTER 1

FOURIER SERIES

INTRODUCTION

Many of the classical partial differential equations with boundary conditions have separated solutions that involve sums of trigonometric functions. This leads to the theory of Fourier series, which is developed here in its own right. This chapter explores the basic properties of Fourier series, including a discussion of convergence and the closely related Sturm-Liouville eigenvalue problem. Basic definitions and examples are given in Sec. 1.1; the next two sections treat more theoretical material and can be omitted without loss of continuity. The basic material resumes in Sec. 1.4 with Parseval's theorem and its applications. The complex Fourier series in Sec. 1.5 are not used until the discussion of Fourier transforms in Chapter 5, but the Sturm-Liouville theory of Sec. 1.6 is used immediately in Chapter 2.

1.1. Definitions and Examples

A *trigonometric series* is a function of the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right) \tag{1.1.1}$$

where A_0, A_1, $B_1, \ldots$ are constants. This is a series of sines and cosines whose frequencies are multiples of a basic angular frequency π/L and whose amplitudes are arbitrary. In this chapter we will explore the possibility of expanding a large class of functions $f(x), -L < x < L$, as trigonometric series. We first prove directly that this set of functions is orthogonal on the interval $-L < x < L$.

1.1.1. Orthogonality relations. In the following discussion the indices m, n assume the values $0, 1, 2, \ldots$.

PROPOSITION 1.1.1. *We have the orthogonality relations*

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases} \tag{1.1.2}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 0 & n = m = 0 \end{cases} \tag{1.1.3}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 \qquad \text{all} \quad m, n \tag{1.1.4}$$

Proof. We use the trigonometric identities

$$\begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\ \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \end{aligned} \tag{1.1.5}$$

Thus to prove (1.1.2), we have, for $n \neq m$,

$$\begin{aligned} \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx &= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right]dx \\ &= \frac{L}{2\pi}\left[\frac{\sin(n-m)\pi x/L)}{n-m}\Big|_{-L}^{L} + \frac{\sin(n+m)\pi x/L)}{n+m}\Big|_{-L}^{L}\right] \\ &= 0 \end{aligned}$$

If $n = m \neq 0$, we have

$$\begin{aligned} \int_{-L}^{L} \cos^2\frac{n\pi x}{L}dx &= \frac{1}{2}\int_{-L}^{L}\left(1+\cos\frac{2n\pi x}{L}\right)dx \\ &= \frac{1}{2}\left(2L + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\Big|_{-L}^{L}\right) \\ &= L \end{aligned}$$

Finally, if $n = m = 0$, the integral is $2L$. This completes the proof of (1.1.2). The proofs of (1.1.3) and (1.1.4) are left as exercises. •

Having established the orthogonality and performed the computation of these integrals, we can now define the Fourier series of a function $f(x)$, $-L < x < L$.

1.1.2. Definition of Fourier coefficients. In order to define the Fourier series of a function, it suffices to define the Fourier coefficients A_n, B_n, which is done as follows.

Definition Let $f(x)$, $-L < x < L$, be a real-valued function. The *Fourier series* of f is the trigonometric series (1.1.1) where (A_n, B_n) are defined by

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{1.1.6}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots \tag{1.1.7}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots \tag{1.1.8}$$

These definitions were suggested in Chapter 0, where we showed that for any orthogonal set $(\varphi_1, \ldots, \varphi_N)$, the minimum of $\|f - \sum_{n=1}^{N} c_n\varphi_n\|^2$ is determined by choosing $(c_1, \ldots, c_N)$ as the Fourier coefficients $\langle f, \varphi_n\rangle / \langle \varphi_n, \varphi_n\rangle, 1 \le n \le N$.

1.1.3. Even functions and odd functions. In order to simplify the computation of Fourier series of many functions encountered in practice, we often exploit symmetry arguments. A function $f(x)$, $-L < x < L$, is *even* if $f(-x) = f(x)$, $-L < x < L$. A function $f(x)$, $-L < x < L$, is *odd* if $f(-x) = -f(x)$, $-L < x < L$. For example, $f(x) = x$, $f(x) = x^3$, and $f(x) = \sin x$ are odd functions, whereas $f(x) = x^2$, $f(x) = x^4$, and $f(x) = \cos x$ are even functions. Of course, many functions are neither even nor odd, for example, $f(x) = x + x^2$. The product of two even functions is an even function, the product of an odd function and an even function is an odd function, and the product of two odd functions is an even function. These properties result from the multiplication facts $(+1)(+1) = +1$, $(-1)(+1) = -1$, and $(-1)(-1) = +1$. If $f(x)$, $-L < x < L$, is an odd function, the integral $\int_{-L}^{L} f(x)dx = 0$. This may be seen in detail by writing

$$\begin{aligned}
\int_{-L}^{0} f(x)dx &= -\int_{L}^{0} f(-t)dt \quad (x = -t, dx = -dt) \\
&= \int_{0}^{L} f(-t)dt \qquad \left(\int_0^L = -\int_L^0\right) \\
&= -\int_{0}^{L} f(t)dt \qquad \text{(oddness)}
\end{aligned}$$

But t is a dummy variable of integration; thus

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = -\int_{0}^{L} f(x)dx + \int_{0}^{L} f(x)dx = 0$$

In a similar fashion it may be shown that if $f(x)$, $-L < x < L$, is an even function, then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.

PROPOSITION 1.1.2. *If $f(x)$, $-L < x < L$, is an even function, then $B_n = 0$, $n = 1, 2, \ldots$. If $f(x)$, $-L < x < L$, is an odd function, then $A_n = 0$, $n = 0, 1, 2, \ldots$.*

Proof. To prove these facts, we first note that $\sin(n\pi x/L)$ is an odd function and $\cos(n\pi x/L)$ is an even function since $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$. Now, if $f(x)$, $-L < x < L$, is an even function, the product $f(x)\sin(n\pi x/L)$ is an odd function and we have $B_n = 0$. If $f(x)$, $-L < x < L$, is an odd function, the product $f(x)\cos(n\pi x/L)$ is an odd function and we have $A_n = 0$. •

EXAMPLE 1.1.1. *Compute the Fourier series of $f(x) = x$, $-L < x < L$.*

Solution. $f(x)$, $-L < x < L$, is an odd function; therefore $A_n = 0$. To compute B_n, we note that $f(x)\sin(n\pi x/L)$ is an even function; thus

$$\begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx \end{aligned}$$

We integrate by parts with $u = x$, $dv = \sin(n\pi x/L)\,dx$. Thus

$$B_n = \frac{2}{L}\left(-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right)$$

The last integral is zero, and we have $B_n = -(2L/n\pi)\cos n\pi = (2L/n\pi)(-1)^{n+1}$. Therefore the Fourier series of $f(x) = x$, $-L < x < L$, is

$$\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L} \quad \bullet$$

EXAMPLE 1.1.2. *Compute the Fourier series of $f(x) = |x|$, $-L < x < L$.*

Solution. $f(x)$, $-L < x < L$, is an even function; therefore $B_n = 0$. To compute A_n, we note that the product $f(x)\cos(n\pi x/L)$ is an even function; thus, for $n \neq 0$,

$$A_n = \frac{1}{L}\int_{-L}^{L} |x|\cos\frac{n\pi x}{L}\,dx \tag{1.1.9}$$

$$= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx \tag{1.1.10}$$

We integrate by parts with $u = x$, $dv = \cos(n\pi x/L)dx$. Thus

$$A_n = \frac{2}{L}\left(x\frac{L}{n\pi}\sin\frac{n\pi x}{L}\bigg|_0^L - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx\right)$$

The first term is zero at both endpoints $x = 0$, $x = L$, while the integral can be evaluated as $\int_0^L \sin(n\pi x/L)\,dx = (L/n\pi)[1-(-1)^n]$. Thus we have $A_n = -(2L/n^2\pi^2)[1-(-1)^n]$ for $n \neq 0$. For $n = 0$, we have $A_0 = (1/L)\int_0^L x\,dx = L/2$. Therefore the Fourier series of $f(x) = |x|$, $-L < x < L$, is

$$\frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi x}{L}$$

This may also be written as

$$(L/2) - (4L/\pi^2)\sum_{m=1}^{\infty}\cos[(2m-1)\pi x/L]/(2m-1)^2$$

by writing $n = 2m-1$ and noting that $1-(-1)^n = 0$ if n is even and $1-(-1)^n = 2$ if n is odd. •

It will be shown in Sec. 1.2 that these Fourier series are convergent and that the equation

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

is valid for $-L < x < L$. We illustrate this graphically for the preceding two examples. To do this, we define the *partial sum of order N* of a trigonometric series as the function

$$f_N(x) = A_0 + \sum_{n=1}^{N}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

In Figs. 1.1.1 and 1.1.2 we give the partial sums for the Fourier series of the preceding two examples.

The method of these two examples may be extended to compute the Fourier series of any polynomial $f(x) = c_0 + c_1x + \cdots + c_kx^k$. To do this, it is sufficient to handle each term separately and integrate by parts. Thus we have the reduction formulas

$$\begin{aligned}
\int_{-L}^{L} x^k \sin\frac{n\pi x}{L}\,dx &= -\frac{Lx^k}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{Lk}{n\pi}\int_{-L}^{L} x^{k-1}\cos\frac{n\pi x}{L}\,dx \\
\int_{-L}^{L} x^k \cos\frac{n\pi x}{L}\,dx &= \frac{Lx^k}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} - \frac{Lk}{n\pi}\int_{-L}^{L} x^{k-1}\sin\frac{n\pi x}{L}\,dx
\end{aligned}$$

Proceeding inductively, we can compute the necessary integrals.

If a function $f(x)$, $-L < x < L$, can be written as a finite trigonometric sum, then its Fourier series is that trigonometric sum. For example, the Fourier

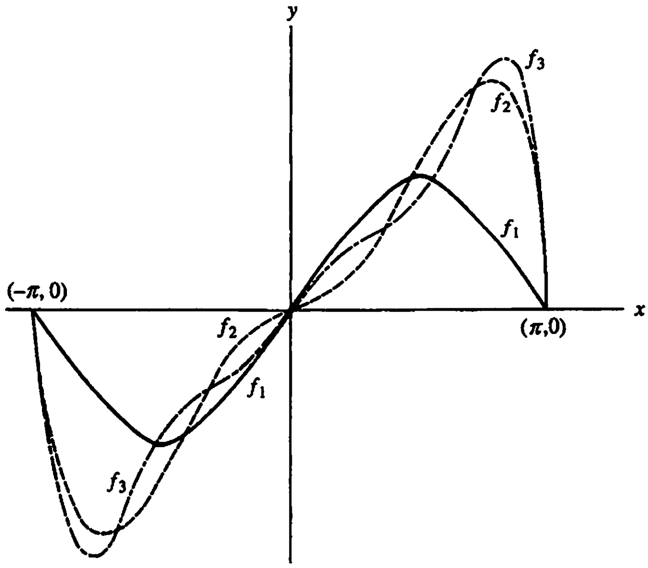


FIGURE 1.1.1 Graphs of the partial sums $f_N(x)$ for $N = 1, 2, 3$ of the Fourier series of $f(x) = x$, $-\pi < x < \pi$.

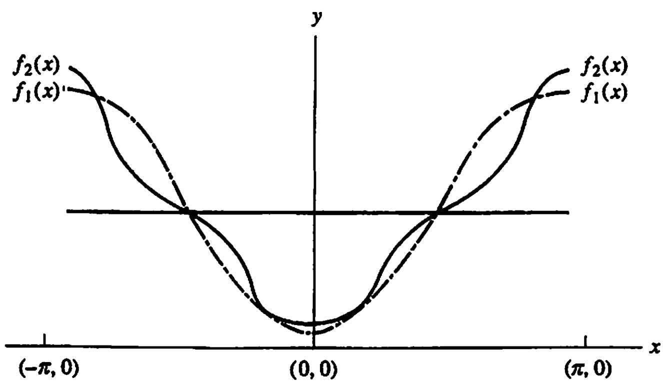


FIGURE 1.1.2 Graphs of the partial sums $f_N(x)$ for $N = 0, 1, 2$ of the Fourier series of $f(x) = |x|$, $-\pi < x < \pi$.

series of $f(x) = \sin^2 x$, $-\pi < x < \pi$, can be obtained by observing that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$; thus $B_n = 0$ for all n, while $A_0 = \frac{1}{2}$, $A_2 = -\frac{1}{2}$, and $A_n = 0$ for $n = 1, 3, 4, 5, \ldots$. It is not necessary to perform any integrations to find the Fourier series in this case.

1.1.4. Periodic functions. We now discuss Fourier series in the context of periodic functions.

Definition A function $f(x)$, $-\infty < x < \infty$, is *2L-periodic* if

$$f(x+2L) = f(x) \qquad -\infty < x < \infty$$

For example, $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ are $2L$-periodic for $n = 1, 2, \ldots$ since

$$\begin{aligned} \sin\frac{n\pi}{L}(x+2L) &= \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\frac{n\pi x}{L} \\ \cos\frac{n\pi}{L}(x+2L) &= \cos\left(\frac{n\pi x}{L} + 2n\pi\right) = \cos\frac{n\pi x}{L} \end{aligned}$$

The sum, difference, or product of any two $2L$-periodic functions is again $2L$-periodic. Therefore any convergent trigonometric series defines a $2L$-periodic function $f(x)$, $-\infty < x < \infty$. Conversely, we can speak of the Fourier series of a $2L$-periodic function $f(x)$, $-\infty < x < \infty$, by restricting x to $-L < x < L$ and computing the Fourier series as we have just done.

EXAMPLE 1.1.3. *Compute the Fourier series of the $2L$-periodic function $f(x) = -1$ if $(2n-1)L < x < 2nL$, $f(x) = 1$ if $2nL < x < (2n+1)L$, $n = 0, \pm1, \pm2, \ldots$.*

Solution. f is an odd function, and thus $A_n = 0$, $B_n = (2/L)\int_0^L \sin n\pi x/L\,dx = (2/L)(L/n\pi)[1-(-1)^n]$. The Fourier series is $(2/\pi)\sum_{n=1}^{\infty}[1-(-1)^n] \times \sin(n\pi x/L)/n$. •

1.1.5. Implementation with Mathematica. Let us redo Example 1.1.1 using Mathematica. The Fourier series of $f(x) = x, -\pi < x < \pi$, was found to be

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin kx}{k}$$

We first define a function of two variables,

```
F[x_,n_]:=2 Sum[(((-1)^(k+1))/k) Sin[k x], {k,1,n}]
```

and a plot-valued function by

```
F[n_]:=Plot[F[x,n],{x,-Pi,Pi}]
```

By typing **Enter**, we record the values of these functions. The correct input can be verified by typing **?F**. To verify the first three terms of the series, move the cursor to a new cell and type **F[x,3]** followed by **Enter**. Mathematica should respond with

```
                       Sin [2x]  Sin[3x]
Out[2] =2(Sin[x]-      ------- + -------)
                          2         3
```

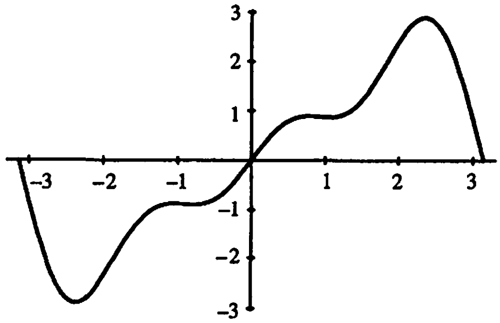


FIGURE 1.1.3 A three-term Fourier series.

To graph the function $F[x, 3]$, type **F[3]** instead of **F[x,3]**, and the result is as shown in Fig. 1.1.3.

Mathematica can also be used to compute the Fourier coefficients of a piecewise smooth function $f(x), -L < x < L$. To do this, we make the following commands:

```
A0[L_,f_]:=(1/(2Pi)) Integrate[f[x], {x,-L,L}]
A[n_,L_,f_]:=(1/(Pi)) Integrate[f[x] Cos[n x], {x,-L,L}]
B[n_,L_,f_]:=(1/(Pi)) Integrate[f[x] Sin[n x], {x,-L,L}]
```

Then we can define a function $f(x)$ in Mathematica and use the above definitions to compute the Fourier coefficients. For example, consider $f(x) = e^x, -L < x < L$. To enter this, we type

```
f[x_]:=E^x
```

and then type

```
A[n,L,f]
```

which produces the output

```
              n     L
          (-1)     E
Out[2]= ----------
                  2  2
                Pi  n
          L(1+ ------)
                   2
                  L
```

1.1.6. Fourier sine and cosine series. Suppose we are given a function $f(x), 0 < x < L$, and we desire a Fourier series representation. To get this, we extend f to the interval $-L < x < L$ and then compute the Fourier coefficients.

There are two natural ways of doing this, giving rise to the Fourier sine series and the Fourier cosine series.

One way of extending f is to define a new function f_O by

$$f_O(x) = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \\ 0 & x = 0 \end{cases} \tag{1.1.11}$$

f_O is called the *odd extension* of f to $(-L, L)$. It is an odd function, and therefore its Fourier coefficients are given as follows:

$$A_n = 0 \qquad n = 0, 1, \ldots \tag{1.1.12}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f_O(x) \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{1.1.13}$$

Therefore we have the *Fourier sine series*

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Another way of extending f to the interval $(-L, L)$ is to define

$$f_E(x) = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \\ 0 & x = 0 \end{cases} \tag{1.1.14}$$

f_E is called the *even extension* of f to $(-L, L)$. It is an even function defined on the interval $(-L, L)$. [Of course, we could define $f_E(0) = \lim_{x\to 0} f(x)$, if this limit exists. The definition $f_E(0) = 0$ is completely arbitrary.] The Fourier coefficients of f_E are as follows:

$$B_n = 0 \qquad n = 1, 2, \ldots \tag{1.1.15}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{L}\int_0^L f(x)dx \tag{1.1.16}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f_E(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \tag{1.1.17}$$

Therefore we have the *Fourier cosine series*

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \qquad 0 \le x \le L$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)dx$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \ldots$$

EXAMPLE 1.1.4. *Compute the Fourier sine series of $f(x) = 1$, $0 < x < L$.*

Solution. We have

$$B_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = -\frac{2L}{Ln\pi}\cos\frac{n\pi x}{L}\bigg|_0^L = \frac{2}{n\pi}[1 - (-1)^n]$$

The Fourier sine series is

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi x}{L} \quad \bullet$$

We now give an alternative method for computing the Fourier sine series of certain functions that satisfy *boundary conditions.* Let $f(x)$, $0 \le x \le L$, be a function with $f(0) = 0$, $f(L) = 0$, and $f''(x)$ continuous for $0 \le x \le L$. Then

$$\begin{aligned} B_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{n\pi}f(x)\cos\frac{n\pi x}{L}\bigg|_L^0 + \frac{2}{n\pi}\int_0^L f'(x)\cos\frac{n\pi x}{L}\,dx \end{aligned} \tag{1.1.18}$$

The first term is zero, and the second term can be integrated again by parts, with the result

$$B_n = -\left(\frac{L}{n\pi}\right)^2 \frac{2}{L}\int_0^L f''(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore the Fourier sine series of $f(x)$, $0 < x < L$, is obtained from the Fourier sine series of $f''(x)$, $0 < x < L$, by multiplication of the nth term of the series by $-(L/n\pi)^2$.

EXAMPLE 1.1.5. *Find the Fourier sine series of $f(x) = x^3 - L^2x$, $0 < x < L$.*

Solution. The function satisfies $f(0) = 0$, $f(L) = 0$ with $f''(x) = 6x$. The Fourier sine series of $6x$ is $(12L/\pi)\sum_1^{\infty}(-1)^{n+1}\sin(n\pi x/L)/n$. Therefore the Fourier sine series of $f(x)$ is $(12L^3/\pi^3)\sum_1^{\infty}(-1)^n\sin(n\pi x/L)/n^3$. •

EXERCISES 1.1

In Exercises 1 to 10, compute the Fourier series of the indicated functions.

1. $f(x) = x^2$, $-L < x < L$
2. $f(x) = x^3$, $-L < x < L$
3. $f(x) = |x|^3$, $-L < x < L$
4. $f(x) = e^x$, $-L < x < L$
5. $f(x) = \sin^2 2x$, $-\pi < x < \pi$

6. $f(x) = \cos^3 x,\ -\pi < x < \pi$
7. $f(x) = 0$ if $-L < x < 0$ and $f(x) = 1$ if $0 \le x < L$
8. $f(x) = 0$ if $-L < x < 0$ and $f(x) = x$ if $0 \le x < L$
9. $f(x) = 0$ if $-\pi < x < 0$ and $f(x) = \sin x$ if $0 \le x < \pi$
10. $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x}),\ -\pi < x < \pi$
11. Prove the orthogonality relations (1.1.3). [*Hint:* Use the trigonometric identities (1.1.5).]
12. Prove the orthogonality relations (1.1.4). [*Hint:* Use the trigonometric identities (1.1.5).]
13. Prove the following facts about even and odd functions:
    (a) The product of two even functions is even.
    (b) The product of two odd functions is even.
    (c) The product of an even function and an odd function is odd.
    (d) Which of statements (a), (b), (c) remains true if the word "product" is replaced by "sum"?
14. Let f be an arbitrary function. Show that there is an odd function f_1 and an even function f_2 such that $f = f_1 + f_2$.
15. Which of the following functions are even, odd, or neither?
    (a) $f(x) = x^3 - 3x$ (b) $f(x) = x^2 + 4$
    (c) $f(x) = \cos 3x$ (d) $f(x) = x^3 - 3x^2$
    (e) $f(x) = \sin x - 3x^5$ (f) $f(x) = |x| \sin x$
    (g) $f(x) = x^2 - \cos x$ (h) $f(x) = \cos^3 x$
16. Find the Fourier sine series for the following functions:
    (a) $f(x) = x,\ 0 \le x \le L$ (b) $f(x) = x^2,\ 0 \le x \le L$
    (c) $f(x) = e^x,\ 0 \le x \le L$ (d) $f(x) = x^3,\ 0 \le x \le L$
    (e) $f(x) = \sin x,\ 0 \le x \le L$ (f) $f(x) = \cos x,\ 0 \le x \le L$
17. Find the Fourier cosine series for the functions in Exercise 16.
18. Let $f(x)$, $-L < x < L$, be an odd function that satisfies the symmetry condition

$$f(L - x) = f(x)$$

Show that

$$\begin{aligned} A_n &= 0 \quad \text{for all } n \\ B_n &= 0 \quad \text{for all even } n \end{aligned}$$

19. Let $f(x)$, $-L < x < L$, be an odd function that satisfies the symmetry condition

$$f(L - x) = -f(x)$$

Show that

$$\begin{aligned} A_n &= 0 \quad \text{for all } n \\ B_n &= 0 \quad \text{for all odd } n \end{aligned}$$

20. A function $f(x), 0 < x < \pi/2$, is to be expanded into a Fourier series

$$f(x) = A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

By extending f to $-\pi < x < \pi$ in four different ways, give four different prescriptions for finding the Fourier coefficients $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$. (*Hint:* There are two choices for extending f to $0 < x < \pi$ and two more choices for further extending f to $-\pi < x < \pi$.)
21. Illustrate the expansions of Exercise 20 with $f(x) = 1$, $0 < x < \pi/2$. Find the four different Fourier series.

For each of the functions in Exercises 22 to 29 state whether or not it is periodic and find the smallest period.

22. $f(x) = \sin \pi x$
23. $f(x) = \sin 2x + \sin 3x$
24. $f(x) = \sin 4x + \cos 6x$
25. $f(x) = \sin x + \sin \pi x$
26. $f(x) = x - [x]$ ($[x] =$ integer part of x)
27. $f(x) = \tan x$
28. $f(x) = \sum_{n=1}^{\infty}(-1)^n x^{2n}/(2n)!$
29. $f(x) = \sin x^2$
30. Compute the Fourier sine series of $f(x) = x^2 - Lx$, $0 < x < L$.
31. Compute the Fourier sine series of $f(x) = x^4 - 2Lx^3 + L^3x$, $0 < x < L$.
32. Let $f(x)$, $-L < x < L$, be an even function. Show that

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

33. Show that the derivative of an even function is an odd function.
34. Show that the derivative of an odd function is an even function.

1.2. Convergence of Fourier Series[1]

In this section we discuss the validity of the equation

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L}\right)$$

where (A_n, B_n) are the Fourier coefficients of the function $f(x)$, $-L < x < L$. For simplicity in writing, we take $L = \pi$ in the exposition; all results obtained can be transformed to the interval $-L < x < L$ by the change of variable $x' = \pi x/L$.

[1]This section treats theoretical material and can be omitted without loss of continuity.

1.2.1. Piecewise smooth functions. Recall that a function f is *continuous* at x if $\lim_{y\to x} f(y) = f(x)$. Not all Fourier series converge, even if we impose the restriction that their functions are continuous. In fact, there exist continuous functions on $[-\pi, \pi]$ whose Fourier series diverge at an infinite number of points! We therefore need to focus our attention on another class of functions, the so-called piecewise smooth functions. We first define the concept of a piecewise continuous function.

Definition A function $f(x)$, $a < x < b$, is *piecewise continuous* if there is a finite set of points $a = x_0 < x_1 < \cdots < x_p < x_{p+1} = b$ such that

$$f \text{ is continuous at } x \neq x_i, \qquad i = 1, \ldots, p \tag{1.2.1}$$

$$\lim_{\epsilon\to 0} f(x_i + \epsilon) \text{ exists} \qquad i = 0, \ldots, p \tag{1.2.2}$$

$$\lim_{\epsilon\to 0} f(x_i - \epsilon) \text{ exists} \quad i = 1, \ldots, p+1 \tag{1.2.3}$$

The limit (1.2.2) is denoted $f(x_i+0)$ and is called the *right-hand limit*. Likewise, the limit (1.2.3) is denoted $f(x_i - 0)$ and is called the *left-hand limit*. These are supposed to be finite.

Definition A function $f(x)$, $a < x < b$, is said to be *piecewise smooth* if f and all of its derivatives are piecewise continuous.

Of course, we assume that the subdivision points $x_0 < x_1 < \cdots < x_{p+1}$ are the same for f and all of its derivatives. With this definition, the derivative of a piecewise smooth function is again piecewise smooth.

If $f(x)$, $a < x < b$, is piecewise smooth, then $f'(x)$ exists except for $x = x_1, \ldots, x_p$. This is the *piecewise derivative* of f. Many of the usual operations with ordinary derivatives are valid for piecewise derivatives; the sum, difference, and product rules are valid except at the subdivision points $(x_1, \ldots, x_p)$. The quotient rule is also valid unless the denominator is zero. The fundamental theorem of calculus must be modified for piecewise smooth functions to the form

$$f(b-0) - f(a+0) = \int_a^b f'(x)dx + \sum_{i=1}^{p}[f(x_i+0) - f(x_i-0)]$$

Indeed, on each interval (x_i, x_{i+1}) we may apply the ordinary fundamental theorem of calculus in the form

$$f(x_{i+1} - 0) - f(x_i + 0) = \int_{x_i}^{x_{i+1}} f'(x)dx$$

Adding these equations for $i = 0, 1, \ldots, p$ gives the result.

If the piecewise smooth function $f(x)$, $a < x < b$, is also *continuous*, then the fundamental theorem of calculus may be applied in its usual form,

$$f(b-0) - f(a+0) = \int_a^b f'(x)dx$$

With these rules in mind, we may operate freely with piecewise smooth functions.

EXAMPLE 1.2.1.

$$f(x) = |x| \qquad -\pi < x < \pi$$

We take $x_0 = -\pi$, $x_1 = 0$, $x_2 = \pi$. Here f is continuous on the entire interval. f' is piecewise continuous, with $f'(0+0) = 1$, $f'(0-0) = -1$. All higher derivatives are zero; hence $f(x), -\pi < x < \pi$, is piecewise smooth.

EXAMPLE 1.2.2.

$$f(x) = \begin{cases} x^2 & -\pi < x < 0 \\ x^2 + 1 & 0 \leq x < \pi \end{cases}$$

In this example f is continuous, with the exception of the point $x = 0$, where we have $f(0+0) = 1$ and $f(0-0) = 0$. All higher derivatives are piecewise continuous on $(-\pi, \pi)$, so $f(x), -\pi < x < \pi$, is piecewise smooth.

EXAMPLE 1.2.3.

$$f(x) = x|x| \qquad -\pi < x < \pi$$

In this case f and f' are continuous. f'' is continuous everywhere except at $x = 0$, where we have $f''(0+0) = 2$ and $f''(0-0) = -2$. All higher derivatives are zero; thus $f(x), -\pi < x < \pi$, is piecewise smooth.

EXAMPLE 1.2.4.

$$f(x) = x^2 \sin\frac{1}{x} \qquad -\pi < x < \pi$$

f is continuous on $(-\pi, \pi)$. f' is continuous on $(-\pi, \pi)$ with the exception of the point $x = 0$. However, $f'(0+0)$ and $f'(0-0)$ do not exist, so $f(x), -\pi < x < \pi$, is piecewise continuous but *is not piecewise smooth.*

EXAMPLE 1.2.5.

$$f(x) = \frac{1}{x^2 - \pi^2} \qquad -\pi < x < \pi$$

In this case $f(x), -\pi < x < \pi$, is continuous, but it is not piecewise continuous since $f(-\pi+0)$ and $f(\pi-0)$ are not finite. In particular, $f(x), -\pi < x < \pi$, is not piecewise smooth.

When working with piecewise smooth functions, we may omit the definition of $f(x)$ at the subdivision points $x_0, x_1, \ldots, x_{p+1}$. This causes no difficulty in the discussion of Fourier series, since the Fourier coefficients A_n, B_n are defined as integrals, which are insensitive to the value of $f(x)$ at a finite number of points. More precisely, if $f_1(x) = f_2(x)$, except for $x = x_0, x_1, \ldots, x_{p+1}$, then $\int_a^b f_1(x)dx = \int_a^b f_2(x)dx$. Therefore we see that the *Fourier coefficients do not depend on any of the numbers* $f(x_0), \ldots, f(x_{p+1})$.

Suppose $f(x), -\pi < x < \pi$, is piecewise smooth. We define the 2π-*periodic extension* of f by setting

$$f(x + 2n\pi) = f(x) \qquad \text{where } x \in (-\pi, \pi)$$

and n is an integer (positive or negative).

It is left as an exercise to show that the 2π-periodic extension of f is piecewise smooth on any open interval and that it is periodic with period 2π. It is also left as an exercise to show that

$$\int_c^d f(x)dx = \int_a^b f(x)dx \qquad \text{if } d - c = 2\pi = b - a$$

where f is *any* 2π-periodic function.

Let $f(x)$, $-\pi < x < \pi$, be a piecewise smooth function and let $\bar{f}(x)$, $-\infty < x < \infty$, be the 2π-periodic extension of f; $\bar{f}$ is a 2π-periodic function with $\bar{f}(x) = f(x)$ for $-\pi < x < \pi$.

The following theorem relates the convergence of a Fourier series to the normalized values of the function

THEOREM 1.1. *(Convergence theorem). Let $f(x), -\pi < x < \pi$, be piecewise smooth. Then the Fourier series of f converges for all x to the value $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$, where $\bar{f}$ is the 2π-periodic extension of f.*

From the periodicity, we see that the left-hand limit $\bar{f}(\pi - 0)$ is equal to the left-hand limit $\bar{f}(-\pi - 0)$, with a corresponding statement for the right-hand limit. Therefore the average of the left- and right hand-limits at the endpoints agrees with the common average of the function at the endpoints; in symbols,

$$\begin{aligned}
\frac{1}{2}[\bar{f}(-\pi - 0) + \bar{f}(-\pi + 0)] &= \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)] \\
\frac{1}{2}[\bar{f}(\pi - 0) + \bar{f}(\pi + 0)] &= \frac{1}{2}[f(-\pi + 0) + f(\pi - 0)]
\end{aligned}$$

The restriction to the interval $-\pi < x < \pi$ is of no significance. It has been made here so that, instead of writing $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$, we may write $\cos mx$ and $\sin mx$.

Before proceeding with the proof, we need two lemmas.

Lemma 1 (Riemann). If f and f' are piecewise continuous on (a, b), then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx = 0$$

Proof. First we write

$$\int_a^b f(x)\sin\lambda x\,dx = \sum_{i=0}^{p}\int_{x_i}^{x_{i+1}} f(x)\sin\lambda x\,dx$$

It remains to show that

$$\lim_{\lambda\to\infty}\int_{x_i}^{x_{i+1}} f(x)\sin\lambda x\,dx = 0$$

For this we integrate by parts, with $u = f(x)$, $dv = \sin\lambda x\,dx$. Thus

$$\int_{x_i}^{x_{i+1}} f(x)\sin\lambda x\,dx = \frac{-f(x)\cos\lambda x}{\lambda}\Bigg|_{x_i}^{x_{i+1}} + \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} f'(x)\cos\lambda x\,dx$$

Each of these tends to zero when $\lambda\to\infty$, completing the proof of Lemma 1. •

We wish to examine the limit as $N\to\infty$ of

$$f_N(x)=A_0+\sum_{m=1}^{N}(A_m\cos mx + B_m\sin mx)$$

Using the definitions of A_0, A_m, B_m given in Sec. 1.1, formulas (1.1.6), (1.1.7), (1.1.8), we have

$$\begin{aligned} f_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{m=1}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos mt\cos mx + \sin mt\sin mx)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \sum_{m=1}^{N}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos m(t-x)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2}+\sum_{m=1}^{N}\cos m(t-x)\right]dt \end{aligned}$$

Clearly, it would be useful to be able to write

$$\frac{1}{2}+\sum_{m=1}^{N}\cos m(t-x)$$

in a more compact form. Therefore we formulate a second lemma.

Lemma 2. For any α real, $\alpha\neq 0, \pm 2\pi, \ldots$, we have

$$\frac{1}{2}+\cos\alpha+\cdots+\cos N\alpha = \frac{\sin(N+\frac{1}{2})\alpha}{2\sin\frac{1}{2}\alpha}$$

Proof. Setting $S=\frac{1}{2}+\cos\alpha+\cdots+\cos N\alpha$, we have

$$S\sin\alpha = \frac{1}{2}\sin\alpha+\sin\alpha\cos\alpha+\cdots+\sin\alpha\cos N\alpha$$

From the addition formulas

$$\begin{aligned}\sin(a+b) &= \sin a\cos b + \cos a\sin b\\ \sin(a-b) &= \sin a\cos b - \cos a\sin b\end{aligned}$$

we have

$$\cos a\sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)]$$

so that

$$\begin{aligned}S\sin\alpha &= \frac{1}{2}[\sin\alpha + \sin 2\alpha - 0 + \sin 3\alpha - \sin\alpha\\ &\quad + \cdots + \sin(N+1)\alpha - \sin(N-1)\alpha]\\ &= \frac{1}{2}[\sin N\alpha + \sin(N+1)\alpha]\end{aligned}$$

To complete the proof, we average the addition formula as follows:

$$\frac{1}{2}[\sin(a+b) + \sin(a-b)] = \sin a\cos b$$

Setting $a+b=(N+1)\alpha$, $a-b=N\alpha$, we take $a=(N+\frac{1}{2})\alpha$, $b=\frac{1}{2}\alpha$, so that

$$\frac{1}{2}[\sin N\alpha + \sin(N+1)\alpha] = \sin\left(N+\frac{1}{2}\right)\alpha\,\cos\frac{1}{2}\alpha$$

and

$$S = \frac{\sin(N+\frac{1}{2})\alpha\cos\frac{1}{2}\alpha}{\sin\alpha}$$

Substituting the identity $\sin\alpha = 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$ completes the proof of Lemma 2. (For a shorter proof of Lemma 2, using complex numbers, see Exercise 13 at the end of this section.) •

In view of Lemma 2, we can write

$$f_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(N+\frac{1}{2})(t-x)}{2\sin\frac{1}{2}(t-x)}\,dt$$

This form is preferable because it makes no mention of the Fourier coefficients $\{A_n\}$, $\{B_n\}$.

1.2.2. Dirichlet kernel. To proceed further, we make the definition

$$D_N(u) = \frac{\sin(N+\frac{1}{2})u}{2\pi\sin u/2} \qquad u \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

and by continuity we define $D_N(u) = (2N+1)/2\pi, u = 0, \pm 2\pi, \pm 4\pi, \ldots$.

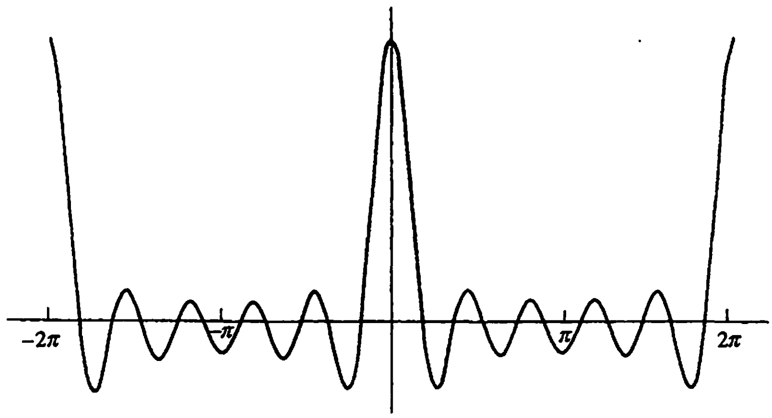


FIGURE 1.2.1 The Dirichlet kernel $D_N(u)$ for $N = 5$.

D_N is the *Dirichlet kernel*, an even, 2π-periodic function. From Lemma 2 we see that

$$\int_0^{\pi} D_N(u)du = \frac{1}{2} = \int_{-\pi}^{0} D_N(u)du$$

From Fig. 1.2.1 we see that $D_N(u)$ behaves roughly like a periodic function with period $2\pi/N$, except in the neighborhood of $u = 0, \pm 2\pi, \ldots$, where it is peaked. The most important property of the Dirichlet kernel is that it provides an explicit representation of the Fourier partial sum, through the formula

$$\boxed{f_N(x) = \int_{-\pi}^{\pi} f(t)D_N(t-x)\,dt} \tag{1.2.4}$$

1.2.3. Proof of convergence. To complete the proof of Theorem 1.1, we extend f to $\bar{f}$, a 2π-periodic function. Therefore the product $D_N(t-x)\bar{f}(t)$ is also a 2π-periodic function of t for each x. We now write

$$\begin{aligned}
f_N(x) &= \int_{-\pi}^{\pi} \bar{f}(t)D_N(t-x)dt \\
&= \int_{-\pi-x}^{\pi-x} \bar{f}(x+u)D_N(u)du \qquad\qquad t-x=u \\
&= \int_{-\pi}^{\pi} \bar{f}(x+u)D_N(u)du \qquad\qquad \text{periodicity} \\
&= \left\{\int_{-\pi}^{0} + \int_{0}^{\pi}\right\} \bar{f}(x+u)D_N(u)du
\end{aligned}$$

We will analyze the two integrals separately and show that

$$\lim_{N\to\infty}\int_0^{\pi} \bar{f}(x+u)D_N(u)du = \frac{1}{2}\bar{f}(x+0) \tag{1.2.5}$$

$$\lim_{N\to\infty}\int_{-\pi}^{0} \bar{f}(x+u)D_N(u)du = \frac{1}{2}\bar{f}(x-0) \tag{1.2.6}$$

from which the result will follow. We carry out the analysis of only the first integral in detail; the second is identical in every respect. Define

$$g(u) \doteq [\bar{f}(x+u) - \bar{f}(x+0)]/u$$

Then

$$\int_0^{\pi}[\bar{f}(x+u) - \bar{f}(x+0)]D_N(u)du = \frac{1}{\pi}\lim_{N\to\infty}\int_0^{\pi} g(u)U(u)\sin\left(N+\frac{1}{2}\right)u\,du$$

where

$$\begin{aligned} U(u) &= \frac{u}{2\sin u/2} \qquad u \neq 0 \\ U(0) &= 1 \end{aligned}$$

Using L'Hospital's rule, we see that the function $U(u)$ is continuous and has a continuous derivative, $-\pi \le u \le \pi$. Similarly, we can use L'Hospital's rule to compute the limits

$$\lim_{u\downarrow 0} g(u) = \bar{f}'(x+0) \tag{1.2.7}$$

$$\lim_{u\downarrow 0} g'(u) = \frac{1}{2}\bar{f}''(x+0) \tag{1.2.8}$$

Therefore $g(u)$ is piecewise continuous with a piecewise continuous derivative. But $U(u)$ has a continuous derivative, and therefore the product $g(u)U(u)$ also has a piecewise continuous derivative. Applying Lemma 1, we have proved that

$$\lim_{N\to\infty}\int_0^{\pi} g(u)U(u)\sin\left(N+\frac{1}{2}\right)u\,du = 0$$

Writing this in terms of $\bar{f}$, we have

$$\lim_{N\to\infty}\int_0^{\pi} \bar{f}(x+u)D_N(u)du = \bar{f}(x+0)\lim_{N\to\infty}\int_0^{\pi} D_N(u)du = \frac{1}{2}\bar{f}(x+0)$$

which was to be proved. •

An examination of the graph of $D_N(u)$ (Fig. 1.2.1) helps to give an intuitive motivation of the proof. Since $\int_{-\pi}^{\pi} D_N(u)du = 1$, the graph suggests that, as N gets large, the area tends to concentrate around $u = 0$, so that $\int_{-\pi}^{\pi} f(u)D_N(u)du$ tends to pick off the values of $f(u)$ near $u = 0$. Thus

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} f(u)D_N(u)du = \frac{1}{2}[f(0+0) + f(0-0)]$$

for functions $f(x)$ that are piecewise smooth.

Having proved the convergence of the Fourier series, we can now obtain many useful conclusions. Referring to the first two examples in Sec. 1.1, we have the convergent Fourier series

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx = x \qquad -\pi < x < \pi$$

$$\frac{\pi}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos nx = |x| \qquad -\pi < x < \pi$$

Both of these examples are continuous functions, for which $f(x+0) = f(x-0) = f(x)$ for all x, $-\pi < x < \pi$. However, the periodic extension is not continuous in the first case, where $f(x) = x$, $-\pi < x < \pi$.

As an example of a discontinuous function, we have the convergent Fourier series

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0 \\ -1 & -\pi < x < 0 \end{cases}$$

These can also be used to obtain various numerical series. Taking $x = 0$ in the Fourier series for $|x|$, we have $0 = \pi/2 - (2/\pi)(2 + \frac{2}{9} + \frac{2}{25} + \cdots)$, $\pi^2/8 = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$. Similarly, taking $x = \pi/2$ in the third example, we obtain $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

EXERCISES 1.2

1. Determine whether or not the indicated function is piecewise smooth.
   (a) $f(x) = |x|^{3/2}$, $-2 < x < 2$
   (b) $f(x) = [x] - x$, $0 < x < 3$ ($[x]$ = integer part of x)
   (c) $f(x) = x^4 \sin(1/x)$, $-1 < x < 1$
   (d) $f(x) = e^{-(1/x^2)}$, $-1 < x < 1$
2. Let $f(x) = x^2 \sin(1/x)$.
   (a) Show that $\lim_{x\to 0} f(x) = 0$.
   (b) Graph $f(x), -\pi < x < \pi$.
   (c) Show that $f'(0+0)$ does not exist by considering $f'(h)$ as $h \to 0$ through the values $1/2n\pi$ and $1/(2n+1)\pi$, $n = 1, 2, \ldots$.
3. Let f and g be piecewise smooth on (a, b).
   (a) Show that $f + g$ is piecewise smooth on (a, b).
   (b) Show that $f \cdot g$ is piecewise smooth on (a, b).
   (c) What restrictions must be made on g in order that f/g be piecewise smooth on (a, b)?

4. Let $\bar{f}$ be the 2π-periodic extension of the piecewise smooth function $f(x)$, $-\pi < x < \pi$.
   (a) Show that $\bar{f}(x), -\infty < x < \infty$, is piecewise smooth.
   (b) Show that $\bar{f}$ is 2π-periodic.
   (c) Show that

$$\int_c^d \bar{f}(x)dx = \int_a^b \bar{f}(x)dx \qquad \text{if } d - c = 2\pi = b - a$$

5. Define $U(0) = 1$ and

$$U(u) = \frac{u}{2\sin(u/2)} \qquad -\pi \le u \le \pi$$

Show that $U(u)$ is continuous and has a continuous derivative for $-\pi \le u \le \pi$. (*Hint:* Use L'Hospital's rule.)

6. Let $f(x)$, $a < x < b$, be a piecewise smooth function. Let $g(u) = [f(x+u) - f(x+0)]/u$ for $u \neq 0$. Show that $g(0+0) = f'(x+0)$, $g(0-0) = f'(x-0)$. (*Hint:* Use L'Hospital's rule.)
7. Let $g(u)$ be defined as in Exercise 6. Show that $g'(0+0) = \frac{1}{2}f''(x+0)$, $g'(0-0) = \frac{1}{2}f''(x-0)$.
8. Prove that $D_N(u)$ is even and 2π-periodic.
9. Use Lemma 1 and the properties of the Dirichlet kernel to compute the following limits:
   (a) $\lim_{N\to\infty} \int_{-\pi/2}^{\pi/2} D_N(u)du$ (b) $\lim_{N\to\infty} \int_0^{\pi/2} D_N(u)du$
   (c) $\lim_{N\to\infty} \int_{-\pi/6}^{\pi/6} D_N(u)du$ (d) $\lim_{N\to\infty} \int_{\pi/2}^{\pi} D_N(u)du$
10. What is the maximum value of $D_N(u)$, $-\pi \le u \le \pi$?
11. Find all solutions of the equation $D_N(u) = 0$.
12. Find all solutions of the equation $D_N'(u) = 0$.
13. There is another way of establishing Lemma 2. Recall that $e^{ix} = \cos x + i\sin x$.
   (a) Show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

   (b) Prove Lemma 2 using part (a) and the fact that for any complex number $r \neq 1$,

$$1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1} \qquad r \neq 1$$

14. This exercise establishes the formula

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

(a) Let

$$f(u) = \left(\frac{1}{2\sin u/2} - \frac{1}{u}\right) \qquad u \neq 0 \qquad f(0) = 0$$

Show that f, f' are continuous on $(0, \pi)$. (The only trouble occurs at $u = 0$. Use L'Hospital's rule to show that the appropriate limits are finite.)

(b) Use Lemma 1 to conclude that

$$\lim_{N\to\infty} \int_0^\pi \left[\frac{1}{2\sin u/2} - \frac{1}{u}\right] \sin(N + \frac{1}{2})u \, du = 0$$

(c) Hence show that

$$\lim_{N\to\infty} \int_0^\pi D_N(u)du = \lim_{N\to\infty} \int_0^\pi \frac{\sin(N+\frac{1}{2})u}{u} du$$

(d) Make the appropriate substitution in the second definite integral and recall the appropriate facts about $D_N(u)$ to conclude that

$$\lim_{N\to\infty} \int_0^{(N+1/2)\pi} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(e) If $(N - \frac{1}{2})\pi \leq X \leq (N + \frac{1}{2})\pi$, show that

$$\int_0^X \frac{\sin x}{x} dx = \int_0^{(N+1/2)\pi} \frac{\sin x}{x} dx + \epsilon_X$$

where $|\epsilon_X| \leq 1/(N - \frac{1}{2})$. Conclude that the improper integral converges to $\pi/2$ when $X \to \infty$.

15. (a) Set $x = \pi/2$ in the Fourier series for $f(x) = x$, $-\pi < x < \pi$, to obtain the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(b) Set $x = \pi/4$ in the series part (a) to obtain

$$\frac{\pi}{4} = \sqrt{2}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots\right) - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

(c) Conclude from part (b) that

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \cdots$$

(d) If we set $x = \pi$ in the series in (a), we find that the series sums to zero. Why doesn't this contradict $f(x) = x$?

16. (a) Show that

$$x^2 = \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \cdots + (-1)^m \frac{4}{m^2} \cos mx + \cdots$$

for $-\pi \leq x \leq \pi$.

(b) Setting $x = 0$ in (a), find the sum

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m^2}$$

(c) What is

$$\sum_{m=1}^{\infty} \frac{1}{m^2}$$

[*Hint:* Set $x = \pi$ in part (a).]

(d) What is

$$\sum_{m\,\mathrm{odd}} \frac{1}{m^2}$$

[*Hint:* Add (b) and (c).]

17. Let $f(x) = x$, $-\pi < x < \pi$. What is the sum of the Fourier series for $x = -\pi$, $x = \pi$?
18. Let $f(x) = e^x$, $-\pi < x < \pi$. What is the sum of the Fourier series for $x = -\pi$, $x = \pi$?
19. Let $f(x)$, $g(x)$ be piecewise smooth functions for $a < x < b$. Show that

$$\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = -\sum_{i=1}^{p} [f(x_i + 0)g(x_i + 0) - f(x_i - 0)g(x_i - 0)] + f(b-0)g(b-0) - f(a+0)g(a+0)$$

20. Use Exercise 19 to prove the following integration-by-parts formula for piecewise smooth functions:

$$\begin{aligned}\int_a^b f(x)g'(x)dx = {} & f(b-0)g(b-0) - f(a+0)g(a+0) - \int_a^b f'(x)g(x)dx \\ & - \sum_{i=1}^{p} g(x_i - 0)[f(x_i+0) - f(x_i - 0)] \\ & - \sum_{i=1}^{p} f(x_i - 0)[g(x_i + 0) - g(x_i - 0)] \\ & - \sum_{i=1}^{p} [f(x_i + 0) - f(x_i - 0)][g(x_i + 0) - g(x_i - 0)]\end{aligned}$$

21. By examining the proof of Theorem 1.1, show that the conclusion is valid if f, f', f'' are piecewise continuous.
22. On the basis of Exercise 21, for which $n \geq 1$, can we assert that the Fourier series of $x^n \sin 1/x$ is convergent for all x, $-\pi < x < \pi$?

23. Let $f(x)$, $-\pi < x < \pi$, be a piecewise smooth function with Fourier coefficients A_n, B_n. Apply Exercise 20 with $a = -\pi$, $b = \pi$, $g'(x) = \cos nx$ to find an asymptotic formula for A_n, B_n, $n \to \infty$.

1.3. Uniform Convergence and the Gibbs Phenomenon[2]

We have seen that the Fourier series of a piecewise smooth function converges to the function except at points of discontinuity, where it converges to the average of the function's left- and right-hand limits. Since we are interested in approximating functions by partial sums of their Fourier series, it is of interest how the Fourier series converge near a discontinuity, that is, how the partial sums of Fourier series behave near discontinuities of their functions. We turn first to an example.

1.3.1. Example of Gibbs overshoot. Consider the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

The cosine coefficients are all zero (f is odd), and the sine coefficients are given by

$$B_n = \frac{2}{\pi}\int_0^\pi \sin nx\, dx = \frac{2}{n\pi}[1 - (-1)^n] \qquad n = 1, 2, \ldots$$

The partial sum of the Fourier series is therefore

$$f_{2n}(x) = f_{2n-1}(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2n-1)x}{2n-1}\right]$$

From the graph of Fig. 1.3.1 we see that, just before the discontinuity, the partial sums overshoot the right- and left-hand limits and then slope rapidly toward their mean. On the interval $-\pi \le x \le \pi$, f_1 has one maximum and one minimum, f_3 has three maxima and three minima, f_5 has five maxima and five minima, etc. We can actually calculate the overshoot by computing the derivative

$$f'_{2n-1}(x) = \frac{4}{\pi}[\cos x + \cos 3x + \cos 5x + \cdots + \cos(2n-1)x] \tag{1.3.1}$$

and solving the equation $f'_{2n-1}(x) = 0$.

To solve this equation, we multiply (1.3.1) by $\sin x$ and use the identity

$$\sin x \cos kx = \frac{1}{2}[\sin(k+1)x - \sin(k-1)x]$$

and get

$$\begin{aligned} \pi \sin x\, f'_{2n-1}(x) &= 2\left\{\sin 2x + \sum_{k=1}^{n-1}[\sin 2(k+1)x - \sin 2kx]\right\} \\ &= 2\sin 2nx \end{aligned}$$

[2]This section treats theoretical material and can be omitted without loss of continuity.

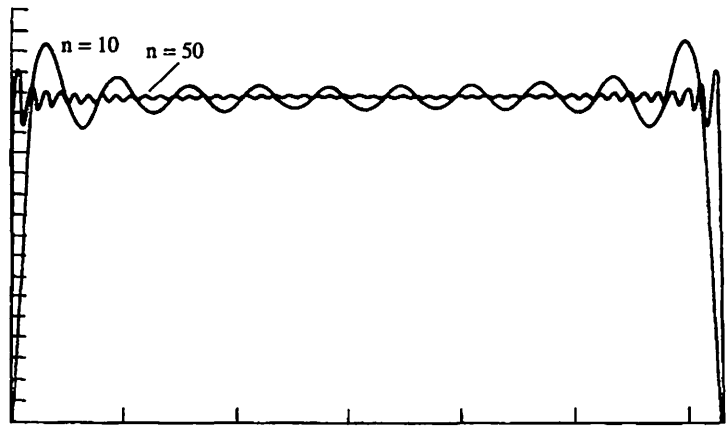


FIGURE 1.3.1 The Gibbs phenomenon for $n = 10$ and $n = 50$.

Therefore, the extrema occur at the points

$$2nx = \pm\pi, \pm 2\pi, \dots, \pm 2n\pi$$

These points are equally spaced in $[-\pi, \pi]$. It is the maximum closest to the discontinuity (i.e., when $x = \pi/2n$) that is of interest, so we wish to compute

$$f_{2n-1}\left(\frac{\pi}{2n}\right) = \frac{4}{\pi}\left[\sin\frac{\pi}{2n} + \frac{1}{3}\sin\frac{3\pi}{2n} + \cdots + \frac{1}{2n-1}\sin\frac{(2n-1)\pi}{2n}\right]$$

for large n. The technique we will use for evaluating this sum consists of rewriting the sum so that it looks like the approximating sum of a Riemann integral and then evaluating the integral. Our answer will be exact when $n \uparrow \infty$ and so should give a good approximation for large n.

The function whose integral we will approximate is $g(x) = (\sin x)/x$. Consider the partition of $[0, \pi]$, given by the points $\{x_k\}$, where

$$\begin{aligned} x_k &= \frac{\pi k}{n} \qquad k = 1, \dots, n \\ \Delta x_k &= \frac{\pi}{n} \end{aligned}$$

If we choose the midpoints x_k' of each of these intervals as our sampling points, then we have

$$\sum_{k=1}^{n} g(x_k')\Delta x_k = \frac{\sin\pi/2n}{\pi/2n}\frac{\pi}{n} + \cdots + \frac{\sin(2n-1)\pi/2n}{(2n-1)\pi/2n}\frac{\pi}{n} \to \int_0^\pi \frac{\sin x}{x}\,dx$$

If we rearrange our sum, we see that it equals

$$\frac{2n}{\pi}\frac{\pi}{n}\left[\sin\frac{\pi}{2n}+\frac{\sin 3\pi/2n}{3}+\cdots+\frac{\sin(2n-1)\pi/2n}{2n-1}\right]=\frac{\pi}{2}f_{2n-1}\left(\frac{\pi}{2n}\right)$$

Therefore the *limit of the overshoot* is given by

$$\lim_{n\uparrow\infty} f_{2n-1}\left(\frac{\pi}{2n}\right)=\frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx$$

We can approximate the integral numerically as follows:

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots$$

so

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots$$

and

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx &= \frac{2}{\pi}\int_0^\pi\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\right)dx+\cdots\\
&= \frac{2}{\pi}\left(\pi-\frac{\pi^3}{18}+\frac{\pi^5}{600}-\frac{\pi^7}{35{,}280}\right)+\cdots\\
&= 2-\frac{\pi^2}{9}+\frac{\pi^4}{300}-\frac{\pi^6}{17{,}640}+\cdots\\
&= 2-1.11+0.33-0.04+\cdots\\
&= 1.18 \text{ to two decimal places}
\end{aligned}$$

This means that if we stand at any one point, we will land on the graph of $f(x)$ in the limit $n\uparrow\infty$. However, if we ride the crest of the worst point possible for each n, then we will never reach the graph of $f(x)$. When $n\uparrow\infty$, we will be left dangling approximately 1.18 units above the origin. This behavior can be described by saying that the partial sums *do not* converge *uniformly* to $f(x)$ (i.e., the *entire* curve is not arbitrarily close to the graph of f for sufficiently large n). Rather, they converge to the graph indicated in Fig. 1.3.2. This is known as the *Gibbs phenomenon.* Notice that the overshoot of 1.18 is 9 percent of the jump made at the discontinuity. This is characteristic of the overshoot due to any discontinuity in any piecewise smooth function f. In fact, we have the following general fact, whose proof is omitted.

Let f be piecewise smooth on $(-\pi,\pi)$. Then the amount of overshoot near a discontinuity, due to the Gibbs phenomenon, is approximately equal to

$$0.09|f(x_0+0)-f(x_0-0)|$$

for large n.

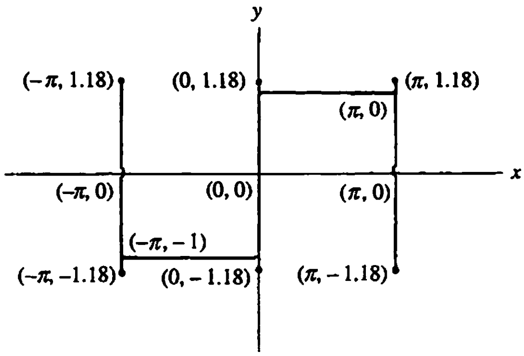


FIGURE 1.3.2 Limiting graph in Gibbs' phenomenon.

1.3.2. Implementation with Mathematica. The graphs of the Gibbs phenomenon can be easily produced using Mathematica. We will illustrate this with the function

$$f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x < \pi \end{cases}$$

To implement this in Mathematica, we first define a step function by means of the "If" function:

```
u[a_,x_]:=If[a<x,1,0]
```

With this definition, the function f can be written

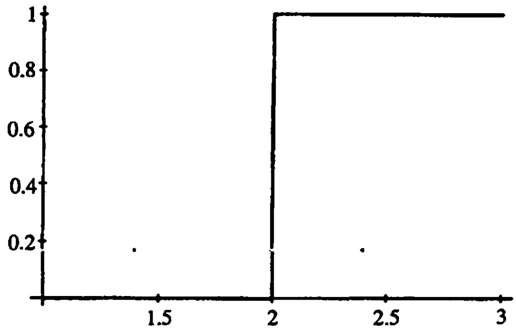
```
f[x_]:=1 - 2u[-Pi,x]+2u[0,x]
```

To see this in more detail, note that the function **If** takes three arguments; the first argument is a condition, the second argument is the value of the function when the condition is satisfied, and the third argument is the value of the function if the condition is not satisfied. In the case at hand, we see that if $-\pi < x < 0$, then the first condition is met but not the second, so that $f(x) = 1-2+0 = -1$. If $0 < x < \pi$, then both conditions are satisfied, so that $f(x) = 1 - 2 + 2 = 1$, as required. In case $x = 0$ only the first condition is satisfied, so that $f(0) = 1 - 2 + 0 = -1$, as required.

This function can be plotted in Mathematica by means of the commmand

```
U[a_]:=Plot[u[a,x],{x,a-1,a+1}]
```

For example, the graph of **u[2,x]** can be obtained by typing **U[2]**:



If we want to graph the Fourier series of f using Mathematica, we first recall the Fourier series representation for the partial sums:

$$f_{2n-1}(x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin((2k-1)x)}{2k-1}$$

To implement this in Mathematica, we define a function of two variables as follows:

```
f[n_,x_]:=(4/Pi) Sum[(1/(2 k -1)) Sin[(2k-1) x],{k,1,n}]
```

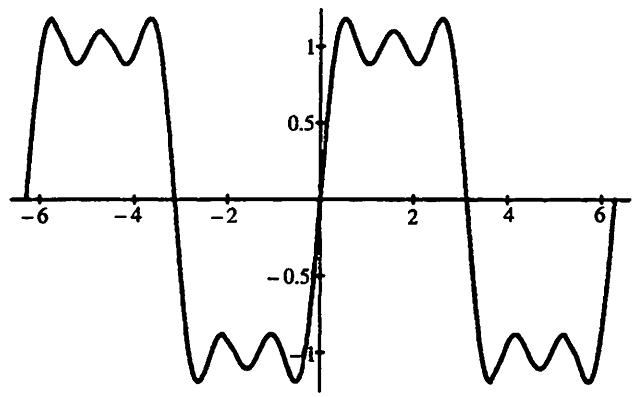
For example, if we now type **f[3,x]**, we obtain the output

```
                      sin[3x]    Sin[5x]
        4(Sin[x]  + -------+ --------)
                         3          5
Out[4]=----------------------------------
                       Pi
```

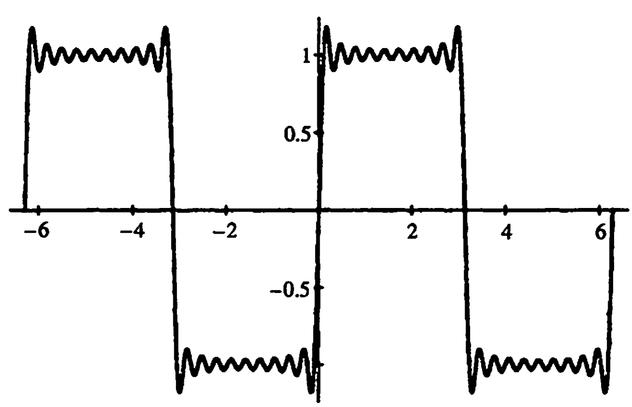
To graph the partial sum, we define a function as follows:

```
fgraph[n_]:=Plot[f[n,x],{x,-2Pi, 2Pi}]
```

If we type **fgraph[3]**, we obtain



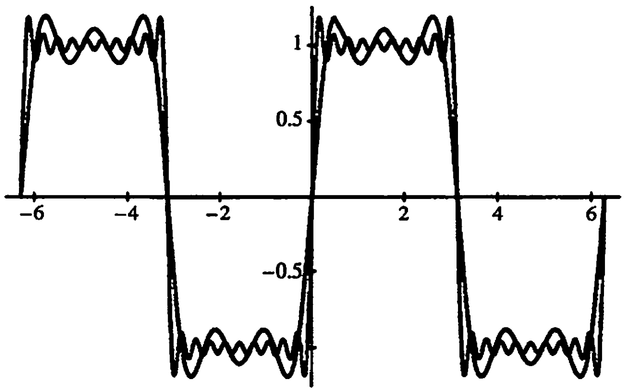
whereas if we type **fgraph[10]** we obtain



In order to display the two graphs simultaneously, we type

```
Plot[{f[3,x], f[10,x]},{x,-2Pi,2Pi}]
```

to obtain



1.3.3. Uniform and nonuniform convergence. In many problems it is important to avoid the Gibbs phenomenon—in other words, to be sure that the function $f(x)$ is well approximated by the partial sum $f_n(x)$ at all points of the interval $-L \le x \le L$. Recall that a sequence of functions $f_n(x)$, $a \le x \le b$, converges *uniformly* to a limit function $f(x)$, $a \le x \le b$, if

$$|f_n(x) - f(x)| \le \epsilon_n \qquad a \le x \le b,\ n = 1, 2, \ldots$$

where

$$\lim_{n\to\infty} \epsilon_n = 0$$

This is clearly violated in the Gibbs phenomenon, for in the previous example $\lim_{n\to\infty}[f_{2n-1}(\pi/2n) - f(\pi/2n)] = 0.18\ldots.$

1.3.4. Two criteria for uniform convergence. We shall give two general criteria for uniform convergence. The first of these can be tested on the series, while the second can be tested on the function.

PROPOSITION 1.3.1. **(First criterion for uniform convergence).** *Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Suppose that the Fourier coefficients $\{A_n\}$, $\{B_n\}$ satisfy*

$$\sum_{n=1}^{\infty}(|A_n| + |B_n|) < \infty$$

Then the Fourier series converges uniformly.

For example, $\sum_{n=1}^{\infty}(\sin nx)/n^2$ is a uniformly convergent Fourier series.

PROPOSITION 1.3.2. **(Second criterion for uniform convergence).** *Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Suppose in addition that*

$$f \text{ is continuous} \qquad -L < x < L \text{ and } f(-L+0) = f(L-0)$$

Then the Fourier series converges uniformly.

For example, $f(x) = |x|$ has a uniformly convergent Fourier series.

Within the class of piecewise smooth functions, these criteria are necessary and sufficient: If the Fourier series of a piecewise smooth function converges uniformly, then f is continuous, $f(-L+0) = f(L-0)$, and $\sum_{n=1}^{\infty}(|A_n| + |B_n|) < \infty$. Once we leave the domain of piecewise smooth functions, the theory becomes much more complicated; for example, the Fourier series $\sum_{n=2}^{\infty}(\sin nx)/(n \log n)$ is known to be uniformly convergent,[3] but it does not satisfy the first criterion. Of course the sum of this series must be a *continuous* function by the general properties of uniform convergence.

1.3.5. Differentiation of Fourier series. We now give a general criterion for differentiating a Fourier series.

PROPOSITION 1.3.3. *Let $f(x)$, $-L < x < L$, be a continuous piecewise smooth function with $f(L-0) = f(-L+0)$. Then*

$$\frac{1}{2}[f'(x+0) + f'(x-0)] = \sum_{n=1}^{\infty} \frac{n\pi}{L}\left(B_n \cos\frac{n\pi x}{L} - A_n \sin\frac{n\pi x}{L}\right)$$

Proof. It suffices to apply the convergence theorem to the piecewise smooth function $f'(x), -L < x < L$. Its Fourier coefficients are given by

$$\begin{aligned}
A_0' &= \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L}\left(f(L-0) - f(-L+0)\right) \\
A_n' &= \frac{1}{2L}\int_{-L}^{L} f'(x)\cos(n\pi x/L)\,dx = \frac{n\pi}{L}\int_{-L}^{L} f\,(x)\sin(n\pi x/L)\,dx = \frac{n\pi}{L}B_n \\
B_n' &= \frac{1}{2L}\int_{-L}^{L} f'(x)\sin(n\pi x/L)\,dx = -\frac{n\pi}{L}\int_{-L}^{L} f\,(x)\cos(n\pi x/L)\,dx = -\frac{n\pi}{L}A_n
\end{aligned}$$

where we have integrated by parts and used the continuity of $f(x), -L < x < L$. The result now follows from Theorem 1.1. •

For example, suppose that we want to compute the Fourier series of $f(x) = x^2$, $-L < x < L$. The Fourier series of this even function is of the form $A_0 + \sum_{n=1}^{\infty} A_n \cos nx$, where $\{A_n\}$ are to be determined. From Proposition 1.3.3 we may write

$$2x = -\sum_{n=1}^{\infty} nA_n \sin nx$$

But from Example 1.1.1, Sec. 1.1, we know that

$$2x = 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

[3] A. Zygmund, *Trigonometrical Series*, Dover Publications, New York, 1955, p. 108.

Therefore $A_n = 4(-1)^n/n^2$ for $n = 1, 2, \ldots$. To compute A_0 we must return to the definition $A_0 = (1/2\pi)\int_{-\pi}^{\pi} x^2 dx = \pi^2/3$. Therefore we have the Fourier series

$$x^2 = \pi^2/3 + 4\sum_{n=1}^{\infty}[(-1)^n/n^2]\cos nx \qquad -\pi < x < \pi$$

1.3.6. Integration of Fourier series. The following proposition shows that a Fourier series may be integrated term by term under very general conditions.

PROPOSITION 1.3.4. *Let $f(x)$, $-\pi < x < \pi$, be a piecewise smooth function with Fourier series*

$$A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx)$$

If $-\pi \le x_0 < x \le \pi$, then

$$\begin{aligned}\int_{x_0}^{x} f(u)du &= A_0(x - x_0)\\ &\quad + \sum_{n=1}^{\infty}\left[\frac{A_n}{n}(\sin nx - \sin nx_0) + \frac{B_n}{n}(\cos nx_0 - \cos nx)\right]\end{aligned}$$

Proof. Let $F(x) = \int_{-\pi}^{x}[f(u) - A_0]du$. F is continuous and piecewise smooth with $F(-\pi) = F(\pi)$. Therefore by the basic convergence theorem (Theorem 1.1) we have

$$F(x) = \bar{A}_0 + \sum_{n=1}^{\infty}(\bar{A}_n \cos nx + \bar{B}_n \sin nx) \qquad -\pi \le x \le \pi$$

where $(\bar{A}_n, \bar{B}_n)$ are the Fourier coefficients of F. To compute these, we have, for $n \neq 0$,

$$\begin{aligned}\bar{A}_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\, dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \left\{\int_{-\pi}^{x}[f(u) - A_0]du\right\} dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}[f(u) - A_0]\left(\int_{u}^{\pi}\cos nx\, dx\right) du\\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi}[f(u) - A_0]\frac{\sin nu}{n}\, du\\ &= -\frac{B_n}{n}\end{aligned}$$

In the same fashion, we have

$$\begin{aligned}\bar{B}_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx \\ &= \frac{1}{n\pi}\int_{-\pi}^{\pi}[f(u)-A_0][\cos nu - \cos n\pi]du \\ &= \frac{A_n}{n}\end{aligned}$$

Recalling the definition of $F(x)$, we have proved that

$$\begin{aligned}\int_{-\pi}^{x} f(u)du &= A_0(x+\pi)+\bar{A}_0 \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{n}(A_n \sin nx - B_n \cos nx) \qquad -\pi \le x \le \pi\end{aligned}$$

If we replace x by x_0 and subtract the result, then $\bar{A}_0$ cancels and we have proved the stated result. •

1.3.7. A continuous function with a divergent Fourier series. This example is constructed by a particular grouping of the terms in a special trigonometric series. Explicitly, we define the finite trigonometric sums

$$C_n(x) = \cos(N_n+1)x + \frac{\cos(N_n+2)x}{2} + \cdots + \frac{\cos(N_n+m_n)x}{m_n} \tag{1.3.2}$$

$$D_n(x) = \cos(N_n-1)x + \frac{\cos(N_n-2)x}{2} + \cdots + \frac{\cos(N_n-m_n)x}{m_n} \tag{1.3.3}$$

and the function

$$f(x) = \sum_{n=1}^{\infty}\frac{C_n(x)-D_n(x)}{n^2} \tag{1.3.4}$$

The integers N_n, m_n will be chosen so that

$$|C_n(x)-D_n(x)| \le 8 \qquad -\pi \le x \le \pi, \quad n = 1,2,\ldots \tag{1.3.5}$$

$$\frac{C_n(0)}{n^2} \to \infty \qquad \frac{D_n(0)}{n^2} \to \infty \qquad n \to \infty \tag{1.3.6}$$

$$N_n + m_n < N_{n+1} - m_{n+1} \quad n = 1,2,\ldots \tag{1.3.7}$$

To do this, we use the following two facts:

$$\sum_{k=1}^{n} \frac{1}{k} > \log n \qquad n = 1, 2, \ldots \tag{1.3.8}$$

$$\Big|\sum_{k=1}^{n} \frac{\sin kx}{k}\Big| \le 4 \qquad n = 1, 2, \ldots, \quad -\pi \le x \le \pi \tag{1.3.9}$$

To prove (1.3.5), we use the trigonometric identity

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

to write

$$C_n(x) - D_n(x) = -2 \sin(N_n x) \left(\sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin m_n x}{m_n} \right)$$

From (1.3.9) the second factor is less than or equal to 4, and we have proved (1.3.5).

To prove (1.3.6), we write

$$C_n(0) = D_n(0) = 1 + \frac{1}{2} + \cdots + m_n > \log(m_n)$$

If we choose $m_n = 2^{n^3}$, then $\log(m_n) = n^3 \log 2$, and thus $C_n(0)/n^2$ and $D_n(0)/n^2$ tend to ∞, as required.

To prove (1.3.7) we define $N_1 = 3$ and for $n > 1$, $N_{n+1} - N_n = 2m_{n+1}$. With this choice, it immediately follows that $N_{n+1} - N_n > m_{n+1} + m_n$, as required.

Having defined N_n, m_n, it follows from (1.3.5) that the series (1.3.4) is uniformly convergent and therefore $f(x), -\pi < x < \pi$, is a continuous function. It remains to compute the Fourier series of f.

Since $f(x), -\pi < x < \pi$, is an even function, the Fourier sine coefficients $B_n \equiv 0$. To compute the Fourier cosine coefficients, we may multiply the uniformly convergent series (1.3.4) by $\cos nx$ and integrate on $-\pi < x < \pi$. From (1.3.7) there is exactly one nonzero term corresponding to each integer of the form $n = N_k \pm 1, \ldots, N_k \pm m_k$. These nonzero terms are of the form

$$A_n = \pm \frac{1}{j} \quad \text{if} \quad n = N_k \pm j \qquad 1 \le j \le m_k$$

In particular, the partial sums at $x = 0$ satisfy

$$\begin{aligned}
f_{N_k+m_k}(0) &= (C_1(0) - D_1(0)) + \cdots + \frac{C_k(0) - D_k(0)}{k^2} \\
f_{N_{k+1}}(0) &= (C_1(0) - D_1(0)) + \cdots + \frac{C_k(0) - D_k(0)}{k^2} - \frac{D_{k+1}(0)}{(k+1)^2} \\
f_{N_k+m_k}(0) - f_{N_{k+1}}(0) &= \frac{D_{k+1}(0)}{(k+1)^2}
\end{aligned}$$

If the sequence of partial sums $f_n(0)$ were convergent, it would follow that $\lim(f_{N_k+m_k}(0) - f_{N_{k+1}}(0)) = 0$, which contradicts (1.3.6). Therefore the Fourier series diverges at $x = 0$, which was to be proved.

EXERCISES 1.3

1. Let
$$f_{2n-1}(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{2n-1}\sin(2n-1)x\right]$$
Show that
$$f_{2n-1}\left(\frac{k\pi}{2n}\right) \to \frac{2}{\pi}\int_0^{k\pi}\frac{\sin x}{x}\,dx \qquad k = 1, 2, \ldots$$
[*Hint:* Write the sum for $f_{2n-1}(k\pi/2n)$ as the approximating sum for an appropriate Riemannian integral.]
2. Estimate the integral $\int_0^{k\pi}(\sin x)/x\,dx$ for $k = 2, 3, 4$.
3. Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Show that the first criterion for uniform convergence follows from the Weierstrass M-test (Appendix A.2.).
4. Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Show that $A_n = O(1/n)$, $B_n = O(1/n)$ when $n \uparrow \infty$.
5. Let $f(x)$, $-L < x < L$, be a piecewise smooth function. Let A'_n, B'_n be the Fourier coefficients of f'.
$$\begin{aligned} A'_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx \\ B'_n &= \frac{1}{L}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}$$
If f is continuous and $f(-L+0) = f(L-0)$, show that
$$A'_n = \frac{n\pi}{L}B_n \qquad B'_n = -\frac{n\pi}{L}A_n$$
6. Let $f(x)$, $-L < x < L$, be a continuous piecewise smooth function with $f(-L+0) = f(L-0)$. Use Exercises 4 and 5 to show that $A_n = O(1/n^2)$, $B_n = O(1/n^2)$ when $n \uparrow \infty$.
7. Let $f(x)$, $-L < x < L$, be a continuous piecewise smooth function with $f(-L+0) = f(L-0)$. Use Exercise 6 to prove the second criterion for uniform convergence (Proposition 1.3.2).
8. Use Exercise 5 and the main convergence theorem (Theorem 1.1) to prove the proposition on differentiating a Fourier series.

9. Let $f(x) = \sum_{n=1}^{\infty} e^{-(n^2\pi/L^2)} \sin n\pi x/L$ be the Fourier series of a piecewise smooth function. Show that

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\pi}{L} e^{-(n^2\pi/L^2)} \cos \frac{n\pi x}{L}$$
$$f''(x) = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 e^{-(n^2\pi/L^2)} \sin \frac{n\pi x}{L}$$

10. Consider the Fourier series of $f(x) = x$ found in Example 1.1.1, Sec. 1.1. By formally differentiating the series at $x = 0$, show that it is not valid to differentiate a Fourier series term by term, even if the function is differentiable.
11. Consider the Fourier series of $f(x) = x$ found in Example 1.1.1, Sec. 1.1. By integrating this series, find a series for x^2.
12. Integrate the series of Exercise 11 and compare the result with Example 1.1.5.
13. Among the series for x, x^2, and $x^3 - L^2x$ found in Exercises 10 to 12, which are uniformly convergent?
14. Let $f(x) = x$, $-\pi < x < \pi$. Find the maximum of the partial sum $f_N(x)$ and verify the presence of Gibb's phenomenon.
15. This exercise provides the missing steps in the proof of (1.3.9).

(i) If $0 \le x \le \pi$, establish the identity

$$\sin x + \frac{\sin 2x}{2} + \cdots + \frac{\sin nx}{n} = \int_0^x (\cos t + \cdots + \cos nt)\, dt = \int_0^x \frac{\sin(n + (1/2))t}{2\sin(t/2)}\, dt - \frac{x}{2}$$

(ii) Rewrite the integral in (i) as

$$\int_0^x \sin(n + (1/2))t \left(\frac{1}{2\sin(t/2)} - \frac{1}{t}\right) dt + \int_0^x \frac{\sin(n + (1/2))t}{t}\, dt$$

(iii) Use the inequalities $|\sin\theta - \theta| \le \theta^3/6$, $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi$ to bound the first integral in the form

$$\left|\int_0^x \sin(n + (1/2))t \left(\frac{1}{2\sin(t/2)} - \frac{1}{t}\right) dt\right| \le \frac{\pi}{48}\int_0^x t\, dt = \frac{\pi x^2}{96}$$

(iv) Make the change of variable $u = (n + 1/2)t$ in the second integral to prove that

$$\left|\int_0^x \frac{\sin(n + (1/2))t}{t}\, dt\right| \le \int_0^\pi \frac{\sin u}{u} du = 1.852$$

(v) Conclude that $|\sin x + \cdots + (\sin nx)/n| \le 3.75$ for $0 \le x \le \pi$.

1.4. Parseval's Theorem and Mean Square Error

Having developed the convergence properties of Fourier series, we now turn to some concrete computations that show how Fourier series may be used in various problems.

1.4.1. Statement and proof of Parseval's theorem. The key to these applications is *Parseval's theorem*, a form of the pythagorean theorem that is valid in the setting of Fourier series.

THEOREM 1.2. *(Parseval's theorem). Let $f(x)$, $-L < x < L$, be a piecewise smooth function with Fourier series*

$$A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

Then

$$\boxed{\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(A_n^2 + B_n^2)} \tag{1.4.1}$$

The left side represents the mean square of the function $f(x)$, $-L < x < L$. The right side represents the sum of the squares of the Fourier components in the various coordinate directions $\cos n\pi x/L$, $\sin n\pi x/L$.

Proof. The proof of Parseval's theorem is especially simple if the piecewise smooth function is also continuous with $f(-L+0) = f(L-0)$. In that case we multiply the uniformly convergent Fourier series by $f(x)$ to obtain

$$f(x)^2 = A_0 f(x) + \sum_{n=1}^{\infty}\left[A_n f(x)\cos\frac{n\pi x}{L} + B_n f(x)\sin\frac{n\pi x}{L}\right]$$

This series is also uniformly convergent, and we may integrate term by term for $-L < x < L$, with the result

$$\begin{aligned}\int_{-L}^{L} f(x)^2 dx &= A_0\int_{-L}^{L} f(x)dx \\ &\quad+\sum_{n=1}^{\infty}\left[A_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} + B_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\right]\end{aligned}$$

On the right we recognize the integrals that define the Fourier coefficients A_0, A_n, B_n. Dividing both sides by $2L$, we obtain equation (1.4.1), the desired form of Parseval's theorem in this case. The proof in the general case is outlined in the exercises. •

1.4.2. Application to mean square error. Our first application of Parseval's theorem is to the *mean square error* σ_N^2, defined by

$$\boxed{\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L} [f(x) - f_N(x)]^2 dx} \tag{1.4.2}$$

This number measures the average amount by which $f_N(x)$ differs from $f(x)$. The Fourier series of $f(x) - f_N(x)$ is

$$\sum_{n=N+1}^{\infty} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

and therefore, by Parseval's theorem, we have

$$\frac{1}{2L}\int_{-L}^{L} [f(x) - f_N(x)]^2 dx = \frac{1}{2}\sum_{n=N+1}^{\infty} (A_n^2 + B_n^2)$$

and the formula

$$\boxed{\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty} (A_n^2 + B_n^2)} \tag{1.4.3}$$

The mean square error is half the sum of the squares of the remaining Fourier coefficients. This formula shows, in particular, that the mean square error tends to zero when N tends to infinity.

EXAMPLE 1.4.1. *Let $f(x) = |x|$, $-\pi < x < \pi$. Find the mean square error and give an asymptotic estimate when $N \to \infty$.*

Solution. We have $B_n = 0$, $A_{2m} = 0$, $A_{2m-1} = -4/\pi(2m-1)^2$, so that

$$\begin{aligned}
\sigma_{2N-1}^2 = \sigma_{2N}^2 &= \frac{1}{2}\sum_{n=2N+1}^{\infty} A_n^2 \\
&= \frac{1}{2}\sum_{m=N+1}^{\infty} \left[\frac{4}{\pi(2m-1)^2}\right]^2 \\
&= \frac{8}{\pi^2}\sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4}
\end{aligned}$$

Although we cannot make a closed-form evaluation of this series, we can still make a useful *asymptotic* estimate. To do this, we compare the sum with the integral

$$\frac{8}{\pi^2}\int_N^{\infty} \frac{1}{(2x-1)^4}dx = \frac{4}{3\pi^2}\frac{1}{(2N-1)^3}$$

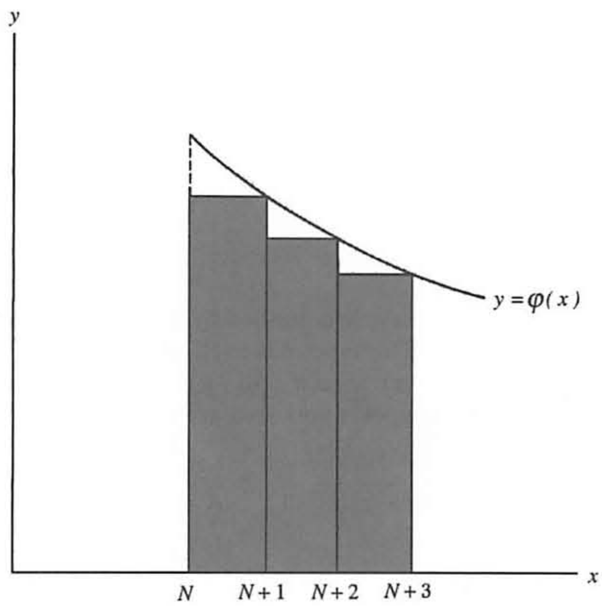


FIGURE 1.4.1 Illustrating the relation $\sum_{m=N+1}^{\infty}\varphi(m) \le \int_N^{\infty}\varphi(x)\,dx$.

Figure 1.4.1 shows the comparison of a sum with an integral. This gives us the useful asymptotic statement

$$\sigma_N^2 = O(N^{-3}) \qquad N \to \infty$$

EXAMPLE 1.4.2. *Let $f(x) = x$, $-\pi < x < \pi$. Find the mean square error and give an asymptotic estimate when $N \to \infty$.*

Solution. We have $A_m = 0$, $B_m = (-1)^{m-1}(2/m)$, and therefore

$$\sigma_N^2 = \frac{1}{2}\sum_{m=N+1}^{\infty}\frac{4}{m^2} = 2\sum_{m=N+1}^{\infty}\frac{1}{m^2}$$

To obtain a useful asymptotic estimate of this sum, we compare it with the integral

$$2\int_N^{\infty}\frac{dx}{x^2} = \frac{2}{N}$$

so that

$$\sigma_N^2 = O(N^{-1}) \qquad N \to \infty \quad \bullet$$

1.4.3. Application to the isoperimetric theorem. We now give an application of Fourier series to geometry, the so-called isoperimetric theorem.

THEOREM 1.3. *Suppose that we have a smooth closed curve in the xy plane that encloses an area A and has perimeter P. Then*

$$P^2 \geq 4\pi A$$

with equality if and only if the curve is a circle.

Proof. Suppose that the curve is described by parametric equations $x = x(t)$, $y = y(t)$ where $-\pi \leq t \leq \pi$. The functions $x(t)$, $y(t)$ are supposed smooth and satisfy the normalization $x(-\pi) = x(\pi)$, $y(-\pi) = y(\pi)$ because the curve is closed. From calculus, the perimeter and area are given by the formulas

$$P = \int_{-\pi}^{\pi} \sqrt{x'(t)^2 + y'(t)^2}\,dt \qquad A = \int_{-\pi}^{\pi} x(t)y'(t)\,dt$$

where $x' = dx/dt$, $y' = dy/dt$. By reparametrizing the curve, we may suppose that $x'(t)^2 + y'(t)^2$ is constant (see Exercise 20); in fact, it must be

$$x'(t)^2 + y'(t)^2 = \frac{P^2}{4\pi^2}$$

Now we introduce the convergent Fourier series

$$\begin{aligned} x(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad -\pi \leq t \leq \pi \\ y(t) &= c_0 + \sum_{n=1}^{\infty}(c_n \cos nt + d_n \sin nt) \qquad -\pi \leq t \leq \pi \end{aligned}$$

Since the functions $x(t)$, $y(t)$ are supposed smooth, we also have the convergent Fourier series

$$\begin{aligned} x'(t) &= \sum_{n=1}^{\infty} n(-a_n \sin nt + b_n \cos nt) \qquad -\pi \leq t \leq \pi \\ y'(t) &= \sum_{n=1}^{\infty} n(-c_n \sin nt + d_n \cos nt) \qquad -\pi \leq t \leq \pi \end{aligned}$$

Applying Parseval's theorem, we have

$$\begin{aligned}
\frac{P^2}{2\pi} &= \int_{-\pi}^{\pi} [x'(t)^2 + y'(t)^2]dt = \pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2 + c_n^2 + d_n^2) \\
A &= \int_{-\pi}^{\pi} x(t)y'(t)dt \\
&= \frac{1}{4}\int_{-\pi}^{\pi} \{[x(t) + y'(t)]^2 - [x(t) - y'(t)]^2\}dt \\
&= \pi \sum_{n=1}^{\infty} n(a_n d_n - b_n c_n)
\end{aligned}$$

Performing the necessary algebraic steps, we have

$$\frac{P^2}{2\pi} - 2A = \pi \sum_{n=1}^{\infty} [n(a_n - d_n)^2 + n(b_n + c_n)^2 + n(n-1)(a_n^2 + b_n^2 + c_n^2 + d_n^2)]$$

The right side is a sum of squares with nonnegative coefficients; thus $P^2/2\pi - 2A \geq 0$. If the sum is zero, then all of the terms are zero; in particular, $a_n^2 + b_n^2 + c_n^2 + d_n^2 = 0$ for $n > 1$ and $a_1 - d_1 = 0$, $b_1 + c_1 = 0$. This means that

$$\begin{aligned}
x(t) &= a_0 + a_1 \cos t - c_1 \sin t \qquad -\pi \leq t \leq \pi \\
y(t) &= c_0 + c_1 \cos t + a_1 \sin t \qquad -\pi \leq t \leq \pi
\end{aligned}$$

which is the equation of a circle of radius $\sqrt{a_1^2 + c_1^2}$ with center at (a_0, c_0). The proof is complete. •

EXERCISES 1.4

Find the mean square errors for the Fourier series of the functions in Exercises 1 to 3.

1. $f(x) = 1$ for $0 < x < \pi, f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$.
2. $f(x) = x^2$, $-\pi \leq x \leq \pi$
3. $f(x) = \sin 10x$, $-\pi < x < \pi$
4. Write out Parseval's theorem for the Fourier series of Exercise 1.
5. Write out Parseval's theorem for the Fourier series of Exercise 2.
6. Show that, in Exercise 1, $\sigma_N^2 = O(N^{-1})$, $N \uparrow \infty$.
7. Show that, in Exercise 2, $\sigma_N^2 = O(N^{-3})$, $N \uparrow \infty$.
8. Let $f(x) = x(\pi - x)$, $0 \leq x \leq \pi$.
   (a) Compute the Fourier sine series of f.
   (b) Compute the Fourier cosine series of f.
   (c) Find the mean square error incurred by using N terms of each series and find asymptotic estimates when $N \to \infty$.
   (d) Which series gives a better mean square approximation of f?

9. Let $f(x)$, $g(x)$, $-L \le x \le L$, be piecewise smooth functions with Fourier series

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right) \\ g(x) &= C_0 + \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi x}{L} + D_n \sin\frac{n\pi x}{L}\right) \end{aligned}$$

Show that

$$\frac{1}{2L}\int_{-L}^{L} f(x)g(x)dx = A_0C_0 + \frac{1}{2}\sum_{n=1}^{\infty}(A_nC_n + B_nD_n)$$

Note that this formula corresponds to the dot product formula

$$(a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})\cdot(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = a_1a_2 + b_1b_2 + c_1c_2$$

for vectors in the three-dimensional space $\mathbf{R}^3$.

10. Let $f(x) = (\cos ax/\sin a\pi)$, $-\pi \le x \le \pi$, where $0 < a < \frac{1}{2}$.
 (a) Find the Fourier series of f.
 (b) Give an asymptotic estimate for the mean square error incurred in approximating f by the first N terms of the Fourier series.
 (c) Apply Parseval's theorem to obtain the following integral formula:

$$\sum_{n=-\infty}^{\infty}(a^2 - n^2)^{-2} = \frac{\pi}{2}(a\sin a\pi)^{-2}\int_{-\pi}^{\pi}\cos^2 ax\,dx$$

 (d) Prove that $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$. [*Hint:* Make a three-term Taylor expansion of part (c) in powers of a and identify the coefficients.]

11. Let $\varphi(x)$ be defined for $x > 0$ with $\varphi(x) > 0$, $\varphi'(x) < 0$, and the integral $\int_1^{\infty}\varphi(x)dx$ convergent.
 (a) Show that

$$\int_{N+1}^{\infty}\varphi(x)dx \le \sum_{n=N+1}^{\infty}\varphi(n) \le \int_{N}^{\infty}\varphi(x)dx$$

 (b) Deduce from this that

$$-\varphi(N) \le \sum_{n=N+1}^{\infty}\varphi(n) - \int_{N}^{\infty}\varphi(x)dx \le 0$$

12. Let $\varphi(x) = 1/x^s$ where $s > 1$.
 (a) Use Exercise 11 to show that

$$\frac{-1}{N^s} \le \sum_{n=N+1}^{\infty}\frac{1}{n^s} - \frac{1}{s-1}\frac{1}{N^{s-1}} \le 0$$

(b) Show that this may be written in the form

$$\sum_{n=N+1}^{\infty} \frac{1}{n^s} = \frac{1}{(s-1)N^{s-1}} \left[1 + O\left(\frac{1}{N}\right)\right] \qquad N \to \infty$$

13. Let σ_N^2 be the mean square error in the Fourier series of $f(x) = x$, $-\pi < x < \pi$. Use Exercise 12 to show that $\sigma_N^2 = (1/N)[1 + O(1/N)]$, $N \to \infty$.
14. Let $\varphi(x) = 1/P(x)$ where $P(x)$ is a polynomial of degree s, $s > 1$. Modify Exercise 11(b) to show that

$$\sum_{n=N+1}^{\infty} \varphi(n) = \int_N^{\infty} \varphi(x)dx \left[1 + O\left(\frac{1}{N}\right)\right] \qquad N \to \infty$$

15. Let σ_N^2 be the mean square error in the Fourier series of $f(x) = |x|$, $-\pi < x < \pi$. Use Exercise 14 to find an asymptotic estimate of the form $\sigma_N^2 = (C/N^s)[1 + O(1/N)]$, $N \to \infty$ for appropriate constants C, s.
16. Let $\varphi(x) = e^{-x}$, $x > 0$. Discuss the validity of the asymptotic estimate

$$\sum_{n=N+1}^{\infty} \varphi(n) = \int_N^{\infty} \varphi(x)dx[1 + O(1/N)] \qquad N \to \infty$$

17. Compute the ratio P^2/A for an equilateral triangle.
18. Compute the ratio P^2/A for a square.
19. Compute the ratio P^2/A for a regular polygon of n sides and compare it with the isoperimetric theorem in the limit when $n \to \infty$.
20. Let $x(t)$, $y(t)$ be smooth functions, $-\pi \le t \le \pi$, with $(x')^2 + (y')^2 \ne 0$. Let $s(t) = \int_{-\pi}^{t} \sqrt{(x')^2 + (y')^2}$, $P = s(\pi)$, $\tilde{t} = -\pi + (2\pi s/P)$, $\tilde{x}(\tilde{t}) = x(t)$, $\tilde{y}(\tilde{t}) = y(t)$. Show that $-\pi \le \tilde{t} \le \pi$ and $(d\tilde{x}/d\tilde{t})^2 + (d\tilde{y}/d\tilde{t})^2 = P^2/4\pi^2$.

The following exercises are designed to lead to a proof of Parseval's theorem for piecewise smooth functions.

21. Let $f(x), -L < x < L$, be a piecewise smooth function. Show that for each $\epsilon > 0$, there is a continuous piecewise smooth function $f^*(x), -L < x < L$, with $f^*(-L+0) = f^*(L-0)$ such that $(1/2L)\int_{-L}^{L}[f(x) - f^*(x)]^2\,dx < \epsilon$. [*Hint:* Across each subdivision point replace f by a linear function on the interval $x_i - h < x < x_i + h$, where h is chosen in terms of ϵ, p and the maximum of $|f(x)|, -L < x < L$.]
22. Let $f(x), -L < x < L$, be a piecewise smooth function and let $f^*(x), -L < x < L$, be the continuous function constructed in the previous exercise. Use Proposition 0.3.2 to show that $||f - f_N|| \le ||f - f_N^*||$, where f_N is the Nth partial sum of the Fourier series for the function $f(x), -L < x < L$, and f_N^* is the Nth partial sum of the Fourier series for the function $f^*(x), -L < x < L$.

23. Use the triangle inequality from Sec. 0.3 to prove the inequality $||f - f_N^*|| \le ||f - f^*|| + ||f^* - f_N^*||$.
24. Show that there is an integer N_0 so that for $N \ge N_0$ we have $||f - f_N^*|| < \epsilon$. [*Hint:* Combine Proposition 0.3.4 with the Parseval theorem already proved for the function $f^*(x), -L < x < L$.]
25. Conclude the validity of Parseval's theorem for the piecewise smooth function $f(x), -L < x < L$.

1.5. Complex Form of Fourier Series

1.5.1. Fourier series and Fourier coefficients. It is often useful to rewrite the formulas of Fourier series using complex numbers. To do this, we begin with Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.5.1}$$

and the immediate consequences

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

We apply these to a Fourier series:

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right) \\ &= A_0 + \frac{1}{2}\sum_{n=1}^{\infty}[(A_n - iB_n)e^{in\pi x/L} + (A_n + iB_n)e^{-(in\pi x/L)}] \end{aligned}$$

Therefore we let $\alpha_n = \frac{1}{2}(A_n - iB_n)$, $n = 1, 2, \ldots$; $\alpha_n = \frac{1}{2}(A_{-n} + iB_{-n})$, $n = -1, -2, \ldots$; and $\alpha_0 = A_0$. With this convention the Fourier series assumes the form

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}} \tag{1.5.2}$$

To obtain integral formulas for the coefficients $\{\alpha_n\}$, we use (1.1.7) and (1.1.8),

$$\begin{aligned} 2\alpha_n = (A_n - iB_n) &= \frac{1}{L}\int_{-L}^{L} f(x) \times \left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)e^{-(in\pi x/L)}dx \end{aligned}$$

with a corresponding formula for the plus sign. When $n = 0$, (1.1.6) shows that α_0 is given appropriately. Thus we have

$$\boxed{\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-(in\pi x/L)}\, dx \qquad n = 0, \pm1, \pm2, \ldots} \tag{1.5.3}$$

1.5.2. Parseval's theorem in complex form. Finally, we retrieve the appropriate form of Parseval's theorem. To do this, multiply (1.5.2) by $f(x)$ and integrate on $(-L, L)$. The result is

$$\boxed{\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2} \tag{1.5.4}$$

1.5.3. Applications and examples. The functions $e^{(in\pi x/L)}$ satisfy an orthogonality relation, which may be written in the form

$$\int_{-L}^{L} e^{(in\pi x/L)} e^{-(im\pi x/L)} dx = \begin{cases} 0 & n \neq m \\ 2L & n = m \end{cases}$$

These may be proved by using Euler's formula and the orthogonality of the trigonometric functions $\cos(n\pi x/L)$, $\sin(n\pi x/L)$. Knowing these orthogonality relations, we can develop the complex form of Fourier series in its own right, without reference to the original formulas of Sec. 1.1.

The theory of Fourier series may also be extended to *complex-valued functions* $f(x)$, $-L < x < L$. These are of the form $f(x) = f_1(x) + if_2(x)$, where f_1, f_2 are real-valued functions. The Fourier coefficients are defined by the same formulas $\alpha_n = (1/2L)\int_{-L}^{L} f(x)e^{-(in\pi x/L)}dx$. If both f_1 and f_2 are piecewise smooth functions, then the complex Fourier series converges for all x to $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$, where $\bar{f}$ is the periodic extension of the piecewise smooth function $f(x)$, $-L < x < L$. This convergence is understood as the limit of the sum $\sum_{-N}^{N}$ when N tends to infinity.

The Fourier coefficients of a real-valued function are characterized by the relation

$$\alpha_{-n} = \bar{\alpha}_n$$

where the bar indicates the *complex conjugate* of a complex number: if $c = a+ib$, then $\bar{c} = a - ib$.

To simplify the computation of complex Fourier series, we indicate some formulas that are of frequent use. If $c = a+ib$ is a complex number, the exponential function $e^{cx} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx)$. From this we have $(d/dx)e^{cx} = ae^{ax}\cos bx - be^{ax}\sin bx + i(ae^{ax}\sin bx + be^{ax}\cos bx) = (a+ib)e^{ax}(\cos bx + i\sin bx) = ce^{cx}$. Hence the differentiation formula

$$\frac{d}{dx}e^{cx} = ce^{cx}$$

is valid for any complex number c.

EXAMPLE 1.5.1. *Compute the complex Fourier series of $f(x) = e^{ax}$, $-\pi < x < \pi$, where a is a real number.*

Solution. The Fourier coefficients are given by the formula

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx$$

Noting that $(d/dx)e^{(a-in)x} = (a-in)e^{(a-in)x}$, we have

$$\begin{aligned}\alpha_n &= \frac{1}{2\pi}\frac{1}{a-in}\left(e^{(a-in)\pi} - e^{(a-in)(-\pi)}\right)\\ &= \frac{1}{2\pi}\frac{1}{a-in}(-1)^n(e^{a\pi} - e^{-a\pi})\\ &= \frac{1}{\pi}\sinh a\pi\frac{(-1)^n(a+in)}{a^2+n^2}\end{aligned}$$

The complex Fourier series of $f(x) = e^{ax}$, $-\pi < x < \pi$, is

$$\frac{1}{\pi}\sinh a\pi\sum_{n=-\infty}^{\infty}\frac{(-1)^n(a+in)}{a^2+n^2}e^{inx} \quad \bullet$$

As our next application of complex Fourier series, we compute the Fourier series of

$$f(x) = \cos^m x \qquad -\pi < x < \pi$$

If we were to use the real form of Fourier series, we would encounter many cumbersome trigonometric identities. With the complex approach, we avoid these. We begin with the identity

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

We expand the mth power, using the binomial theorem:

$$(e^{ix} + e^{-ix})^m = \sum_{j=0}^{m}\binom{m}{j}e^{ijx}e^{-i(m-j)x}$$

Therefore

$$\cos^m x = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}e^{i(2j-m)x}$$

This is the complex form of the Fourier series for $\cos^m x$. As a by-product, we can obtain some useful integrals. To do this, we multiply the previous equation by e^{-inx} and integrate for $-\pi < x < \pi$. By orthogonality all the integrals are zero except when $2j - m - n = 0$, in which case the integral is 2π. In particular, $m + n$ must be even. Therefore we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos x)^m e^{-inx}dx = \begin{cases} 0 & m+n \text{ odd}\\ \frac{1}{2^m}\binom{m}{j} & 0 \le m+n = 2j \le 2m\end{cases}$$

The Fourier series for $\cos^m x$ can also be written in a real form, to obtain familiar trigonometric identities. It is simpler to consider separately the cases m even and m odd. Thus, if $m = 2k+1$, we can group the terms of the Fourier series in pairs: $j = 0$ with $j = m$ and $j = 1$ with $j = m - 1$, etc. To each pair, we apply Euler's formula, with the result

$$\cos^{2k+1} x = \left(\frac{1}{2}\right)^{2k} \left[\cos(2k+1)x + \cdots + \binom{2k+1}{k} \cos x\right]$$

In particular, this gives the identities

$$\begin{aligned} \cos^3 x &= \frac{1}{4}(\cos 3x + 3\cos x) \\ \cos^5 x &= \frac{1}{16}(\cos 5x + 5\cos 3x + 10\cos x) \end{aligned}$$

If m is even, we group the term $j = 0$ with $j = m$, etc., as before and finish with one ungrouped term in the middle. Applying Euler's theorem again, we have, with $m = 2k$,

$$\cos^{2k} x = \left(\frac{1}{2}\right)^{2k} \left[2\cos 2kx + \cdots + 2\binom{2k}{k-1} \cos 2x + \binom{2k}{k}\right]$$

In particular, we retrieve the identities

$$\begin{aligned} \cos^2 x &= \frac{1}{2}(\cos 2x + 1) \\ \cos^4 x &= \frac{1}{8}(\cos 4x + 4\cos 2x + 3) \end{aligned}$$

1.5.4. Fourier series of mass distributions. The theory of Fourier series is especially natural in the case of a *mass distribution*. This is defined by a *mass distribution function* $F(x), -L < x < L$, which can be any increasing function. The left and right limits are denoted by $F(x-0)$ and $F(x+0)$, respectively. The *mass of the interval* $a < x < b$ is defined by $m(a,b) = F(b-0) - F(a+0)$. The mass of a point is defined by $m(\{a\}) = F(a+0) - F(a-0)$.

For example, the Dirac δ distribution of mass m at the point x_0 is defined by setting $F(x) = 0$ for $x < x_0$ and $F(x) = m$ for $x > x_0$. At the other extreme, a mass distribution with density $f(x), -L < x < L$, is defined by the mass distribution function $F(x) = \int_{-L}^{x} f(y)\, dy$.

The Fourier coefficients of a mass distribution function are defined by the integrals

$$\alpha_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} dF(x) \qquad n = 0, \pm 1, \pm 2, \ldots$$

For $n = 0$ this is the total mass per unit length: $\alpha_0 = [F(L-0) - F(-L+0)]/(2L)$. The precise meaning for $n \neq 0$ can be defined by partial integration. If the mass distribution consists of several point masses plus a density, then each of the terms can be done separately.

EXAMPLE 1.5.2. *Find the Fourier coefficients of the mass distribution that consists of a uniform distribution of mass M on the interval $-L < x < L$, together with a Dirac δ distribution of mass m situated at the point $x = 0$.*

Solution. The mass distribution function is linear with a jump at the point $x = 0$. In detail, we have

$$F(x) = \begin{cases} (M/2L)(x+L) & \text{if } -L < x < 0 \\ m + (M/2L)(x+L) & \text{if } 0 < x < L \end{cases}$$

The Fourier coefficients are obtained as

$$\alpha_0 = (1/2L)(m+M), \quad \alpha_n = (m/2L) + (M/2L)\int_{-L}^{L} e^{-in\pi x/L}dx = (m/2L)$$

since the last integral is zero for $n \neq 0$. •

The following theorem shows that the theory of Fourier inversion of mass distributions is especially simple.

THEOREM 1.4. *(Convergence theorem). Suppose that $F(x), -L < x < L$, defines a mass distribution m with Fourier coefficients α_n. Define the Fourier partial sum by*

$$f_N(x) = \sum_{n=-N}^{N} \alpha_n e^{in\pi x/L} \qquad -L < x < L, \quad N = 1, 2, \ldots$$

Then if $a < b$, we have

$$\lim_{N\to\infty} \int_a^b f_N(x)\,dx = m(a,b) + \frac{1}{2}m(\{a\}) + \frac{1}{2}m(\{b\})$$

Proof. We can repeat the steps of the proof of Fourier convergence, noting that

$$\begin{aligned} f_N(x) &= \int_{-L}^{L} D_N(x-y)dF(y) \\ \int_a^b f_N(x)\,dx &= \int_{-L}^{L} \left(\int_a^b D_N(x-y)\,dx \right) dF(y) \end{aligned}$$

where D_N is the Dirichlet kernel introduced in Sec. 1.2. From the properties of the Dirichlet kernel proved there, it follows that

$$\begin{aligned} \lim_{N\to\infty} \int_a^b D_N(x-y)\,dx &= 1 \qquad a < y < b \\ \lim_{N\to\infty} \int_a^b D_N(x-y)\,dx &= 1/2 \quad y = a, b \\ \lim_{N\to\infty} \int_a^b D_N(x-y)\,dx &= 0 \qquad \text{otherwise} \end{aligned}$$

and that the integral is uniformly bounded by a constant. Therefore one may take the limit inside the sign of integration to obtain the result. •

EXERCISES 1.5

1. Verify that the orthogonality relations hold, in the form

$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx = \begin{cases} 0 & \text{if } n \neq m \\ 2L & \text{if } n = m \end{cases}$$

2. Use the formulas in Exercise 1 to prove (1.5.3) from (1.5.2). You may assume that the series (1.5.2) converges uniformly for $-L < x < L$.
3. Use the complex form to find the Fourier series of $f(x) = e^x$, $-L < x < L$.
4. Let $0 < r < 1$, $f(x) = 1/(1 - re^{ix})$, $-\pi < x < \pi$. Find the Fourier series of f. (*Hint:* First expand f as a power series in r.)
5. Use Exercise 4 to derive the real formulas

$$\frac{1 - r\cos x}{1 + r^2 - 2r\cos x} = 1 + \sum_{n=1}^{\infty} r^n \cos nx, \qquad 0 \leq r < 1$$

$$\frac{r\sin x}{1 + r^2 - 2r\cos x} = \sum_{n=1}^{\infty} r^n \sin nx, \qquad 0 \leq r < 1$$

6. Show that the convergence theorem from Sec. 1.2 can be written in complex form as

$$\frac{1}{2}[\tilde{f}(x+0) + \tilde{f}(x-0)] = \lim_{N\to\infty} \sum_{n=-N}^{N} \alpha_n e^{in\pi x/L}$$

7. Show that the unrestricted double limit

$$\lim_{M,N\to\infty} \sum_{n=-M}^{N} \alpha_n e^{in\pi x/L}$$

does not exist in general. (*Hint:* Try Example 1.1.4 at $x = 0$.)

In the following exercises, find the Fourier coefficients of the indicated mass distributions.

8. A mass m at the point x_0.
9. A row of three equally spaced masses of mass $m/3$ at the points $x = -L/2, x = 0, x = L/2$.
10. A uniform distribution of mass M on the interval $-L/2 < x < L/2$.
11. A triangular mass distribution described by the density function $f(x) = M(L - |x|)/L^2, -L < x < L$.

12. Theorem 1.4 in the text gives no information in case $a = b$. Show that in this case
$$m(\{a\}) = \lim_{N\to\infty} \frac{f_N(a)}{2N+1}$$
[*Hint:* Examine the behavior of $D_N(x)/(2N+1)$ when N is large.]
13. Show that the following analogue of Parseval's identity is valid:
$$\lim_{N\to\infty} \frac{\sum_{n=-N}^{N} |\alpha_n|^2}{2N+1} = \sum_a m(\{a\})^2$$
where the sum is over all of the point masses of the mass distribution.
14. A sequence of functions $f_n(x), -\pi < x < \pi$, is said to *converge weakly* to the function $f(x), -\pi < x < \pi$, if for every piecewise smooth function $g(x), -\pi < x < \pi$, we have
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f_n(x)\, g(x)\, dx = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$
Suppose that $f(x), -\pi < x < \pi$, is an arbitrary continuous function with Fourier partial sum $f_n(x), -\pi < x < \pi$. Prove that $f_n(x), -\pi < x < \pi$, converges weakly to $f(x), -\pi < x < \pi$. [*Hint:* First establish the identity $\int_{-\pi}^{\pi} f_n(x) g(x)\, dx = \int_{-\pi}^{\pi} g_n(x) f(x)$ where $g_n(x), -\pi < x < \pi$, is the Fourier partial sum of $g(x), -\pi < x < \pi$.]

1.6. Sturm-Liouville Eigenvalue Problems

Fourier series may be formulated as the orthogonal expansion in terms of functions $\phi(x)$ that are solutions of the differential equation

(1.6.1) $$\phi''(x) + \lambda\,\phi(x) = 0$$

on the interval $-L < x < L$ and that satisfy the periodic boundary conditions
$$\phi(-L) = \phi(L) \qquad \phi'(-L) = \phi'(L)$$
Indeed, the functions $\phi(x) = \sin(n\pi x/L)$ and $\phi(x) = \cos(n\pi x/L)$ satisfy these conditions with the value $\lambda = (n\pi/L)^2$.

More generally, we can study the solutions of the differential equation (1.6.1) that satisfy other sets of boundary conditions arising in problems of heat conduction and wave propagation. The general *two-point boundary condition* on the interval $a \le x \le b$ is written

(1.6.2) $$\cos\alpha\,\phi(a) - L\sin\alpha\,\phi'(a) = 0$$

(1.6.3) $$\cos\beta\,\phi(b) + L\sin\beta\,\phi'(b) = 0$$

where $L = b - a$ and α, β are dimensionless parameters that may be assumed to satisfy $0 \le \alpha < \pi, 0 \le \beta < \pi$. The number λ is called an *eigenvalue* and $\phi(x)$ is called an *eigenfunction* of the Sturm-Liouville (S-L) eigenvalue problem defined by (1.6.1), (1.6.2), and (1.6.3). Clearly, $\phi(x) \equiv 0$ is always a solution of the Sturm-Liouville eigenvalue problem, the so-called trivial solution. A solution $\phi(x)$ of (1.6.1), (1.6.2), and (1.6.3) that is not identically zero is called a *nontrivial solution.*

1.6.1. Examples of Sturm-Liouville eigenvalue problems. Fourier sine series and Fourier cosine series both arise from Sturm-Liouville problems with a two-point boundary condition on the interval $0 < x < L$. In the first case we use $\alpha = 0$, $\beta = 0$, corresponding to the boundary conditions $\phi(0) = 0, \phi(L) = 0$; in the second case we use $\alpha = \pi/2$, $\beta = \pi/2$ corresponding to the boundary conditions $\phi'(0) = 0, \phi'(L) = 0$.

The following worked examples demonstrate that no other solutions exist. In order to simplify the writing, we ignore arbitrary constants that may occur in the nontrivial solutions.

EXAMPLE 1.6.1. ($\alpha = 0$, $\beta = 0$) *Find all nontrivial solutions of (1.6.1) on the interval* $0 < x < L$ *satisfying the boundary conditions* $\phi(0) = 0, \phi(L) = 0$.

Solution. We consider separately the cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

In case $\lambda = 0$, the general solution of (1.6.1) is $\phi(x) = Ax+B$. The boundary conditions further require that $0 = \phi(0) = B$, $0 = \phi(L) = AL + B$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda = -\mu^2 < 0$, the general solution of (1.6.1) is $\phi(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions further require that $0 = A + B$, $0 = Ae^{\mu L} + Be^{-\mu L}$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda > 0$, the general solution is $\phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$. The boundary conditions further require that $0 = A$, $0 = A\cos(L\sqrt{\lambda}) + B\sin(L\sqrt{\lambda})$. A nontrivial solution is obtained by taking $B \neq 0$, $L\sqrt{\lambda} = n\pi$, where $n = 1, 2, \ldots$. Therefore we have found all of the eigenvalues and eigenfunctions, in the form

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \phi_n(x) = \sin\left(\frac{n\pi\, x}{L}\right) \qquad n = 1, 2, \ldots \quad \bullet$$

EXAMPLE 1.6.2. ($\alpha = \pi/2$, $\beta = \pi/2$) *Find all nontrivial solutions of (1.6.1) on the interval* $0 < x < L$ *satisfying the boundary conditions* $\phi'(0) = 0, \phi'(L) = 0$.

Solution. We consider separately the cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

In case $\lambda = 0$, the general solution of (1.6.1) is $\phi(x) = Ax+B$. The boundary conditions further require that $0 = \phi'(0) = A$, $0 = \phi'(L) = A$, which gives a nontrivial solution if and only if $A = 0$ and B is nonzero.

In case $\lambda = -\mu^2 < 0$, the general solution of (1.6.1) is $\phi(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions further require that $0 = \mu A - \mu B$, $0 = A\mu e^{\mu L} - B\mu e^{-\mu L}$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda > 0$, the general solution is $\phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$. The boundary conditions further require that $0 = \phi'(0) = B\sqrt{\lambda}$, $0 = \phi'(L) =$

$-A\sqrt{\lambda}\sin L\sqrt{\lambda} + B\sqrt{\lambda}\cos(L\sqrt{\lambda})$. A nontrivial solution is obtained by taking $B = 0$, $L\sqrt{\lambda} = n\pi$, where $n = 1, 2, \ldots$. Therefore we have found all of the eigenvalues and eigenfunctions, in the form

$$\lambda_0 = 0,\ \phi_0(x) = 1 \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2,\ \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, \ldots \quad \bullet$$

1.6.2. Some general properties of S-L eigenvalue problems. The solutions of Sturm-Liouville eigenvalue problems with two-point boundary conditions have some general properties, which are summarized in the following theorem.

THEOREM 1.5. *Consider the Sturm-Liouville eigenvalue problem represented by (1.6.1), (1.6.2), and (1.6.3).*

1. *Suppose that $\phi(x), \psi(x)$ are nontrivial solutions of (1.6.1)-(1.6.3) with the same eigenvalue λ. Then there is a constant $C \neq 0$ such that*
$$\phi(x) = C\psi(x)$$
2. *Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions of (1.6.1)-(1.6.3) with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal:*
$$\int_a^b \phi_1(x)\,\phi_2(x)\,dx = 0$$

Proof.

1. First consider the case $\alpha = 0$, where the boundary condition at the left end requires $\phi(a) = 0$, $\psi(a) = 0$. Both $\phi(x)$ and $\psi(x)$ satisfy the same second-order linear homogeneous differential equation, and so does any linear combination. We set

$$f(x) = \psi'(a)\phi(x) - \phi'(a)\psi(x)$$

The function $f(x), a < x < b$, also satisfies (1.6.1) and the initial conditions $f(a) = 0, f'(a) = 0$. This requires that $f(x) \equiv 0$. But if $\psi'(a) = 0$ (resp. $\phi'(a) = 0$), then $\psi(x) \equiv 0$ (resp. $\phi(x) \equiv 0$), a contradiction, so that we have proved (1) with the value $C = \phi'(a)/\psi'(a)$.

In the general case $\alpha \neq 0$, we set

$$f(x) = \psi(a)\phi(x) - \phi(a)\psi(x)$$

The function $f(x), a < x < b$, also satisfies (1.6.1) and the initial conditions $f(a) = 0, f'(a) = 0$. This requires that $f(x) \equiv 0$. But if $\psi(a) = 0$ (resp. $\phi(a) = 0$), then from (1.6.2) it follows that $\psi'(a) = 0$ (resp. $\phi'(a) = 0$), so that $\phi(x) \equiv 0$ (resp. $\psi(x) \equiv 0$), a contradiction. We have proved the theorem with the value $C = \phi(a)/\psi(a)$.

2. To prove the orthogonality, we write (1.6.1) for $\phi_1(x)$:

$$\phi_1''(x) + \lambda_1\phi_1(x) = 0 \tag{1.6.4}$$

Multiply (1.6.4) by $\phi_2(x)$ and integrate on the interval $a < x < b$:

$$\int_a^b \phi_2(x)\phi_1''(x)\,dx + \lambda_1 \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

The first integral can be integrated by parts, to obtain

$$\phi_2(x)\phi_1'(x)|_{x=a}^{x=b} - \int_a^b \phi_1'(x)\phi_2'(x)\,dx + \lambda_1 \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

Now we interchange the roles of (ϕ_1, λ_1) and (ϕ_2, λ_2) to obtain

$$\phi_1(x)\phi_2'(x)|_{x=a}^{x=b} - \int_a^b \phi_2'(x)\phi_1'(x)\,dx + \lambda_2 \int_a^b \phi_2(x)\phi_1(x)\,dx = 0$$

When we subtract these two equations, the first integrals cancel, and we are left with

$$\left(\phi_2(x)\phi_1'(x) - \phi_1(x)\phi_2'(x)\right)|_{x=a}^{x=b} + (\lambda_1 - \lambda_2) \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

From the boundary conditions, we conclude that the endpoint terms contribute zero, so we are left with the statement

$$(\lambda_1 - \lambda_2) \int_a^b \phi_1(x)\phi_2(x)\,dx = 0$$

But we have assumed that $\lambda_1 - \lambda_2 \neq 0$; hence we conclude the required orthogonality. •

1.6.3. Example of transcendental eigenvalues. The next example illustrates the possibility of numerical/graphical determination of the eigenvalues.

EXAMPLE 1.6.3. ($\alpha = 0$, $0 < \beta < \pi/2$) *Find all nontrivial solutions of (1.6.1) on the interval* $0 < x < L$ *satisfying the boundary conditions* $\phi(0) = 0, h\,\phi(L) + \phi'(L) = 0$, *where* $h > 0$.

Solution. In case $\lambda = 0$, the general solution of (1.6.1) is $\phi(x) = Ax + B$. The boundary conditions further require that $0 = \phi(0) = B$, $0 = h\,\phi(L) + \phi'(L) = h(AL + B) + A = A(1 + hL)$, which requires that $A = 0, B = 0$—hence a trivial solution.

In case $\lambda = -\mu^2 < 0$, the general solution of (1.6.1) is $\phi(x) = Ae^{\mu x} + Be^{-\mu x}$. The boundary conditions further require that $0 = A + B$, $0 = h(Ae^{\mu L} + Be^{-\mu L}) + (A\mu e^{\mu L} - B\mu e^{-\mu L})$, which is satisfied if and only if $(A, B) = (0, 0)$.

In case $\lambda > 0$, the general solution is $\phi(x) = A\cos(x\sqrt{\lambda}) + B\sin(x\sqrt{\lambda})$. The boundary conditions further require that $0 = \phi(0) = A$, $0 = h\,\phi(L) + \phi'(L) = hB\sin(L\sqrt{\lambda}) + B\sqrt{\lambda}\cos(L\sqrt{\lambda})$. Clearly, neither term can be zero, so we can divide and obtain a nontrivial solution if and only if λ satisfies the equation

$$\cot(L\sqrt{\lambda}) = -\frac{h}{\sqrt{\lambda}} = -\frac{hL}{L\sqrt{\lambda}} \tag{1.6.5}$$

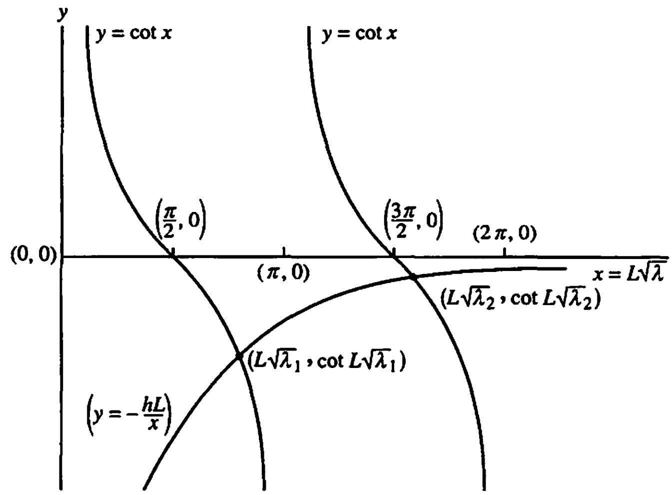


FIGURE 1.6.1 Graphical solution of $\cot(L\sqrt{\lambda}) = -h/\sqrt{\lambda}$.

Therefore we have found all eigenfunctions in the form

$$\phi_n(x) = \sin(x\sqrt{\lambda_n}) \qquad n = 1, 2, \dots$$

where the eigenvalues λ_n are determined by solving (1.6.5). •

From the graph of the cotangent function (Fig. 1.6.1), it is seen that the eigenvalues satisfy the inequalities

$$\frac{\pi}{2} < L\sqrt{\lambda_1} < \pi, \; \frac{3\pi}{2} < L\sqrt{\lambda_2} < 2\pi, \quad L\sqrt{\lambda_n} - \left(n - \frac{1}{2}\right)\pi \to 0, \; n \to \infty$$

It is possible to make a more refined asymptotic analysis of the eigenvalues as follows. Writing $L\sqrt{\lambda} = (n - (1/2))\pi + \epsilon_n$, we invoke the Taylor expansion of the cotangent function about the point $(n - (1/2))\pi$:

$$\cot\left((n - (1/2))\pi + \epsilon\right) = -\epsilon + O(\epsilon^3) \qquad \epsilon \to 0$$

Substituting in (1.6.5), we find that

$$-\epsilon_n + O(\epsilon^3) = -\frac{hL}{(n - (1/2))\pi + \epsilon_n}$$

from which we conclude that $\epsilon_n = -hL/n\pi + O(1/n^2)$ and we get the asymptotic formula

$$L\sqrt{\lambda_n} = (n - (1/2))\pi - \frac{hL}{n\pi} + O(1/n^2) \qquad n \to \infty$$

1.6.4. Further properties: completeness and positivity. By analogy with Fourier series, we may expect to be able to expand a piecewise smooth function in a series of Sturm-Liouville eigenfunctions in the form

$$f(x) \sim \sum_{n=1}^{\infty} A_n \phi_n(x) \tag{1.6.6}$$

where the Fourier coefficients are defined by

$$A_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\int_a^b \phi_n(x)^2\,dx} \qquad n = 1, 2, \ldots \tag{1.6.7}$$

The following theorem shows that we may always expect a complete set of eigenfunctions for the Sturm-Liouville eigenvalue problem.

THEOREM 1.6. *There exist an infinite sequence of solutions $\lambda_n, \phi_n(x)$ of the Sturm-Liouville eigenvalue problem defined by (1.6.1)-(1.6.3) that possess the following properties.*

- $\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \to \pi/L, \quad n \to \infty$
- *If $f(x), a < x < b$, is a piecewise smooth function, the series (1.6.6) converges to $f(x+0)/2 + f(x-0)/2$, $a < x < b$.*
- *Parseval's relation holds, in the form*

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b \phi_n(x)^2\,dx = \int_a^b f(x)^2\,dx$$

The proof will not be given here, but can be found in more advanced texts of analysis.[4]

A final point of detail regarding Sturm-Liouville eigenvalue problems is the question of *positivity* of the eigenvalues. From the previous theorem, we see that we must have $\lambda_n > 0$ for all large n, but it may happen that in some cases $\lambda_1 \le 0$—for example, $\lambda_1 = 0$ in case $\alpha = \pi/2, \beta = \pi/2$. The following *sufficient* condition is easily proved.

THEOREM 1.7. *Suppose that the parameters α, β satisfy the inequalities $0 \le \alpha < \pi/2$, $0 \le \beta < \pi/2$. Then all eigenvalues of the Sturm-Liouville eigenvalue problem (1.6.1) with the boundary conditions (1.6.2),(1.6.3) satisfy $\lambda_n > 0$.*

Proof. Suppose that $\phi(x)$ is a nontrivial solution of the Sturm-Liouville problem (1.6.1)-(1.6.3). We multiply (1.6.1) by $\phi(x)$ and integrate on the interval $a \le x \le b$:

$$\int_a^b \phi(x)\phi''(x)\,dx + \lambda \int_a^b \phi(x)^2\,dx = 0$$

[4]See, e.g., G. Birkhoff and G. C. Rota, *Ordinary Differential Equations*, Ginn, Lexington, MA, 1962.

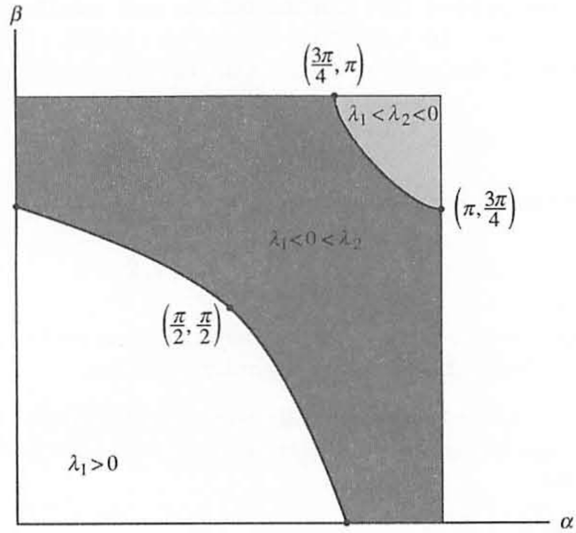


FIGURE 1.6.2 Regions of positive and negative eigenvalues.

The first integral can be integrated by parts, which leads to the identity

$$\lambda \int_a^b \phi(x)^2\,dx = \int_a^b \phi'(x)^2\,dx + \phi(a)\phi'(a) - \phi(b)\phi'(b)$$

The new integral on the right-hand side is strictly positive, since otherwise $\phi(x)$ would be a constant function, which is possible if and only if $\alpha = \pi/2$, $\beta = \pi/2$, which is excluded. On the other hand, we can rewrite the boundary conditions in the form $\phi(a) = L\,\tan\alpha\,\phi'(a)$, $\phi(b) = -L\,\tan\beta\,\phi'(b)$, which leads to

$$\lambda \int_a^b \phi(x)^2\,dx > L\phi'(a)^2\tan\alpha + L\phi'(b)^2\tan\beta \geq 0$$

since α and β both lie in the first quadrant $0 \leq \alpha, \beta < \pi/2$. •

We emphasize that the previous theorem only provides a sufficient condition for the positivity of the eigenvalues. In order to obtain more precise results, we can plot the set of points (α, β) for which $\lambda_1 > 0$. Figure 1.6.2 shows that this region contains the square $0 \leq \alpha, \beta < \pi/2$ and is bounded by a curve whose equation is $\sin(\alpha+\beta)+\cos\alpha\cos\beta = 0$. This curve passes through the three points $(\alpha, \beta) = (3\pi/4, 0)$, $(\pi/2, \pi/2)$, and $(0, 3\pi/4)$. The complete analysis of negative eigenvalues is described next. Further details are described in the exercises.

We now present the complete analysis of the existence of negative eigenvalues for the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3). If $\lambda = -\mu^2 < 0$ is a negative eigenvalue, then the corresponding eigenfunction must be of the form

$$\phi(x) = A\sinh\mu x + B\cosh\mu x$$

We may assume, without loss of generality, that $\mu > 0$. Applying the boundary conditions (1.6.2), (1.6.3) yields the two simultaneous linear equations

$$\begin{aligned}
\cos\alpha\,(A\sinh\mu a + B\cosh\mu a) - L\sin\alpha\,(A\mu\cosh\mu a + B\mu\sinh\mu a) &= 0\\
\cos\beta\,(A\sinh\mu b + B\cosh\mu b) + L\sin\beta\,(A\mu\cosh\mu b + B\mu\sinh\mu b) &= 0
\end{aligned}$$

For a nontrivial solution we must have $(A, B) \neq (0,0)$, which can happen if and only if the determinant of the coefficients is zero. After some algebra, this is written

$$\frac{\tanh\mu L}{\mu L} = -\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta + (L\mu)^2\sin\alpha\sin\beta} \tag{1.6.8}$$

We consider four separate cases:

(i) $0 < \alpha < \pi/2$, $0 < \beta < \pi/2$
(ii) $0 < \alpha < \pi/2 < \beta < \pi$
(iii) $0 < \beta < \pi/2 < \alpha < \pi$
(iv) $\pi/2 < \alpha < \pi$, $\pi/2 < \beta < \pi$

In case (i), the left side of (1.6.8) is positive, while the right side is negative for $\mu > 0$; hence there are no solutions—in accord with Theorem 1.7.

In case (ii), the denominator of the right side of (1.6.8) is zero when $\mu L = \sqrt{|\cot\alpha\cot\beta|}$, yielding a vertical asymptote, to the right of which the right side of (1.6.8) is negative. The number of solutions to (1.6.8) depends on the initial value of the right side at $\mu = 0$, which is seen to be

$$-\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} \tag{1.6.9}$$

We consider two subcases:

(iia) $\sin(\alpha+\beta) + \cos\alpha\cos\beta > 0$
(iib) $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$

In subcase (iia) the initial value (1.6.9) is greater than 1 and the right side of (1.6.8) increases to infinity, whereas the left side remains less than 1 and tends to zero. Hence the graphs do not intersect, and we have no solution. In subcase (iib) the initial value (1.6.9) is less than 1 and the right side of (1.6.8) increases to infinity, so that the graphs must intersect at some point to the left of the vertical asymptote. Hence there exists exactly one solution μ_1 that satisfies $0 < \mu_1 L < \sqrt{|\cot\alpha\cot\beta|}$.

Case (iii) is identical to (ii) with the roles of α and β interchanged; hence the analysis is identical.

For case (iv) we rewrite (1.6.8) in the form

$$\tanh \nu = \frac{A\nu}{B + C\nu^2} \qquad \nu = L\mu \tag{1.6.10}$$

Note that the function $\nu \to A\nu/(B+C\nu^2)$ begins from the origin; it rises steadily to a maximum value, strictly larger than 1, at $\nu = \sqrt{B/C} = \sqrt{|\cot\alpha\cot\beta|}$, and then steadily decreases to zero. The number of solutions depends on the slope at $\nu = 0$, leading again to the consideration of subcases:

(iva) $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$
(ivb) $\sin(\alpha+\beta) + \cos\alpha\cos\beta > 0$

In subcase (iva) the slope of the right side of (1.6.10) at $\nu = 0$ is greater than 1, the slope of the hyperbolic tangent; hence we have no intersection to the left of the maximum. To the right of the maximum the right side of (1.6.10) tends to zero; hence there is exactly one intersection with the graph of the hyperbolic tangent.

In subcase (ivb) the slope of the right side of (1.6.10) at $\nu = 0$ is less than the slope of the hyperbolic tangent; therefore initially it lies below the hyperbolic tangent. But at the maximum the order is reversed; hence there is precisely one solution to the left of the maximum. To the right of the maximum the right side of (1.6.10) tends steadily to zero, whereas the hyperbolic tangent tends to 1; hence there is another solution to the right.

Summarizing the preceding analysis, we have the following breakdown:

- There are no negative eigenvalues if either $0 < \alpha < \pi/2$ and $\sin(\alpha+\beta)+\cos\alpha\cos\beta > 0$ *or* $0 < \beta < \pi/2$ and $\sin(\alpha+\beta)+\cos\alpha\cos\beta > 0$.
- There is precisely one negative eigenvalue if $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$. This is in the interval $0 < L\sqrt{-\lambda} < \sqrt{|\cot\alpha\cot\beta|}$.
- There are precisely two negative eigenvalues if $\pi/2 < \alpha < \pi$, $\pi/2 < \beta < \pi$, and $\sin(\alpha+\beta) + \cos\alpha\cos\beta > 0$. The first one satisfies $0 < L\sqrt{-\lambda_1} < \sqrt{|\cot\alpha\cot\beta|}$ while the second one satisfies $L\sqrt{-\lambda_2} > \sqrt{|\cot\alpha\cot\beta|}$.

In other words, the equation $\sin(\alpha+\beta) + \cos\alpha\cos\beta = 0$ defines two curves that divide the square $0 < \alpha < \pi$, $0 < \beta < \pi$ into three regions, corresponding to two, one, or zero negative eigenvalues. This is depicted in Fig. 1.6.2, where the unshaded region corresponds to no negative eigenvalues, the darker shaded region corresponds to one negative eigenvalue, and the lighter shaded region corresponds to two negative eigenvalues.

1.6.5. General Sturm-Liouville problems. Many of the properties of the eigenfunctions of the simple differential equation $\phi''(x) + \lambda\phi(x) = 0$ are shared by the eigenfunctions of the more general equation

$$[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0 \qquad a < x < b \tag{1.6.11}$$

where $s(x)$, $\rho(x)$, $q(x)$ are given functions on the interval $a < x < b$ with $\rho(x) > 0$. We have already studied the special case $s(x) \equiv 1$, $\rho(x) \equiv 1$, $q(x) = 0$. The new feature here is that the eigenfunctions will satisfy a property of *weighted orthogonality* with respect to the weight function $\rho(x), a < x < b$.

As before, we also need to consider boundary conditions at the endpoints $x = a$, $x = b$. These are written in the form (1.6.2)-(1.6.3), exactly as in the previous cases. We state and prove the corresponding orthogonality properties of the Sturm-Liouville eigenfunctions.

THEOREM 1.8. *Consider the Sturm-Liouville problem (1.6.11),(1.6.2)-(1.6.3). Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal with respect to the weight function $\rho(x), a < x < b$:*

$$\int_a^b \phi_1(x)\,\phi_2(x)\rho(x)\,dx = 0$$

If the two eigenfunctions belong to the same eigenvalue $\lambda_1 = \lambda_2$, then the eigenfunctions must be proportional: $\phi_2(x) = C\phi_1(x)$ for some constant C.

Proof. Write the Sturm-Liouville equation satisfied by ϕ_1:

$$[s\phi_1']' + (\lambda_1\rho - q)\phi_1 = 0$$

Multiply this equation by ϕ_2 and integrate the resulting equation on the interval $a < x < b$:

$$\int_a^b \phi_2(x)(s\phi_1'(x))'\,dx + \int_a^b \phi_2(x)(\lambda_1\rho(x) - q(x))\phi_1(x)\,dx = 0$$

The first integral can be integrated by parts to yield

(1.6.12)

$$\phi_2(x)s(x)\phi_1'(x)|_a^b - \int_a^b \phi_2'(x)s(x)\phi_1'(x)\,dx + \int_a^b \phi_2(x)(\lambda_1\rho(x) - q(x))\phi_1(x)\,dx = 0$$

Now we interchange the roles of $\phi_1(x)$ and $\phi_2(x)$ to yield

(1.6.13)

$$\phi_1(x)s(x)\phi_2'(x)|_a^b - \int_a^b \phi_1'(x)s(x)\phi_2'(x)\,dx + \int_a^b \phi_1(x)(\lambda_2\rho(x) - q(x))\phi_2(x)\,dx = 0$$

When we subtract (1.6.12) and (1.6.13) and apply the boundary conditions, all of the terms cancel except for the final integrals. This yields the statement that $(\lambda_1 - \lambda_2)\int_a^b \phi_1(x)\phi_2(x)\rho(x)\,dx = 0$; if $\lambda_1 - \lambda_2 \neq 0$, it follows that ϕ_1 and ϕ_2 must be orthogonal with respect to the weight function ρ, which was to be proved. •

EXAMPLE 1.6.4. *Find the orthogonality relation for eigenfunctions of the Bessel equation of order zero: $(x\phi')' + \lambda x\phi = 0$.*

Solution. In this case we have $s(x) = x$, $\rho(x) = x$, $q(x) = 0$. If $\phi_1(x)$ and $\phi_2(x)$ both satisfy the same two-point boundary conditions with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form $\int_a^b \phi_1(x)\phi_2(x)\,x\,dx = 0$. •

EXAMPLE 1.6.5. *Find the orthogonality relation for eigenfunctions of the Bessel equation of order m:* $(x\phi')' + (\lambda x - m^2/x)\phi = 0$.

Solution. In this case we have $s(x) = x$, $\rho(x) = x$, $q(x) = m^2/x$. If $\phi_1(x)$ and $\phi_2(x)$ both satisfy the same two-point boundary conditions with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form $\int_a^b \phi_1(x)\phi_2(x)\,x\,dx = 0$. •

The orthogonality asserted in Theorem 1.8 also applies in the case of other types of boundary conditions, specifically

Periodic boundary conditions: $s(a) = s(b)$, $\phi(a) = \phi(b)$, $\phi'(a) = \phi'(b)$

Singular Sturm-Liouville problems: $s(a) = 0$, $s(b) = 0$

In each of these cases we simply need to verify that the boundary term is zero. In detail,

$$s(x)\left(\phi_1(x)\phi_2'(x) - \phi_1'(x)\phi_2(x)\right)\Big|_a^b = 0$$

EXAMPLE 1.6.6. *Verify the orthogonality of eigenfunctions for the Legendre equation* $[(1-x^2)\phi']' + \lambda\phi = 0$, *where* $-1 < x < 1$.

Solution. This is a singular Sturm-Liouville problem with $s(x) = (1-x^2)$, $\rho(x) = 1$, $q(x) = 0$, since $s(1) = 0$, $s(-1) = 0$. The weight function is $\rho(x) = 1$, so that the orthogonality relation is $\int_{-1}^1 \phi_1(x)\phi_2(x)\,dx = 0$. •

In some cases we may have a singular Sturm-Liouville problem with respect to one end. In that case we require only that the boundary condition be satisfied at the nonsingular end, where $s(x) \neq 0$. The Bessel equation on the interval $0 < x < b$ provides an example of this type.

EXAMPLE 1.6.7. *Find the orthogonality relation for eigenfunctions of the Bessel equation of order m:* $(x\phi')' + (\lambda x - m^2/x)\phi = 0$ *on the interval* $0 < x < b$.

Solution. In this case we have $s(x) = x$, $\rho(x) = x$, $q(x) = m^2/x$. If $\phi_1(x)$ and $\phi_2(x)$ both satisfy the same separable boundary conditions at $x = b$ with different eigenvalues $\lambda_1 \neq \lambda_2$, then we must have the orthogonality in the form $\int_0^b \phi_1(x)\phi_2(x)\,x\,dx = 0$. •

The case of periodic boundary conditions can be applied to give a new proof of the orthogonality of $\sin(n\pi x/L)$, $\cos(n\pi x/L)$ as follows.

EXAMPLE 1.6.8. *Consider the Sturm-Liouville eigenvalue problem for the equation $\phi'' + \lambda\phi = 0$ on the interval $-L < x < L$ with the periodic boundary conditions $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$. Find the eigenfunctions and the associated orthogonality relation for $\lambda > 0$.*

Solution. The general solution of the equation $\phi'' + \lambda\phi = 0$ with $\lambda > 0$ is $\phi(x) = A\cos x\sqrt{\lambda} + B\sin x\sqrt{\lambda}$. The periodic boundary conditions translate into the following system of two simultaneous linear equations:

$$\begin{aligned} A\cos L\sqrt{\lambda} - B\sin L\sqrt{\lambda} &= A\cos L\sqrt{\lambda} + B\sin L\sqrt{\lambda} \\ -\sqrt{\lambda}A\cos L\sqrt{\lambda} - \sqrt{\lambda}B\sin L\sqrt{\lambda} &= -A\cos L\sqrt{\lambda} + B\sin L\sqrt{\lambda} \end{aligned}$$

This system has a nontrivial solution if and only if $\sin L\sqrt{\lambda} = 0$, namely, $L\sqrt{\lambda} = n\pi$. The eigenfunctions are of the form $\phi_n(x) = A\cos(n\pi x/L) + B\sin(n\pi x/L)$, and the orthogonality relation is $\int_{-L}^{L}\phi_m(x)\phi_n(x)\,dx = 0$ if $m \neq n$. •

1.6.6. Complex-valued eigenfunctions and eigenvalues. In the above discussion of Sturm-Liouville eigenvalue problems, it has been tacitly assumed that both the eigenvalue and eigenfunction are real-valued. We now demonstrate that this leads to no loss of generality.

PROPOSITION 1.6.1. *Suppose that $\phi(x)$ is a complex-valued function and λ is a (possibly) complex number that satisfies the Sturm-Liouville equation (1.6.11) where $s(x), \rho(x), q(x)$ are real-valued functions. Suppose further that $\phi(x)$ satisfies one of the above boundary conditions. Then λ is a real number, and both the real and imaginary parts of $\phi(x)$ are eigenfunctions of the Sturm-Liouville eigenvalue problem.*

Proof. We multiply the Sturm-Liouville equation (1.6.11) by the complex conjugate of $\phi(x)$ and integrate over the basic interval:

$$\int_a^b \bar{\phi}(x)[s(x)\phi'(x)]'\,dx + \int_a^b [\lambda\rho(x) - q(x)]\bar{\phi}(x)\phi(x)\,dx = 0$$

Similarly,

$$\int_a^b \phi(x)[s(x)\bar{\phi}'(x)]'\,dx + \int_a^b [\bar{\lambda}\rho(x) - q(x)]\phi(x)\bar{\phi}(x)\,dx = 0$$

We subtract these and apply integration by parts on each of the first terms as follows:

$$\int_a^b \left(\bar{\phi}(x)[s(x)\phi'(x)]' - \phi(x)[s(x)\bar{\phi}'(x)]'\right)dx = s(x)[\bar{\phi}(x)\phi'(x) - \bar{\phi}'(x)\phi(x)|_a^b$$

But the boundary conditions imply that this term is zero. When we subtract the second terms, the result is

$$(\lambda - \bar{\lambda})\int_a^b \rho(x)|\phi(x)|^2\,dx = 0$$

which proves that the imaginary part of λ is zero; in other words, λ must be a real number. Writing $\phi(x) = u(x) + iv(x)$, we see that both $u(x)$ and $v(x)$ satisfy the same Sturm-Liouville equation that was satisfied by the complex function $\phi(x)$, which was to be proved. •

EXAMPLE 1.6.9. *Consider the Sturm-Liouville eigenvalue problem for the equation $\phi''(x) + \lambda\phi(x) = 0$. Find the complex-valued eigenfunctions satisfying the periodic boundary conditions $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$.*

Solution. From the previous work, all of the real-valued solutions are written $\sin(n\pi x/L)$, $\cos(n\pi x/L)$ with the eigenvalue $\lambda = (n\pi/L)^2$, where $n = 0, 1, 2 \ldots$. The corresponding complex-valued functions may be written

$$\phi(x) = e^{in\pi x/L} \qquad \phi(x) = e^{-in\pi x/L} \quad \bullet$$

By contrast, it should be noted that in the case of two-point boundary conditions, Theorem 1.5 implies that the real and imaginary parts of a complex eigenfunction must be proportional to one another; put differently, any complex eigenfunction is a complex multiple of a real-valued eigenfunction.

EXERCISES 1.6

In Exercises 1–6, find the eigenvalues and eigenfunctions of the Sturm-Liouville eigenvalue problem (1.6.1).

1. $\phi(0) = 0, \phi'(L) = 0$
2. $\phi'(0) - h\phi(0) = 0, \phi'(L) + h\phi(L) = 0,\ h > 0$
3. $\phi'(0) = 0, \phi(L) = 0$
4. $\phi(0) = \phi(L), \phi'(0) = \phi'(L)$
5. $\phi(0) = 0, \phi'(L) - \phi(L) = 0$
6. $\phi'(0) - \phi(0) = 0, \phi'(L) = 0$
7. Show that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem defined by (1.6.1)–(1.6.3) if and only if the parameters α, β satisfy the relation $\sin(\alpha+\beta)+\cos\alpha\cos\beta = 0$, which can be written in the form $\tan\alpha+\tan\beta = -1$ when $\alpha \neq \pi/2, \beta \neq \pi/2$.
8. Suppose the boundary conditions (1.6.2), (1.6.3) are written in the form $h_1\phi(0) - \phi'(0) = 0$, $h_2\phi(L) + \phi'(L) = 0$. Show that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem if and only if the parameters h_1, h_2 satisfy the equation of the two-sheeted hyperbola: $h_1 + h_2 + Lh_1h_2 = 0$.
9. On the basis of the results in this section, how many negative eigenvalues exist for the Sturm-Liouville problem (1.6.1)–(1.6.3) in the following cases?
   (a) $\alpha = \pi/4, \beta = \pi/2$
   (b) $\alpha = \pi/4, \beta = 3\pi/4$
   (c) $\alpha = 7\pi/8, \beta = 7\pi/8$

10. Suppose $\alpha = 0$ and $0 \leq \beta < 3\pi/4$. Show directly that all eigenvalues of the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3) satisfy $\lambda_n > 0, n = 1, 2, \ldots$. [*Hint:* If $\phi(x) = A\sinh(\mu(x-a))$ is an eigenfunction satisfying the boundary condition at $x = a$, find a transcendental equation for μ and show that it has no solution. Also check $\lambda = 0$ separately.]
11. Suppose that $\beta = 0$ and $0 \leq \alpha < 3\pi/4$. Show directly that all eigenvalues of the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3) satisfy $\lambda_n > 0, n = 1, 2, \ldots$. [*Hint:* Use instead $\phi(x) = A\sinh(\mu(x-b))$ to find the appropriate transcendental equation.]
12. Show that the Sturm-Liouville eigenvalue problem (1.6.1)–(1.6.3) has a negative eigenvalue if and only if the the parameters α, β satisfy the inequality $\sin(\alpha+\beta) + \cos\alpha\cos\beta < 0$. [*Hint:* If $\phi(x) = A\sinh(\mu x) + B\cosh(\mu x)$ is an eigenfunction, show that μ must be a solution of the transcendental equation
$$\tanh(\mu L) = -L\mu\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta + (L\mu)^2\sin\alpha\sin\beta}$$
and that this equation will have a nonzero solution if and only if the slope at $\mu = 0$ is larger than 1.]
13. With reference to the generalized Sturm-Liouville problem, let L be the linear differential operator defined by $L\varphi = (s\varphi')' - q\varphi$. Prove the *Lagrange identity* $\varphi_2 L\varphi_1 - \varphi_1 L\varphi_2 = (s(\varphi_1'\varphi_2 - \varphi_1\varphi_2'))$, where φ_1, φ_2 are twice-differentiable functions.
14. Use the Lagrange identity to give an alternative proof of Theorem 1.8.
15. Show that if $s(x) \geq 0$, $q(x) \geq 0$, then all eigenvalues of the generalized Sturm-Liouville problem with the two-point boundary conditions $\varphi(a) = 0$, $\varphi(b) = 0$ satisfy $\lambda_n \geq 0$. [*Hint:* Apply the Lagrange identity with $\varphi_2 = 1$.]

CHAPTER 2

BOUNDARY-VALUE PROBLEMS IN RECTANGULAR COORDINATES

INTRODUCTION

In this chapter we will derive the general form of the heat equation and the wave equation for the vibrating string. These PDEs will eventually be solved in regions with rectangular, cylindrical, and spherical boundaries. In this chapter we focus attention on the case of rectangular boundaries, where we can use the usual cartesian coordinates (x, y, z), coupled with trigonometric Fourier series, which were introduced in Chapter 1. Regions with cylindrical or spherical boundaries will be treated in Chapter 3 and Chapter 4, respectively.

2.1. The Heat Equation

In this and the next two sections we will apply Fourier series to some typical problems of heat conduction. These are concerned with the flow of heat—specifically, with representing changes in temperature as a function of space and time. We denote by $u(x, y, z; t)$ the temperature measured at the point (x, y, z) at the time instant t. We suppose that u is a smooth function of $(x, y, z; t)$ and will proceed to determine a partial differential equation for u.

2.1.1. Fourier's law of heat conduction. We consider a solid material that occupies a portion of three-dimensional space. A basic quantity of importance is the *heat current density* $\mathbf{q}(\mathbf{x}; t)$. This vector quantity represents the rate of heat flow at the point $\mathbf{x} = (x, y, z)$. If $\mathbf{n}$ is any unit vector, the scalar quantity $\mathbf{q} \cdot \mathbf{n}$ is called the *heat flux* in the direction $\mathbf{n}$. It measures the rate of heat flow per unit time per unit area across a plane with normal vector $\mathbf{n}$. *Fourier's law* states that

$$\mathbf{q} = -k \operatorname{grad} u$$

where k is the thermal conductivity of the material. From calculus we know that grad u points in the direction of the maximum increase of u. Since heat is expected to flow from warmer to cooler regions, we insert the minus sign in Fourier's law. Thus $\mathbf{q}$ points in the direction of maximum *decrease* of u and $|\mathbf{q}|$ is the rate of heat flow in that direction.

2.1.2. Derivation of the heat equation. During a small time interval $(t, t+\Delta t)$ heat flows through the material and may also be generated by internal sources, at a rate $s(\mathbf{x}, t)$. Therefore the amount of heat that enters any region R of the material within the time interval $(t, t+\Delta t)$ is, to first order in Δt, given by

$$Q = \left(-\iint_{\partial R} \mathbf{q} \cdot \mathbf{n}\, dS + \iiint_R s\, dV \right) \Delta t + O(|\Delta t|^2)$$

where $\mathbf{n}$ is the outward-pointing normal vector, ∂R denotes the boundary of R, and the minus sign is in front of the surface integral because $\mathbf{q} \cdot \mathbf{n}\, dS$ is the density of heat flowing *out* of the surface element dS per unit time.

On the other hand, this heat Q has the effect of raising the temperature by the amount $u_t \Delta t$, to first order in Δt. Therefore we can write

$$Q = \iiint_R c\rho u_t\, dV\, \Delta t + O(|\Delta t|^2)$$

where c is the heat capacity per unit mass and ρ is the mass density of the material. Equating these, dividing by Δt, and letting $\Delta t \to 0$, we have the *continuity equation*

$$\iiint_R c\rho u_t\, dV = -\iint_{\partial R} \mathbf{q} \cdot \mathbf{n}\; dS + \iiint_R s\, dV$$

This equation is valid for any region, no matter how large or small. In particular, we take a small spherical region R about the point (x, y, z), divide by the volume, and take the limit when the diameter of the sphere tends to zero. The surface integral can be handled using the divergence theorem,

$$\iint_{\partial R} \mathbf{q} \cdot \mathbf{n}\, dS = \iiint_R (\operatorname{div} \mathbf{q}) dV$$

and we obtain the differential form of the continuity equation:

$$c\rho u_t = \operatorname{div}(k \operatorname{grad} u) + s$$

This is the general form of the heat equation.

In most problems k is independent of $\mathbf{x}$, and we can bring it outside and thus obtain the heat equation in the form

$$\boxed{u_t = K \operatorname{div}(\operatorname{grad} u) + r = K\nabla^2 u + r} \tag{2.1.1}$$

where $K = k/c\rho$ and $r = s/c\rho$ are the renormalized conductivity and source terms, respectively. K is called the *thermal diffusivity* of the material. The *Laplacian* of a function u is defined by

$$\nabla^2 u = \operatorname{div}(\operatorname{grad} u) = u_{xx} + u_{yy} + u_{zz}$$

Remark. We can derive the heat equation without using the divergence theorem, by the following direct argument. Let R be the rectangular box defined by the inequalities $x_1 \le x \le x_2$, $y_1 \le y \le y_2$, $z_1 \le z \le z_2$, and let q^x, q^y, q^z be the components of the heat current density vector. Then

$$\begin{aligned}\int\int_{\partial R} \mathbf{q}\cdot\mathbf{n}\,dS &= \int_{z_1}^{z_2}\int_{y_1}^{y_2}[q^x(x_2,y,z)-q^x(x_1,y,z)]dy\,dz \\ &+\int_{z_1}^{z_2}\int_{x_1}^{x_2}[q^y(x,y_2,z)-q^y(x,y_1,z)]dx\,dz \\ &+\int_{y_1}^{y_2}\int_{x_1}^{x_2}[q^z(x,y,z_2)-q^z(x,y,z_1)]dx\,dy\end{aligned}$$

We must show that

$$\frac{1}{(x_2-x_1)(y_2-y_1)(z_2-z_1)}\int\int_{\partial R}\mathbf{q}\cdot\mathbf{n}\,dS$$

tends to div $\mathbf{q} = (q_x^x + q_y^y + q_z^z)(x_1,y_1,z_1)$ when $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$. To do this, we consider each of the three integrals separately. For the first integral we have to examine

$$\frac{1}{(y_2-y_1)(z_2-z_1)}\int_{z_1}^{z_2}\int_{y_1}^{y_2}\frac{q^x(x_2,y,z)-q^x(x_1,y,z)}{x_2-x_1}\,dy\,dz$$

When $x_2 \to x_1$, the integrand tends to $q_x^x(x_1,y,z)$, a continuous function. When $y_2 \to y_1$, $z_2 \to z_1$, the resulting integral tends to $q_x^x(x_1,y_1,z_1)$. The same result is obtained if we first let $y_2 \to y_1$, $z_2 \to z_1$. The second integral, where q^x is replaced by q^y, tends to $q_y^y(x_1,y_1,z_1)$ when $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$ in any order, and similarly for the third integral. This proves that $\int\int_R \mathbf{q}\cdot\mathbf{n}\,dS$, divided by the volume of the box R, tends to div $\mathbf{q}$ when the sides tend to zero, in any order. Referring to the continuity equation and letting $x_2 \to x_1$, $y_2 \to y_1$, $z_2 \to z_1$, we have proved that $c\rho u_t(x_1,y_1,z_1) = -\operatorname{div}\mathbf{q}(x_1,y_1,z_1) + s(x_1,y_1,z_1)$, which was to be shown.

2.1.3. Boundary conditions. The heat equation describes the flow of heat within the solid material. To completely determine the time evolution of temperature, we must also consider boundary conditions of various forms. For example, if the material is in contact with an ice-water bath, it is natural to suppose that $u = 32°\mathrm{F}$ on the boundary. Alternatively, we can imagine that the heat flux across the boundary is given; therefore by Fourier's law the appropriate boundary condition is of the type $\nabla u\cdot\mathbf{n} = a$, a given function on the boundary. For example, an insulated surface would necessitate $\nabla u\cdot\mathbf{n} = 0$ on the boundary. A third type of boundary condition results from Newton's law of cooling, written in the form

$$\mathbf{q}\cdot\mathbf{n} = h(u-T)$$

The heat flux across the boundary is proportional to the difference between the temperature u of the body and the temperature T of the surrounding medium.

2.1.4. Steady-state solutions in a slab. An important class of solutions of the heat equation are the *steady-state solutions.* This means that $\partial u/\partial t = 0$ or that u is a function of (x, y, z), independent of t. Thus we must have $K\nabla^2 u + r = 0$, a form of Poisson's equation. If in addition there are no internal sources of heat, then we have $r = 0$ and u satisfies Laplace's equation $\nabla^2 u = 0$. We restate this as follows.

PROPOSITION 2.1.1. *Steady-state solutions of the heat equation, with no internal heat sources, are solutions of Laplace's equation.*

Thus, Laplace's equation is a special case of the heat equation.

In the next three sections we will make a detailed study of the heat equation in a slab, defined by the inequalities $0 < z < L$, $-\infty < x < \infty$, $-\infty < y < \infty$. This mathematical model is appropriate for a wall of thickness L, where we ignore the variations of temperature in the x, y directions. The boundary conditions at the surfaces $z = 0$ and $z = L$ reflect the thermal properties of the inside (resp. outside) of the wall.

EXAMPLE 2.1.1. *Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$ satisfying the boundary conditions $u(x, y, 0) = T_1$, $(\partial u/\partial z + hu)(x, y, L) = 0$, where T_1 and h are positive constants.*

Solution. Steady-state solutions of the heat equation are solutions of Laplace's equation, $u_{xx} + u_{yy} + u_{zz} = 0$. Since the boundary conditions are independent of (x, y), we look for the solution in the form $u(x, y, z) = U(z)$, independent of (x, y). Thus U must satisfy $U''(z) = 0$, whose general solution is $U(z) = A + Bz$. The boundary condition at $z = 0$ requires $T_1 = A$, while the boundary condition at $z = L$ requires $B + h(A + BL) = 0$. Thus $B(1 + hL) = -hA = -hT_1$, and the solution is $U(z) = T_1 - hT_1 z/(1 + hL)$. •

In many problems it is important to compute the flux through the faces of the slab. From our earlier discussion, the flux is given by $-k\nabla u \cdot \mathbf{n}$; here $\mathbf{n} = (0, 0, 1)$ for the upper face and $\mathbf{n} = (0, 0, -1)$ for the lower face. Thus in Example 2.1.1, the flux from the upper face is $-k\partial U/\partial z = khT_1/(1 + hL)$, while the flux from the lower face is $k\partial U/\partial z = -khT_1/(1 + hL)$. •

We now consider an example with internal heat sources.

EXAMPLE 2.1.2. *Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$ satisfying the boundary conditions $u(x, y, 0) = T_1$, $(\partial u/\partial z + hu)(x, y, L) = 0$, where r, K, h, and T_1 are positive constants. Find the flux through the upper and lower faces.*

Solution. The boundary conditions are independent of (x, y); hence we look for the solution in the form $u(x, y, z) = U(z)$, independent of (x, y). Thus U must satisfy $KU''(z) + r = 0$, whose general solution is $U(z) = -rz^2/2K + A + Bz$.

The boundary condition at $z = 0$ requires $T_1 = A$, while the boundary condition at $z = L$ requires $-rL/K + B + h(-rL^2/2K + A + BL) = 0$. Thus $B(hL+1) = rL/K + hrL^2/2K - hT_1$. The solution is $U(z) = -rz^2/2K + T_1 + Bz$, where $B(1+b) = (rL/K)(1+\frac{1}{2}b) - hT_1$ and the *Biot modulus* b is defined as $b = hL$. The flux through the upper face is $-kU'(L) = krL/K - kB$. The flux through the lower face is $kU'(0) = kB$. •

In some cases the steady-state solution is not uniquely determined by the boundary conditions. For example, the heat equation $u_t = K\nabla^2 u$ with the boundary conditions $u_z(x, y, 0) = 0$, $u_z(x, y, L) = 0$ has the solution $U(z) = A$ for *any* constant A. This phenomenon of nonuniqueness is equivalent to the statement that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem with the associated homogeneous boundary conditions. Indeed, if we have two different steady-state solutions $U_1(z)$, $U_2(z)$ with the same nonhomogeneous boundary conditions, then the difference $U(z) = U_1(z) - U_2(z)$ is a nonzero solution of the homogeneous equation $U''(z) = 0$, satisfying the homogeneous boundary conditions. This is exactly the statement that $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem with these homogeneous boundary conditions. We will come back to this point in Sec. 2.3.

2.1.5. Time-periodic solutions. Another important class of solutions of the heat equation are the *periodic solutions*. These correspond to a stationary regime, where the solution exists for all time, $-\infty < t < \infty$. Typically the solution is specified by a boundary condition of boundedness. We illustrate with the following problem from geophysics.

The temperature at the surface of the earth is a given periodic function of time, and we seek the temperature z units below the surface. We assume that there are no internal heat sources and the thermal diffusivity is constant throughout the earth.

To formulate this problem, we suppose that the earth is flat and that the surface is given by the equation $z = 0$. (In Chapter 4 we show that the flat earth is a valid approximation for shallow depths.) The temperature on the surface is independent of location and depends only on time. Therefore we must solve the problem

$$\begin{aligned} u_t &= Ku_{zz} && z > 0,\ -\infty < t < \infty \\ u(0;t) &= u_0(t) && -\infty < t < \infty \end{aligned}$$

where $u_0(t)$ is periodic with period τ. In addition we require that the temperature be bounded,

$$|u(z;t)| \le M$$

since we do not expect that the temperature variations within the earth will exceed the variations on the surface.

To solve this problem, we first look for complex separated solutions, of the form

$$u(z;t) = Z(z)T(t)$$

Since the heat equation has real coefficients, the real and imaginary parts of a complex-valued solution are again solutions. Thus we may allow $Z(z)$, $T(t)$ to be complex-valued. Substituting into the heat equation, we have

$$\frac{KZ''(z)}{Z(z)} = \frac{T'(t)}{T(t)}$$

Both sides must be a constant, which we call $-\lambda$. Thus we have the ordinary differential equations

$$\begin{aligned} T'(t) + \lambda T(t) &= 0 \\ Z''(z) + \frac{\lambda}{K} Z(z) &= 0 \end{aligned}$$

The first equation has the solution $T(t) = e^{-\lambda t}$. Since we require bounded solutions for $-\infty < t < \infty$, λ must be pure imaginary, $\lambda = i\beta$ with β real. To solve the second equation, we try $Z(z) = e^{\gamma z}$. Thus we must have $\gamma^2 e^{\gamma z} + (\lambda/K)e^{\gamma z} = 0$, yielding the quadratic equation

$$\gamma^2 + \frac{i\beta}{K} = 0$$

In the case where $\beta > 0$, this has two solutions:

$$\gamma = \pm(-1+i)\sqrt{\frac{\beta}{2K}}$$

Since we require bounded solutions for $z > 0$, we must take the solution with $\operatorname{Re}\gamma < 0$, that is, the plus sign. Therefore we have the *complex separated solutions*

$$e^{-i\beta t} e^{(-1+i)z\sqrt{\beta/2K}}$$

Taking the real and imaginary parts, we have the real solutions

$$e^{-cz}\cos(\beta t - cz), \qquad e^{-cz}\sin(\beta t - cz), \qquad c = \sqrt{\beta/2K}$$

(If $\beta < 0$, it can be shown that no new solutions are obtained.) We refer to these as the *quasi-separated solutions.*

To solve the original problem, we suppose that the boundary temperature has been expanded as a Fourier series.

$$u_0(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2n\pi t}{\tau} + B_n \sin\frac{2n\pi t}{\tau}\right)$$

We take $\beta_n = 2n\pi/\tau$, $c_n = \sqrt{n\pi/K\tau}$ in the quasi-separated solutions just developed to obtain the solution in the form

$$u(z;t) = A_0 + \sum_{n=1}^{\infty} e^{-c_n z}[A_n\cos(\beta_n t - c_n z) + B_n \sin(\beta_n t - c_n z)]$$

To verify that this is indeed a rigorous solution to the original problem, we may suppose that A_n, B_n are bounded by some constant. Then it may be shown that the formal series for u_z, u_{zz}, u_t converge uniformly, and hence u indeed satisfies the heat equation.

EXAMPLE 2.1.3. *Solve the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, with the boundary condition*

$$u(0;t) = A_0 + A_1 \cos \frac{2\pi t}{\tau}$$

where A_0, A_1, and τ are positive constants. Graph the solution as a function of t for $z\sqrt{\pi/K\tau} = 0$, $\pi/2$, π, $3\pi/2$, 2π and $0 \le t \le \tau$.

Solution. Referring to the general solution just obtained, we let $B_n = 0$ for $n \ge 1$ and $A_n = 0$ for $n \ge 2$. The solution is

$$u(z;t) = A_0 + A_1 e^{-c_1 z} \cos\left(\frac{2\pi t}{\tau} - z\sqrt{\frac{\pi}{K\tau}}\right)$$

In Fig. 2.1.1 we plot the temperature as a function of time for the depths indicated.

2.1.6. Applications to geophysics. This theory can be used to study the seasonal variations of temperature within the earth. For $z = 0$, the maximum of $u(z;t)$ is attained at $t = 0, \pm\tau, \pm 2\tau, \ldots$. For $z = \sqrt{\pi K\tau}$, $u(z;t)$ attains its *minimum* value for the same times, $t = 0, \pm\tau, \pm 2\tau, \ldots$. Stated differently, when it is summer on the earth's surface, it is winter at a depth of $z = \sqrt{\pi K\tau}$.

EXAMPLE 2.1.4. *Suppose that $K = 2 \times 10^{-3}\text{cm}^2/\text{s}$, $\tau = 3.15 \times 10^7\text{s}$. Find the depth necessary for a change from summer to winter.*

Solution. We have $\sqrt{\pi K\tau} = 4.45 \times 10^2$ cm. Therefore when it is summer on the earth's surface, it is winter at a depth of 4.4 meters. •

This theory can also be used to estimate the thermal diffusivity of the earth. To do this, we define the *amplitude variation* of the solution $u(z;t)$ as

$$A(z) = \max_{-\infty<t<\infty} u(z;t) - \min_{-\infty<t<\infty} u(z;t)$$

By measuring $A(z)$ at different depths, we may determine the diffusivity K. Indeed, using the solution obtained in Example 2.1.3, we have $\max u(z;t) = A_0 + A_1 e^{-c_1 z}$, $\min u(z;t) = A_0 - A_1 e^{-c_1 z}$, and thus $A(z) = 2A_1 e^{-c_1 z}$, $A(z)/A(0) = e^{-c_1 z}$. Let z_1 be the depth for which $e^{-c_1 z} = \frac{1}{2}$. Since $c_1 = \sqrt{\pi/K\tau}$, we have $\sqrt{\pi/K\tau}\, z_1 = \ln 2$, $K = \pi z_1^2/\tau(\ln 2)^2$.

EXAMPLE 2.1.5. *Estimate the thermal diffusivity of the earth if the summer-winter amplitude variation decreases by a factor of 2 at a depth of 1.3 meters.*

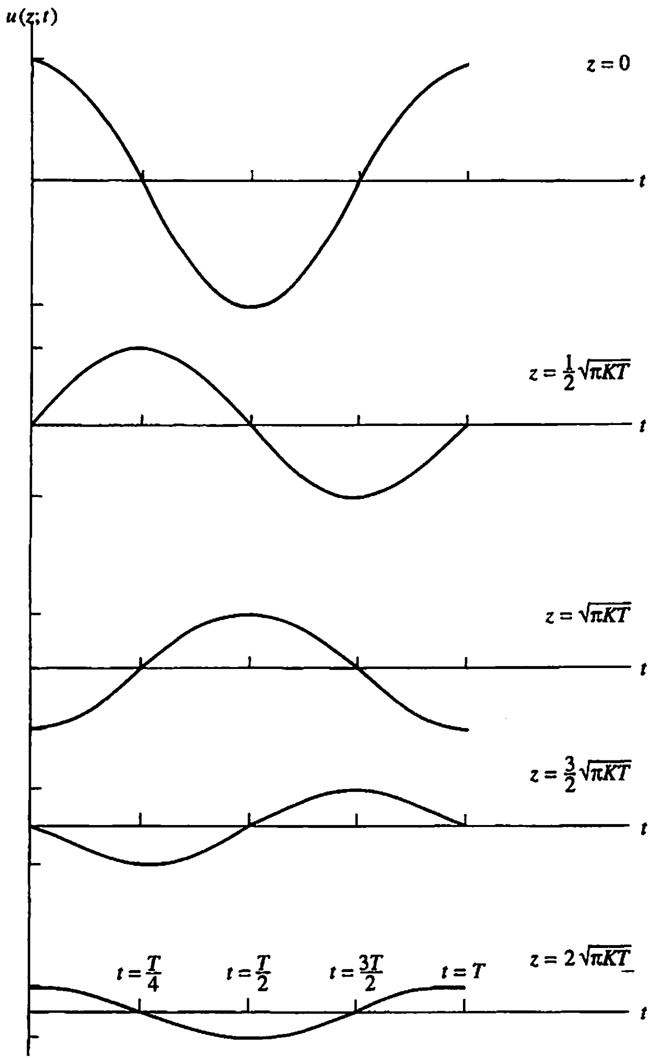


FIGURE 2.1.1 Temperature as a function of time at different depths.

Solution. We take $\tau = (365)(24)(3600) = 3.15 \times 10^7\,\text{s}$, $z_1 = 1.3$ m. Thus

$$K = \frac{\pi(1.3)^2}{(3.15 \times 10^7)(0.69)^2} = 3.5 \times 10^{-7}\,\text{m}^2/\text{s} \quad \bullet$$

2.1.7. Implementation with Mathematica. We can use Mathematica to do a three-dimensional plot of the bounded function $u(z;t)$ that satisfies the heat equation

$$u_t = K\, u_{zz}, \qquad z > 0, -\infty < t < \infty$$

with the boundary condition

$$u(0;t) = \cos\frac{2\pi t}{T}$$

From Example 2.1.3, the solution is

$$u(z;t) = e^{-c_1 z}\cos\left(\frac{2\pi t}{T} - c_1\, z\right), \qquad c_1 = \sqrt{\frac{\pi}{KT}}$$

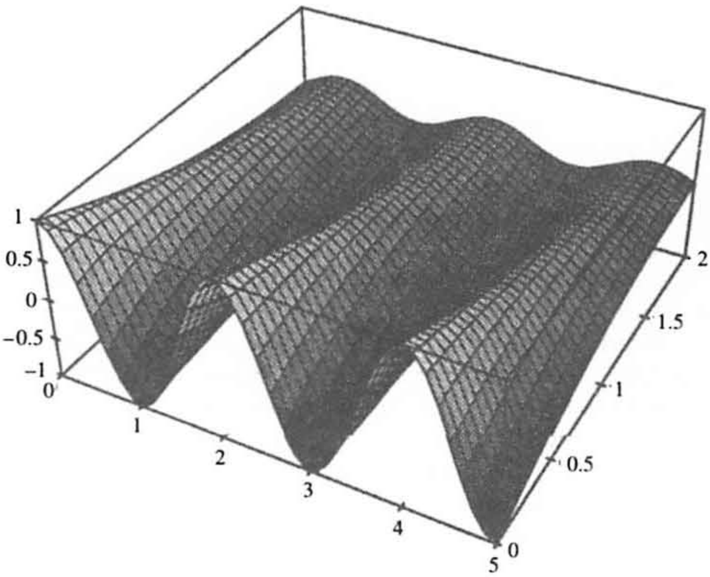
This function can be defined in Mathematica using the command

```
u[z_,t_,K_,T_]:=E^(-z Sqrt[Pi/(KT)])*Cos[2 Pi t/T - z Sqrt[Pi/(KT)]]
```

In the following graph we have chosen the parameter values $T = K = 2$; the independent variables range over the intervals $0 \le z \le 2$, $0 \le t \le 5$. The plot is realized by typing

```
Plot3D[u[z,t,2,2],{t,0,5},{z,0,2},PlotPoints->40,PlotRange->{-1,1}]
```

to yield



At the front of this graph, moving from left to right, we see the change of seasons at the surface of the earth, while at the back of the graph, moving from left to right, we see the change of seasons at a depth of 2 feet.

EXERCISES 2.1

1. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, satisfying the boundary conditions $u(x, y, 0) = T_1$, $u(x, y, L) = T_2$, where T_1 and T_2 are positive constants.
2. For the solution found in Exercise 1, find the flux through the upper face $z = L$.
3. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, satisfying the boundary conditions $(\partial u/\partial z)(x, y, 0) = \Phi$, $u(x, y, L) = T_0$, where Φ and T_0 are positive constants.
4. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, satisfying the following boundary conditions: $[k(\partial u/\partial z) - h(u - T_0)](x, y, 0) = 0$, $[k(\partial u/\partial z) + h(u - T_1)](x, y, L) = 0$.
5. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u - \beta(u - T_3)$ in the slab $0 < z < L$, satisfying the boundary conditions $u(x, y, 0) = T_1$, $u(x, y, L) = T_2$ where T_1, T_2, T_3, and β are positive constants.
6. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$, satisfying the boundary conditions $(\partial u/\partial z)(x, y, 0) = 0$, $u(x, y, L) = T_1$ where K, r, and T_1 are positive constants. Find the flux through the face $z = L$.
7. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$, satisfying the boundary conditions $u(x, y, 0) = T_1$, $u(x, y, L) = T_2$, where K, r, T_1, and T_2 are positive constants. If $T_1 = T_2$, show that the flux across the plane $z = \frac{1}{2}L$ is zero.
8. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r(z)$ in the slab $0 < z < L$, satisfying the boundary condition $u(x, y, 0) = 0$, $u(x, y, L) = 0$, where $r(z) = r_0$ for $L/3 < z < 2L/3$, $r(z) = 0$ for $0 < z < L/3$ and $2L/3 < z < L$, and r_0 and K are positive constants. (*Hint:* Although u is not smooth, it may be supposed that u and u_z are both continuous.)
9. A wall of thickness 25 cm has outside temperature $-10°$C and inside temperature 18°C. The conductivity is $k = 0.0016$ cal/s-cm-°C and there are no internal heat sources. Find the steady-state heat flux through the outer wall, per unit area.
10. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the half-space $z > 0$ for $-\infty < t < \infty$ satisfying the conditions $|u(z;t)| \leq M$, $u(0;t) = A_0 + A_1 \cos 2\pi t/\tau_1 + A_2 \cos 2\pi t/\tau_2$, where A_0, A_1, A_2, τ_1, and τ_2 are positive constants.

11. Let $u(z;t) = e^{-cz}\cos(\beta t - cz)$, where β and c are constants. Show that u satisfies the heat equation $u_t = Ku_{zz}$ if and only if $c^2 = \beta/2K$.

Exercises 12 to 14 require the solution of the heat equation in the slab $0 < z < L$, where one face is maintained at temperature zero. Thus we have the boundary-value problem

$$\begin{aligned} u_t &= Ku_{zz} && 0 < z < L, -\infty < t < \infty \\ u(0;t) &= A_0 + A_1\cos(2\pi t/\tau) && -\infty < t < \infty \\ u(L;t) &= 0 && -\infty < t < \infty \end{aligned}$$

12. Find all complex separated solutions satisfying the heat equation that are of the form $u(z;t) = e^{\gamma z}e^{i\beta t}$, where β is positive.
13. By taking the real and imaginary parts of the complex-valued solutions found in Exercise 12, show that we have the quasi-separated solutions

$$\begin{aligned} u(z;t) &= e^{cz}\cos(\beta t + cz) && u(z;t) = e^{-cz}\cos(\beta t - cz) \\ u(z;t) &= e^{cz}\sin(\beta t + cz) && u(z;t) = e^{-cz}\sin(\beta t - cz) \end{aligned}$$

where $c = \sqrt{\beta/2K}$.
14. By taking suitable linear combinations of the quasi-separated solutions found in Exercise 12 and steady-state solutions, solve the boundary-value problem in the slab $0 < z < L$.
15. Suppose that the daily temperature variation at the earth's surface is a periodic function $\varphi(t) = A_0 + A_1\cos(2\pi t/\tau)$. Find the depth necessary for a change from maximum to minimum daily temperature if $K = 2 \times 10^{-3}$ cm^2/s and $\tau = 24 \times 3600$ s.
16. Find the bounded solution of the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, satisfying the boundary conditions $u(0;t) = 1$ for $0 < t < \frac{1}{2}\tau$, $u(0;t) = -1$ for $\frac{1}{2}\tau < t < \tau$, where $u(0;t)$ is periodic with period τ.
17. Find the bounded solution of the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, satisfying the boundary condition $u_z(0;t) = A_1\cos\beta t$, where β and A_1 are positive constants.
18. Find the bounded solution of the heat equation $u_t = Ku_{zz}$ for $z > 0$, $-\infty < t < \infty$, satisfying the boundary condition $u_z(0;t) - hu(0;t) = A_1\cos\beta t$, where h, β, and A_1 are positive constants.
19. For the solution found in Exercise 14, find the limit of $u(z;t)$ when $L \to \infty$ and compare it with the solution for Example 2.1.3.
20. Find the steady-state solution of the heat equation $u_t = K\nabla^2 u + r$ in the slab $0 < z < L$ satisfying the boundary conditions $u_z(0;t) = h[u(0;t) - T_1]$, $u_z(L;t) = -h[u(L;t) - T_2]$, where r, h, T_1, and T_2 are positive constants.
21. For which values of the constants K, r, Φ_1, and Φ_2 does there exist a steady-state solution of the equation $u_t = K\nabla^2 u + r$ satisfying the boundary conditions $u_z(x,y,0;t) = \Phi_1$, $u_z(x,y,L;t) = \Phi_2$?

2.2. Homogeneous Boundary Conditions on a Slab

Many problems in mathematical physics and engineering involve a partial differential equation with initial conditions and boundary conditions. In this section we consider the case of homogeneous boundary conditions for the heat equation in the slab $0 < z < L$. In Sec. 2.3 we will consider the general nonhomogeneous boundary condition.

A homogeneous boundary condition at $z = 0$ has one of the following forms:

$$u(0;t) = 0 \quad \text{or} \quad u_z(0;t) = 0 \quad \text{or} \quad u_z(0;t) = hu(0;t)$$

where h is a nonzero constant that has the dimension of length^{-1}. All three of these may be included in the following succinct form:

$$\cos\alpha\, u(0;t) - L \sin\alpha\, u_z(0;t) = 0 \tag{2.2.1}$$

where the dimensionless parameter α satisfies $0 \le \alpha < \pi$. When $\alpha = 0$ we have the first boundary condition, $u(0;t) = 0$; when $\alpha = \pi/2$ we have the second boundary condition, $u_z(0;t) = 0$; and when $\cot\alpha = hL$ we have the third boundary condition, $u_z(0;t) = hu(0;t)$. Similarly, the general homogeneous boundary condition at $z = L$ is written in the form

$$\cos\beta\, u(L;t) + L \sin\beta\, u_z(L;t) = 0 \tag{2.2.2}$$

where $0 \le \beta < \pi$. The constant β is not related to α, in general.

2.2.1. Separated solutions with boundary conditions. We now discuss separated solutions of the heat equation $u_t = Ku_{zz}$ with the homogeneous boundary conditions (2.2.1) and (2.2.2). A separated solution of the heat equation is written

$$u(z;t) = \phi(z)T(t)$$

Substituting in the heat equation $u_t = Ku_{zz}$, we obtain

$$\phi(z)T'(t) = K\phi''(z)T(t)$$

Dividing by $K\phi(z)T(t)$, we obtain $T'(t)/KT(t) = \phi''(z)/\phi(z)$. The left side depends on t alone, and the right side depends on z alone; therefore each is a constant, which we call $-\lambda$. Thus we have the ordinary differential equations

$$T'(t) + \lambda K T(t) = 0 \tag{2.2.3}$$

$$\phi''(z) + \lambda\phi(z) = 0 \tag{2.2.4}$$

Equation (2.2.3) has the solution $T(t) = e^{-\lambda K t}$, which is never zero. To the second equation, (2.2.4), we must add the boundary conditions (2.2.1) and (2.2.2). The product $u(z;t) = \phi(z)T(t)$ satisfies (2.2.1) if and only if $\phi(z)$ satisfies the boundary condition $\cos\alpha\phi(0) - L\sin\alpha\phi'(0) = 0$. Similarly, $u(z;t)$ satisfies (2.2.2) if and only if $\phi(z)$ satisfies the boundary condition $\cos\beta\phi(L) + L\sin\beta\phi'(L) = 0$. This leads us to the following proposition.

PROPOSITION 2.2.1. *The separated solutions of the heat equation* $u_t = Ku_{zz}$ *with the boundary conditions (2.2.1) and (2.2.2) are of the form* $u_n(z;t) = e^{-\lambda_n Kt}\phi_n(z)$ *where* λ_n *is an eigenvalue and* $\phi_n(z)$ *is an eigenfunction of the Sturm-Liouville eigenvalue problem* $\phi''(z) + \lambda\phi(z) = 0$ *with the boundary conditions* $\cos\alpha\,\phi(0) - L\sin\alpha\,\phi'(0) = 0$, $\cos\beta\,\phi(L) + L\sin\beta\,\phi'(L) = 0$. *These eigenfunctions satisfy the orthogonality relation* $\int_0^L \phi_n(z)\phi_m(z)dz = 0$ *for* $m \neq n$.

Our first example corresponds to a slab with both faces maintained at temperature zero.

EXAMPLE 2.2.1. *Find all the separated solutions of the heat equation* $u_t = Ku_{zz}$ *for* $0 < z < L$ *satisfying the boundary conditions* $u(0;t) = 0$, $u(L;t) = 0$.

Solution. The associated Sturm-Liouville problem is $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi(L) = 0$. In Sec. 1.6, we found that the solutions are $\phi_n(z) = \sin(n\pi z/L)$, $\lambda_n = (n\pi/L)^2$. Thus we have the separated solutions

$$u_n(z;t) = \sin(n\pi z/L)\, e^{-(n\pi/L)^2 Kt}, \qquad n = 1, 2, \ldots \quad \bullet$$

The next example corresponds to a slab with one face insulated and the other face maintained at temperature zero.

EXAMPLE 2.2.2. *Find all the separated solutions of the heat equation* $u_t = Ku_{zz}$ *for* $0 < z < L$ *satisfying the boundary conditions* $u(0;t) = 0$, $u_z(L;t) = 0$.

Solution. The associated Sturm-Liouville problem is $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi'(L) = 0$. For $\lambda = 0$ the general solution of the differential equation is $\phi(z) = Az + B$. The first boundary condition requires $B = 0$, while the second boundary condition requires $A = 0$. Hence $\lambda = 0$ is not an eigenvalue. For $\lambda = -\mu^2 < 0$ the general solution satisfying the first boundary condition is $\phi(z) = A\sinh(\mu z)$, but this satisfies the second boundary condition if and only if $A = 0$; hence $\lambda < 0$ is not a possible eigenvalue. For $\lambda > 0$ the general solution of the differential equation is $\phi(z) = A\sin z\sqrt{\lambda} + B\cos z\sqrt{\lambda}$. The first boundary condition requires that $B = 0$, while the second boundary condition requires that $A\cos L\sqrt{\lambda} = 0$. For a nonzero solution we must take $L\sqrt{\lambda} = (n - \frac{1}{2})\pi$, $n = 1, 2, \ldots$. Therefore the solutions are $\phi_n(z) = \sin(n - \frac{1}{2})\pi z/L$, $\lambda_n = (n - \frac{1}{2})^2\pi^2/L^2$. The separated solutions of the heat equation are

$$u_n(z;t) = \sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left[-\left(n - \frac{1}{2}\right)^2 \pi^2 \frac{Kt}{L^2}\right], \quad n = 1, 2, \ldots \quad \bullet$$

2.2.2. Solution of the initial-value problem in a slab. Having obtained the separated solutions of the heat equation with homogeneous boundary conditions, we can solve the following *initial-value problem:*

$$\begin{aligned} u_t &= K u_{zz} && t>0, 0<z<L \\ \cos\alpha\, u(0;t) - L\sin\alpha\, u_z(0;t) &= 0 && t>0 \\ \cos\beta\, u(L;t) + L\sin\beta\, u_z(L;t) &= 0 && t>0 \\ u(z;0) &= f(z) && 0<z<L \end{aligned}$$

where $f(z), 0<z<L$, is a piecewise smooth function.

To solve this initial-value problem, we first expand $f(z)$ in a series of eigenfunctions of the Sturm-Liouville problem, in the form

$$f(z) = \sum_{n=1}^{\infty} A_n \phi_n(z) \qquad 0<z<L$$

[If f is discontinuous at z, the series converges to $\frac{1}{2}f(z+0)+\frac{1}{2}f(z-0)$.] The formal solution of the initial-value problem is given by the series

$$\boxed{u(z;t) = \sum_{n=1}^{\infty} A_n \phi_n(z) e^{-\lambda_n K t}} \tag{2.2.5}$$

The solution has been written as a superposition of separated solutions of the heat equation satisfying the indicated homogeneous boundary conditions. The Fourier coefficients A_n are obtained from the orthogonality relations by the formulas

$$\int_0^L f(z)\phi_n(z)dz = A_n \int_0^L \phi_n(z)^2 dz \qquad n=1,2,\ldots$$

To prove that the formal solution (2.2.5) is a rigorous solution of the heat equation, we must check that, for each $t>0$, the series for u, u_z, u_{zz}, and u_t are uniformly convergent for $0 \le z \le L$. This can be shown for each type of boundary condition we consider.

EXAMPLE 2.2.3. *Solve the initial-value problem $u_t = Ku_{zz}$ for $t>0$, $0<z<L$, with the boundary conditions $u(0;t)=0$, $u(L;t)=0$ and the initial condition $u(z;0)=1$.*

Solution. The separated solutions of the heat equation satisfying the boundary conditions are $\sin(n\pi z/L)\, e^{-(n\pi/L)^2 Kt}$, $n=1,2,\ldots$. To satisfy the initial condition, we must expand the function $f(z)=1$ in a Fourier sine series. The Fourier coefficients are given by

$$A_n \int_0^L \sin^2 \frac{n\pi z}{L}\, dz = \int_0^L \sin\frac{n\pi z}{L}\, dz = \frac{L}{n\pi}[1-(-1)^n]$$

Thus $A_n = (2/n\pi)[1-(-1)^n]$ and the solution is

$$u(z;t) = \frac{2}{\pi}\sum_{1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi z}{L}\,e^{-(n\pi/L)^2Kt}$$

For $t > 0$ and $0 \le z \le L$, this series converges uniformly, owing to the exponential factor. Likewise, the series for u_z, u_{zz}, and u_t converge uniformly for $0 \le z \le L$ and each $t > 0$. Thus u is a rigorous solution of the heat equation. •

2.2.3. Asymptotic behavior and relaxation time. In Example 2.2.3 we obtained a *transient solution* of the heat equation, meaning that $u(z;t)$ tends to zero when t tends to infinity. To analyze this more generally, we assume that the boundary conditions are $u(0;t) = 0$, $u(L;t) = 0$ and the initial condition is $u(z;0) = f(z)$, a piecewise smooth function. The solution is

$$u(z;t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L}\,e^{-(n\pi/L)^2Kt}$$

where A_n are the Fourier sine coefficients of the piecewise smooth function $f(z)$, $0 < z < L$. Thus

$$A_n = \frac{2}{L}\int_0^L f(z)\sin(n\pi z/L)\,dz \qquad \text{and} \qquad |A_n| \le 2M$$

where M is the maximum of $|f(z)|$, $0 < z < L$. Writing $a = \pi^2K/L^2$ and noting that $|\sin n\pi z/L| \le 1$, we have

$$|u(z;t)| \le 2M\sum_{n=1}^{\infty} e^{-n^2at}$$

But $n^2 \ge n$ for $n \ge 1$, and thus $e^{-n^2at} \le e^{-nat} = (e^{-at})^n$. Hence

$$\begin{aligned}|u(z;t)| &\le 2M\sum_{n=1}^{\infty}(e^{-at})^n\\ &= 2M\frac{e^{-at}}{1-e^{-at}}\end{aligned}$$

where we have used the formula for the sum of a geometric series $\sum_{n=1}^{\infty}\gamma^n = \gamma/(1-\gamma)$, $0 \le \gamma < 1$. When $t \to \infty$, $e^{-at} \to 0$, and we have shown that

$$u(z;t) = O(e^{-at}) \qquad t \to \infty$$

In particular $u(z;t) \to 0$ when $t \to \infty$, which means that $u(z;t)$ is a transient solution.

We define the *relaxation time* τ by the formula

$$\frac{1}{\tau} = -\lim_{t\to\infty}\frac{1}{t}\ln|u(z;t)|$$

provided that the limit exists and is independent of z, $0 < z < L$. For transient solutions of the heat equation, the relaxation time can be computed explicitly from the first nonzero term of the series solution. The following theorem extends the previous example to the general set of homogeneous boundary conditions.

THEOREM 2.1. *For the heat equation $u_t = Ku_{zz}$ with the boundary conditions (2.2.1) and (2.2.2), suppose that all eigenvalues λ_n are positive. Then $u(z;t) = \sum_{n=1}^{\infty} A_n\phi_n(z)e^{-\lambda_n Kt}$ is a transient solution of the heat equation, and the relaxation time is given by $\tau = 1/\lambda_1 K$ if $A_1 \neq 0$.*

EXAMPLE 2.2.4. *Compute the relaxation time for the solution*

$$u(z;t) = \sum_{n=1}^{\infty} A_n \sin(n\pi z/L)\, e^{-(n\pi/L)^2 Kt}$$

Solution. We write

$$u(z;t) = A_1 \sin\frac{\pi z}{L}\, e^{-(\pi/L)^2 Kt} + \sum_{n=2}^{\infty} A_n \sin\frac{n\pi z}{L}\, e^{-(n\pi/L)^2 Kt}$$

From the preceding analysis the last series is $O(e^{-(4\pi^2 Kt/L^2)})$ when $t \to \infty$. If $A_1 \neq 0$, we may write

$$\begin{aligned} u(z;t) &= A_1 \sin\frac{\pi z}{L}\, e^{-(\pi/L)^2 Kt}[1 + O(e^{-(3\pi^2 Kt/L^2)})] \\ \ln|u(z;t)| &= \ln|A_1| + \ln\sin\frac{\pi z}{L} - (\pi/L)^2 Kt + O(e^{-(3\pi^2 Kt/L^2)}) \end{aligned}$$

Thus $\lim_{t\to\infty} t^{-1}\ln|u(z;t)| = -\pi^2 K/L^2$. We have proved that $\tau = L^2/\pi^2 K$ provided that $A_1 \neq 0$. •

This analysis of relaxation time shows that, for large t, the solution $u(z;t)$ is well approximated by the first term of the series. This can also be seen graphically, by plotting the function $z \to u(z;t)$ for various values of t. When t is small, the solution is close to the initial function $f(z)$. As t increases, the solution tends to zero and assumes the shape of a sine curve, corresponding to the first term of the series solution. The graphs in Fig. 2.2.1 plot the solution of the initial-value problem $u_t = 2u_{zz}$ for $0 < z < \pi$, with the boundary conditions $u(z;0) = 0$, $u(\pi;0) = 0$ and the initial conditions $u(z;0) = 2z$ for the times $t = 0$, 0.005, 0.01, 0.05, 0.1, 0.2, 0.3, 0.5, 0.7, and 0.8.

2.2.4. Uniqueness of solutions. We now discuss the *uniqueness* of the solution of the initial-value problem. We have found a solution as a series of separated solutions, but it is conceivable that by another method we might produce a distinct solution of the heat equation with the same initial conditions and boundary conditions. We shall prove that this is impossible. To be specific, we take the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$.

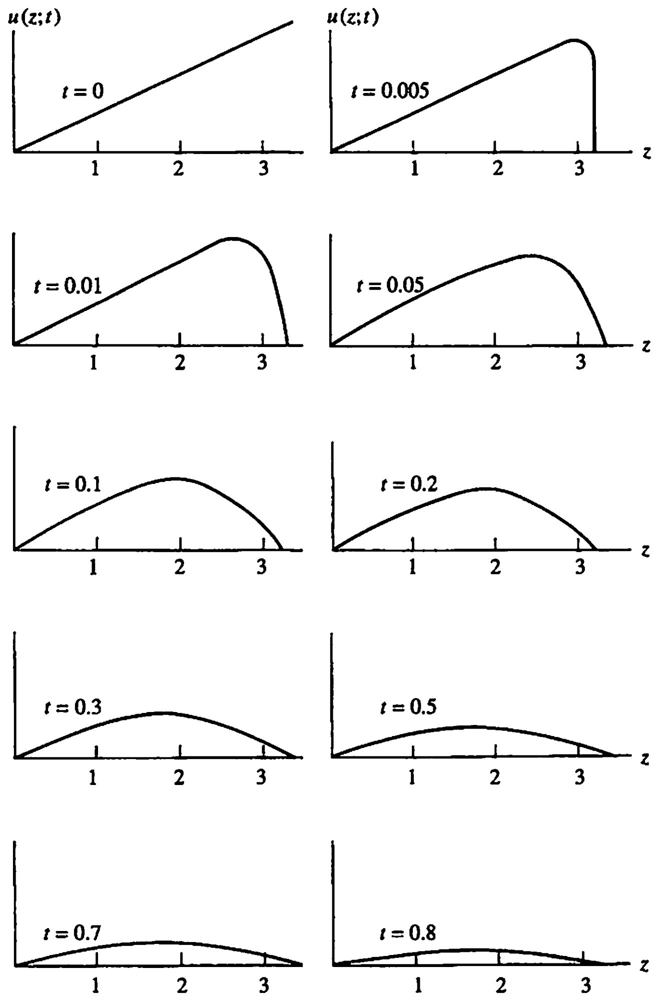


FIGURE 2.2.1 Solution of the heat equation at 10 different times.

For this purpose, suppose that u_1 and u_2 are two solutions with the same initial and boundary conditions, and set $u = u_1 - u_2$. Then u satisfies the heat equation with zero boundary conditions and zero initial conditions. Let

$$w(t) = \frac{1}{2}\int_0^L u(z;t)^2 dz$$

Then

$$w'(t) = \int_0^L u(z;t)u_t(z;t)dz \tag{2.2.6}$$

$$= K\int_0^L u(z;t)u_{zz}(z;t)dz \tag{2.2.7}$$

$$= Ku(z;t)u_z(z;t)|_0^L - K\int_0^L u_z(z;t)^2 dz \tag{2.2.8}$$

where we have used the heat equation to obtain (2.2.7) and integration by parts to obtain (2.2.8). Using the boundary conditions, we see that the first term in (2.2.8) is zero. Therefore we must have

$$w'(t) = -K\int_0^L u_z(z;t)^2 dz$$

But K is a positive constant and $u_z(z;t)^2 \ge 0$, since squares of real numbers are greater than or equal to zero. Thus we have both

$$w'(t) \le 0 \quad \text{and} \quad w(t) \ge 0$$

But $u(z;0) = 0$, which means that $w(0) = 0$. To complete the proof, we use the fundamental theorem of calculus:

$$w(t) = w(0) + \int_0^t w'(s)ds \le 0$$

Since $w(t) \ge 0$, we are forced to conclude that $w(t) \equiv 0$, which means that $u(z;t) = 0$ for each t, that is, $u_1(z;t) = u_2(z;t)$. Hence we have proved uniqueness of the solution.

The careful reader will notice that we have used the boundary conditions only to show that $uu_z \,|_0^L = 0$. Hence our proof applies also to other boundary conditions, for example, $u_z(0) = 0$, $u_z(L) = 0$.

2.2.5. Examples of transcendental eigenvalues. In certain cases we must solve the heat equation with the homogeneous boundary conditions

$$u(0;t) = 0, \qquad u_z(L,t) + hu(L;t) = 0 \tag{2.2.9}$$

where h is a positive constant. We will see that the eigenvalues are obtained by solving a transcendental equation. The separated solutions of the problem are of the form $u(z;t) = \phi(z)T(t)$, where $T(t) = e^{-\lambda K t}$, λ is an eigenvalue, and $\phi(z)$ is an eigenfunction of the Sturm-Liouville problem $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi'(L) + h\,\phi(L) = 0$. This was solved as Example 1.6.3 in Sec. 1.6, where we found the solutions $\phi(z) = B\sin(z\sqrt{\lambda})$, where λ is determined as a solution of the transcendental equation

$$\sqrt{\lambda}\cos(L\sqrt{\lambda}) + h\sin(L\sqrt{\lambda}) = 0 \tag{2.2.10}$$

Although (2.2.10) cannot be solved in closed form explicitly, a graphical solution can be determined by plotting the cotangent function against the reciprocal function. Indeed, we must have $\sin(L\sqrt{\lambda}) \neq 0$ for any solution of (2.2.10) [otherwise $\cos(L\sqrt{\lambda}) = \pm 1$, which does not satisfy (2.2.10)]. Hence (2.2.10) can be written in the form

$$\cot(L\sqrt{\lambda}) = -\frac{h}{\sqrt{\lambda}} = -\frac{hL}{L\sqrt{\lambda}} \tag{2.2.11}$$

From Fig. 2.2.2 it is clear that the smallest eigenvalue, λ_1, satisfies $\pi/2 < L\sqrt{\lambda_1} < \pi$ and that the higher eigenvalues satisfy $(n - \frac{1}{2})\pi < L\sqrt{\lambda_n} < n\pi$, with $L\sqrt{\lambda_n} \to (n - \frac{1}{2})\pi$ when $n \to \infty$. The discussion is complete.

It is interesting to examine the heat equation with the boundary conditions

$$u(0;t) = 0, \quad u_z(L;t) - hu(L;t) = 0$$

where $h > 0$. Although this does not correspond to a physically realistic boundary condition, the mathematical analysis reveals some new features. Following the previous discussion, we look for separated solutions in the form

$$u(z;t) = \phi(z)T(t)$$

Substituting in the heat equation, we obtain the ordinary differential equations

$$\begin{aligned} T'(t) + \lambda K T(t) &= 0 \\ \phi''(z) + \lambda\phi(z) &= 0 \end{aligned}$$

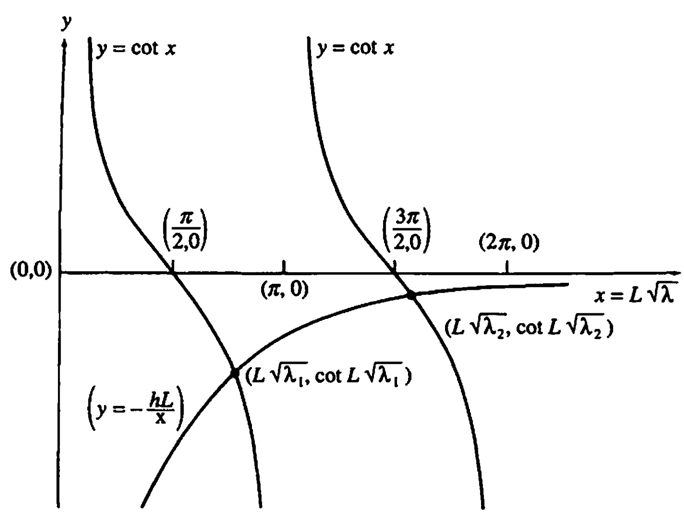


FIGURE 2.2.2 Graphical solution of the transcendental equation $\cot(L\sqrt{\lambda}) = -h/\sqrt{\lambda}$.

with the boundary conditions

$$\phi(0) = 0, \qquad \phi'(L) = h\phi(L)$$

The general solution satisfying the first boundary condition is of the form

$$\phi(z) = \begin{cases} B \sinh z\sqrt{-\lambda} & \lambda < 0 \\ Bz & \lambda = 0 \\ B \sin z\sqrt{\lambda} & \lambda > 0 \end{cases}$$

For the second boundary condition we consider three cases:

Case 1: $\lambda < 0$, $B\sqrt{-\lambda} \cosh L\sqrt{-\lambda} = hB \sinh L\sqrt{-\lambda}$
Case 2: $\lambda = 0$, $B = hBL$
Case 3: $\lambda > 0$, $B\sqrt{\lambda} \cos L\sqrt{\lambda} = hB \sin L\sqrt{-\lambda}$

In each case we desire a nontrivial solution; hence $B \neq 0$. Dividing by B, we can rewrite these three equations.

Case 1: $\lambda < 0$, $\tanh L\sqrt{-\lambda} = L\sqrt{-\lambda}/Lh$
Case 2: $\lambda = 0$, $Lh = 1$
Case 3: $\lambda > 0$, $\tan L\sqrt{\lambda} = L\sqrt{\lambda}/Lh$.

For $\lambda < 0$, we examine the graph of $y = \tanh x$ (Fig. 2.2.3). If $Lh \leq 1$, we see from the graph that the line $y = x/Lh$ does not intersect the curve $y = \tanh x$ for $x > 0$; hence there are no solutions for $\lambda < 0$, $Lh \leq 1$. If $Lh > 1$, the line $y = x/Lh$ intersects the curve $y = \tanh x$ in exactly one place $x > 0$; hence there is one solution $\lambda < 0$ if $Lh > 1$.

For $\lambda = 0$, we have a solution if and only if $Lh = 1$.

For $\lambda > 0$, we examine the graph of $y = \tan x$, shown in Fig. 2.2.4. From the graph we see that the line $y = x/Lh$ intersects the curve $y = \tan x$ infinitely many times. If $Lh < 1$, the first such intersection occurs for $0 < L\sqrt{\lambda} < \pi/2$; otherwise the first intersection occurs for $\pi < L\sqrt{\lambda} < 3\pi/2$.

Summarizing, we have the following:

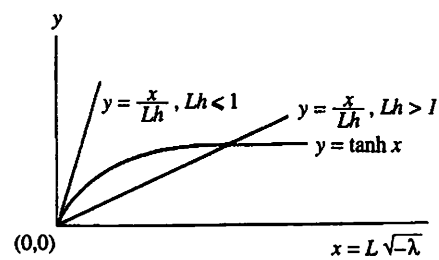


FIGURE 2.2.3 Graphical solution of the transcendental equation $\tanh L\sqrt{\lambda} = \sqrt{\lambda}/h$.

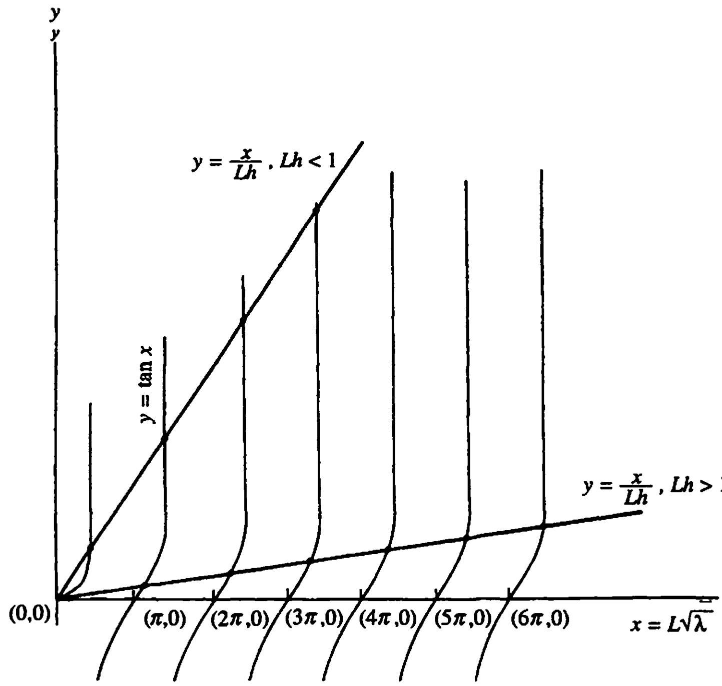


FIGURE 2.2.4 Graphical solution of the transcendental equation $\tan L\sqrt{\lambda} = \sqrt{\lambda}/h$.

$Lh < 1$: All separated solutions are of the form

$$
\begin{aligned}
u_z(z;t) &= B_n \sin z\sqrt{\lambda_n} e^{-\lambda_n K t} \qquad && n = 1, 2, \ldots \\
\frac{(n-1)^2\pi^2}{L^2} &< \lambda_n < \frac{(n-\frac{1}{2})^2\pi^2}{L^2} && \tan L\sqrt{\lambda_n} = \frac{\sqrt{\lambda_n}}{h}
\end{aligned}
$$

$Lh = 1$: There is one separated solution of the form

$$u_1(z;t) = B_1 z, \qquad \lambda_1 = 0$$

All other separated solutions are of the form

$$
\begin{aligned}
u_n(z;t) &= B_n \sin z\sqrt{\lambda_n} e^{-\lambda_n K t} \qquad && n = 2, 3, \ldots \\
\frac{(n-1)^2\pi^2}{L^2} &< \lambda_n < \frac{(n-\frac{1}{2})^2\pi^2}{L^2} && \tan L\sqrt{\lambda_n} = \frac{\sqrt{\lambda_n}}{h}
\end{aligned}
$$

$Lh > 1$: There is one separated solution of the form

$$u_1(z;t) = B_1 \sinh z\sqrt{-\lambda_1} e^{-\lambda_1 K t} \qquad \lambda_1 < 0, \ \tanh L\sqrt{-\lambda_1} = \frac{\sqrt{-\lambda_1}}{h}$$

All other separated solutions are of the form

$$u_n(z;t) = B_n \sin z\sqrt{\lambda_n}e^{-\lambda_n Kt} \qquad n = 2, 3, \ldots$$
$$\frac{(n-1)^2\pi^2}{L^2} < \lambda_n < \frac{(n-\frac{1}{2})^2\pi^2}{L^2}, \qquad \tan L\sqrt{\lambda_n} = \frac{\sqrt{\lambda_n}}{h}$$

Therefore we have found all the separated solutions of the heat equation with the boundary conditions (2.2.9). We emphasize that the eigenvalues λ_n must be determined graphically or by a numerical method. There is no elementary formula for the solution of the transcendental equation $\tan L\sqrt{\lambda_n} = \sqrt{\lambda_n}/h$.

EXERCISES 2.2

1. Solve the initial-value problem $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$ and the initial condition $u(z;0) = z$, $0 < z < L$.
2. Solve the initial-value problem $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$ and the initial conditions $u(z;0) = T$ for $0 < z < \frac{1}{2}L$, $u(z;0) = 0$ for $\frac{1}{2}L < z < L$, where T is a positive constant.
3. Solve the initial-value problem $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0;t) = 0$, $u_z(L;t) = 0$ and initial condition $u(z;0) = 3\sin \pi z/2L + 5\sin 3\pi z/2L$.
4. Find all of the separated solutions of the heat equation $u_t = Ku_{zz}$ satisfying the boundary conditions $u_z(0;t) = 0$, $u_z(L;t) = 0$.
5. Solve the initial-value problem $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u_z(0;t) = 0$, $u_z(L;t) = 0$ and the initial condition $u(z;0) = z$, $0 < z < L$.
6. Solve the initial-value problem $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u_z(0;t) = 0$, $u_z(L;t) = 0$ and the initial condition $u(z;0) = 3 + 4\cos \pi z/L + 7\cos 3\pi z/L$, $0 < z < L$.
7. Consider the heat equation

$$u_t = Ku_{zz} \qquad t > 0, 0 < z < L$$
$$u(0;t) = 0 \qquad u_z(L;t) = -hu(L,t)$$

where $h > 0$. Show that *all* separated solutions are of the form

$$u_n(z;t) = B_n \sin z\sqrt{\lambda_n}e^{-\lambda_n Kt}$$

where $\lambda_n > 0$ are the solutions of

$$\tan L\sqrt{\lambda} = -\frac{\sqrt{\lambda}}{h}$$

8. By direct computation, show that

$$\int_0^L \sin z\sqrt{\lambda_n} \sin z\sqrt{\lambda_m} dz = 0 \qquad n \neq m$$

where $\{\lambda_n\}$ are the solutions obtained in Exercise 7.

9. Solve the initial-value problem $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0;t) = 0$, $u_z(L;t) = -hu(L;t)$ and the initial condition $u(z;0) = 1$, where h is a positive constant.
10. Find all of the separated solutions of the heat equation $u_t = Ku_{zz}$ satisfying the boundary conditions $u_z(0;t) = hu(0;t)$, $u_z(L;t) + hu(L;t) = 0$, where h is a positive constant.
11. Solve the heat equation $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$, with the boundary conditions $u_z(0;t) = hu(0;t)$, $u_z(L;t) + hu(L;t) = 0$ and the initial condition $u(z;0) = 1$, where h is a positive constant.
12. Find the relaxation time for the solution found in Exercise 1.
13. Find the relaxation time for the solution found in Exercise 2.
14. Find the relaxation time for the solution found in Exercise 3.
15. Find the relaxation time for the solution found in Exercise 9. (It may be expressed in terms of the solution of a certain transcendental equation.)
16. In Exercise 2, suppose that $K = 0.15\,\text{cm}^2/\text{s}$, $L = 40$ cm, $T = 100$°C. Compute the relaxation time and $u(20;t)$ for $t = 0.1, 1.0, 10.0, 100$ minutes.
17. In Exercise 3, suppose that $K = 0.15\ \text{cm}^2/\text{s}$, $L = 40\,\text{cm}$, $T = 100$°C. Compute the relaxation time and $u(20;t)$ for $t = 0.1, 1.0\ 10.0, 100$ minutes.

Exercises 18 to 20 treat heat flow in a circular ring of circumference L.

18. Find all of the separated solutions of the heat equation $u_t = Ku_{zz}$ satisfying the periodic boundary conditions $u(0;t) = u(L;t)$, $u_z(0;t) = u_z(L;t)$.
19. Solve the heat equation $u_t = Ku_{zz}$ with the periodic boundary conditions $u(0;t) = u(L;t)$, $u_z(0;t) = u_z(L;t)$ and the initial conditions $u(z;0) = 100$ if $0 < z < \frac{1}{2}L$ and $u(z;0) = 0$ if $\frac{1}{2}L < z < L$.
20. For the solution of Exercise 19, find the relaxation time. Compare your result with the relaxation time for heat flow in a slab of width L, with zero boundary conditions, found in Exercise 13.
21. Extend the uniqueness proof in this section to the heat equation with boundary conditions $u_z(0;t) = 0$, $u_z(L;t) = 0$.
22. Extend the uniqueness proof in this section to the heat equation with the boundary conditions $u_z(0,t) - h_1\, u(0;t) = 0$, $u_z(L;t) + h_2\, u(L;t) = 0$, where h_1 and h_2 are *positive* constants.

2.3. Nonhomogeneous Boundary Conditions

In this section we give the complete treatment of initial-value problems for the heat equation in the slab $0 < z < L$, with general boundary conditions. Our analysis is based on a *five-stage method*, which will also apply to initial-value problems for the heat equation in other coordinate systems, studied in Chapters 3 and 4.

2.3.1. Statement of problem. An initial-value problem for the heat equation in the slab $0 < z < L$ is the following set of four conditions:

$$u_t = Ku_{zz} + r \tag{2.3.1}$$

$$\cos\alpha\, u(0;t) - L\sin\alpha\, u_z(0;t) = T_1 \tag{2.3.2}$$

$$\cos\beta\, u(L;t) + L\sin\beta\, u_z(L;t) = T_2 \tag{2.3.3}$$

$$u(z;0) = f(z) \tag{2.3.4}$$

where $f(z), 0 < z < L$, is a piecewise smooth function and α, β, r, T_1, T_2, and K are constants. We seek the solution $u(z;t)$ for all $t > 0$, $0 < z < L$.

The heat equation (2.3.1) is nonhomogeneous, owing to the source term r. The boundary conditions (2.3.2) and (2.3.3) are nonhomogeneous, owing to the constants T_1 and T_2.

2.3.2. Five-stage method of solution. *Stage 1: Steady-state solution.* We first ignore the initial conditions and look for a function $U(z)$ that satisfies the heat equation (2.3.1) and the boundary conditions (2.3.2) and (2.3.3). Thus we must have $KU''(z) + r = 0$, whose general solution is $U(z) = -rz^2/2K + A + Bz$. The boundary conditions determine the constants A and B, as we have shown by examples in Sec. 2.1, except in the case where $\lambda = 0$ is an eigenvalue of the associated homogeneous boundary conditions. In this case we take the most general steady-state solution; the ambiguity will be resolved in stage 3.

Stage 2: Transformation of the problem. Having obtained the steady-state solution, we use the subtraction principle to transform the problem to a heat equation with no internal sources and homogeneous boundary conditions. To do this, we define a new unknown function,

$$v(z;t) = u(z;t) - U(z)$$

We have $v_t(z;t) = u_t(z;t)$, $v_z(z;t) = u_z(z;t) - U'(z)$, $v_{zz}(z;t) = u_{zz}(z;t) - U''(z)$. Thus $v_t - Kv_{zz} = u_t - K[u_{zz} - U''(z)] = Ku_{zz} + r - Ku_{zz} - r = 0$. Likewise, v satisfies the boundary conditions (2.3.2) and (2.3.3) with $T_1 = 0$, $T_2 = 0$. Thus we have

$$v_t = Kv_{zz} \tag{2.3.5}$$

$$\cos\alpha\, v(0;t) - L\sin\alpha\, v_z(0;t) = 0 \tag{2.3.6}$$

$$\cos\beta\, v(L;t) + L\sin\beta\, v_z(L;t) = 0 \tag{2.3.7}$$

$$v(z;0) = f(z) - U(z) \tag{2.3.8}$$

Thus $v(z;t)$ satisfies a homogeneous equation with homogeneous boundary conditions and a new initial condition. This type of problem was treated in Sec. 2.2.

Stage 3: Separation of variables. To determine the new unknown function $v(z;t)$, we use a superposition of separated solutions of the heat equation with homogeneous boundary conditions (2.3.6) and (2.3.7).

$$v(z;t) = \sum_{n=1}^{\infty} A_n \phi_n(z) e^{-\lambda_n K t}$$

The coefficients A_n are determined by expanding the initial condition $f(z)-U(z)$ in a series of eigenfunctions $\sum_{n=1}^{\infty} A_n \phi_n(z)$. Equivalently, they may be computed from the integrals

$$\int_0^L [f(z) - U(z)]\phi_n(z)dz = A_n \int_0^L \phi_n(z)^2 dz \qquad n = 1, 2, \ldots$$

The formal solution of the initial-value problem is

$$\boxed{u(z;t) = U(z) + \sum_{n=1}^{\infty} A_n \phi_n(z) e^{-\lambda_n K t}} \tag{2.3.9}$$

If $\lambda = 0$ is an eigenvalue of the Sturm-Liouville problem for $\phi''(z) + \lambda\phi(z) = 0$ with the associated homogeneous boundary conditions, the steady-state solution $U(z)$ is determined uniquely by requiring that $U(z) - v(z)$ be orthogonal to the eigenfunction $\phi_1(z)$, for which $\lambda_1 = 0$. We take $A_1 = 0$ in this case.

Stage 4: Verification of the solution. At this point it is appropriate to verify that the formal solution (2.3.9) is unique and indeed satisfies the initial-value problem. To illustrate the proof, we assume that $\alpha = 0$, $\beta = 0$ where the eigenfunctions are $\phi_n(z) = \sin(n\pi z/L)$ and the eigenvalues are $\lambda_n = (n\pi/L)^2$. Then the series for u, u_z, u_{zz}, u_t are

$$\begin{aligned}
u: &\quad U(z) + \sum_{n=1}^{\infty} A_n \sin(n\pi z/L) e^{-\lambda_n K t} \\
u_z: &\quad U'(z) + \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \cos(n\pi z/L) e^{-\lambda_n K t} \\
u_{zz}: &\quad U''(z) - \sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \sin(n\pi z/L)\, e^{-\lambda_n K t} \\
u_t: &\quad -\sum_{n=1}^{\infty} A_n K \left(\frac{n\pi}{L}\right)^2 \sin(n\pi z/L)\, e^{-\lambda_n K t}
\end{aligned}$$

We have $|A_n| \le 2M$, where M is the maximum of $|f(z) - U(z)|$. Therefore, for each $t > 0$, each of these series is uniformly convergent for $0 \le z \le L$. Hence the colons become equality and u satisfies the heat equation $u_t = K u_{zz} + r$, together with the boundary conditions.

If u_1 and u_2 are two solutions of the problem, their difference $u = u_1 - u_2$ must satisfy the heat equation with zero initial and boundary conditions. But we have already shown in Sec. 2.2 that such a function is zero. Hence we have proved the uniqueness of our solution.

Stage 5: Asymptotic behavior. We now discuss the convergence of $u(z;t)$ to the steady-state solution $U(z)$ when $t \to \infty$. While this may be obvious on physical grounds, the mathematical analysis has not been given.

To do this, we assume that all eigenvalues are positive, $\lambda_n > 0$, for $n = 1, 2, \ldots$. The basic fact about convergence is

$$u(z;t) - U(z) = O(e^{-\lambda_1 K t}) \qquad t \to \infty$$

We illustrate the proof in this case of zero boundary conditions $u(0;t) = 0$, $u(L;t) = 0$. In this case $\lambda_n = (n\pi/L)^2$, $|A_n| \le 2M$, and we have

$$\begin{aligned}
|u(z;t) - U(z)| &= \left|\sum_{n=1}^{\infty} A_n \sin(n\pi z/L)\, e^{-(n\pi/L)^2 K t}\right| \\
&\le \sum_{n=1}^{\infty} 2M(e^{-at})^{n^2} \qquad\qquad \left(a = \frac{\pi^2 K}{L^2}\right) \\
&\le 2M \sum_{n=1}^{\infty} (e^{-at})^n \\
&= \frac{2Me^{-at}}{1 - e^{-at}}
\end{aligned}$$

The denominator tends to 1 when $t \to \infty$, and we have demonstrated that $u(z;t) - U(z) = O(e^{-at})$, $t \to \infty$, which was to be shown.

Finally we discuss the *relaxation time.* This is the number τ defined by the equation

$$\frac{1}{\tau} = -\lim_{t\to\infty} \frac{1}{t} \ln |u(z;t) - U(z)|$$

provided that the limit exists and is independent of z, $0 < z < L$.

The basic fact about relaxation time is that

$$\boxed{\tau = \frac{1}{\lambda_1 K}} \tag{2.3.10}$$

provided that $A_1 \neq 0$.[1] We illustrate the computation in the case of zero boundary conditions $u(0;t) = 0$, $u(L;t) = 0$. In this case

$$\begin{aligned}
u(z;t) - U(z) &= A_1 \sin\frac{\pi z}{L} e^{-(\pi^2 Kt/L^2)} + \sum_{n=2}^{\infty} A_n \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt} \\
&= A_1 \sin\frac{\pi z}{L} e^{-(\pi^2 Kt/L^2)} + O(e^{-(4\pi^2 Kt/L^2)}) \\
&= A_1 \sin\frac{\pi z}{L} e^{-(\pi^2 Kt/L^2)}[1 + O(e^{-(3\pi^2 Kt/L^2)})] \\
\ln|u(z;t) - U(z)| &= \ln\left|A_1 \sin\frac{\pi z}{L}\right| - \frac{\pi^2 Kt}{L^2} + \ln[1 + O(e^{-(3\pi^2 Kt/L^2)})]
\end{aligned}$$

The final term tends to zero when $t \to \infty$, and the first term is independent of t. Dividing by t, we see that

$$\lim_{t\to\infty} \frac{1}{t} \ln|u(z;t) - U(z)| = -\frac{\pi^2 K}{L^2}$$

Therefore the relaxation time is given by $\tau = L^2/\pi^2 K$, provided $A_1 \neq 0$. This makes good physical sense, for the larger the conductivity K, the smaller the relaxation time. Likewise the larger the width of the slab L, the larger the relaxation time. (If $A_1 = 0$ and $A_2 \neq 0$, then $\tau = L^2/4\pi^2 K$.)

This concludes our analysis of the initial-value problem (2.3.1)–(2.3.4). We will use this method repeatedly for solving problems with nonhomogeneous boundary conditions.

The following examples are worked in detail in order to illustrate the five-stage method.

EXAMPLE 2.3.1. *Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0;t) = T_1$, $u_z(L;t) = 0$ and the initial condition $u(z;0) = T_3$, where T_1 and T_3 are positive constants.*

Solution. We use the five-stage method outlined above.

Stage 1: Steady-state solution. In this case the steady-state solution satisfies $U_{zz} = 0$, $U(0) = T_1$, $U_z(L) = 0$. The general solution is $U(z) = Az + B$, and the boundary conditions give $B = T_1$, $A = 0$, so the steady-state solution is

$$U(z) = T_1$$

Stage 2: Transformation of the problem. Again we set $v(z;t) = u(z;t) - U(z)$. Then $v(z;t)$ satisfies

(1′) $v_t = Kv_{zz}$
(2′) $v(0;t) = 0$, $t > 0$
(3′) $v_z(L;t) = 0$, $t > 0$
(4′) $v(z;0) = T_3 - T_1$

[1] If $A_1 = 0$ and $A_2 \neq 0$, then $\tau = 1/(\lambda_2 K)$.

The boundary conditions (2′) and (3′) are *homogeneous*; this means that the superposition principle can be applied.

Stage 3: Separation of variables. We look for separated solutions

$$v(z;t) = \phi(z)T(t)$$

that satisfy the heat equation and the homogeneous boundary conditions $v(0;t) = 0$; $v_z(L;t) = 0$. The heat equation requires that

$$\frac{\phi''(z)}{\phi(z)} = \frac{T'(t)}{KT(t)}$$

and hence both sides equal the constant $-\lambda$. Thus

$$T'(t) + K\lambda T(t) = 0 \tag{2.3.11}$$

$$\phi''(z) + \lambda\phi(z) = 0 \tag{2.3.12}$$

Equation (2.3.11) requires that

$$T(t) = Ce^{-\lambda Kt}$$

for some constant C. Equation (2.3.12) must be solved taking into account the boundary conditions. Separating the three cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, we have

$$\begin{aligned}
\phi(z) &= A\cos z\sqrt{\lambda} + B\sin z\sqrt{\lambda} && (\lambda > 0)\\
\phi(z) &= A + Bz && (\lambda = 0)\\
\phi(z) &= A\cosh z\sqrt{-\lambda} + B\sinh z\sqrt{-\lambda} && (\lambda < 0)
\end{aligned}$$

We now apply the boundary conditions. In the case where $\lambda > 0$, this means that $A = 0$, $B\sqrt{\lambda}\cos L\sqrt{\lambda} = 0$, so that for a nonzero solution we must have

$$L\sqrt{\lambda} = \left(n - \frac{1}{2}\right)\pi \qquad n = 1, 2, \ldots$$

In the case where $\lambda = 0$, we must have $A = 0$, $B = 0$, which is a zero solution. Finally, in the case where $\lambda < 0$, the boundary conditions require $A = 0$, $B\sqrt{-\lambda}\cosh L\sqrt{-\lambda} = 0$, again a zero solution.

The separated solutions of the heat equation that satisfy the homogeneous boundary are therefore of the form

$$\sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left\{-\left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2 Kt\right\} \qquad n = 1, 2, \ldots$$

As before, the superposition principle shows that any function of the form

$$v(z;t) = \sum_{n=1}^{\infty} A_n \sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right)\exp\left\{-\left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2 Kt\right\}$$

is a solution of the heat equation with homogeneous boundary conditions. To satisfy the new initial conditions, we set $t = 0$ and obtain

$$T_3 - T_1 = \sum_{n=1}^{\infty} A_n \sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left((2n-1)\frac{\pi z}{2L}\right)$$

This Sturm-Liouville series on $0 < z < L$ can be thought of as a Fourier sine series on $0 < z < 2L$ for which the even-numbered sine terms are absent. To determine the coefficients A_n, we use the orthogonality relations, with the result

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L (T_3 - T_1)\sin\left(\left(n - \frac{1}{2}\right)\frac{\pi z}{L}\right) dz \\ &= \frac{2}{\pi}\frac{T_3 - T_1}{n - \frac{1}{2}} \qquad n = 1, 2, \ldots \end{aligned}$$

Therefore the solution to the original problem is

$$u(z;t) = T_1 + \frac{2}{\pi}(T_3 - T_1)\sum_{n=1}^{\infty}\frac{\sin(n-\frac{1}{2})(\pi z/L)}{n - \frac{1}{2}} \exp\left\{-\left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2 Kt\right\}$$

Stage 4: Verification of the solution. As before, it can be verified that $u(z;t)$ satisfies the heat equation, initial conditions, and boundary conditions. We leave the verification as an exercise.

Stage 5: Asymptotic behavior. To determine the asymptotic behavior as $t \to \infty$, we note that $|A_n| \le 4/\pi(T_3 - T_1)$, $(n - \frac{1}{2})^2 \ge \frac{1}{2}(n - \frac{1}{2})$, and therefore

$$\begin{aligned} |u(z;t) - U(z)| &\le \frac{4}{\pi}(T_3 - T_1)\sum_{n=1}^{\infty}\exp\left[-\frac{1}{2}\left(n - \frac{1}{2}\right)\frac{\pi^2}{L^2}Kt\right] \\ &= \frac{4}{\pi}(T_3 - T_1)\frac{\exp(-\pi^2 Kt/4L^2)}{1 - \exp(-\pi^2 Kt/2L^2)} \end{aligned}$$

Hence we see, as before, that $u(z;t) \to U(z)$ when $t \to \infty$. The relaxation time is given by $\tau = 4L^2/\pi^2 K$, provided that $T_1 \ne T_3$. •

Our next example corresponds to a slab where one face exchanges heat by convection and the other face is maintained at a fixed temperature.

EXAMPLE 2.3.2. *Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0;t) = T_1$, $u_z(L;t) = -h[u(L;t) - T_2]$ and the initial condition $u(z;0) = T_3$, where h, T_1, T_2 and T_3 are positive constants.*

Solution. We use the five-stage method.

Stage 1. We look for the steady-state solution $U(z)$ that satisfies $KU'' = 0$ and the boundary conditions $U(0) = T_1$, $U'(L) = -h[U(L) - T_2]$. The general solution of the equation is $U(z) = A + Bz$. The first boundary condition requires $A = T_1$, while the second boundary condition requires $B = -h(A + BL - T_2)$. Solving these, we have the steady-state solution $U(z) = T_1 + hz(T_2 - T_1)/(1 + hL)$.

Stage 2. We use the steady-state solution to transform to a homogeneous equation with homogeneous boundary conditions. Thus setting $v(z;t) = u(z;t) - U(z)$, we have $v_t = Kv_{zz}$ with the boundary conditions $v(0;t) = 0$, $v_z(L;t) + hv(L;t) = 0$ and the initial condition $v(z;0) = T_3 - U(z)$.

Stage 3. The separated solutions of the heat equation $v_t = Kv_{zz}$ with the homogeneous boundary conditions are of the form $v(z;t) = e^{-\lambda Kt}\phi(z)$, where λ is an eigenvalue and $\phi(z)$ is an eigenfunction of the Sturm-Liouville problem $\phi''(z) + \lambda\phi(z) = 0$ with the boundary conditions $\phi(0) = 0$, $\phi'(L) + h\phi(L) = 0$. These boundary conditions satisfy the positivity condition of Sec. 1.6 for Sturm-Liouville eigenvalue problems, and thus we know that $\lambda > 0$. The general solution of the differential equation is $\phi(z) = A\sin z\sqrt{\lambda} + B\cos z\sqrt{\lambda}$. The boundary condition at $z = 0$ requires $B = 0$, while the boundary condition at $z = L$ requires $A\sqrt{\lambda}\cos L\sqrt{\lambda} + Ah\sin L\sqrt{\lambda} = 0$. We obtain a nonzero solution by taking $A \neq 0$; thus λ must be a solution of the transcendental equation $\sqrt{\lambda}\cos L\sqrt{\lambda} + h\sin L\sqrt{\lambda} = 0$. These eigenvalues may be obtained graphically by examining the graph of the cotangent function (Fig. 2.3.1). Therefore the separated solutions of the heat equation with homogeneous boundary conditions are $\sin z\sqrt{\lambda_n}e^{-\lambda_n Kt}$, where the λ_n are determined as before.

The initial-value problem for $v(z;t)$ is solved by a superposition of separated solutions.

$$v(z;t) = \sum_{n=1}^{\infty} A_n \sin z\sqrt{\lambda_n}e^{-\lambda_n Kt}$$

The Fourier coefficients are obtained from the expansion of

$$T_3 - U(z) = \sum_{n=1}^{\infty} A_n \sin z\sqrt{\lambda_n} \qquad 0 < z < L$$

Using orthogonality, we have the integral formulas

$$\int_0^L [T_3 - U(z)]\sin z\sqrt{\lambda_n}\,dz = A_n \int_0^L \sin^2(z\sqrt{\lambda_n})\,dz \qquad n = 1, 2, \ldots$$

To compute these integrals, we may use the integration formulas

$$\begin{aligned}
\int_0^L \sin z\sqrt{\lambda}\,dz &= \frac{1 - \cos L\sqrt{\lambda}}{\sqrt{\lambda}} \\
\int_0^L z\sin z\sqrt{\lambda}\,dz &= \frac{-L\cos L\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{\sin L\sqrt{\lambda}}{\lambda} \\
\int_0^L \sin^2 z\sqrt{\lambda}\,dz &= \frac{1}{2}\left(L - \frac{\sin 2L\sqrt{\lambda}}{2\sqrt{\lambda}}\right)
\end{aligned}$$

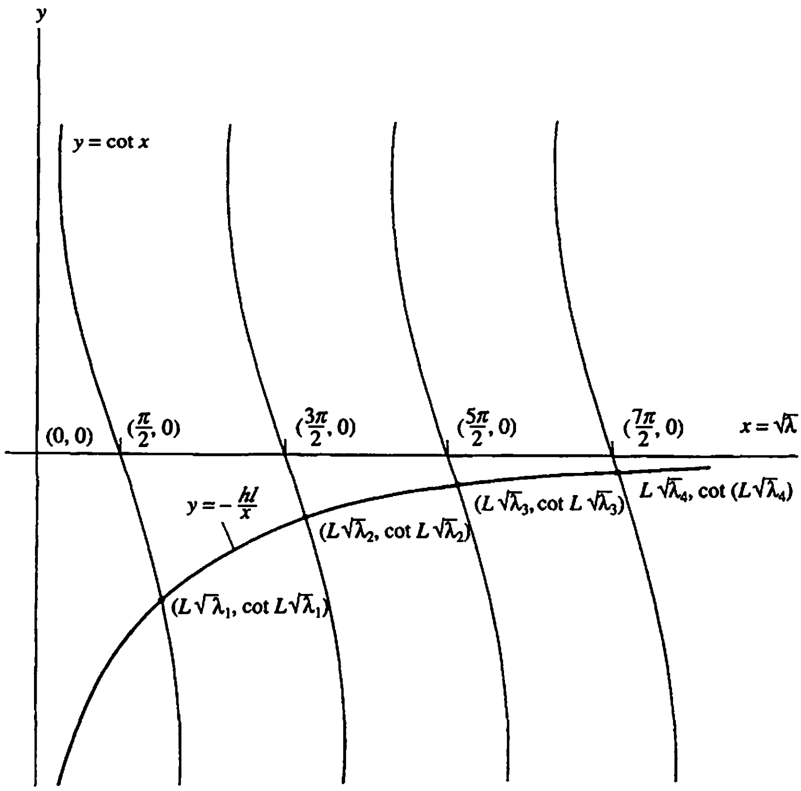


FIGURE 2.3.1 Graphical solution of the transcendental equation $\cot L\sqrt{\lambda} = -h/\sqrt{\lambda}$.

By making the necessary substitutions and using the transcendental equation for λ_n, we obtain the Fourier coefficients in the form

$$\begin{aligned}\frac{1}{2}A_n(\lambda_n L + h\sin^2 L\sqrt{\lambda_n}) &= (T_3 - T_1)(\sqrt{\lambda_n} + h\sin L\sqrt{\lambda_n}) \\ &\quad + h(T_1 - T_2)\sin L\sqrt{\lambda_n}\end{aligned}$$

The formal solution of the initial-value problem is

$$u(z;t) = U(z) + \sum_{n=1}^{\infty} A_n \sin z\sqrt{\lambda_n} e^{-\lambda_n K t}$$

where the A_n have just been obtained and the λ_n are determined from the transcendental equation $\sqrt{\lambda_n}\cos L\sqrt{\lambda_n} + h\sin L\sqrt{\lambda_n} = 0$.

Stage 4: Verification of the solution. From the graph of the cotangent function, we see that $L\sqrt{\lambda_n} - (n-\frac{1}{2})\pi \to 0$ when $n \to \infty$. From the formula for A_n, we see that A_n tends to zero when $n \to \infty$; in particular, $|A_n| \le M$ for some constant M. These estimates permit us to conclude that, for each $t > 0$, the following series are uniformly convergent for $0 \le z \le L$:

$$\sum_{n=1}^{\infty} A_n \sin(z\sqrt{\lambda_n})e^{-\lambda_n Kt}$$
$$\sum_{n=1}^{\infty} A_n \sqrt{\lambda_n}\cos(z\sqrt{\lambda_n})e^{-\lambda_n Kt}$$
$$\sum_{n=1}^{\infty} A_n \lambda_n \sin(z\sqrt{\lambda_n})e^{-\lambda_n Kt}$$

Therefore $u(z;t)$ is a differentiable function and $u(z;t)$ satisfies the heat equation $u_t = Ku_{zz}$ for $t > 0$, $0 < z < L$.

Stage 5: Asymptotic behavior. When $t \to \infty$, $u(z;t)$ tends to the steady-state solution $U(z)$. The relaxation time is obtained from the first term of the series. Thus, if $T_3 - T_1$ and $T_1 - T_2$ are both positive, we have $\tau = 1/\lambda_1 K$, where λ_1 is obtained from the graph of the cotangent function in Fig. 2.3.1. •

2.3.3. Temporally nonhomogeneous problems. The methods used to study the general boundary-value problem for the one-dimensional heat equation in the temporally homogeneous case can also be used to study problems with explicit time dependence. The most general problem of this type is of the form

$$\begin{aligned} u_t - Ku_{zz} &= r(z;t) \\ \cos\alpha\, u(0;t) - L\sin\alpha\, u_z(0;t) &= T_1(t) \\ \cos\beta\, u(L;t) + L\sin\beta\, u_z(L;t) &= T_2(t) \\ u(z;0) &= f(z) \end{aligned}$$

Here $r(z;t)$, $f(z)$, $T_1(t)$, and $T_2(t)$ are given functions, assumed to be piecewise smooth in each variable.

To solve a problem of this type, we first consider the case of homogeneous boundary conditions, that is, $T_1(t) \equiv 0$, $T_2(t) \equiv 0$. The solution is sought in the form of a series of eigenfunctions of the homogeneous problem. Thus

$$r(z;t) = \sum_{n=1}^{\infty} r_n(t)\phi_n(z), \qquad f(z) = \sum_{n=1}^{\infty} f_n\phi_n(z), \qquad u(z;t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(z)$$

where $\phi_n(z)$ are normalized eigenfunctions of the Sturm-Liouville eigenvalue problem with the associated boundary conditions. The expansion coefficients $r_n(t)$, f_n, and $u_n(t)$ are obtained from the orthogonality of the eigenfunctions as the generalized Fourier coefficients:

$$r_n(t) = \int_0^L r(z;t)\phi_n(z)\,dz, \quad f_n = \int_0^L f(z)\phi_n(z)\,dz, \quad u_n(t) = \int_0^L u(z;t)\phi_n(z)\,dz$$

Substituting the series for $u(z;t)$ into the nonhomogeneous heat equation, we have

$$u_t - Ku_{zz} = \sum_{n=1}^{\infty}[u_n'(t) + K\lambda_n u_n(t)]\phi_n(z) = \sum_{n=1}^{\infty} r_n(t)\phi_n(z)$$

Therefore we choose $u_n(t)$ as the solution of the ordinary differential equation $u_n'(t) + K\lambda_n u_n(t) = r_n(t)$, with the initial conditions $u_n(0) = f_n$. This first-order ordinary differential equation is easily solved in the form

$$u_n(t) = f_n e^{-\lambda_n Kt} + \int_0^t r_n(s)e^{-\lambda_n K(t-s)}\,ds$$

It is not difficult to show that the above series obtained converges uniformly for $0 < z < L$ together with the differentiated series for u_z, u_{zz}, u_t and that the function obtained satisfies the equation $u_t = Ku_{zz} + r(z;t)$.

To solve the original problem with $T_1(t)$ and $T_2(t)$ nonzero, we reduce to homogeneous boundary conditions by defining a new function,

$$v(z;t) = u(z;t) - [A(t) + z\,B(t)]$$

In order for $u(z;t)$ to solve the stated equation, we must have

$$\begin{aligned} v_t - Kv_{zz} &= r(z;t) - [A'(t) + z\,B'(t)] \\ v(z;0) &= f(z) - [A(0) + z\,B(0)] \end{aligned}$$

The functions $A(t)$, $B(t)$ are chosen so that the linear function $A(t) + z\,B(t)$ satisfies the nonhomogeneous boundary conditions of the given problem. This requires that

$$\begin{aligned} \cos\alpha\,A(t) - L\sin\alpha\,B(t) &= T_1(t) \\ \cos\beta[A(t) + L\,B(t)] + L\sin\beta\,B(t) &= T_2(t) \end{aligned}$$

This system of linear equations has a unique solution if and only if the determinant of the coefficients is nonzero—$\cos\alpha\sin\beta + \sin\alpha\cos\beta + \cos\alpha\cos\beta \neq 0$, which is equivalent to the statement that $\lambda = 0$ is *not* an eigenvalue of the Sturm-Liouville problem with the associated homogeneous boundary conditions. Assuming this, we can determine $A(t)$, $B(t)$ and infer that the function $v(z;t)$ so defined satisfies the homogeneous boundary conditions. This leads to the trial solution in the form

$$v(z;t) = \sum_{n=1}^{\infty} v_n(t)\phi_n(z)$$

This can be done as above, once we determine the right side. We must have

$$\begin{aligned} v_t - Kv_{zz} &= r(z;t) - [A'(t) + zB'(t)] \\ v(z;0) &= f(z) -.[A(0) + z\,B(0)] \end{aligned}$$

To solve the problem, we must find the generalized Fourier series of the linear function $a + bz$,

$$a + bz = \sum_{n=1}^{\infty}(a\langle 1, \phi_n\rangle + b\langle z, \phi_n\rangle)\phi_n(z)$$

where we have used the inner product notation for the generalized Fourier coefficients. Replacing $r_n(t)$ and v_n suitably, we are led to the solution

$$\begin{aligned} v_n(t) &= e^{-\lambda_n K t}[v_n - A(0)\langle 1, \phi_n\rangle - B(0)\langle z, \phi_n\rangle] \\ &\quad + \int_0^t e^{-\lambda_n K(t-s)}[r_n(s) - A'(s)\langle 1, \phi_n\rangle - B'(s)\langle z, \phi_n\rangle]\, ds \end{aligned}$$

EXAMPLE 2.3.3. *Find the solution of the heat equation $u_t - K u_{zz} = 0$ with the boundary conditions $u(0;t) = a_0 + b_0 t$, $u(L;t) = a_1 + b_1 t$ and the initial conditions $u(z;0) = 0$.*

Solution. In this case the associated homogeneous boundary conditions are $u(0;t) = 0$, $u(L;t) = 0$, for which $\lambda = 0$ is not an eigenvalue. Therefore we can determine the functions $A(t)$ and $B(t)$ uniquely to yield $U(z;t) = a_0 + b_0 t + (z/L)[a_1 - a_0 + (b_1 - b_0)t]$, so that the new function $v(z;t)$ satisfies the nonhomogeneous heat equation $v_t - K v_{zz} = -[b_0 + (z/L)(b_1 - b_0)]$ with $v(z;0) = -a_0 - (z/L)(a_1 - a_0)$. The appropriate Fourier series is

$$b_0 + (b_1 - b_0)\left(\frac{z}{L}\right) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{b_0}{n}[1 - (-1)^n] + \frac{b_1 - b_0}{n}(-1)^{n+1}\right)\sin\frac{n\pi z}{L}$$

The coefficients $v_n(t)$ are given by

$$\begin{aligned} v_n(t) &= \frac{2}{\pi}e^{-\lambda_n K t}\left(-\frac{a_0}{n}[1 - (-1)^n] - (a_1 - a_0)(-1)^{n+1}\right) \\ &\quad - \frac{2}{\pi}\frac{1 - e^{-\lambda_n K t}}{\lambda_n K}\left(\frac{b_0}{n}[1 - (-1)^n] + \frac{b_1 - b_0}{n}(-1)^{n+1}\right) \end{aligned}$$

The solution of the given problem is $u(z;t) = U(z;t) + \sum_{n=1}^{\infty} v_n(t)\phi_n(z)$ when we make the appropriate substitutions for $U(z;t)$ and $v_n(t)$ from above. •

EXERCISES 2.3

1. Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0;t) = T_1$, $u_z(L;t) = \Phi_2$ and the initial condition $u(z;0) = T_3$, where T_1, Φ_2, and T_3 are positive constants. Find the relaxation time.
2. Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u_z(0;t) = 0$, $u_z(L;t) + h[u(L;t) - T_2] = 0$ and the initial condition $u(z;0) = T_3$, where h, T_2, and T_3 are positive constants.
3. Solve the initial-value problem for the heat equation $u_t = Ku_{zz} + r$ with the boundary conditions $u(0;t) = T_1$, $u(L;t) = T_2$ and the initial condition $u(z;0) = T_3$, where r, T_1, T_2, and T_3 are positive constants. Find the relaxation time.
4. Consider heat flow in an infinite slab $0 \le z \le L$, with the no-flux boundary conditions $u_z = 0$ at $z = 0$, $z = L$ and the initial condition $u = 273 + 96(2L - 4z)$. Find the solution of this initial-value problem and determine the relaxation time. (*Warning:* The steady-state solution cannot be determined from the boundary conditions alone.)
5. Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u_z(0;t) = \Phi$, $u_z(L;t) = \Phi$ and the initial condition $u(z;0) = T_3$, where Φ and T_3 are positive constants.
6. Consider heat flow in the infinite slab $0 < z < L$ with a source that generates heat at a rate per unit volume that is directly proportional to the distance from the face $z = 0$. Initially the temperature is zero throughout the slab; both faces are maintained at that temperature forever.

   (a) Show that this leads to the equation $u_t = Ku_{zz} + az$ with $u(0;t) = 0$, $u(L;t) = 0$.

   (b) Find the steady-state solution U, the full solution u, and the relaxation time τ.
7. Consider heat flow in an infinite slab $0 < z < L$ that is initially at temperature zero. The face $z = 0$ is maintained at temperature zero, and the temperature at the face $z = L$ increases linearly with time.

   (a) Show that this leads to the heat equation $u_t = Ku_{zz}$ with $u(0;t) = 0$, $u(L;t) = At$, $u(z;0) = 0$.

   (b) Show that the solution may be obtained in the form $u(z;t) = Azt/L + v(z;t)$, where $v(z;t)$ is a suitable Fourier sine series.
8. If a slender wire is placed in a medium that exchanges heat with the surroundings, the "linear law of heat transfer" dictates the equation $u_t = Ku_{zz} - bu$, where b is a positive constant. Assume that the ends of the wire are insulated and that initially the temperature $u = T_0$.

   (a) Find the steady-state solution of the problem.

   (b) Find the solution of the full initial-boundary-value problem. [*Hint:* The function $w(z;t) = e^{bt}u(z;t)$ solves a known problem.]

9. With reference to the treatment of temporally nonhomogeneous problems, suppose that the functions $r(z;t)$, $v(z)$, $T_1(t)$ and $T_2(t)$ are uniformly bounded for $t > 0$, $0 < z < L$. Prove that the series defining the functions $v(z;t)$, $v_t(z;t)$, $v_z(z;t)$, and $v_{zz}(z;t)$ all converge uniformly for $0 < z < L$ for each fixed $t > 0$ and that $v(z;t)$ satisfies the appropriate heat equation.
10. Find the solution of the nonhomogeneous heat equation $u_t - Ku_{zz} = 1 - e^{-t}$ with the boundary conditions $u(0;t) = 0$, $u_z(L;t) = 0$ and the initial conditions $u(z;0) = 100$.
11. Find the solution of the nonhomogeneous heat equation

$$\begin{aligned} u_t &= K u_{zz} + ve^{-at} \sin \frac{\pi z}{L}, \qquad 0 < z < L, t > 0 \\ u(z;0) &= 0 = u(0;t) = u(L;t) \end{aligned}$$

where a, v, and K are constants.
12. Find the solution of the nonhomogeneous heat equation

$$\begin{aligned} u_t &= K u_{zz} + ve^{-at} \sin \frac{\pi z}{L}, \qquad 0 < z < L, t > 0 \\ u(z;0) &= 0 = u(0;t) = u_z(L;t) \end{aligned}$$

where a, v, and K are constants.

2.4. The Vibrating String

In this section we derive and solve the equation of the vibrating string. This equation is the one-dimensional form of the wave equation, which occurs throughout many branches of mathematical physics.

2.4.1. Derivation of the equation. Imagine a perfectly flexible elastic string, which at rest is stretched between two fixed pegs. For convenience we take a system of rectangular coordinates, so that the pegs are at the points $(0,0,0)$ and $(L,0,0)$. The points of the string are labeled by a parameter s, $0 \le s \le L$ (see Fig. 2.4.1). The motion of the string is described by a vector function $\mathbf{r}(s;t) = (X(s;t), Y(s;t), Z(s;t))$, which gives the rectangular coordinates of the string point s at the time instant t. Thus the vector $\partial\mathbf{r}/\partial s$ is tangent to the string at the point s (Fig. 2.4.2). The vector $\partial\mathbf{r}/\partial t$ is the instantaneous velocity of the string at the point s, while the vector $\partial^2\mathbf{r}/\partial t^2$ is the instantaneous acceleration of the string at point s.

FIGURE 2.4.1 Vibrating string in equilibrium position.

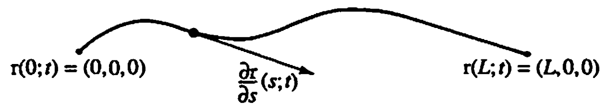


FIGURE 2.4.2 Vibrating string in motion-tangent vector.

We now determine a system of partial differential equations satisfied by the functions $X(s;t)$, $Y(s;t)$, $Z(s;t)$. To do this, we apply Newton's second law of motion, stating that the force on any segment of the string is the time derivative of the momentum of that segment. The mass of the string is given by a *mass density function* $\rho(s)$, $0 \le s \le L$. This means that

$$\int_a^b \rho(s)ds = \text{mass of the segment of the string for which } a \le s \le b$$

Likewise,

$$\int_a^b \rho(s)\frac{\partial \mathbf{r}}{\partial t}(s;t)ds = \text{momentum of the segment of the string for which } a \le s \le b$$

A typical segment of the string for which $a \le s \le b$ is subject to contact forces exerted at a and b by the rest of the string and to external forces (such as gravity) from the environment. These external forces may be written as a vector function $\mathbf{F}(s;t) = (F_1(s;t), F_2(s;t), F_3(s;t))$, representing the force per unit mass acting on the point s of the string. To describe the contact forces, we introduce the *tension* $T(s;t)$. The segment of the string with $b \le s \le L$ exerts a contact force at b, on the segment $a \le s \le b$ of the string, of the form

$$T^+(b;t)\frac{(\partial\mathbf{r}/\partial s)(b;t)}{|(\partial\mathbf{r}/\partial s)(b;t)|} = \begin{array}{l}\text{force on the segment } a \le s \le b\\ \text{due to the segment } b \le s \le L\end{array}$$

This means that the force is directed along the tangent to the string, as illustrated in Fig. 2.4.3. This property is a mathematical statement of the assumption that the string is perfectly flexible. Similarly, we set

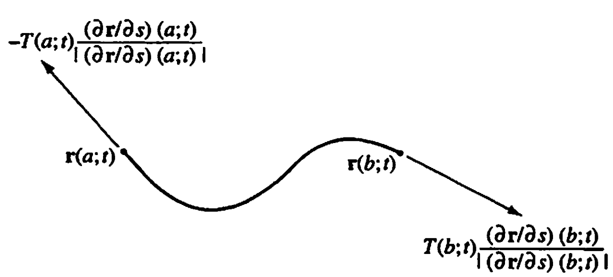


FIGURE 2.4.3 Contact forces on vibrating string.

$$-T^-(a;t)\frac{(\partial\mathbf{r}/\partial s)(a;t)}{|(\partial\mathbf{r}/\partial s)(a;t)|} = \begin{array}{l}\text{force on the segment } a \le s \le b\\ \text{due to the segment } 0 \le s \le a\end{array}$$

The minus sign enters for geometric reasons, which will become clear momentarily. The force on the segment $a \le s \le b$ is the external force plus these contact forces. Applying Newton's second law, we have

$$\int_a^b \rho(s)\frac{\partial^2\mathbf{r}}{\partial t^2}(s;t)ds = \int_a^b \rho(s)\mathbf{F}(s;t)ds$$
$$+T^+(b;t)\frac{(\partial\mathbf{r}/\partial s)(b;t)}{|(\partial\mathbf{r}/\partial s)(b;t)|} - T^-(a;t)\frac{(\partial\mathbf{r}/\partial s)(a;t)}{|(\partial\mathbf{r}/\partial s)(a;t)|}$$

This is a vector integral equation, equivalent to three scalar integral equations. Now we let $b \to a$. In the limit the integrals vanish and we find that $T^+(a;t) = T^-(a;t)$. We accordingly drop the plus and minus signs and write $T(s;t)$ for the common value, called the *tension* at $(s;t)$.

To write these integral equations as differential equations, we differentiate with respect to b, set $b = s$, and obtain the following differential form of the equation of the vibrating string.

$$\boxed{\rho(s)\frac{\partial^2\mathbf{r}}{\partial t^2}(s;t) = \rho(s)\mathbf{F}(s;t) + \frac{\partial}{\partial s}\left[T(s;t)\frac{(\partial\mathbf{r}/\partial s)(s;t)}{|(\partial\mathbf{r}/\partial s)(s;t)|}\right]} \tag{2.4.1}$$

The vector equation (2.4.1) is equivalent to three scalar equations for the four variables X, Y, Z, T. We obtain a determinate system by saying how the tension $T(s;t)$ is influenced by the stretch factor $|(\partial\mathbf{r}/\partial s)(s;t)|$. For each elastic material, there is a well-defined function N expressing this dependence by

$$T(s;t) = N\left(\left|\frac{\partial\mathbf{r}}{\partial s}(s;t)\right|, s\right) \tag{2.4.2}$$

The equation obtained by substituting (2.4.2) into (2.4.1) is a rigorous consequence of Newton's second law of motion. However, this equation is extremely difficult to solve. Therefore we will make assumptions to obtain simplified equations that may be solved. The alert reader will note that some of the steps taken may be difficult to justify within our treatment. The essential point is that *the solutions of the simplified equations can be shown to be close, in an appropriate sense, to the solutions of the exact equations (2.4.1) and (2.4.2).* For more information the reader is referred to the book by H. F. Weinberger[2] or to the article by S. Antman.[3]

[2] H. Weinberger, *A First Course in Partial Differential Equations*, Ginn, Blaisdell, Waltham, MA, 1965.

[3] S. Antman, "The Equations for the Large Vibrations of Strings," *American Mathematical Monthly*, vol. 87, pp. 359–370, 1980.

2.4.2. Linearized model. To obtain the simplified equations, we will look for solutions that describe *small vibrations*. This means that

$$X(s;t) = s + \epsilon x(s;t) \tag{2.4.3}$$
$$Y(s;t) = \epsilon y(s;t) \tag{2.4.4}$$
$$Z(s;t) = \epsilon z(s;t) \tag{2.4.5}$$
$$T(s;t) = T_0 + \epsilon T_1(s;t) \tag{2.4.6}$$
$$\mathbf{F}(s;t) = \epsilon \mathbf{f}(s;t) \tag{2.4.7}$$

The parameter ϵ may be thought of as a rough measure of the maximum displacement of the string from its neutral position $X = s$, $Y = 0$, $Z = 0$. The expressions T_0 and T_1 may be found by substituting the equations (2.4.3), (2.4.4), and (2.4.5) into (2.4.1) and (2.4.2) and discarding higher powers of ϵ. In particular, T_0 is the tension of the string in its neutral position $X = s$, $Y = 0$, $Z = 0$.

With the assumptions (2.4.3), (2.4.4), and (2.4.5), we have $\partial X/\partial s = 1 + \epsilon(\partial x/\partial s)$, $\partial Y/\partial s = \epsilon(\partial y/\partial s)$, $\partial Z/\partial s = \epsilon(\partial z/\partial s)$; thus

$$\begin{aligned}\left|\frac{\partial \mathbf{r}}{\partial s}\right|^2 &= \left(1+\epsilon\frac{\partial x}{\partial s}\right)^2 + \left(\epsilon\frac{\partial y}{\partial s}\right)^2 + \left(\epsilon\frac{\partial z}{\partial s}\right)^2 \\ &= 1 + 2\epsilon\left(\frac{\partial x}{\partial s}\right) + \epsilon^2\left[\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2\right]\end{aligned}$$

Taking the square root and using the Taylor expansion $(1+a)^{1/2} = 1+\frac{1}{2}a+O(a^2)$, we have

$$\left|\frac{\partial \mathbf{r}}{\partial s}\right| = 1 + \epsilon\frac{\partial x}{\partial s} + O(\epsilon^2)$$

Thus

$$\frac{\partial \mathbf{r}/\partial s}{|\partial \mathbf{r}/\partial s|} = \left(1 + O(\epsilon^2), \epsilon\frac{\partial y}{\partial s} + O(\epsilon^2), \epsilon\frac{\partial z}{\partial s} + O(\epsilon^2)\right)$$

If we insert these into (2.4.1) and use (2.4.3), (2.4.4), and (2.4.5), we obtain the following equations for the x, y, and z components:

$$\begin{aligned}\epsilon\rho(s)\frac{\partial^2 x}{\partial t^2} &= \epsilon\rho(s)f_1(s;t) + \frac{\partial}{\partial s}\{[T_0 + \epsilon T_1(s;t)][1 + O(\epsilon^2)]\} \\ \epsilon\rho(s)\frac{\partial^2 y}{\partial t^2} &= \epsilon\rho(s)f_2(s;t) + \frac{\partial}{\partial s}\left\{[T_0 + \epsilon T_1(s;t)]\left[\epsilon\frac{\partial y}{\partial s} + O(\epsilon^2)\right]\right\} \\ \epsilon\rho(s)\frac{\partial^2 z}{\partial t^2} &= \epsilon\rho(s)f_3(s;t) + \frac{\partial}{\partial s}\left\{[T_0 + \epsilon T_1(s;t)]\left[\epsilon\frac{\partial z}{\partial s} + O(\epsilon^2)\right]\right\}\end{aligned}$$

Equating the coefficients of ϵ in each of these three equations, we obtain the following simplified equations for the longitudinal and transverse vibrations:

$$\begin{aligned}
\rho(s)\frac{\partial^2 x}{\partial t^2} &= \rho(s)f_1(s;t) + \frac{\partial}{\partial s}T_1(s;t) \\
\rho(s)\frac{\partial^2 y}{\partial t^2} &= \rho(s)f_2(s;t) + T_0\frac{\partial^2 y}{\partial s^2} \\
\rho(s)\frac{\partial^2 z}{\partial t^2} &= \rho(s)f_3(s;t) + T_0\frac{\partial^2 z}{\partial s^2}
\end{aligned}$$

We shall only be interested in the transverse vibrations, obtained by solving the second and third equations. These are of the same form, the so-called *one-dimensional wave equation*

$$\boxed{\frac{\partial^2 y}{\partial t^2} = f(s;t) + \frac{T_0}{\rho(s)}\frac{\partial^2 y}{\partial s^2}} \tag{2.4.8}$$

EXAMPLE 2.4.1. *Suppose that $\rho(s) = \rho$ and $f(s;t) = g$, independent of (s,t). Find the steady-state solution of the wave equation (2.4.8) satisfying the boundary conditions $y(0) = 0$, $y(L) = 0$.*

Solution. The function $y(s)$ must satisfy the ordinary differential equation

$$0 = g + \frac{T_0}{\rho}y''(s)$$

The general solution of this equation is $y(s) = -(\rho g/2T_0)s^2 + As + B$, where A and B are arbitrary constants. Applying the boundary conditions, we have $0 = B$, $0 = -(\rho g/2T_0)L^2 + AL + B$. The solution is $y(s) = (\rho g/2T_0)(Ls - s^2)$. •

2.4.3. Motion of the plucked string. Now we turn to time-dependent solutions of the wave equation, specifically, the problem of the *plucked string*. We shall suppose that the string is uniform $[\rho(s) = \rho]$ and no outside forces are present $[f(s,t) = 0]$. We let

$$c^2 = \frac{T_0}{\rho}$$

which has the dimension (velocity)^2. The wave equation is now written as $y_{tt} = c^2 y_{ss}$. The initial position of the string is supposed to be given by a function $f_1(s)$, $0 \le s \le L$, while the initial velocity is zero. Thus we have the problem

$$\begin{aligned}
y_{tt}(s;t) &= c^2 y_{ss} \\
y(0;t) &= 0 = y(L;t) \\
y(s;0) &= f_1(s) \\
y_t(s;0) &= 0
\end{aligned}$$

In Example 0.2.7 we showed that the separated solutions that satisfy the boundary conditions and the second initial condition are $\sin(n\pi s/L)\cos(n\pi ct/L)$, $n = 1, 2, \ldots$. Therefore we may obtain a formal solution by the superposition principle as

$$\boxed{y(s;t) = \sum_{n=1}^{\infty} B_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi s}{L}} \tag{2.4.9}$$

To find the coefficients B_n, we set $t = 0$. This requires that

$$f_1(s) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L} \tag{2.4.10}$$

Therefore B_n is the nth Fourier sine coefficient of f_1.

$$B_n = \frac{2}{L}\int_0^L f_1(s)\sin\frac{n\pi s}{L}\,ds \tag{2.4.11}$$

If $f_1(s)$, $0 < s < L$, is continuous and piecewise smooth, the Fourier series (2.4.9) converges for all s to the odd periodic extension of f_1, denoted $\bar{f}_1$. Therefore, to solve the problem, we have a simple rule: Given f_1, compute B_n from (2.4.11) and substitute into (2.4.9) to solve the problem. This is called the *Fourier representation* of the solution. We illustrate with the problem of the symmetric plucked string.

EXAMPLE 2.4.2. *Solve the vibrating string problem in the case where*

$$f_1(s) = \begin{cases} s & 0 < s < L/2 \\ L - s & L/2 < s < L \end{cases}$$

Solution. To compute the Fourier coefficients, we notice that f_1 is even about $s = L/2$, whereas $\sin n\pi s/L$ is even (resp. odd) about $s = L/2$ if n is odd (resp. even). Therefore $B_n = 0$ if n is even. If n is odd, we have

$$\begin{aligned}
B_n &= \frac{2}{L}\int_0^L f_1(s)\sin\frac{n\pi s}{L}\,ds \\
&= \frac{4}{L}\int_0^{L/2} f_1(s)\sin\frac{n\pi s}{L}\,ds \\
&= \frac{4}{L}\int_0^{L/2} s\sin\frac{n\pi s}{L}\,ds \\
&= \frac{4}{L}\frac{Ls}{n\pi}\cos\frac{n\pi s}{L}\Big|_{L/2}^{0} + \frac{4}{n\pi}\int_0^{L/2}\cos\frac{n\pi s}{L}\,ds \\
&= \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}$$

If n is even, $B_n = 0$, while if n is odd, we write $n = 2m - 1$, $\sin n\pi/2 = (-1)^{m+1}$. Therefore we have solved the problem.

$$y(s;t) = \frac{4L}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \cos(2m-1)\frac{\pi ct}{L} \sin(2m-1)\frac{\pi s}{L} \quad \bullet$$

The Fourier representation (2.4.9) displays the solution as a Fourier sine series for each time t. The nth term of this series is a purely harmonic vibration of frequency $n\pi c/L$ and amplitude B_n. Hence this form of the solution affords a natural analogy between the vibrating string and an infinite system of harmonic oscillators.

2.4.4. Acoustic interpretation. In the theory of acoustics, the numbers $\omega_n = n\pi c/L$ are interpreted as the *frequencies* of purely harmonic vibrations. It is a characteristic feature of the vibrating string that these numbers are multiples of a common frequency $\omega_1 = \pi c/L$, which is called the *fundamental frequency.* The higher frequencies are called *overtones*, which strengthen the quality of the sound.

If the initial position of the string is given by $y(s;0) = C \sin n\pi s/L$, then the string will vibrate at the corresponding frequency $\omega_n = n\pi c/L$ when released from rest. In practice this initial condition is rarely met; instead we often have initial conditions that result from striking or bowing the string. In these cases both the fundamental frequency and many overtones will be present in the resultant vibration, which is written as a superposition of several purely harmonic vibrations. The resulting impulses that are transmitted are characteristic of the particular stringed instrument. We list below typical solutions for both a piano string and a violin string, corresponding to the fundamental frequency of 440 cycles/s (concert A).

$$y(s;t) = \sum_{n=1}^{8} A_n \sin\frac{n\pi s}{L} \cos\frac{n\pi ct}{L}$$

	Piano	Violin
A_1	1	1
A_2	0.20	1
A_3	0.25	0.45
A_4	0.10	0.50
A_5	0.08	1.00
A_6	0.00	0.03
A_7	0.00	0.03
A_8	0.00	0.03

According to these data, the piano vibration is much closer to a purely harmonic vibration than is the violin vibration. To see this numerically, we may use the formula for the mean square error, which was developed in Sec. 1.2.4; thus

$$\frac{1}{L}\int_0^L \left[y(s;t) - A_1 \sin\frac{\pi s}{L}\cos\frac{\pi ct}{L}\right]^2 ds = \frac{1}{2}\sum_{n=2}^{8} A_n^2 \cos^2\frac{n\pi ct}{L}$$

In the case of the piano vibration, we have

$$\sum_{n=2}^{8} A_n^2 = (0.20)^2 + (0.25)^2 + (0.10)^2 + (0.08)^2 = 0.1189$$

whereas in the case of the violin vibration we have

$$\begin{aligned}\sum_{n=2}^{8} A_n^2 &= 1 + (0.45)^2 + (0.50)^2 + (1.00)^2 + (0.03)^2 + (0.03)^2 + (0.03)^2 \\ &= 2.4552\end{aligned}$$

To obtain a meaningful comparison, we define the *fractional mean square error* as

$$\bar{\sigma} = \frac{\sum_{n=2}^{8} A_n^2}{\sum_{n=1}^{8} A_n^2}$$

For the piano vibration we have $\bar{\sigma}^2 = 0.1189/1.1189 = 0.1063$, whereas for the violin vibration we have $\bar{\sigma}^2 = 2.4552/3.4552 = 0.7106$, nearly seven times as large.[4]

2.4.5. Explicit (d'Alembert) representation. Returning to the mathematics, we now discuss some disadvantages of the Fourier representation (2.4.9). On the one hand it is difficult to verify that $y(s;t)$ actually is a solution of the wave equation. Consider, for example, the computation of the second time derivative y_{tt} of the solution obtained in Example 2.4.2. Proceeding formally, we encounter the series

$$\frac{4c^2}{L}\sum_{m=1}^{\infty}(-1)^{m+1}\cos(2m-1)\frac{\pi ct}{L}\sin(2m-1)\frac{\pi s}{L}$$

We have lost the factor $1/(2m-1)^2$, which ensured convergence of the series for $y(s;t)$. The new series converges for no value of t.

A second disadvantage of the Fourier representation is that it provides little geometric insight into the motion of the vibrating string. We expect that an initial disturbance will be propagated as some sort of wave motion, but the Fourier representation does not show this directly.

[4]Data obtained from D. Halliday and R. Resnick, *University Physics*, 3d ed., 1977, John Wiley & Sons, New York.

To overcome these difficulties, we will rewrite the solution (2.4.9) in a form that avoids the Fourier coefficients B_n. To do this, we use the trigonometric identity $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ and proceed formally.

$$\begin{aligned} y(s;t) &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L}\cos\frac{n\pi ct}{L} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} B_n\left[\sin\frac{n\pi}{L}(s+ct)+\sin\frac{n\pi}{L}(s-ct)\right] \end{aligned}$$

We recognize the Fourier sine series for $\bar{f}_1(s+ct)$, $\bar{f}_1(s-ct)$, where $\bar{f}_1$ is the odd, $2L$-periodic extension of $f_1(s)$, $0<s<L$. Therefore we have the *explicit representation*

$$\boxed{y(s;t) = \frac{1}{2}[\bar{f}_1(s+ct)+\bar{f}_1(s-ct)]} \tag{2.4.12}$$

In physical terms, we have written $y(s;t)$ as a sum of two traveling waves, one moving to the right with speed c and the other moving to the left with speed c. This will enable us to obtain graphical representations of the solution.

Using the representation (2.4.12), we now verify that $y(s;t)$ satisfies the wave equation. Indeed, whenever $\bar{f}_1'$, $\bar{f}_1''$ exist, we have

$$\begin{aligned} y_t &= \frac{1}{2}c\bar{f}_1'(s+ct) - \frac{1}{2}c\bar{f}_1'(s-ct) \\ y_{tt} &= \frac{1}{2}c^2\bar{f}_1''(s+ct) + \frac{1}{2}c^2\bar{f}_1''(s-ct) \\ y_s &= \frac{1}{2}\bar{f}_1'(s+ct) + \frac{1}{2}\bar{f}_1'(s-ct) \\ y_{ss} &= \frac{1}{2}\bar{f}_1''(s+ct) + \frac{1}{2}\bar{f}_1''(s-ct) \end{aligned}$$

Clearly, $y_{tt} = c^2 y_{ss}$. To check the boundary conditions, we have

$$y(0;t) = \frac{1}{2}[\bar{f}_1(ct)+\bar{f}_1(-ct)] = 0$$

since $\bar{f}$ is odd, and

$$\begin{aligned} y(L;t) &= \frac{1}{2}[\bar{f}(L+ct)+\bar{f}(L-ct)] \\ &= \frac{1}{2}[\bar{f}(L+ct)-\bar{f}(-L+ct)] \\ &= \frac{1}{2}[\bar{f}(L+ct)-\bar{f}(L+ct)] \\ &= 0 \end{aligned}$$

where we have used the oddness and $2L$-periodicity of $\bar{f}$. The initial condition $y(s;0) = \bar{f}(s)$ is satisfied everywhere, while $y_t(s;0) = 0$ wherever $\bar{f}'$ is defined. This completes the validation of the solution.

The second application of (2.4.12) is to obtain a picture of the motion of the vibrating string. We will now illustrate this in the case of the asymmetric plucked string, where

$$f_1(s) = \begin{cases} s(L-s_0) & 0 < s < s_0 \\ s_0(L-s) & s_0 < s < L \end{cases}$$

When we extend f_1 as an odd periodic function, we obtain the graph depicted in Fig. 2.4.4. The extended function satisfies $\bar{f}_1(-s) = -\bar{f}_1(s)$, $\bar{f}_1(s+2L) = \bar{f}_1(s)$ for any s. We substitute this into (2.4.12) and get

$$y(s;t) = \frac{1}{2}[\bar{f}_1(s+ct) + \bar{f}_1(s-ct)]$$

To obtain a picture of the motion of the plucked string, we must average the left and right translates of $\bar{f}_1$. Since the average of two linear functions is again linear, it suffices to plot five points in the interval $0 \le s \le L$: the two endpoints where the string is fixed, the interior point where $\bar{f}_1(s+ct) = \bar{f}_1(s-ct)$, and the two interior points where $\bar{f}_1'(s+ct)$ changes sign.

The diagrams in Fig. 2.4.5 give a motion picture of the plucked string during the half-period $0 \le t \le L/c$. For convenience we take $s_0 = 3L/4$. At $t = 0$, we have the odd periodic function $\bar{f}_1(s)$ with vertices at $A(s = 3L/4)$, $B(s = 5L/4)$, and $C(s = -3L/4)$. These points, which determine the discontinuities of $\bar{f}_1'$, move along the axis and are labeled $A_\pm$, $B_\pm$, $C_\pm$. For $t = L/8c$, we have A_+, A_- as the only vertices with $0 \le s \le L$. When $t = L/4c$, A_- has arrived at the middle of the interval and the plucked string is symmetric about $s = L/2$. The vertex A_+ is replaced by B_-. When $L/4c \le t \le 3L/4c$, part of the string goes below the axis, until it reaches the symmetric configuration shown at $t = 3L/4c$. When $t = 3L/4c$, the A_- vertex disappears from the interval and is replaced by C_+; this results in the symmetric configuration completely below the axis. Finally, when $t = L/c$, the string has completed one-half of its period and is congruent to its initial position with vertex at $s = L/4$.

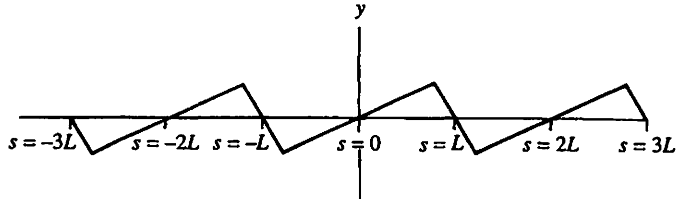


FIGURE 2.4.4 The odd, $2L$-periodic extension of $f_1(s)$, $0 < s < L$.

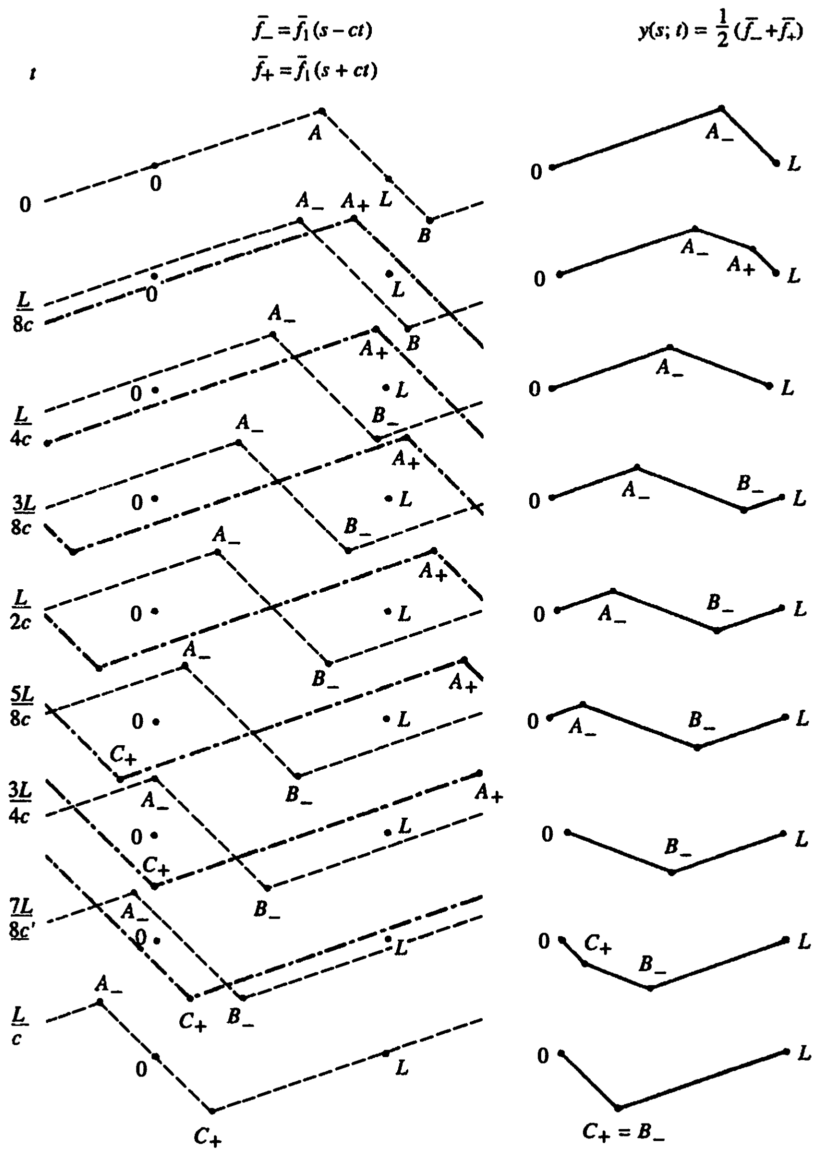


FIGURE 2.4.5 Successive positions of the asymmetric plucked string.

2.4.6. Motion of the struck string. We now solve the problem of the vibrating string, starting from equilibrium, with nonzero initial velocity. Thus we must solve

$$\begin{array}{lll} y_{tt}(s;t) &= c^2 y_{ss} & t>0, 0<s<L \\ y(0;t) &= 0 = y(L;t) & t>0 \\ y(s;0) &= 0 & 0<s<L \\ y_t(s;0) &= f_2(s) & 0<s<L \end{array}$$

The initial velocity profile $f_2(s)$ is unspecified for the moment.

To solve this problem, we begin with the separated solutions that satisfy the wave equation, the boundary conditions, and the first initial condition. These are of the form

$$\sin\frac{n\pi s}{L}\sin\frac{n\pi ct}{L}$$

To satisfy the second initial condition, we try a superposition of these:

$$\boxed{y(s;t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L}\sin\frac{n\pi ct}{L}} \tag{2.4.13}$$

This is the Fourier representation of the solution. The coefficients $\{B_n\}$ are determined by differentiating this series with respect to t and setting $t = 0$; thus

$$y_t(s;t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\frac{n\pi s}{L}\cos\frac{n\pi ct}{L}$$

Setting $t = 0$, we must have

$$y_t(s;0) = f_2(s) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\frac{n\pi s}{L} \tag{2.4.14}$$

In other words, $(n\pi c/L)B_n$ is the nth Fourier sine coefficient of $f_2(s)$, $0 < s < L$:

$$\frac{n\pi c}{L} B_n = \frac{2}{L}\int_0^L f_2(s)\sin\frac{n\pi s}{L}\,ds \qquad n = 1, 2, \ldots$$

If $f_2(s)$, $0 < s < L$, is continuous and piecewise smooth, the Fourier series (2.4.14) converges for all s to the odd periodic extension of f_2, denoted $\bar{f}_2$. This completes the Fourier representation of the solution.

To obtain an explicit representation, we apply the trigonometric identity $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ to the Fourier representation. Thus

$$y(s;t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\cos\frac{n\pi}{L}(s-ct) - \cos\frac{n\pi}{L}(s+ct)\right]$$

We use calculus to rewrite this as

$$
\begin{aligned}
y(s;t) &= \frac{1}{2}\sum_{n=1}^{\infty} B_n \frac{n\pi}{L}\int_{s-ct}^{s+ct} \sin\frac{n\pi z}{L}\,dz \\
&= \frac{1}{2}\int_{s-ct}^{s+ct}\left\{\sum_{n=1}^{\infty} B_n \frac{n\pi}{L}\sin\frac{n\pi z}{L}\right\} dz
\end{aligned}
$$

where we have formally interchanged the summation and integration. Aside from the factor c, we recognize the formula in braces as the Fourier sine series for $\bar{f}_2$, the odd, $2L$-periodic extension of $f_2(s)$, $0 < s < L$. Thus we have the *explicit representation*

$$
\boxed{y(s;t) = \frac{1}{2c}\int_{s-ct}^{s+ct} \bar{f}_2(z)dz} \tag{2.4.15}
$$

The formula (2.4.15) defines a solution of the wave equation and satisfies the boundary conditions and both initial conditions. In particular cases this formula can be used to graph the solution of the wave equation.

EXAMPLE 2.4.3. *Graph the solution of the vibrating string if*

$$
f_2(s) = \begin{cases} 0 & 0 < s < L/4 \\ 1 & L/4 < s < 3L/4 \\ 0 & 3L/4 < s < L \end{cases}
$$

Solution. To graph the solution, we first extend f_2 as an odd, $2L$-periodic function. From formula (2.4.15), we have $y_s(s;t) = (1/2c)[\bar{f}_2(s+ct) - \bar{f}_2(s-ct)]$. This quantity is 0, $\pm 1/2c$, $\pm 1/c$. For each t, $y(s;t)$ is linear on each segment on which y_s is constant. Finally we have the set of graphs shown in Fig. 2.4.6 of the solution for $t = 0$, $L/8c$, $L/4c$, $3L/8c$, $L/2c$. •

2.4.7. d'Alembert's general solution. The explicit solutions just obtained can be combined to obtain a solution of the wave equation for the general initial conditions $y(s;0) = f_1(s)$, $y_t(s;0) = f_2(s)$. For this purpose, consider the function

$$
\boxed{y(s;t) = \frac{1}{2}[f_1(s+ct) + f_1(s-ct)] + \frac{1}{2c}\int_{s-ct}^{s+ct} f_2(z)\,dz} \tag{2.4.16}
$$

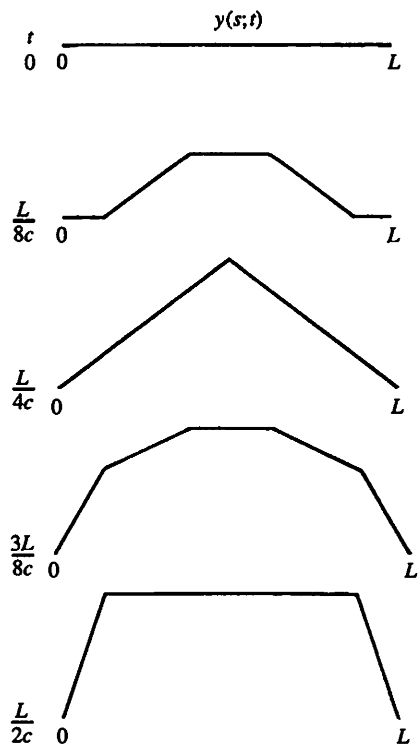


FIGURE 2.4.6 Successive positions of the struck string.

Suppose that f_1, f_1', f_1'', f_2, and f_2' are continuous functions. Then

$$\begin{aligned}
y_t &= \frac{c}{2}[f_1'(s+ct) - f_1'(s-ct)] + \frac{1}{2}[f_2(s+ct) + f_2(s-ct)] \\
y_{tt} &= \frac{c^2}{2}[f_1''(s+ct) + f_1''(s-ct)] + \frac{c}{2}[f_2'(s+ct) - f_2'(s-ct)] \\
y_s &= \frac{1}{2}[f_1'(s+ct) + f_1'(s-ct)] + \frac{1}{2c}[f_2(s+ct) - f_2(s-ct)] \\
y_{ss} &= \frac{1}{2}[f_1''(s+ct) + f_1''(s-ct)] + \frac{1}{2c}[f_2'(s+ct) - f_2'(s-ct)]
\end{aligned}$$

We observe that $y(s;0) = f_1(s)$, $y_t(s;0) = f_2(s)$, and that $y(s;t)$ satisfies the wave equation $y_{tt} = c^2 y_{ss}$ for all $(s;t)$. We have made no use of boundary conditions or periodicity considerations. This general solution is called *d'Alembert's solution* of the wave equation. We summarize the result as a proposition.

PROPOSITION 2.4.1. *Let f_1, f_2 be continuous functions with continuous derivatives f_1', f_1'', f_2'. The d'Alembert formula (2.4.16) gives a solution of the wave equation $y_{tt} = c^2 y_{ss}$ valid for all $t > 0$, $-\infty < s < \infty$, and satisfies the initial conditions $y(s;0) = f_1(s)$, $y_t(s;0) = f_2(s)$.*

The careful reader will note that Examples 2.4.2 and 2.4.3 do not satisfy the hypotheses of the proposition. Therefore this proposition, although mathematically rigorous, excludes examples of physical interest. In order to improve this point of the theory, mathematicians have extended the concept of *solution* to include functions $y(s;t)$ for which some of the indicated partial derivatives may not exist. The basic idea is that a function $y(x;t)$ is a weak solution if there exists a sequence of (twice-differentiable) solutions $y_n(x;t)$ so that $y(x,t) = \lim_{n\to\infty} y_n(x;t)$ for each (x,t). For example, $y_n(x;t)$ can be chosen as the partial sum of the Fourier series. This concept of *weak solution* is discussed further by Weinberger.[5]

2.4.8. Vibrating string with external forcing. In the remainder of this section, we consider a vibrating string with a time-dependent external force, using the method of Fourier series. As our first model we consider the problem

$$\begin{aligned} y_{tt} - c^2 y_{ss} &= g(s)\cos\omega t \qquad 0 < s < L, t > 0 \\ y(0;t) &= 0 = y(L;t) \qquad t > 0 \end{aligned}$$

where $g(s)$, $0 < s < L$, is a piecewise smooth function and ω is a positive constant.

We look for a particular solution in the form

$$y(s;t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi s}{L}$$

(The general solution can be found by adding a solution of the homogeneous equation, which has already been discussed.) To find the coefficient functions $A_n(t)$, we substitute in the differential equation and use the Fourier sine expansion of $g(s)$.

$$g(s) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L}$$

$$\sum_{n=1}^{\infty}\left[A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t)\right]\sin\frac{n\pi s}{L} = \cos\omega t \sum_{n=1}^{\infty} B_n \sin\frac{n\pi s}{L}$$

We choose $A_n(t)$ to be solutions of the ordinary differential equations

$$A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t) = B_n \cos\omega t$$

[5] Weinberger, op. cit.

If $\omega \neq n\pi c/L$ for any n, a particular solution is

$$A_n(t) = A_n \cos\omega t$$
$$A_n\left[-\omega^2 + \left(\frac{n\pi c}{L}\right)^2\right] = B_n$$

The formal solution of the problem is

$$\boxed{y(s;t) = \cos\omega t \sum_{n=1}^{\infty} \frac{B_n \sin(n\pi s/L)}{(n\pi c/L)^2 - \omega^2}} \tag{2.4.17}$$

The solution is a periodic function of time with the same period. The series for $y(s;t)$ converges uniformly for $0 \le s \le L$. If $g(s)$ is continuous and satisfies the boundary conditions, the differentiated series for y_s, y_{ss}, y_t, y_{tt} also converge uniformly for $0 \le s \le L$, and $y(s;t)$ is a solution of the problem.

EXAMPLE 2.4.4. *Find a particular solution of the problem $y_{tt} - c^2 y_{ss} = A\cos\omega t$, satisfying the boundary conditions $y(0;t) = 0 = (L;t)$, where A and ω are positive constants with $\omega \neq n\pi c/L$.*

Solution. In this case we use the Fourier sine series expansion of the constant function

$$1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi s}{L} \qquad 0 < s < L$$

Thus $B_n = (2/n\pi)[1-(-1)^n]$, and the solution is

$$y(s;t) = \frac{2A}{\pi}\cos\omega t \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n[(n\pi c/L)^2 - \omega^2]} \sin\frac{n\pi s}{L}$$

The series for y_s, y_{ss}, y_t, y_{tt} converge uniformly for $\delta \le s \le L - \delta$ for any $\delta > 0$, and this implies that $y(s;t)$ is a solution of the equation. •

EXAMPLE 2.4.5. *Find a particular solution of the problem $y_{tt} - c^2 y_{ss} = A\cos\omega t$, satisfying the boundary conditions $y(0;t) = 0 = y(L;t)$, where A and ω are positive constants with $\omega = N\pi c/L$ for some integer N.*

Solution. We look for a particular solution in the form

$$y(s;t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi s}{L}$$

Repeating the analysis of the previous cases, we choose $A_n(t)$ as a solution of the ordinary differential equation

$$A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t) = A\frac{1-(-1)^n}{n}\cos\omega t$$

If $n \neq N$, this is solved as before.

To solve the equation for $n = N$, we use the following observation. For any $\omega' \neq \omega$,

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \cos \omega' t = [\omega^2 - (\omega')^2] \cos \omega' t$$

while

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \cos \omega t = 0$$

Therefore

$$\left(\frac{d^2}{dt^2} + \omega^2\right) \frac{\cos \omega' t - \cos \omega t}{\omega^2 - (\omega')^2} = \cos \omega' t$$

Taking the limit $\omega' \to \omega$, we have

$$\lim_{\omega' \to \omega} \frac{\cos \omega' t - \cos \omega t}{\omega^2 - (\omega')^2} = \frac{t}{2\omega} \sin \omega t$$

Thus we have the solution

$$y(s;t) = A\frac{1-(-1)^N}{N} \frac{t}{2\omega} \sin \omega t \sin \frac{N\pi s}{L} + \frac{2}{\pi} A \cos \omega t \sum_{n \neq N} \frac{\sin(n\pi s/L)}{(n\pi c/L)^2 - \omega^2} \frac{1-(-1)^n}{n}$$

This solution is not a periodic function of time, but oscillates with increasing amplitude as time progresses. This is the phenomenon of *resonance*. •

EXERCISES 2.4

1. Consider the initial-value problem for the symmetric plucked string.

$$\begin{aligned} y_{tt} &= c^2 y_{ss} & & t > 0, 0 < s < L \\ y(0;t) &= 0 = y(L;t) & & t > 0 \\ y(s;0) &= s & & 0 < s \leq L/2 \\ y(s;0) &= L - s & & L/2 < s < L \\ y_t(s;0) &= 0 & & 0 < s < L \end{aligned}$$

   Make a graphical representation of the solution for $ct = L/4$, $L/2$, $3L/4$, L. At what time is $y(s;t) = 0$ for all $0 < s < L$?
2. Let $y(s;t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct/L) \sin(n\pi s/L)$ be a solution of the vibrating string problem. Suppose that the string is further constrained at its midpoint, so that $y(L/2, t) = 0$ for all t. What condition does this impose on the coefficients B_n?
3. Let $y(s,t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct/L) \sin(n\pi s/L)$ be a solution of the vibrating string problem. Suppose that the string is constrained so that $y(L/3, t) = 0$ for all t. What condition does this impose on the coefficients B_n?
4. The *energy* of a vibrating string of tension T_0 and density $\rho = m/L$ is defined by

$$E = \frac{1}{2} \int_0^L (\rho y_t^2 + T_0 y_s^2) ds$$

Let

$$y(s;t) = \sum_{n=1}^{\infty} (\tilde{A}_n \cos \omega_n t + \tilde{B}_n \sin \omega_n t) \sin \frac{n\pi s}{L}$$

be a solution of the wave equation with $\omega_n = n\pi c/L$. Use Parseval's theorem to write E as an infinite series involving $\tilde{A}_n$, $\tilde{B}_n$ and verify the law of conservation of energy.

5. Let $y(s;t)$ be a solution of the wave equation $y_{tt} = c^2 y_{ss}$ satisfying the boundary conditions $y(0;t) = 0 = y(L;t)$. By differentiating under the integral sign, show directly that $dE/dt = 0$, where E is the energy defined in Exercise 4.
6. Consider the initial-value problem for the plucked string of Example 2.4.2. Compute the total energy corresponding to normal modes $n \neq 1$ and show that this is less than half of the total energy.
7. Suppose that $f_2(s), -\infty < s < \infty$, is an odd, $2L$-periodic function. Define $y(s;t) = (1/2c) \int_{s-ct}^{s+ct} f_2(z)\, dz$; show that $y(0;t) = 0$, $y(L;t) = 0$ for all t.
8. Let $f(s)$, $-L < s < L$, be a piecewise smooth function. Extend f to a $2L$-periodic function defined for $-\infty < s < \infty$. Show that the resulting function is piecewise smooth on every interval $a < s < b$.
9. Let $f(s)$, $-L \leq s \leq L$, be an odd function. Extend f to a $2L$-periodic function defined for $-\infty < s < \infty$. Show that the resulting function is again odd.
10. Let $f(s)$, $-\infty < s < \infty$, be an odd function with the property that $f(L - s) = f(s)$. Show that $f(s + L/2) + f(s - L/2) = 0$ for all s.
11. Let $f(s)$, $0 \leq s < L$, satisfy $f(s) = f(L - s)$. Let y be the solution (2.4.12) satisfying $y(s;0) = f(s)$, $y_t(s;0) = 0$. Show that $y(s, L/2c) = 0$ for $0 < s < L$.
12. Show that the Fourier series solution obtained in Example 2.4.2 converges uniformly for $0 \leq s \leq L$.
13. Consider the following initial-value problem for the wave equation $y_{tt} = c^2 y_{ss}$ for $t > 0$, $0 < s < L$: $y(0;t) = y(L;t) = 0$ for $t > 0$; $y(s;0) = 0$ and $y_t(s;0) = 1$ for $0 < s < L$.
    (a) Find the Fourier representation of the solution.
    (b) Find the explicit representation of the solution and graph the solution for $t = 0$, $L/4c$, $L/2c$, $3L/4c$, L/c.

Exercises 14 to 16 are designed to review the techniques from calculus that are used in establishing d'Alembert's formula. Recall that the fundamental theorem of calculus states that $(d/dx) \int_0^s f(z)dz = f(x)$ for any continuous function f. The chain rule for differentiating composite functions states that $(d/dx)F(G(x)) = F'(G(x))G'(x)$.

14. Let f be a continuous function and set $F(x) = \int_0^x f(z)\, dz$. Show that $\int_{x-ct}^{x+ct} f(z)\, dz = F(x + ct) - F(x - ct)$.

15. Use the chain rule and the fundamental theorem of calculus to show that $(d/dx)\int_{x-ct}^{x+ct} f(z)\,dz = f(x+ct) - f(x-ct)$.
16. Use the chain rule and the fundamental theorem of calculus to show that $(d/dt)\int_{x-ct}^{x+ct} f(z)\,dz = cf(x+ct) + cf(x-ct)$.
17. The voltage $u(x;t)$ in a transmission cable is known to satisfy the partial differential equation $u_{tt} + 2au_t + a^2u = c^2u_{xx}$, where a and c are positive constants. Let $y(x;t) = e^{at}u(x;t)$ and show that y satisfies the wave equation $y_{tt} = c^2y_{xx}$.
18. Use Exercise 17 and d'Alembert's formula to solve the initial-value problem $u_{tt}+2au_t+a^2u = c^2u_{xx}$ for $t > 0$, $-\infty < x < \infty$, with the initial conditions $u(x;0) = g_1(x)$, $u_t(x;0) = 0$.
19. Use Exercise 17 and d'Alembert's formula to solve the initial-value problem $u_{tt}+2au_t+a^2u = c^2u_{xx}$ for $t > 0$, $-\infty < x < \infty$, with the initial condtions $u(x;0) = 0$, $u_t(x;0) = g_2(x)$.
20. A vibrating string with friction in a periodic force field is described by the equation $y_{tt} + 2ay_t - c^2y_{ss} = A\cos\omega t$ and boundary conditions $y(0;t) = 0 = y(L;t)$, where A, a, and ω are positive constants. Find a particular solution that is also periodic in time.

2.5. Applications of Multiple Fourier Series

In this section we consider boundary-value problems in rectangular coordinates (x, y, z) where more than one of these variables appears in the solution. This is in contrast to the previous sections, where the solution depended on z alone. We will solve initial-value problems for the heat equation, boundary-value problems for Laplace's equation, and the wave equation for a vibrating membrane.

The key idea in our work is a *double Fourier series.* To illustrate this, we display a double Fourier sine series,

$$\sum_{m,n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}$$

which may be used in problems involving a rectangle or column, described by the inequalities $0 < x < L_1$, $0 < y < L_2$. Similarly, a double Fourier cosine series is of the form $\sum A_{mn}\cos(m\pi x/L_1)\cos(n\pi y/L_2)$. Clearly we could consider other combinations of these, where we mix sines and cosines, for example. All double series of this type are of the form $\sum_{m,n}\phi_m(x)\psi_n(y)$, where ϕ_m, ψ_n are the eigenfunctions of a Sturm-Liouville eigenvalue problem. Accordingly, the corresponding functions of two variables obey suitable orthogonality relations.

For example, in the case of double Fourier sine series, we have

$$\int_0^{L_2}\int_0^{L_1} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sin\frac{m'\pi x}{L_1}\sin\frac{n'\pi y}{L_2}\,dx\,dy = \begin{cases} (L_1L_2)/4 & \text{if both } m = m' \text{ and } n = n' \\ 0 & \text{otherwise} \end{cases}$$

Many functions can be written as sums of multiple Fourier series. In the case of double Fourier sine series in the rectangle $0 < x < L_1$, $0 < y < L_2$, we have expansion formulas

$$\begin{aligned} f(x,y) &= \sum_{m,n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2} \\ A_{mn} &= \frac{4}{L_1L_2}\int_0^{L_2}\int_0^{L_1} f(x,y)\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\,dx\,dy \end{aligned}$$

If $f(x,y)$ is a smooth function in the rectangle, the series converges to $f(x,y)$ for $0 < x < L_1$, $0 < y < L_2$.

2.5.1. The heat equation (homogeneous boundary conditions). As our first application of double Fourier series, we consider the heat equation in a rectangular column $0 < x < L_1$, $0 < y < L_2$, with the homogeneous boundary conditions that $u = 0$ on all four sides of the column, $x = 0$, $x = L_1$, $y = 0$, $y = L_2$. The separated solutions, which depend on (x, y, t), are of the form

$$u(x,y;t) = \phi_1(x)\phi_2(y)T(t)$$

Substituting in the heat equation and dividing by Ku, we have

$$\frac{T'(t)}{KT(t)} = \frac{\phi_1''(x)}{\phi_1(x)} + \frac{\phi_2''(y)}{\phi_2(y)}$$

The left side depends on t and the right side depends on (x, y). Therefore each side is a constant, which we call $-\lambda$. Applying the same argument to the right side, we see that both ϕ_1''/ϕ_1 and ϕ_2''/ϕ_2 are constants, to be called $-\mu_1$ and $-\mu_2$, respectively. Therefore we have the ordinary differential equations

$$\begin{aligned} T'(t) + \lambda KT(t) &= 0 \\ \phi_1''(x) + \mu_1\phi_1(x) &= 0 \\ \phi_2''(y) + \mu_2\phi_2(y) &= 0 \end{aligned}$$

where $\lambda = \mu_1 + \mu_2$. From the boundary conditions we must have $\phi_1(0) = 0$, $\phi_1(L_1) = 0$, $\phi_2(0) = 0$, $\phi_2(L_2) = 0$. The solutions of these Sturm-Liouville problems are $\phi_1(x) = \sin(m\pi x/L_1)$, $\mu_1 = (m\pi/L_1)^2$, $\phi_2(y) = \sin(n\pi y/L_2)$, $\mu_2 = (n\pi/L_2)^2$; we have $T(t) = e^{-\lambda Kt}$, where $\lambda = (m\pi/L_1)^2 + (n\pi/L_2)^2$. Thus we have the separated solutions of the heat equation in the column, with zero boundary conditions:

$$\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}e^{-\lambda_{mn}Kt} \qquad m,n=1,2,\ldots$$
$$\lambda_{mn}=\left(\frac{m\pi}{L_1}\right)^2+\left(\frac{n\pi}{L_2}\right)^2$$

The indices m, n are independent of one another. A general solution of the heat equation with zero boundary conditions is obtained as a superposition:

$$\boxed{u(x,y;t)=\sum_{m,n=1}^{\infty}B_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}e^{-\lambda_{mn}Kt}} \tag{2.5.1}$$

We can use these to solve initial-value problems for the heat equation.

EXAMPLE 2.5.1. *Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$, with the boundary conditions $u(0,y;t)=0$, $u(L_1,y;t)=0$, $u(x,0;t)=0$, $u(x,L_2;t)=0$ and the initial condition $u(x,y;0)=1$, for $0<x<L_1$, $0<y<L_2$. Find the relaxation time.*

Solution. We look for the solution as a sum of separated solutions (2.5.1). The Fourier coefficients B_{mn} are obtained by setting $t=0$ and using the initial conditions. Thus we have

$$\begin{aligned}
1 &= \sum_{m,n=1}^{\infty}B_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\\
B_{mn} &= \frac{4}{L_1L_2}\int_0^{L_2}\int_0^{L_1}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\,dx\,dy\\
&= \frac{4}{\pi^2}\frac{[1-(-1)^m][1-(-1)^n]}{mn}
\end{aligned}$$

The solution is

$$u(x,y;t)=\frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{1-(-1)^m}{m}\frac{1-(-1)^n}{n}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}e^{-\lambda_{mn}Kt}$$

For each $t>0$, the series for u, u_x, u_y, u_{xx}, u_{yy}, u_t converge uniformly for $0\le x\le L_1$, $0\le y\le L_2$, and hence u is a rigorous solution of the heat equation. The relaxation time is given by the first term of the series, when $m=1$, $n=1$. Thus $[(\pi^2/L_1^2)+(\pi^2/L_2^2)]K\tau=1$, and the relaxation time is $\tau = L_1^2L_2^2/K\pi^2(L_1^2+L_2^2)$. •

Initial-value problems for the heat equation in a three-dimensional cube can be handled similarly, using Fourier series in three variables. For example, if we have the cube $0<x<L$, $0<y<L$, $0<z<L$, then we find the separated solutions in the form $\sin(m\pi x/L)\sin(n\pi y/L)\sin(p\pi z/L)e^{-\lambda Kt}$, where (m,n,p) are independent indices and $\lambda=(m\pi/L)^2+(n\pi/L)^2+(p\pi/L)^2$. The initial

conditions determine an expansion in a triple Fourier series and hence lead to a series solution of the initial-value problem.

2.5.2. Laplace's equation. We now turn to Laplace's equation $\nabla^2 u = 0$ in rectangular coordinates. To find separated solutions, we assume the form $u(x, y, z) = \phi_1(x)\phi_2(y)\phi_3(z)$ and substitute in $\nabla^2 u = 0$, with the result

$$0 = \frac{\phi_1''(x)}{\phi_1(x)} + \frac{\phi_2''(y)}{\phi_2(y)} + \frac{\phi_3''(z)}{\phi_3(z)}$$

By the methods of separation of variables, each of these must be constant, and we have the ordinary differential equations

$$\phi_1''(x) + \mu_1\phi_1(x) = 0 \tag{2.5.2}$$

$$\phi_2''(y) + \mu_2\phi_2(y) = 0 \tag{2.5.3}$$

$$\phi_3''(z) + \mu_3\phi_3(z) = 0 \tag{2.5.4}$$

where the separation constants (μ_1, μ_2, μ_3) must obey the relation $\mu_1+\mu_2+\mu_3 = 0$. To proceed further, we must know the form of the boundary conditions. The method will become clear from the following examples.

EXAMPLE 2.5.2. *Find the separated solutions $u(x, y)$ of Laplace's equation in the column $0 < x < L_1$, $0 < y < L_2$ satisfying the boundary conditions $u(0, y) = 0$, $u(L_1, y) = 0$.*

Solution. Since u depends on (x, y), we take $\mu_3 = 0$, $\mu_1 + \mu_2 = 0$. The boundary conditions require that we solve the Sturm-Liouville problem $\phi_1''(x) + \mu_1\phi_1(x) = 0$, $\phi_1(0) = 0$, $\phi_1(L_1) = 0$, whose solution is $\phi_1(x) = \sin(n\pi x/L_1)$. Thus $\mu_2 = -\mu_1 = -(n\pi/L_1)^2$, and the equation for ϕ_2 is $\phi_2'' - (n\pi/L_1)^2\phi_2 = 0$, whose solution is $\phi_2(y) = A\cosh(n\pi y/L_1) + B\sinh(n\pi y/L_1)$. The separated solutions are $u(x, y) = \sin(n\pi x/L_1)(A\cosh(n\pi y/L_1) + B\sinh(n\pi y/L_1))$, $n = 1, 2, \ldots$ •

Once we have determined the separated solutions of Laplace's equation, we may solve boundary-value problems for Laplace's equation by the methods of Fourier series. The success of the method depends on considering one side at a time.

EXAMPLE 2.5.3. *Solve Laplace's equation in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u(0, y) = 0$, $u(L_1, y) = 0$, $u(x, 0) = 0$, $u(x, L_2) = 1$.*

Solution. In Example 2.5.2 we found the separated solutions satisfying the first two boundary conditions,

$$u(x, y) = \sin\frac{n\pi x}{L_1}\left(A\cosh\frac{n\pi y}{L_1} + B\sinh\frac{n\pi y}{L_1}\right)$$

The third boundary condition requires $0 = A\sin n\pi x/L_1$, $0 < x < L_1$; hence $A = 0$. We look for the solution as a superposition

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L_1} \sinh\frac{n\pi y}{L_1}$$

The fourth boundary condition requires that

$$1 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L_1} \sinh\frac{n\pi L_2}{L_1}$$

which is a Fourier sine series in x. But we know the Fourier sine expansion of

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin\frac{n\pi x}{L_1}$$

and therefore

$$B_n \sinh\frac{n\pi L_2}{L_1} = \frac{2}{\pi}\frac{1-(-1)^n}{n}$$

and the solution is

$$u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1-(-1)^n]\sin(n\pi x/L_1)\sinh(n\pi y/L_1)}{n\sinh(n\pi L_2/L_1)} \quad \bullet$$

We now outline the procedure for solving Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ with four nonzero boundary conditions: $u(x,0) = T_1$, $u(x,L_2) = T_2$, $u(0,y) = T_3$, $u(L_1,y) = T_4$. Following Example 2.5.3, we can obtain $u_2(x,y)$, the solution of the problem when T_1, T_3, T_4 are replaced by zero. By interchanging the roles of x and y, we can similarly obtain $u_4(x,y)$, the solution of the problem with T_1, T_2, T_3 replaced by zero.

$$u_4(x,y) = \frac{2T_4}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\sin(n\pi y/L_2)\sinh(n\pi x/L_2)}{n\sinh(n\pi L_1/L_2)}$$

The remainder of the solution can be obtained by the substitutions $x \to L_1 - x$ and $y \to L_2 - y$; thus

$$u_3(x,y) = \frac{2T_3}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\sinh(n\pi(L_1-x)/L_2)\sin(n\pi y/L_2)}{n\sinh(n\pi L_1/L_2)}$$

with $u_1(x,y)$ obtained similarly. Adding these four functions gives a solution of Laplace's equation that satisfies all four boundary conditions; thus we have $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$. This can be illustrated as in Fig. 2.5.1.

In the preceding examples the solution of Laplace's equation in the column $0 < x < L_1$, $0 < y < L_2$ was written as a sum of ordinary Fourier series. If we consider problems in a cube, we encounter double Fourier series.

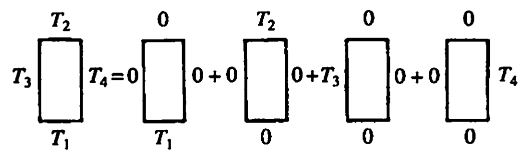


FIGURE 2.5.1 Solution of Laplace's equation by superposition.

EXAMPLE 2.5.4. *Find the separated solutions of Laplace's equation in the cube* $0 < x < L$, $0 < y < L$, $0 < z < L$ *satisfying the boundary conditions* $u(0, y, z) = 0$, $u(L, y, z) = 0$, $u(x, L, z) = 0$.

Solution. Referring to (2.5.2), we have the Sturm-Liouville problems $\phi_1'' + \mu_1\phi_1 = 0$, $\phi_2'' + \mu_2\phi_2 = 0$ with the boundary conditions $\phi_1(0) = 0$, $\phi_1(L_1) = 0$, $\phi_2(0) = 0$, $\phi_2(L_2) = 0$. Thus $\phi_1(x) = \sin(m\pi x/L_1)$, $\phi_2(y) = \sin(n\pi y/L_2)$, $\mu_1 = -(m\pi/L_1)^2$, $\mu_2 = -(n\pi/L_2)^2$, and $\mu_3 = -(\mu_1 + \mu_2)$. Thus

$$\phi_3(z) = A\cosh\frac{\pi z}{L}\sqrt{m^2+n^2} + B\sinh\frac{\pi z}{L}\sqrt{m^2+n^2}$$

and the separated solutions are

$$\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L}\left(A\cosh\frac{\pi z}{L}\sqrt{m^2+n^2} + B\sinh\frac{\pi z}{L}\sqrt{m^2+n^2}\right) \quad \bullet$$

These may be used to solve a boundary-value problem, as in the case of the rectangular column.

EXAMPLE 2.5.5. *Solve Laplace's equation* $\nabla^2 u = 0$ *in the cube* $0 < x < L$, $0 < y < L$, $0 < z < L$, *with the boundary conditions* $u(0, y, z) = 0$, $u(L, y, z) = 0$, $u(x, 0, z) = 0$, $u(x, L, z) = 0$, $u(x, y, 0) = 0$, $u(x, y, L) = 1$.

Solution. Proceeding as in Example 2.5.3, we take $A = 0$ in the separated solutions found in Example 2.5.4 and obtain the solution $u(x, y, z)$ as

$$\frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n]\sin(m\pi x/L)\,\sin(n\pi y/L)\,\sinh(\pi z/L)\sqrt{m^2+n^2}}{mn\,\sinh(\pi\sqrt{m^2+n^2})} \quad \bullet$$

2.5.3. The heat equation (nonhomogeneous boundary conditions). We now combine the above methods to solve initial-value problems for the heat equation where the boundary conditions are *nonhomogeneous*. To do this we use the five-stage method developed in Sec. 2.3. In stage 1 we obtain the steady-state solution by solving Laplace's equation; this solution has been discussed above. In stage 2 we use this steady-state solution to transform the problem to homogeneous boundary conditions, for which the separated solutions have been obtained. In stage 3 we form a superposition of these and use the initial conditions to find the Fourier coefficients. In stage 4 we verify the solution, and in stage 5 we obtain the relaxation time.

EXAMPLE 2.5.6. *Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$, with the boundary conditions $u(0,y;t) = 0$, $u(L_1,y;t) = 0$, $u(x,0;t) = 0$, $u(x,L_2;t) = T_2$ and the initial condition $u(x,y;0) = T_1$, where T_1 and T_2 are positive constants.*

Solution. The steady-state solution, denoted $U(x,y)$, satisfies Laplace's equation $\nabla^2 U = 0$ with the indicated boundary conditions. This was solved in Example 2.5.3.

$$U(x,y) = \frac{2T_2}{\pi}\sum_{m=1}^{\infty}[1-(-1)^m]\frac{\sin(m\pi x/L_1)\sinh(m\pi y/L_1)}{m\sinh(m\pi L_2/L_1)}$$

Letting $v(x,y;t) = u(x,y;t) - U(x,y)$, we have $v_t = K\nabla^2 v$ with $v = 0$ on all four sides, while $v(x,y;0) = T_1 - U(x,y)$. We look for $v(x,y;t)$ as a superposition of separated solutions with zero boundary conditions.

$$v(x,y;t) = \sum_{m,n} B_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}e^{-\lambda_{mn}Kt}$$

The Fourier coefficients B_{mn} are obtained by setting $t = 0$; thus

$$T_1 - U(x,y) = \sum_{m,n=1}^{\infty} B_{mn}\sin(m\pi x/L_1)\sin(n\pi y/L_2)$$

To obtain the required coefficients B_{mn}, we begin with the Fourier sine expansion of the hyperbolic sine.

$$\sinh\frac{a\pi y}{L_2} = \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin\frac{n\pi y}{L_2} \qquad 0 < y < L_2$$

Letting $a = mL_2/L_1$ and substituting in the formula for $U(x,y)$, we have the double Fourier sine series

$$U(x,y) = \frac{4T_2}{\pi^2}\sum_{m,n=1}^{\infty}\frac{[1-(-1)^m]n(-1)^{n+1}}{m[n^2+(mL_2/L_1)^2]}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

Likewise, the double Fourier sine expansion of T_1 is

$$T_1 = \frac{4T_1}{\pi^2}\sum_{m,n=1}^{\infty}\frac{1-(-1)^m}{m}\frac{1-(-1)^n}{n}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

From these we can obtain B_{mn} as the coefficient of $\sin(m\pi x/L_1)\sin(n\pi y/L_2)$. The solution of the original boundary-value problem is

$$u(x,y;t) = U(x,y) + \sum_{m,n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}e^{-\lambda_{mn}Kt}$$

For each $t > 0$, the series for u, u_x, u_{xx}, u_y, u_{yy}, u_t are uniformly convergent for $0 < x < L_1$, $0 < y < L_2$; thus $u(x,y;t)$ is a rigorous solution of the boundary-value problem. The relaxation time is given by the first term of the series, $(\pi^2/L_1^2 + \pi^2/L_2^2)K\tau = 1$ if $B_{11} \neq 0$. •

2.5.4. The wave equation (nodal lines). We now turn to an example involving the wave equation. The small transverse vibrations of a tightly stretched membrane are governed by the two-dimensional wave equation. This has the form

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

where $u(x, y; t)$ denotes the transverse displacement of the membrane from its equilibrium position and c is a positive constant. The membrane is supposed to cover the rectangle $0 < x < L_1$, $0 < y < L_2$ with the edges fixed; thus $u(x, y; t) = 0$ for $x = 0$, $x = L_1$, $y = 0$, $y = L_2$.

As a first step we find the separated solutions of the wave equation that satisfy these boundary conditions. These have the form

$$u(x, y; t) = \phi_1(x)\phi_2(y)T(t)$$

Substituting in the wave equation and separating variables, we have

$$\frac{T''(t)}{c^2T(t)} = \frac{\phi_1''(x)}{\phi_1(x)} + \frac{\phi_2''(y)}{\phi_2(y)}$$

The left side depends on t alone, while the right side depends on (x, y); thus each is a constant, say $-\lambda$. Introducing further separation constants μ_1 and μ_2, we have the ordinary differential equations

$$\begin{aligned} T''(t) + \lambda c^2 T(t) &= 0 \\ \phi_1''(x) + \mu_1\phi_1(x) &= 0 \\ \phi_2''(y) + \mu_2\phi_2(y) &= 0 \end{aligned}$$

with $\lambda = \mu_1 + \mu_2$. The boundary conditions require $\phi_1(0) = 0$, $\phi_1(L_1) = 0$, $\phi_2(0) = 0$, $\phi_2(L_2) = 0$. These Sturm-Liouville problems have the solutions $\phi_1(x) = \sin(m\pi x/L_1)$, $\phi_2(y) = \sin(n\pi y/L_2)$, $\mu_1 = (m\pi/L_1)^2$, $\mu_2 = (n\pi/L_2)^2$. Thus $\lambda > 0$, and we can write $T(t) = A\cos(ct\sqrt{\lambda}) + B\sin(ct\sqrt{\lambda})$. We now have the separated solutions of the rectangular membrane

$$u_{mn}(x, y; t) = \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right) \tag{2.5.5}$$

$$\omega_{mn} = c\left[\left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2\right]^{1/2} \tag{2.5.6}$$

The constants A_{mn} and B_{mn} can be chosen to fit various initial conditions by using the superposition principle and the methods of double Fourier series.

EXAMPLE 2.5.7. *Solve the initial-value problem for the vibrating membrane with the initial conditions* $u(x, y; 0) = 0$, $u_t(x, y; 0) = 1$.

Solution. We look for the solution as a superposition of separated solutions:

$$u(x,y;t) = \sum_{m,n} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right)$$

Setting $t = 0$, it follows that $0 = \sum_{m,n} A_{mn}\sin(m\pi x/L_1)\sin(n\pi y/L_2)$; thus $A_{mn} = 0$. Differentiating and setting $t = 0$, we must have

$$1 = \sum_{m,n} B_{mn}\omega_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

Thus $B_{mn}\omega_{mn} = (2/\pi)^2[1-(-1)^m][1-(-1)^n]/mn$, and we have found the formal solution of the problem:

$$u(x,y;t) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n]}{mn\omega_{mn}}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sin\omega_{mn}t \quad \bullet$$

In contrast with the heat equation, the double Fourier series solutions obtained do not converge sufficiently fast to verify the convergence of the series for the various derivatives u_x, u_{xx}, u_y, u_{yy}, u_t, u_{tt}. Therefore we usually restrict attention to solutions that contain only a finite number of terms. In particular, it is interesting to examine the separated solutions obtained previously. Indeed, these solutions have an important physical interpretation as *standing waves*. The profile is given by the function $\sin(m\pi x/L_1)\sin(n\pi y/L_2)$. This function undergoes a periodic oscillation with period $\tau = 2\pi/\omega_{mn}$, owing to the time dependence $A\cos\omega_{mn}t + B\sin\omega_{mn}t$. The membrane can be divided into various zones, depending on the sign of u; these zones are divided by curves, which are called

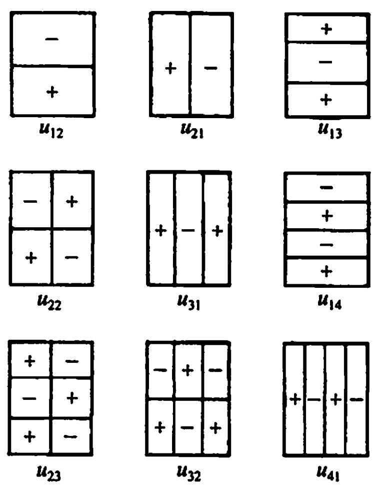


FIGURE 2.5.2 Nodal lines of a rectangular membrane.

(m,n)	ω_{mn}	$u_{mn}(x,y)$
$(1,1)$	$(\pi c/L)\sqrt{2}$	$\sin(\pi x/L)\sin(\pi y/L)$
$(2,1)$	$(\pi c/L)\sqrt{5}$	$\sin(2\pi x/L)\sin(\pi y/L)$
$(1,2)$	$(\pi c/L)\sqrt{5}$	$\sin(\pi x/L)\sin(2\pi y/L)$
$(2,2)$	$(\pi c/L)\sqrt{8}$	$\sin(2\pi x/L)\sin(2\pi y/L)$
$(3,1)$	$(\pi c/L)\sqrt{10}$	$\sin(3\pi x/L)\sin(\pi y/L)$
$(1,3)$	$(\pi c/L)\sqrt{10}$	$\sin(\pi x/L)\sin(3\pi y/L)$
$(3,2)$	$(\pi c/L)\sqrt{13}$	$\sin(3\pi x/L)\sin(2\pi y/L)$
$(2,3)$	$(\pi c/L)\sqrt{13}$	$\sin(2\pi x/L)\sin(3\pi y/L)$
$(4,1)$	$(\pi c/L)\sqrt{17}$	$\sin(4\pi x/L)\sin(\pi y/L)$
$(1,4)$	$(\pi c/L)\sqrt{17}$	$\sin(\pi x/L)\sin(4\pi y/L)$

TABLE 2.5.1

nodal lines. We illustrate in Fig. 2.5.2 the nodal lines for some of the separated solutions we have just found.

We now consider in more detail the vibrating *square* membrane with $0 < x < L$, $0 < y < L$. Thus we take $L_1 = L$, $L_2 = L$. The first 10 frequencies of the separated solutions are listed in Table 2.5.1.

We distinguish between *simple frequencies* and *multiple frequencies*. For example, $\omega_{11} = (\pi c/L)\sqrt{2}$ is a simple frequency, whereas $\omega_{12} = (\pi c/L)\sqrt{5}$ is a multiple frequency, of multiplicity 2. We may obtain solutions with a more complex nodal structure by taking sums of solutions corresponding to a multiple frequency. For example,

$$\begin{aligned} u_{12} - u_{21} &= \sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} - \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L} \\ &= 2\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\left(\cos\frac{\pi y}{L} - \cos\frac{\pi x}{L}\right) \end{aligned}$$

Thus $u_{12} - u_{21} = 0$ for $x = y$, since the factor $\cos(\pi y/L) - \cos(\pi x/L) = 0$ on that line. If we consider the multiple frequency $\omega_{13} = (\pi c/L)\sqrt{10}$, and study the difference $u_{13} - u_{31}$, it may be shown that this function is zero along both lines $x + y = L$ and $y = x$. These possibilities are illustrated in Fig. 2.5.3. More complex diagrams may be obtained by considering higher values of (m,n).

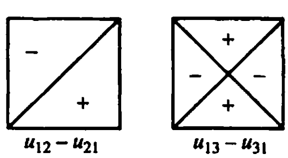


FIGURE 2.5.3 Diagonal nodal lines of a square membrane.

2.5.5. Multiplicities of the eigenvalues. One may inquire as to the possibility of predicting in advance which frequencies are simple and which are multiple—in particular, to determine the multiplicity of the latter. This can be done using a number-theoretic method involving the factorization of the eigenvalue $\lambda = m^2 + n^2$. In order to illustrate this, we list below in Table 2.5.2 all of the eigenvalues $\lambda \leq 100$ in increasing order, together with their factorizations and multiplicities. The sum of the multiplicities is 69, which is the number of normal modes whose frequencies are less than or equal to 10. In addition to the previously observed eigenvalues of multiplicity 2, we now observe eigenvalues of multiplicity 3 and eigenvalues of multiplicity 4. Indeed, the eigenvalue $\lambda = 50$ has three linearly independent eigenfunctions, corresponding to the pairs $(m, n) = (5, 5), (7, 1)$, and $(1, 7)$. The eigenvalue $\lambda = 65$ has four linearly independent eigenfunctions, corresponding to the pairs $(m, n) = (7, 4), (4, 7), (8, 1)$, and $(1, 8)$, while the eigenvalue $\lambda = 85$ has four linearly independent eigenfunctions, corresponding to the pairs $(m, n) = (9, 2), (2, 9), (7, 6)$, and $(6, 7)$. Altogether, there are 6 simple eigenvalues, 26 eigenvalues of multiplicity 2, 1 eigenvalue of multiplicity 3, and 2 eigenvalues of multiplicity 4.

In order to understand the apparently erratic nature of the individual multiplicities, it suffices to distinguish three different sets of prime numbers that may enter into the factorization of λ:

- The prime 2
- Odd primes of the form $4n + 1$, i.e., $5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97$
- Odd primes of the form $4n - 1$, i.e., $3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83$

Inspection of the table suggests that the multiplicity is related to the number of prime factors of the second type that occur in the factorization. For example, the eigenvalue $\lambda = 25 = 5^2$ is of multiplicity 2 and has only one prime, $p = 5$, in its factorization, whereas the eigenvalue $\lambda = 65 = 5 \cdot 13$ is of multiplcity 4 and has the two different primes, $p = 5$ and $p = 13$, in its factorization, leading to the higher multiplicity. Similarly, the eigenvalue $\lambda = 85 = 5 \cdot 17$ is also of multiplicity 4 and has the two different primes, $p = 5$ and $p = 17$, in its factorization. For a deeper understanding of the multiplicities, one must examine the factorization in the domain of *gaussian integers*, which are complex numbers of the form $m + ni$, where $m, n = 0 \pm 1, \pm 2, \ldots$. The interested reader is referred to a book on number theory.[6]

Table 2.5.2 becomes somewhat simplified when we consider the eigenvalues of the 45-45-90 triangle. These are the numbers of the form $\lambda = m^2 + n^2$ with $m > n$, as shown in Table 2.5.3.

The sum of the multiplicities is 31, which is the number of normal modes whose frequencies are less than or equal to 10. In this case one sees clearly that

[6]G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed. Oxford University Press, Oxford, 1979.

$\lambda = \omega^2 = m^2 + n^2$	$N(\lambda)$	Factorization
$2 = 1^2 + 1^2$	1	2
$5 = 2^2 + 1^2$	2	5
$8 = 2^2 + 2^2$	1	2^3
$10 = 3^2 + 1^2$	2	$2 \cdot 5$
$13 = 3^2 + 2^2$	2	13
$17 = 4^2 + 1^2$	2	17
$18 = 3^2 + 3^2$	1	$2 \cdot 3^2$
$20 = 4^2 + 2^2$	2	$2^2 \cdot 5$
$25 = 4^2 + 3^2$	2	5^2
$26 = 5^2 + 1^2$	2	$2 \cdot 13$
$29 = 5^2 + 2^2$	2	29
$32 = 4^2 + 4^2$	1	2^5
$34 = 5^2 + 3^2$	2	$2 \cdot 17$
$37 = 6^2 + 1^2$	2	37
$40 = 6^2 + 2^2$	2	$2^3 \cdot 5$
$41 = 5^2 + 4^2$	2	41
$45 = 6^2 + 3^2$	2	$3^2 \cdot 5$
$50 = 5^2 + 5^2$	3	$2 \cdot 5^2$
$= 7^2 + 1^2$		
$52 = 6^2 + 4^2$	2	$2^2 \cdot 13$
$53 = 7^2 + 2^2$	2	53
$58 = 7^2 + 3^2$	2	$2 \cdot 29$
$61 = 6^2 + 5^2$	2	61
$65 = 7^2 + 4^2$	4	$5 \cdot 13$
$= 8^2 + 1^2$		
$68 = 8^2 + 2^2$	2	$2^2 \cdot 17$
$72 = 6^2 + 6^2$	1	$2^3 \cdot 3^2$
$73 = 8^2 + 3^2$	2	73
$74 = 7^2 + 5^2$	2	$2 \cdot 37$
$80 = 8^2 + 4^2$	2	$2^4 \cdot 5$
$82 = 9^2 + 1^2$	2	$2 \cdot 41$
$85 = 9^2 + 2^2$	4	$5 \cdot 17$
$= 7^2 + 6^2$		
$89 = 8^2 + 5^2$	2	89
$90 = 9^2 + 3^2$	2	$2 \cdot 3^2 \cdot 5$
$97 = 9^2 + 4^2$	2	97
$98 = 7^2 + 7^2$	1	$2 \cdot 7^2$
$100 = 8^2 + 6^2$	2	$2^2 \cdot 5^2$

TABLE 2.5.2

$\lambda = \omega^2 = m^2 + n^2$	$N(\lambda)$	Factorization
$5 = 2^2 + 1^2$	1	5
$10 = 3^2 + 1^2$	1	$2 \cdot 5$
$13 = 3^2 + 2^2$	1	13
$17 = 4^2 + 1^2$	1	17
$20 = 4^2 + 2^2$	1	$2^2 \cdot 5$
$25 = 4^2 + 3^2$	1	5^2
$26 = 5^2 + 1^2$	1	$2 \cdot 13$
$29 = 5^2 + 2^2$	1	29
$34 = 5^2 + 3^2$	1	$2 \cdot 17$
$37 = 6^2 + 1^2$	1	37
$40 = 6^2 + 2^2$	1	$2^3 \cdot 5$
$41 = 5^2 + 4^2$	1	41
$45 = 6^2 + 3^2$	1	$3^2 \cdot 5$
$50 = 7^2 + 1^2$	1	$2 \cdot 5^2$
$52 = 6^2 + 4^2$	1	$2^2 \cdot 13$
$53 = 7^2 + 2^2$	1	53
$58 = 7^2 + 3^2$	1	$2 \cdot 29$
$61 = 6^2 + 5^2$	1	61
$65 = 7^2 + 4^2$	2	$5 \cdot 13$
$= 8^2 + 1^2$		
$68 = 8^2 + 2^2$	1	$2^2 \cdot 17$
$73 = 8^2 + 3^2$	1	73
$74 = 7^2 + 5^2$	1	$2 \cdot 37$
$80 = 8^2 + 4^2$	1	$2^4 \cdot 5$
$82 = 9^2 + 1^2$	1	$2 \cdot 41$
$85 = 9^2 + 2^2$	2	$5 \cdot 17$
$= 7^2 + 6^2$		
$89 = 8^2 + 5^2$	1	89
$90 = 9^2 + 3^2$	1	$2 \cdot 3^2 \cdot 5$
$97 = 9^2 + 4^2$	1	97
$100 = 8^2 + 6^2$	1	$2^2 \cdot 5^2$

TABLE 2.5.3

the only multiple eigenvalues are those that are products of distinct primes of the form $4n + 1$.

2.5.6. Implementation with Mathematica. One can use Mathematica to obtain three-dimensional graphs of the rectangular drumhead. We apply the command **Plot3D** to the formula (2.5.5) with $A = 1$, $B = 0$, $L_1 = L_2 = \pi$, and $c = 1$, so that

$$v_{mn}(x, y; t) = \sin mx \sin ny \cos(\sqrt{m^2 + n^2}\, t)$$

This is written in Mathematica as

```
v[m_,n_,x_,y_,t_]=Sin[m x]Sin[n y]Cos[Sqrt[m^2+n^2]t]
```

To obtain a three-dimensional graph of the solutions described in Fig. 2.5.2, we define a plot-valued function with

```
vv[m_,n_,t_]=Plot3D[Sin[m x]Sin[n y]Cos[Sqrt[m^2+n^2]t],
          {x,0,Pi},{y,0,Pi},PlotPoints->40]
```

To obtain a three-dimensional graph of the solutions described in Fig. 2.5.3, we define

```
vV[m_,n_,t_]=Plot3D[(Sin[m x]Sin[n y]-Sin[n x]Sin[m y])*
          Cos[Sqrt[m^2+n^2]t],{x,0,Pi},{y,0,Pi},PlotPoints->40]
```

The examples in Fig. 2.5.4 are obtained by typing the commands

```
vv[1,2,0]          vv[2,2,0]
vv[1,4,0]          vv[2,5,0]
vV[1,2,0]          vV[1,4,0]
```

2.5.7. Application to Poisson's equation. Poisson's equation $\nabla^2 u = -\rho$ is very similar to Laplace's equation. We look for a particular solution U that does not necessarily satisfy all of the boundary conditions. The function $v = u - U$ then satisfies Laplace's equation with some new boundary conditions. The solution v can be found by the above method for Laplace's equation. We illustrate with a simple example.

EXAMPLE 2.5.8. *Find the solution of Poisson's equation $\nabla^2 u = -1$ in the rectangle $0 < x < L_1$, $0 < y < L_2$ satisfying the boundary conditions that $u = 0$ on all four sides of the rectangle.*

Solution. A particular solution depending on x alone satisfies $u_{xx} = -1$; thus $U(x, y) = \frac{1}{2}x(L_1 - x)$ satisfies the equation and the boundary conditions at $x = 0$, $x = L_1$. The function $v = u - U$ satisfies Laplace's equation $\nabla^2 v = 0$ with the boundary conditions that $v = 0$ when $x = 0$, $x = L_1$ and $v = -\frac{1}{2}x(L_1 - x)$ when $y = 0, y = L_2$. This is sought as a series of separated solutions in the form

$$v(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L_1} \left(A_n \sinh \frac{n\pi(L_2 - y)}{L_1} + B_n \sinh \frac{n\pi y}{L_1} \right)$$

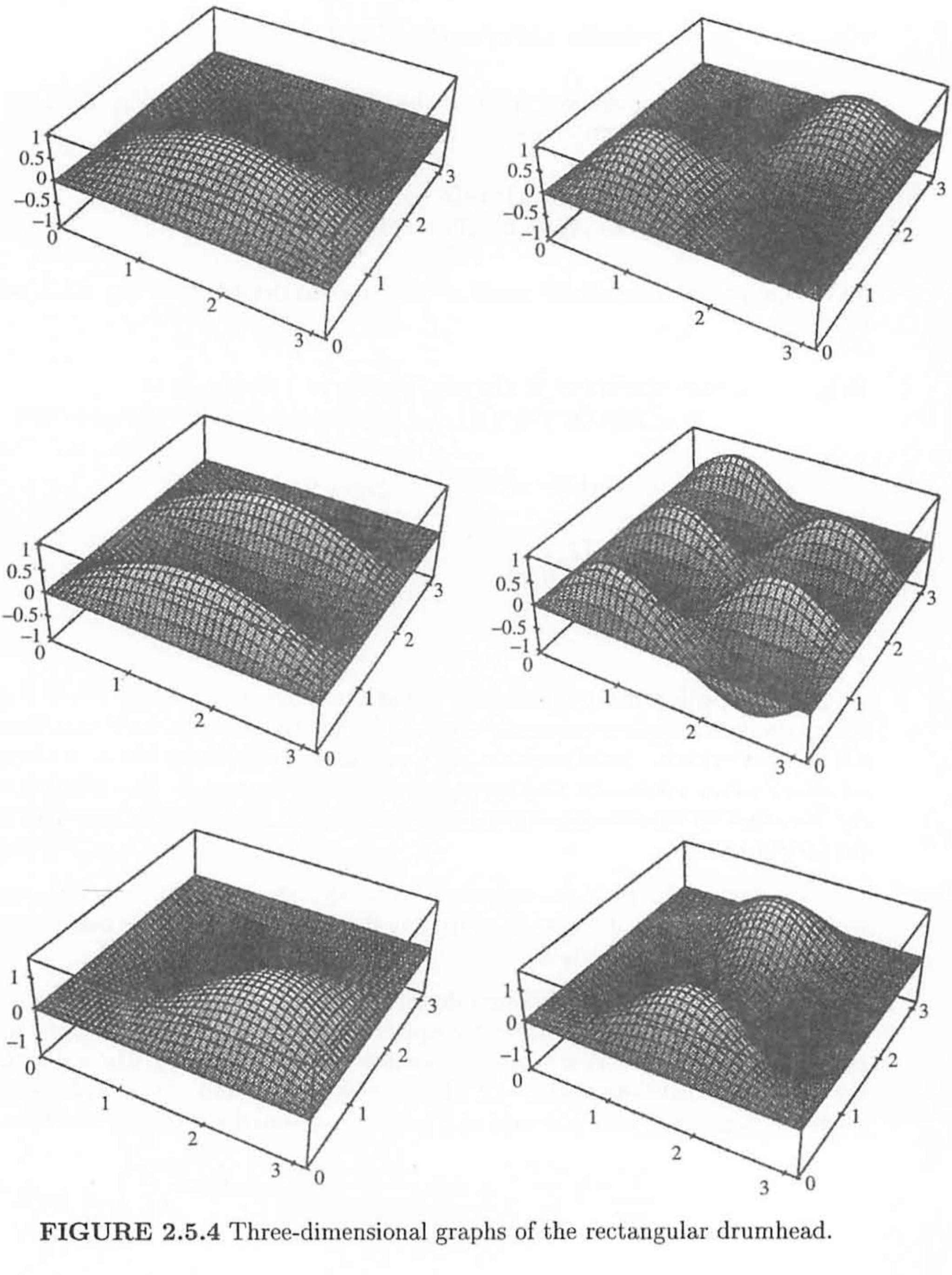


FIGURE 2.5.4 Three-dimensional graphs of the rectangular drumhead.

To satisfy the remaining boundary conditions, we must have

$$-\frac{1}{2}x(L_1 - x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L_1}\sinh\frac{n\pi L_2}{L_1}$$

$$-\frac{1}{2}x(L_1 - x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L_1}\sinh\frac{n\pi L_2}{L_1}$$

The Fourier series of $x(L_1 - x)$ is $(8L_1^2/\pi^3)\sum_{n\,\mathrm{odd}} n^{-3}\sin(n\pi x/L_1)$. Equating coefficients gives

$$A_n = B_n = -\frac{4L_1^2}{n^3\pi^3\sinh(n\pi L_2/L_1)}$$

for n odd and $A_n = 0$, $B_n = 0$ for n even. This leads to

$$v(x,y) = -\frac{4L_1^2}{\pi^3}\sum_{n\,\mathrm{odd}}\frac{\sin(n\pi x/L_1)}{n^3\sinh(n\pi L_2/L_1)}\left[\sinh\frac{n\pi y}{L_1} + \sinh\frac{n\pi(L_2 - y)}{L_1}\right]$$

Combining this with the Fourier series for U, we may write the solution of Poisson's equation as a single series in the form

$$u(x,y) = \frac{4L_1^2}{\pi^3}\sum_{n\,\mathrm{odd}} n^{-3}\sin\frac{n\pi x}{L_1}\left[2 - \frac{\sinh(n\pi y/L_1) + \sinh(n\pi(L_2 - y)/L_1)}{\sinh(n\pi L_2/L_1)}\right] \quad \bullet$$

We now compare this with another method of solution.

EXAMPLE 2.5.9. *Find the solution of Poisson's equation $\nabla^2 u = -1$ in the rectangle $0 < x < L_1$, $0 < y < L_2$ in the form of a double Fourier sine series*

$$u(x,y) = \sum_{m,n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

Solution. The indicated sine products form a complete set of functions in the rectangle and satisfy the required boundary conditions. Therefore it remains to satisfy Poisson's equation. For this purpose we write

$$\nabla^2 u = u_{xx} + u_{yy} = \sum_{m,n=1}^{\infty} A_{mn}\left[\left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2\right]\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

The Fourier series of the constant function is

$$1 = \frac{4}{\pi}\sum_{m,n\,\mathrm{odd}}\frac{1}{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}$$

The two series will be equal if and only if we choose

$$A_{mn} = \frac{4}{mn\pi[(m\pi/L_1)^2 + (n\pi/L_2)^2]} \quad \bullet$$

It is interesting to compare the two forms of the series solution obtained for Poisson's equation. In the first example, the nth term of the series for $v(x,y)$ tends to zero at the exponential rate, where the exponent is the larger of $e^{-n\pi y/L_1}$ and $e^{-n\pi(L_2-y)/L_1}$. In the second case, the double Fourier series tends to zero at an algebraic rate, according to $1/mn(m^2+n^2)$. The series for u_x, u_{xx}, u_y, u_{yy} converge at an even slower rate. Thus, for the purposes of numerical computation, the first series is superior to the second series.

EXERCISES 2.5

1. Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u(0,y;t) = 0$, $u_x(L_1,y,t) = 0$, $u(x,0;t) = 0$, $u_y(x,L_2;t) = 0$ and the initial condition $u(x,y;0) = 1$. Find the relaxation time.
2. Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u(0,y;t) = 0$, $u(L_1,y;t) = 0$, $u_y(x,0,t) = 0$, $u_y(x,L_2;t) = 0$ and the initial condition $u(x,y;0) = 1$. Find the relaxation time.
3. Find separated solutions of Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ satisfying the boundary conditions $u_x(0,y) = 0$, $u_x(L_1,y) = 0$.
4. Solve Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u_x(0,y) = 0$, $u_x(L_1,y) = 0$, $u(x,0) = 0$, $u(x,L_2) = x$.
5. Solve Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u_x(0,y) = 0$, $u_x(L_1,y) = 0$, $u(x,0) = 0$, $u(x,L_2) = 1$.
6. Solve Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u_x(0,y) = 0$, $u_x(L_1,y) = 0$, $u(x,0) = T_1$, $u(x,L_2) = T_2$, where T_1 and T_2 are constants.
7. Find separated solutions of Laplace's equation $\nabla^2 u = 0$ in the cube $0 < x < L$, $0 < y < L$, $0 < z < L$ satisfying the boundary conditions $u(0,y,z) = 0$, $u_x(L,y,z) = 0$, $u(x,0,z) = 0$, $u_y(x,L,z) = 0$.
8. Find the separated solutions of Laplace's equation $\nabla^2 u = 0$ in the cube $0 < x < L$, $0 < y < L$, $0 < z < L$ satisfying the boundary conditions $u_x(0,y,z) = 0$, $u_x(L,y,z) = 0$, $u_y(x,0,z) = 0$, $u_y(x,L,z) = 0$.
9. Solve Laplace's equation $\nabla^2 u = 0$ in the cube $0 < x < L$, $0 < y < L$, $0 < z < L$ with the boundary conditions

$$\begin{array}{lll} u(0,y,z) = 0 & u_x(L,y,z) = 0 & u(x,0,z) = 0 \\ u_y(x,L,z) = 0 & u(x,y,0) = 0 & u(x,y,L) = 1 \end{array}$$

10. Solve Laplace's equation $\nabla^2 u = 0$ in the cube $0 < x < L$, $0 < y < L$, $0 < z < L$, with the boundary conditions

$$\begin{array}{lll} u_x(0,y,z) = 0 & u_x(L,y,z) = 0 & u_y(x,0,z) = 0 \\ u_y(x,L,z) = 0 & u_z(x,y,0) = 0 & u(x,y,L) = 1 \end{array}$$

11. Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u_x(0,y;t) = 0$, $u_x(L_1,y;t) = 0$, $u(x,0;t) = T_1$, $u(x,L_2;t) = T_2$ and the initial condition $u(x,y;0) = T_3$, where T_1, T_2 and T_3 are constants.
12. Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the square column $0 < x < L$, $0 < y < L$ with the boundary conditions $u(0,y;t) = 0$, $u(L,y;t) = 0$, $u(x,0;t) = 0$, $u(x,L;t) = T_1$ and the initial condition $u(x,y;0) = 0$. Find the relaxation time.
13. Solve the initial-value problem for the vibrating membrane in the square $0 < x < L$, $0 < y < L$ with $u(x,y;0) = 3\sin(\pi x/L)\sin(2\pi y/L) + 4\sin(3\pi x/L)\sin(5\pi y/L)$, $u_t(x,y;0) = 0$.
14. Find the separated solutions of the wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$ in the square $0 < x < L$, $0 < y < L$ with the boundary conditions $u_x(0,y;t) = 0$, $u_x(L,y;t) = 0$, $u_y(x,0;t) = 0$, $u_y(x,L;t) = 0$ and the initial conditions $u(x,y;0) = 0$.
15. Find the first 10 frequencies of the separated solutions found in Exercise 14.
16. A vibrating membrane in the shape of an isosceles right triangle covers the region $0 < y < x < L$. Show that $u_{mn} - u_{nm}$ satisfies the wave equation with zero boundary conditions, where $m < n$ and u_{mn} is given by (2.5.5) with $L_1 = L_2$.
17. Find the first 10 frequencies of the separated solutions found in Exercise 16.
18. Solve the initial-value problem for the wave equation on the isosceles right triangle $0 < y < x < L$ with zero boundary conditions and the initial conditions $u(x,y,0) = 0$, $u_t(x,y,0) = 1$.
19. Consider a vibrating membrane covering the equilateral triangle $0 < y < x\sqrt{3}$, $0 < y < \sqrt{3}(L-x)$. Let $d_1 = y$, $d_2 = \frac{1}{2}(x\sqrt{3}-y)$, $d_3 = \frac{1}{2}[\sqrt{3}(L-x)-y]$ be the distance from the point (x,y) to the ith side of the triangle, $i = 1,2,3$. For $n = 1,2,\ldots$ let

$$u_n(x,y;t) = \left(\sin\frac{4\pi n d_1}{L\sqrt{3}} + \sin\frac{4\pi n d_2}{L\sqrt{3}} + \sin\frac{4\pi n d_3}{L\sqrt{3}}\right)\cos\omega t$$

Show that u_n satisfies the wave equation with zero boundary conditions if ω is suitably chosen. (*Hint:* To check the boundary conditions you may use the fact that $d_1 + d_2 + d_3 = L\sqrt{3}/2$.)
20. Let $n = 1$ in Exercise 19. Show that if $u_1(x,y;0) = 0$ and $d_1 > 0$, $d_2 > 0$, then $d_3 = 0$.
21. Use Exercise 20 to show that $u_1(x,y;0) \neq 0$ inside the equilateral triangle.

22. Let $n = 2$ in Exercise 19. Show that $u_2(x, y; 0) = 0$ along the lines $d_1 = L\sqrt{3}/4$, $d_2 = L\sqrt{3}/4$, $d_3 = L\sqrt{3}/4$ and draw a diagram.
23. Let $n = 3$ in Exercise 19. Plot the nodal lines along which $u_3(x, y; 0) = 0$.
24. In Example 2.5.6 compute B_{11}. Show that $B_{11} = 0$ if and only if $2T_1/L_1^2 = T_2/(L_1^2 + L_2^2)$. Show that, for a square column, this is the statement that the initial temperature is the average of the boundary temperatures.
25. Find all of the eigenvalues $\lambda = m^2 + n^2$ in the range $101 < \lambda \leq 200$ and their multiplicities for the square $0 < x < \pi$, $0 < y < \pi$ corresponding to the boundary conditions that $u = 0$ on all four sides.
26. Find all of the eigenvalues $\lambda = m^2 + n^2$ in the range $101 < \lambda \leq 200$ and their multiplicities for the 45-45-90 triangle $0 < x < y < \pi$, corresponding to the boundary conditions that $u = 0$ on all four sides.
27. Find all of the eigenvalues $\lambda = m^2 + n^2$ in the range $0 \leq \lambda \leq 100$ and their multiplicities for the square $0 < x < \pi$, $0 < y < \pi$ corresponding to the boundary conditions that the normal derivative $\partial u/\partial n = 0$ on all four sides.

CHAPTER 3

BOUNDARY-VALUE PROBLEMS IN CYLINDRICAL COORDINATES

INTRODUCTION

In this chapter we consider boundary-value problems in regions with circular or cylindrical boundaries. Section 3.1 is devoted to Laplace's equation in a circle, which can be solved in terms of trigonometric Fourier series. Then we develop the properties of Bessel functions in Sec. 3.2, in order to solve more complicated problems. These problems include the vibrating drumhead, in Sec. 3.3, and heat flow in a cylinder, in Secs. 3.4 and 3.5.

3.1. Laplace's Equation and Applications

3.1.1. Laplacian in cylindrical coordinates. As a first step we express the Laplacian ∇^2 in cylindrical coordinates. Recall the equations of transformation between rectangular and cylindrical coordinates:

$$x = \rho\cos\varphi \tag{3.1.1}$$

$$y = \rho\sin\varphi \tag{3.1.2}$$

$$z = z \tag{3.1.3}$$

These are simply polar coordinates in the xy plane, where we have saved the more conventional letter r for the three-dimensional distance and the more conventional letter θ for the three-dimensional polar angle in Chapter 4.

Let $u(x, y, z)$ be a smooth function and $U(\rho, \varphi, z)$ the corresponding function in cylindrical coordinates: $U(\rho, \varphi, z) = u(\rho\cos\varphi, \rho\sin\varphi, z)$. We wish to express $u_{xx} + u_{yy}$ in terms of the partial derivatives $U_{\rho\rho}$, $U_{\varphi\varphi}$, U_ρ, U_φ.

We begin with the chain rule for partial derivatives:

$$\begin{aligned} u_x &= \frac{\partial u}{\partial x} = \frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial x} \\ u_y &= \frac{\partial u}{\partial y} = \frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial U}{\partial \varphi}\frac{\partial \varphi}{\partial y} \end{aligned}$$

Therefore we must determine $\partial\rho/\partial x$, $\partial\varphi/\partial x$, $\partial\rho/\partial y$, $\partial\varphi/\partial y$. From (3.1.1) and (3.1.2) we have $\rho^2 = x^2 + y^2$, so that

$$2\rho\frac{\partial\rho}{\partial x} = 2x \qquad 2\rho\frac{\partial\rho}{\partial y} = 2y$$

which are solved to yield

$$\frac{\partial\rho}{\partial x} = \frac{x}{\rho} = \cos\varphi \qquad \frac{\partial\rho}{\partial y} = \frac{y}{\rho} = \sin\varphi$$

Differentiating both sides of (3.1.2) with respect to x, we have

$$\begin{aligned} 0 &= \frac{\partial\rho}{\partial x}\sin\varphi + \rho\cos\varphi\frac{\partial\varphi}{\partial x} \\ &= \cos\varphi\sin\varphi + \rho\cos\varphi\frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial x} &= -\frac{\sin\varphi}{\rho} \end{aligned}$$

Differentiating both sides of (3.1.1) with respect to y, we have

$$\begin{aligned} 0 &= \frac{\partial\rho}{\partial y}\cos\varphi - \rho\sin\varphi\frac{\partial\varphi}{\partial y} \\ &= \sin\varphi\cos\varphi - \rho\sin\varphi\frac{\partial\varphi}{\partial y} \\ \frac{\partial\varphi}{\partial y} &= \frac{\cos\varphi}{\rho} \end{aligned}$$

Therefore we can express the action of the partial derivative operators $\partial/\partial x$ and $\partial/\partial y$ in cylindrical coordinates as

$$\frac{\partial u}{\partial x} = \cos\varphi\frac{\partial U}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial U}{\partial\varphi} \tag{3.1.4}$$

$$\frac{\partial u}{\partial y} = \sin\varphi\frac{\partial U}{\partial\rho} + \frac{\cos\varphi}{\rho}\frac{\partial U}{\partial\varphi} \tag{3.1.5}$$

Next we apply them again to obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \cos^2\varphi\frac{\partial^2 U}{\partial\rho^2} + \frac{2\cos\varphi\sin\varphi}{\rho^2}\frac{\partial U}{\partial\varphi} - \frac{2\sin\varphi\cos\varphi}{\rho}\frac{\partial^2 U}{\partial\rho\partial\varphi} \\ &\quad + \frac{\sin^2\varphi}{\rho}\frac{\partial U}{\partial\rho} + \frac{\sin^2\varphi}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2} \end{aligned} \tag{3.1.6}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \sin^2\varphi\frac{\partial^2 U}{\partial\rho^2} - \frac{2\sin\varphi\cos\varphi}{\rho^2}\frac{\partial U}{\partial\varphi} + \frac{2\sin\varphi\cos\varphi}{\rho}\frac{\partial^2 U}{\partial\rho\partial\varphi} \\ &\quad + \frac{\cos^2\varphi}{\rho}\frac{\partial U}{\partial\rho} + \frac{\cos^2\varphi}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2} \end{aligned} \tag{3.1.7}$$

Adding (3.1.6) and (3.1.7) gives

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 U}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial U}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2}$$

so that the Laplacian becomes

$$\boxed{\nabla^2 u=\frac{\partial^2 U}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial U}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2}+\frac{\partial^2 U}{\partial z^2}} \tag{3.1.8}$$

EXAMPLE 3.1.1. *Compute* $\nabla^2[x(x^2+y^2)^3]$.

Solution. The function $u=x(x^2+y^2)^3$ is expressed in cylindrical coordinates as $U=\rho^7\cos\varphi$; we have $U_\rho=7\rho^6\cos\varphi$, $U_{\rho\rho}=42\rho^5\cos\varphi$, $U_{\varphi\varphi}=-\rho^7\cos\varphi$. Therefore, the Laplacian is given by $\nabla^2 u=48\rho^5\cos\varphi=48x(x^2+y^2)^2$. •

The correspondence $u\to U$ produces a smooth function $U(\rho,\varphi,z)$ for every smooth function $u(x,y,z)$. But many smooth functions of (ρ,φ,z) do not arise in this manner. For example, $U=\rho$ is a function of (x,y,z), but it is not smooth since the partial derivative $\partial\rho/\partial x$ is undefined at $\rho=0$. The example $U=\varphi$ does not correspond to a smooth function of (x,y,z) since φ changes by 2π when we make a 360° rotation about the z-axis and return to the same point. These theoretical difficulties need not hinder us in our work if we check that each solution we obtain in cylindrical coordinates corresponds to a smooth function of (x,y,z). For example, $U=\rho^n\cos n\varphi$ can be written as a polynomial in (x,y) if n is an integer and therefore is a smooth function; if n is not an integer, U is not a smooth function of (x,y). With these precautions in mind, we now formulate and solve some boundary-value problems in cylindrical coordinates. By abuse of notation, we write the solution as $u=u(\rho,\varphi,z)$, assumed to be a smooth function of (x,y,z).

3.1.2. Separated solutions of Laplace's equation in ρ,φ. As our first application of (3.1.8), we obtain separated solutions of Laplace's equation in cylindrical coordinates, defined for $\rho>0$, $-\pi\le\varphi\le\pi$, and independent of z. Assuming a solution of the form $u(\rho,\varphi)=R(\rho)\Phi(\varphi)$, we substitute in the equation $\nabla^2 u=0$, with the result

$$\begin{aligned}0&=u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\varphi\varphi}\\&=R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}R\Phi''\end{aligned}$$

Dividing by $R\Phi$ and multiplying by ρ^2, we have

$$0=\rho^2\frac{R''+(1/\rho)R'}{R}+\frac{\Phi''}{\Phi}$$

The first term depends only on ρ and the second only on φ; therefore both are constant. This leads to the ordinary differential equations

$$\Phi'' + \lambda\Phi = 0, \qquad \Phi(-\pi) = \Phi(\pi), \qquad \Phi'(-\pi) = \Phi'(\pi) \tag{3.1.9}$$

$$R'' + \frac{1}{\rho}R' - \frac{\lambda}{\rho^2}R = 0 \tag{3.1.10}$$

where λ is the separation constant. Φ must satisfy the indicated periodic boundary conditions because u is supposed to be a smooth (single-valued) function of (x, y). The solution to the Sturm-Liouville problem (3.1.9) was obtained in Sec. 1.6 with the result

$$\lambda = m^2, \qquad \Phi(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi \qquad m = 0, 1, 2, \ldots$$

Equation (3.1.10) is a form of Euler's equidimensional equation. For $m \neq 0$, it has solutions $R(\rho) = \rho^m$, ρ^{-m}; for $m = 0$, the solutions are $R(\rho) = 1$, $\ln\rho$. Combining these, we get the following separated solutions of Laplace's equation:

$$u(\rho, \varphi) = \begin{cases} \rho^m (A_m \cos m\varphi + B_m \sin m\varphi) & m = 1, 2, \ldots \\ A_0 + B_0 \ln\rho & m = 0 \\ \rho^{-m}(C_m \cos m\varphi + D_m \sin m\varphi) & m = 1, 2, \ldots \end{cases} \tag{3.1.11}$$

One may note that the first set of solutions correspond to smooth functions of (x, y). These polynomial solutions of Laplace's equation are listed in the following table.

m	$\rho^m \cos m\varphi$	$\rho^m \sin m\varphi$
0	1	0
1	x	y
2	$x^2 - y^2$	$2xy$
3	$x^3 - 3xy^2$	$3xy^2 - y^3$
4	$x^4 - 6x^2y^2 + y^4$	$4x^3y - 4xy^3$

The logarithmic solution and the solutions containing negative powers are not smooth functions of (x, y), since they become infinite when $\rho \to 0$. Nevertheless they may be used to solve boundary-value problems in the *exterior* of a circle or cylinder.

3.1.3. Application to boundary-value problems. In the following two examples we consider boundary conditions that do not depend upon φ.

EXAMPLE 3.1.2. *Find the solution of Laplace's equation in the region $\rho_1 < \rho < \rho_2$, with the boundary conditions $u(\rho_1) = T_1$, $u(\rho_2) = T_2$, where T_1 and T_2 are constants. Solve for the average temperature $u(\rho) = (T_1 + T_2)/2$.*

Solution. Since the boundary conditions are independent of φ, we use the previous separated solutions with $m = 0$. Thus

$$u(\rho) = A_0 + B_0 \ln \rho$$

To satisfy the boundary conditions, we must have

$$\begin{aligned} T_1 &= A_0 + B_0 \ln \rho_1 \\ T_2 &= A_0 + B_0 \ln \rho_2 \end{aligned}$$

Solving these simultaneous linear equations yields

$$B_0 = \frac{T_2 - T_1}{\ln(\rho_2/\rho_1)} \qquad A_0 = T_1 - (T_2 - T_1)\frac{\ln \rho_1}{\ln(\rho_2/\rho_1)}$$

The solution can be written in the form

$$\begin{aligned} u(\rho) &= T_1 + (T_2 - T_1)\frac{\ln(\rho/\rho_1)}{\ln(\rho_2/\rho_1)} \\ &= T_1\frac{\ln(\rho_2/\rho)}{\ln(\rho_2/\rho_1)} + T_2\frac{\ln(\rho/\rho_1)}{\ln(\rho_2/\rho_1)} \end{aligned}$$

This example shows that the average temperature is not assumed at the average radius $\rho = \frac{1}{2}(\rho_1 + \rho_2)$ but instead at the geometric mean $\rho = (\rho_1\rho_2)^{1/2}$. Indeed, $u((\rho_1\rho_2)^{1/2}) = T_1 + \frac{1}{2}(T_2 - T_1) = \frac{1}{2}(T_1 + T_2)$. •

In many problems of practical interest, it is required to compute the steady-state flux.

EXAMPLE 3.1.3. *Two concentric cylinders of radii $\rho_1 = 10$ cm and $\rho_2 = 50$ cm are maintained at the temperatures $T_1 = 100°C$ and $T_2 = 0°C$. find the steady-state flux from the outer cylinder if the conductivity is $k = 0.35$ cal/s-cm-°C.*

Solution. The flux is given by

$$\begin{aligned} -k\frac{\partial u}{\partial \rho}|_{\rho=\rho_2} &= \frac{-k}{\rho_2}\frac{T_2 - T_1}{\ln(\rho_2/\rho_1)} \\ &= (0.35)(100)/(50 \ln 5) = 0.435\,\text{cal/s-cm}^2 \quad \bullet \end{aligned}$$

We now use separation of variables to solve the boundary-value problem for Laplace's equation when the boundary values depend on φ. We have the problem

$$\begin{aligned} \nabla^2 u = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} &= 0, && \rho_1 < \rho < \rho_2,\ -\pi \le \varphi \le \pi \\ u(\rho_1, \varphi) &= T_1(\varphi) && -\pi \le \varphi \le \pi \\ u(\rho_2, \varphi) &= T_2(\varphi) && -\pi \le \varphi \le \pi \end{aligned}$$

$T_1(\varphi)$ and $T_2(\varphi)$ are piecewise smooth functions that give the temperature on the inner and outer cylinders. We will obtain the solution of Laplace's equation in the form

$$
\begin{aligned}
(3.1.12) \qquad u(\rho,\varphi) &= A_0 + B_0 \ln\rho + \sum_{m=1}^{\infty} \rho^m (A_m \cos m\varphi + B_m \sin m\varphi) \\
&\quad + \sum_{m=1}^{\infty} \rho^{-m} (C_m \cos m\varphi + D_m \sin m\varphi)
\end{aligned}
$$

To satisfy the boundary conditions, we must have

$$
\begin{aligned}
T_1(\varphi) &= A_0 + B_0 \ln\rho_1 + \sum_{m=1}^{\infty} (\rho_1^m A_m + \rho_1^{-m} C_m) \cos m\varphi \\
&\quad + \sum_{m=1}^{\infty} (\rho_1^m B_m + \rho_1^{-m} D_m) \sin m\varphi \\
T_2(\varphi) &= A_0 + B_0 \ln\rho_2 + \sum_{m=1}^{\infty} (\rho_2^m A_m + \rho_2^{-m} C_m) \cos m\varphi \\
&\quad + \sum_{m=1}^{\infty} (\rho_2^m B_m + \rho_2^{-m} D_m) \sin m\varphi
\end{aligned}
$$

Using the orthogonality of the functions $\{1, \cos m\varphi, \sin m\varphi\}$, we can obtain the coefficients by the Fourier formulas.

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} T_1(\varphi)\,d\varphi &= A_0 + B_0 \ln\rho_1 \\
\frac{1}{2\pi}\int_{-\pi}^{\pi} T_2(\varphi)\,d\varphi &= A_0 + B_0 \ln\rho_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} T_1(\varphi)\cos m\varphi\,d\varphi &= \rho_1^m A_m + \rho_1^{-m} C_m \qquad m = 1, 2, \ldots \\
\frac{1}{\pi}\int_{-\pi}^{\pi} T_2(\varphi)\cos m\varphi\,d\varphi &= \rho_2^m A_m + \rho_2^{-m} C_m \qquad m = 1, 2, \ldots \\
\frac{1}{\pi}\int_{-\pi}^{\pi} T_1(\varphi)\sin m\varphi\,d\varphi &= \rho_1^m B_m + \rho_1^{-m} D_m \qquad m = 1, 2, \ldots \\
\frac{1}{\pi}\int_{-\pi}^{\pi} T_2(\varphi)\sin m\varphi\,d\varphi &= \rho_2^m B_m + \rho_2^{-m} D_m \qquad m = 1, 2, \ldots
\end{aligned}
$$

These simultaneous equations can be solved to obtain the coefficients A_m, B_m, C_m, D_m.

EXAMPLE 3.1.4. *Solve Laplace's equation in the cylinder* $1 < \rho < 2$ *with the boundary conditions* $u(1,\varphi) = 0$ *and* $u(2,\varphi) = 1$ *if* $0 < \varphi < \pi$ *and* $u(2,\varphi) = -1$ *if* $-\pi < \varphi < 0$.

Solution. In this case the six equations become $A_0 + B_0 \ln \rho_1 = 0$, $A_0 + B_0 \ln \rho_2 = 0$, $A_m + C_m = 0$, $2^m A_m + 2^{-m} C_m = 0$, $B_m + D_m = 0$, $2^m B_m + 2^{-m} D_m = 2[1 - (-1)^m]/m\pi$. This gives $A_0 = 0$, $B_0 = 0$, $A_m = 0$, $C_m = 0$, $B_m = -D_m = 2[1 - (-1)^m]/m\pi(2^m - 2^{-m})$. The solution is

$$u(\rho, \varphi) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\rho^m - \rho^{-m}}{2^m - 2^{-m}} \frac{1 - (-1)^m}{m} \sin m\varphi \qquad 1 < \rho < 2, -\pi \leq \varphi \leq \pi \quad \bullet$$

3.1.4. Regularity. It is not difficult to show that the formal solution (3.1.12) of Laplace's equation is indeed a smooth function. To simplify the writing, we consider the case $\rho_1 = 0$, when we solve the problem in the interior of the cylinder $0 \leq \rho < \rho_2$. The formal solution is

$$u(\rho, \varphi) = A_0 + \sum_{m=1}^{\infty} \rho^m (A_m \cos m\varphi + B_m \sin m\varphi) \qquad 0 < \rho < \rho_2 \tag{3.1.13}$$

where $\rho_2^m A_m$ and $\rho_2^m B_m$ are the Fourier coefficients of the boundary function $T(\varphi)$, $-\pi < \varphi < \pi$. This function is piecewise smooth and therefore bounded by a constant M. This means that we must have

$$|\rho_2^m A_m| \leq 2M, \qquad |\rho_2^m B_m| \leq 2M \qquad m = 1, 2, \ldots$$

Therefore the terms of the series (3.1.13) are no larger then $4M(\rho/\rho_2)^m$. But this is the general term of a convergent series for $\rho < \rho_2$. This also shows that the series (3.1.13) is uniformly convergent for $0 \leq \rho \leq \rho_0$, where ρ_0 is any number less than ρ_2. Furthermore, each term in the series $\rho^m \cos m\varphi$ and $\rho^m \sin m\varphi$ is a smooth function of (x, y). The partial derivatives are easily found to be

$$\begin{aligned} \frac{\partial}{\partial x} (\rho^m \cos m\varphi) &= m\rho^{m-1} \cos(m-1)\varphi \\ \frac{\partial}{\partial x} (\rho^m \sin m\varphi) &= m\rho^{m-1} \sin(m-1)\varphi \end{aligned}$$

The general term of the series for $\partial u/\partial x$ is no larger than $(4Mm/R)(\rho/\rho_2)^{m-1}$. Therefore this series is also uniformly convergent for $0 \leq \rho \leq \rho_0$, where ρ_0 is any number less than ρ_2. Continuing in this way, we see that the series for u_{xx}, u_y, u_{yy} are uniformly convergent, and therefore we can differentiate term by term and verify that u satisfies Laplace's equation.

3.1.5. Uniqueness of solutions. The solution of Laplace's equation in a bounded region is *unique*. To be specific, suppose we have two solutions u_1, u_2 of Laplace's equation $\nabla^2 u = 0$ in the region $0 \leq \rho < R$, with the same boundary values $u_1(R, \varphi) = u_2(R, \varphi)$ for $-\pi \leq \varphi \leq \pi$. The function $u = u_1 - u_2$ satisfies Laplace's equation $\nabla^2 u = 0$ in the region $0 \leq \rho < R$, with boundary value $u(R, \varphi) = 0$. We may apply Green's theorem $\oint (M\,dx + N\,dy) = \iint (N_x - M_y) dx\,dy$, where the line integral is taken over the circle $\rho = R$ and the double integral is taken over the disc $\rho < R$, with $M = -uu_y$, $N = uu_x$. The line integral is zero, since $u = 0$ on the circle, while the integrand in the double integral is $N_x - M_y = (uu_x)_x + (uu_y)_y = uu_{xx} + (u_x)^2 + uu_{yy} + (u_y)^2 = u(u_{xx} + u_{yy}) + (u_x^2 + u_y^2)$. The first parenthetical term is zero since $\nabla^2 u = 0$, and we are left with $0 = \iint (u_x^2 + u_y^2)dx\,dy$. Both u_x^2 and u_y^2 are nonnegative, and their integrals are zero; hence they must both be zero. This means that $u(x, y)$ must be a constant. But $u = 0$ on the circle, which proves that $u_1 = u_2$—the desired uniqueness result.

3.1.6. Exterior problems. The separated solutions of Laplace's equation can be used to solve boundary-value problems in the *exterior* of a cylinder. Suppose that we wish to determine the solution $u(\rho, \varphi)$ of Laplace's equation $\nabla^2 u = 0$ for $\rho > R$ satisfying the boundary condition $u(R, \varphi) = T(\varphi)$, a given piecewise smooth function. We require in addition that the solution be bounded: $|u(\rho, \varphi)| \leq M$ for some constant M; otherwise we may have nonuniqueness. For example, the function $u_1(\rho, \varphi) = 1 + (\rho/R)\cos\varphi - (R/\rho)\cos\varphi$ satisfies Laplace's equation in the exterior of the cylinder $\rho > R$ and $u_1(R, \varphi) = 1$. The function $u_2(\rho, \varphi) \equiv 1$ also satisfies Laplace's equation with the same boundary values.

EXAMPLE 3.1.5. *Find the bounded solution of Laplace's equation $\nabla^2 u = 0$ in the exterior $\rho > R$ satisfying the boundary conditions $u(R, \varphi) = 1$ if $0 < \varphi < \pi$, $u(R, \varphi) = -1$ if $-\pi < \varphi < 0$.*

Solution. To ensure boundedness, we take a sum of separated solutions of the form

$$u(\rho, \varphi) = A_0 + \sum_{n=1}^{\infty} \rho^{-n}(A_n \cos n\varphi + B_n \sin n\varphi)$$

To satisfy the boundary conditions, we must have $A_n = 0$, $R^{-n}B_n = (2/n\pi)[1 - (-1)^n]$. The solution is

$$u(\rho, \varphi) = \frac{2}{\pi}\sum_{n=1}^{\infty}[1 - (-1)^n]\frac{R^n}{n\rho^n}\sin n\varphi \quad \bullet$$

3.1.7. Wedge domains. Often we encounter boundary-value problems for Laplace's equation in a *wedge domain*, of the form $0 < \rho < R$, $0 < \varphi < \alpha$, where $\alpha < 2\pi$. In this case the separated solutions are still of the form $\rho^n(A\cos n\varphi + B\sin n\varphi)$, but n is no longer necessarily an integer. The allowed values of n will depend on the wedge opening α and the nature of the boundary conditions that are imposed at $\varphi = 0$, $\varphi = \alpha$.

EXAMPLE 3.1.6. *Find the solution of Laplace's equation in the wedge domain $0 < \rho < 1$, $0 < \varphi < \alpha$ satisfying the boundary conditions $u(\rho, 0) = 0$, $u(\rho, \alpha) = 0$ for $0 < \rho < 1$, and $u(1, \varphi) = 1$ for $0 < \varphi < \alpha$.*

Solution. For the separated solutions $\rho^n(A\cos n\varphi + B\sin n\varphi)$ the boundary conditions at $\varphi = 0$ and $\varphi = \alpha$ require that $A = 0$, $\sin n\alpha = 0$. Therefore $n = m\pi/\alpha$ for $m = 1, 2, 3, \ldots$. To satisfy the boundary condition at $\rho = 1$, we try a sum of separated solutions: $u(\rho,\varphi) = \sum_{m=1}^{\infty} B_m \rho^{m\pi/\alpha} \sin m\pi\varphi/\alpha$. We must have $1 = \sum_{m=1}^{\infty} B_m \sin m\pi\varphi/\alpha$, $0 < \varphi < \alpha$; therefore $B_m = (2/m\pi)[1-(-1)^m]$. The solution is $u(\rho,\varphi) = (2/\pi)\sum_{m=1}^{\infty}[1-(-1)^m]m^{-1}\rho^{m\pi/\alpha} \sin m\pi\varphi/\alpha$. •

3.1.8. Neumann problems. In all of the preceding applications we have solved Laplace's equation with *Dirichlet* boundary conditions, meaning that the value of $u(R,\varphi)$ is given. We can also solve problems with *Neumann* boundary conditions where $\partial u/\partial\rho = G(\varphi)$ is a given piecewise smooth function. This problem features nonuniqueness: if $u(\rho,\varphi)$ is a solution, then $u(\rho,\varphi) + C$ is also a solution for any constant C. To ensure a unique solution, we may require (i) that $u = 0$ when $\rho = 0$ or (ii) that $\int_{-\pi}^{\pi} u(\rho,\varphi)d\varphi = 0$. In addition $G(\varphi)$ must satisfy the condition $\int_{-\pi}^{\pi} G(\varphi)d\varphi = 0$. This is not too surprising since the solutions of Laplace's equation represent steady-state temperature distributions and $\partial u/\partial\rho = G(\varphi)$ is proportional to the flux across the boundary of the cylinder. It is natural to expect that, in steady state, the total flux across the boundary is zero.

To solve Neumann problems for Laplace's equation in the cylinder $0 \le \rho < R$, we try a sum of separated solutions of the form

$$u(\rho,\varphi) = \sum_{n=1}^{\infty} \rho^n (A_n \cos n\varphi + B_n \sin n\varphi)$$

The boundary conditions require that $nR^{n-1}A_n$ and $nR^{n-1}B_n$ are the Fourier coefficients of the piecewise smooth function $G(\varphi)$, $-\pi < \varphi < \pi$.

EXAMPLE 3.1.7. *Find the solution of Laplace's equation in the cylinder* $0 < \rho < R$ *and satisfying the Neumann boundary condition* $\partial u/\partial\rho = 1$ *for* $0 < \varphi < \pi$ *and* $\partial u/\partial\rho = -1$ *for* $-\pi < \varphi < 0$.

Solution. The solution is sought in the form $u(\rho,\varphi) = \sum_{n=1}^{\infty} \rho^n (A_n \cos n\varphi + B_n \sin n\varphi)$. The boundary conditions require that $A_n = 0$ and $nR^{n-1}B_n = (2/n\pi)[1-(-1)^n]$. The solution is

$$u(\rho,\varphi) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{R}{n^2}[1-(-1)^n]\left(\frac{\rho}{R}\right)^n \sin n\varphi \quad \bullet$$

3.1.9. Explicit representation by Poisson's formula. We have obtained the Fourier representation of the solution of Laplace's equation in the cylinder $0 \le \rho < R$ by means of formula (3.1.13). We now show that this can be converted to an explicit representation, the *Poisson integral formula.* To do this, we recall the formulas for the Fourier coefficients, written with the integration variable ψ:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} T(\psi)d\psi \\ R^m A_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\psi)\cos m\psi\, d\psi \qquad m = 1, 2, \ldots \\ R^m B_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} T(\psi)\sin m\psi\, d\psi \qquad m = 1, 2, \ldots \end{aligned}$$

We transform the expression $A_m \cos m\varphi + B_m \sin m\varphi$ by writing

$$\begin{aligned} \pi R^m(A_m \cos m\varphi &+ B_m \sin m\varphi) \\ &= \int_{-\pi}^{\pi} T(\psi)(\cos m\psi \cos m\varphi + \sin m\psi \sin m\varphi)d\psi \\ &= \int_{-\pi}^{\pi} T(\psi)\cos m(\psi - \varphi)d\psi \qquad m = 1, 2, \ldots \end{aligned}$$

Substituting this in (3.1.13), we have

$$u(\rho, \varphi) = \frac{1}{\pi}\int_{-\pi}^{\pi} T(\psi)\left[\frac{1}{2} + \sum_{m=1}^{\infty}\left(\frac{\rho}{R}\right)^m \cos m(\psi - \varphi)\right] d\psi, \qquad 0 \le \rho < R.$$

The inner sum was evaluated in the exercises for Sec. 1.5, with the result

$$\frac{1}{2} + \sum_{m=1}^{\infty}\left(\frac{\rho}{R}\right)^m \cos m(\psi - \varphi) = \frac{R^2 - \rho^2}{2[R^2 + \rho^2 - 2\rho R\cos(\psi - \varphi)]}$$

This gives the Poisson integral formula

(3.1.14)

$$\boxed{u(\rho, \varphi) = (2\pi)^{-1}\int_{-\pi}^{\pi} \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\psi - \varphi)} T(\psi)d\psi \qquad 0 \le \rho < R}$$

The Poisson integral formula has some important theoretical consequences for solutions of Laplace's equation. We list these facts in the form of a proposition.

PROPOSITION 3.1.1. *Let $u(\rho, \varphi)$ be a solution of Laplace's equation in the cylinder $0 \le \rho < R$, represented by the Poisson integral formula (3.1.14), where $T(\psi)$ is a given continuous function for $-\pi \le \psi \le \pi$. Then*

1. $u(\rho, \varphi)|_{\rho=0} = (2\pi)^{-1}\int_{-\pi}^{\pi} T(\psi)d\psi$, *the average of T.*
2. $u(\rho, \varphi) \le \max_{-\pi\le\psi\le\pi} T(\psi)$
3. *If $u(\rho_0, \varphi_0) = \max_{-\pi\le\psi\le\pi} T(\psi)$ for some $\rho_0 < R$, $-\pi \le \varphi_0 \le \pi$, then $u(\rho, \varphi)$ is a constant for all $0 \le \rho \le R$, $-\pi \le \varphi \le \pi$.*

Proof. Taking $\rho = 0$ in the Poisson integral formula gives property 1. To prove properties 2 and 3, let $M = \max_{-\pi \le \psi \le \pi} T(\psi)$; integrating the uniformly convergent Fourier series $\frac{1}{2} + \sum_{n=1}^{\infty} (\rho/R)^n \cos n(\psi - \varphi)$ term by term for $-\pi \le \psi \le \pi$, we see that the total integral of the Poisson kernel is 1. Thus for any (ρ, φ) we have

$$2\pi[M - u(\rho, \varphi)] = \int_{-\pi}^{\pi} [M - T(\psi)] \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\psi - \varphi)}\, d\psi$$

The integrand is a nonnegative continuous function for $-\pi \le \psi \le \pi$. Therefore $M - u(\rho, \varphi) \ge 0$, and we have proved 2. If this is zero for some (ρ_0, φ_0), then the integrand must be zero for all ψ; thus $T(\psi) = M$ for all ψ, $-\pi \le \psi \le \pi$. Referring back to Poisson's formula, we see that $u(\rho, \varphi) = M$ for all $0 \le \rho < R$, $-\pi \le \varphi \le \pi$. •

EXERCISES 3.1

In Exercises 1 to 5 use formula (3.1.8) to compute the indicated quantities.

1. $\nabla^2(\rho^4 \cos 2\varphi)$
2. $\nabla^2(\rho^2 \cos 2\varphi)$
3. $\nabla^2(\rho^n)$, $n = 1, 2, \ldots$
4. $\nabla^2(\rho^n \cos m\varphi)$, $m, n = 1, 2, \ldots$
5. $\nabla^2(e^{\rho} \cos \varphi)$
6. Which of the functions in Exercises 1 to 5 corresponds to a smooth function of (x, y)?
7. Show that formula (3.1.8) can be written in the form
$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$$
8. Let $u = f(\sqrt{x^2 + y^2})$, where f is a smooth function. Show that $\nabla^2 u = f'' + (1/\rho) f' = (1/\rho)(\rho f')'$.
9. Use Exercise 8 to find the general solution of the equation $\nabla^2 f(\rho) = 0$.
10. Use Exercise 8 to find the general solution of the equation $\nabla^2 f(\rho) = -1$.
11. Find the solution of the equation $\nabla^2 f(\rho) = 0$ satisfying the boundary conditions $f(1) = 3$, $f(2) = 5$.
12. Find the solution of the equation $\nabla^2 f(\rho) = -1$ satisfying the boundary conditions $f(1) = 0$, $f(2) = 1$.
13. Find the solution $u(\rho, \varphi)$ of Laplace's equation in the cylinder $0 \le \rho < R$ satisfying the boundary conditions $u(R, \varphi) = 1 + \cos 2\varphi + 3 \sin 3\varphi$, $-\pi \le \varphi \le \pi$.
14. Find the solution $u(\rho, \varphi)$ of Laplace's equation in the cylindrical region $1 < \rho < 2$ satisfying the boundary conditions $u(1, \varphi) = \cos 2\varphi$, $u(2, \varphi) = 1$ for $-\pi \le \varphi \le \pi$.

15. Find the solution $u(\rho, \varphi)$ of Laplace's equation in the cylindrical region $1 < \rho < 2$ satisfying the boundary conditions $u(1, \varphi) \equiv 0$, $u(2, \varphi) = 0$ for $-\pi < \varphi < 0$ and $u(2, \varphi) = 1$ for $0 < \varphi < \pi$.
16. Find the bounded solution of Laplace's equation in the exterior of the cylinder $\rho > R$ satisfying the boundary condition $u(R, \varphi) = 3 + 4\cos 2\varphi + 5\sin 3\varphi$, $-\pi \le \varphi \le \pi$.
17. Let $u(\rho, \varphi)$ be the bounded solution of Laplace's equation in the exterior of the cylinder $\rho > R$ with the boundary condition $u(R, \varphi) = G(\varphi)$, a given piecewise smooth function. Show formally that

$$u(\rho, \varphi) \to \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\varphi) d\varphi \qquad \text{when } \rho \to \infty$$

18. Find the separated solutions of Laplace's equation in the wedge domain $0 < \rho < 1$, $0 < \varphi < \pi/2$ satisfying the boundary conditions $u(\rho, 0) = 0$, $u(\rho, \pi/2) = 0$.
19. Use the separated solutions found in Exercise 18 to solve Laplace's equation in the wedge domain $0 < \rho < 1$, $0 < \varphi < \pi/2$ with the boundary conditions $u(\rho, 0) = 0$, $u(\rho, \pi/2) = 0$ for $0 < \rho < 1$ and $u(1, \varphi) = 1$ for $0 < \varphi < \pi/2$.
20. Find separated solutions of Laplace's equation in the wedge domain $0 < \rho < 1$, $0 < \varphi < \pi$ satisfying the boundary conditions $u_\varphi(\rho, 0) = 0$, $u_\varphi(\rho, \pi) = 0$ for $0 < \rho < 1$.
21. Use the separated solutions found in Exercise 20 to find the solution of Laplace's equation in the wedge domain $0 < \rho < 1$, $0 < \varphi < \pi$ satisfying the boundary conditions $u_\varphi(\rho, 0) = 0$, $u_\varphi(\rho, \pi) = 0$ for $0 < \rho < 1$ and $u(1, \varphi) = \varphi(\pi - \varphi)$ for $0 < \varphi < \pi$.
22. Let

$$u(\rho) = T_1 + (T_2 - T_1)\frac{\ln \rho - \ln \rho_1}{\ln \rho_2 - \ln \rho_1}$$

be the solution of Laplace's equation in the cylindrical region $\rho_1 < \rho < \rho_2$.

(a) Show that $u(\rho) \to T_2$ if $\rho_1 \to 0$ and ρ is a fixed number with $0 < \rho \le \rho_2$.

(b) Show that $u(\rho) \to T_1$ if $\rho_2 \to \infty$ and ρ is a fixed number with $\rho \ge \rho_1$.

23. Let $u(\rho, \varphi)$ be the solution of Laplace's equation in the cylinder $0 \le \rho \le R$ with the boundary condition $u(R, \varphi) = T$ if $0 < \varphi < \pi$ and $u(R, \varphi) = 0$ if $-\pi < \varphi < 0$.

(a) Show that $u(\rho, 0) = \frac{1}{2}T$ for $0 \le \rho < R$.

(b) Show that $u(\rho, \pi/2) = \frac{1}{2}T + (2T/\pi)\tan^{-1} \rho/R$ for $0 \le \rho < R$.

24. Let $u(\rho, \varphi)$ be a solution of Laplace's equation in the cylinder $0 \le \rho < R$ represented by the Poisson integral formula (3.1.14) with $0 \le G(\varphi)$.

(a) Show that

$$\frac{R-\rho}{R+\rho}u(0,\varphi) \le u(\rho,\varphi) \le \frac{R+\rho}{R-\rho}u(0,\varphi) \qquad 0 \le \rho \le R$$

(b) Use the inequality to prove that

$$\left|\frac{1}{u}\frac{\partial u}{\partial \rho}\right|_{\rho=0} \le \frac{2}{R}$$

25. Find the solution of Laplace's equation in the cylinder $\rho_1 < \rho < \rho_2$ satisfying the boundary conditions $\partial u/\partial\rho + h_2(u - T_2) = 0$ at $\rho = \rho_2$ and $\partial u/\partial\rho - h_1(u - T_1) = 0$ at $\rho = \rho_1$, where h_1, h_2, T_1, T_2 are positive constants.
26. Find the solution of Laplace's equation in the region $0 \le \rho < R$ satisfying the (Robin) boundary condition

$$\frac{\partial u}{\partial \rho}(R,\varphi) + h(u(R,\varphi) - T(\varphi)) = 0$$

where $h > 0$ and $T(\varphi)$ is a piecewise smooth function with Fourier series

$$T(\phi) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\varphi + b_n \sin n\varphi)$$

3.2. Bessel Functions

To treat more general boundary-value problems in cylindrical coordinates, we need to study the properties of Bessel functions. The reader may be familiar with many properties of Bessel functions from previous work in ordinary differential equations. In this case much of this section can be read quickly and used for later reference when we study the applications to boundary-value problems. Our treatment, which requires no previous knowledge of Bessel functions, is especially designed for the applications to boundary-value problems in cylindrical coordinates in this chapter and for applications to boundary-value problems in spherical coordinates in Chapter 4. For these applications, the important facts about Bessel functions are contained in Proposition 3.2.6 and the examples that follow.

3.2.1. Bessel's equation. Bessel functions originate as solutions of the following equation containing three parameters (d, λ, μ):

$$\boxed{y'' + (d-1)\frac{y'}{\rho} + \left(\lambda - \frac{\mu}{\rho^2}\right)y = 0} \tag{3.2.1}$$

The parameters are assumed to satisfy the restrictions $d \ge 1$, $\mu \ge 0$. Because of their typical origins, we call the parameter d the *dimension*, the parameter λ the *eigenvalue*, and the parameter μ the *angular index*. The special case $\lambda = 0$ is the Euler equidimensional equation. For solutions in cylindrical coordinates we shall need $d = 2$, while for solutions in spherical coordinates we shall need $d = 3$. For solutions with circular symmetry in cylindrical coordinates we take $\mu = 0$, whereas for solutions without circular symmetry in cylindrical coordinates we shall need $\mu = m^2 = 1, 4, 9, \ldots$. The corresponding solutions in spherical coordinates require $\mu = k(k+1) = 2, 6, 12, \ldots$. An extremely special but prototypical case of (3.2.1) is the familiar equation $y'' + \lambda y = 0$, which is the case $d = 1$, $\mu = 0$.

3.2.2. The power series solution of Bessel's equation. The Bessel equation becomes singular at $\rho = 0$; therefore we cannot expect two linearly independent solutions that remain bounded when $\rho \to 0$. There is always one solution that remains bounded when $\rho \to 0$. To find it, we will follow the method of Frobenius and look for the solution as a power series

$$y = \rho^\gamma \sum_{n=0}^{\infty} a_n \rho^n = \sum_{n=0}^{\infty} a_n \rho^{n+\gamma} \tag{3.2.2}$$

where $\gamma, a_0, a_1, \ldots$ are to be determined. To find these, we substitute the series (3.2.2) into (3.2.1) and rewrite the result as a single power series. Thus

$$y' = \sum_{n=0}^{\infty} (n+\gamma) a_n \rho^{n+\gamma-1} \tag{3.2.3}$$

$$y'' = \sum_{n=0}^{\infty} (n+\gamma)(n+\gamma-1) a_n \rho^{n+\gamma-2} \tag{3.2.4}$$

$$y'' + \frac{d-1}{\rho} y' - \frac{\mu}{\rho^2} y = \sum_{n=0}^{\infty} [(n+\gamma)(n+\gamma+d-2) - \mu] a_n \rho^{n+\gamma-2}$$

In order for this to be equal to the series for $-\lambda y$, the two series must agree, term by term. The series for λy begins with the power ρ^γ, while the above series begins with $\rho^{\gamma-2}$. Therefore we must have

$$0 = (\gamma(\gamma+d-2) - \mu)\, a_0 \tag{3.2.5}$$

$$0 = ((1+\gamma)(\gamma+d-1) - \mu)\, a_1 \tag{3.2.6}$$

$$0 = ((n+\gamma)(n+\gamma+d-2) - \mu)\, a_n + \lambda a_{n-2} \qquad n = 2, 3, \ldots \tag{3.2.7}$$

We obtain a nonzero solution by taking

$$a_0 \neq 0, \qquad a_1 = 0$$

$$\gamma = 1 - \frac{d}{2} + \left[\mu + \left(\frac{d}{2} - 1\right)^2\right]^{1/2} \tag{3.2.8}$$

The exponent γ is *nonnegative* and is the largest root of the indicial equation $\gamma(\gamma+d-2)-\mu=0$, from (3.2.5). To determine a_n, $n \ge 2$, we use the indicial equation to write

$$(n+\gamma)(n+\gamma+d-2)-\mu = n(n+2\gamma+d-2)$$

Thus (3.2.7) becomes

$$n(n+2\gamma+d-2)a_n + \lambda a_{n-2} = 0, \qquad n = 2,3,\dots$$

$a_1 = 0$ requires $a_3 = 0 = a_5 = \cdots$, while

$$\begin{aligned}
a_2 &= \frac{-\lambda}{2(d+2\gamma)}a_0 \\
a_4 &= \frac{-\lambda}{2(d+2\gamma)}\frac{-\lambda}{4(d+2\gamma+2)}a_0 \\
&\vdots \\
a_{2n} &= \frac{(-\lambda)^n}{2(d+2\gamma)4(d+2\gamma+2)\cdots 2n(d+2\gamma+2n-2)}a_0 \qquad n = 1,2,\dots
\end{aligned}$$

Hence we have obtained the sought-after function,

$$(3.2.9)\quad y(\rho) = a_0\rho^\gamma\left[1+\sum_{n=1}^{\infty}\frac{(-\lambda)^n\rho^{2n}}{2(d+2\gamma)4(d+2\gamma+2)\cdots 2n(d+2\gamma+2n-2)}\right]$$

We may check convergence of the series (3.2.9) by the ratio test. The ratio of two consecutive terms is $a_{2n+2}\rho^{2n+2}/a_{2n}\rho^{2n} = -\lambda\rho^2/(2n+2)(d+2\gamma+2n)$. For any ρ this tends to zero when $n \to \infty$. Therefore the series (3.2.9) converges for all ρ. By a similar use of the ratio test, it may be shown that the differentiated series (3.2.3), (3.2.4) converge for all ρ. This convergence is uniform on all finite intervals, and therefore we may differentiate the series term by term and verify that (3.2.9) is a solution of Bessel's equation.

In case $d = 2$ and $\sqrt{\mu} = m$ is an integer, there is a standard choice of a_0, which we shall adopt. From (3.2.8) we see that $\gamma = m$; therefore the formula for a_{2n} simplifies to

$$\begin{aligned}
a_{2n} &= \frac{(-\lambda)^n a_0}{2(2+2m)4(4+2m)\cdots 2n(2n+2m)} \qquad n = 1,2,\dots \\
&= \frac{(-\lambda)^n a_0}{2^{2n}n!(1+m)\cdots(n+m)}
\end{aligned}$$

We follow established usage and choose $a_0 = 1/m!2^m$, $\lambda = 1$. This leads to the definition

$$(3.2.10)\qquad \boxed{J_m(x) = \frac{x^m}{2^m m!}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}n!(1+m)\cdots(n+m)}\right]}$$

This may also be written in the form

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+m}}{2^{m+2n}(m+n)!n!} \tag{3.2.11}$$

If m is not an integer, we *define* $m! = \int_0^\infty t^m e^{-t}\,dt$, a convergent improper integral for $m > -1$. This is often denoted by $\Gamma(m+1)$, the so-called gamma function. Integration by parts shows

$$\begin{aligned} \Gamma(m+1) &= \int_0^\infty t^m e^{-t}\,dt \\ &= -\int_0^\infty t^m\,d(e^{-t}) \\ &= m\int_0^\infty t^{m-1}e^{-t}\,dt \\ &= m\Gamma(m) \end{aligned}$$

The fundamental property of factorials is preserved: $(m+1)! = (m+1)m!$ for $m > -1$. We now *define* the Bessel function $J_m(x)$ for arbitrary $m > -1$ by formula (3.2.10). The series converges for all x and is a solution of the Bessel equation (3.2.1) with $d = 2$, $\lambda = 1$. The formula (3.2.11) is also valid for arbitrary $m > -1$, since the factorials have the property $(m+n)! = (m+n)\cdots(m+1)m!$ for $n = 0, 1, 2, \ldots$ and arbitrary $m > -1$.

EXAMPLE 3.2.1. *Find the power series solution of the Bessel equation with* $d = 2$, $\mu = 0$, $\lambda > 0$.

Solution. In this case we have $\gamma = 0$,

$$a_{2n} = \frac{(-\lambda)^n}{2^{2n}(n!)^2}a_0$$

The solution is

$$\begin{aligned} y(\rho) &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n a_0}{2^{2n}(n!)^2}\rho^{2n} \\ &= a_0\sum_{n=1}^{\infty}\frac{(-1)^n(\rho\sqrt{\lambda})^{2n}}{2^{2n}(n!)^2} \\ &= a_0 J_0(\rho\sqrt{\lambda}) \quad \bullet \end{aligned}$$

EXAMPLE 3.2.2. *Find the power series solution of the Bessel equation with* $d = 3$, $\mu = 0$, $\lambda > 0$.

Solution. From (3.2.8) we have $\gamma = -\frac{1}{2} + \frac{1}{2} = 0$,

$$a_{2n} = \frac{(-\lambda)^n a_0}{2\cdot 3\cdot 4\cdot 5\cdots 2n(2n+1)}$$

The solution is

$$\begin{aligned} y(\rho) &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n a_0 \rho^{2n}}{(2n+1)!} \\ &= \frac{a_0}{\rho\sqrt{\lambda}} \sum_{n=0}^{\infty} \frac{(-1)^n (\rho\sqrt{\lambda})^{2n+1}}{(2n+1)!} \\ &= a_0 \frac{\sin \rho\sqrt{\lambda}}{\rho\sqrt{\lambda}} \end{aligned}$$

In this example the solution of Bessel's equation is an elementary function. •

In case $d \neq 2$ the general solution (3.2.9) can be expressed in terms of the standard Bessel function J_m for a suitable value of m provided that $\lambda > 0$.

PROPOSITION 3.2.1. *Suppose that $\lambda > 0$. Then the power series solution (3.2.9) is related to J_m by*

$$\boxed{\frac{y(\rho)}{\rho^{\gamma}} = a_0 2^m m! \frac{J_m(\rho\sqrt{\lambda})}{(\rho\sqrt{\lambda})^m}}$$

where

$$m = \gamma + \frac{d-2}{2} = \sqrt{\mu + \frac{(d-2)^2}{4}}$$

Proof. With this choice of m, the denominator of (3.2.9) is

$$\begin{aligned} 2(d+2\gamma)4\cdots 2n(d+2\gamma+2n-2) &= 2(2+2m)4(4+2m)\cdots 2n(2n+2m) \\ &= 2^{2n} n!(1+m)(2+m)\cdots(n+m) \end{aligned}$$

which is precisely the denominator of (3.2.10). The numerator is $(-\lambda\rho^2)^n = (-1)^n(\rho\sqrt{\lambda})^{2n}$. Dividing by ρ^{γ} completes the identification. •

In some problems we encounter Bessel's equation with $\lambda < 0$. To treat such problems, we define the *modified Bessel function* by

$$I_m(x) = i^{-m} J_m(ix) = \frac{x^m}{2^m m!}\left[1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^{2n} n!(1+m)\cdots(n+m)}\right]$$

where i is the imaginary unit, $i^2 = -1$. $I_m(x)$ is a real-valued function.

EXAMPLE 3.2.3. *Find the power series solution of Bessel's equation with $d = 2$, $m = 0$, $\lambda = -c^2 < 0$.*

Solution. We have $\gamma = 0$,

$$a_{2n} = \frac{c^{2n} a_0}{2^{2n}(n!)^2} \rho^{2n}$$

The solution is

$$\begin{aligned} y &= \sum_{n=1}^{\infty} \frac{c^{2n} a_0}{2^{2n}(n!)^2} \rho^{2n} \\ &= a_0 I_0(c\rho) \quad \bullet \end{aligned}$$

3.2.3. Integral representation of Bessel functions. In many problems the power series is not the most efficient representation of Bessel functions. For example, if we wish to determine the asymptotic behavior of $J_m(x)$ when $x \to \infty$, the power series provides no useful information. For these purposes we will prove the following integral formulas:

$$\boxed{J_m(x) = \frac{i^{-m}}{2\pi} \int_{-\pi}^{\pi} e^{ix\cos\theta} e^{-im\theta} d\theta \qquad m = 0, 1, 2, \ldots} \tag{3.2.12}$$

In other words, $i^m J_m(x)$ is the mth Fourier coefficient of the complex function $\theta \to e^{ix\cos\theta}$. Since we know that J_m is a real function, the imaginary part of this integral is zero, and we can obtain an equivalent real form. For example, when $m = 0$, we can write

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x\cos\theta) d\theta$$

To prove (3.2.12), we expand $e^{ix\cos\theta}$ in a power series.

$$e^{ix\cos\theta} = \sum_{n=0}^{\infty} \frac{(ix\cos\theta)^n}{n!}$$

For a fixed x, this series converges uniformly for $-\pi \le \theta \le \pi$. When we multiply by $e^{-im\theta}$, we still have uniform convergence, and we can therefore integrate term by term, with the result

$$\int_{-\pi}^{\pi} e^{ix\cos\theta} e^{-im\theta} d\theta = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_{-\pi}^{\pi} \cos^n\theta\, e^{-im\theta} d\theta.$$

This integral was worked out in Sec. 1.5, where we found a nonzero value only for $m + n$ even, $0 \le m + n \le 2n$, in particular $n \ge m$. For fixed m, the nonzero coefficients are obtained when $n = m, m + 2, m + 4, \ldots$. Introducing a new summation variable j through the equation $n = m + 2j$, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}e^{-im\theta}d\theta &= \sum_{j=0}^{\infty}\frac{(ix)^{m+2j}}{(m+2j)!}\frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos\theta)^{m+2j}e^{-im\theta}d\theta \\
&= \sum_{j=0}^{\infty}\frac{(ix)^{m+2j}}{(m+2j)!}\frac{1}{2^{m+2j}}\binom{m+2j}{j} \\
&= \sum_{j=0}^{\infty}\frac{(ix)^{m+2j}}{2^{m+2j}j!(m+j)!} \\
&= i^m x^m \sum_{j=0}^{\infty}\frac{(-x^2)^j}{2^{m+2j}j!(m+j)!} \\
&= i^m J_m(x)
\end{aligned}$$

This completes the proof of (3.2.12).

EXAMPLE 3.2.4. *Show that* $|J_m(x)| \le 1$ *for* $m = 0, 1, 2, \ldots$.

Solution. The function $e^{ix\cos\theta}e^{-im\theta}$ has absolute value 1. Therefore the integral (3.2.12) has absolute value no greater than 1. •

We will now prove a differentiation formula. Beginning with the integral representation (3.2.12), we have

$$\begin{aligned}
2\pi i^m J_m'(x) &= \int_{-\pi}^{\pi} e^{ix\cos\theta}e^{-im\theta} i\cos\theta\, d\theta \\
&= \frac{i}{2}\int_{-\pi}^{\pi} e^{ix\cos\theta}(e^{-i(m-1)\theta} + e^{-i(m+1)\theta})d\theta \qquad m = 0, 1, 2, \ldots
\end{aligned}$$

If $m \ge 1$, we may use (3.2.12) to rewrite this as

$$2\pi i^m J_m'(x) = \frac{i}{2}[2\pi i^{m-1}J_{m-1}(x) + 2\pi i^{m+1}J_{m+1}(x)]$$

Thus we have proved the *differentiation formula*

$$\text{(3.2.13)} \qquad \boxed{J_m'(x) = \tfrac{1}{2}[J_{m-1}(x) - J_{m+1}(x)] \qquad m = 1, 2, \ldots}$$

If $m = 0$, we use (3.2.12) to write

$$\begin{aligned}
2\pi(J_0' + J_1)(x) &= \int_{-\pi}^{\pi}(i\cos\theta\, e^{ix\cos\theta} - ie^{-i\theta}e^{ix\cos\theta})d\theta \\
&= -\int_{-\pi}^{\pi}\sin\theta\, e^{ix\cos\theta}d\theta \\
&= \int_{-\pi}^{\pi}\sin\theta\, \cos(x\cos\theta)d\theta \\
&= 0
\end{aligned}$$

where we have used the fact that the integral is real and the final integrand is an odd function of θ, $-\pi < \theta < \pi$. Thus

$$J_0'(x) = -J_1(x) \tag{3.2.14}$$

We now use integration by parts to find another useful formula, known as the recurrence formula. From (3.2.12) we have

$$\begin{aligned}
2\pi i^m J_m(x) &= \int_{-\pi}^{\pi} e^{ix\cos\theta} d\frac{e^{-im\theta}}{-im} \qquad m = 1, 2, \ldots \\
&= e^{ix\cos\theta}\, \frac{e^{-im\theta}}{-im}\Big|_{-\pi}^{\pi} - \frac{1}{m}\int_{-\pi}^{\pi} e^{ix\cos\theta} x \sin\theta e^{-im\theta} d\theta \\
&= -\frac{x}{m}\int_{-\pi}^{\pi} e^{ix\cos\theta} \frac{e^{-i(m-1)\theta} - e^{-i(m+1)\theta}}{2i}\, d\theta \\
&= \frac{ix}{2m}\left[2\pi i^{m-1} J_{m-1}(x) - 2\pi i^{m+1} J_{m+1}(x)\right]
\end{aligned}$$

In the second line we have used periodicity to discard the first term. Therefore we have the *recurrence formula*

$$\boxed{J_m(x) = (x/2m)\,[J_{m-1}(x) + J_{m+1}(x)] \qquad m = 1, 2, \ldots} \tag{3.2.15}$$

This formula allows us to compute J_{m+1} in terms of J_m and J_{m-1}. Combining this with (3.2.10) and (3.2.11), adding, and subtracting, we obtain the *differentiation formulas*

$$J_m'(x) + \frac{m}{x} J_m(x) = J_{m-1}(x) \quad m = 1, 2, \ldots \tag{3.2.16}$$

$$J_m'(x) - \frac{m}{x} J_m(x) = -J_{m+1}(x) \quad m = 0, 1, 2, \ldots \tag{3.2.17}$$

Using the integrating factors $x^{\pm m}$, these can be rewritten in the form

$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x) \qquad m = 1, 2 \ldots \tag{3.2.18}$$

$$\frac{d}{dx}[x^{-m} J_m(x)] = -x^{-m} J_{m+1}(x) \qquad m = 0, 1, 2 \ldots \tag{3.2.19}$$

These formulas can be used to reduce certain integrals that occur in the normalization of the Bessel functions. For example, with $m = 1$ we have

$$\frac{d}{dx}(xJ_1) = x\, J_0$$

Integrating this for $0 \le x \le R$, we have

$$RJ_1(R) = \int_0^R x J_0(x) dx$$

3.2.4. The second solution of Bessel's equation. Since Bessel's equation becomes singular at $\rho = 0$, we cannot expect two linearly independent solutions in the form of power series. Let $y_1(\rho) = \sum_{n=0}^{\infty} a_n \rho^{n+\gamma}$ be the solution found above. To find the second solution $y_2(\rho)$, we use the method of reduction of order: let $v = y_2/y_1$ and find a first-order equation satisfied by v. Thus

$$\begin{aligned} y_2' &= v y_1' + v' y_1 \\ y_2'' &= v y_1'' + 2v' y_1' + v'' y_1 \end{aligned}$$

Assuming y_2 is a solution, we must have

$$\begin{aligned} 0 &= y_2'' + \frac{d-1}{\rho} y_2' + \left(\lambda - \frac{\mu}{\rho^2}\right) y_2 \\ &= v \left[y_1'' + \frac{d-1}{\rho} y_1' + \left(\lambda - \frac{\mu}{\rho^2}\right) y_1 \right] + 2v' y_1' + v'' y_1 + \frac{d-1}{\rho} (v' y_1) \end{aligned}$$

We require that v be a solution of the equation

$$v'' y_1 + v' \left(2y_1' + \frac{d-1}{\rho} y_1 \right) = 0$$

This is a first-order linear equation for v', which may be solved with the integrating factor $y_1 \rho^{d-1}$. Thus we obtain a solution by writing

$$\begin{aligned} (y_1^2 \rho^{d-1} v')' &= 0 \\ y_1^2 \rho^{d-1} v' &= c \neq 0 \\ v(\rho) &= \int_\rho^1 \frac{c}{y_1^2 \rho^{d-1}} \, d\rho \end{aligned}$$

The second constant of integration gives a multiple of y_1 and hence is omitted.

The integration begins at $\rho > 0$ since the integrand becomes infinite when $\rho \to 0$. To see this, recall that $y_1(\rho) \sim \rho^\gamma$, $\rho \to 0$. Therefore $y_1^2 \rho^{d-1} \sim \rho^{2\gamma + d - 1}$. From (3.2.8),

$$\begin{aligned} 2\gamma + d - 1 &= 2 \left\{ 1 - \frac{d}{2} + \left[\left(\frac{d}{2} - 1 \right)^2 + \mu \right]^{1/2} \right\} + d - 1 \\ &= 1 + 2 \left[\left(\frac{d}{2} - 1 \right)^2 + \mu \right]^{1/2} \end{aligned}$$

This is greater than or equal to 1, and hence the integral for $v(\rho)$ diverges when $\rho \to 0$.

To study this more precisely, we consider separate cases. If $d = 2$ and $\mu = 0$, then $\gamma = 0$ and we have

$$v(\rho) \sim c \int_\rho^1 \frac{d\rho}{\rho} = -c \log \rho$$

$$y_2(\rho) = v(\rho)y_1(\rho) \sim -c\log\rho, \qquad \rho \to 0 \qquad d = 2\mu = 0 \tag{3.2.20}$$

If $d \neq 2$ or $\mu \neq 0$, then $2\gamma + d - 1 > 1$ and

$$v(\rho) \sim c\frac{\rho^{2-2\gamma-d}}{2-2\gamma-d} \qquad \rho \to 0$$

But $y_1(\rho) \sim \rho^\gamma$, $\rho \to 0$. Therefore

$$y_2(\rho) \sim c\rho^{-(\gamma+d-2)} \qquad \rho \to 0 \qquad d > 2 \text{ or } \mu > 0 \tag{3.2.21}$$

But $\gamma + d - 2 = (d/2 - 1) + [\mu + (d/2-1)^2]^{1/2}$, which is positive if $d > 2$ or $d = 2$, $\mu \neq 0$.

To summarize, we have found a second, linearly independent solution of Bessel's equation

$$\boxed{y_2(\rho) = y_1(\rho)\int_\rho^1 \frac{d\rho'}{y_1(\rho')^2(\rho')^{d-1}}} \tag{3.2.22}$$

and $y_2(\rho)$ becomes infinite when $\rho \to 0$ according to (3.2.20), (3.2.21).

3.2.5. Zeros of the Bessel function J_0. For this purpose we write the series for J_0 in the form

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{4} + \sum_{p=1}^{\infty}\left\{\frac{x^{4p}}{2^{4p}[(2p)!]^2} - \frac{x^{4p+2}}{2^{4p+2}[(2p+1)!]^2}\right\} \\ &= 1 - \frac{x^2}{4} + \sum_{p=1}^{\infty}\frac{x^{4p}}{2^{4p}[(2p)!]^2}\left[1 - \frac{x^2}{4(2p+1)^2}\right] \end{aligned}$$

If $0 \le x \le 2$, all of the terms in the summation are positive since

$$x^2 \le 4 < 4(2p+1)^2 \qquad p = 1, 2, \ldots$$

Therefore we have the inequality

$$J_0(x) > 1 - \frac{x^2}{4} \qquad 0 \le x \le 2$$

In particular $J_0(2) > 0$.

On the other hand, we may write the series in the form

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \sum_{p=1}^{\infty}\left\{\frac{x^{4p+2}}{2^{4p+2}[(2p+1)!]^2} - \frac{x^{4p+4}}{2^{4p+4}[(2p+2)!]^2}\right\} \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \sum_{p=1}^{\infty}\left\{\frac{x^{4p+2}}{2^{4p+2}[(2p+1)!]^2}\left(1 - \frac{x^2}{4(2p+2)^2}\right)\right\} \end{aligned}$$

If $0 \le x \le 3$, all of the terms in the summation are positive, since

$$x^2 \le 9 < 4(2p+2)^2 \qquad p = 1, 2, \ldots$$

Therefore we have the inequality

$$J_0(x) < 1 - \frac{x^2}{4} + \frac{x^4}{64} \qquad 0 \le x \le 3$$

In particular $J_0(2\sqrt{2}) < 1 - \frac{8}{4} + \frac{64}{64} = 0$.

Now we can apply the intermediate-value theorem. J_0 is a continuous function with $J_0(2) > 0 > J_0(2\sqrt{2})$. Therefore $J_0(x) = 0$ for some x with $2 < x < 2\sqrt{2}$. But $J_0(x) > 0$ for $0 < x < 2$. Therefore we have proved that the smallest solution x_1 of the equation $J_0(x) = 0$ satisfies $2 < x_1 < 2\sqrt{2} = 2.828$.

The zeros of J_0 have been computed numerically. The first five are listed in the following table, together with the successive differences.

n	x_n	$x_{n+1} - x_n$
1	2.404	
2	5.520	3.116
3	8.654	3.134
4	11.792	3.138
5	14.931	3.139

Using this information, it is possible to sketch a graph of the function $y = J_0(x)$. (See Fig. 3.2.1.)

3.2.6. Asymptotic behavior and zeros of Bessel functions. Solutions of the Bessel equation (3.2.1) in case $\lambda > 0$ can be expected to behave asymptotically like trigonometric functions. This can be understood either (i) as $\rho \to +\infty$

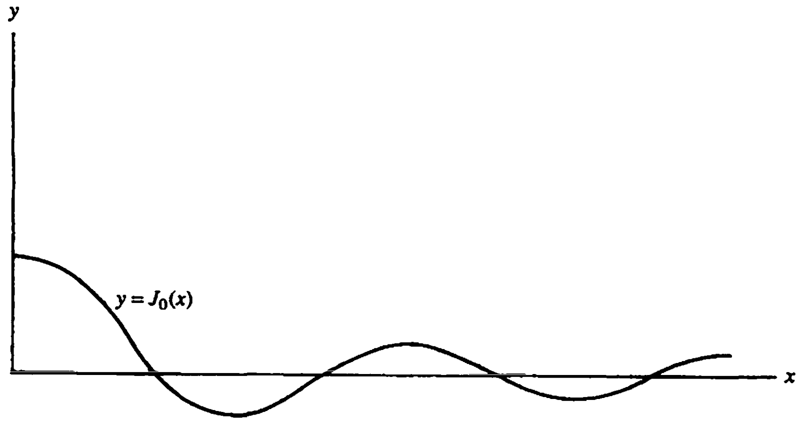


FIGURE 3.2.1 Graph of the Bessel function $y = J_0(x)$.

or (ii) as $\lambda \to +\infty$. In either limiting case the Bessel equation resembles the simple harmonic equation $y'' + \lambda y = 0$, whose solutions are trigonometric functions if $\lambda > 0$.

To see this in detail, we begin by removing the first derivative term in (3.2.1) by defining the new function $z(\rho) = \rho^{(d-1)/2}y(\rho)$. A straightforward computation shows that z satisfies the differential equation

$$z'' + \left(\lambda - \frac{C}{\rho^2}\right) z = 0, \qquad C = \mu + (d-1)(d-3)/4 \tag{3.2.23}$$

The second reduction consists of defining an *amplitude* $A(\rho)$ and a *phase* $\theta(\rho)$ by means of the equations

$$\sqrt{\lambda}z(\rho) = A(\rho)\sin\theta(\rho), \qquad z'(\rho) = A(\rho)\cos\theta(\rho) \tag{3.2.24}$$

Computing the required derivatives reveals

$$\theta'(\rho) = \sqrt{\lambda} - \frac{C\sin^2\theta}{\rho^2\sqrt{\lambda}} \tag{3.2.25}$$

$$\frac{A'(\rho)}{A(\rho)} = \frac{C\sin\theta(\rho)\cos\theta(\rho)}{\rho^2\sqrt{\lambda}} \tag{3.2.26}$$

It will be shown that asymptotically $\theta(\rho)$ behaves as a linear function and $A(\rho)$ behaves as a constant function. This is embodied in the following proposition.

PROPOSITION 3.2.2.

1. Suppose that $y(\rho)$ is a solution of (3.2.1) with fixed parameters $d \geq 1$, $\mu \geq 0$, $\lambda > 0$, defined for $\rho > 0$. Then there exist constants A_∞, θ_∞ so that when $\rho \to \infty$

$$\theta(\rho) = \rho\sqrt{\lambda} - \theta_\infty + O(1/\rho), \quad A(\rho) = A_\infty + O(1/\rho)$$

In particular,

$$\rho^{(d-1)/2}y(\rho) = A_\infty \sin(\rho\sqrt{\lambda} - \theta_\infty) + O(1/\rho)$$

$$\left(\rho^{(d-1)/2}y(\rho)\right)' = \sqrt{\lambda}A_\infty \cos(\rho\sqrt{\lambda} - \theta_\infty) + O(1/\rho)$$

2. Suppose that $d \geq 1$, $\mu \geq 0$ are fixed parameters and that $y = y(\rho, \lambda)$ is the solution of the Bessel equation (3.2.1) with initial conditions $y(\rho_0, \lambda)$, $y'(\rho_0, \lambda)$, where $\rho_0 > 0$. Then for any fixed $\rho > \rho_0$ we have, when $\lambda \to \infty$,

$$\begin{aligned} \theta(\rho, \lambda) &= \theta(\rho_0, \lambda) + \sqrt{\lambda}(\rho - \rho_0) + O(1/\sqrt{\lambda}) \\ A(\rho, \lambda) &= A(\rho_0, \lambda) + O(1/\sqrt{\lambda}) \end{aligned}$$

In particular,

$$\begin{aligned}
\rho^{(d-1)/2}y(\rho,\lambda) &= A(\rho_0)\sin\left(\theta(\rho_0)+(\rho-\rho_0)\sqrt{\lambda}\right)+O\left(\frac{1}{\sqrt{\lambda}}\right)\\
\left(\rho^{(d-1)/2}y(\rho,\lambda)\right)' &= A(\rho_0)\sqrt{\lambda}\cos\left(\theta(\rho_0)+(\rho-\rho_0)\sqrt{\lambda}\right)+O\left(\frac{1}{\sqrt{\lambda}}\right)
\end{aligned}$$

Proof. To prove the first statement, we integrate (3.2.25) on the interval (ρ_0, ρ) to obtain

$$\theta(\rho)-\theta(\rho_0)=\sqrt{\lambda}(\rho-\rho_0)-\frac{C}{\sqrt{\lambda}}\int_{\rho_0}^{\rho}\frac{\sin^2\theta(u)}{u^2}\,du \tag{3.2.27}$$

The integral tends to a limiting value when $\rho \to \infty$ with an error $O(1/\rho)$, which proves the asymptotic form of $\theta(\rho)$. Similarly, (3.2.26) is integrated to obtain

$$A(\rho)=A(\rho_0)\exp\left(\int_{\rho_0}^{\rho}\frac{C\sin\theta(u)\cos\theta(u)}{u^2\sqrt{\lambda}}\,du\right) \tag{3.2.28}$$

The integral in the exponent tends to a finite limit, with remainder $(1/\rho)$, which proves the asserted form of $A(\rho)$. To prove the second statement, we return to (3.2.27), noting that the integral term is $O(1/\sqrt{\lambda})$, so that the form of $\theta(\rho,\lambda)$ follows immediately. Similarly, reference to (3.2.28) yields the form of $A(\rho,\lambda)$. •

The preceding asymptotic formulas contain two constants, the *asymptotic amplitude* A_∞ and the *asymptotic phase* θ_∞. They depend on the specific solution, as well as the parameters (d,λ,μ). [In Chapter 6 we shall make an exact determination of these constants for the power series solutions $J_m(x)$ of the Bessel equation with $d=2$.]

We can use the above methods to determine the zeros of solutions of the Bessel equation (3.2.1).

PROPOSITION 3.2.3.

1. Suppose that $y(\rho)$ is a solution of (3.2.1) with fixed parameters $d \geq 1$, $\mu \geq 0$, $\lambda > 0$, defined for $\rho > 0$. Then there exist $\rho_1 < \rho_2 < \cdots < \rho_n \to \infty$ so that $y(\rho_n) = 0$. They further satisfy the asymptotic behavior $\rho_{n+1}-\rho_n \to \pi/\sqrt{\lambda}$ when $n \to \infty$.

2. Suppose that $d \geq 1$, $\mu \geq 0$ are fixed parameters and that $y = y(\rho,\lambda)$ is the solution of Bessel's equation (3.2.1) with initial conditions $y(\rho_0,\lambda)$, $y'(\rho_0,\lambda)$ where $\rho_0 > 0$. Then for any $\rho > \rho_0$ there exist $\lambda_1(\rho) < \lambda_2(\rho) < \cdots < \lambda_n(\rho) \to \infty$ so that $y(\lambda_n(\rho),\rho) = 0$ for $n \geq 1$. They further satisfy the asymptotic behavior $\sqrt{\lambda_{n+1}(\rho)} - \sqrt{\lambda_n(\rho)} \to \pi/(\rho-\rho_0)$ when $n \to \infty$.

Proof. We note from (3.2.24) that the zeros of $z(\rho)$ occur precisely when $\sin(\theta(\rho)) = 0$, or equivalently $\theta(\rho) = n\pi$ for some $n = 1, 2, \ldots$.

In case 1, we can solve the equation $\theta(\rho_n) = n\pi$, since $\rho \to \theta(\rho)$ is unbounded when $\rho \to \infty$. From Proposition 3.2.2, the successive zeros satisfy

$$\begin{aligned}\pi &= \theta(\rho_{n+1}) - \theta(\rho_n)\\ &= \sqrt{\lambda}(\rho_{n+1} - \rho_n) + O(1/\rho)\end{aligned}$$

hence $\rho_{n+1} - \rho_n \to \pi/\sqrt{\lambda}$, as required. In case 2, we note that $\theta(\rho, \lambda)$ depends continuously on λ and increases to $+\infty$ when $\lambda \to \infty$. Therefore we can uniquely solve the equation $\theta(\rho, \lambda) = n\pi$ to uniquely determine $\lambda_n = \lambda_n(\rho)$. The successive zeros satisfy

$$\begin{aligned}\pi &= \theta(\rho, \lambda_{n+1}) - \theta(\rho, \lambda_n)\\ &= \sqrt{\lambda_{n+1}}(\rho - \rho_0) - \sqrt{\lambda_n}(\rho - \rho_0) + O(1/\sqrt{\lambda})\end{aligned}$$

from which the result follows. •

Remark. The preceding method of proof yields precise information on the spacing of the zeros of the Bessel function $J_m(\rho)$. Setting $\lambda = 1$, $d = 2$, $\mu = m^2$, we have $C = m^2 - (1/4)$, and (3.2.27) reads

$$\theta(\rho) - \theta(\rho_0) = (\rho - \rho_0) - \left(m^2 - (1/4)\right) \int_{\rho_0}^{\rho} \frac{\sin^2 \theta(u)}{u^2}\, du$$

Evaluating this at two consecutive zeros $\rho_n < \rho_{n+1}$ yields

$$\rho_{n+1} - \rho_n = \pi + \left(m^2 - (1/4)\right) \int_{\rho_n}^{\rho_{n+1}} \frac{\sin^2 \theta(u)}{u^2}\, du$$

In particular, if $m^2 > 1/4$, then $\rho_{n+1} - \rho_n > \pi$. If $m^2 < 1/4$, then $\rho_{n+1} - \rho_n < \pi$, while if $m^2 = 1/4$, then $\rho_{n+1} - \rho_n = \pi$. For example, the spacing between the successive zeros of $J_0(\rho)$ is less than π.

By specializing to the case $d = 2$, $\lambda = 1$, we can state a result that summarizes the computations.

PROPOSITION 3.2.4. *The equation $J_m(x) = 0$ has infinitely many positive solutions $\{x_n\}$, $n = 1, 2, \ldots$. They satisfy*

$$\begin{aligned}\lim_{n\to\infty} x_n &= \infty\\ \lim_{n\to\infty} (x_{n+1} - x_n) &= \pi\end{aligned}$$

In many problems it is important to have information about the zeros of $\cos\beta\, J_m(x) + \sin\beta\, x J_m'(x)$, where $0 < \beta \le \pi/2$.

PROPOSITION 3.2.5. *For any $0 < \beta \le \pi/2$, the equation $\cos\beta\, J_m(x) + \sin\beta\, x J_m'(x)$ has infinitely many positive solutions $\{x_n\}$, $n = 1, 2, \ldots$. They satisfy*

$$\lim_{n\to\infty} x_n = \infty \tag{3.2.29}$$

$$\lim_{n\to\infty} (x_{n+1} - x_n) = \pi \tag{3.2.30}$$

Proof. We use the phase plane representation $\sqrt{x}J_m(x) = R(x)\sin\theta(x)$, $(\sqrt{x}J_m)'(x) = R(x)\cos\theta(x)$. The equation $\cos\beta J_m(x) + x\sin\beta J_m'(x) = 0$ is equivalent to

$$\cot\theta(x) = \frac{\frac{1}{2} - \cot\beta}{x} \tag{3.2.31}$$

From the graph of the cotangent function it is seen that (3.2.31) has a unique solution x_n satisfying $n\pi < \theta(x_n) < (n+1)\pi$. Furthermore, $\theta(x_n) - n\pi \to 0$ when $n \to \infty$. On the other hand, $\theta(x_{n+1}) - \theta(x_n) = \pi + 0(1/n)$. Combining these, we have proved (3.2.29). •

3.2.7. Fourier-Bessel series. In many problems it is important to expand a given function in a series of the form $\sum_{n=1}^{\infty} A_n J_m(xx_n)$, where m is a fixed positive number and $\{x_n\}$ are determined from a suitable boundary condition. The boundary condition might be $J_m(x_n) = 0$ or $J_m'(x_n) = 0$, or that some linear combination of these equals zero. To study series of this type, we first derive the orthogonality properties of the functions $J_m(xx_n)$.

PROPOSITION 3.2.6. *Let $\{x_n\}$ be the nonnegative solutions of the equation*

$$\cos\beta\, J_m(x_n) + \sin\beta\, x_n J_m'(x_n) = 0 \tag{3.2.32}$$

where $m \geq 0$ and $0 \leq \beta \leq \pi/2$. Then

$$\int_0^1 J_m(xx_{n_1})J_m(xx_{n_2})x\,dx = 0 \qquad n_1 \neq n_2 \tag{3.2.33}$$

$$\int_0^1 J_m(xx_n)^2 x\,dx = \frac{1}{2}J_{m+1}(x_n)^2 \qquad \text{if } \beta = 0 \tag{3.2.34}$$

$$\int_0^1 J_m(xx_n)^2 x\,dx = \frac{(x_n^2 - m^2 + \cot^2\beta)J_m(x_n)^2}{2x_n^2} \qquad \text{if } 0 < \beta \leq \pi/2 \tag{3.2.35}$$

Proof. If $y(x) = J_m(xx_n)$, then $y'(x) = x_n J_m'(xx_n)$. In this notation the equation for x_n becomes $\cos\beta\, y(1) + \sin\beta\, y'(1) = 0$. The Bessel equation can be rewritten in the form

$$(xy')' + \left(xx_n^2 - \frac{m^2}{x}\right)y = 0 \tag{3.2.36}$$

Taking $y = y_1(x) = J_m(xx_{n_1})$ and multiplying by $y_2(x)$, we obtain, upon integration by parts,

$$y_1'(1)y_2(1) - \int_0^1 xy_1'(x)y_2'(x)\,dx + \int_0^1 \left(xx_{n_1}^2 - \frac{m^2}{x^2}\right) y_1(x)y_2(x)\,dx = 0$$

Interchanging the roles of y_1 and y_2 and subtracting the resulting equations leaves

$$(y_1'y_2 - y_1y_2')|_{x=1} + (x_{n_1}^2 - x_{n_2}^2)\int_0^1 xy_1(x)y_2(x)\,dx = 0$$

But (3.2.32) requires that the first term be zero; hence the second term is also zero. But $x_{n_1}^2 - x_{n_2}^2 \neq 0$; therefore we conclude that the integral is zero, which was to be shown. Thus we have proved (3.2.33).

To compute the integrals (3.2.34), we multiply (3.2.36) by xy' to obtain

$$[(xy')^2]' + (x_n^2 x^2 - m^2)(y^2)' = 0 \tag{3.2.37}$$

Integrating (3.2.37) for $0 < x < 1$ and integrating the second term by parts, we have

$$y'(1)^2 + (x_n^2 - m^2)y(1)^2 - 2x_n^2 \int_0^1 xy(x)^2 dx = 0 \tag{3.2.38}$$

If $\beta = 0$, the boundary condition is $y(1) = 0$, which gives

$$2x_n^2 \int_0^1 xy(x)^2 dx = y'(1)^2 = x_n^2 J_m'(x_n)^2 = x_n^2 J_{m+1}(x_n)^2$$

To handle the case $0 < \beta \leq \pi/2$, solve for $y'(1)$ in the form $y'(1) = -\cot\beta y(1)$. Substituting this in (3.2.38), we have

$$2x_n^2 \int_0^1 xy(x)^2 dx = y(1)^2(\cot^2\beta + x_n^2 - m^2)$$

Noting that $y(1) = J_m(x_n)$, we have the required form. •

These orthogonality relations permit us to compute the coefficients in the expansion of a piecewise smooth function $f(x)$, $0 < x < 1$, in a series of the form

$$f(x) = \sum_{n=1}^{\infty} A_n J_m(xx_n) \qquad 0 < x < 1 \tag{3.2.39}$$

where $\{x_n\}$ are the nonnegative solutions of $\cos\beta\, J_m(x) + \sin\beta\, xJ_m'(x) = 0$. This is called a *Fourier-Bessel expansion.* To obtain $\{A_n\}$, we proceed formally; we multiply the equation by $J_m(xx_n)$ and integrate with respect to the weight $x\,dx$ from 0 to 1. This gives the formula

$$\int_0^1 f(x) J_m(xx_n) x\, dx = A_n \int_0^1 J_m(xx_n)^2 x\, dx \qquad n = 1, 2, \ldots \tag{3.2.40}$$

We state without proof a theorem concerning this expansion.[1]

THEOREM 3.1. *Let $m \geq 0$, $0 \leq \beta \leq \pi/2$, and let $\{x_n : n \geq 1\}$ be the nonnegative solutions of (3.2.32). If $f(x)$, $0 < x < 1$, is a piecewise smooth function, define $\{A_n : n \geq 1\}$ by (3.2.40). Then the series $\sum_{n=1}^{\infty} A_n J_m(xx_n)$ converges for each x, $0 \leq x \leq 1$, and the sum is $\frac{1}{2}[f(x+0) + f(x-0)]$ if $0 < x < 1$.*

[1]See H. F. Weinberger, *A First Course in Partial Differential Equations*, Ginn, Blaisdell, Waltham, MA, 1965, pp. 176–178.

One may note that the sum of the series will have the value $f(0+0)$ at $x = 0$.

It is important to note under what conditions we may have $x_1 = 0$. Since $y(x) = J_m(xx_1)$ must be nonzero, this implies that $0 \neq J_m(0)$, that is, $m = 0$. We must also have the boundary condition $\cos\beta J_m(0) = 0$, which requires $\beta = \pi/2$. Conversely, the function $y = J_0(xx_1)$ is a solution of the Bessel equation satisfying the boundary condition $\cos\beta J_0(x_1) + \sin\beta x_1 J_0'(x_1) = 0$ if $x_1 = 0$. We record this as a proposition.

PROPOSITION 3.2.7. *If $m > 0$ or $\beta \neq \pi/2$, then $x_n > 0$ for all $n \geq 1$. If $m = 0$ and $\beta = \pi/2$, then $x_1 = 0$ and $x_n > 0$ for all $n \geq 2$.*

We now pass to some concrete examples of Fourier-Bessel expansions.

EXAMPLE 3.2.5. *Compute the Fourier-Bessel expansion of the function $f(x) = 1$, $0 < x < 1$, where $m = 0$, $\beta = 0$.*

Solution. We have $\sum_{n=1} A_n J_0(xx_n)$, where $J_0(x_n) = 0$ and

$$\int_0^1 xJ_0(xx_n)dx = A_n \int_0^1 J_0(xx_n)^2 x\,dx \qquad n = 1, 2, \ldots$$

To compute the first integral, we use (3.2.18) with $m = 1$. With the substitution $t = xx_n$, we have

$$\int_0^1 xJ_0(xx_n)dx = \frac{1}{x_n^2}\int_0^{x_n} tJ_0(t)dt = \frac{1}{x_n} J_1(x_n)$$

The integral $\int_0^1 J_0(xx_n)^2 x\,dx$ was already shown to be $\frac{1}{2}J_1(x_n)^2$, by (3.2.34). The required expansion is

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0(xx_n)}{x_n J_1(x_n)} \qquad 0 < x < 1 \quad \bullet$$

EXAMPLE 3.2.6. *Compute the Fourier-Bessel expansion of the function $f(x) = 1 - x^2$, $0 < x < 1$, where $m = 0$, $\beta = 0$.*

Solution. We have $1 - x^2 = \sum_{n=1}^{\infty} A_n J_0(xx_n)$, where $J_0(x_n) = 0$, and

$$\int_0^1 (1 - x^2)J_0(xx_n)x\,dx = A_n \int_0^1 J_0(xx_n)^2 x\,dx$$

To compute the first integral, let $t = xx_n$ and use the formulas (3.2.18)–(3.2.19) in the forms $(d/dt)(tJ_1) = (tJ_0)$, $(d/dt)(J_0) = -J_1$. Thus

$$\begin{aligned}\int_0^1 (1-x^2)J_0(xx_n)x\,dx &= \frac{1}{x_n^4}\int_0^{x_n}(x_n^2-t^2)tJ_0(t)dt \\ &= \frac{2}{x_n^4}\int_0^{x_n} t^2J_1(t)dt \\ &= \frac{4}{x_n^4}\int_0^{x_n} tJ_0(t)dt \\ &= \frac{4}{x_n^3}J_1(x_n)\end{aligned}$$

from Example 3.2.5 and integration by parts. The second integral, $\int_0^1 J_0(xx_n)^2x\,dx$, was shown to be $\frac{1}{2}J_1(x_n)^2$. The required expansion is therefore

$$1-x^2 = 8\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n^3J_1(x_n)} \qquad 0<x<1 \quad \bullet$$

These examples suggest a general method for computing the Fourier-Bessel expansion of certain polynomials. Let $P_0(x) = \frac{1}{2}$ and let $P_{2n}(x)$ be the polynomial of degree $2n$ that satisfies

$$(xP_{2n}')' = -xP_{2n-2}, \qquad P_{2n}(1)=0 \qquad n=1,2,3,\ldots$$

For example, $P_2(x) = (1-x^2)/8$, $P_4(x) = \frac{1}{128}(3-4x^2+x^4)$. To find the Fourier-Bessel expansion, we write $\lambda = x_n^2$, $y(x) = J_0(xx_n)$, $y'(x) = x_nJ_0'(xx_n)$, $(xy')' + \lambda(xy) = 0$. Multiplying the Bessel equation by $P_{2n}(x)$ and integrating, we have

$$\begin{aligned}\lambda\int_0^1 P_{2n}(x)xy(x)dx &= -\int_0^1 P_{2n}(x)(xy')'(x)dx \\ &= \int_0^1 P_{2n}'(x)(xy')(x)dx \\ &= -\int_0^1 [xP_{2n}'(x)]'y(x)dx \\ &= \int_0^1 P_{2n-2}(x)xy(x)dx\end{aligned}$$

Therefore the Fourier-Bessel coefficients of P_{2n} are obtained from those of P_{2n-2} upon division by $\lambda = x_n^2$. For example, beginning with the expansion

$$P_0(x) = \frac{1}{2} = \sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_nJ_1(x_n)} \qquad\qquad 0<x<1$$

we have

$$P_2(x) = \frac{1-x^2}{8} = \sum_{n=1}^{\infty} \frac{J_0(xx_n)}{x_n^3 J_1(x_n)} \qquad 0<x<1$$

$$P_4(x) = \frac{1}{128}(3-4x^2+x^4) = \sum_{n=1}^{\infty} \frac{J_0(xx_n)}{x_n^5 J_1(x_n)} \qquad 0<x<1$$

In some problems it is necessary to find the Fourier-Bessel expansion of discontinuous functions. The following example gives a typical case.

EXAMPLE 3.2.7. *Let $f(x)$, $0<x<1$, be defined by $f(x)=1$ for $0<x<\frac{1}{2}$ and $f(x)=0$ for $\frac{1}{2}<x<1$. Find the Fourier-Bessel expansion of $f(x)$, $0<x<1$, with $m=0$, $\beta=0$.*

Solution. The desired expansion is of the form $f(x) = \sum_{n=1}^{\infty} A_n J_0(xx_n)$, where $J_0(x_n)=0$. The coefficients are determined by orthogonality—leading to $A_n \int_0^1 J_0(xx_n)^2 dx = \int_0^1 f(x)J_0(xx_n)x\,dx = \int_0^{1/2} J_0(xx_n)x\,dx$. The first integral was evaluated in Proposition 3.2.6 as $\frac{1}{2}J_1(x_n)^2$. To evaluate the second integral, we make the substitution $t=xx_n$ and find

$$\int_0^{1/2} J_0(xx_n)x\,dx = (1/x_n^2)\int_0^{x_n/2} tJ_0(t)dt = (1/x_n^2)tJ_1(t)\big|_{t=0}^{t=x_n/2} = \left(\frac{1}{2}x_n\right) J_1\left(\frac{1}{2}x_n\right)$$

Therefore $A_n = J_1(\frac{1}{2}x_n)/[x_n J_1(x_n)^2]$ and the required expansion is

$$f(x) = \sum_{n=1}^{\infty} \frac{J_1(\frac{1}{2}x_n)}{x_n J_1(x_n)^2} J_0(xx_n), \qquad 0<x<1, x\neq \frac{1}{2} \quad \bullet$$

In all these examples of Fourier-Bessel expansions, we have taken $m>0$, $\beta=0$. This is well suited to problems with radial symmetry ($m=0$) or where the boundary conditions do not involve the derivative ($\beta=0$). In the following examples, we give Fourier-Bessel expansions with $m>0$ or $\beta>0$.

EXAMPLE 3.2.8. *For $m>0$, $\beta=0$, compute the Fourier-Bessel expansion of the function $f(x)=x^m$, $0<x<1$.*

Solution. The Fourier coefficients $\{A_n\}$ are given by $A_n\int_0^1 J_m(xx_n)^2 x\,dx = \int_0^1 x^{m+1}J_m(xx_n)dx$. The first integral is given by (3.2.34): $\int_0^1 J_m(xx_n)^2x\,dx = \frac{1}{2}J_{m+1}(x_n)^2$. To compute the second integral, we use the first differentiation formula (3.2.16) with m replaced by $m+1$.

$$\frac{d}{dt}\left[t^{m+1}J_{m+1}(t)\right] = t^{m+1}J_m(t)$$

Thus

$$\begin{aligned}\int_0^1 x^{m+1} J_m(xx_n)dx &= \frac{1}{x_n^{m+2}} \int_0^{x_n} t^{m+1} J_m(t)dt \\ &= \frac{1}{x_n^{m+2}} x_n^{m+1} J_{m+1}(x_n) \\ &= \frac{1}{x_n} J_{m+1}(x_n)\end{aligned}$$

Hence $A_n = 2/x_n J_{m+1}(x_n)$, and we have the expansion

$$x^m = 2\sum_{n=1}^{\infty} \frac{J_m(xx_n)}{x_n J_{m+1}(x_n)} \qquad x_n = x_n^{(m)} \qquad 0 < x < 1 \quad \bullet$$

EXAMPLE 3.2.9. *For $m = 0$, $0 < \beta \le \pi/2$, compute the Fourier-Bessel expansion of the functions $f(x) = 1$ and $f(x) = 1 - x^2$.*

Solution. The first expansion is of the form $1 = \sum_{n=1}^{\infty} A_n J_0(xx_n)$ where $\cos\beta\, J_0(x_n) + \sin\beta\, x_n J_0'(x_n) = 0$. From the orthogonality relations we must have $A_n \int_0^1 J_0(xx_n)^2 x\,dx = \int_0^1 x J_0(xx_n)dx$. From (3.2.34) the first integral is $(x_n^2 + \cot^2\beta) J_0(x_n)^2/2x_n^2$. To compute the second integral, we use the differentiation formula (3.2.16) with $m = 1$: $(d/dt)(tJ_1) = tJ_0$. Thus $\int_0^1 x J_0(xx_n)dx = (1/x_n^2)\int_0^{x_n} tJ_0(t)dt = J_1(x_n)/x_n$. From the boundary condition we have $x_n J_1(x_n) = -x_n J_0'(x_n) = \cot\beta J_0(x_n)$ and the expansion

$$1 = 2\cot\beta \sum_{n=1}^{\infty} \frac{J_0(xx_n)}{(x_n^2 + \cot^2\beta) J_0(x_n)} \qquad 0 < x < 1$$

The expansion of $f(x) = 1 - x^2$ is handled similarly, using integration by parts to reduce the integral $\int_0^1 (1 - x^2) J_0(xx_n) x\,dx$ to the integral that has already been computed. The result is

$$1 - x^2 = 8\cot\beta \sum_{n=1}^{\infty} \frac{J_0(xx_n)}{x_n^2 (x_n^2 + \cot^2\beta) J_0(x_n)} \qquad 0 < x < 1 \quad \bullet$$

3.2.8. Implementation with Mathematica. In Mathematica the command for the Bessel function $J_n(x)$ is **BesselJ[n,x]**. For example, if we want to obtain the power series expansion for $J_0(x)$ at $x = 0$ up to and including terms with x^8, we can type

```
In[1]= Series[BesselJ[0,x],{x,0,8}]
```

to yield the output

```
              2    4     6       8
             x    x     x       x             9
Out[1] = 1 - -- + -- - ---- + ------  +  O[x]
             4    64   2304   147456
```

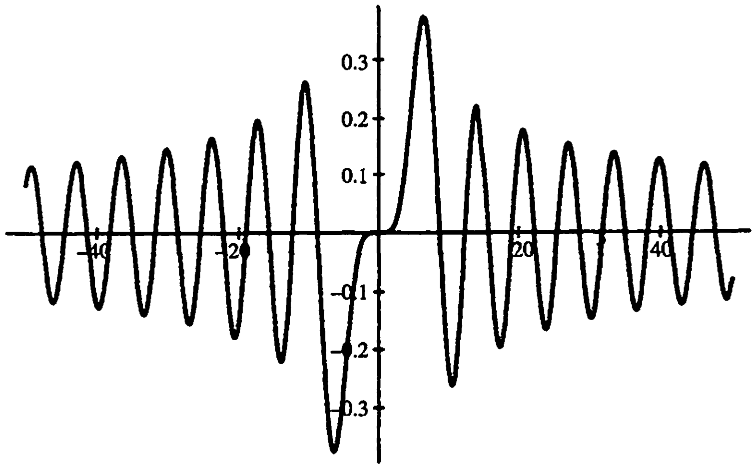


FIGURE 3.2.2 Graph of the Bessel function $J_5(x)$.

Note that the odd terms are zero, so that we obtain only five nonzero coefficients in this case.

The second linearly independent solution of Bessel's equation is denoted **BesselY[n,x]** in Mathematica. We can ask Mathematica to solve Bessel's equation of order 2 by typing

```
DSolve[ x^2 y''[x] + x y'[x] + (x^2-4)y[x]==0, y[x],x]
```

to yield the output

```
{{y[x]-> BesselY[2,x] C[1] + BesselJ[2,x]C[2]}}
```

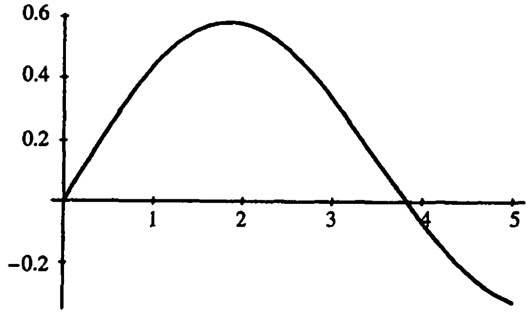
Graphs of Bessel functions can be obtained using Mathematica in much the same way that we plotted trigonometric functions and Legendre functions. In order to obtain the available options, simply type **??Plot** to obtain these options together with their default values. For example, the option **PlotPoints** has a default value of 25, but for a highly oscillatory function such as $J_5(x)$ the number should be increased from 25 to 40. In order to obtain the plot shown in Fig. 3.2.2, we type

```
Plot[BesselJ[5,x],{x,-50,50},PlotPoints->40]
```

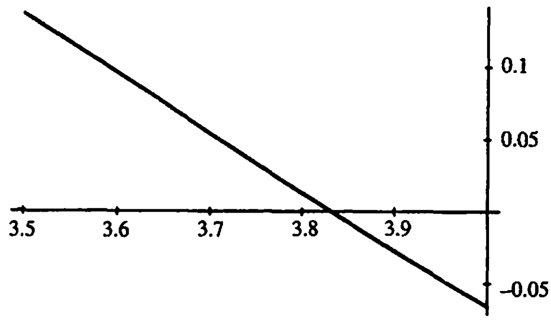
The graphing capabilities of Mathematica can be effectively used to find the zeros of Bessel functions. Let's find the first positive zero of J_1, which is the smallest positive number that satisfies $J_1(x) = 0$; call it x_1^1. To begin, we define a plot-valued function **jj** by

```
jj[a_,b_]:=Plot[BesselJ[1,x],{x,a,b}]
```

For example, **jj[0,5]** yields



This shows that x_1^1 lies between 3.5 and 4. To obtain a more accurate estimate, type **jj[3.5,4]** to obtain the graph



which shows that x_1^1 lies between 3.8 and 3.85. At the next stage we would type **jj[3.8,3.85]** and learn that x_1^1 is slightly more than 3.8317.

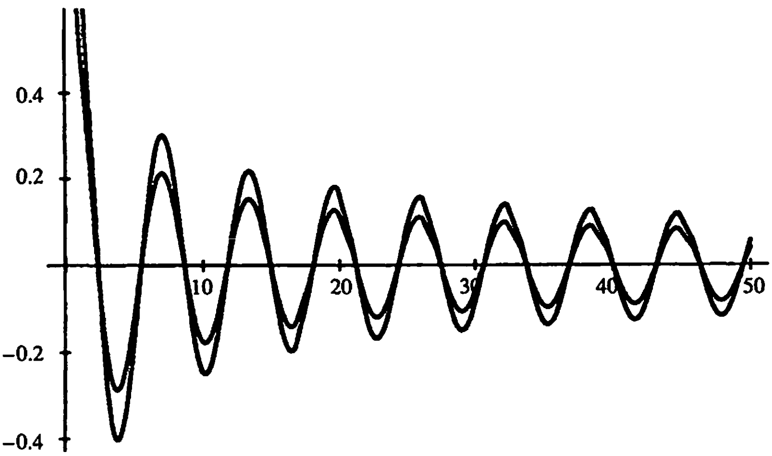
The asymptotic behavior of the Bessel function $J_0(x)$ can be illustrated effectively with Mathematica. From the results of chapter 6, it is found that

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right)\right) \qquad x \to \infty$$

If we type

```
Plot[{BesselJ[0,x],(2/Pi x)^(1/2) Cos[x -Pi/4]},{x,.1,50}]
```

we obtain



One can also illustrate the Fourier-Bessel expansions with Mathematica, based on the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2 \\ 0 & \text{if } 1/2 < x < 1 \end{cases}$$

The Fourier-Bessel expansion was obtained in Example 3.2.7,

$$f(x) = \sum_{n=1}^{\infty} \frac{J_1(x_n/2)}{x_n J_1(x_n)^2} J_0(x\, x_n)$$

where x_n is the nth zero of the Bessel function $J_0(x)$. These can be obtained from tables and incorporated as a list. For example, for the first 20 zeros, we write

```
X= {2.405,  5.520,  8.654, 11.792, 14.931,
   18.071, 21.212, 24.352, 27.493, 30.635,
   33.776, 36.972, 40.058, 43.200, 46.863,
   49.482, 52.624, 55.766, 58.907, 62.049}
```

The kth zero is thus represented as **X[[k]]**. The nth partial sum of the Fourier-Bessel series is described as follows in Mathematica:

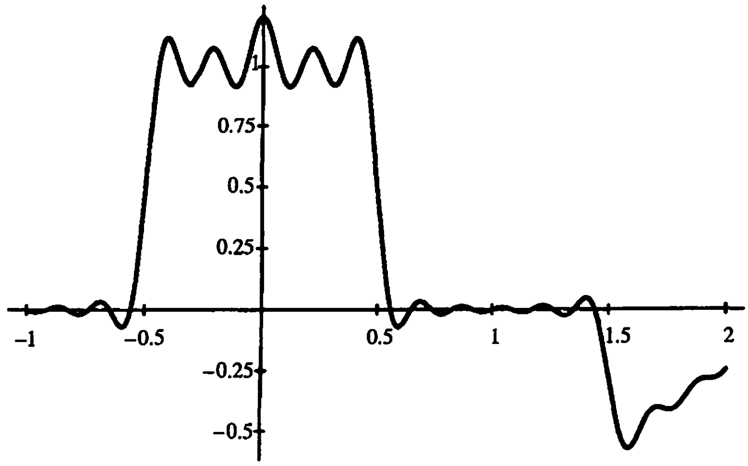
```
f[x_,n_]:=Sum[
   BesselJ[1, Release[X[[k]]]/2]    BesselJ[0, x Release[X[[k]]]]
   /(Release[X[[k]]] BesselJ[1, Release[X[[k]]]]^2),{k,n}]
```

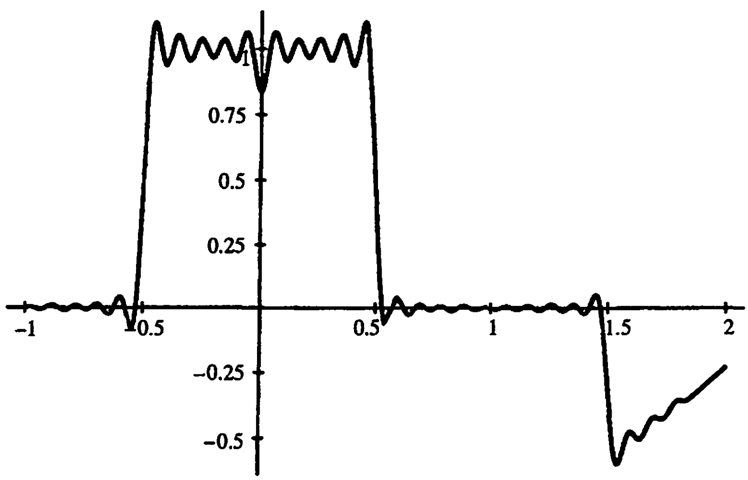
These partial sums are defined for all real values of x, despite the fact that the original function is defined only on the interval $0 \le x < 1$. The following two graphs are obtained by typing, respectively,

```
Plot[f[x,10],{x,-1,2},PlotPoints->40]
```

and

```
Plot[f[x,20],{x,-1,2},PlotPoints->40]
```





It is interesting to note that the partial sums oscillate more strongly in the neighborhoods of $x = 0$ and $x = \pm 1/2$ than elsewhere. The latter oscillations are identical to those encountered with Gibbs' phenomenon in Chapter 1, where we compute the partial sums of a Fourier series in the neighborhood of a discontinuity. The oscillations near $x = 0$ are of a different character, related to a *slower rate of convergence* of the series. At the points $x \neq 0$ the rate of convergence is $O(1/n), n \to \infty$, whereas at $x = 0$ the rate of convergence can be shown to be $O(1/\sqrt{n})$ by an explicit analysis of the Bessel functions. This is a phenomenon that is not present for one-dimensional Fourier series, but occurs when we consider the expansions of certain radially symmetric functions in two and three variables.

EXERCISES 3.2

1. Show that $(-\frac{1}{2})! = \int_0^\infty t^{-1/2}e^{-t}dt = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$. (You may assume it is known that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.)
2. Show that $(\frac{1}{2})! = \frac{1}{2}\sqrt{\pi}$, $(\frac{3}{2})! = \frac{3}{4}\sqrt{\pi}$.
3. Show that $(n+\frac{1}{2})! = \{[(2n+1)!]/(2^{2n+1}n!)\}\sqrt{\pi}$, $n = 0,1,2,\ldots$.
4. Show that $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin x$.
5. Show that $J_{-1/2}(x) = \sqrt{2/\pi x}\,\cos x$.
6. Use the ratio test to show that the series (3.2.3) converges for all x.
7. Use the ratio test to show that the series (3.2.4) converges for all x.
8. Write down the first four nonzero terms in the power series expansions of $J_0(x)$ and $J_1(x)$.
9. Use Example 3.2.4 and Eq. (3.2.13) to show that $|J'_m(t)| \le 1$ for $m = 1,2,\ldots$.
10. Let $m > 0$, not necessarily an integer. Prove the differentiation formula $J'_m(x) = \frac{1}{2}[J_{m-1}(x) - J_{m+1}(x)]$ directly from the power series definition (3.2.10).
11. Let $m > 0$, not necessarily an integer. Prove the recurrence formula $J_{m+1}(x) = (2m/x)J_m(x) - J_{m-1}(x)$ directly from the power series definition (3.2.10).
12. Let $m > 0$, not necessarily an integer. Use Exercises 10 and 11 to verify the formula $xJ'_m(x) = mJ_m(x) - xJ_{m+1}(x)$.
13. Let $m > 0$, not necessarily an integer. Use Exercises 10 and 11 to verify the formula $xJ'_m(x) = xJ_{m-1}(x) - mJ_m(x)$.
14. Let $m > 0$, not necessarily an integer. Prove the differentiation formulas
$$\frac{d}{dx}x^m J_m(x) = x^m J_{m-1}(x) \quad \text{and} \quad \frac{d}{dx}x^{-m}J_m(x) = -x^{-m}J_{m+1}(x)$$
15. Let x_n be a solution of the equation $J_m(x_n) = 0$, $m > 0$. Use Exercises 12 and 13 to show that $J'_m(x_n) = J_{m-1}(x_n) = -J_{m+1}(x_n)$.
16. Use Exercises 4 and 14 to show that
$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$
17. Let $\theta(\rho)$ and $A(\rho)$ be functions that satisfy the equations (3.2.25) and (3.2.26). Show that $z(\rho) = \lambda^{-1/2}A(\rho)\sin\theta(\rho)$ satisfies the equation $z'' + [\lambda - (C/\rho^2)]z = 0$.
18. Show that $\int_{-\pi}^{\pi} e^{ix\cos\theta}e^{-im\theta}d\theta = \int_{-\pi}^{\pi} e^{ix\cos\theta}e^{im\theta}d\theta$ for $m = 0,1,2,\ldots$.

19. Use Exercise 18 and the integral representation of $J_m(x)$ to show that $\int_{-\pi}^{\pi} e^{ix\cos\theta} e^{im\theta} d\theta = 2\pi i^{|m|} J_{|m|}(x)$ for $m = 0, \pm 1, \pm 2, \ldots$.
20. Use Exercise 19 and the properties of complex Fourier series to show that $e^{ix\cos\theta} = \sum_{-\infty}^{\infty} i^{|m|} J_{|m|}(x) e^{im\theta}$.
21. Use Exercise 20 to show that

$$\begin{aligned} \cos(x\cos\theta) &= J_0(x) - 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta \cdots \\ -\sin(x\cos\theta) &= -2J_1(x)\cos\theta + 2J_3(x)\cos 3\theta - 2J_5(x)\cos 5\theta \cdots \end{aligned}$$

22. Use Exercise 21 and Parseval's theorem to show that

$$1 = J_0(x)^2 + 2\sum_{m=1}^{\infty} J_m(x)^2$$

23. Show that for $m = 0, 1, 2, \ldots$

$$\begin{aligned} (-1)^m J_{2m}(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\cos 2m\theta\, d\theta \\ (-1)^m J_{2m+1}(x) &= \frac{1}{\pi}\int_0^{\pi} \sin(x\cos\theta)\cos(2m+1)\theta\, d\theta \end{aligned}$$

24. Show that

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \cdots \quad \text{and} \quad 0 = J_1(x) + J_3(x) + J_5(x) + \cdots$$

25. Show that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x)\cdots \quad \text{and} \quad \sin x = 2J_1(x) - 2J_3(x) + 2J_5(x)$$

26. Show that for any fixed x, $\lim_{m\to\infty} J_m(x) = 0$. (*Hint:* Apply the Riemann lemma from Sec. 1.2.)
27. Show that for each $p = 1, 2, \ldots$, $\lim_{m\to\infty} m^p J_m(x) = 0$. (*Hint:* Integrate by parts in Exercise 23.)
28. Let $f(x) = (1-x^2)^2$, $0 < x < 1$. Find the coefficients $\{A_n\}$ in the Fourier-Bessel expansion $f(x) = \sum_{n=1}^{\infty} A_n J_0(x x_n)$, where $J_0(x_n) = 0$. [*Hint:* Write $f(x)$ as a linear combination of $P_2(x)$ and $P_4(x)$.]
29. Find $P_6(x)$, the solution of $(xP_6')' = -xP_4$, $P_6(1) = 0$.
30. Let $f(x) = (1-x^2)^3, 0 < x < 1$. Find the coefficients $\{A_n\}$ in the Fourier-Bessel expansion $f(x) = \sum_{n=1}^{\infty} A_n J_0(x x_n)$, where $J_0(x_n) = 0$.
31. Compute $P_8(x)$, $P_{10}(x)$.
32. Let $f(\rho) = \rho$, $0 < \rho < 1$. Find the coefficients $\{A_n\}$ in the Fourier-Bessel expansion $f(\rho) = \sum_{n=1}^{\infty} A_n J_1(\rho x_n)$, where $J_1(x_n) = 0$.
33. Obtain the sum of the series $F_3(\rho) = \sum_{n=1}^{\infty} J_1(\rho x_n)/x_n^3 J_2(x_n)$, where $J_1(x_n) = 0$. [*Hint:* Use Exercise 32 to show that $\nabla^2(F_3(\rho)\cos\varphi) = -(\rho/2)\cos\varphi$.]
34. Obtain the sum of the series $F_5(\rho) = \sum_{n=1}^{\infty} J_1(\rho x_n)/x_n^5 J_2(x_n)$, where $J_1(x_n) = 0$.

35. Let $y(\rho)$ be a solution of the Bessel equation (3.2.1). Define a new function by $z(\rho) = \rho^{-s}y(\rho^r)$, where r and s are constants with $r > 0$. Show that $z(\rho)$ satisfies the differential equation

$$z''(\rho) + \frac{2s+1+r(d-2)}{\rho} z'(\rho) + \left[\lambda r^2 \rho^{(s+2)(r-1)} - \frac{r^2m^2+s^2+rs(d-2)}{\rho^2}\right] z(\rho) = 0$$

[*Hint:* Let $t = \rho^r$, $y(t) = t^{s/r}z(t^{1/r})$ and express $y'(t)$, $y''(t)$ in terms of z, z', z''.]

3.3. The Vibrating Drumhead

As a first application of Bessel functions, we study the small transverse vibrations of a circular membrane whose perimeter is fixed. This gives a mathematical model of a drumhead and is closely related to the rectangular membrane that was treated in Sec. 2.5.

3.3.1. Wave equation in polar coordinates. To be specific, suppose that the drumhead occupies the disc $x^2 + y^2 \le a^2$. Let $u(x, y; t)$ be the displacement of the point (x, y) at the time instant t. By an argument entirely similar to the derivation of the one-dimensional wave equation, it is seen that $u(x, y; t)$ satisfies the partial differential equation

$$u_{tt} = c^2(u_{xx} + u_{yy}) \tag{3.3.1}$$

where c is a positive constant, expressible in terms of the mass, area, and tension of the drumhead. The wave equation (3.3.1) is second order in time, and therefore it is natural to specify two initial conditions:

$$u(x, y; 0) = u_1(x, y) \quad \text{if} \quad x^2 + y^2 < a^2 \tag{3.3.2}$$

$$u_t(x, y; 0) = u_2(x, y) \quad \text{if} \quad x^2 + y^2 < a^2 \tag{3.3.3}$$

These correspond to the initial position and velocity of the drumhead. Finally, we have the boundary condition

$$u(x, y; t) = 0 \qquad \text{if} \quad x^2 + y^2 = a^2. \tag{3.3.4}$$

This means that the perimeter of the drumhead is fixed during the motion. The equation (3.3.1) and boundary conditions (3.3.4) are both homogeneous; therefore we can immediately proceed to look for separated solutions of the wave equation. To do this, we take polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. By abuse of notation, we let $u(\rho, \varphi; t)$ denote the displacement in polar coordinates. The wave equation takes the form

$$u_{tt} = c^2\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi}\right) \tag{3.3.5}$$

We look for separated solutions in the form

$$u(\rho,\varphi;t) = R(\rho)\Phi(\varphi)T(t) \tag{3.3.6}$$

Substituting into (3.3.5) and dividing by c^2u, we have

$$\frac{T''(t)}{c^2T(t)} = \frac{R''(\rho) + (1/\rho)R'(\rho)}{R(\rho)} + \frac{\Phi''(\varphi)}{\rho^2\Phi(\varphi)} \tag{3.3.7}$$

The left member depends only on t, while the right member depends only on (ρ,φ). Therefore each side equals the same constant, which we call $-\lambda$. This is rewritten in the form

$$T''(t) + \lambda c^2 T(t) = 0 \tag{3.3.8}$$

The remaining equations take the form

$$\rho^2\left[\lambda + \frac{R''(\rho) + (1/\rho)R'(\rho)}{R(\rho)}\right] = -\frac{\Phi''(\varphi)}{\Phi(\varphi)}$$

The right member depends only on φ and the left member depends only on ρ; therefore each equals the same constant μ. We rewrite this in the form

$$\Phi''(\varphi) + \mu\Phi(\varphi) = 0 \tag{3.3.9}$$

$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(\lambda - \frac{\mu}{\rho^2}\right)R(\rho) = 0 \tag{3.3.10}$$

Equations (3.3.8) to (3.3.10) are the ordinary differential equations whose solutions describe the vibrating membrane.

First we treat (3.3.9). The membrane occupies the disc whose equation in polar coordinates is $0 \le \rho < a$, $-\pi \le \varphi \le \pi$. Therefore Φ must be a smooth periodic function.

$$\Phi(-\pi) = \Phi(\pi)$$
$$\Phi'(-\pi) = \Phi'(\pi)$$

The general solution of (3.3.9) can be analyzed according to three cases. This is a Sturm-Liouville problem with periodic boundary conditions, which will now be solved. We have

Case 1: $\mu > 0$, $\Phi(\varphi) = A\cos\varphi\sqrt{\mu} + B\sin\varphi\sqrt{\mu}$
Case 2: $\mu = 0$, $\Phi(\varphi) = A + B\varphi$
Case 3: $\mu < 0$, $\Phi(\varphi) = A\cosh\varphi\sqrt{-\mu} + B\sinh\varphi\sqrt{-\mu}$

In case 1 the boundary conditions require that

$$A\cos(-\pi\sqrt{\mu}) + B\sin(-\pi\sqrt{\mu}) = A\cos\pi\sqrt{\mu} + B\sin\pi\sqrt{\mu}$$

$$-A\sqrt{\mu}\sin(-\pi\sqrt{\mu}) + B\sqrt{\mu}\cos(-\pi\sqrt{\mu}) = -A\sqrt{\mu}\sin\pi\sqrt{\mu} + B\sqrt{\mu}\cos\pi\sqrt{\mu}$$

The first equation states that $B\sin\pi\sqrt{\mu} = 0$ and the second equation states that $A\sqrt{\mu}\sin\pi\sqrt{\mu} = 0$. These are clearly satisfied if $\sqrt{\mu} = 1, 2, \ldots$. If $\sqrt{\mu}$ is not an integer, then we must have $A = B = 0$, yielding the trivial solution $\Phi(\varphi) = 0$. Thus we have found the general solution in case 1:

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi \qquad m = 1, 2, 3, \ldots \tag{3.3.11}$$

In case 2 the boundary conditions require that $B = 0$; hence we obtain the additional solution $\Phi(\varphi) = A$. This can be included in (3.3.11) if we let $m = 0$. In case 3 the boundary conditions cannot be satisfied. (This is left as an exercise.)

Returning to (3.3.10) with $\mu = m^2\,(m = 0, 1, 2, \ldots)$, we have

$$\begin{aligned} R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(\lambda - \frac{m^2}{\rho^2}\right)R(\rho) &= 0 \qquad 0 \le \rho \le a \\ R(a) &= 0 \end{aligned}$$

From the discussion of Bessel functions in Sec. 3.2, the solution is

$$R(\rho) = J_m(\rho\sqrt{\lambda})$$

where λ is chosen so that $J_m(a\sqrt{\lambda}) = 0$. Thus

$$a\sqrt{\lambda} = x_n^{(m)}$$

where $x_n^{(m)}$ are the positive zeros of the Bessel function J_m. Finally the time dependence is obtained by solving (3.3.8).

$$T(t) = \tilde{A}\cos ct\sqrt{\lambda} + \tilde{B}\sin ct\sqrt{\lambda} \qquad \lambda = \frac{(x_n^{(m)})^2}{a^2}$$

Therefore we have found separated solutions of the form

$$\boxed{u(\rho, \varphi; t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\left(\tilde{A}\cos\frac{ctx_n^{(m)}}{a} + \tilde{B}\sin\frac{ctx_n^{(m)}}{a}\right)} \tag{3.3.12}$$

3.3.2. Solution of initial-value problems. By taking a superposition of separated solutions, we may satisfy the initial conditions (3.3.2)–(3.3.3). These solutions become quite complicated; hence it is instructive to briefly study the separated solutions for small values of m, n. At time $t = 0$, the drumhead can be divided into zones, depending on whether u is positive or negative. The curves that divide the zones are called *nodal lines*. For concreteness we take $\tilde{A} = 1$, $\tilde{B} = 0$, $A = 1$, $B = 0$. Thus for $n = 1$, we have the diagrams of Fig. 3.3.1. For $n = 2$ the Bessel function has an interior zero, and the drumhead appears as in Fig. 3.3.2.

For larger values of (m, n) the diagrams become successively more complex. As time progresses, each of these profiles is multiplied by $\cos(ctx_n^{(m)}/a)$ a periodic function. But when we form a superposition of these separated solutions, the resulting solution is *no longer periodic*. Indeed, the numbers $x_n^{(m)}$ are not multiples

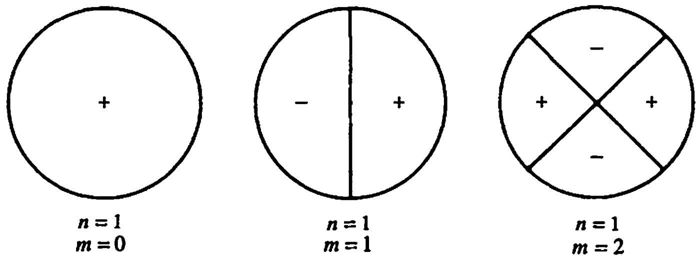


FIGURE 3.3.1 Nodal lines of a vibrating drumhead for $n = 1$.

of a fixed fundamental frequency, as was the case for the vibrating string, where we had separated solutions of the form

$$\sin \frac{n\pi x}{L}\left(\widetilde{A}\cos\frac{n\pi ct}{L} + \widetilde{B}\sin\frac{n\pi ct}{L}\right)$$

We now use the separated solutions (3.3.12) to solve the initial-value problem (3.3.2).

EXAMPLE 3.3.1. *Find the solution of the vibrating membrane problem in the case where* $u_2(\rho,\varphi) = 0$.

Solution. In this case we use the separated solutions (3.3.12), which satisfy the second initial condition, $\widetilde{B} = 0$. We write the solution as a formal sum

$$u(\rho,\varphi;t) = \sum_{m,n} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\cos\frac{ct x_n^{(m)}}{a}$$

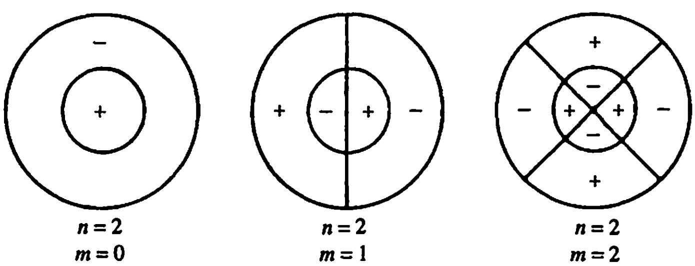


FIGURE 3.3.2 Nodal lines of a vibrating drumhead for $n = 2$.

The first initial condition requires that

$$\begin{aligned} u_1(\rho,\varphi) &= u(\rho,\varphi;0) \\ &= \sum_{m,n} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi) \\ &= \sum_{m=0}^{\infty}\cos m\varphi\sum_{n=1}^{\infty}A_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right) + \sum_{m=1}^{\infty}\sin m\varphi\sum_{n=1}^{\infty}B_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right) \end{aligned}$$

This is a Fourier series in φ, whose coefficients are Fourier-Bessel series with $\beta = 0$, $m = 0, 1, 2, \ldots$. Therefore, to solve the problem, we must expand $u_1(\rho,\varphi)$ in a series of this type and identify the coefficients A_{mn}, B_{mn}. •

The next example illustrates a specific case.

EXAMPLE 3.3.2. *Find the solution of the vibrating membrane problem in the case where* $u_2(\rho,\varphi) = 0$, $u_1(\rho,\varphi) = a^2 - \rho^2$, $0 < \rho < a$.

Solution. From Sec. 3.2 we have the Fourier-Bessel expansion

$$1 - x^2 = 8\sum_{n=1}^{\infty}\frac{J_0(x x_n^{(0)})}{(x_n^{(0)})^3 J_1(x_n^{(0)})} \qquad 0 < x < 1$$

Making the substitution $x = \rho/a$, we have the required expansion

$$u_1(\rho,\varphi) = a^2 - \rho^2 = 8a^2\sum_{n=1}^{\infty}\frac{J_0(\rho x_n^{(0)}/a)}{(x_n^{(0)})^3 J_1(x_n^{(0)})}$$

We see that $B_{mn} = 0$ for all m, n and $A_{mn} = 0$ for $m > 0$, while $A_{0n} = 8a^2/(x_n^{(0)})^3 J_1(x_n^{(0)})$. The solution of the problem is

$$u(\rho,\varphi;t) = 8a^2\sum_{n=1}^{\infty}\frac{J_0(\rho x_n^{(0)}/a)}{(x_n^{(0)})^3 J_1(x_n^{(0)})}\cos\frac{ctx_n^{(0)}}{a} \quad \bullet$$

EXAMPLE 3.3.3. *Find the solution of the vibrating membrane problem in the case where* $u_2(\rho,\varphi) = 0$, $u_1(\rho,\varphi) = J_3(\rho x_1^{(3)}/a)\cos 3\varphi$.

Solution. In this case the initial data are already written as a Fourier-Bessel series as in the previous example, with $A_{31} = 1$ and all other coefficients zero. Therefore the solution of the problem is

$$u(\rho,\varphi;t) = J_3\left(\frac{\rho x_1^{(3)}}{a}\right)\cos 3\varphi\cos\frac{ctx_1^{(3)}}{a} \quad \bullet$$

3.3.3. Implementation with Mathematica. We can use Mathematica to draw three-dimensional graphs of the vibrating drum, beginning with formula (3.3.12). To be specific, we take $c = 1$, $a = 1$, $A = \tilde{A} = 1$, and $B = \tilde{B} = 0$. Then the formula becomes

$$u(\rho, \varphi; t) = J_m(\rho x_n^{(m)}) \cos(m\varphi) \cos(t x_n^{(m)})$$

In Mathematica, this is written

```
u[m_,x_,rho_,phi_,t_]:=BesselJ[m,rho x]Cos[m phi]Cos[t x]
```

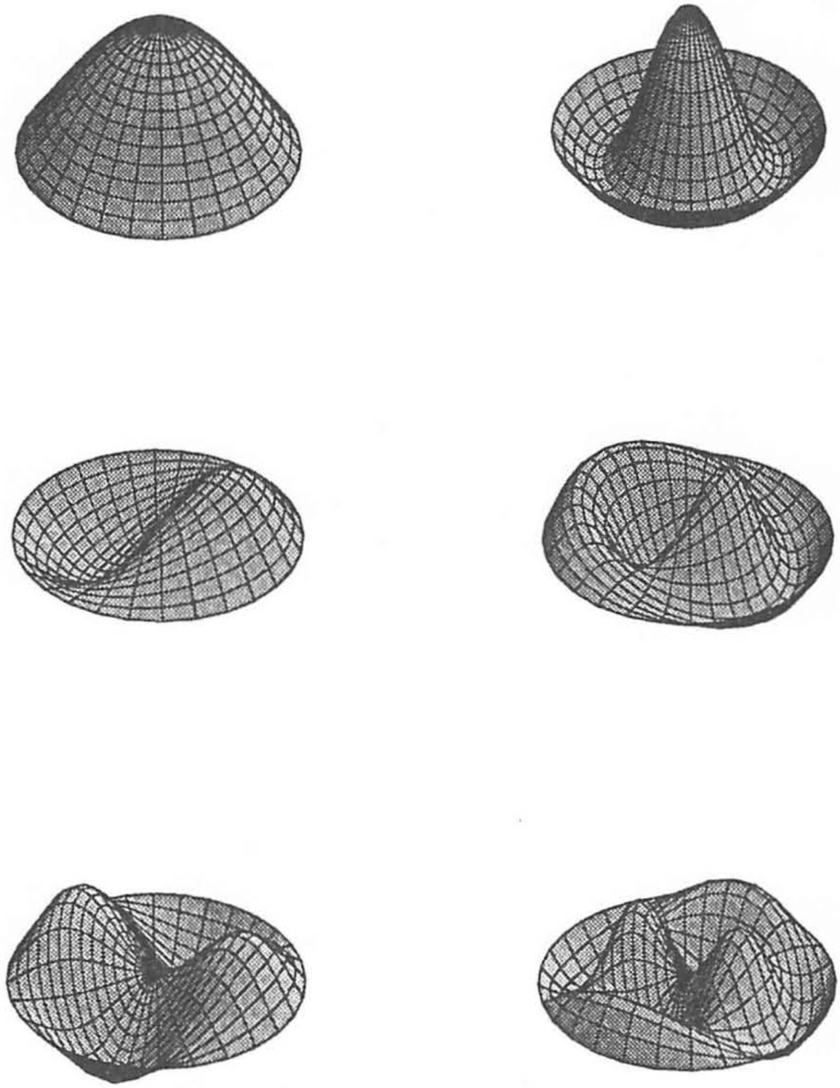
To graph u, we need the command **CylindricalPlot3D**, which is part of the package **NewParametricPlot3D**. We use this to define a plot-valued function **uu**, defined as

```
uu[m_,x_,t_]=CylindricalPlot3D[u[m,x,rho,phi,t],
              {rho,0,1},{phi,0,2Pi,Pi/15}, boxed->False]
```

Then the three-dimensional nodal plots corresponding to Fig. 3.3.1 and Fig. 3.3.2 are generated by the commands

```
uu[0,2.40482,0]        uu[1,3.83171,0]        uu[2,5.13562,0]
uu[0,5.52007,0]        uu[1,7.01559,0]        uu[2,8.41724,0]
```

and given as follows:



EXERCISES 3.3

1. Suppose the drumhead is under the influence of gravity. The wave equation takes the form

$$u_{tt} = c^2(u_{xx} + u_{yy}) - g \qquad x^2 + y^2 \le a^2$$

Find the steady-state solution of the form $u = U(\rho)$ that satisfies the boundary condition $U(a) = 0$.
2. Let $f(\varphi) = A\cosh\varphi\sqrt{-\mu} + B\sinh\varphi\sqrt{-\mu}$, $\mu < 0$. Show that if f satisfies the boundary conditions $f(0) = f(2\pi)$, $f'(0) = f'(2\pi)$, then $A = B = 0$.
3. Sketch the drumhead profiles for $n = 3, 4$.
4. Find the solution of the vibrating membrane problem in the case where $u_1(\rho,\varphi) = F_1(\rho)$, $u_2(\rho,\varphi) = 0$.
5. Find the solution of the vibrating membrane problem in the case where $u_1(\rho,\varphi) = 0$ and $u_2(\rho,\varphi) = F_2(\rho)$.
6. Find the solution of the vibrating membrane problem in the case where $u_1(\rho,\varphi) = 0$ and $u_2(\rho,\varphi) = 1$, $0 < \rho < a$.
7. Find the solution of the vibrating membrane problem in the case where $u_1(\rho,\varphi) = 0$ and $u_2(\rho,\varphi) = a^2 - \rho^2$, $0 < \rho < a$.
8. Find the solution of the vibrating membrane problem in the case where $u_1(\rho,\varphi) = 0$, $u_2(\rho,\varphi) = J_3(\rho x_1^{(3)}/a)\cos 3\varphi$.
9. Consider a membrane in the shape of a half-circle $0 \le \varphi \le \pi$, $0 \le \rho \le a$. Show that the separated solutions of the wave equation with zero boundary conditions have the form

$$u(\rho,\varphi;t) = \left(A\cos\frac{ctx_n^{(m)}}{a} + B\sin\frac{ctx_n^{(m)}}{a}\right) J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\sin m\varphi$$

where $m = 1, 2, \ldots$ and $x_n^{(m)}$ are the positive zeros of the Bessel function J_m.

3.4. Heat Flow in the Infinite Cylinder

In this section and the next we study initial-value problems for the heat equation in cylindrical coordinates. To solve these problems, we shall combine the five-stage method of Chapter 2 with the separated solutions expressed in terms of Bessel functions. The present section is devoted to solutions in the infinite cylinder $0 \le \rho < \rho_{\max}$, $-\infty < z < \infty$, $-\pi \le \varphi \le \pi$, which are independent of z. In the next section we consider the case of the finite cylinder $0 < z < L$, where the solutions depend upon z.

3.4.1. Separated solutions. To begin, we look for separated solutions of the heat equation in cylindrical coordinates, independent of z. We have

$$u(\rho, \varphi; t) = R(\rho)\Phi(\varphi)T(t) \tag{3.4.1}$$

$$u_t = K\nabla^2 u = K\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi}\right) \tag{3.4.2}$$

Substituting (3.4.1) into (3.4.2) and dividing by Ku, we have

$$\frac{T'(t)}{KT(t)} = \frac{R'' + (1/\rho)R'}{R} + \frac{(1/\rho^2)\Phi''}{\Phi}$$

The right side depends on (ρ, φ), whereas the left side depends on t. Therefore each is a constant, which we call $-\lambda$. Thus

$$T'(t) + \lambda K T(t) = 0 \tag{3.4.3}$$

Likewise the ratio Φ''/Φ is a constant, which we call $-\mu$. Thus we have the equations

$$\Phi'' + \mu\Phi = 0 \tag{3.4.4}$$

$$R'' + \frac{1}{\rho}R' + \left(\lambda - \frac{\mu}{\rho^2}\right)R = 0 \tag{3.4.5}$$

Equation (3.4.3) is solved by $T(t) = e^{-\lambda K t}$. Equation (3.4.4) with the periodic boundary conditions $\Phi(-\pi) = \Phi(\pi)$, $\Phi'(-\pi) = \Phi'(\pi)$ is solved by $\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi$, $m = 0, 1, 2, \ldots$. Equation (3.4.5) is a form of Bessel's equation with $d = 2$. The power series solution is $J_m(\rho\sqrt{\lambda})$, and the second solution may be obtained from (3.2.22). Thus we have obtained the separated solutions of the heat equation in cylindrical coordinates.

$$u(\rho, \varphi; t) = J_m(\rho\sqrt{\lambda})(A\cos m\varphi + B\sin m\varphi)e^{-\lambda K t} \tag{3.4.6}$$

The separation constant λ is obtained from the boundary conditions of the problem.

3.4.2. Initial-value problems in a cylinder. In this subsection we present four worked examples of heat flow problems in the infinite cylinder $0 < \rho < \rho_{\max}$, $-\pi < \phi < \pi$, $-\infty < z < \infty$.

EXAMPLE 3.4.1. *Find the solution of the heat equation in the infinite cylinder $0 \le \rho < \rho_{\max}$, satisfying the boundary condition $u(\rho_{\max}, \varphi; t) = 0$ and the initial condition $u(\rho, \varphi; 0) = f(\rho, \varphi)$.*

Solution. The steady-state solution is zero. The solution must be finite at $\rho = 0$, so we do not take the second solution of Bessel's equation. The required separated solutions are $J_m(\rho\sqrt{\lambda})(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt}$. The boundary condition requires that $J_m(\rho_{\max}\sqrt{\lambda}) = 0$; thus $\rho_{\max}\sqrt{\lambda} = x_n^{(m)}$, a positive zero of the Bessel function J_m. The solution takes the form

$$u(\rho,\varphi;t) = \sum_{m,n} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{\max}}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\exp\left[\frac{-(x_n^{(m)})^2 Kt}{\rho_{\max}^2}\right]$$

To satisfy the initial condition, we must have

$$f(\rho,\varphi) = \sum_{m,n} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{\max}}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)$$

This is a Fourier series in $(\cos m\varphi, \sin m\varphi)$, whose coefficients are Fourier-Bessel expansions with $\beta = 0$, $m = 0, 1, 2, \ldots$. The problem is completely solved once we have an expanded $f(\rho,\varphi)$ in a series of this type. •

The next example gives a specific case.

EXAMPLE 3.4.2. *Find the solution of the heat equation in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max},\varphi;t) = 0$ and the initial condition $u(\rho,\varphi;0) = 1$. Find the relaxation time.*

Solution. We use the method of the previous example with $f(\rho,\varphi) = 1$. In this case, we use the Fourier-Bessel expansion

$$1 = 2\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n J_1(x_n)} = 2\sum_{n=1}^{\infty}\frac{J_0(\rho x_n/\rho_{\max})}{x_n J_1(x_n)} \qquad 0 < \rho < \rho_{\max}$$

where $J_0(x_n) = 0$ and we have made the substitution $x = \rho/\rho_{\max}$. Therefore the solution of the initial-value problem is

$$u(\rho,\varphi;t) = 2\sum_{n=1}^{\infty}\frac{J_0(\rho x_n/\rho_{\max})}{x_n J_1(x_n)}\exp\left[\frac{-x_n^2 Kt}{\rho_{\max}^2}\right]$$

For each $t > 0$, the series for u, u_ρ, $u_{\rho\rho}$, u_t converge uniformly; hence u is a rigorous solution. When $t \to \infty$, the solution tends to zero, the steady-state solution. The relaxation time can be computed from the first term of the series.

$$\begin{aligned} -\frac{1}{\tau} &= \lim_{t\to\infty}\frac{1}{t}\ln|u(\rho;\varphi;t)| = -\frac{x_1^2 K}{\rho_{\max}^2} \\ \tau &= \frac{\rho_{\max}^2}{Kx_1^2} = 0.1729\frac{\rho_{\max}^2}{K} \end{aligned}$$ •

In the next example we add a nonhomogeneous boundary condition and a nonhomogeneous term to the heat equation, representing a heat source. To solve problems of this type, we first find the steady-state solution, according to the five-stage method of Chapter 2.

EXAMPLE 3.4.3. *Find the solution of the following heat equation in the infinite cylinder* $0 \le \rho < \rho_{\max}$*:*

$$\begin{aligned} u_t &= K\nabla^2 u + \sigma & & t > 0, 0 \le \rho < \rho_{\max}, -\pi \le \varphi \le \pi \\ u(\rho_{\max}, \varphi; t) &= T_1 & & t > 0, -\pi \le \varphi \le \pi \\ u(\rho, \varphi; 0) &= T_2 & & 0 < \rho < \rho_{\max}, -\pi \le \varphi \le \pi \end{aligned}$$

where σ, T_1, *and* T_2 *are positive constants.*

Solution.

Stage 1. To find the steady-state solution, we try $U = U(\rho)$, independent of φ (since the boundary condition is independent of φ). We obtain the ordinary differential equation $K[U'' + (1/\rho)U'] + \sigma = 0$ with the boundary condition $U(\rho_{\max}) = T_1$. The general solution of the equation is $U = -\sigma\rho^2/4K + A + B\log\rho$. We must have $U(\rho_{\max}) = T_1$, $U(0)$ finite; hence $B = 0$, $A = T_1 + (\sigma\rho_{\max}^2/4K)$. The steady-state solution is

$$U(\rho) = T_1 + \frac{\sigma}{4K}(\rho_{\max}^2 - \rho^2)$$

Stage 2. We use the steady-state solution to transform to a homogeneous equation with homogeneous boundary conditions. Thus, letting $v = u - U$, we have $v_t = K\nabla^2 v$, $v(\rho_{\max}, \varphi; t) = 0$, $v(\rho, \varphi; 0) = T_2 - U(\rho)$.

Stage 3. We look for v as a sum of separated solutions of the homogeneous equation satisfying the homogeneous boundary conditions. These were found in Example 3.4.1:

$$v(\rho, \varphi; t) = \sum_{m,n} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{\max}}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\exp\left[-\frac{(x_n^{(m)})^2 Kt}{\rho_{\max}^2}\right]$$

To satisfy the initial condition, we must have

$$T_2 - U(\rho) = v(\rho, \varphi; 0) = \sum_{m,n} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{\max}}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)$$

To obtain the required expansion of $T_2 - U(\rho)$, recall the Fourier-Bessel expansions from Sec. 3.2.

$$\begin{aligned} 1 &= 2\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n J_1(x_n)} & & 0 < x < 1, J_0(x_n) = 0 \\ 1 - x^2 &= 8\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n^3 J_1(x_n)} & & 0 < x < 1, J_0(x_n) = 0 \end{aligned}$$

Letting $x = \rho/\rho_{\max}$, we have

$$\begin{aligned} T_2 - U(\rho) &= (T_2 - T_1) - \frac{\sigma\rho_{\max}^2}{4K}\left(1 - \frac{\rho^2}{\rho_{\max}^2}\right) \\ &= \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right) \qquad 0 < \rho < \rho_{\max} \end{aligned}$$

where

$$A_n = \frac{2(T_2 - T_1)/x_n - 2\sigma\rho_{\max}^2/Kx_n^3}{J_1(x_n)} \qquad n = 1, 2, \ldots$$

The solution of the original problem is therefore

$$u(\rho, \varphi; t) = U(\rho) + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right) \exp\left[-\frac{x_n^2 Kt}{\rho_{\max}^2}\right] \qquad J_0(x_n) = 0$$

Stage 4. For each $t > 0$, the series for u, u_ρ, $u_{\rho\rho}$, u_t converge uniformly for $0 \le \rho \le \rho_{\max}$; hence u satisfies the heat equation.

Stage 5. If $A_1 \neq 0$, the relaxation time is given by the first term of the series: $\tau = \rho_{\max}^2/(x_1^2 K)$. •

The next example illustrates the possibility of boundary conditions that involve radiation of heat to the exterior of the cylinder.

EXAMPLE 3.4.4. *Find the solution of the following heat equation in the infinite cylinder* $0 \le \rho < \rho_{\max}$*:*

$$\begin{aligned} u_t &= K\nabla^2 u + \sigma \qquad t > 0, 0 \le \rho < \rho_{\max} \\ -ku_\rho|_{\rho=\rho_{\max}} &= h(u - T_1)|_{\rho=\rho_{\max}} \\ u(\rho, \varphi; 0) &= T_2 \end{aligned}$$

Here σ, K, k, h, T_1, *and* T_2 *are positive constants.*

Solution.

Stage 1. We look for the steady-state solution in the form $U = U(\rho)$. We obtain the ordinary differential equation

$$K\left[U'' + \frac{1}{\rho}U'\right] + \sigma = 0$$

with the boundary condition $-kU'(\rho_{\max}) = h[U(\rho_{\max}) - T_1]$. The general solution is $U(\rho) = -(\sigma\rho^2/4K) + A + B\log\rho$. The condition $U(0)$ finite requires $B = 0$, while the boundary condition requires

$$k\sigma\rho_{\max}/2K = h[A - (\sigma\rho_{\max}^2/4K) - T_1], \quad A = T_1 + \sigma\rho_{\max}k/2hK + \sigma\rho_{\max}^2/4K$$

The steady-state solution is

$$U(\rho) = T_1 + \frac{\sigma\rho_{\max}k}{2hK} + \frac{\sigma}{4K}(\rho_{\max}^2 - \rho^2)$$

Stage 2. We introduce the function $v = u - U$, which satisfies the homogeneous equation $v_t = K\nabla^2 v$ with the homogeneous boundary condition $-kv_\rho \,|_{\rho=\rho_{\max}} = hv\,|_{\rho=\rho_{\max}}$.

Stage 3. Previously we found the separated solutions of the heat equation $J_m(\rho\sqrt{\lambda})(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)e^{-\lambda Kt}$. To satisfy the new boundary condition, we must have $k\sqrt{\lambda}J'_m(\rho_{\max}\sqrt{\lambda}) + hJ_m(\rho_{\max}\sqrt{\lambda}) = 0$. It was shown in Sec. 3.2 that this equation has infinitely many solutions $\rho_{\max}\sqrt{\lambda} = x_n^{(m)}$, $n = 1, 2, \ldots$. To satisfy the initial condition, we take a sum of these separated solutions. Since the initial condition is independent of φ, we may write

$$v = v(\rho; t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right)\exp\left[-\frac{x_n^2 Kt}{\rho_{\max}^2}\right]$$

To satisfy the initial condition, we must have

$$T_2 - T_1 - \frac{\sigma\rho_{\max}k}{2hK} - \frac{\sigma}{4K}(\rho_{\max}^2 - \rho^2) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right)$$

The Fourier coefficients $\{A_n\}$ are obtained from the Fourier-Bessel expansion of Example 3.2.9 with $\cot\beta = h\rho_{\max}/k$.

$$\begin{aligned} 1 &= \frac{2k\rho_{\max}}{h}\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{[x_n^2 + (h\rho_{\max}/k)^2]J_0(x_n)} \qquad 0 < x < 1 \\ 1 - x^2 &= \frac{8k\rho_{\max}}{h}\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n^2[x_n^2 + (h\rho_{\max}/k)^2]J_0(x_n)} \qquad 0 < x < 1 \end{aligned}$$

Making the substitution $x = \rho/\rho_{\max}$, we have

$$A_n = \frac{2}{[x_n^2 + (h\rho_{\max}/k)^2]J_0(x_n)}\left(T_2 - T_1 - \frac{\sigma\rho_{\max}k}{2hK} - \frac{\sigma\rho_{\max}^2}{Kx_n^2}\right)$$

The solution of the original problem is written in the form

$$u(\rho; t) = U(\rho) + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right)\exp\left[-\frac{x_n^2 Kt}{\rho_{\max}{}^2}\right]$$

Stage 4. For $t > 0$, the series for u, u_ρ, $u_{\rho\rho}$, and u_t converge uniformly for $0 \le \rho \le \rho_{\max}$. Hence u is a rigorous solution of the heat equation.

Stage 5. When $t \to \infty$, $u(\rho; t)$ tends to $U(\rho)$, the steady-state solution. The relaxation time $\tau = \rho_{\max}^2/x_1^2 K$, where $(k/\rho_{\max})x_1 J'_0(x_1) + hJ_0(x_1) = 0$, $A_1 \ne 0$. •

3.4.3. Initial-value problems between two cylinders. As our final application we consider heat flow between two cylinders of inner radius ρ_1 and outer radius ρ_2. To solve problems of this type, we need both solutions of Bessel's equation. We consider the following initial-boundary-value problem:

$$\begin{aligned} u_t &= K\nabla^2 u & & t>0, \rho_1<\rho<\rho_2, -\pi \le \varphi \le \pi \\ u(\rho_1,\varphi;t) &= T_1 & & t>0, -\pi\le\varphi\le\pi \\ u(\rho_2,\varphi;t) &= T_2 & & t>0, -\pi\le\varphi\le\pi \\ u(\rho,\varphi;0) &= f(\rho,\varphi) & & \rho_1<\rho<\rho_2, -\pi\le\varphi\le\pi \end{aligned}$$

Stage 1. The steady-state equation is $\nabla^2 u = 0$, with the above boundary conditions. Since these are independent of φ, we look for the steady-state solution in the form $U = U(\rho)$. The general solution of $\nabla^2 u = 0$ is $U = A + B\ln\rho$. The boundary conditions require $A + B\ln\rho_1 = T_1$, $A + B\ln\rho_2 = T_2$; thus

$$U(\rho) = T_1 + (T_2 - T_1)\frac{\ln\rho - \ln\rho_1}{\ln\rho_2 - \ln\rho_1}$$

Stage 2. We use the steady-state solution to transform to homogeneous boundary conditions. Thus, letting $v = u-U$, we have $v_t = K\nabla^2 v$, $v(\rho_1,\varphi;t) = 0$, $v(\rho_2,\varphi;t) = 0$, $v(\rho,\varphi;0) = f(\rho,\varphi) - U(\rho)$.

Stage 3. The separated solutions of the heat equation in cylindrical coordinates are $R(\rho)(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt}$, where $m = 0,1,2,\ldots$ and $R(\rho)$ is a solution of Bessel's equation with $d = 2$ satisfying the boundary conditions $f_1(\rho_1) = 0$, $f_1(\rho_2) = 0$. These are obtained in the form

$$R(\rho,\lambda) = J_m(\rho\sqrt{\lambda})Y_m(\rho_1\sqrt{\lambda}) - J_m(\rho_1\sqrt{\lambda})Y_m(\rho\sqrt{\lambda})$$

Indeed, this combination is a solution of the Bessel equation and satisfies the boundary condition $R(\rho_1,\lambda) = 0$. The theory of Sec. 1.6 guarantees an infinite number of eigenvalues $\lambda_n^{(m)}$, so that $R(\rho_2,\lambda_n^{(m)}) = 0$. The eigenvalues and normalized eigenfunctions $R_n^m(\rho)$ satisfy the asymptotic relations

$$\begin{aligned} \sqrt{\lambda_n^{(m)}} &= n\pi/(\rho_2-\rho_1) + O(1/n) \quad n\to\infty \\ \sqrt{\rho}R_n^m(\rho) &= \left(\frac{2}{\rho_2-\rho_1}\right)^{1/2}\sin\frac{n\pi(\rho-\rho_1)}{\rho_2-\rho_1} + O\left(\frac{1}{n}\right) \quad n\to\infty \end{aligned}$$

The solution is written in the form

$$v(\rho,\varphi;t) = \sum_{m,n} R_n^m(\rho)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)e^{-\lambda_n^{(m)}Kt}$$

The eigenfunctions R_n^m are orthogonal with respect to the weight $\rho\,d\rho$ for different values of n, meaning that

$$\int_{\rho_1}^{\rho_2} R_{n_1}^m R_{n_2}^m \rho\,d\rho = 0 \qquad n_1 \ne n_2$$

Therefore we can obtain the Fourier coefficients for $m \neq 0$ by the formulas

$$\int_{-\pi}^{\pi}\int_{\rho_1}^{\rho_2} [f(\rho,\varphi) - U(\rho)]R_n^{(m)}(\rho)\cos m\varphi\,\rho\,d\rho\,d\varphi = \pi\,A_{mn}\int_{\rho_1}^{\rho_2} R_n^{(m)}(\rho)^2\rho\,d\rho$$

$$\int_{-\pi}^{\pi}\int_{\rho_1}^{\rho_2} [f(\rho,\varphi) - U(\rho)]R_n^{(m)}(\rho)\sin m\varphi\,\rho\,d\rho\,d\varphi = \pi\,B_{mn}\int_{\rho_1}^{\rho_2} R_n^{(m)}(\rho)^2\rho\,d\rho$$

If $m = 0$ one must replace π by 2π on the right sides.

Stage 4. The formal solution to our problem is

$$u(\rho,\varphi;t) = U(\rho) + \sum_{m,n} R_n^{(m)}(\rho)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)e^{-\lambda_n^{(m)}Kt}$$

where the Fourier coefficients A_{mn}, B_{mn} are obtained in stage 3. To verify that this series converges and represents a solution of the heat equation, we consider the special case of radially symmetric solutions, where $f(\rho,\varphi) = f(\rho)$, independent of φ. In this case the solution assumes the simpler form

$$u(\rho;t) = U(\rho) + \sum_{n=1}^{\infty} A_n R_n(\rho)e^{-\lambda_n Kt}$$

where $\lambda_n = \lambda_n^{(0)}$, $R_n = R_n^{(0)}$, $A_n = A_{0n}$. Suppose $|f(\rho)| \le M$, a constant. The Fourier coefficients can be estimated by

$$\begin{aligned} 2\pi|A_n| &= \left|\int_{\rho_1}^{\rho_2} [f(\rho) - U(\rho)]R_n(\rho)\rho\,d\rho\right| \\ &\le M_2\int_{\rho_1}^{\rho_2} \rho|R_n(\rho)|d\rho \\ &\le M_2\left(\frac{\rho_2^2 - \rho_1^2}{2}\right)^{1/2} \qquad M_2 = M + T_1 + T_2 \end{aligned}$$

where we have used Schwarz's inequality for integrals and the normalization $\int_{\rho_1}^{\rho_2} R_n(\rho)^2\rho\,d\rho = 1$. Therefore the terms of the series $\sum_{n=1}^{\infty} A_n R_n(\rho)e^{-\lambda_n Kt}$ are bounded by the terms of the series $M_3\sum_{n=1}^{\infty}(e^{-at})^n$, $a = \pi K/(\rho_2 - \rho_1)$, $M_3 = M_2(1 + \rho_2/\rho_1)^{1/2}$. Likewise the series for u_ρ, $u_{\rho\rho}$, and u_t are bounded by convergent numerical series and hence are uniformly convergent.

Stage 5. When $t \to \infty$, the solution $u(\rho,\varphi;t)$ tends to the steady-state solution $U(\rho)$. To compute the relaxation time, we again restrict attention to radially symmetric solutions, where $A_{mn} = 0$ for $m \neq 0$ and $B_{mn} = 0$. In this case the eigenvalues $\{\lambda_n\}$ have been tabulated numerically for various values of the ratio ρ_1/ρ_2. The following table lists some representative values.[2]

[2]Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1972, p. 415.

ρ_1/ρ_2	$\rho_1\sqrt{\lambda_1}$	$\rho_1\sqrt{\lambda_2}$	$\rho_1\sqrt{\lambda_3}$
0.8	12.56	25.13	37.70
0.6	4.70	9.42	14.13
0.4	2.07	4.18	6.27
0.2	0.76	0.69	2.35
0.1	0.33	0.53	1.04

For example, if $\rho_1 = 3$, $\rho_2 = 5$, then we can read from the second line of the table that $\lambda_1^0 \approx 2.45$, $\lambda_2^0 \approx 9.86$, $\lambda_3^0 = 22.18$. The relaxation time can be obtained from the first term of the series for $u(\rho;t)$; thus $\lambda_1 K\tau = 1$. For example, if $\rho_1 = 3$, $\rho_2 = 5$, then $\tau = 1/2.45K = 0.41/K$ to two decimal places.

3.4.4. Implementation with Mathematica. We can use Mathematica to numerically determine the eigenvalues of the heat flow problem in an annular region. We begin with the separated solutions

$$u(\rho,\varphi;t) = R(\rho)(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt}$$

where

$$R(\rho) = R(\rho,\lambda) = J_m(\rho\sqrt{\lambda})Y_m(\rho_1\sqrt{\lambda}) - J_m(\rho_1\sqrt{\lambda})Y_m(\rho\sqrt{\lambda})$$

The eigenvalue λ must be chosen so that $R(\rho_2,\lambda) = 0$. This requires that

$$J_m(\rho_2\sqrt{\lambda})Y_m(\rho_1\sqrt{\lambda}) - J_m(\rho_1\sqrt{\lambda})Y_m(\rho_2\sqrt{\lambda}) = 0$$

In order to render this in Mathematica, we consider the example $\rho_1 = 3, \rho_2 = 5$. We define

```
In[1]=a:=BesselJ[m,5 p] BesselY[m, 3 p]-BesselJ[m,3 p] BesselY[m, 5 p]
```

To determine the first three eigenvalues, we look for the first zero corresponding to $m = 0, 1, 2$. To determine the numerical values, we can use Mathematica's **FindRoot** command:

```
In[2]:=eigs=Table[FindRoot[a/.m->i,{p,1}],{i,0,2}]
Out[2]={{p->1.56569},{p->1.58602},{p->1.64543}}
```

3.4.5. Time-periodic heat flow in the cylinder. It is also possible to find solutions of the heat equation in the infinite cylinder corresponding to a time-periodic boundary condition, as we did for the slab in Sec. 2.1. Specifically, we have the boundary-value problem

$$\begin{aligned} u_t &= K\nabla^2 u = K\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho\right) && -\infty < t < \infty, 0 \le \rho \le a \\ u(a;t) &= A_0 + A_1\cos\omega t && -\infty < t < \infty \end{aligned}$$

We look for complex separated solutions in the form

$$u(\rho;t) = R(\rho)e^{-i\omega t}$$

Substituting in the heat equation and separating the terms, we find

$$-i\frac{\omega}{K} = \frac{R''(\rho) + R'(\rho)/\rho}{R}$$

Therefore $R(\rho)$ must be a solution of the equation

$$R'' + \frac{R'}{\rho} + i(\omega/K)R = 0$$

This is a form of Bessel's equation with a purely imaginary eigenvalue parameter: $\lambda = i\omega/K$. The solution that remains regular at $\rho = 0$ is the Bessel function

$$R(\rho) = J_0(\rho\sqrt{\lambda}) = J_0(\rho c(1+i)) \qquad c = \sqrt{\omega/2K}$$

In order to find the (real-valued) solution that satisfies the boundary condition, we take the real part and obtain the solution of the boundary-value problem in the form

$$u(\rho;t) = A_0 + A_1 \operatorname{Re}\left(e^{-i\omega t}\frac{J_0(\rho c(1+i))}{J_0(ac(1+i))}\right)$$

This can be related to the one-dimensional solution in the limiting case when a is large and $z := a - \rho$ is fixed. This requires knowledge of the asymptotic behavior of the Bessel function for complex arguments:[3]

$$J_0((1+i)s) = \sqrt{\frac{1}{2\pi s}}e^{s}e^{is}e^{i\pi/8}(1 + O(1/s)) \qquad s \to \infty$$

Applying this with $s = \rho c$, $s = ac$ and simplifying, we get the result

$$\begin{aligned}\frac{J_0((\rho c(1+i))}{J_0((ac(1+i))} &\doteq \frac{e^{\rho c}e^{i\rho c}}{e^{ac}e^{iac}}(1 + O(1/a)) \\ &= e^{-cz}e^{-icz}(1 + O(1/a))\end{aligned}$$

Hence to within terms of order a^{-1},

$$\begin{aligned}u(\rho;t) &= A_0 + A_1 \operatorname{Re}\left(e^{-cz}e^{-i(\omega t - cz)}\right) \\ &= A_0 + A_1 e^{-cz}\cos(\omega t - cz)\end{aligned}$$

which is the solution found in Sec 2.1 for the slab in rectangular coordinates.

We close by remarking that the periodic solution obtained above is *unique*. More precisely, suppose that $\tilde{u}(\rho;t)$ is another time-periodic solution satisfying the same boundary conditions. Then the difference $v = u - \tilde{u}$ is a time-periodic

[3] Ibid.

solution of the heat equation in the cylinder with $v(a;t) = 0$ for all t. But any solution is represented as

$$v(\rho;t) = \sum_{n=1}^{\infty} A_n J_0(\rho x_n/a) e^{-\lambda_n K t}$$

This function will be time-periodic if and only if $A_n \equiv 0$—hence $v(\rho;t) \equiv 0$, which was to be proved.

EXERCISES 3.4

1. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max}, \varphi; t) = 0$ and the initial condition $u(\rho, \varphi; 0) = \rho_{\max}^2 - \rho^2$.
2. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max}, \varphi; t) = 1$ and the initial condition $u(\rho, \varphi; 0) = 0$. Find the relaxation time.
3. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max}, \varphi; t) = 1 + \frac{1}{2}\cos\varphi$ and the initial condition $u(\rho, \varphi; 0) = 0$. (*Hint:* Use Example 3.2.8.)
4. Find the solution of the heat equation $u_t = K\nabla^2 u + \sigma$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max}, \varphi; t) = T_1$ and the initial condition $u(\rho, \varphi; 0) = T_2(1 - \rho^2/\rho_{\max}^2)$.
5. Find the separated solutions of the heat equation $u_t = K\nabla^2 u$ in the infinite half-cylinder $0 \le \rho < \rho_{\max}$, $0 < \varphi < \pi$ satisfying the boundary conditions $u(\rho, 0; t) = 0$, $u(\rho, \pi; t) = 0$, $u(\rho_{\max}, \varphi; t) = 0$.
6. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite half-cylinder $0 \le \rho < \rho_{\max}$, $0 < \varphi < \pi$ satisfying the boundary conditions $u(\rho, 0; t) = 0$, $u(\rho, \pi; t) = 0$, $u(\rho_{\max}, \varphi; t) = 0$ and the initial conditions $u(\rho, \varphi; 0) = f(\rho)$.
7. Consider heat flow in the region $\rho_1 < \rho < \rho_2$ with $\rho_1 = 3\,\text{cm}$, $\rho_2 = 15\,\text{cm}$. The boundary conditions are $T_1 = 0°\text{C}$, $T_2 = 100°\text{C}$. Find the steady-state solution and the relaxation time.
8. Consider heat flow in the cylinder $0 \le \rho < 2$, where the surface $\rho = 2$ is maintained at 100°C. At $t = 0$, we have $u = 0$ for $0 \le \rho < 1$ and $u = 50°\text{C}$ for $1 \le \rho < 2$. Find the solution $u = u(\rho; t)$ for all $t > 0$, $0 \le \rho < 2$. (*Hint:* Use the method of Example 3.2.7.)
9. Consider the heat flow in the cylinder $0 \le \rho < R$, where the surface $\rho = R$ is *insulated*, $\partial u/\partial\rho = 0$ at $\rho = R$. Find the separated solutions of the heat equation that satisfy this boundary condition. Solve the problem in the case where $u(\rho; 0) = 100°\text{C}$. (*Hint:* $x_1 = 0$ from Proposition 3.2.7.)
10. Find the solution $u(\rho, \varphi; t)$ of the heat equation $u_t = K\nabla^2 u$ in the infinite half-cylinder $0 \le \rho < \rho_{\max}$, $0 < \varphi < \pi$ satisfying the boundary conditions

$u(\rho, 0; t) = 0 = u(\rho, \pi; t) = 0$ for $0 \le \rho < \rho_{\max}$, $t > 0$, $u(\rho_{\max}, \varphi; t) = 0$, and the initial conditions $u(\rho, \varphi; 0) = \rho \sin \varphi$.

11. Find the solution $u(\rho, \varphi; t)$ of the heat equation $u_t = K\nabla_u^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max}, \varphi; t) = 0$ for $-\pi \le \varphi \le \pi$, $t > 0$ and the initial condition $u(\rho, \varphi; 0) = \rho^2 \cos 2\varphi$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$.
12. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $-ku_\rho|_{\rho=\rho_{\max}} = h(u - T_1)|_{\rho=\rho_{\max}}$ and the initial condition $u(\rho, \varphi; 0) = T_2(1 - \rho^2/\rho_{\max}^2)$, where K, h, k, T_1, and T_2 are positive constants.
13. If $u(\rho; t)$ is the time-periodic solution of the heat equation in the cylinder $0 \le \rho < a$, with $u(a; t) = A_0 + A_1 \cos(\omega t)$, find an asymptotic formula for $u(0; t)$ in the limiting case when $a \to \infty$.

3.5. Heat Flow in the Finite Cylinder

In this section we modify the analysis of the previous section to study heat flow in the finite cylinder $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$, $0 < z < L$. This will include the solution of Laplace's equation as a special case.

3.5.1. Separated solutions. To begin, we look for separated solutions of the heat equation in cylindrical coordinates, in the form

$$u(\rho, \varphi, z; t) = R(\rho)\Phi(\varphi)Z(z)T(t) \tag{3.5.1}$$

$$u_t = K\nabla^2 u = K\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} + u_{zz}\right) \tag{3.5.2}$$

Substituting (3.5.1) into (3.5.2) and dividing by Ku, we have

$$\frac{T'(t)}{KT(t)} = \frac{R'' + (1/\rho)R'}{R} + \frac{(1/\rho^2)\Phi''}{\Phi} + \frac{Z''}{Z}$$

The right side depends on (ρ, φ, z), whereas the left side depends on t. Therefore each is a constant, which we call $-\lambda$. Thus

$$T'(t) + \lambda K T(t) = 0 \tag{3.5.3}$$

Likewise the ratios Φ''/Φ and Z''/Z are both constants, which we call $-\mu$ and $-\nu$, respectively. Thus we have the equations

$$\Phi'' + \mu\Phi = 0 \tag{3.5.4}$$

$$Z'' + \nu Z = 0 \tag{3.5.5}$$

$$R'' + \frac{1}{\rho}R' + \left(\gamma - \frac{\mu}{\rho^2}\right)R = 0 \tag{3.5.6}$$

where we have introduced an additional separation constant γ that is related to the others by the equation

$$\gamma = \lambda - \nu \tag{3.5.7}$$

Two of the above equations can be solved without further reference to the boundary conditions, namely, (3.5.3) and (3.5.4). Apart from a constant, the general solution of (3.5.3) is

$$T(t) = e^{-\lambda K t}$$

whereas the general solution of (3.5.4) with the periodic boundary conditions $\Phi(-\pi) = \Phi(\pi)$, $\Phi'(-\pi) = \Phi'(\pi)$ is

$$\mu = m^2, \quad \Phi(\varphi) = A\cos m\varphi + B\sin m\varphi \qquad m = 0, 1, 2, \ldots$$

Apparently there are four separation constants, λ, μ, ν, and γ. But these are linked by (3.5.7); hence there are only three *independent* separation constants. These may be chosen to satisfy various boundary conditions. The choice $\lambda = 0$ corresponds to Laplace's equation, which will be treated in the following subsection.

3.5.2. Solution of Laplace's equation. Solutions of Laplace's equation are solutions of the heat equation that do not depend on t. In terms of the separation constants introduced above, this means that $\lambda = 0$; in other words, (3.5.7) becomes the equation

$$\gamma = -\nu \tag{3.5.8}$$

We can satisfy various boundary conditions by suitable choices of μ, γ, and ν. In order to do this, we consider three types of solutions

Type I: $\nabla^2 u = 0$, $u = 0$ on $z = 0$, $u = 0$ on $z = L$.
Type II: $\nabla^2 u = 0$, $u = 0$ on $z = 0$, $u = 0$ on $\rho = \rho_{\max}$.
Type III: $\nabla^2 u = 0$, $u = 0$ on $z = L$, $u = 0$ on $\rho = \rho_{\max}$.

The type I solutions are zero on the upper and lower circular faces of the cylinder. The solutions of type II are zero on the lower circular face and also on the curvilinear boundary, while the solutions of type III are zero on the upper circular face and on the curvilinear boundary. See Fig. 3.5.1. The general solution of Laplace's equation in the finite cylinder can be represented as a sum of solutions of these three types:

$$u = u_I + u_{II} + u_{III}$$

We now find the separated solutions of Laplace's equation of types I, II, and III.

Type I. In this case the function $Z(z)$ must satisfy the ordinary differential equation (3.5.5) with the boundary conditions $Z(0) = 0 = Z(L)$. This is a

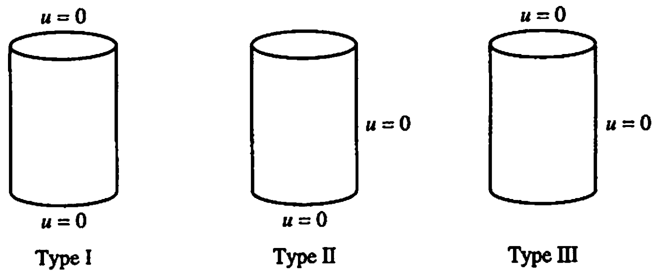


FIGURE 3.5.1 Three types of solutions of Laplace's equation.

Sturm-Liouville eigenvalue problem that has been solved previously, with the eigenfunctions and eigenvalues

$$Z(z) = \sin(k\pi z/L), \qquad \nu = (k\pi/L)^2 \quad k = 1, 2, \ldots$$

Referring to (3.5.8), we see that $\gamma = -\nu = -(k\pi/L)^2$ and therefore $R(\rho)$ must be a solution of the Bessel equation

$$R'' + \frac{R'}{\rho} - \left((k\pi/L)^2 + \frac{m^2}{\rho^2}\right) R = 0$$

whose general solution apart from a constant is the modified Bessel function

$$R(\rho) = I_m(k\pi\rho/L) \qquad m = 0, 1, 2 \ldots$$

Hence we have the general separated solution of type I:

$$u_I(\rho, \varphi, z) = I_m(k\pi\rho/L)\,(A_m \cos m\varphi + B_m \sin m\varphi) \sin(k\pi z/L) \tag{3.5.9}$$

Type II. In this case the function $R(\rho)$ must satisfy the Bessel equation (3.5.6) with the boundary conditions $R(\rho_{\max}) = 0$. The general solution apart from a constant is

$$R(\rho) = J_m(\rho x_n/\rho_{\max}) \qquad m = 0, 1, 2, \ldots$$

where x_n is a zero of the Bessel function J_m and where we make the identification $\gamma = (x_n/\rho_{\max})^2$ and hence from (3.5.8) $\nu = -\gamma = -(x_n/\rho_{\max})^2$. Returning to (3.5.5), we see that $Z(z)$ must satisfy this ordinary differential equation with the boundary condition $Z(0) = 0$. Apart from a constant, the general solution can be written as

$$Z(z) = \sinh(z\sqrt{-\nu}) = \sinh(x_n z/\rho_{\max})$$

Hence we have the general separated solution of type II:

(3.5.10) $u_{II}(\rho, \varphi, z) = J_m(\rho x_n/\rho_{\max})\,(A_m \cos m\varphi + B_m \sin m\varphi)\sinh(x_n z/\rho_{\max})$

Type III. As in the case of type II, the function $R(\rho)$ must satisfy the ordinary Bessel equation (3.5.4) with the boundary conditions $R(\rho_{\max}) = 0$. Apart from a constant, the general solution is

$$R(\rho) = J_m(\rho x_n/\rho_{\max}) \qquad m = 0, 1, 2, \ldots$$

where x_n is a zero of the Bessel function J_m and where we make the identification $\gamma = (x_n/\rho_{\max})^2$ and hence from (3.5.8) $\nu = -\gamma = -(x_n/\rho_{\max})^2$. Returning to (3.5.5), we see that $Z(z)$ must satisfy this ordinary differential equation with the boundary condition $Z(L) = 0$. The general solution, apart from a constant, can be written as

$$Z(z) = \sinh((L - z)\sqrt{-\nu}) = \sinh(x_n(L - z)/\rho_{\max})$$

Hence we have the general separated solution of type III:

(3.5.11)
$$u_{III}(\rho, \varphi, z) = J_m(\rho x_n/\rho_{\max})\,(A_m \cos m\varphi + B_m \sin m\varphi)\sinh(x_n(L - z)/\rho_{\max})$$

These separated solutions can be combined to solve various boundary-value problems for Laplace's equation.

EXAMPLE 3.5.1. *Find the solution of Laplace's equation in the finite cylinder $0 < z < L$, $0 < \rho < \rho_{\max}$ satisfying the boundary condition that $u = 1$ for $\rho = \rho_{\max}$ and $u = 0$ for $z = 0$, $z = L$.*

Solution. We use the separated solutions of type I. Since the boundary conditions are independent of φ, it suffices to take $m = 0$. A general sum of separated solutions of type I with $m = 0$ is written

$$u(\rho, z) = \sum_{k=1}^{\infty} A_k I_0(k\pi\rho/L)\sin(k\pi z/L)$$

In order to satisfy the boundary condition at $\rho = \rho_{\max}$, we must have

$$1 = \sum_{k=1}^{\infty} A_k I_0(k\pi\rho_{\max}/L)\sin(k\pi z/L)$$

This is a Fourier sine series for the function $f(z) = 1, 0 < z < L$, whose expansion is

$$1 = \frac{4}{\pi}\sum_{k\,\mathrm{odd}} \frac{\sin(k\pi z/L)}{k}$$

which allows one to solve for the coefficient A_k and obtain the solution of Laplace's equation

$$u(\rho, z) = \frac{4}{\pi} \sum_{k \text{ odd}} \frac{\sin(k\pi z/L) I_0(k\pi\rho/L)}{k\, I_0(k\pi\rho_{\max}/L)} \quad \bullet$$

EXAMPLE 3.5.2. *Find the solution of Laplace's equation in the finite cylinder* $0 < z < L$, $0 < \rho < \rho_{\max}$ *satisfying the boundary condition that* $u = 1$ *for* $z = L$ *and* $u = 0$ *for* $z = 0$, $\rho = \rho_{\max}$.

Solution. We use the separated solutions of type II. Since the boundary conditions are independent of φ, it suffices to take $m = 0$. A general sum of separated solutions of type II with $m = 0$ is written

$$u(\rho, z) = \sum_{n=1}^{\infty} A_n J_0(\rho x_n/\rho_{\max}) \sinh(x_n z/\rho_{\max})$$

In order to satisfy the boundary condition at $z = L$, we must have

$$1 = \sum_{n=1}^{\infty} A_n J_0(\rho x_n/\rho_{\max}) \sinh(x_n L/\rho_{\max})$$

This is a Fourier-Bessel series for the function $f(\rho) = 1, 0 \le \rho < \rho_{\max}$ whose expansion is

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/\rho_{\max})}{x_n\, J_1(x_n)}$$

which allows one to solve for the coefficient A_n and obtain the solution of Laplace's equation

$$u(\rho, z) = 2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/\rho_{\max})}{x_n J_1(x_n) \sinh(x_n L/\rho_{\max})} \quad \bullet$$

3.5.3. Solutions of the heat equation with zero boundary conditions. In this subsection we first determine the separated solutions of the heat equation that are zero on the the boundary of the finite cylinder. Referring to the separation of variables introduced in (3.5.1), we see that we must have $Z(0) = 0 = Z(L)$ and $R(\rho_{\max}) = 0$. Hence, apart from a constant,

$$R(\rho) = J_m(\rho x_n/\rho_{\max}), \qquad Z(z) = \sin(k\pi z/L)$$

and we have the separated solutions of the heat equation with zero boundary conditions:

$$u(\rho, \varphi, z; t) = J_m(\rho x_n/\rho_{\max})\,(A_m \cos m\varphi + B_m \sin m\varphi) \sin(k\pi z/L) e^{-\lambda K t}$$

where

$$\lambda = \lambda_{kn} = (k\pi/L)^2 + (x_n/\rho_{\max})^2$$

These may be used to solve various initial-value problems for the heat equation with zero boundary conditions.

EXAMPLE 3.5.3. *Solve the heat equation* $u_t = K\nabla^2 u$ *in the finite cylinder* $0 < z < L$, $0 \le \rho < \rho_{\max}$ *with the initial conditions* $u(\rho, \varphi, z; 0) = 1$ *and the boundary conditions that* $u = 0$ *on the top, bottom, and lateral boundaries of the cylinder. Find the relaxation time.*

Solution. Since the boundary conditions are independent of φ, we look for a solution with $m = 0$. The general sum of solutions of this type is written

$$u(\rho, z; t) = \sum_{k,n} A_{kn} J_0(\rho x_n/\rho_{\max}) \sin(k\pi z/L) e^{-\lambda_{kn} K t}$$

The initial conditions require that

$$1 = \sum_{k,n} A_{kn} J_0(\rho x_n/\rho_{\max}) \sin(k\pi z/L)$$

This is a Fourier sine series for the function $f_1(z) = 1$, multiplied by a Fourier-Bessel series for the function $f_2(\rho) = 1$. The appropriate expansion is

$$1 = \frac{8}{\pi} \sum_{k\,\mathrm{odd}, n\ge 1} \frac{\sin(k\pi z/L)\, J_0(\rho x_n/\rho_{\max})}{k x_n J_1(x_n)}$$

so that the solution of the heat equation is

$$u(\rho, z; t) = \frac{8}{\pi} \sum_{k\,\mathrm{odd}, n\ge 1} \frac{\sin(k\pi z/L)\, J_0(\rho x_n/\rho_{\max})}{k x_n J_1(x_n)} e^{-\lambda_{kn} K t}$$

The relaxation time τ is found from the first nonzero term of the series; thus

$$\lambda_{11} K \tau = 1, \qquad \tau^{-1} = K\lambda_{11} = K\left((\pi/L)^2 + (x_1/\rho_{\max})^2\right) \quad \bullet$$

3.5.4. General initial-value problems for the heat equation. We can combine the methods of the previous two subsections to treat a general initial-value problem for the heat equation in the finite cylinder, with nonhomogeneous boundary conditions. This has the form

$$\begin{aligned} u_t &= K\nabla^2 u + r \\ u(\rho, \varphi, 0, t) &= T_1(\rho, \varphi) \\ u(\rho, \varphi, L; t) &= T_2(\rho, \varphi) \\ u(\rho_{\max}, \varphi, z; t) &= T_3(z, \varphi) \\ u(\rho, \varphi, z; 0) &= f(\rho, \varphi, z) \end{aligned} \tag{3.5.12}$$

Note that we have included a source term, as well as the most general nonhomogeneous boundary conditions. The analysis proceeds by means of the five-stage method. In stage 1 we find a steady-state solution; the function $U_0(\rho) =$

$(r/K)(\rho_{\max}^2 - \rho^2)$ can be used to replace the source term by $r = 0$, at the expense of changing the initial and boundary conditions. Then we solve Laplace's equation with the new boundary conditions to obtain the steady-state solution $U(\rho, \varphi, z)$. In stage 2 we define $v(\rho, \varphi, z; t) = u(\rho, \varphi, z; t) - U(\rho, \varphi, z)$, which satisfies the heat equation with $r = 0$ and with zero boundary conditions. This is written as a sum of separated solutions:

$$v(\rho, \varphi, z; t) = \sum_{k,m,n} J_m(\rho x_n/\rho_{\max})(A_{kmn}\cos m\varphi + B_{kmn}\sin m\varphi)\sin(k\pi z/L)e^{-\lambda_{kmn}Kt}$$

The coefficients A_{kmn} and B_{kmn} are determined by expansion of the function

$$f(\rho, \varphi, z) - U(\rho, \varphi, z) = \sum_{k,m,n} J_m(\rho x_n/\rho_{\max})(A_{kmn}\cos m\varphi + B_{kmn}\sin m\varphi)\sin(k\pi z/L)$$

This gives the formal series solution of the problem

$$u(\rho, \varphi, z; t) = U(\rho, \varphi, z) + v(\rho, \varphi, z; t)$$

Stage 4 consists of verifying that the series that define u, u_t, u_z, u_{zz}, u_ρ, $u_{\rho\rho}$, u_φ, $u_{\varphi\varphi}$ are uniformly convergent and thus u is a smooth function of (ρ, φ, z, t) that satisfies the heat equation. Stage 5 consists of the asymptotic analysis:

$$u(\rho, \varphi, z; t) - U(\rho, \varphi, z) = O(e^{-\lambda_{111}Kt}) \qquad t \to \infty$$

For large time the solution tends to the steady-state solution and the relaxation time τ is given by the first nonzero term of the series, $\lambda_{111}K\tau = 1$, provided that the corresponding Fourier coefficient is nonzero.

EXERCISES 3.5

1. Find the solution $u(\rho, \varphi, z)$ of Laplace's equation $\nabla^2 u = 0$ in the finite cylinder $0 \le \rho < \rho_{\max}$, $0 < z < L$ satisfying the boundary conditions $u(\rho_{\max}, \varphi, z) = 0$ for $-\pi \le \varphi \le \pi$, $0 < z < L$; $u(\rho, \varphi, 0) = 1$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$; $u(\rho, \varphi, L) = 0$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$.
2. Find the solution of Laplace's equation in the finite cylinder $0 \le \rho < \rho_{\max}$, $0 < z < L$ satisfying the boundary conditions $u(\rho_{\max}, \varphi, z) = 0$ for $-\pi \le \varphi \le \pi$, $0 < z < L$; $u(\rho, \varphi, 0) = T_1$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$; $u(\rho, \varphi, L) = T_2$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$.
3. Find all of the separated solutions $u(\rho, \varphi, z)$ of Laplace's equation $\nabla^2 u = 0$ in the finite cylinder $0 \le \rho < \rho_{\max}$, $0 < z < L$ satisfying the boundary conditions $u(\rho, \varphi, 0) = 0 = u_z(\rho, \varphi, L)$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi \le \pi$.
4. Find the solution $u(\rho, \varphi, z)$ of Laplace's equation $\nabla^2 u = 0$ in the finite cylinder $0 \le \rho < \rho_{\max}$, $0 < z < L$ satisfying the boundary conditions $u(\rho, \varphi, 0) = 0 = u_z(\rho, \varphi, L)$ for $0 \le \rho < \rho_{\max}$, $-\pi \le \varphi < \pi$; $u(\rho_{\max}, \varphi, z) = 1$ for $-\pi \le \varphi \le \pi$, $0 < z < L$.

5. Find the solution $u(\rho, \varphi, z)$ of Laplace's equation $\nabla^2 u = 0$ in the finite cylinder $0 \leq \rho < \rho_{\max}$, $0 < z < L$ satisfying the boundary conditions $u(\rho, \varphi, 0) = 0 = u(\rho, \varphi, L)$ for $0 \leq \rho < \rho_{\max}$, $-\pi \leq \varphi < \pi$; $u(\rho_{\max}, \varphi, z) = z$ for $-\pi \leq \varphi \leq \pi$, $0 < z < L$.
6. (a) Find all of the separated solutions of the heat equation in the finite cylinder $0 < z < L$, $0 < \rho < \rho_{\max}$ satisfying the boundary conditions that $u_z = 0$ on the top and bottom and $u_\rho = 0$ on the side.

   (b) Use these to write the general solution of an initial-value problem with these boundary conditions.
7. (a) Find all of the separated solutions of the heat equation in the finite cylinder $0 < z < L$, $0 < \rho < \rho_{\max}$ satisfying the boundary conditions that $u = 0$ on the top and bottom and $u_\rho = 0$ on the side.

   (b) Use these to write the general solution of an initial-value problem with these boundary conditions.

CHAPTER 4

BOUNDARY-VALUE PROBLEMS IN SPHERICAL COORDINATES

INTRODUCTION

In this chapter we consider boundary-value problems in regions with spherical boundaries. Section 4.1 treats some spherically symmetric problems, which can be reduced to one dimension and solved in terms of elementary functions. In Sec. 4.2 we develop the properties of Legendre functions and spherical Bessel functions. These are applied in Sec. 4.3 to the solution of more general boundary-value problems in spherical coordinates.

4.1. Spherically Symmetric Solutions

Recall that spherical coordinates are defined by the formulas

(4.1.1)
$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

The three-dimensional distance r satisfies $r \geq 0$, $r^2 = x^2 + y^2 + z^2$. The polar angle θ measures the colatitude, i.e., the angle that the vector (x, y, z) makes with the positive z-axis. The azimuthal angle φ measures the longitude, i.e., the angle that the vector $(x, y, 0)$ makes with the positive x-axis in the xy plane. By convention $0 \leq \theta \leq \pi$ and $-\pi \leq \varphi \leq \pi$, where $\theta = 0$ and $\theta = \pi$ correspond to the positive (resp. negative) z-axis and where the variable φ is undefined. This is illustrated in Fig. 4.1.1.

Let $u(x, y, z)$ be a smooth function and $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$, the Laplacian of u. A new function U is defined by $U(r, \theta, \varphi) = u(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$. As in the discussion of cylindrical coordinates, we emphasize that the mapping $u \to U$ produces a smooth function of (r, θ, φ) for every choice of the smooth function $u(x, y, z)$, but there are many smooth functions of (r, θ, φ) that do not arise this way. For example, $U(r) = r^2$ comes from the smooth function $u(x, y, z) = x^2 + y^2 + z^2$, whereas $U(r) = r$ does not correspond to a smooth function of (x, y, z). We expect that solutions produced in spherical coordinates will correspond to smooth functions of (x, y, z), which frequently provides a convenient consistency check of symbolic computations.

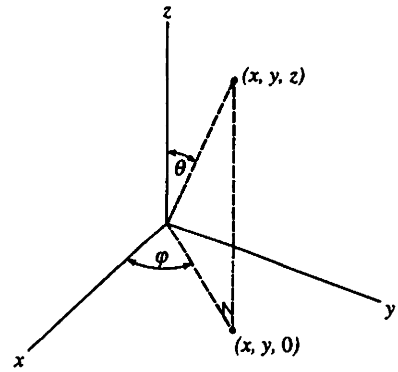


FIGURE 4.1.1 Spherical coordinates.

4.1.1. Laplacian in spherical coordinates. We wish to compute $\nabla^2 u$ in terms of U_{rr}, U_r, $U_{\theta\theta}$, etc. In order to simplify the exposition, we shall write instead u_{rr}, u_r, u_θ for the indicated partial derivatives. This simplification will cause no confusion.

To compute $\nabla^2 u$, we first recall the result from Sec. 3.1 for the Laplacian in cylindrical coordinates:

$$u_{xx} + u_{yy} = u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} \tag{4.1.2}$$

The polar coordinates (ρ, φ) obey the relations

$$\begin{aligned} x &= \rho \cos \varphi \\ y &= \rho \sin \varphi \end{aligned}$$

From the formulas for z and x, we have

$$\begin{aligned} z &= r \cos \theta \\ \rho &= r \sin \theta \end{aligned}$$

Therefore (z, ρ) *are obtained from* (r, θ) *in exactly the same way that* (x, y) *are obtained from* (ρ, φ). Thus we must have

$$u_{zz} + u_{\rho\rho} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \tag{4.1.3}$$

Adding (4.1.2) and (4.1.3), we have

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} \tag{4.1.4}$$

It remains to compute u_ρ. From the chain rule for partial derivatives, we have

$$u_\rho = u_r \frac{\partial r}{\partial \rho} + u_\theta \frac{\partial \theta}{\partial \rho} + u_\varphi \frac{\partial \varphi}{\partial \rho}$$

The transformation from cylindrical coordinates to spherical coordinates gives $r = (\rho^2 + z^2)^{1/2}$, $\theta = \tan^{-1} \rho/z$, $\varphi = \varphi$. From this it follows that $\partial r/\partial \rho = \rho/r$, $\partial\theta/\partial\rho = (\cos\theta)/r$, $\partial\varphi/\partial\rho = 0$. Substituting these in the equation for u_ρ, we have

$$u_\rho = u_r \frac{\rho}{r} + u_\theta \frac{\cos\theta}{r}$$

Substituting this into (4.1.4) and rearranging yields the result

$$\boxed{\nabla^2 u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\varphi\varphi}\right)} \tag{4.1.5}$$

This is the required formula.

EXAMPLE 4.1.1. *Compute* $\nabla^2(x^2 + y^2 + z^2)^{5/2}$.

Solution. If we were to use the definition of ∇^2, this would be a tedious computation. Noting that the function is expressed in spherical coordinates as r^5, we have

$$\begin{aligned} \nabla^2(r^5) &= 20r^3 + \frac{2}{r} 5r^4 \\ &= 30r^3 \quad \bullet \end{aligned}$$

4.1.2. Time-periodic heat flow: Applications to geophysics. As a first application of spherical coordinates, we reconsider the geophysics problem from Sec. 2.1. Now we assume that the earth is a perfect sphere of radius a and, as before, that the surface temperature is a periodic function of time. Thus we have the boundary-value problem

$$\begin{aligned} u_t &= K\nabla^2 u \qquad && -\infty < t < \infty,\ 0 \le x^2 + y^2 + z^2 < a^2 \\ u(x, y, z; t) &= u_0(t) && -\infty < t < \infty,\ x^2 + y^2 + z^2 = a^2 \end{aligned} \tag{4.1.6}$$

Taking spherical coordinates, we look for the solution in the form $u = u(r; t)$. Thus we must have

$$\begin{aligned} u_t &= K\left(u_{rr} + \frac{2}{r} u_r\right) \quad && -\infty < t < \infty, 0 \le r < a \\ u(a; t) &= u_0(t) && -\infty < t < \infty \end{aligned}$$

This can be reduced to the heat equation in one dimension by introducing the new function

$$w(r; t) = ru(r; t)$$

Thus $w_t = ru_t$, $w_r = ru_r + u$, $w_{rr} = ru_{rr} + 2u_r$. Multiplying the heat equation by r and making these substitutions, we have

$$
\begin{aligned}
w_t &= Kw_{rr} && -\infty < t < \infty, 0 \le r < a \\
w(a;t) &= au_0(t) && -\infty < t < \infty \\
w(0;t) &= 0 && -\infty < t < \infty
\end{aligned}
\tag{4.1.7}
$$

The final boundary condition comes from the fact that the temperature at the center of the sphere must be finite for all time.

The result of the above transformations is to *reduce the three-dimensional spherically symmetric heat flow problem to a one-dimensional heat flow problem in a slab with zero temperature on one face.*[1] This problem can be solved by looking for complex separated solutions

$$w(r;t) = e^{i\omega t}e^{\gamma r}$$

Substituting in the heat equation (4.1.7), we find that $\gamma^2 = i\omega/K$; hence $\gamma = \pm\sqrt{\omega/2K}(1+i)$, which yields the complex separated solutions

$$e^{-cr}e^{i(\omega t - cr)}, \qquad e^{cr}e^{i(\omega t + cr)}, \qquad c = \sqrt{\frac{\omega}{2K}}$$

To match the boundary conditions, we assume that $u_0(t)$ has been expanded as a Fourier series.

$$
\begin{aligned}
u_0(t) &= A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2\pi nt}{T} + B_n \sin\frac{2\pi nt}{T}\right) \\
&= \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi int/T}
\end{aligned}
$$

Therefore it suffices to solve the problem with $u_0(t) = e^{i\omega t}$ and to take the real and imaginary parts. For this we try a linear combination of complex separated solutions

$$w(r;t) = C_1 e^{-cr}e^{i(\omega t - cr)} + C_2 e^{cr}e^{i(\omega t + cr)}$$

The boundary conditions at $r = 0$ and $r = a$ require that

$$
\begin{aligned}
0 &= C_1 e^{i\omega t} + C_2 e^{i\omega t} \\
ae^{i\omega t} &= C_1 e^{-ca}e^{i(\omega t - ca)} + C_2 e^{ca}e^{i(\omega t + ca)}
\end{aligned}
$$

Thus $C_1 + C_2 = 0$, $a = C_1 e^{-ca}e^{-ica} + C_2 e^{ca}e^{ica}$. Solving these two equations simultaneously and simplifying, we have

$$w(r;t) = ae^{i\omega t}\frac{e^{c(1+i)r} - e^{-c(1+i)r}}{e^{c(1+i)a} - e^{-c(1+i)a}}$$

[1]This general principle applies to all the applications discussed in this section.

The original problem is therefore solved by

$$u(r;t) = \frac{a}{r}\sum_{n=-\infty}^{\infty} \alpha_n e^{i\omega_n t}\left(\frac{e^{c_n(1+i)r} - e^{-c_n(1+i)r}}{e^{c_n(1+i)a} - e^{-c_n(1+i)a}}\right)$$

where we have set $\omega_n = 2\pi n/T$, $c_n = \sqrt{\pi n/KT}$.

EXAMPLE 4.1.2. *Find the solution of the heat equation $u_t = K\nabla^2 u$ for $-\infty < t < \infty$, in the sphere $0 < r < a$, satisfying the boundary condition $u(a;t) = A_0 + A_1\cos\omega t$. Find $u(0;t)$.*

Solution. We take $\alpha_0 = A_0$, $\alpha_1 = \alpha_{-1} = \frac{1}{2}A_1$. The solution is

$$u(r;t) = A_0 + \frac{aA_1}{r}\ \mathrm{Re}\ e^{i\omega t}\frac{e^{c(1+i)r} - e^{-c(1+i)r}}{e^{c(1+i)a} - e^{-c(1+i)a}} \tag{4.1.8}$$

where $c = \sqrt{\omega/2K}$. To find $u(0;t)$, we may appeal to L'Hospital's rule to take the limit when $r \to 0$. Thus

$$u(0;t) = \lim_{r\to 0} u(r;t) = A_0 + aA_1\ \mathrm{Re}\ e^{i\omega t}\frac{2c(1+i)}{e^{c(1+i)a} - e^{-c(1+i)a}} \quad \bullet$$

It is instructive to compare the solution (4.1.8) of Example 4.1.2 with the solution for the flat earth in Chapter 2. To do this, we let $z = a - r$ and consider the limit when $a \to \infty$ with z fixed. This corresponds to observing the temperature at a shallow depth. Removing a factor of $e^{c(1+i)a}$ from the denominator and removing the factor $e^{c(1+i)r}$ from the numerator, we have

$$u(r;t) = A_0 + \frac{aA_1}{r}\ \mathrm{Re}\ e^{c(r-a)}e^{i[\omega t + c(r-a)]}\frac{1 - e^{-2cr(1+i)}}{1 - e^{-2ca(1+i)}} \tag{4.1.9}$$

If $a \to \infty$, $r \to \infty$ with $z = a - r$ fixed, the final fraction tends to 1 and

$$\lim_{a\to\infty} u(r;t) = A_0 + A_1\ \mathrm{Re}\ e^{-cz}e^{i(\omega t - cz)} = A_0 + A_1 e^{-cz}\cos(\omega t - cz) \tag{4.1.10}$$

This is exactly the solution we found in Sec. 2.1, Example 2.1.3 for the flat earth. Hence we see that, for shallow depths, the earth's surface can be assumed flat. To make a numerical estimate, note that the solution (4.1.9) differs from the solution (4.1.10) in two respects: (i) the factor a/r and (ii) the exponential terms in the numerator and denominator of the final fraction. Assuming the approximate values $a = 6400$ km, $z = 1.6$ km, $K = 2 \times 10^{-3}$ cm, $T = 3.15 \times 10^7$ s, we have $c = \sqrt{\omega/2K}$, $cr = 1.02 \times 10^7$, so that e^{-cr} is negligibly small. On the other hand, $a/r = 0.99974$, so that if we replace a/r by 1, we incur less than 0.1 percent error in the solution.

This concludes the discussion of the time-periodic solutions of the heat equation.

4.1.3. Initial-value problem for heat flow in a sphere. As a second application of spherical coordinates we consider the following initial-value problem for heat flow in a sphere:

$$\begin{aligned} u_t &= K\nabla^2 u & t>0, 0\le r<a \\ u(a;t) &= T_1 & t>0 \\ u(r;0) &= f(r) & 0\le r<a \end{aligned} \tag{4.1.11}$$

where $f(r), 0 \le r < a$, is a given piecewise smooth function. This problem is solved by the five-stage method introduced in Chapter 2. The steady-state solution that satisfies the heat equation and boundary condition is $U = T_1$. Subtracting this, we have the transformed problem with T_1 replaced by zero and $f(r)$ replaced by $f(r) - T_1$. Introducing the new function $w = r(u - U)$, the problem reduces to

$$\begin{aligned} w_t &= Kw_{rr} & t>0, 0\le r<a \\ w(a;t) &= 0 & t>0 \\ w(0;t) &= 0 & t>0 \\ w(r;0) &= r[f(r)-T_1] & 0\le r<a \end{aligned} \tag{4.1.12}$$

This is a one-dimensional problem for heat flow in a slab, for which we know the separated solutions:

$$w(r;t) = \sin\frac{n\pi r}{a}\, e^{-(n\pi/a)^2Kt} \qquad n = 1,2,\dots$$

Therefore, by superposition, we have solved the original problem in the form

$$\boxed{u(r;t) = T_1 + \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi r}{a}\, e^{-(n\pi/a)^2Kt}} \tag{4.1.13}$$

The coefficients B_n can be obtained by setting $t = 0$ and using the formulas from Fourier series. Thus,

$$\begin{aligned} f(r) &= T_1 + \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\frac{n\pi r}{a} & 0<r<a \\ B_n &= \frac{2}{a}\int_0^a r[f(r)-T_1]\sin\frac{n\pi r}{a}\,dr & n=1,2,\dots \end{aligned} \tag{4.1.14}$$

If $f(r), 0 \le r < a$, is piecewise smooth, the series (4.1.13) converges for $0 \le r \le a$ and u satisfies the heat equation, initial conditions, and boundary conditions. The relaxation time can be found by taking the first term of the series; thus $\tau = a^2/\pi^2 K$ if $B_1 \ne 0$.

The temperature at the center of the sphere is found by noting that $\lim_{r\to 0}(1/r)\sin n\pi r/a = n\pi/a$. Thus,

$$u(0;t) = \lim_{r\to 0} u(r;t) = T_1 + \sum_{n=1}^{\infty} \frac{n\pi}{a} B_n e^{-(n\pi/a)^2 Kt}$$

In using this to compute the temperature at the center for small times, we must be careful in estimating the sum of the series. Indeed, we expect on physical grounds (and it can be proved mathematically) that $u(0;t)$ is no larger than T_1 and the maximum of f, and no smaller than T_1 and the minimum of f. This should be reflected in practical computations based on the series solution we have just found.

EXAMPLE 4.1.3. *Let $f(r) = T_2$, a constant. Find the solution $u(r;t)$. For the numerical values $K = 0.03$, $a = 0.5$, $T_1 = 0$, $T_2 = 100$, find the relaxation time and estimate $u(0;t)$ for $t = 5$, $t = 1$, $t = 0.1$.*

Solution. In this case the formula (4.1.14) gives

$$B_n = \frac{2(T_2 - T_1)}{a} \int_0^a r \sin \frac{n\pi r}{a}\, dr$$

These integrals were computed in Example 1.1.1, with the result

$$\frac{2}{a} \int_0^a r \sin \frac{n\pi r}{a}\, dr = \frac{(-1)^{n+1}}{n} \frac{2a}{\pi}$$

Therefore, from (4.1.13), we have the solution

$$u(r;t) = T_1 + \frac{2a(T_2 - T_1)}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi r}{a} e^{-(n\pi/a)^2 Kt}$$

At the center we have

$$u(0;t) = T_1 + 2(T_2 - T_1) \sum_{n=1}^{\infty} (-1)^{n+1} e^{-(n\pi/a)^2 Kt}$$

For $K = 0.03$, $a = 0.5$, we have $\pi^2 K/a^2 = 1.18$, to two decimals. The relaxation time is $\tau = 1/1.18 = 0.85$, to two decimals. For $t = 5$, the first term of the series is 0.0027 and the remaining terms are less than 10^{-11}. This leads to the estimate $u(0;t) = 0.54$. For $t = 1$, the first two terms of the series are $0.3073 - 0.0089$ and the remaining terms are less than 10^{-4}. This leads to the estimate $u(0;1) = 59.68$, to two decimal places. For $t = 0.1$, the first five terms of the series are $0.8887 - 0.6237 + 0.3458 - 0.1514 + 0.0523 = 0.5117$. If we use this to estimate the temperature, we have $u(0;0.1) = 102.34$, to two decimals, a physically unrealistic result. To obtain a more realistic result,. we may average the fourth and fifth partial sums of the series. This leads to the estimate $u(0;0.1) = 97.11$, to two decimals. •

At the end of this section we will discuss in some greater detail the preceding convergence problems.

As our next application, we consider the problem of a sphere that exchanges heat with the outside according to Newton's law of cooling, i.e., the heat flux across the boundary is proportional to the difference between the surface temperature and the outside temperature. In addition, we assume that heat is produced at a constant rate σ. Problems of this type occur when we consider apples placed in a refrigerator.

The mathematical problem is expressed as follows:

$$
\begin{aligned}
u_t &= K\nabla^2 u + \sigma && 0 \le r < a, t > 0 \\
-k\frac{\partial u}{\partial r}(a;t) &= h[u(a;t) - T_1] && t > 0 \\
u(r;0) &= f(r) && 0 \le r < a
\end{aligned}
\tag{4.1.15}
$$

where $h > 0$, $k > 0$, and $f(r), 0 \le r < a$, is a piecewise smooth function. The initial and boundary conditions are independent of (θ, φ), and therefore we may expect the solution in the form $u = u(r;t)$. Letting $w(r;t) = ru(r;t)$, we have $w_t = ru_t$, $w_r = ru_r + u$, $w_{rr} = ru_{rr} + 2u_r = r\nabla^2 u$. We multiply the equations (4.1.15) by r. In terms of $w(r;t)$ the problem becomes

$$
\begin{aligned}
w_t &= Kw_{rr} + \sigma r && 0 \le r < a, t > 0 \\
-k\left(w_r - \frac{w}{r}\right)(a;t) &= h[w(a;t) - aT_1] && t > 0 \\
w(0;t) &= 0 && t > 0 \\
w(r;0) &= r\,f(r) && 0 \le r < a.
\end{aligned}
\tag{4.1.16}
$$

This boundary-value problem for w is a one-dimensional problem that can be solved by the five-stage method of Chapter 2. Note that we have the additional boundary condition at $r = 0$.

Stage 1. The steady-state equation is $KW_{rr} + \sigma r = 0$ with the two boundary conditions at $r = 0$, $r = a$. The general solution of this ordinary differential equation is

$$W(r) = -\frac{\sigma r^3}{6K} + A + Br$$

where A and B are arbitrary constants. The boundary condition $W(0) = 0$

requires that $A = 0$. To analyze the boundary condition at $r = a$, we write

$$\begin{aligned} W'(r) &= -\frac{\sigma r^2}{2K} + B \\ W'(r) - \frac{W(r)}{r} &= B - \frac{\sigma r^2}{2K} - B + \frac{\sigma r^2}{6K} \\ &= -\frac{\sigma r^2}{3K} \\ h[W(r) - rT_1] &= h\left(Br - \frac{\sigma r^3}{6K} - rT_1\right) \end{aligned}$$

Therefore B is determined by the equation

$$-k\left(-\frac{\sigma a^2}{3K}\right) = h\left(Ba - \frac{\sigma a^3}{6K} - aT_1\right)$$

The solution is

$$W(r) = \frac{r\sigma}{6K}(a^2 - r^2) + r\left(T_1 + \frac{\sigma ak}{3hK}\right) \tag{4.1.17}$$

We can use this to compute the total flux out of the sphere in steady state. In terms of the original temperature function u, we have

$$-k\left.\frac{\partial u}{\partial r}\right|_{r=a} = h(u - T_1)|_{r=a} = \left.\frac{h(w - rT_1)}{r}\right|_{r=a} = \frac{\sigma a}{3}$$

Multiplying this by the surface area $4\pi a^2$, we have the total flux $\sigma 4\pi a^3/3$. This agrees with physical intuition, since the only way that heat can flow across the boundary in steady state is from the source term σ.

Stage 2. We use the steady-state solution to transform the problem. Letting $v(r;t) = w(r;t) - W(r)$, we have the equation for v:

$$\begin{aligned} v_t &= Kv_{rr} & & 0 \le r < a, t > 0 \\ v(0;t) &= 0 & & t > 0 \\ -kv_r(a;t) &= hv(a;t) - \frac{k}{a}v(a;t) & & t > 0 \\ v(r;0) &= r\,f(r) - W(r) & & 0 \le r < a \end{aligned}$$

Stage 3. The separated solutions of the problem for v can be easily obtained. Writing $v(r;t) = R(r)T(t)$, we have the equations $T' + \lambda KT = 0$, $R'' + \lambda R = 0$ with the boundary conditions $R(0) = 0$, $R'(a) = (1/a - h/k)R(a)$. The first equation can be solved within a constant by $T(t) = e^{-\lambda Kt}$. The equation for $R(r)$ is a Sturm-Liouville eigenvalue problem,

$$R'' + \lambda R = 0, \qquad R(0) = 0, \quad k\,R'(a) + \left(h - \frac{k}{a}\right)R(a) = 0$$

This will be solved as in Example 1.6.3.

If $\lambda = 0$ is an eigenvalue, then the corresponding eigenfunction satisfying the first boundary condition must be linear: $R(r) = r$; this will satisfy the second boundary condition if and only if $k + (h - k/a)a = 0$, which implies $h = 0$—a contradiction if $h > 0$.

If an eigenvalue satisfies $\lambda = -\mu^2 < 0$, then the corresponding eigenfunction satisfying the first boundary condition must be, to within a constant, $R(r) = \sinh \mu r$. This will satisfy the second boundary condition if and only if $\tanh(a\mu) = ka\mu/(k - ah)$, which has no solution $\mu \neq 0$ if $h > 0$.

Therefore all of the eigenvalues are positive and the corresponding eigenfunctions are, to within a constant multiple,

$$R_n(r) = \sin r\sqrt{\lambda_n}$$

The eigenvalues $\{\lambda_n\}$ are solutions of the equation $a\sqrt{\lambda_n}\cot(a\sqrt{\lambda_n}) = 1-(ha/k)$. They may be obtained graphically when we are given the numerical values of a, h, and k. For large n they have the asymptotic behavior $a\sqrt{\lambda_n} = n\pi + O(1)$, $n \to \infty$.

We can write the superposition of separated solutions as

$$v(r;t) = \sum_{n=1}^{\infty} A_n \sin r\sqrt{\lambda_n}\, e^{-\lambda_n Kt} \tag{4.1.18}$$

From Theorem 1.5, the eigenfunctions must be orthogonal. Thus

$$\int_0^a \sin r\sqrt{\lambda_n} \sin r\sqrt{\lambda_m}\, dr = 0 \qquad n \neq m$$

The normalization can be computed as the integral

$$\begin{aligned}\int_0^a \sin^2 r\sqrt{\lambda_n}\, dr &= \frac{1}{2}\int_0^a (1 - \cos 2r\sqrt{\lambda_n})\, dr \\ &= \frac{1}{2}\left(a - \frac{\sin 2a\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\right)\end{aligned}$$

The Fourier coefficients A_n can be obtained using these relations. Thus by setting $t = 0$ in (4.1.18), multiplying by $\sin r\sqrt{\lambda_n}$, and integrating, we have

$$\int_0^a [rf(r) - W(r)] \sin r\sqrt{\lambda_n}\, dr = A_n \int_0^a \sin^2 r\sqrt{\lambda_n}\, dr \tag{4.1.19}$$

EXAMPLE 4.1.4. *Find the Fourier coefficients in the case where $f(r) = T_2$, a constant.*

Solution. In this case we must compute the integral

$$\int_0^a [rT_2 - W(r)] \sin r\sqrt{\lambda_n}\, dr = \int_0^a \left[r\left(T_2 - T_1 - \frac{\sigma ak}{3hK}\right) - \frac{r\sigma}{6K}(a^2 - r^2)\right] \sin r\sqrt{\lambda_n}\, dr$$

We use the integrals

$$\int_0^a r\sin r\sqrt{\lambda}\,dr = \frac{\sin a\sqrt{\lambda}}{\lambda} - \frac{a\cos a\sqrt{\lambda}}{\sqrt{\lambda}}$$

$$\int_0^a r(a^2-r^2)\sin r\sqrt{\lambda}\,dr = \frac{6\sin a\sqrt{\lambda}}{\lambda^2} - \frac{6a\cos a\sqrt{\lambda}}{\lambda^{3/2}} - \frac{2a^2\sin a\sqrt{\lambda}}{\lambda}$$

We substitute these in the above formulas and use the relation $a\sqrt{\lambda}\cot a\sqrt{\lambda} = 1 - ha/k$, with the result

$$\begin{aligned}\int_0^a [rT_2 - W(r)]\sin r\sqrt{\lambda}\,dr &= \left(T_2 - T_1 - \frac{\sigma ak}{3Kh}\right)\frac{\sin a\sqrt{\lambda}}{\lambda}\frac{ha}{k} \\ &\quad - \frac{\sigma}{K}\frac{\sin a\sqrt{\lambda}}{\lambda^2}\left\{\frac{ha}{k} - \frac{1}{3}\left[1 - \left(\frac{ha}{k}\right)^2\right]\right\}\end{aligned}$$

Stage 4. We have obtained the formal solution of the problem (4.1.15) as

$$\boxed{u(r;t) = U(r) + \frac{1}{r}\sum_{n=1}^{\infty} A_n \sin r\sqrt{\lambda_n}\, e^{-\lambda_n Kt}} \tag{4.1.20}$$

where the steady-state solution $U(r) = (\sigma/6K)(a^2-r^2)+T_1+\sigma ak/3Kh$; the eigenvalues $\{\lambda_n\}$ are determined from the transcendental equation $a\sqrt{\lambda_n}\cot a\sqrt{\lambda_n} = 1 - ha/k$, and the Fourier coefficients $\{A_n\}$ are obtained from (4.1.19). We have $A_n = O(1)$ and $a\sqrt{\lambda_n} = n\pi + O(1)$ when $n \to \infty$. Therefore for each $t > 0$, the series (4.1.20) and the differentiated series for u_r, u_{rr}, and u_t all converge uniformly for $0 \le r \le a$. Thus $u(r;t)$ is a rigorous solution of the heat equation.

Stage 5. When $t \to \infty$, the solution $u(r;t)$ tends to the steady-state solution $U(r)$. We use the method from Chapter 2 to estimate the rate of approach; thus

$$\frac{1}{r}\sum_{n=1}^{\infty} A_n \sin r\sqrt{\lambda_n} e^{-\lambda_n Kt} = O(e^{-\lambda_1 Kt}) \qquad t \to \infty$$

Therefore $u(r;t) - U(r) = O(e^{-\lambda_1 Kt})$, $t \to \infty$. Finally we compute the relaxation time by noting that

$$u(r;t) - U(r) = A_1 \frac{\sin r\sqrt{\lambda_1}}{r} e^{-\lambda_1 Kt} + O(e^{-\lambda_2 Kt}) \qquad t \to \infty$$

If $A_1 \neq 0$, the relaxation time is given by

$$\tau = \frac{1}{\lambda_1 K} \tag{4.1.21}$$

To obtain numerical estimates of the relaxation time, we may appeal to the graphs in Fig. 4.1.2. •

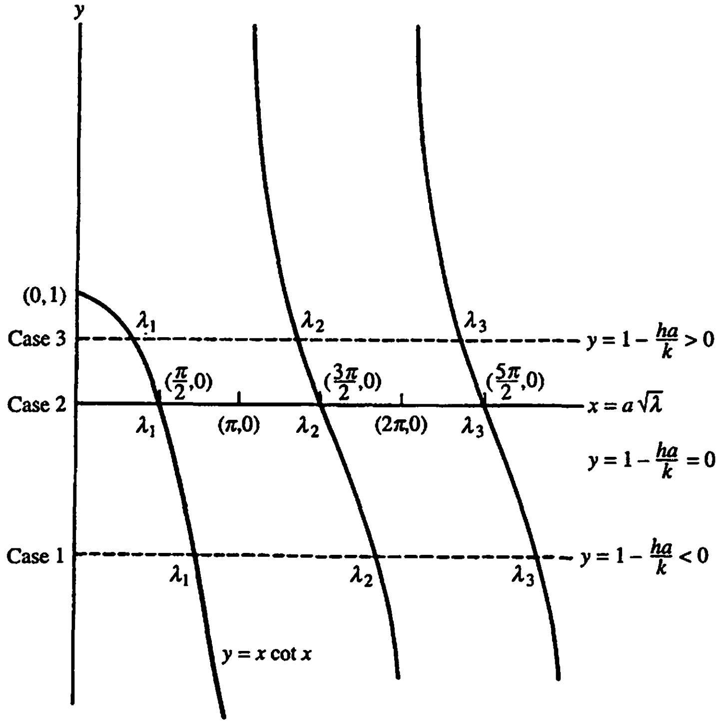


FIGURE 4.1.2 Determination of the eigenvalues λ_n as solutions of the transcendental equation $a\sqrt{\lambda}\cot(a\sqrt{\lambda}) = 1 - ha/k$.
Case 1: $-ha/k < 0$. Case 2: $1 - ha/k = 0$. Case 3: $1 - ha/k > 0$.

EXAMPLE 4.1.5. *If $K = 0.30$, $a = 0.50$, $h = 0,08$, and $k = 0.303$, find the relaxation time.*

Solution. We have $1 - ha/k = 0.868$. The smallest solution of the equation $x \cot x = 0.868$ is $x = 0.62$; thus $\sqrt{\lambda_1} = 0.62/0.50 = 1.24$. The relaxation time is $\tau = 1/(0.30)(1.24)^2$, about 2 seconds. •

The heat equation can be used to study the cooling of an apple placed into a refrigerator. Assuming a perfect sphere of radius $a = 2$ in, diffusivity $K = 0.720$ in^2/h, and Biot modulus $ha/k = 1.0$, we have, for the first eigenvalue $a\sqrt{\lambda_1} = \pi/2$, $\sqrt{\lambda_1} = 0.7854$ and the relaxation time $\tau = 1/\lambda_1 K = 1/(0.720)(0.7854)^2$, about 2 hours. From this we expect that within 10 hours the apple will be within 1 percent of the ambient refrigerator temperature, relative to its initial temperature.

4.1.4. The three-dimensional wave equation. We close this section by remarking that the preceding techniques can also be used to find spherically symmetric solutions of the three-dimensional wave equation

$$u_{tt} = c^2\nabla^2 u = c^2\left(u_{rr} + \frac{2}{r}\,u_r\right)$$

which is satisfied by the scalar potential function of electromagnetic theory or by the small pressure variations of a gas. To obtain solutions, we let $w(r;t) = ru(r;t)$ and obtain the equation for w:

$$w_{tt} = c^2 w_{rr}$$

This is the one-dimensional wave equation that was encountered in the discussion of the vibrating string in Sec. 2.4, where we found the separated solutions $w(r;t) = (A\sin kr + B\cos kr)(C\cos kct + D\sin kct)$. The function w must satisfy the additional boundary condition that $w(0;t) = 0$. This gives the separated solutions of the wave equation in the form

$$u(r;t) = \frac{1}{r}\sin kr(C\cos kct + D\sin kct) \tag{4.1.22}$$

These may be used to solve initial-value problems for the three-dimensional wave equation.

EXAMPLE 4.1.6. *Find the solution of the wave equation $u_{tt} = c^2\nabla^2 u$ in the sphere $0 \le r < a$ satisfying the initial conditions that $u(r;0) = C$, $u_t(r;0) = 0$ and the boundary condition that $u(a;t) = 0$, where C is a constant.*

Solution. We look for the solution as a superposition of the separated solutions (4.1.22) that satisfy the boundary condition:

$$u(r;t) = \sum_{n=1}^{\infty}\frac{\sin n\pi r/a}{r}\left(C_n\cos\frac{n\pi ct}{a} + D_n\sin\frac{n\pi ct}{a}\right) \tag{4.1.23}$$

The second initial condition requires that $D_n = 0$, while the first initial condition requires that C_n be obtained as the Fourier coefficients in the expansion

$$C = \sum_{n=1}^{\infty} C_n\frac{\sin n\pi r/a}{r}$$

From Example 1.1.1 this requires that $C_n = (2aC/n\pi)(-1)^{n+1}$, and therefore the solution of the boundary-value problem is obtained as

$$u(r;t) = \frac{2aC}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{\sin n\pi r/a}{r}\cos\left(\frac{n\pi ct}{a}\right) \quad \bullet \tag{4.1.24}$$

We can also use these methods to solve boundary-value problems for the wave equation with time-dependent boundary conditions, as illustrated in the next example.

EXAMPLE 4.1.7. *Find the solution of the wave equation $u_{tt} = c^2\nabla^2 u$ in the sphere $0 \le r < a$ satisfying the initial conditions that $u(r;0) = 0$, $u_t(r;0) = 0$ and the boundary condition that $u(a;t) = E\cos(\omega t)$, where E and ω are positive constants.*

Solution. In order to satisfy the wave equation, we begin with a separated solution (4.1.22). In order to satisfy the boundary condition, we must choose $kc = \omega$, $D = 0$, $(C\sin ka)/a = E$. This is possible provided that $\sin(\omega a/c) \neq 0$; in other words, $\omega a/c \neq n\pi$, $n = 1, 2, \ldots$. If this is satisfied, then we have a particular solution

$$U(r;t) = E\left(\frac{a\sin\omega r/c}{r\sin\omega a/c}\right)\cos\omega t \qquad \omega \neq \frac{n\pi c}{a}$$

We look for the solution of the initial-value problem as the above particular solution plus a sum of separated solutions with zero boundary conditions:

$$u(r;t) = E\left(\frac{a\sin\omega r/c}{r\sin\omega a/c}\right)\cos\omega t + \sum_{n=1}^{\infty}\frac{\sin n\pi r/a}{r}\left(C_n\cos\frac{n\pi ct}{a} + D_n\sin\frac{n\pi ct}{a}\right)$$

The second initial condition is satisfied by taking $D_n = 0$. To satisfy the first initial condition, we must have

$$0 = E\left(\frac{a\sin\omega r/c}{r\sin\omega a/c}\right) + \sum_{n=1}^{\infty} C_n\frac{\sin n\pi r/a}{r}$$

We must find the Fourier sine series expansion of the function $\sin(\omega r/c)$. This is easily obtained from the integrals

$$\int_0^a \sin\frac{\omega r}{c}\sin\frac{n\pi r}{a}\,dr = \frac{n\pi}{a}\frac{(-1)^n\sin\omega a/c}{(\omega/c)^2 - (n\pi/a)^2} \qquad \omega \neq \frac{n\pi c}{a}$$

which are now substituted into the series for $u(r;t)$ to obtain the result

$$u(r;t) = E\left(\frac{a\sin\omega r/c}{r\sin\omega a/c}\right)\cos\omega t \tag{4.1.25}$$

$$-\frac{2E}{a}\sin\frac{\omega a}{c}\sum_{n=1}^{\infty}\frac{n\pi}{a}\frac{(-1)^n}{(\omega/c)^2 - (n\pi/a)^2}\frac{\sin\pi r/a}{r}\cos\left(\frac{n\pi ct}{a}\right) \qquad \omega \neq \frac{n\pi c}{a}$$

If $\omega = n\pi c/a$ for some $n = 1, 2 \ldots$, then we can find the solution as a limiting case of (4.1.25). A particular solution satisfying the boundary conditions is found as

$$U(r;t) = E(-1)^n\left(\cos\frac{n\pi r}{a}\cos\frac{n\pi ct}{a} - \frac{ct}{r}\sin\frac{n\pi r}{a}\sin\frac{n\pi ct}{a}\right)$$

The solution $u(r;t)$ of the initial-value problem is obtained by examining the other terms in the above series solution. •

4.1.5. Convergence of series in three dimensions. We add a note of warning, prompted by the discussion of Example 4.1.3, where we found that the solution of the heat equation in spherical coordinates was badly approximated by the series solution for small time. This is closely related to the phenomenon of *nonlocalization*, which is particular to certain series of separated solutions in three dimensions. The following proposition is noted.

PROPOSITION 4.1.1. *Suppose that $\phi_n(r)$ are the eigenfunctions of the three-dimensional Sturm-Liouville problem*

$$\phi''(r) + \frac{2}{r}\phi'(r) + \lambda\phi = 0 \qquad 0 \le r < a$$

with the boundary condition $\phi(a) = 0$.

Let $f(r), 0 \le r < a$, be a piecewise smooth function with Fourier coefficients

$$A_n = \frac{\int_0^a f(r)\phi_n(r)\, r^2\, dr}{\int_0^a \phi_n(r)^2 r^2\, dr}$$

Then the orthogonal series $\sum_{n\ge 1} A_n\phi_n(r)$ is convergent for $0 < r < a$ to the value $(f(r+0) + f(r-0))/2$. At $r = 0$ the convergence takes place if and only if f satisfies the boundary condition $f(a) = 0$.

Proof. Explicitly, the eigenfunctions and Fourier coefficients are

$$\begin{aligned} \phi_n(r) &= \frac{1}{r}\sin\left(\frac{n\pi r}{a}\right) & n \ge 1 \\ A_n &= \frac{2}{a}\int_0^a r\, f(r)\sin\left(\frac{n\pi r}{a}\right) dr & n \ge 1 \end{aligned}$$

The convergence for $0 < r < a$ follows from Theorem 1.1 in Chapter 1, applied to the function $r\, f(r)$, whose Fourier sine series is precisely the above orthogonal series, multiplied by the factor r.

At $r = 0$ the eigenfunctions are defined by continuity: $\phi_n(0) = n\pi/a$. Integration by parts reveals that

$$\begin{aligned} A_n &= -\frac{2}{n\pi}\int_0^a r\, f(r)\frac{d}{dr}\left(\cos\left(\frac{n\pi r}{a}\right)\right) dr \\ &= (-1)^{n+1}\frac{2a\, f(a)}{n\pi} + \frac{2}{n\pi}\int_0^a \cos\left(\frac{n\pi r}{a}\right)\frac{d}{dr}(r\, f(r))\, dr \\ A_n\phi_n(0) &= 2(-1)^{n+1} f(a) + \frac{2}{a}\int_0^a \cos\left(\frac{n\pi r}{a}\right)\frac{d}{dr}(r\, f(r))\, dr \end{aligned}$$

The final integral is the Fourier cosine coefficient of the function $(rf)'$ and thus provides the general term of a convergent series. However, the first term will yield a divergent series, unless $f(a) = 0$. Indeed, the contribution of the first term is $2(-1)^{n+1} f(a)$, which is divergent unless $f(a) = 0$. •

In other words, this is the phenomenon of *nonlocalization*: the convergence of the series at $r = 0$ depends on the behavior of the function at the distant point $r = a$. This is a particular feature of certain orthogonal expansions in several variables, and is not present in one-dimensional Fourier series.

EXERCISES 4.1

In Exercises 1 to 7, use formula (4.1.5) to compute the indicated expressions.

1. $\nabla^2(r^3)$
2. $\nabla^2(r^2 \sin^3 \theta)$
3. $\nabla^2(r)$
4. $\nabla^2(\theta)$
5. $\nabla^2(r^2 \sin^2 \theta \cos 2\varphi)$
6. $\nabla^2(e^{3r})$
7. $\nabla^2(r^n)$, $n = 1, 2, \ldots$
8. Show that $\nabla^2[f(r)] = (1/r)(rf)''$ where $f(r)$ is a smooth function of one variable.
9. Show that the general solution of the equation $\nabla^2[f(r)] = 0$ is $f(r) = A + B/r$ for arbitrary constants A, B.
10. Show that the general solution of the equation $\nabla^2[f(r)] = -1$ is $f(r) = A + B/r - r^2/6$ for arbitrary constants A, B.
11. Solve the equation $\nabla^2[f(r)] = -1$ with the boundary condition $f(a) = 0$ and $f(0)$ finite.
12. Solve the equation $\nabla^2[f(r)] = -r^2$ with the boundary condition $f(a) = 0$ and $f(0)$ finite.
13. Solve the equation $\nabla^2[f(r)] = -r^4$ with the boundary condition $f(a) = 0$ and $f(0)$ finite.
14. Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u$, $-\infty < t < \infty$, in the sphere $0 \le r < a$ satisfying the boundary condition $u(a;t) = 3\cos 2t$.
15. Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u$, $-\infty < t < \infty$, in the sphere $0 \le r < a$ satisfying the boundary condition $u(a;t) = A_1 \cos \omega(t - t_0)$ where A_1, ω, and t_0 are positive constants.
16. Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u$, $-\infty < t < \infty$, in the sphere $0 \le r < a$ satisfying the boundary condition $u(a;t) = 2\cos 3t + 5\cos \pi t$.
17. Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u + \sigma$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a;t) = T_1$ and the initial condition $u(r;0) = T_2$. Use the five-stage method, and find the relaxation time.
18. Find the solution $u(r;t)$ of the heat equation $u_t = K\nabla^2 u$ in the sphere $0 \le r < 2a$ satisfying the boundary conditions $u(2a;t) = T_1$ and the initial conditions $u(r;0) = T_2$ for $0 \le r < a$ and $u(r;0) = 0$ for $a \le r < 2a$.

19. Suppose a ball 0.5 m in diameter, initially at 100°C, is placed into a refrigerator that instantaneously cools the surface to 0°C. Find the approximate temperature at the center at $t = 5$ minutes if the ball is made of (a) iron ($K = 0.15$ cgs unit) and (b) concrete ($K = 0.005$ cgs unit).
20. Consider the heat equation $u_t = K\nabla^2 u$ in the sphere $0 \le r < a$ with the boundary condition $\partial u/\partial r = 0$ at $r = a$. Find the separated solutions of the form $u(r;t) = f(r)g(t)$.
21. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the sphere $0 \le r < a$ with the boundary condition $\partial u/\partial r = 0$ at $r = a$ and the initial condition $u(r;0) = T_1$ for $0 \le r < \frac{1}{2}a$ and $u(r;0) = 0$ for $\frac{1}{2}a \le r < a$.
22. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the sphere $0 \le r < a$ with the boundary condition $-k(\partial u/\partial r) = (k/a)(u - T_1)$ at $r = a$ and the initial condition $u(r;0) = T_2$, $0 \le r < a$.
23. A green pea of radius $a = 0.25$ in, diffusivity $K = 0.75$ in^2/h, and Biot modulus $ha/k = 1.0$ is removed from a freezer and allowed to thaw. Find the relaxation time.
24. This problem is designed to show that $a\sqrt{\lambda_n} = (n - \frac{1}{2})\pi + O(1/n)$, $n \to \infty$, for the solutions of the equation $a\sqrt{\lambda_n} \cot a\sqrt{\lambda_n} = b$, $b = 1 - ah/k$.

    (a) From the graph in Fig. 4.1.2 show that $(n-1)\pi < a\sqrt{\lambda_n} < n\pi$, $n = 1, 2, 3, \ldots$.

    (b) Defining new variables ϵ, y by $a\sqrt{\lambda_n} = (n - \frac{1}{2})\pi + \epsilon$, $y = 1/(n - \frac{1}{2})\pi$, show that $by = (1 + \epsilon y)\tan\epsilon$ and $-\pi/2 < \epsilon < \pi/2$.

    (c) Use the method of implicit differentiation to show that the solution $\epsilon(y)$ with $\epsilon(0) = 0$ has $\epsilon'(0) = b$.

    (d) Conclude that $a\sqrt{\lambda_n} = (n - \frac{1}{2})\pi + b/n + O(1/n^2)$, $n \to \infty$.
25. Let $u = u(r;t)$ be a solution of the three-dimensional wave equation $u_{tt} = c^2\nabla^2 u$ and let $w(r;t) = ru(r;t)$. Show that w is a solution of the one-dimensional wave equation $w_{tt} = c^2 w_{rr}$.
26. Solve the following initial-value problem for the three-dimensional wave equation $u_{tt} = c^2\nabla^2 u$ in a sphere of radius a: $u(a;t) = 0$, $u(r;0) = 0$, $u_t(r;0) = (A/r)\sin n\pi r/a$, where $A > 0$ and $n = 1, 2, \ldots$.
27. Discuss the presence or absence of nonlocalization for the expansion in eigenfunctions of the Sturm-Liouville problem $\phi'' + (2/r)\phi'(r) + \lambda\phi = 0$, $0 \le r < a$, with the boundary condition that $\phi'(a) = 0$.

4.2. Legendre Functions and Spherical Bessel Functions

In the previous section we obtained separated solutions with radial symmetry by reducing to a one-dimensional problem. In the present section we deal with the general separated solutions, which involve new classes of special functions.

4.2.1. Separated solutions in spherical coordinates. Having obtained the form of $\nabla^2 u$, we can obtain general solutions of the heat equation in spherical coordinates. A fundamental first step is the construction of separated solutions.

$$u(r,\theta,\varphi;t) = R(r)\Theta(\theta)\Phi(\varphi)T(t) \tag{4.2.1}$$

Substituting this into the heat equation in spherical coordinates, we obtain

(4.2.2)
$$\frac{1}{R(r)}\left[R''(r) + \frac{2}{r}R'(r)\right] + \frac{1}{r^2\Theta(\theta)}\left[\Theta''(\theta) + \cot\theta\,\Theta'(\theta)\right] + \frac{\csc^2\theta\Phi''(\varphi)}{r^2\Phi(\varphi)} - \frac{T'(t)}{KT(t)} = 0$$

The first three terms are independent of t, whereas the final term is a function of t alone and therefore a constant, to be called $-\lambda$. Thus

$$\boxed{T'(t) + \lambda K\,T(t) = 0} \tag{4.2.3}$$

Multiplying (4.2.2) by r^2, we see that the second and third terms are independent of r, while the remainder depends upon r. Therefore we introduce a new separation constant $-\mu$. This produces the equation

$$\frac{1}{\Theta(\theta)}\left[\Theta''(\theta) + \cot\theta\,\Theta'(\theta)\right] + \csc^2\theta\frac{\Phi''(\varphi)}{\Phi(\varphi)} = -\mu \tag{4.2.4}$$

Multiplying (4.2.4) by $\sin^2\theta$, we see that the term $\Phi''(\varphi)/\Phi(\varphi)$ is independent of θ, whereas the remainder of (4.2.4) depends upon θ. Introducing a new separation constant $-\nu$, we have the equations

$$\boxed{\Phi''(\varphi) + \nu\Phi(\varphi) = 0} \tag{4.2.5}$$

and

$$\boxed{\Theta''(\theta) + \cot\theta\,\Theta'(\theta) + (\mu - \nu\csc^2\theta)\Theta(\theta) = 0} \tag{4.2.6}$$

Finally, the remaining part of (4.2.2) depends on r alone. Therefore we have the equation

$$\boxed{R''(r) + (2/r)\,R'(r) + (\lambda - (\mu/r^2))\,R(r) = 0} \tag{4.2.7}$$

Thus we have reduced the heat equation to four ordinary differential equations: (4.2.3), (4.2.5), (4.2.6), (4.2.7). These involve the three separation constants (λ,μ,ν), whose values will be determined below.

The solution of (4.2.3) is straightforward:

$$T(t) = e^{-\lambda K t}$$

which is identical to the form obtained previously in rectangular and cylindrical coordinates.

We are now ready to solve the angular equations (4.2.5) and (4.2.6). The product $\Theta(\theta)\Phi(\varphi)$ is called a *spherical harmonic.* It is our goal in this section to obtain a complete system of spherical harmonics.

Equation (4.2.5) is straightforward. Indeed, from the physical meaning of φ, we have the periodic boundary conditions $\Phi(-\pi) = \Phi(\pi)$, $\Phi'(-\pi) = \Phi'(\pi)$. Therefore

$$\begin{aligned} \Phi(\varphi) &= A\cos m\varphi + B\sin m\varphi \qquad m = 0, 1, 2, \ldots \\ \nu &= m^2 \end{aligned}$$

Equation (4.2.6) is the *associated Legendre equation.* To obtain solutions of this, we introduce the dimensionless variable $s = \cos\theta$. With this notation,

$$\frac{d\Theta}{d\theta} = -\sin\theta\frac{d\Theta}{ds} \qquad \text{and} \qquad \frac{d^2\Theta}{d\theta^2} = \sin^2\theta\frac{d^2\Theta}{ds^2} - \cos\theta\frac{d\Theta}{ds}$$

Thus (4.2.6) becomes

$$\boxed{(1-s^2)\frac{d^2\Theta}{ds^2} - 2s\frac{d\Theta}{ds} + \left(\mu - \frac{m^2}{1-s^2}\right)\Theta = 0} \tag{4.2.8}$$

This is a second-order differential equation with $s = 0$ an ordinary point; therefore we can expect a power series solution, of the form $\Theta(s) = \sum_{n=0}^{\infty} a_n s^n$.

4.2.2. Legendre polynomials. We first consider the case where $m = 0$, which is called the *Legendre equation:*

$$\boxed{(1-s^2)\Theta'' - 2s\Theta' + \mu\Theta = 0} \tag{4.2.9}$$

Assuming a power series solution $\Theta(s) = \sum_{n=0}^{\infty} a_n s^n$, we have

$$\Theta'(s) = \sum_{n=0}^{\infty} n a_n s^{n-1}, \qquad \Theta''(s) = \sum_{n=0}^{\infty} n(n-1)a_n s^{n-2}$$

Substituting these in (4.2.9), we have

$$\begin{aligned} 0 &= (1-s^2)\sum_{n=0}^{\infty} n(n-1)a_n s^{n-2} - 2s\sum_{n=0}^{\infty} n a_n s^{n-1} + \mu\sum_{n=0}^{\infty} a_n s^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n s^{n-2} + \sum_{n=0}^{\infty}[\mu - 2n - n(n-1)]a_n s^n \end{aligned}$$

The first sum is the same as $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}s^n$. Therefore

$$0 = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}s^n + \sum_{n=0}^{\infty}[\mu - n(n+1)]a_n s^n$$

Since a power series is zero if and only if all coefficients are zero, we must have

$$(n+2)(n+1)a_{n+2} + [\mu - n(n+1)]a_n = 0 \qquad n = 0, 1, 2, \ldots$$

This yields the *recurrence relation*

$$\boxed{a_{n+2} = \frac{n(n+1)-\mu}{(n+2)(n+1)} a_n \qquad n = 0, 1, 2, \ldots} \tag{4.2.10}$$

The constants a_0, a_1 may be chosen arbitrarily, and, for any value of μ, we obtain two linearly independent power series solutions of the Legendre equation. If μ is of the form $k(k+1)$ for some integer k, then the recurrence relation (4.2.10) shows that $a_n = 0$ for $n = k+2, k+4, \ldots$. Therefore we may obtain a polynomial solution in the following way: if k is even, choose $a_1 = 0$, $a_0 \neq 0$. Then from (4.2.10), $a_n = 0$ for $n = 3, 5, \ldots$. Combining this with the fact that $k+2$ is even, we see that $a_n = 0$ for all $n \geq 1 + k$; in other words, $\Theta(s)$ is a polynomial of degree k. In case k is odd, we choose $a_0 = 0$, $a_1 \neq 0$ and conclude in the same way that $a_n = 0$ for $n \geq 1+k$. In either case we conclude that *if* $\mu = k(k+1)$ *for an integer* k*, then* a_0*,* a_1 *can be chosen so that* $\Theta(s)$ *is a polynomial of degree* k. This polynomial, denoted $P_k(s)$, is called the *Legendre polynomial* of degree k.

In order to uniquely define $P_k(s)$, we require a standard choice of a_0, a_1. Although by no means canonical, we follow established conventions and choose these so that a_k, the coefficient of s^k, is $(2k)!/2^k(k!)^2$. We will now show that this implies the easily remembered fact that $P_k(1) = 1$.

EXAMPLE 4.2.1. *Compute* $P_2(s)$.

Solution. We take $k = 2$, $\mu = 2 \cdot 3 = 6$, and $a_2 = 4!/2^2(2!)^2 = \frac{3}{2}$. Taking $n = 0$ in (4.2.10) yields $a_2 = -\frac{6}{2}a_0$; hence $a_0 = -\frac{1}{2}$. Thus $P_2(s) = \frac{3}{2}s^2 - \frac{1}{2}$. •

We list the few Legendre polynomials and the corresponding values of $\mu = k(k+1)$.

k	μ	$P_k(s)$
0	0	1
1	2	s
2	6	$(3s^2-1)/2$
3	12	$(5s^3-3s)/2$
4	20	$(35s^4-30s^2+3)/8$

The Legendre polynomials satisfy the following orthogonality relations, which are a special case of the general orthogonality relations satisfied by the eigenfunctions of Sturm-Liouville eigenvalue problems studied in Chapter 1, Sec. 1.6.

PROPOSITION 4.2.1. *If* $k_1 \neq k_2$*, then*

$$\int_{-1}^{1} P_{k_1}(s) P_{k_2}(s)\, ds = 0 \qquad k_1 \neq k_2 \tag{4.2.11}$$

Proof. To prove the orthogonality, we write (4.2.9) for $P_{k_1}(s)$:

$$((1-s^2)P'_{k_1})' + k_1(k_1+1)\,P_{k_1}(s) = 0 \tag{4.2.12}$$

Multiply (4.2.12) by $P_{k_2}(s)$ and integrate on the interval $-1 < s < 1$.

$$\int_{-1}^{1} P_{k_2}(s)\left((1-s^2)P'_{k_1}(s)\right)'\,ds + k_1(k_1+1)\int_{-1}^{1} P_{k_2}(s)P_{k_1}(s)\,ds = 0$$

The first integral can be integrated by parts, and the endpoint terms vanish, leading to

$$-\int_{-1}^{1}(1-s^2)\,P'_{k_1}(s)P'_{k_2}(s)\,ds + k_1(k_1+1)\int_{-1}^{1} P_{k_1}(s)\,P_{k_2}(s)\,ds = 0$$

Now we interchange the roles of (k_1, P_{k_1}) and (k_2, P_{k_2}) to obtain

$$-\int_{-1}^{1}(1-s^2)\,P'_{k_1}(s)P'_{k_2}(s)\,ds + k_2(k_2+1)\int_{-1}^{1} P_k(s)\,P_l(s)\,ds = 0$$

When we subtract these two equations, the first integrals cancel, leading to

$$(k_1(k_1+1) - k_2(k_2+1))\int_{-1}^{1} P_{k_1}(s)\,P_{k_2}(s)\,ds = 0$$

But we have assumed $k_1 \neq k_2$; hence we conclude the required orthogonality. •

It is also important to recognize the orthogonality relation in spherical coordinates. Making the substitution $s = \cos\theta$, $ds = -\sin\theta\,d\theta$, we are led to

$$\int_0^{\pi} P_{k_1}(\cos\theta)P_{k_2}(\cos\theta)\sin\theta\,d\theta = 0 \qquad k_1 \neq k_2 \tag{4.2.13}$$

The graphs in Fig. 4.2.1 give the Legendre polynomials for $k = 0, 1, 2, 3, 4$.

We now use the orthogonality of Legendre polynomials to obtain an important representation of $P_k(s)$, known as *Rodrigues' formula.* To do this, we note that any polynomial Q of degree k can be written as a finite sum $\sum_{j=0}^{k} c_j P_j(s)$. Indeed, $P_k(s)$ is a polynomial of degree k and thus, by choosing the coefficient c_k suitably, we can arrange that $Q(s) - c_k P_k(s)$ is a polynomial of degree $k-1$. Continuing this iteratively, we arrive at the desired representation. We now apply this observation to $Q(s) = (d/ds)^k (s^2-1)^k$, a polynomial of degree k. Thus

$$\left(\frac{d}{ds}\right)^k (s^2-1)^k = \sum_{j=0}^{k} c_j P_j(s) \tag{4.2.14}$$

By orthogonality,

$$\int_{-1}^{1} P_j(s)\left(\frac{d}{ds}\right)^k (s^2-1)^k\,ds = c_j\int_{-1}^{1} P_j(s)^2 ds \qquad 0 \le j \le k \tag{4.2.15}$$

The left side can be integrated by parts k times. The endpoint terms vanish and, if $j < k$, $(d/ds)^k P_j(s) = 0$. Therefore $c_j = 0$ for $j < k$. To compute c_k, recall

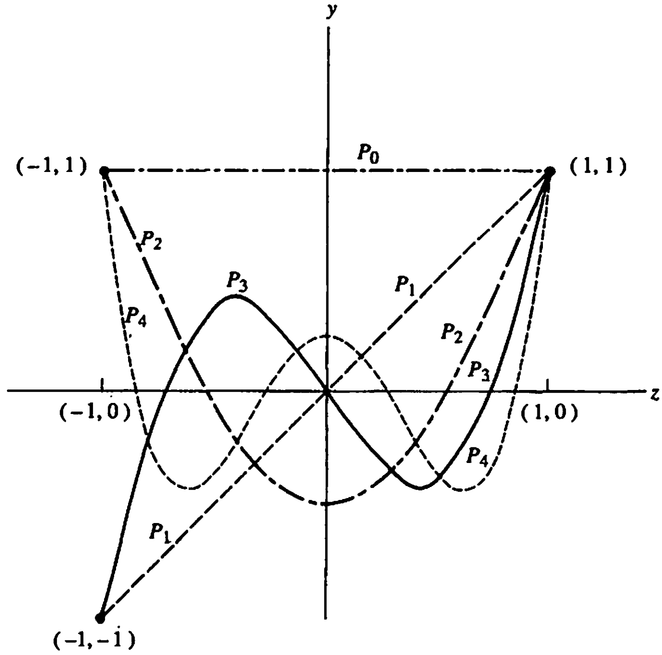


FIGURE 4.2.1 Graphs of the Legendre polynomials $y = P_k(s)$ for $k = 0, 1, 2, 3, 4$ and $-1 \le s \le 1$.

that the coefficient of s^k in $P_k(s)$ was taken to be $(2k)!/2^k(k!)^2$. On the other hand, the coefficient of s^k in $(d/ds)^k(s^2-1)^k$ is $2k(2k-1)\cdots(k+1) = (2k)!/k!$. Comparing the two, we see that $c_k(2k)!/2^k(k!)^2 = (2k)!/k!$; thus $c_k = 2^k k!$ Hence we have proved *Rodrigues' formula*,

$$\boxed{P_k(s) = \frac{1}{2^k k!}\left(\frac{d}{ds}\right)^k (s^2-1)^k} \tag{4.2.16}$$

This will now be used to compute the integrals that appear on the right side of (4.2.15). For this purpose we again integrate by parts k times and use

$(d/ds)^{2k}(s^2-1)^k = (2k)!$ to write

$$\begin{aligned}
(2^k k!)^2 \int_{-1}^{1} P_k(s)^2 ds &= \int_{-1}^{1} \left[\left(\frac{d}{ds}\right)^k (s^2-1)^k\right]^2 ds \\
&= (-1)^k \int_{-1}^{1} (s^2-1)^k \left(\frac{d}{ds}\right)^{2k} (s^2-1)^k ds \\
&= (2k)! \int_{-1}^{1} (1-s^2)^k \, ds
\end{aligned}$$

To do the final integral, denote its value by d_k. Then

$$d_k - d_{k-1} = -\int_{-1}^{1} s^2(1-s^2)^{k-1} ds = \frac{1}{2k}\int_{-1}^{1} s d(1-s^2)^k = -\frac{d_k}{2k}$$

Thus we have the recurrence relation $d_k/d_{k-1} = 2k/(2k+1)$ and the initial value $d_0 = 2$. Iterating these, we have

$$d_k = \frac{2k(2k-2)\cdots 4\cdot 2}{(2k+1)(2k-1)\cdots 5\cdot 3}2 = \frac{2^{2k+1}(k!)^2}{(2k+1)(2k)!}$$

Substituting this in the previous equation, we find that

$$\boxed{\int_{-1}^{1} P_k(s)^2 ds = \frac{2}{2k+1}} \tag{4.2.17}$$

The Legendre polynomials satisfy a three-term recurrence relation, which will now be derived. To do this, note that for any $n \geq 2$, $sP_{n-1}(s)$ is a polynomial of degree n with no term of degree $n-1$, and hence can be written as a linear combination

$$sP_{n-1}(s) = c_n^n P_n(s) + \sum_{j=0}^{n-2} c_j^n P_j(s) \quad n = 2, 3, \ldots$$

c_n^n is obtained from the coefficient of s^n; on the left side we have

$$\begin{aligned}
sP_{n-1}(s) &= s\frac{(2n-2)!}{2^{n-1}[(n-1)!]^2}s^{n-1} + \text{lower-order terms} \\
&= \frac{n}{2n-1}\frac{(2n)!}{2^n(n!)^2}s^n + \text{lower-order terms} \\
&= \frac{n}{2n-1}P_n(s) + \text{lower-order terms}
\end{aligned}$$

so that $c_n^n = n/(2n-1)$. To obtain the other coefficients, we have, by orthogonality,

$$\frac{2}{2j+1}c_j^n = \int_{-1}^{1} s\, P_{n-1}(s)\, P_j(s)\, ds \qquad 0 \leq j \leq n-2$$

From the above discussion we have

$$sP_j(s) = \frac{j+1}{2j+1}P_{j+1}(s) + \sum_{k=0}^{j-1} c_k^{j+1} P_k(s) \qquad 0 \le j \le n-2$$

In particular, if we take $j = n-2$, we have

$$\frac{2}{2(n-2)+1}c_{n-2}^n = \frac{n-1}{2n-3}\frac{2}{2n-1}$$

whereas the lower-order coefficients c_j^n are zero if $j < n-2$, by the orthogonality of the Legendre polynomials. Hence we have proved the recurrence relation in the form

$$\boxed{(2n-1)sP_{n-1}(s) = nP_n(s) + (n-1)P_{n-2}(s) \qquad n = 2,3,\ldots} \tag{4.2.18}$$

4.2.3. Legendre polynomial expansions. The integrals (4.2.17) can be used to compute the coefficients in the expansion of a function in a series of Legendre polynomials. Suppose that

$$f(s) = \sum_{k=0}^{\infty} A_k P_k(s)$$

and that the series converges uniformly for $-1 \le s \le 1$. Multiply by $P_j(s)$, integrate term by term, and use orthogonality to obtain

$$\int_{-1}^{1} f(s)P_j(s)ds = A_j \int_{-1}^{1} P_j(s)^2 ds = \frac{2A_j}{2j+1}$$

This formula motivates the following theorem, which can be used to expand an "arbitrary function" in a series of Legendre polynomials.

THEOREM 4.1. *Let $f(s)$, $-1 < s < 1$, be a piecewise smooth function. Let*

$$A_k = \frac{1}{2}(2k+1)\int_{-1}^{1} f(s)P_k(s)ds \qquad k = 0,1,2,\ldots \tag{4.2.19}$$

Then

$$\sum_{k=0}^{\infty} A_k P_k(s) = \frac{1}{2}[f(s+0) + f(s-0)] \qquad -1 < s < 1 \tag{4.2.20}$$

At $s = 1$ (resp. $s = -1$), the series converges to $f(1-0)$ (resp. $f(-1+0)$).

The proof of this theorem will not be given. We now give an example of the computation of the coefficients.

EXAMPLE 4.2.2. *Let*

$$f(s) = \begin{cases} 1 & a < s < b, -1 \le a < b \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

Find the expansion of $f(s)$ in a series of Legendre polynomials.

Solution. To find the coefficients A_k, $k \geq 1$, we write Legendre's equation in the form $[(1-s^2)P_k']' + k(k+1)P_k = 0$. Integrating this equation for $a < s < b$, we find that

$$(1-s^2)P_k'(s)\big|_a^b + k(k+1)\int_a^b P_k(s)\,ds = 0$$

Therefore

$$2A_k = \frac{2k+1}{k(k+1)}\left[(1-a^2)P_k'(a) - (1-b^2)P_k'(b)\right] \qquad k \geq 1$$

The coefficient A_0 is obtained from (4.2.19) as $A_0 = \frac{1}{2}(b-a)$. •

On the interval $0 < s < 1$, it is possible to expand a piecewise smooth function $f(s)$ in a Legendre series of the form $\sum_{n=1}^{\infty} A_{2n+1}P_{2n+1}(s)$. To do this, we extend f as an *odd* function, defined for $-1 < s < 1$, by setting $f(-s) = -f(s)$. Then the product $f(s)P_k(s)$ is an odd function for $k = 0, 2, 4, \ldots$ and an even function for $k = 1, 3, 5, \ldots$. Therefore we have

$$\begin{aligned} A_k &= \frac{1}{2}(2k+1)\int_{-1}^{1} f(s)P_k(s)ds \\ &= \begin{cases} 0 & k = 0,2,4,\ldots \\ (2k+1)\int_0^1 f(s)P_k(s)ds & k = 1,3,5,\ldots \end{cases} \end{aligned} \quad \bullet$$

EXAMPLE 4.2.3. *Expand $f(s) = 1$ in a series of the form $\sum_{n=0}^{\infty} A_{2n+1}P_{2n+1}(s)$.*

Solution. We have

$$\int_0^1 P_k(s)ds = \frac{1}{k(k+1)}P_k'(0)$$

Therefore

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \frac{4n+3}{(2n+1)(2n+2)} P_{2n+1}'(0)P_{2n+1}(s) \qquad 0 < s < 1 \\ &= \frac{3}{2}P_1(s) - \frac{7}{8}P_3(s) + \frac{11}{16}P_5(s)\cdots \end{aligned} \quad \bullet$$

4.2.4. Implementation with Mathematica. The Legendre polynmials can be generated by applying the three-term recurrence formula (4.2.18). We write this as

$$P_n(s) = \frac{1}{n}\left((2n-1)sP_{n-1}(s) - (n-1)P_{n-2}(s)\right) \qquad n = 2, 3, \ldots$$

and recall that $P_0(s) = 1$, $P_1(s) = s$. This is translated into Mathematica as follows.

```
LP[0,s_]:=1
LP[1,s_]:=s
LP[n_,s_]:=(1/n)((2n-1)s LP[n-1,s] -(n-1)LP[n-2,s])
```

This defines a function **LP** of the two variables **n** and **s**. If we now type **LP[5,s]** and simplify the resulting expression using **Simplify**, we obtain

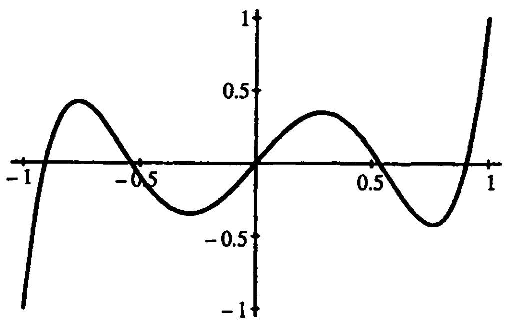
```
         s(15 - 70 s^2 + 63 s^4)
Out[5]= ----------------------
                   8
```

To graph **LP[5,s]**on the interval $-1 < s < 1$, we type

```
Plot[LP[5,s],{s,-1,1}];
```

The semicolon at the end of the line was typed in order to keep Mathematica from printing the word "Graphics" after the plot.

The result is



Now we use Mathematica to compute the integrals that occur in the orthogonality relations of the Legendre polynomials, namely,

$$\int_{-1}^{1} P_n(s)^2\, ds = \int_{-1}^{1} \mathbf{LP[n,s]}^2\, ds$$

For example, the command

```
Integrate[LP[5,s]^2,{s,-1,1}]
```

yields

```
         2
Out[6]= --
        11
```

which agrees with the value computed in (4.2.17).

One can also use Mathematica to verify that the Legendre polynomials produced here are indeed solutions of the Legendre equation

$$(1 - s^2)y'' - 2\,s\,y' + n(n+1)y = 0$$

Considering the case $n = 20$ as a test case, we define

```
y[s_]:=LP[20,s]
```

and apply the differentiation operator **D**. The first and second derivatives of **y[s]** are denoted **D[y[s],s]** and **D[y[s],{s,2}]**.These are combined into

```
(1-s^2)D[y[s],{s,2}]-2 s D[y[s],s]+20 21 y[s]
```

Further application of the **Simplify** command shows that the resultant expression is zero.

Finally, we note that Mathematica has a built-in routine for producing the Legendre polynomials, namely, **LegendreP[n,s]**. For example, typing **LegendreP[10,s]** produces the output

```
-63 + 3465 s^2 -30030 s^4 + 90090 s^6 - 109395 s^8 + 46189 s^10
---------------------------------------------------------------
                              256
```

4.2.5. Associated Legendre functions. Returning to the theory, we now consider the associated Legendre equation (4.2.6) for $m \neq 0$. To solve this equation, we write the equation for $\nu = 0$:

$$\Theta''(\theta) + \cot\theta\,\Theta'(\theta) + k(k+1)\Theta(\theta) = 0$$

Differentiating with respect to θ yields

$$\Theta'''(\theta) + \cot\theta\,\Theta''(\theta) + \left[k(k+1) - \frac{1}{\sin^2\theta}\right]\Theta'(\theta) = 0$$

Comparing this with (4.2.6), we see that we have obtained a solution for the value $\mu = k(k+1)$, $m = 1$. This is called the *associated Legendre function* $P_{k,1}(\cos\theta)$.

To obtain associated Legendre functions for higher values of m, we can use the formula

$$P_{k,m}(\cos\theta) = \sin^m\theta\, P_k^{(m)}(\cos\theta) \qquad m = 1, 2, \ldots, k \tag{4.2.21}$$

To prove (4.2.21), we differentiate the Legendre equation (4.2.9) m times with respect to s. Thus

$$\begin{aligned} 0 &= (1-s^2)\Theta'' - 2s\Theta' + k(k+1)\Theta \\ 0 &= (1-s^2)\Theta^{(m+2)} - 2ms\,\Theta^{(m+1)} - m(m-1)\Theta^{(m)} \\ &\quad - 2\left(s\,\Theta^{(m+1)} + m\,\Theta^{(m)}\right) + k(k+1)\Theta^{(m)} \end{aligned}$$

Hence

$$0 = (1-s^2)\Theta^{(m+2)} - (2m+2)s\Theta^{(m+1)} + [k(k+1) - m(m+1)]\Theta^{(m)} \tag{4.2.22}$$

k	m	$P_{k,m}(\cos\theta)$	$P_{k,m}(\cos\theta)\cos m\varphi$	$P_{k,m}(\cos\theta)\sin m\varphi$
0	0	1	1	0
1	0	$\cos\theta$	z	0
1	1	$\sin\theta$	x	y
2	0	$(3\cos^2\theta - 1)/2$	$(2z^2 - x^2 - y^2)/2$	0
2	1	$3\cos\theta\sin\theta$	$3yz$	$3xz$
2	2	$3\sin^2\theta$	$3(x^2 - y^2)$	$6xy$
3	0	$(5\cos^3\theta - 3\cos\theta)/2$	$(2z^3 - 3x^2z - 3y^2z)/2$	0
3	1	$(15\cos^2\theta - 3\sin\theta)/2$	$(12xz^2 - 3x^3 - 3xy^2)/2$	$(12yz^2 - 3x^2y - 3y^3)/2$
3	2	$15\cos\theta\sin^2\theta$	$15(x^2 - y^2)z$	$30xyz$
3	3	$15\sin^3\theta$	$15(x^3 - 3xy^2)$	$15(3x^2y - y^3)$

TABLE 4.2.1 Associated Legendre functions

Now let $g(s) = (1 - s^2)^{m/2}\Theta^{(m)}(s)$. We have

$$\begin{aligned}
g'(s) &= (1 - s^2)^{m/2}\Theta^{(m+1)} - ms(1 - s^2)^{m/2-1}\Theta^{(m)} \\
(1 - s^2)g' &= (1 - s^2)^{1+m/2}\Theta^{(m+1)} - ms(1 - s^2)^{m/2}\Theta^{(m)} \\
((1 - s^2)g')' &= (1 - s^2)^{1+m/2}\Theta^{(m+2)} - (m+2)s(1 - s^2)^{m/2}\Theta^{(m+1)} \\
&\quad - ms(1 - s^2)^{m/2}\Theta^{(m+1)} - m\Theta^{(m)}(1 - s^2)^{m/2-1}[1 - (m+1)s^2]
\end{aligned}$$

We use (4.2.22) and the definition of g to write

$$\begin{aligned}
[(1 - s^2)g']' &= (1 - s^2)^{m/2}[m(m+1) - k(k+1)]\Theta^{(m)} \\
&\quad - m[1 - (m+1)s^2]\Theta^{(m)}(1 - s^2)^{m/2-1} \\
&= [m(m+1) - k(k+1)]g - \frac{m[1 - (m+1)s^2]g}{1 - s^2} \\
&= \left[-k(k+1) + \frac{m^2}{1 - s^2}\right] g
\end{aligned}$$

Therefore g satisfies the associated Legendre equation (4.2.8) with $\mu = k(k+1)$, which was to be proved.

Since $P_k(s)$ is a polynomial of degree k, this procedure yields a nonzero function $g(s)$, provided that $m \le k$. This can be written in the form

$$\boxed{P_{k,m}(s) = (1 - s^2)^{m/2}\left(\frac{d}{ds}\right)^m P_k(s)} \tag{4.2.23}$$

The associated Legendre functions $P_{k,m}$ for $k = 0, 1, 2, 3$ are listed in Table 4.2.1, together with the corresponding harmonic polynomials in (x, y, z).

The associated Legendre functions satisfy the following orthogonality relation:

$$\int_{-1}^{1} P_{k_1,m}(s)P_{k_2,m}(s)ds = 0 \qquad k_1 \ne k_2 \tag{4.2.24}$$

The proof is exactly the same as for the orthogonality relation (4.2.11) for Legendre polynomials, and is therefore omitted.

To obtain the normalization coefficients, we use the notation $D = d/ds$ and write

$$\begin{aligned}\int_{-1}^{1} P_{k,m}(s)^2 ds &= \int_{-1}^{1} [(1-s^2)^{m/2} D^m P_k(s)]^2 ds \\ &= \int_{-1}^{1} (1-s^2)^m D^m P_k(s) D^m P_k(s) ds \\ &= (-1)^m \int_{-1}^{1} P_k(s) D^m (1-s^2)^m D^m P_k(s) ds\end{aligned}$$

where we have used the definition of $P_{k,m}$ and integrated by parts m times. The next step is to write the coefficient of $P_k(s)$ as a multiple of $P_k(s)$ plus lower-order terms. To do this, we recall that $P_k(s) = a_k s^k$ plus lower-order terms. Therefore $D^m P_k(s) = [k!a_k/(k-m)!]s^{k-m}$ plus lower-order terms, $(1-s^2)^m D^m P_k(s) = (-1)^m [k!a_k/(k-m)!]s^{k+m}$ plus lower-order terms, $D^m(1-s^2)^m D^m P_k(s) = (-1)^m \times [(k+m)!/(k-m)!]a_k s^k$ plus lower-order terms $= (-1)^m[(k+m)!/(k-m)!]P_k(s)$ plus lower-order terms. These lower-order terms can be written as linear combinations of the Legendre polynomials $P_n(s)$ with $n \le k-1$. By orthogonality all of the resulting integrals are zero. But the integral of $P_k(s)^2$ was shown to be $2/(2k+1)$. Therefore we have proved that

$$\int_{-1}^{1} P_{k,m}(s)^2 ds = \frac{(k+m)!}{(k-m)!} \frac{2}{2k+1} \qquad 0 \le m \le k \tag{4.2.25}$$

4.2.6. Spherical Bessel functions. We now turn to the analysis of the radial equation (4.2.7). This is a form of the general Bessel equation (3.2.1) with the parameters $d = 3$, $\mu = k(k+1)$. According to the theory from Sec. 3.2, this equation has a power series solution of the form $\sum_{n=0}^{\infty} a_n r^{\gamma+n}$, where $\gamma = -\frac{1}{2} + (k + \frac{1}{2}) = k$. If $\lambda > 0$ this is the *spherical Bessel function* of order k.

$$R(r) = j_k(r\sqrt{\lambda}) = r^k \left(a_0 + \sum_{n=1}^{\infty} a_n r^n \right)$$

From the results of Sec. 3.2, this function is expressed in terms of the standard Bessel function J_m by the relation

$$\frac{j_k(r)}{r^k} = a_0 \frac{J_m(r)}{r^m} \qquad m = k + \frac{1}{2}$$

The constant is determined by requiring that $j_0(0) = 1$, which leads to the choice $a_0 = \sqrt{\pi/2}$ and the definition

$$\boxed{j_k(r) = \left(\frac{\pi}{2r}\right)^{1/2} J_{k+1/2}(r)} \tag{4.2.26}$$

The orthogonality and normalization of the spherical Bessel function is a special case of the general theory of Bessel functions, which was treated in Chapter 3. To discuss this, we assume a boundary condition of the form

$$\cos\beta f(a) + a\sin\beta f'(a) = 0 \qquad 0 \le \beta \le \frac{\pi}{2} \tag{4.2.27}$$

Then we can summarize the orthogonality properties as follows.

PROPOSITION 4.2.2. *Let $j_k(r\sqrt{\lambda})$ and $j_k(r\sqrt{\lambda'})$ be two solutions of the spherical Bessel equation satisfying the boundary conditions (4.2.27). Then we have*

$$\int_0^a j_k(r\sqrt{\lambda})j_k(r\sqrt{\lambda'})r^2dr = 0 \qquad 0 \le \beta \le \frac{\pi}{2}, \lambda \ne \lambda' \tag{4.2.28}$$

(4.2.29)

$$\int_0^a j_k(r\sqrt{\lambda})^2r^2dr = \begin{cases} (\pi/4)a^2J'_{k+1/2}(a\sqrt{\lambda})^2 & \beta = \pi/2 \\ \pi a^2\left(\lambda + \cot^2\beta - m^2/a^2\right)J_{k+1/2}(a\sqrt{\lambda})^2 & 0 \le \beta < \pi/2 \end{cases}$$

We list below the spherical Bessel functions for $k = 0, 1, 2$.

k	$j_k(r)$
0	$(1/r)\sin r$
1	$(1/r^2)\sin r - (1/r)\cos r$
2	$(3/r^3 - 1/r)\sin r - (3/r^2)\cos r$

If $\lambda = 0$, the radial equation (4.2.7) is

$$R'' + \frac{2}{r}R' + \frac{k(k+1)}{r^2}R = 0$$

This is a form of Euler's equidimensional equation. We obtain the general solution by trying $R(r) = r^\gamma$, which leads to the quadratic equation $\gamma(\gamma+1)-k(k+1) = 0$. The solutions of this equation are $\gamma = k$, $\gamma = -(k+1)$. The first solution can be used to solve problems for Laplace's equation in the interior of a sphere, whereas the second solution can be used to solve Laplace equations in the exterior of a sphere.

Summary of results. Summarizing the results of this section, we have found the separated solutions of the heat equation in spherical coordinates:

$$u(r,\theta,\varphi;t) = j_k(r\sqrt{\lambda})P_{k,m}(\cos\theta)(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt} \qquad \lambda > 0$$

If $\lambda = 0$, we have the separated solutions of Laplace's equation

$$u(r,\theta,\varphi) = r^kP_{k,m}(\cos\theta)(A\cos m\varphi + B\sin m\varphi) \qquad \lambda = 0 \tag{4.2.30}$$

Separated solutions of the wave equation in spherical coordinates are written as

$$u(r,\theta,\varphi;t) = j_k(r\sqrt{\lambda})P_{k,m}(\cos\theta)(A\cos m\varphi + B\sin m\varphi)(C\cos ct\sqrt{\lambda} + D\sin ct\sqrt{\lambda})$$

All of the above solutions are valid throughout all of space. If we need solutions defined only in the exterior of a sphere, then the analysis of the radial equation allows us to choose the second solution of Bessel's equation, which becomes infinite when $r \to 0$. This function is written $R(r) = n_k(r\sqrt{\lambda})$, $\lambda > 0$. Similarly, when $\lambda = 0$, we may choose $R(r) = r^{-(k+1)}$. Thus we have the separated solutions of the heat and Laplace equations, valid for $r \neq 0$:

$$u(r,\theta,\varphi;t) = \begin{cases} n_k(r\sqrt{\lambda})P_{k,m}(\cos\theta)(A\cos m\varphi + B\sin m\varphi)e^{-\lambda Kt} & \lambda > 0 \\ r^{-(k+1)}P_{k,m}(\cos\theta)(A\cos m\varphi + B\sin m\varphi) & \lambda = 0 \end{cases}$$

Solutions of the wave equation in the exterior of the sphere are written

$$u(r,\theta,\varphi;t) = n_k(r\sqrt{\lambda})P_{k,m}(\cos\theta)(A\cos m\varphi + B\sin m\varphi)(C\cos ct\sqrt{\lambda} + D\sin ct\sqrt{\lambda})$$

It is important to note that the above separated solutions in (r,θ,φ) can be regarded as smooth solutions in (x,y,z). To see this, we first examine the solutions of Laplace's equation (4.2.30) in case $m = 0$.

If $k = 2n$ is even, then the polynomial P_k contains only even terms, so that

$$r^k P_k(\cos\theta) = r^{2n}P_{2n}(z/r) = r^{2n}\left(a_0 + \cdots + a_{2n}(z/r)^{2n}\right)$$

which is clearly a polynomial in the variables z^2, r^2, homogeneous of degree $2n$. Noting that $r^2 = x^2 + y^2 + z^2$, we see that we have a polynomial in the variables x, y, z, homogeneous of degree $2n$.

If $k = 2n+1$ is odd, then

$$\begin{aligned} r^k P_k(\cos\theta) &= r^{2n+1}P_{2n+1}(z/r) \\ &= r^{2n+1}\left(a_1(z/r) + \cdots + a_{2n+1}(z/r)^{2n+1}\right) \\ &= z\left(a_1 r^{2n} + \cdots + a_{2n+1}z^{2n}\right) \end{aligned}$$

which is clearly a polynomial in the variables (x,y,z), homogeneous of degree $2n+1$. This completes the analysis in case $m = 0$.

We can perform a similar analysis in case $m > 0$ by appealing to the form (4.2.23) of the associated Legendre function: $P_{km}(\cos\theta) = \sin^m\theta\, Q_{k-m}(\cos\theta)$, where Q_{k-m} is a polynomial of degree $k - m$. Recalling from Sec. 3.1 that $\rho^m\cos m\varphi$, $\rho^m\sin m\varphi$ are homogeneous polynomials of degree m in (x,y), we write

$$\begin{aligned} r^k P_{km}(\cos\theta)\cos m\varphi &= r^k\sin^m\theta\, Q_{k-m}(\cos\theta)\cos m\varphi \\ &= \left(r^{k-m}Q_{k-m}(\cos\theta)\right)\left(\rho^m\cos m\varphi\right) \end{aligned}$$

$$\begin{aligned} r^k P_{km}(\cos\theta)\sin m\varphi &= r^k\sin^m\theta\, Q_{k-m}(\cos\theta)\sin m\varphi \\ &= \left(r^{k-m}Q_{k-m}(\cos\theta)\right)\left(\rho^m\sin m\varphi\right) \end{aligned}$$

From the analysis in case $m = 0$, we see that the first factor is a homogeneous polynomial of degree $k - m$ in (x, y, z), whereas the second factors are homogeneous polynomials of degree m in (x, y). The product is clearly a homogeneous polynomial of degree k in (x, y, z).

This completes the analysis of Laplace's equation, where we have shown that the solid harmonics defined by (4.2.30) are homogeneous polynomials of degree k:

$$r^k P_{km}(\cos\theta)(A\cos m\varphi + B\sin m\varphi) = \sum_{a+b+c=k} A_{abc}x^a y^b z^c$$

A similar analysis can be applied to the separated solution of the heat equation in spherical coordinates. For example, we may write

$$u(r,\theta,\varphi;t) = \frac{j_k(r\sqrt{\lambda})}{r^k}\left(r^k P_{km}(\cos\theta)(A\cos m\varphi + B\sin m\varphi)\right)e^{-\lambda K t}$$

The first factor is a power series in $r^2 = x^2 + y^2 + z^2$ while the second factor is a polynomial—in particular, a smooth function of (x, y, z). This is multiplied by the exponential, which is also a smooth function of t. Hence the resulting separated solution is a smooth function of (x, y, z, t).

EXERCISES 4.2

1. Compute $P_1(0)$, $P_2(0)$, $P_3(0)$, $P_4(0)$.
2. Compute $P_1'(0)$, $P_2'(0)$, $P_3'(0)$, $P_4'(0)$.
3. Write down the Legendre polynomials $P_5(s)$, $P_6(s)$.
4. Show that $P_k(s)$ is an even function if $k = 0, 2, 4, \ldots$.
5. Show that $P_k(s)$ is an odd function if $k = 1, 3, 5, \ldots$.
6. Show that $P_k'(s)$ is an odd function if $k = 0, 2, 4, \ldots$.
7. Show that $P_k'(s)$ is an even function if $k = 1, 3, 5, \ldots$.
8. Use Rodrigues' formula to show that $P_k(1) = 1$ for $k = 0, 1, 2, 3, 4, \ldots$.
9. Use Exercises 4, 5, and 8 to show that $P_k(-1) = (-1)^k$ for $k = 0, 1, 2, 3, 4$.
10. It is known that $P_k(s)$ has exactly k zeros on the interval $-1 \le s \le 1$. Find these zeros for $k = 0, 1, 2, 3, 4$.
11. Let $f(s)$, $-1 < s < 1$, be a piecewise smooth function and define $a_k = (2k+1)/2\int_{-1}^{1} f(s)P_k(s)ds$.
    (a) If f is odd, show that $a_{2n} = 0$, $n = 0, 1, 2, \ldots$.
    (b) If f is even, show that $a_{2n+1} = 0$.
12. Let $f(s) = 0$ for $-1 < s < 0$ and $f(s) = 1$ for $0 < s < 1$. Find the expansion of $f(s)$ in a series of Legendre polynomials.
13. Let $f(s) = -1$ for $-1 < s < 0$ and $f(s) = 1$ for $0 < s < 1$. Find the expansion of $f(s)$ in a series of Legendre polynomials.
14. Let $f(s) = 1$ if $-\frac{1}{2} < s < \frac{1}{2}$ and $f(s) = 0$ otherwise. Find the first four terms in the expansion of $f(s)$ in a series of Legendre polynomials.
15. Write down the associated Legendre functions $P_{41}(s)$, $P_{42}(s)$, $P_{43}(s)$, $P_{44}(s)$.

16. Show that $P_{kk}(\cos\theta) = c_k(\sin\theta)^k$ for $k = 0, 1, 2, 3, \ldots$, $0 \le \theta \le \pi$, where c_k is a suitable constant.
17. Use Rodrigues' formula to show that

$$P'_{2n+1}(0) = (-1)^n \frac{(2n+2)!}{n!(n+1)!2^{2n+1}}, \qquad P'_{2n}(0) = 0$$

for $n = 0, 1, 2, \ldots$.
18. Let $f(s)$, $-1 < s < 1$, be a function with n continuous derivatives. Use Rodrigues' formula and integration by parts to show that the coefficients in the Legendre expansion of $f(s)$ can be written in the form

$$A_k = \frac{(-1)^k}{2^k k!} \int_{-1}^{1} (s^2-1)^k f^{(k)}(s)ds \qquad 0 \le k \le n$$

19. Use Exercise 18 to find the Legendre expansion of s^2, s^3, s^4.
20. Derive the generating function of Legendre polynomials: If $-1 < t < 1$, $-1 \le s \le 1$, then

$$(1 - 2ts + t^2)^{-1/2} = \sum_{k=0}^{\infty} t^k P_k(s)$$

Use the following steps:
(a) Write down the binomial series for $(1-\alpha)^{-1/2}$, $-1 < \alpha < 1$.
(b) Let $\alpha = 2st - t^2$ and use the binomial theorem to expand α^k.
(c) Rearrange the resulting double series to identify the coefficient of t^k.
21. Complete the details of the reduction of the spherical Bessel equation to the ordinary Bessel functions.

4.3. Laplace's Equation in Spherical Coordinates

In this section we consider boundary-value problems for Laplace's equation

$$0 = \nabla^2 u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\left(u_{\theta\theta} + \cot\theta\, u_\theta + \csc^2\theta\, u_{\varphi\varphi}\right)$$

From the results in Sec. 4.2 we have the following separated solutions of Laplace's equation in spherical coordinates:

$$(4.3.1)\quad u(r,\theta,\varphi) = (A_1 r^k + A_2 r^{-(k+1)}) P_{k,m}(\cos\theta)(A_3 \cos m\varphi + A_4 \sin m\varphi)$$
$$k = 0, 1, 2, \ldots; m = 0, 1, \ldots, k$$

The constants A_1, A_2, A_3, A_4 can be specialized to solve various boundary-value problems.

4.3.1. Boundary-value problems in a sphere. As a first application we find the solution of $\nabla^2 u = 0$ inside the sphere $0 \le r \le a$ satisfying the boundary condition $u(a,\theta,\varphi) = G(\theta)$, where $G(\theta)$ is a given piecewise smooth function. To do this, we note that, since we are looking for solutions inside the sphere, we need only consider separated solutions with $A_2 = 0$; otherwise the solution would not be defined at $r = 0$. Furthermore, the boundary condition is independent of φ; therefore we need only the separated solutions for $m = 0$. This leads to the choice

$$u(r,\theta) = \sum_{k=0}^{\infty} B_k r^k P_k(\cos\theta)$$

The boundary condition requires that $u(a,\theta) = \sum_{k=0}^{\infty} B_k a^k P_k(\cos\theta)$. By the orthogonality of Legendre polynomials, we must have

$$\int_0^{\pi} u(a,\theta) P_k(\cos\theta)\sin\theta\, d\theta = B_k a^k \int_0^{\pi} P_k(\cos\theta)^2 \sin\theta\, d\theta = B_k a^k \frac{2}{2k+1}$$

Therefore the coefficients B_k must be taken to be

$$B_k = \frac{k+\frac{1}{2}}{a^k}\int_0^{\pi} G(\theta) P_k(\cos\theta)\sin\theta\, d\theta$$

EXAMPLE 4.3.1. *Find the solution $u(r,\theta)$ of Laplace's equation in the sphere $0 \le r \le a$ satisfying $u(a,\theta) = 1$ if $0 < \theta < \pi/2$, $u(a,\theta) = 0$ if $\pi/2 < \theta < \pi$. Show that $u(r,\pi/2) = \frac{1}{2}$ for $0 \le r \le a$.*

Solution. In this case we can apply the above method with

$$B_k = \frac{k+\frac{1}{2}}{a^k}\int_0^{\pi/2} P_k(\cos\theta)\sin\theta\, d\theta \qquad k = 0,1,2,\ldots$$

This integral was evaluated in Sec. 4.2, with the result

$$\int_0^{\pi/2} P_k(\cos\theta)\sin\theta\, d\theta = \frac{P_k'(0)}{k(k+1)} \qquad k = 1,2,\ldots$$

Therefore $B_0 = \frac{1}{2}$, $B_k = [(k+\frac{1}{2})/k(k+1)]P_k'(0)$. The solution to the problem is

$$u(r,\theta) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{r}{a}\right)^k P_k'(0)\frac{k+\frac{1}{2}}{k(k+1)} P_k(\cos\theta)$$

$P_k(0) = 0$ if k is odd, while $P_k'(0) = 0$ if k is even. This shows that $u(r,\pi/2) = \frac{1}{2}$. ●

In some problems we can avoid computation of the integrals in the definition of B_k, when the boundary function $G(\theta)$ can be written as a sum of Legendre polynomials.

EXAMPLE 4.3.2. *Find the solution $u(r,\theta)$ of Laplace's equation in the sphere $0 \le r \le a$, satisfying $u(a,\theta) = 1 + 3\cos\theta + 3\cos^2\theta$.*

Solution. The boundary condition can be written in the form

$$\begin{aligned} u(a,\theta) = G(\theta) &= 2 + 3\cos\theta + 3\cos^2\theta - 1 \\ &= 2P_0(\cos\theta) + 3P_1(\cos\theta) + 2P_2(\cos\theta) \end{aligned}$$

Therefore the required solution to Laplace's equation is

$$u(r,\theta) = 2\left(\frac{r}{a}\right)^2 P_2(\cos\theta) + 3\frac{r}{a}P_1(\cos\theta) + 2P_0(\cos\theta) \quad \bullet$$

If the boundary conditions depend on (θ,φ), then we must use the separated solutions with $m \neq 0$. Consider, for example, the boundary-value problem for Laplace's equation inside the sphere $0 \le r \le a$, with $u(a,\theta,\varphi) = G(\theta,\varphi)$, a given function. We look for solutions in the form

$$u(r,\theta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} r^k P_{km}(\cos\theta)(A_{km}\cos m\varphi + B_{km}\sin m\varphi)$$

EXAMPLE 4.3.3. *Find the solution $u(r,\theta,\varphi)$ of Laplace's equation in the sphere $0 \le r \le a$, satisfying*

$$u(a,\theta,\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\theta\sin\varphi$$

Solution. From the table of associated Legendre functions in Sec. 4.2, we have $P_{11} = \sin\theta$, $P_{21} = 3\sin\theta\cos\theta$. Thus $u(a,\theta,\varphi) = \cos\varphi P_{11} + \frac{1}{3}\sin\varphi P_{21}$. Comparing this with the separated solutions in (4.3.1), we must have

$$u(r,\theta,\varphi) = rP_{11}(\cos\theta)\cos\varphi + \frac{r^2}{3}P_{21}(\cos\theta)\sin\varphi \quad \bullet$$

If $u(a,\theta,\varphi)$ is not already expressed as a sum of spherical harmonics, then we must compute integrals in order to solve the problem. Using orthogonality of the associated Legendre functions, we have for $m \neq 0$

$$\int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)P_{km}(\cos\theta)\cos m\varphi\sin\theta\,d\theta\,d\varphi = \pi A_{km}a^k\int_0^{\pi} P_{km}(\cos\theta)^2\sin\theta\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)P_{km}(\cos\theta)\sin m\varphi\sin\theta\,d\theta\,d\varphi = \pi B_{km}a^k\int_0^{\pi} P_{km}(\cos\theta)^2\sin\theta\,d\theta$$

while for $m = 0$ we must replace π by 2π.

4.3.2. Boundary-value problems exterior to a sphere. We can also use the separated solutions (4.3.1) to solve Laplace's equation in the exterior of a sphere, $r \ge a$. In this case we impose the requirement that $u(r,\theta,\varphi) \to 0$ when $r \to \infty$. This ensures the uniqueness of the solution. For example, the function $u_1(r) = 1 - a/r$ satisfies Laplace's equation for $r \ge a$ and is zero on the sphere $r = a$. The function $u_2(r) \equiv 0$ satisfies these same conditions. With this in mind we choose $A_1 = 0$ in the separated solutions (4.3.1) and get the general exterior solution

$$u(r,\theta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} r^{-(k+1)} P_{km}(\cos\theta)(A_{km}\cos m\varphi + B_{km}\sin m\varphi)$$

EXAMPLE 4.3.4. *Find the solution of Laplace's equation outside the sphere $r = a$ and satisfying $u(a,\theta) = 1$ if $0 < \theta < \pi/2$ and $u(a,\theta) = 0$ if $\pi/2 < \theta < \pi$.*

Solution. Since the boundary condition is independent of φ, we may take the separated solutions with $m = 0$; thus

$$u(r,\theta) = \sum_{k=0}^{\infty} A_k r^{-(k+1)} P_k(\cos\theta)$$

The Legendre expansion of $u(a,\theta)$ was obtained in Example 4.3.1.

$$u(a,\theta) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{k+\frac{1}{2}}{k(k+1)} P_k'(0) P_k(\cos\theta)$$

Therefore the required solution is

$$u(r,\theta) = \frac{1}{2}\frac{a}{r} + \sum_{k=1}^{\infty} \left(\frac{a}{r}\right)^{k+1} \frac{k+\frac{1}{2}}{k(k+1)} P_k'(0) P_k(\cos\theta) \quad \bullet$$

We now consider a more general boundary condition for Laplace's equation in the sphere $r \le a$.

$$\cos\alpha\, u(a,\theta,\varphi) + \sin\alpha \frac{\partial u}{\partial r}(a,\theta,\varphi) = G(\theta,\varphi)$$

Here α is a constant that may assume values between 0 and $\pi/2$; $\alpha = 0$ corresponds to the *Dirichlet problem*, which has already been solved; $\alpha = \pi/2$ corresponds to the *Neumann problem* where $\partial u/\partial r$ is specified on the boundary. The intermediate values of α correspond to the so-called mixed problem. Physically this occurs when we have free radiation of heat according to Newton's law of cooling.

To solve the problem, we begin with the series of separated solutions

$$u(r,\theta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} r^k P_{km}(\cos\theta)(A_{km}\cos m\varphi + B_{km}\sin m\varphi)$$

The boundary condition requires that

$$G(\theta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} [(a^k\cos\alpha + ka^{k-1}\sin\alpha) P_{km}(\cos\theta)(A_{km}\cos m\varphi + B_{km}\sin m\varphi)]$$

To obtain A_{km}, B_{km}, we expand $G(\theta,\varphi)$ in a series of spherical harmonics and equate coefficients. If

$$G(\theta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} P_{km}(\cos\theta)(\overline{A}_{km}\cos m\varphi + \overline{B}_{km}\sin m\varphi)$$

then we must have

$$\begin{aligned} \overline{A}_{km} &= (a^k\cos\alpha + ka^{k-1}\sin\alpha)A_{km} \qquad k=0,1,2,\ldots;0\le m\le k \\ \overline{B}_{km} &= (a^k\cos\alpha + ka^{k-1}\sin\alpha)B_{km} \qquad k=0,1,2,\ldots;0\le m\le k \end{aligned}$$

The coefficients of A_{km}, B_{km} are nonzero unless both $\alpha = 0$ and $k = 0$. Thus we may solve for A_{km} and B_{km} and obtain the formal solution of the problem. If $\alpha = 0$, then we see that the coefficient of A_{00} is undetermined; to uniquely specify the solution, we take $A_{00} = 0$. We summarize these computations in the following proposition.

PROPOSITION 4.3.1. *If $0 < \alpha \le \pi/2$, the solution of Laplace's equation in the sphere $0 \le r < a$ with the boundary condition $\cos\alpha\, u + \sin\alpha\, \partial u/\partial r = G$ is uniquely determined by the above procedure. If $\alpha = \pi/2$, then G must satisfy the additional condition $\int_0^{2\pi}\int_0^{\pi} G(\theta,\varphi)\sin\theta\, d\theta\, d\varphi = 0$. In this case $u(r,\theta,\varphi)$ is uniquely determined by requiring $u(0,\theta,\varphi) = 0$.*

EXAMPLE 4.3.5. *Find the solution of $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $\partial u/\partial r + u = \cos\theta$.*

Solution. According to the above procedure, we have $\alpha = \pi/4$ and $G(\theta,\varphi) = (1/\sqrt{2})\cos\theta$. Assuming a solution of the form $u = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta)$, we must have $\cos\theta = \sum_{k=0}^{\infty}(k+a)a^{k-1}A_kP_k(\cos\theta)$; thus $A_k = 0$ for $k \ne 1$ and $1 = (k+a)A_k$. Thus the solution is $u(r,\theta) = (r\cos\theta)/(1+a)$. •

The formal solutions we have obtained may be established as rigorous solutions with little difficulty. To be specific, we consider the Laplace equation in the sphere $0 \le r < a$ with the boundary condition $u(a;0) = G(\theta)$, a given piecewise smooth function for $0 < \theta < \pi$. The formal solution to the problem was found to be

$$u(r,\theta) = \sum_{k=0}^{\infty} A_k \left(\frac{r}{a}\right)^k P_k(\cos\theta) \qquad 0 \le r \le a \tag{4.3.2}$$

where

$$A_k = \left(k+\frac{1}{2}\right)\int_0^{\pi} G(\theta)P_k(\cos\theta)\sin\theta\, d\theta \qquad k = 0,1,2,\ldots$$

The Legendre polynomials satisfy $|P_k| \le 1$, $|P_k'| \le k^2$, $|P_k''| \le k^4$. To show that $u_r = \sum_{k=0}^{\infty} A_k k(r/a)^{k-1}P_k(\cos\theta)$, we note that the terms of this series are no larger than $M(2k+1)/(r/a)^{k-1}$, where M is the maximum of $G(\theta)$, $0 \le \theta \le \pi$. If $r_0 < a$, these terms are no larger than $M(2k+1)/[(a+r_0)/2a]^{k-1}$ for $0 \le r \le$

$\frac{1}{2}(a + r_0)$. By the Weierstrass M-test, the series for u_r is uniformly convergent for $0 \le r \le \frac{1}{2}(a + r_0)$. In a similar fashion it may be shown that the series for u_{rr}, u_θ, $u_{\theta\theta}$ are also uniformly convergent. Thus $u(r, \theta)$ defined by (4.3.2) is in fact a rigorous solution of Laplace's equation in the sphere $0 \le r < a$.

4.3.3. Applications to potential theory. Solutions of Laplace's equation also arise in problems of electrostatic or gravitational attraction, where we are given a distribution of mass or charge and are required to compute the resultant potential energy function.

The potential energy corresponding to the density function $F(x, y, z)$ is defined by the integral

$$u(x, y, z) = \frac{1}{4\pi} \iiint \frac{F(x', y', z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} dx'dy'dz'$$

where the integral is extended over all of three-dimensional space. In Chapter 8 it will be shown that this integral is the solution of *Poisson's equation*

$$\nabla^2 u = -F$$

In particular if $F(x, y, z) = 0$ outside the sphere $r = a$, then u is a solution of Laplace's equation there. However, we will not make direct use of this fact.

On the other hand, from the previous subsection, we have the following general formula for solutions of Laplace's equation in the exterior of the sphere:

$$u(r, \theta, \varphi) = \frac{A_{00}}{r} + \sum_{k=1}^{\infty} \frac{1}{r^{k+1}} \sum_{m=0}^{k} P_{km}(\cos\theta) \left(A_{km} \cos m\varphi + B_{km} \sin m\varphi\right) \tag{4.3.3}$$

The next result shows that the coefficients A_{km}, B_{km} can be obtained as the *moments* of the density function $F(x, y, z)$. In order to simplify the notation, we let $Y_{k0} = r^k P_k$, $Y_{km}^{\cos} = r^k P_{km} \cos m\varphi$, and $Y_{km}^{\sin} = r^k P_{km} \sin m\varphi$ be the harmonic polynomials introduced above.

THEOREM 4.2. *Suppose that the function $u(r, \theta, \varphi)$ satisfies $\nabla^2 u = -F$ and is represented by (4.3.3) in the exterior of the sphere $r = a$. Then the coefficients are obtained as*

$$A_{00} = \frac{1}{4\pi} \iiint F \, dV \tag{4.3.4}$$

$$A_{k0} = \frac{1}{4\pi} \iiint F \, Y_{k0} \, dV \quad k \ge 1 \tag{4.3.5}$$

$$\frac{(k+m)!}{(k-m)!} A_{km} = \frac{1}{2\pi} \iiint F \, Y_{km}^{\cos} \, dV \quad 1 \le m \le k \tag{4.3.6}$$

$$\frac{(k+m)!}{(k-m)!} B_{km} = \frac{1}{2\pi} \iiint F \, Y_{km}^{\sin} \, dV \quad 1 \le m \le k \tag{4.3.7}$$

where the integrals are taken over the solid sphere $0 \le r \le a$.

Proof. We use the divergence theorem in the sphere $0 \le r \le a$, first applied to the equation $\operatorname{div}(\operatorname{grad} u) = -F$:

$$\iint_{r=a} \operatorname{grad} u \cdot \mathbf{n}\, dS = -\iiint_{r\le a} F\, dx\, dy\, dz$$

The integrand on the left side is $\partial u/\partial r = -A_{00}/r^2 + \cdots$. From the orthogonality relations for Legendre functions, the higher terms integrate to zero on the sphere, so that we obtain

$$-\frac{A_{00}}{a^2} 4\pi a^2 = -\iiint_{r<a} F\, dx\, dy\, dz$$

which proves (4.3.4). To prove (4.3.5) and (4.3.6), we consider the solid harmonic $Y_{km} = r^k P_{km} \cos m\varphi$ and write

$$\begin{aligned}
\operatorname{div}(Y_{km} \operatorname{grad} u) &= Y_{km}\nabla^2 u + \operatorname{grad} Y_{km} \cdot \operatorname{grad} u = -FY_{km} + \operatorname{grad} Y_{km} \cdot \operatorname{grad} u \\
\operatorname{div}(u \operatorname{grad} Y_{km}) &= u\nabla^2 Y_{km} + \operatorname{grad} Y_{km} \cdot \operatorname{grad} u = \operatorname{grad} Y_{km} \cdot \operatorname{grad} u
\end{aligned}$$

Hence

$$FY_{km} = \operatorname{div}(u \operatorname{grad} Y_{km} - Y_{km} \operatorname{grad} u)$$

to which we apply the divergence theorem to obtain

$$\iiint_{0\le r\le a} F\, Y_{km}\, dx\, dy\, dz = \iint_{r=a} \left(u\frac{\partial Y_{km}}{\partial r} - Y_{km}\frac{\partial u}{\partial r} \right) dS$$

The terms on the right side are easily computed, using

$$\begin{aligned}
\frac{\partial Y_{km}}{\partial r} &= kr^{k-1} P_{km} \cos m\varphi \\
\frac{\partial u}{\partial r} &= -\frac{A_0}{r^2} - \sum_{k=1}^{\infty} \frac{k+1}{r^{k+2}} \sum_{m=0}^{k} P_{km}(\cos\theta)\, (A_{km} \cos m\varphi + B_{km} \sin m\varphi)
\end{aligned}$$

If we collect terms and use the orthogonality and normalization of the Legendre functions, we obtain

$$\begin{aligned}
\iiint_{0\le r\le a} F\, Y_{km}\, dx\, dy\, dz &= (2k+1)A_{km} \int_0^{\pi}\int_0^{2\pi} P_{km}^2(\cos\theta) \sin\theta \cos^2 m\varphi\, d\theta\, d\varphi \\
&= 2\pi\, A_{km} \frac{(k+m)!}{(k-m)!}
\end{aligned}$$

which completes the required identification of A_{km}. The proof for B_{km} is entirely similar.

These techniques can be used to compute the gravitational potential of an oblate spheroid, as an approximation to the earth's shape.

EXAMPLE 4.3.6. *Suppose the mass of a solid body is uniformly distributed over the region defined in spherical coordinates by the inequalities*

$$0 \le r \le a + b\sin\theta \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$

where $0 < b < a$. *Find the coefficients* $A_{00}, A_{10}, A_{11}, A_{20}, .., B_{22}$ *in the expansion (4.3.3) of the potential function.*

Solution. From the data, the density function in spherical coordinates is

$$F(r,\theta,\varphi) = \begin{cases} \delta & \text{if } 0 \le r \le a + b\sin\theta, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

The coefficient A_{00} is computed as

$$\begin{aligned} A_{00} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{a+b\sin\theta} \delta r^2 \sin\theta\, dr\, d\theta\, d\varphi \\ &= \frac{\delta}{6}\int_0^{\pi}(a+b\sin\theta)^3 \sin\theta\, d\theta \\ &= \frac{\delta}{6}\left(2a^3 + \frac{3}{2}\pi a^2 b + 4ab^2 + \frac{3\pi b^3}{8}\right) \end{aligned}$$

This is the total mass of the solid body. The next coefficient is

$$\begin{aligned} A_{10} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{a+b\cos\theta} \delta\, \cos\theta\, r^3 \sin\theta\, dr\, d\theta\, d\varphi \\ &= \frac{\delta}{8}\int_0^{\pi}(a+b\sin\theta)^4 \cos\theta \sin\theta\, d\theta = 0 \end{aligned}$$

since the integrand is an odd function with respect to the transformation $\theta \to \pi - \theta$. Similarly, the integrals that define A_{11}, B_{11}, A_{21}, A_{22}, B_{21}, B_{22} are also zero, since they contain integrals of the form $\int_0^{2\pi} \cos m\varphi\, d\varphi$, $\int_0^{2\pi} \sin m\varphi\, d\varphi$, which are zero if $m = 1, 2$. It remains to compute

$$\begin{aligned} A_{20} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{a+b\sin\theta} \delta P_2(\cos\theta) r^4 \sin\theta\, dr\, d\theta\, d\varphi \\ &= \frac{\delta}{20}\int_0^{\pi}(a+b\sin\theta)^5(3\cos^2\theta - 1)\sin\theta\, d\theta \end{aligned}$$

Using the integration formulas

$$\int_0^{\pi} \sin^{2m}\theta \cos^{2n}\theta\, d\theta = (m - 1/2)!(n - 1/2)!/(m+n)!$$

we obtain

$$A_{20} = -\frac{\delta}{20}\left[\frac{5a^4 b\pi}{8} + \frac{16a^3b^2}{3} + \frac{15a^2b^3\pi}{8} + \frac{64ab^4}{21} + \frac{25b^5\pi}{128}\right]$$

If $b << a$, this will be small compared with A_0, the principal term in the expansion of (4.3.3).

EXERCISES 4.3

1. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a, \theta) = 3 + 4\cos\theta + 2\cos^2\theta$.
2. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a, \theta) = 1 + \cos 2\theta$.
3. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a, \theta) = (5 - 4\cos\theta)^{-1/2}$. *Hint:* Use the generating function for Legendre polynomials

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad -1 < t < 1, \quad -1 \le x \le 1$$

4. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary conditions $u(a, \theta) = 1$ if $0 < \theta < \pi/2$ and $u(a; \theta) = 0$ otherwise.
5. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary conditions $u(a, \theta) = 1$ if $0 < \theta < \pi/2$ and $u(a, \theta) = -1$ if $\pi/2 < \theta < \pi$. Find $u(r, \pi/2)$.
6. Show that the functions $r^{2n-1} P_{2n-1}(\cos\theta)$, $n = 1, 2, \ldots$, satisfy Laplace's equation in the hemisphere $0 \le r < a$, $0 < \theta < \pi/2$, with the boundary condition $u(r, \pi/2) = 0$, $0 \le r < a$.
7. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the hemisphere $0 \le r < a$, $0 < \theta < \pi/2$ satisfying the boundary conditions $u(r, \pi/2) = 0$, $0 \le r < a$, and $u(a, \theta) = 4\cos\theta + 2\cos^3\theta$, $0 < \theta < \pi/2$.
8. Find the solution of Laplace's equation in the hemisphere $0 \le r < a$, $0 < \theta < \pi/2$ satisfying the boundary conditions $u(r, \pi/2) = 0$ for $0 < r < a$ and $u(a; \theta) = 1$ for $0 < \theta < \pi/2$.
9. Find the solution of Laplace's equation $\nabla^2 u = 0$ outside the sphere $r \ge a$ satisfying the boundary condition $u(a; \theta) = 1 + 2\cos\theta + \cos^4\theta$, $0 < \theta < \pi$.
10. Find the solution of Laplace's equation $\nabla^2 u = 0$ outside the sphere $r \ge a$ satisfying the boundary condition $u(a; \theta) = 1$ if $0 < \theta < \pi/2$ and $u(a, \theta) = -1$ if $\pi/2 < \theta < \pi$.
11. Find the solution of Laplace's equation $\nabla^2 u = 0$ outside the sphere $r \ge a$ with the boundary condition $(\partial u/\partial r)(a, \theta) = \cos\theta + 3\cos^3\theta$, $0 < \theta < \pi$.
12. Find the solution of Laplace's equation $\nabla^2 u = 0$ outside the sphere $r \ge a$ with the boundary condition $(\partial u/\partial r)(a, \theta) + u(a, \theta) = 3\cos\theta$, $0 < \theta < \pi$.
13. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a, \theta, \varphi) = \sin\theta\cos\varphi + \sin^2\theta\sin 2\varphi$.
14. Find the solution of Laplace's equation $\nabla^2 u = 0$ outside the sphere $r > a$ satisfying the boundary condition $u(a, \theta, \varphi) = \sin\theta\cos\varphi + \sin^2\theta\sin 2\varphi$.
15. Let $u(r, \theta, \varphi)$ be the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$, with the boundary condition $u(a, \theta, \varphi) = G(\theta, \varphi)$ for $0 < \theta < \pi$, $0 < \varphi < 2\pi$. Show that $u(0, \theta, \varphi) = (1/4\pi)\int_0^{2\pi}\int_0^{\pi} G(\theta, \varphi)\sin\theta\, d\theta\, d\varphi$.

16. Let $u(r, \theta, \varphi)$ be the solution of Laplace's equation $\nabla^2 u = 0$ outside the sphere $r \geq a$, with the boundary condition $u(a, \theta, \varphi) = G(\theta, \varphi)$ for $0 < \theta < \pi$, $0 < \varphi < 2\pi$. Show that, when $r \to \infty$,

$$u(r, \theta, \varphi) = \frac{1}{4\pi r} \int_0^{2\pi} \int_0^{\pi} G(\theta, \varphi) \sin\theta \, d\theta \, d\varphi + O\left(\frac{1}{r^2}\right)$$

17. A solid hemisphere of uniform density δ occupies the region defined by the inequalities $0 \leq r \leq a$, $0 \leq \theta \leq \pi/2$, $0 \leq \varphi \leq 2\pi$. Find all of the coefficients in the representation (4.3.3) of the potential function.
18. A spherical slab of uniform density δ occupies the region defined by the inequalities $0 \leq r \leq a$, $\pi/6 \leq \theta \leq 5\pi/6$, $0 \leq \varphi \leq 2\pi$. Find all of the coefficients in the representation (4.3.3) of the potential function.

CHAPTER 5

FOURIER TRANSFORMS AND APPLICATIONS

INTRODUCTION

In previous chapters we obtained the solution of initial- and boundary-value problems in bounded regions, obtaining the solution in terms of Fourier series of separated solutions. In this chapter we consider problems in unbounded regions, which require a continuous superposition of separated solutions, leading to the notion of a Fourier *integral* representation. In many cases this can be rewritten to give an explicit representation as an integral transform of the initial/boundary data with a standard function, the *fundamental solution* of the problem.

In Sec. 5.1 we develop the properties of the Fourier transform in its own right. The following sections present the applications to the heat equation, Laplace's equation, the wave equation, and the telegraph equation.

5.1. Basic Properties of the Fourier Transform

5.1.1. Passage from Fourier series to Fourier integrals. To motivate the definition of the Fourier transform, we look for a suitable generalization of the Fourier series to infinite intervals. We assume that $f(x), -\infty < x < \infty$, is a real- or complex-valued piecewise smooth function and that all of the relevant integrals can be suitably defined. By restricting $f(x)$ to the interval $-L < x < L$, we can form the complex Fourier series:

$$f(x) = \sum_{-\infty}^{\infty} \alpha_n e^{(in\pi x/L)} \tag{5.1.1}$$

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-(in\pi x/L)}dx \tag{5.1.2}$$

Define a discrete variable μ_n by

$$\mu_n = \frac{n\pi}{L} \qquad n = 0, \pm 1, \ldots$$

$$\Delta\mu_n = \mu_{n+1} - \mu_n = \frac{\pi}{L}$$

The partial Fourier transform of f on the interval $-L < x < L$ is defined by

$$F_L(\mu) = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{-i\mu x}\,dx \tag{5.1.3}$$

In this notation, formulas (5.1.1) and (5.1.2) can be written in the form

$$f(x) = \sum_{-\infty}^{\infty} F_L(\mu_n) e^{i\mu_n x} \Delta\mu_n \tag{5.1.4}$$

$$\alpha_n = F_L(\mu_n)\Delta\mu_n \tag{5.1.5}$$

In this form we take the limit $L \to \infty$; (5.1.4) is an approximating sum for an (improper) Riemann integral. Taking the limit formally, (5.1.4) and (5.1.3) become

$$\boxed{f(x) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x} d\mu} \tag{5.1.6}$$

$$\boxed{F(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx} \tag{5.1.7}$$

This completes the heuristic derivation of the Fourier transform formulas. $F(\mu)$, defined by (5.1.7), is the *Fourier transform* of $f(x)$. Formula (5.1.6) is the *Fourier inversion formula.*

In addition to these formulas, we derive the analogue of Parseval's equality. To obtain this, we recall from (5.1.1) and (5.1.5)

$$\int_{-L}^{L} |f(x)|^2 \, dx = 2L \sum_{-\infty}^{\infty} |\alpha_n|^2 = 2\pi \sum_{-\infty}^{\infty} |F_L(\mu_n)|^2 \Delta\mu_n$$

Taking the limit $L \to \infty$, we have *Parseval's theorem for Fourier transforms*:

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 2\pi \int_{-\infty}^{\infty} |F(\mu)|^2 \, d\mu} \tag{5.1.8}$$

The preceding derivations, although intuitively attractive, must be interpreted correctly. In particular, the integral (5.1.6) must be understood as the limit of the integral on $(-M, M)$ when $M \to \infty$, a so-called Cauchy principal value. This is clarified in the next subsection, where we give the mathematically rigorous definitions and theorems.

In case the function $f(x), -\infty < x < \infty$, is real-valued, the Fourier inversion formula (5.1.6) can be written in terms of the real-valued transforms

$$A(\mu) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \mu x \, dx, \qquad B(\mu) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \mu x \, dx$$

Note that $A(\mu)$ is an even function and that $B(\mu)$ is an odd function, with $A(\mu) - iB(\mu) = 2F(\mu)$.

When we substitute this in (5.1.6) and simplify, we obtain

$$f(x) = \int_0^{\infty} (A(\mu) \cos \mu x + B(\mu) \sin \mu x) d\mu$$

This is analogous to the formula for the Fourier series of real-valued functions from Chapter 1. The appropriate version of Parseval's theorem is

$$\int_{-\infty}^{\infty} f(x)^2\,dx = \pi \int_0^{\infty} [A(\mu)^2 + B(\mu)^2]\,d\mu$$

5.1.2. Definition and properties of the Fourier transform. The Fourier transform of a complex-valued function $f(x), -\infty < x < \infty$, is defined by the integral (5.1.7). The following theorem gives precise conditions for the existence of the integrals and for the Fourier representation formula (5.1.6)

THEOREM 5.1. **(Convergence theorem for Fourier transforms.)** *Let $f(x)$, $-\infty < x < \infty$, be piecewise smooth on each finite interval such that $\int_{-\infty}^{\infty}|f(x)|dx < \infty$. Define the Fourier transform $F(\mu)$ by (5.1.7). Then for each x,*

$$\lim_{M\to\infty} \int_{-M}^{M} F(\mu)e^{i\mu x}d\mu = \frac{1}{2}f(x+0) + \frac{1}{2}f(x-0) \tag{5.1.9}$$

We emphasize that the limit (5.1.9) is a *Cauchy principal value*. The integral over the infinite interval $-\infty < \mu < \infty$ will not be absolutely convergent in general, as we will see through simple examples. Furthermore, one cannot in general take the double limit of the partial integrals $\int_{-M_1}^{M_2}$ when M_1, $M_2 \to \infty$ in any order.

We now give some examples of computation of Fourier transforms.

EXAMPLE 5.1.1. *Find the Fourier transform of the "square wave"*

$$f(x) = \begin{cases} 0 & x < a \\ 1 & a \le x \le b \\ 0 & x > b \end{cases}$$

and illustrate the convergence theorem.

Solution. For $\mu = 0$, (5.1.7) shows that $F(0) = (b-a)/2\pi$. Otherwise, for $\mu \neq 0$ we have

$$\begin{aligned} F(\mu) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\mu x}dx \\ &= \frac{1}{2\pi}\int_a^b e^{-i\mu x}dx \\ &= \frac{e^{-i\mu b} - e^{-i\mu a}}{-2\pi i\mu} \end{aligned}$$

The convergence theorem equation (5.1.9) states that

$$\lim_{M\to\infty}\int_{-M}^{M}\frac{(e^{-i\mu b}-e^{-i\mu a})e^{i\mu x}}{-2\pi i\mu}\,d\mu=\begin{cases}0 & x<a\\ \frac{1}{2} & x=a\\ 1 & a<x<b\\ \frac{1}{2} & x=b\\ 0 & x>b\end{cases}$$

This integral is not absolutely convergent, and the separate integrals $\int_0^M$ and $\int_{-M}^0$ do not have limits in general. •

The next theorem gives some general properties of the Fourier transform.

THEOREM 5.2. *The Fourier transform enjoys the following properties.*

- *(Linearity) If F_1 is the Fourier transform of f_1 and F_2 is the Fourier transform of f_2, then $a_1F_1 + a_2F_2$ is the Fourier transform of $a_1f_1 + a_2f_2$, for any choice of the complex constants a_1, a_2.*

- *(Multiplication and convolution) If F_1 is the Fourier transform of f_1 and F_2 is the Fourier transform of f_2, then F_1F_2 is the Fourier transform of $(f_1*f_2)/2\pi$, where the convolution of two functions is defined by the integral*

$$(f_1 * f_2)(x) = \int_{-\infty}^{\infty} f_1(y)f_2(x-y)\,dy$$

- *(Differentiation and multiplication) If F is the Fourier transform of f, then $i\mu F(\mu)$ is the Fourier transform of $f'(x)$ and $F'(\mu)$ is the Fourier transform of $-ixf(x)$.*

- *(Translation and phase factor) If F is the Fourier transform of f, then $e^{-ia\mu}F(\mu)$ is the Fourier transform of $f(x-a)$ and $F(\mu+b)$ is the Fourier transform of $e^{ibx}f(x)$.*

- *(Parseval's theorem) If F is the Fourier transform of f, then (5.1.8) holds, provided that at least one of the integrals converges.*

The detailed proofs are given at the end of the section. We note that additional examples can be obtained by applying Theorem 5.2 to Example 5.1.1. For example, the linearity property can be used to compute the Fourier transform of any piecewise constant function, defined by $f(x) = C_i$ if $a_i < x < b_i$, which is a sum of square wave functions. The convolution property can be used to compute the Fourier transform of the tentlike function defined by $f(x) = 2 - |x|$ if $-2 < x < 2$ and $f(x) = 0$ otherwise. Indeed, $f = g * g$, where $g(x) = 1$ for

$-1 < x < 1$ and $g(x) = 0$ otherwise. The Fourier transform of g is $(\sin\mu)/2\pi\mu$, so that the Fourier transform of f is $(\sin^2\mu)/2\pi\mu^2$. The first differentiation property cannot be applied to Example 5.1.1, since the square wave fails to have a derivative at the points $x = a$, $x = b$. The second differentiation property can be used to compute the Fourier transform of the function defined by $f(x) = x, a < x < b$, and $f(x) = 0$ otherwise.

The next example is of central importance in applications of the heat equation and will be done in complete detail.

EXAMPLE 5.1.2. *Find the Fourier transform of the normal density function*

$$f(x) = \frac{\exp[-(x-m)^2/2\sigma^2]}{\sqrt{2\pi\sigma^2}}$$

Solution. To compute the Fourier transform $F(\mu)$, we make the change of variable $\xi = (x-m)/\sigma$ and expand $e^{-i\mu\sigma\xi}$ in a Taylor series. Thus

$$\begin{aligned} 2\pi F(\mu) &= \int_{-\infty}^{\infty} \frac{\exp[-(x-m)^2/2\sigma^2]}{\sqrt{2\pi\sigma^2}}\, e^{-i\mu x}\, dx \\ &= \int_{-\infty}^{\infty} \frac{e^{-(\xi^2/2)}}{\sqrt{2\pi}}\, e^{-i\mu(m+\sigma\xi)} d\xi \\ &= e^{-i\mu m} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(\xi^2/2)}}{\sqrt{2\pi}}\, \frac{(-i\mu\sigma\xi)^n}{n!}\, d\xi \end{aligned}$$

This interchange of summation and integration will be fully justified at the end of the calculation. To evaluate each of these integrals, we use mathematical induction. Let

$$I_n = \int_{-\infty}^{\infty} \xi^n \frac{e^{-(\xi^2/2)}}{\sqrt{2\pi}}\, d\xi \qquad n = 0, 1, 2, \ldots$$

Thus

$$2\pi F(\mu) = e^{-i\mu m} \sum_{n=0}^{\infty} \frac{(-i\mu\sigma)^n}{n!} I_n$$

For $n = 0$, we have a classical calculation using polar coordinates. Indeed

$$\begin{aligned} I_0^2 &= \left(\int_{-\infty}^{\infty} \frac{e^{-(\xi^2/2)}}{\sqrt{2\pi}}\, d\xi \right)^2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(\xi_1^2/2)} e^{-(\xi_2^2/2)}}{2\pi}\, d\xi_1 d\xi_2 \\ &= \int_0^{2\pi} \int_0^{\infty} \frac{e^{-(r^2/2)}}{2\pi}\, r\, dr\, d\theta \\ &= 1 \end{aligned}$$

Hence $I_0 = 1$. To compute I_n for $n > 0$, we integrate by parts.

$$\begin{aligned}\sqrt{2\pi} I_n &= -\int_{-\infty}^{\infty} \xi^{n-1} d(e^{-(\xi^2/2)}) \\ &= (n-1)\int_{-\infty}^{\infty} \xi^{n-2} e^{-(\xi^2/2)} d\xi \\ &= (n-1)\sqrt{2\pi} I_{n-2}\end{aligned}$$

Thus we have the recurrence formula $I_n = (n-1)I_{n-2}$. Finally, we notice that $I_n = 0$ whenever n is odd, since $\xi^n e^{-(\xi^2/2)}$ is an odd function in that case. Putting these facts together, we have

$$\begin{aligned}I_{2n} &= (2n-1)I_{2n-2} \\ &= (2n-1)(2n-3)I_{2n-4} \\ &\vdots \\ &= (2n-1)(2n-3)\cdots 3\cdot 1\end{aligned}$$

To obtain a more compact form, we write

$$\begin{aligned}(2n-1)(2n-3)\cdots 3\cdot 1 &= \frac{(2n)(2n-1)(2n-2)\cdots 3\cdot 2\cdot 1}{(2n)(2n-2)\cdots 4\cdot 2} \\ &= \frac{(2n)!}{2^n n!}\end{aligned}$$

Finally, we have

$$\begin{aligned}2\pi F(\mu) &= e^{-im\mu}\sum_{n=0}^{\infty} \frac{(-i\mu\sigma)^{2n}}{(2n)!}\,\frac{(2n)!}{2^n n!} \\ &= e^{-im\mu}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mu^2\sigma^2}{2}\right)^n \\ &= e^{-im\mu}e^{-(\mu^2\sigma^2/2)}\end{aligned}$$

Thus we have the Fourier transform pair

$$\boxed{f(x) = \frac{\exp[-(x-m)^2/2\sigma^2]}{(2\pi\sigma^2)^{1/2}}, \qquad F(\mu) = \frac{\exp[-im\mu - \mu^2\sigma^2/2]}{2\pi}} \tag{5.1.10}$$ •

This completes the formal calculation of the Fourier transform. We note that the graph of $y = f(x)$ is a bell-shaped curve of unit total area, centered around $x = m$. The parameter σ measures the spread of the graph about its midpoint.

These relations are expressed by the formulas

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\,dx \\
0 &= \int_{-\infty}^{\infty} (x-m)\,f(x)\,dx \\
\sigma^2 &= \int_{-\infty}^{\infty} (x-m)^2\,f(x)\,dx
\end{aligned}
$$

On the other hand, the graph of $y = |F(\mu)|$ is also a bell-shaped curve centered about $\mu = 0$, where the variance parameter appears in the numerator of the exponent. This means that the spread of its graph about the midpoint is *inversely proportional* to σ^2. This is a specific expression of the *uncertainty principle*, which will be proved later in this section. The larger σ^2 is, the more closely concentrated the values of $F(\mu)$ will be about $\mu = 0$.

We now justify the interchange of summation and integration in the solution of Example 5.1.2. To do this, we consider the Taylor remainder formula for the complex exponential function:

$$e^{itx} = \sum_{k=0}^{N} \frac{(itx)^k}{k!} + R_N(tx) \qquad \text{where} \quad R_N(u) = \frac{i^{N+1}}{N!}\int_0^u (u-s)^N e^{is}\,ds$$

Integrating this finite sum with respect to the normal density, we have

$$\int_{-\infty}^{\infty} e^{-x^2/2} e^{itx}\,dx = \sum_{k=0}^{N} \frac{1}{k!}\int_{-\infty}^{\infty} e^{-x^2/2}\,(itx)^k\,dx + \int_{-\infty}^{\infty} e^{-x^2/2} R_N(tx)\,dx$$

We must show that the final integral tends to zero when $N \to \infty$. To do this, we estimate the Taylor remainder by noting that $|e^{is}| = 1$; hence

$$|R_N(u)| \le \frac{1}{N!}\left|\int_0^u (u-s)^N\,ds\right| = \frac{|u|^{N+1}}{(N+1)!}$$

Therefore

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} e^{-x^2/2} R_{2N-1}(tx)\,dx\right| &\le \int_{-\infty}^{\infty} e^{-x^2/2}\frac{|tx|^{2N}}{(2N)!}\,dx \\
&= \sqrt{2\pi}\,|t|^{2N}\frac{2^N}{N!}
\end{aligned}
$$

which tends to zero when $N \to \infty$, completing the proof.

Computation of certain Fourier transforms may be facilitated by the integral formula

$$\int_0^{\infty} x^n e^{-cx}\,dx = \frac{n!}{c^{n+1}} \tag{5.1.11}$$

where c may be a complex number with positive real part and $n > -1$. (In case n is not a positive integer, $n!$ must be interpreted by the gamma function.)

EXAMPLE 5.1.3. *Let* $f(x) = x^n e^{-ax}$ *for* $x > 0$ *and* $f(x) = 0$ *for* $x < 0$ *where* $a > 0, n > -1$. *Find the Fourier transform* $F(\mu)$.

Solution. We apply (5.1.11) with $c = a + i\mu$ to obtain

$$\begin{aligned} 2\pi F(\mu) &= \int_0^\infty f(x) e^{-i\mu x}\, dx \\ &= \int_0^\infty x^n e^{-ax} e^{-i\mu x}\, dx \\ &= \int_0^\infty x^n e^{-x(a+i\mu)}\, dx \\ &= \frac{n!}{(a+i\mu)^{n+1}} \end{aligned}$$

The Fourier transform is given by

$$F(\mu) = \frac{n!}{2\pi(a+i\mu)^{n+1}} \quad \bullet$$

EXAMPLE 5.1.4. *Let* $f(x) = x^n e^{-ax}\cos(bx)$ *for* $x > 0$ *and* $f(x) = 0$ *for* $x < 0$, *where* $a > 0$, $b > 0$, $n > -1$. *Find the Fourier transform* $F(\mu)$.

Solution. We write $\cos bx = (e^{ibx} + e^{-ibx})/2$ and apply (5.1.11) twice, with $c = a + i\mu \pm ib$:

$$\begin{aligned} 2\pi F(\mu) &= \int_0^\infty x^n e^{-ax}\cos(bx) e^{-i\mu x} dx \\ &= \frac{1}{2}\int_0^\infty x^n (e^{-x(a+i\mu-ib)} + e^{-x(a-i\mu+ib)}) dx \\ &= \frac{1}{2}\left(\frac{n!}{(a+i\mu-ib)^{n+1}} + \frac{n!}{(a+i\mu+ib)^{n+1}}\right) \end{aligned}$$

The Fourier transform is

$$F(\mu) = \frac{n!}{4\pi}\left(\frac{1}{(a+i\mu-ib)^{n+1}} - \frac{1}{(a+i\mu+ib)^{n+1}}\right)$$

This can be simplified by elementary algebra if necessary. •

The next example illustrates the application of (5.1.11) to the computation of a Fourier transform of a function that is nonzero everywhere.

EXAMPLE 5.1.5. *Let* $f(x) = e^{-a|x|}$, *where* $a > 0$. *Find the Fourier transform* $F(\mu)$.

Solution. Explicitly, f is defined by the formula $f(x) = e^{ax}$ when $x < 0$ and $f(x) = e^{-ax}$ when $x > 0$. Therefore

$$2\pi F(\mu) = \int_{-\infty}^{0} e^{ax}e^{-i\mu x}\,dx + \int_{0}^{\infty} e^{-ax}e^{-i\mu x}\,dx$$

From (5.1.11), the second integral has the value $1/(a+i\mu)$. The first integral can be computed using the substitution $y = -x$ to obtain

$$\int_{-\infty}^{0} e^{ax}e^{-i\mu x}\,dx = \int_{0}^{\infty} e^{-ay}e^{i\mu y}\,dy = \frac{1}{a-i\mu}$$

Hence

$$2\pi F(\mu) = \frac{1}{a-i\mu} + \frac{1}{a+i\mu} = \frac{2a}{a^2+\mu^2}$$

so that we have the Fourier transform pair

$$\text{(5.1.12)} \qquad \boxed{f(x) = e^{-a|x|}, \qquad F(\mu) = \frac{a}{\pi(a^2+\mu^2)}} \qquad \bullet$$

Often new Fourier transforms can be found from old ones by interchanging the roles of x and μ.

EXAMPLE 5.1.6. *Find the Fourier transform of $f(x) = a/(a^2 + x^2)$, where $a > 0$.*

Solution. We have shown in Example 5.1.5 that the Fourier transform of $e^{-a|x|}$ is $(a/\pi)/(a^2 + \mu^2)$. Applying the convergence theorem for Fourier transforms, we have

$$e^{-a|x|} = \lim_{M\to\infty} \frac{a}{\pi} \int_{-M}^{M} e^{i\mu x} \frac{1}{a^2+\mu^2}\,d\mu$$

an absolutely convergent improper integral. Changing the roles of x and μ, we obtain the new Fourier transform pair

$$\text{(5.1.13)} \qquad \boxed{f(x) = a/(a^2+x^2), \qquad F(\mu) = \frac{1}{2}e^{-a|\mu|}} \qquad \bullet$$

5.1.3. Fourier sine and cosine transforms. The Fourier sine transform and the Fourier cosine transform arise when we specialize the Fourier transform to functions defined only for $x > 0$.

In detail, let $f(x)$ be defined for $x > 0$. We extend f to negative x by defining $f(-x) = f(x)$. Taking the Fourier transform of this even function, we have

$$\begin{aligned}
2\pi F(\mu) &= \int_{-\infty}^{\infty} f(x)e^{-i\mu x}\,dx \\
&= \int_0^{\infty} f(x)e^{-i\mu x}dx + \int_{-\infty}^{0} f(-x)e^{-i\mu x}\,dx \\
&= \int_0^{\infty} f(x)(e^{-i\mu x} + e^{i\mu x})dx \\
&= 2\int_0^{\infty} f(x)\cos\mu x\,dx
\end{aligned}$$

Therefore the Fourier transform is also an even function. Hence from (5.1.6),

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}\,d\mu \\
&= \int_0^{\infty} F(\mu)e^{i\mu x}\,d\mu + \int_{-\infty}^{0} F(-\mu)e^{i\mu x}\,d\mu \\
&= \int_0^{\infty} F(\mu)(e^{i\mu x} + e^{-i\mu x})d\mu \\
&= 2\int_0^{\infty} F(\mu)\cos\mu x\,d\mu
\end{aligned}$$

Writing $F_c(\mu) = 2F(\mu)$, we have the *Fourier cosine formulas*

$$\boxed{f(x) = \int_0^{\infty} F_c(\mu)\cos\mu x\,d\mu, \qquad F_c(\mu) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos\mu x\,dx} \tag{5.1.14}$$

$F_c(\mu)$ is the *Fourier cosine transform* of $f(x)$, $-\infty < x < \infty$. This can also be described by the boundary condition $f'(0) = 0$, which is satisfied whenever the Fourier transform satisfies $\int_0^{\infty} \mu|F_s(\mu)|d\mu < \infty$.

To obtain the Fourier sine transform, we again begin with a function f, defined for $x > 0$, and extend it as an odd function: $f(-x) = -f(x)$. Following the same steps as for the cosine formulas, we obtain the *Fourier sine formulas*

$$\boxed{f(x) = \int_0^{\infty} F_s(\mu)\sin\mu x\,d\mu, \qquad F_s(\mu) = \frac{2}{\pi}\int_0^{\infty} f(x)\sin\mu x\,dx} \tag{5.1.15}$$

$F_s(\mu)$ is the *Fourier sine transform* of $f(x)$, $-\infty < x < \infty$. This can also be described by the boundary condition $f(0) = 0$.

EXAMPLE 5.1.7. *Let $f(x) = e^{-ax}$, where $a > 0$. Find the Fourier cosine transform $F_c(\mu)$ and the Fourier sine transform $F_s(\mu)$.*

Solution. We have

$$\int_0^\infty e^{-ax}e^{i\mu x}\,dx = \int_0^\infty e^{-x(a-i\mu)}\,dx = \frac{1}{a-i\mu} = \frac{a+i\mu}{a^2+\mu^2}$$

Taking the real and imaginary parts, we have the integrals

$$\int_0^\infty e^{-ax}\cos\mu x\,dx = \frac{a}{a^2+\mu^2}, \qquad \int_0^\infty e^{-ax}\sin\mu x\,dx = \frac{\mu}{a^2+\mu^2}$$

Therefore

$$F_c(\mu) = \frac{2a}{\pi(a^2+\mu^2)}, \qquad F_s(\mu) = \frac{2\mu}{\pi(a^2+\mu^2)} \quad \bullet$$

5.1.4. Generalized h-transform. The Fourier cosine transform is naturally associated with functions defined for $x > 0$ and satisfying the boundary condition $f'(0) = 0$, whereas the Fourier sine transform is naturally associated with functions satisfying the boundary condition $f(0) = 0$. It is possible to combine these and thus define a transform that is naturally associated with functions defined for $x > 0$ and satisfying the boundary condition $f'(0) = hf(0)$. To do this, suppose that $h > 0$ and that $f(x), x > 0$, is piecewise smooth and integrable. The idea is to extend $f'(x) - hf(x)$ as an odd function. This is done explicitly by defining $\tilde f(x), -\infty < x < \infty$, by the formula $\tilde f(x) = f(-x) - 2he^{hx}\int_0^{-x} e^{hy}f(y)\,dy$ for $x < 0$, while $\tilde f(x) = f(x)$ for $x > 0$. Then $\tilde f$ is integrable, and the Fourier transform of $\tilde f$ is given by

$$\tilde F(\mu) = \frac{-i\mu F_c(\mu) - ihF_s(\mu)}{2(h-i\mu)} = \frac{1}{2}[\tilde A(\mu) - i\tilde B(\mu)]$$

where F_c and F_s denote the Fourier cosine and Fourier sine transforms of $f(x), 0 < x < \infty$, and

$$\tilde A(\mu) = \frac{\mu^2F_c(\mu) + h\mu F_s(\mu)}{h^2+\mu^2}, \qquad \tilde B(\mu) = \frac{h\mu F_c(\mu) + h^2F_s(\mu)}{h^2+\mu^2}$$

This leads to the inversion formula for $x > 0$:

$$\lim_{M\to\infty}\int_0^M \frac{\mu\cos\mu x + h\sin\mu x}{\mu^2+h^2}(\mu F_c(\mu) + hF_s(\mu))d\mu = \frac{1}{2}f(x+0) + \frac{1}{2}f(x-0)$$

The *kernel function* $\mu\cos\mu x + h\sin\mu x$ satisfies the boundary condition $f'(0) = hf(0)$ for every $\mu > 0$, while the general function f is written as a continuous superposition of these. In the limiting case $h \to 0$ we retrieve the Fourier cosine inversion formula where $\tilde f'(0) = 0$, whereas in the limiting case $h \to \infty$ we retrieve the Fourier sine inversion formula where $\tilde f(0) = 0$. The intermediate cases correspond to the boundary condition $\tilde f'(0) - h\tilde f(0) = 0$. The proofs of these statements are left for the exercises.

EXAMPLE 5.1.8. *Let $f(x) = e^{-ax}$ for $x > 0$ where $a > 0$. Find the formulas for the generalized h-transform.*

Solution. The Fourier sine and cosine transforms are given by the formulas $F_c(\mu) = (2/\pi)a/(a^2+\mu^2)$, $F_s(\mu) = (2/\pi)\mu/(a^2+\mu^2)$. Substituting these above, we have for $x > 0$

$$\frac{2}{\pi}\int_0^\infty \frac{\mu\cos\mu x + h\sin\mu x}{\mu^2+h^2}\left(\mu a/(a^2+\mu^2) + h\mu/(a^2+\mu^2)\right)d\mu = e^{-ax} \quad \bullet$$

5.1.5. Fourier transforms in several variables. The preceding formulas and theorems for Fourier transforms in one variable can be extended to functions of two or three variables. In the following paragraphs we sketch the extension to three variables (x, y, z).

The Fourier transform of a complex-valued function $f(x, y, z)$ is defined by the improper integral

$$F(\mu_1,\mu_2,\mu_3) = \frac{1}{(2\pi)^3}\iiint f(x,y,z)e^{-i(\mu_1x+\mu_2y+\mu_3z)}dx\,dy\,dz$$

where the integration is performed over the entire three-dimensional space $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. With this definition, one obtains the properties of linearity, differentiation, convolution, and translation, which were described in detail for the one-dimensional case in Theorem 5.2. If the Fourier transform $F(\mu_1,\mu_2,\mu_3)$ is also absolutely integrable, then the original function f can be recovered as the integral

$$f(x,y,z) = \iiint F(\mu_1,\mu_2,\mu_3)e^{i(\mu_1x+\mu_2y+\mu_3z)}d\mu_1\,d\mu_2\,d\mu_3$$

which is understood as the limit, when $M \to \infty$, of the integral on the solid ball defined by $\mu_1^2+\mu_2^2+\mu_3^2 \le M^2$. Examples of Fourier transforms in three variables can be easily obtained from the one-dimensional case by separation of variables as follows: if f_i, $i = 1, 2, 3$, are functions with Fourier transforms F_i, then the Fourier transform of the function $f_1(x)f_2(y)f_3(z)$ is the function $F_1(\mu_1)F_2(\mu_2)F_3(\mu_3)$. We illustrate with the three-dimensional normal density function.

EXAMPLE 5.1.9. *Find the three-dimensional Fourier transform of*

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}}e^{-(x^2+y^2+z^2)/2}$$

Solution. From Example 5.1.2 we recall the Fourier transform of the one-dimensional normal density function. Applying this three times, we obtain the three-dimensional Fourier transform pair

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}}e^{-(x^2+y^2+z^2)/2}, \qquad F(\mu_1,\mu_2,\mu_3) = \frac{1}{(2\pi)^3}e^{-(\mu_1^2+\mu_2^2+\mu_3^2)/2} \quad \bullet$$

The next example, which arises in the study of the wave equation in Sec. 5.4, is not of the above separable type.

EXAMPLE 5.1.10. *Find the three-dimensional Fourier transform of the function*

$$f(x,y,z) = \begin{cases} 1 & \text{if } x^2+y^2+z^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

Solution. The Fourier transform is given by the integral

$$(2\pi)^3 F(\mu_1,\mu_2,\mu_3) = \iiint_{x^2+y^2+z^2\le R^2} e^{-i(\mu_1 x+\mu_2 y+\mu_3 z)}\,dx\,dy\,dz$$

If $|\mu| = 0$, this integral is the volume of the sphere: $(2\pi)^3 F(0,0,0) = (4\pi R^3)/3$. If $|\mu| \neq 0$, we can evaluate this integral by taking a system of spherical coordinates (r,θ,φ) so that the polar axis $\theta = 0$ points along the vector (μ_1,μ_2,μ_3). Then $\mu_1 x + \mu_2 y + \mu_3 z = |\mu| r\cos\theta$, $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$, and the integral is

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} e^{-i|\mu|r\cos\theta} r^2 \sin\theta\,dr\,d\theta\,d\varphi &= \frac{4\pi}{|\mu|}\int_0^R r\,\sin(r|\mu|)dr \\ &= \frac{4\pi}{|\mu|}\left(\frac{\sin(R|\mu|) - R|\mu|\cos(R|\mu|)}{|\mu|^2}\right) \quad \bullet\end{aligned}$$

The result of Example 5.1.10 can be used to find a simple formula for the integral of $e^{i(\mu_1 x+\mu_2 y+\mu_3 z)}$ on the surface of the sphere $|x| = r$. Indeed, the volume integral and surface integrals are related by

$$\iiint_{x^2+y^2+z^2\le R^2} e^{i(\mu_1 x+\mu_2 y+\mu_3 z)}\,dx\,dy\,dz = \int_0^R \left(\iint_{x^2+y^2+z^2=r^2} e^{i(\mu_1 x+\mu_2 y+\mu_3 z)}\,dS\right)dr$$

Therefore

$$\begin{aligned}\iint_{x^2+y^2+z^2=R^2} e^{i(\mu_1 x+\mu_2 y+\mu_3 z)}\,dS &= \frac{d}{dR}\iiint_{x^2+y^2+z^2\le R^2} e^{i(\mu_1 x+\mu_2 y+\mu_3 z)}\,dx\,dy\,dz \\ &= \frac{4\pi}{|\mu|^3}\frac{d}{dR}(\sin R|\mu| - R|\mu|\cos R|\mu|) \qquad (5.1.16)\\ &= 4\pi R^2 \frac{\sin R|\mu|}{R|\mu|}\end{aligned}$$

This formula will be used in Sec. 5.3 to solve the three-dimensional wave equation.

5.1.6. The uncertainty principle. We have noted in Example 5.1.2 that if a normal density function is highly peaked about its midpoint, then the Fourier transform will be widely spread about its midpoint. In order to make this quantitative, we define the *dispersion about zero* of a complex-valued function $f(x), -\infty < x < \infty$ by the formula

$$D_0(f) = \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2\,dx}{\int_{-\infty}^{\infty} |f(x)|^2\,dx}$$

This is defined whenever the relevant integrals are finite. For example if $f(x) = e^{-x^2/2\sigma^2}$, then $D_0(f) = \sigma^2/2$, $D_0(F) = 1/(2\sigma^2)$ and the product is $D_0(f)D_0(F) = 1/4$. In the general case we have the following inequality.

PROPOSITION 5.1.1. **(Uncertainty principle).** *Let $f(x), -\infty < x < \infty$, be a complex-valued function with Fourier transform $F(\mu)$, for which the integrals defining $D_0(f)$, $D_0(F)$ are finite. Then we have the inequality*

$$D_0(f)D_0(F) \geq \frac{1}{4}$$

Equality holds if and only if f is a normal density function centered at $x = 0$; in detail, $f(x) = C_1 e^{-x^2/2\sigma^2}$, $F(\mu) = C_2 e^{-\sigma^2\mu^2/2}$ for suitable constants C_1, C_2, σ^2.

Remark. The term "uncertainty principle" comes from the interpretation that we cannot localize both $f(x)$ and $F(\mu)$ in their respective spaces. If $f(x)$ is localized about $x = 0$, then $D_0(f)$ will be small; the uncertainty principle then asserts that $D_0(F)$ will be correspondingly large, indicating a lack of localization about $\mu = 0$.

Proof. Both the numerator and denominator of the expressions defining $D_0(f)$ and $D_0(F)$ may be transformed by Parseval's theorem. In this way one is led to examine a corresponding integral involving $F'(\mu)$. Specifically, we write the real part of the integral of $\mu\bar{F}F'$ in two different ways. On the one hand,

$$2\,\mathrm{Re}\int \mu\bar{F}F'\,d\mu = \int\mu(F'\bar{F} + F\bar{F}')d\mu = \int\mu(F\bar{F})'d\mu = \int\mu(|F|^2)'d\mu = -\int |F|^2\,d\mu$$

where we have integrated by parts in the last step. On the other hand,

(5.1.17)

$$-\mathrm{Re}\int \mu\bar{F}F'd\mu \leq \left|\int \mu\bar{F}(\mu)F'(\mu)\,d\mu\right| \leq \left(\int |\mu\bar{F}(\mu)|^2\,d\mu\right)^{1/2}\left(\int |F'(\mu)|^2\,d\mu\right)^{1/2}$$

where we have applied the Schwarz inequality to the functions $F'(\mu)$ and $\mu F(\mu)$. Now we apply Parseval's theorem twice, recalling that the Fourier transform of $xf(x)$ is $iF'(\mu)$:

$$\int |F(\mu)|^2\,d\mu = \frac{1}{2\pi}\int |f(x)|^2\,dx, \qquad \int |F'(\mu)|^2\,d\mu = \frac{1}{2\pi}\int x^2|f(x)|^2\,dx$$

Squaring both sides of (5.1.17) and making these substitutions gives the desired result, in the form $(1/4)\int|f|^2\int|F|^2 \leq \int|xf|^2\int|\mu F|^2$.

In case equality occurs in (5.1.17), we obtain two conditions: (i) the imaginary part of $\int\mu\bar{F}F'd\mu$ must be zero, and (ii) F must satisfy the differential equation $F'(\mu) = -A\mu F(\mu)$ for some complex constant A. The differential equation has the general solution $F(\mu) = Ce^{-A\mu^2/2}$ for some complex constant C. This function will yield a finite value of $D_0(F)$ if and only if $\operatorname{Re} A > 0$. To show that $\operatorname{Im} A = 0$, we write $A = \alpha + i\beta$ and compute $F'(\mu) = -CA\mu e^{-A\mu^2/2}$, $\bar{F}(\mu) = \bar{C}e^{-\bar{A}\mu^2/2}$:

$$\int \mu \bar{F}(\mu) F'(\mu) d\mu = -|C|^2 A \int \mu^2 e^{-\alpha\mu^2} d\mu$$

The imaginary part of the integral is zero if and only if $\beta = \operatorname{Im} A = 0$, which was to be proved, where we have made the identifications $A = \sigma^2$, $C = C_1$. •

5.1.7. Proof of convergence. We close this section by giving the proof of Theorem 5.1, the convergence theorem for Fourier transforms. We are given a function $f(x)$, $-\infty < x < \infty$, that is piecewise smooth on each finite interval and for which $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent. Defining the Fourier transform by (5.1.7), we make the following transformations:

$$\begin{aligned} 2\pi \int_{-M}^{M} F(\mu) e^{i\mu x} d\mu &= \int_{-M}^{M} \left(\int_{-\infty}^{\infty} e^{-i\mu\xi} f(\xi) d\xi \right) e^{i\mu x} d\mu \\ &= \int_{-\infty}^{\infty} \left(\int_{-M}^{M} e^{i\mu(x-\xi)} d\mu \right) f(\xi) d\xi \\ &= 2 \int_{-\infty}^{\infty} \frac{\sin M(x-\xi)}{(x-\xi)} f(\xi) d\xi \\ &= 2 \int_{-\infty}^{\infty} \frac{\sin M\eta}{\eta} f(x+\eta) d\eta \end{aligned}$$

where we have interchanged the order of integration and made the substitution $\eta = \xi - x$. Using the Riemann lemma of Chapter 1, Sec. 1.2, it follows that the integral $\int_{|\eta| \geq \delta} (\sin M\eta/\eta) f(x+\eta) d\eta$ tends to zero when M tends to infinity, for any positive number δ. It remains to analyze the integral for $-\delta \leq \eta \leq \delta$. Another use of the Riemann lemma shows that each of the integrals

$$2 \int_{0}^{\delta} \frac{\sin M\eta}{\eta} [f(x+\eta) - f(x+0)] d\eta$$

and

$$2 \int_{-\delta}^{0} \frac{\sin M\eta}{\eta} [f(x+\eta) - f(x-0)] d\eta$$

tends to zero when M tends to infinity. Therefore

$$\begin{aligned} \lim_{M\to\infty} 2 \int_{-\delta}^{\delta} \frac{\sin M\eta}{\eta} f(x+\eta) d\eta &= [f(x+0) + f(x-0)] \lim_{M\to\infty} \int_{-\delta}^{\delta} \frac{\sin M\eta}{\eta} d\eta \\ &= \pi [f(x+0) + f(x-0)] \end{aligned}$$

Dividing by 2π we have the desired result. •

The unrestricted improper integral in (5.1.6) does not exist. An example is provided in Exercise 26.

EXERCISES 5.1

In Exercises 1 to 10 find the Fourier transforms of the indicated functions.

1. $f(x) = 1$ for $-2 < x < 2$ and $f(x) = 0$ otherwise
2. $f(x) = -4$ for $-1 < x < 0$, $f(x) = 4$ for $0 < x < 1$ and $f(x) = 0$ otherwise
3. $f(x) = e^{-3x}$ for $x > 0$ and $f(x) = e^{2x}$ for $x < 0$
4. $f(x) = xe^{-|x|}$
5. $f(x) = \cos x\, e^{-|x|}$
6. $f(x) = \cos^2 x\, e^{-|x|}$
7. $f(x) = 2x/(1+x^2)^2$
8. $f(x) = \exp[-(x^2+3x)/2]$
9. $f(x) = \cos x\, e^{-x^2/2}$
10. $f(x) = xe^{-x^2/2}$
11. Suppose that $f(x)$, $-\infty < x < \infty$, is continuous and piecewise smooth on every finite interval, and both $\int_{-\infty}^{\infty} |f(x)|dx$ and $\int_{-\infty}^{\infty} |f'(x)|dx$ are absolutely convergent. Using integration by parts, show that the Fourier transform of f' is $i\mu F(\mu)$.
12. Apply Exercise 11 to Exercises 4 to 10 to obtain additional examples of Fourier transforms.
13. Let $f(x) = xe^{-x}$ for $x > 0$. Find the Fourier cosine transform $F_c(\mu)$ and the Fourier sine transform $F_s(\mu)$.
14. Complete the derivation of the Fourier sine formulas (5.1.15).
15. Let $a > 0$. Show that the Fourier transform of $f(ax)$ is $F(\mu/a)/a$.
16. Show that the Fourier transform of $f(x-a)$ is $e^{-ia\mu}F(\mu)$.

Use Exercises 15 and 16 to find the Fourier transforms of the following functions.

17. $f(x) = 1/[1+(x-3)^2]$
18. $f(x) = e^{-(x-2)^2/2}$
19. $f(x) = e^{-3|x-2|}$
20. $f(x) = \sin 2x\, e^{-2x}$ for $x > 0$ and $f(x) = 0$ otherwise
21. $f(x) = e^{-2x}$ for $x > 0$ and $f(x) = 0$ otherwise
22. Compute $D_0(f)D_0(F)$ for the following functions.
    (a) $f(x) = xe^{-x^2/2}$
    (b) $f(x) = x^2e^{-x^2/2}$
    (c) $f(x) = e^{-(x-1)^2/2}$
23. If $f(x), -\infty < x < \infty$, is a complex-valued function with Fourier transform $F(\mu)$, let $f_{a,m}(x) = e^{imx}f(x-a)$, where a, m are real constants. Show that the Fourier transform of $f_{a,m}$ is $e^{-iam}F_{m,a}$.
24. The *dispersion about a* of a complex-valued function f is defined by

$$D_a(f) = \frac{\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2\, dx}{\int_{-\infty}^{\infty} |f(x)|^2\, dx}$$

Show that $D_a(f) = D_0(f_{a,m})$ and $D_m(F) = D_0(F_{m,a})$ for any (a, m).

25. Combine the previous two exercises with the uncertainty principle to show that for any (a, m) and any f, we have the inequality $D_a(f)D_m(F) \geq 1/4$ with equality if and only if

$$f(x) = C_1 e^{-A^{-1}(x-a)^2/2}, \qquad F(\mu) = C_2 e^{-A(\mu-m)^2/2}$$

for suitable constants A, C_1, and C_2.

26. Let $f(x) = e^{-x}$ for $x > 0$ and $f(x) = -e^{-x}$ for $x < 0$. Show that the unrestricted double limit

$$\lim_{M_1, M_2 \to \infty} \int_{-M_1}^{M_2} F(\mu) e^{i\mu x}\, d\mu$$

does not exist for all x. [*Hint:* Examine the integrals $\int_0^\infty F(\mu)e^{i\mu x}\, dx$ and $\int_{-\infty}^0 F(\mu)e^{i\mu x}\, dx$ separately.]

27. (*Fourier transforms in real form*) Suppose that $f(x), -\infty < x < \infty$, is a real-valued function with $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ and is piecewise smooth on each finite interval. Define the real-valued functions $A(\mu)$, $B(\mu)$ by

$$A(\mu) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos \mu x\, dx, \qquad B(\mu) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin \mu x\, dx$$

Show that we have, for each x,

$$\lim_{M \to \infty} \int_0^M (A(\mu)\cos \mu x + B(\mu) \sin \mu x)\, d\mu = \frac{1}{2} f(x+0) + \frac{1}{2} f(x-0)$$

28. (*Generalized h-transform*) Suppose that $h > 0$ and that $f(x), x > 0$, is piecewise smooth and integrable. Define $\tilde{f}(x), -\infty < x < \infty$, by the formula $\tilde{f}(x) = f(-x) - 2he^{hx} \int_0^{-x} e^{hy} f(y)\, dy$ for $x < 0$, while $\tilde{f}(x) = f(x)$ for $x > 0$.

(a) Show that $\tilde{f}$ is integrable and the Fourier transform of $\tilde{f}$ is given by

$$\tilde{F}(\mu) = \frac{-i\mu F_c(\mu) - ihF_s(\mu)}{2(h - i\mu)}$$

where F_c, F_s denote the Fourier cosine and Fourier sine transforms of $f(x), 0 < x < \infty$.

(b) With reference to Exercise 27, show that

$$\tilde{A}(\mu) = \frac{\mu^2 F_c(\mu) + h\mu F_s(\mu)}{h^2 + \mu^2}, \qquad \tilde{B}(\mu) = \frac{h\mu F_c(\mu) + h^2 F_s(\mu)}{h^2 + \mu^2}$$

(c) Conclude that we have the inversion formula

$$\lim_{M\to\infty} \int_0^M \frac{\mu \cos \mu x + h \sin \mu x}{\mu^2 + h^2} (\mu F_c(\mu) + hF_s(\mu)) d\mu$$
$$= \frac{1}{2} f(x+0) + \frac{1}{2} f(x-0)$$

(d) Show that in the limiting case $h \to 0$ we retrieve the Fourier cosine inversion formula where $\tilde{f}'(0) = 0$, whereas in the limiting case $h \to \infty$ we retrieve the Fourier sine inversion formula, where $\tilde{f}(0) = 0$. The intermediate cases correspond to the boundary condition $\tilde{f}'(0) - h\tilde{f}(0) = 0$.

29. Let $f(x) = 1$ for $0 \le x \le L$ and $f(x) = 0$ for $x > L$. Find the h-transform representation of the function $f(x), 0 < x < \infty$, described in the previous exercise.

5.2. Solution of the Heat Equation for an Infinite Rod

We will now see how the Fourier transform applies to the heat equation. For this purpose we consider the following initial-value problem on the entire axis:

$$u_t = Ku_{xx} \qquad t > 0, -\infty < x < \infty \tag{5.2.1}$$

$$u(x;0) = f(x) \qquad -\infty < x < \infty \tag{5.2.2}$$

We will solve this problem by two different methods.

5.2.1. First method: Fourier series and passage to the limit. We consider the entire axis as the limit when $L \to \infty$ of the interval $-L < x < L$ with the periodic boundary conditions $u(L,t) = u(-L,t)$ and $u_x(L,t) = u_x(-L,t)$. Equation (5.2.1) with periodic boundary conditions can be solved by separation of variables. In complex form, the separated solutions are

$$e^{in\pi x/L}e^{-(n\pi/L)^2Kt} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Using the formula for complex Fourier series, we look for the solution as a superposition

$$u_L(x;t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}e^{-(n\pi/L)^2Kt}$$

The initial data and the Fourier coefficients are related by the formulas

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}$$

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$$

To take the limit when $L \to \infty$, we follow the method of Sec. 5.1. We let

$$F_L(\mu) = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{-i\mu x}dx$$

$$\mu_n = \frac{n\pi}{L}$$

$$\Delta\mu_n = \mu_{n+1} - \mu_n = \frac{\pi}{L}$$

Thus the previous formulas can be rewritten as

$$\begin{aligned} \alpha_n &= F_L(\mu_n)\Delta\mu_n \\ u_L(x;t) &= \sum_{n=-\infty}^{\infty} F_L(\mu_n)e^{i\mu_n x}e^{-\mu_n^2 Kt}\Delta\mu_n \end{aligned}$$

Taking the limit $L \to \infty$, we get the formulas

$$\boxed{u(x;t) = \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}e^{-\mu^2 Kt}d\mu} \tag{5.2.3}$$

$$\boxed{F(\mu) = (2\pi)^{-1}\int_{-\infty}^{\infty} f(x)e^{-i\mu x}dx} \tag{5.2.4}$$

This is the *Fourier representation* of the solution of the initial-value problem (5.2.1). To find $u(x;t)$, we first find $F(\mu)$, the Fourier transform of $f(x)$. Then we substitute into the first equation and perform the indicated integration.

EXAMPLE 5.2.1. *Solve (5.2.1) in the case* $f(x) = e^{-x^2/2}$, $-\infty < x < \infty$.

Solution. The integral (5.2.4) can be done explicitly, as in Example 5.1.2, with the result

$$F(\mu) = \frac{e^{-\mu^2/2}}{\sqrt{2\pi}}$$

Substituting this into (5.2.3) and using the same example, we have

$$u(x;t) = \frac{e^{-x^2/(2+4Kt)}}{\sqrt{1+2Kt}} \quad \bullet$$

5.2.2. Second method: Direct solution by Fourier transform. In this method we use the Fourier transform to convert the partial differential equation to an ordinary differential equation. Let $U(\mu;t)$ be the Fourier transform of $u(x;t)$. Thus from (5.1.6) and (5.1.7) we have

$$u(x;t) = \int_{-\infty}^{\infty} U(\mu;t)e^{i\mu x}d\mu \tag{5.2.5}$$

$$U(\mu;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x;t)e^{-i\mu x}dx \tag{5.2.6}$$

Assuming for the moment that the derivatives can be taken under the integral, we have

$$\begin{aligned} u_t(x;t) &= \int_{-\infty}^{\infty} U_t(\mu;t)e^{i\mu x}d\mu \\ u_x(x;t) &= \int_{-\infty}^{\infty} U(\mu;t)i\mu e^{i\mu x}d\mu \\ u_{xx}(x;t) &= \int_{-\infty}^{\infty} U(\mu;t)(i\mu)^2 e^{i\mu x}d\mu \end{aligned}$$

In order that u satisfy the heat equation, we must have

$$0 = u_t - Ku_{xx} = \int_{-\infty}^{\infty} [U_t(\mu;t) + K\mu^2 U(\mu;t)]e^{i\mu x} d\mu$$

Therefore we ask that U be a solution of the ordinary differential equation

$$U_t + K\mu^2 U = 0$$

The initial condition is determined by setting $t = 0$ in (5.2.5). Thus $U(\mu;0)$ must be the Fourier transform of f:

$$U(\mu;0) = F(\mu)$$

Therefore we must have

$$U(\mu;t) = F(\mu)e^{-\mu^2 Kt}$$

Substituting into (5.2.5) gives formula (5.2.3), as in the first method.

5.2.3. Verification of the solution. We now prove that (5.2.3) is a rigorous solution of the initial-value problem (5.2.1).

THEOREM 5.3. *Let $f(x), -\infty > x < \infty$, be a piecewise smooth function with Fourier transform $F(\mu)$.*

1. *The integral (5.2.3) defines a solution of the heat equation $u_t = Ku_{xx}$ for $t > 0, -\infty < x < \infty$.*
2. *If in addition $\int_{-\infty}^{\infty} \mu^2 |F(\mu)| d\mu < \infty$, then $\lim_{t\to 0} u(x;t) = f(x)$ uniformly for $-\infty < x < \infty$.*

Proof. To prove that u satisfies the heat equation, it suffices to prove that

$$u_x(x;t) = \int_{-\infty}^{\infty} F(\mu) i\mu e^{i\mu x} e^{-\mu^2 Kt} d\mu \tag{5.2.7}$$

$$u_{xx}(x;t) = \int_{-\infty}^{\infty} F(\mu)(i\mu)^2 e^{i\mu x} e^{-\mu^2 Kt} d\mu \tag{5.2.8}$$

$$u_t(x;t) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x}(-\mu^2 K e^{-\mu^2 Kt}) d\mu \tag{5.2.9}$$

Once we have proved these formulas, it will apparent that $u_t = Ku_{xx}$. To prove (5.2.7), we write

$$\frac{u(x+h;t) - u(x;t)}{h} - \int_{-\infty}^{\infty} F(\mu) i\mu\, e^{i\mu x} e^{-\mu^2 Kt} d\mu$$
$$= \int_{-\infty}^{\infty} F(\mu) \left(\frac{e^{i\mu(x+h)} - e^{i\mu x} - i\mu h e^{i\mu x}}{h} \right) e^{-\mu^2 Kt} d\mu$$

To estimate this term, notice that from the Taylor remainder formula

$$\left|e^{i\theta} - 1 - i\theta\right| = \left| \int_0^{\theta} (\theta - \phi) e^{i\phi} d\phi \right| \le \left| \int_0^{\theta} (\theta - \phi) d\phi \right| = \theta^2/2$$

Applying this with $\theta = \mu h$, we have

$$\left|\frac{u(x+h;t)-u(x;t)}{h} - \int_{-\infty}^{\infty} F(\mu) i\mu\, e^{i\mu x} e^{-\mu^2 Kt} d\mu\right| \le |h| \int_{-\infty}^{\infty} |F(\mu)|\mu^2 e^{-\mu^2 Kt} d\mu$$

The integral on the right is absolutely convergent. Taking the limit as $h \to 0$, we have proved (5.2.7). It is left as an exercise to prove (5.2.8) and (5.2.9) by the same method. This completed, we have proved that the Fourier representation (5.2.3) gives a rigorous solution of the heat equation. It is a remarkable fact that u is *differentiable to any order*, in spite of the fact that f is assumed only *piecewise* smooth.

To prove that the initial condition is satisfied uniformly, we write

$$u(x;t) - f(x) = \int_{-\infty}^{\infty} \left(e^{-\mu^2 Kt} - 1\right) e^{i\mu x} F(\mu)\, d\mu$$

Now the inequality $|e^{-\theta} - 1| \le |\theta|$, valid for $\theta > 0$, may be applied to the integrand, with the result

$$|u(x;t) - f(x)| \le \int_{-\infty}^{\infty} \mu^2 Kt |F(\mu)|\, d\mu$$

The right side tends to zero when $t \to 0$, which completes the proof. •

5.2.4. Explicit representation by the Gauss-Weierstrass kernel. We now show how to obtain an *explicit representation* of the solution, involving only one integration and not involving the Fourier transform. To do this, we note that (5.2.3) represents $u(x;t)$ as the Fourier integral of the product of the Fourier transform of f with the elementary function $e^{-\mu^2 Kt}$. But the Fourier transform of the normal density function was encountered in Example 5.1.2, where $\sigma^2 = 2Kt$:

$$\int_{-\infty}^{\infty} e^{i\mu(x-\xi)}\, e^{-\mu^2 Kt}\, d\mu = 2\pi \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}}$$

Appealing to the convolution property of Theorem 5.2, we obtain the *explicit representation*

$$\boxed{u(x;t) = \int_{-\infty}^{\infty} f(\xi) \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}}\, d\xi} \tag{5.2.10}$$

We will give an independent proof below that (5.2.10) defines a solution of the initial-value problem (5.2.1). The function $(t, x, \xi) \to e^{-(x-\xi)^2/4Kt}/\sqrt{4\pi Kt}$ is called the *Gauss-Weierstrass kernel* or the *fundamental solution* of the heat equation; for each ξ, this function of (x, t) is a solution, from which the general solution is obtained by a continuous superposition over all possible values of ξ. This is to be contrasted with the Fourier representation (5.2.3), which represents $u(x;t)$ as a continuous superposition of separated solutions.

The explicit representation (5.2.10) is preferable to the Fourier representation (5.2.3) for several reasons: (a) it is computationally more direct, requiring only one integration; (b) it makes sense for many functions f for which the Fourier transform is undefined, for example, any bounded continuous function; and (c) it does not require any smoothness in order to satisfy the initial-value problem. We state the properties as a theorem.

THEOREM 5.4. *Suppose that $f(x), -\infty < x < \infty$, is piecewise continuous and bounded on the entire axis: $|f(x)| \le A$ for $-\infty < x < \infty$. Then the Gauss-Weierstrass integral (5.2.10) defines a solution of the heat equation $u_t = Ku_{xx}$ for $t > 0$ and $\lim_{t\to 0} u(x;t) = \frac{1}{2}f(x+0) + \frac{1}{2}f(x-0)$.*

Proof. By changing the time scale, we may assume that $K = 1/2$. Now a direct calculation shows that

$$\frac{\partial}{\partial x}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right) = -\frac{(x-\xi)}{t}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right)$$

$$2\frac{\partial}{\partial t}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right) = \frac{(x-\xi)^2-t}{t^2}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right)$$

Therefore it suffices to prove that we can differentiate under the integral sign:

$$u_x(x;t) = \int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right) f(\xi)\,d\xi$$

$$u_{xx}(x;t) = \int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right) f(\xi)\,d\xi$$

$$u_t(x;t) = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\frac{e^{-(x-\xi)^2/2t}}{\sqrt{2\pi t}}\right) f(\xi)\,d\xi$$

To prove the first formula, it suffices to do the computation at $x = 0$, by changing the origin of the x-axis. Thus

$$\begin{aligned}\frac{u(h;t)-u(0;t)}{h} &- \int_{-\infty}^{\infty}\frac{\xi}{t}\frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}}f(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty}\left(\frac{e^{-(\xi-h)^2/2t}-e^{-\xi^2/2t}}{h} - \frac{\xi}{t}e^{-\xi^2/2t}\right)\frac{f(\xi)d\xi}{\sqrt{2\pi t}}\end{aligned}$$

For any fixed M, this integral may be written as $\int_{-\infty}^{\infty} = \int_{|\xi|\le M} + \int_{|\xi|>M}$. The integral over $|\xi| \le M$ tends to zero when $h \to 0$. To handle the integral over $|\xi| > M$, we examine the two terms (derivative and difference quotient) separately; in the first case

$$\left|\int_{|\xi|>M} \frac{\xi}{t} e^{-\xi^2/2t} f(\xi)\, d\xi\right| \le A \int_{|\xi|>M} \frac{|\xi|}{t} e^{-\xi^2/2t}\, d\xi = 2\,A\, e^{-M^2/2t}$$

Given $\epsilon > 0$, this can be made less than $\epsilon/2$ by choosing M sufficiently large, independent of h; in the second case

$$\frac{1}{h}\left|\int_{|\xi|>M} \left(e^{-(\xi-h)^2/2t} - e^{-\xi^2/2t}\right) f(\xi)\, d\xi\right| \le A \int_{|\xi|>M} \frac{|\xi|}{t} e^{-\xi^2/2t} d\xi = 2\,A\, e^{-M^2/2t}$$

which can also be made less than $\epsilon/2$ by choosing M sufficiently large, independent of h; Taking the limit when $h \to 0$, we see that for any $\epsilon > 0$

$$\limsup_{h\to 0} \left|\frac{u(h;t) - u(0;t)}{h} - \int_{-\infty}^{\infty} \frac{\xi}{t} e^{-\xi^2/2t} \frac{f(\xi)\, d\xi}{\sqrt{2\pi t}}\right| < \epsilon$$

This holds for any $\epsilon > 0$; hence the indicated limit superior is zero, and we have proved the first differentiation formula. The proofs of the formulas for u_{xx} and u_t are carried out in a similar manner.

To prove the stated form of the initial conditions, we first make the change of variable $z = (x - \xi)/\sqrt{t}$, which leads to

$$u(x;t) = \int_{-\infty}^{\infty} f(x - z\sqrt{t}) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

Given $\epsilon > 0$, there is a $\delta > 0$ so that

$$|f(y) - f(x+0)| < \epsilon, \quad x < y < x + \delta \tag{5.2.11}$$

$$|f(y) - f(x-0)| < \epsilon, \quad x - \delta < y < x \tag{5.2.12}$$

Now we write

$$\int_0^{\infty} f(x - z\sqrt{t}) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz - \frac{1}{2} f(x-0)$$
$$= \left\{\int_0^{\delta/\sqrt{t}} + \int_{\delta/\sqrt{t}}^{\infty}\right\} [f(x - z\sqrt{t}) - f(x-0)] \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

In the first integral we use (5.2.11) and obtain the upper bound

$$\epsilon \int_0^{\delta/\sqrt{t}} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \le \frac{\epsilon}{2}$$

In the second integral we note that $|f(x - z\sqrt{t}) - f(x+0)| \le 2A$; therefore the second integral is bounded by $2A \int_{\delta/\sqrt{t}}^{\infty} e^{-z^2/2} dz$, which tends to zero when $t \to 0$. Therefore

$$\limsup_{t\to 0} \left|\int_0^{\infty} [f(x - z\sqrt{t}) - f(x-0)] \frac{e^{-z^2/2}\, dz}{\sqrt{2\pi}}\right| \le \epsilon$$

Since this holds for every $\epsilon > 0$, we conclude that

$$\lim_{t\to 0}\int_0^\infty f(x - z\sqrt{t})\frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} = \frac{1}{2}f(x-0)$$

In the same fashion, using (5.2.12), it is shown that

$$\lim_{t\to 0}\int_{-\infty}^0 f(x - z\sqrt{t})\frac{e^{-z^2/2}\,dz}{\sqrt{2\pi}} = \frac{1}{2}f(x+0)$$

Adding the two statements gives the required limiting result. •

5.2.5. Some explicit formulas. In case the initial temperature function $f(x)$, $-\infty < x < \infty$, is piecewise constant, the solution $u(x;t)$ of the heat equation can be written in terms of the *normal distribution function* Φ, defined by

$$\Phi(x) = \int_{-\infty}^x e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}$$

This is a continuous increasing function with $\Phi(-\infty) = 0$, $\Phi(0) = \frac{1}{2}$, $\Phi(\infty) = 1$. To apply this function to the heat equation, we proceed as follows: if $a < b$, we make the substitution $z = (x-\xi)/\sqrt{2Kt}$ and write

$$\begin{aligned}
\int_a^b \frac{e^{-(x-\xi)^2/4Kt}}{\sqrt{4\pi Kt}}\,d\xi &= \int_{(x-b)/\sqrt{2Kt}}^{(x-a)/\sqrt{2Kt}} e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\\
&= \int_{-\infty}^{(x-a)/\sqrt{2Kt}} e^{-z^2/2}\frac{dz}{\sqrt{2\pi}} - \int_{-\infty}^{(x-b)/\sqrt{2Kt}} e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\\
&= \Phi\left(\frac{x-a}{\sqrt{2Kt}}\right) - \Phi\left(\frac{x-b}{\sqrt{2Kt}}\right)
\end{aligned}$$

EXAMPLE 5.2.2. *Solve (5.2.1) in the case where*

$$f(x) = \begin{cases} 0 & x < a\\ T & a \le x < b\\ 0 & x > b\end{cases}$$

and discuss $\lim_{t\to 0} u(x;t)$. *Graph the solution at* $t = 0$, $t = 0.01$, $t = 1$, $t = 100$ *if* $K = 1/2$, $a = -1$, $b = 1$, $T = 1$.

Solution. Formula (5.2.10) reduces to

$$\begin{aligned}
u(x;t) &= T\int_a^b \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi Kt}}\,d\xi\\
&= T\int_{(x-b)/\sqrt{2Kt}}^{(x-a)/\sqrt{2Kt}} e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\\
&= T\Phi\left(\frac{x-a}{\sqrt{2Kt}}\right) - T\Phi\left(\frac{x-b}{\sqrt{2Kt}}\right)
\end{aligned}$$

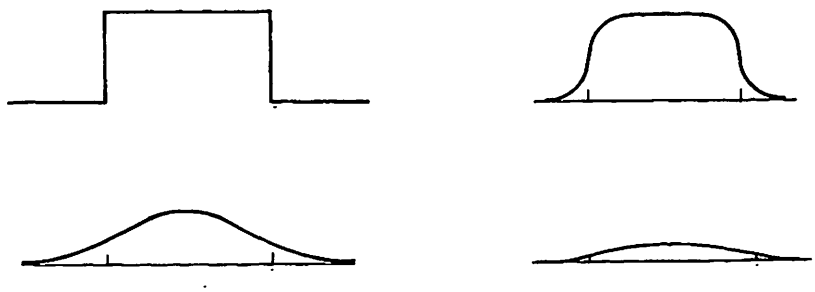


FIGURE 5.2.1 Solution of the heat equation at four different times.

Using the properties $\Phi(-\infty) = 0$, $\Phi(0) = \frac{1}{2}$, $\Phi(\infty) = 1$, we can directly verify the following limits:

$$\lim_{t \downarrow 0} u(x,t) = \begin{cases} 0 & x < a \\ T/2 & x = a \\ T & a < x < b \\ T/2 & x = b \\ 0 & x > b \end{cases}$$

Using tabulated values of $\Phi(a)$, it is possible to obtain accurate graphs of the temperature function $u(x;t)$ for various values of t. The graphs in Fig. 5.2.1 assume that $K = \frac{1}{2}$, $a = -1$, $b = 1$. The values of t are $t = 0$, $t = 0.01$, $t = 1$, $t = 100$.

EXAMPLE 5.2.3. *Two materials of the same conductivity are initially at different temperatures T_1 and T_2. Find the temperature at all later times when heat is allowed to flow between the two materials. Discuss the approach to a steady state.*

Solution. Suppose that the first material occupies the negative axis $x < 0$ and the second occupies the positive axis $x > 0$. Letting $u(x;t)$ denote the temperature at the point x at time t, we have the initial-value problem

$$\begin{aligned} u_t &= K u_{xx} && t > 0, -\infty < x < \infty \\ u(x;0) &= T_1 && x < 0 \\ u(x;0) &= T_2 && x > 0 \end{aligned}$$

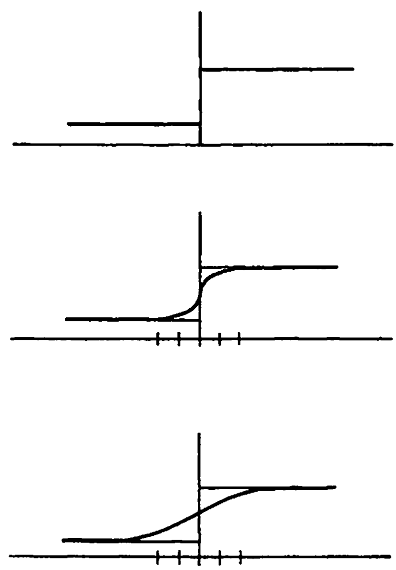


FIGURE 5.2.2 Solution of the heat equation at three different times.

The solution to this problem is given in (5.2.10), by the formula

$$\begin{aligned} u(x;t) &= T_1 \int_{-\infty}^{0} \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi K t}}\, d\xi + T_2 \int_{0}^{\infty} \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi K t}}\, d\xi \\ &= T_1 \int_{x/\sqrt{2Kt}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz + T_2 \int_{-\infty}^{x/\sqrt{2Kt}} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz \\ &= T_1 \left[1 - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right] + T_2 \Phi\left(\frac{x}{\sqrt{2Kt}}\right) \end{aligned}$$

The graphs in Fig. 5.2.2 depict $u(x;t)$ for three values of t when $K = 200$, $T_1 = 25$, $T_2 = 100$.

To study the approach to steady state, we recall that $\lim_{x\to 0} \Phi(x) = \Phi(0) = \frac{1}{2}$. Therefore $\lim_{t\to\infty} u(x;t) = \frac{1}{2}T_1 + \frac{1}{2}T_2$ for any x. To proceed further, we write

$$\begin{aligned} u(x;t) - \left(\frac{1}{2}T_1 + \frac{1}{2}T_2\right) &= T_1 \left[\frac{1}{2} - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right] + T_2 \left[\Phi\left(\frac{x}{\sqrt{2Kt}}\right) - \frac{1}{2}\right] \\ &= (T_2 - T_1) \left[\Phi\left(\frac{x}{\sqrt{2Kt}}\right) - \frac{1}{2}\right] \end{aligned}$$

When $t \to \infty$, we have $\Phi(x/\sqrt{2Kt}) - \frac{1}{2} = (x/\sqrt{4\pi Kt}) + O(1/t)$; from this it follows that $t^{-1}\ln|u(x;t) - (\frac{1}{2}T_1 + \frac{1}{2}T_2)| = O(t^{-1}\ln t)$, which tends to zero. Therefore the relaxation time, as defined in Chapter 2, is infinite. To obtain a concrete numerical estimate of the time necessary to attain steady state, we define τ^* as the solution of

$$\left|u(x;\tau^*) - \left(\frac{1}{2}T_1 + \frac{1}{2}T_2\right)\right| = 0.1|T_1 - T_2|$$

To solve this equation, we must solve the equation $\Phi(s) - \frac{1}{2} = 0.1$ and set $s = x/\sqrt{2K\tau^*}$. From tables of the normal distribution function, we have $s = 0.25$, and thus $\tau^* = (2K)^{-1}(x/0.25)^2$. For example, if $K = 200$ cm^2/s and $x = 10$ cm, then $\tau^* = 4$ s. •

The final example illustrates the possibility of solving the heat equation when the initial data are given by an *unbounded* function.

EXAMPLE 5.2.4. *Find the solution of the heat equation $u_t = Ku_{xx}$ with the initial data $u(x;0) = e^{bx}$, where b is a positive constant.*

Solution. In this case the Gauss-Weierstrass integral can be computed by completing the square, as follows:

$$\begin{aligned}
u(x;t) &= \frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4Kt}e^{b\xi}\,d\xi \\
&= \frac{1}{\sqrt{4\pi Kt}}e^{-x^2/4Kt}\int_{-\infty}^{\infty} e^{x\xi/2Kt}e^{-\xi^2/4Kt}e^{b\xi}\,d\xi \\
&= \frac{1}{\sqrt{4\pi Kt}}e^{-x^2/4Kt}\int_{-\infty}^{\infty} e^{-[\xi^2-2\xi(x+2bKt)]/4Kt}d\xi \\
&= \frac{1}{\sqrt{4\pi Kt}}e^{-x^2/4Kt}e^{(x+2bKt)^2/4Kt}\int_{-\infty}^{\infty} e^{-[\xi-(x+2bKt)]^2/4Kt}d\xi \\
&= e^{bx}e^{b^2Kt} \quad \bullet
\end{aligned}$$

This example shows that the Gauss-Weierstrass integral can be applied to certain unbounded functions. Additional examples of this type are contained in the exercises.

5.2.6. Solutions on a half-line: The method of images. Now we turn to heat flow in the semi-infinite region $x > 0$. In each case we make a suitable extension of the inital data to the complementary region $x < 0$, where the nature of the extension depends on the boundary condition imposed at $x = 0$. (In physical terms, we are creating a fictional temperature distribution on the negative axis.) This is known as the *method of images.* We consider the three boundary conditions separately.

5.2.6.1. *Dirichlet boundary condition at $x = 0$.* We first consider the problem

$$\begin{aligned} u_t &= Ku_{xx} && t > 0, x > 0 \\ u(0;t) &= 0 && t > 0 \\ u(x;0) &= f(x) && x > 0 \end{aligned}$$

To solve this problem, we extend the given initial function $f(x)$ to the entire real axis as an odd function and apply the explicit representation (5.2.10). In detail, we set

$$\tilde{f}(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

so that $\tilde{f}(x)$ is an odd function, defined for $-\infty < x < \infty$. We substitute in (5.2.10) and obtain

$$\begin{aligned} u(x;t) &= \int_{-\infty}^{\infty} \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi Kt}} \tilde{f}(\xi) d\xi \\ &= \left(\int_{-\infty}^{0} + \int_{0}^{\infty} \right) \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi Kt}} \tilde{f}(\xi) d\xi \end{aligned}$$

In the first integral we change ξ to $-\xi$ and use the oddness of $\tilde{f}$. Thus

$$\int_{-\infty}^{0} \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi Kt}} \tilde{f}(\xi) d\xi = -\int_{0}^{\infty} \frac{\exp[-(x+\xi)^2/4Kt]}{\sqrt{4\pi Kt}} f(\xi) d\xi$$

Putting the two together, we have

$$(5.2.13) \qquad \boxed{u(x;t) = \int_0^{\infty} \frac{\exp[-(x-\xi)^2/4Kt] - \exp[-(x+\xi)^2/4Kt]}{\sqrt{4\pi Kt}} f(\xi)\, d\xi}$$

EXAMPLE 5.2.5. *Use the method of images to solve the inital-value problem for the heat equation with $u(x;0) = 1$, with the boundary condition $u(0;t) = 0$.*

Solution. From formula (5.2.13), we have

$$\begin{aligned} u(x;t) &= \int_0^{\infty} \frac{\exp[-(x-\xi)^2/4Kt] - \exp[-(x+\xi)^2/4Kt]}{\sqrt{4\pi Kt}} f(\xi)\, d\xi \\ &= \Phi(x/\sqrt{2Kt}) - \Phi(-x/\sqrt{2Kt}) \quad \bullet \end{aligned}$$

EXAMPLE 5.2.6. *Use the method of images to solve the inital-value problem for the heat equation with $u(x;0) = e^{bx}$, with the boundary condition $u(0;t) = 0$.*

Solution. Applying (5.2.13) directly, we have

$$u(x;t) = \int_0^{\infty} \frac{\exp[-(x-\xi)^2/4Kt] - \exp[-(x+\xi)^2/4Kt]}{\sqrt{4\pi Kt}} e^{b\xi}\, d\xi$$

This can be expressed in terms of the normal distribution function Φ. Making the change of variable $z = (\xi - x)/\sqrt{2Kt}$, the first integral is transformed as follows:

$$\begin{aligned}
\int_0^\infty \frac{\exp[-(x-\xi)^2/4Kt]}{\sqrt{4\pi Kt}} e^{b\xi} d\xi &= e^{bx} \int_{-x/\sqrt{2Kt}}^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}} e^{bz\sqrt{2Kt}}\, dz \\
&= e^{bx} e^{b^2 Kt} \int_{-x/\sqrt{2Kt}}^\infty \frac{e^{-(z-b\sqrt{2Kt})^2/2}}{\sqrt{2\pi}}\, dz \\
&= e^{bx} e^{b^2 Kt} \int_{-x/\sqrt{2Kt}-b\sqrt{2Kt}}^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du \\
&= e^{bx} e^{b^2 Kt} \left(1 - \Phi(-\frac{x}{\sqrt{2Kt}} - b\sqrt{2Kt})\right)
\end{aligned}$$

The second integral is obtained by replacing x by $-x$. Subtracting the two terms, we obtain

$$\begin{aligned}
u(x;t) &= e^{bx} e^{b^2 Kt} \left(1 - \Phi(\frac{-x}{\sqrt{2Kt}} - b\sqrt{2Kt})\right) \\
&- e^{-bx} e^{b^2 Kt} \left(1 - \Phi(\frac{x}{\sqrt{2Kt}} - b\sqrt{2Kt})\right)
\end{aligned}$$

This is the required form of the solution. •

5.2.6.2. *Neumann boundary condition at $x = 0$.* The method of images can be modified to solve the initial-boundary-value problem

$$\begin{aligned}
u_t &= K u_{xx} \qquad && t > 0, x > 0 \\
u_x(0;t) &= 0 && t > 0 \\
u(x;0) &= f(x) && x > 0
\end{aligned}$$

Notice that we have changed only the boundary condition and that otherwise the problem is identical to the previous one. To treat this modification, we extend the initial function as an even function, by the rule

$$\tilde{f}(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

so that $\tilde{f}$ is an even function. The reader is invited to follow the steps of the preceding problem to show that in this case we have the solution formula

$$\boxed{u(x;t) = \int_0^\infty \frac{\exp[-(x-\xi)^2/4Kt] + \exp[-(x+\xi)^2/4Kt]}{\sqrt{4\pi Kt}} f(\xi)\, d\xi} \tag{5.2.14}$$

EXAMPLE 5.2.7. *Use the method of images to solve the inital-value problem for the heat equation with $u(x;0) = 1$, with the boundary condition $u_x(0;t) = 0$.*

Solution. From formula (5.2.14), we have

$$\begin{aligned} u(x;t) &= \int_0^\infty \frac{\exp[-(x-\xi)^2/4Kt] + \exp[-(x+\xi)^2/4Kt]}{\sqrt{4\pi Kt}} f(\xi)\, d\xi \\ &= \Phi(x/\sqrt{2Kt}) + \Phi(-x/\sqrt{2Kt}) \quad \bullet \end{aligned}$$

5.2.6.3. *Mixed boundary condition at* $x = 0$. As a final "variation on the theme," we consider the initial-value problem

$$\begin{aligned} u_t &= K u_{xx} & t > 0, x > 0 \\ u_x(0;t) - hu(0;t) &= 0 & t > 0 \\ u(x;0) &= f(x) & x > 0 \end{aligned}$$

where h is a positive constant; this is the general homogeneous boundary condition of the third kind at $x = 0$. Since $u_x - hu$ is zero when $x = 0$, it is natural to extend the initial data $f(x)$ so that $f'(x) - hf(x)$ is an *odd* function, continuous at $x = 0$. This may be done by solving an ordinary differential equation, and is closely related to the generalized h-transform described in Sec. 5.1.

Assuming that we have extended $f(x), 0 < x < \infty$, to $\tilde{f}(x), -\infty < x < \infty$, so that

$$\tilde{f}'(x) - h\tilde{f}(x) = -\tilde{f}'(-x) + h\tilde{f}(-x)$$

we define

$$u(x;t) = \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4Kt} \tilde{f}(\xi)\, d\xi$$

Clearly, $u(x;t)$ satisfies the heat equation with the initial condition $u(x;0) = \tilde{f}(x)$. To verify the boundary condition, we write

$$\begin{aligned} u_x(0;t) &= \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} \frac{\xi}{2Kt} e^{-\xi^2/4Kt} \tilde{f}(\xi)\, d\xi \\ &= -\frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} \tilde{f}(\xi) d\left(e^{-\xi^2/4Kt}\right) \\ &= \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} \tilde{f}'(\xi) e^{-\xi^2/4Kt}\, d\xi \end{aligned}$$

Therefore

$$u_x(0;t) - hu(0;t) = \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} [\tilde{f}'(\xi) - h\tilde{f}(\xi)] e^{-\xi^2/4Kt}\, d\xi$$

The integrand is an odd function; hence the integral is zero, and we have proved that the boundary condition $u_x - hu = 0$ is satisfied.

To compute $\tilde{f}(x)$ for $x < 0$, we must solve the first-order ordinary differential equation

$$\tilde{f}'(x) - h\,\tilde{f}(x) = g(x) = -f'(-x) + h\,f(-x)$$

This is done by means of the integrating factor e^{-hx}, with the result

$$\tilde{f}(x) = \tilde{f}(0)e^{hx} + \int_0^x e^{h(x-\xi)}g(\xi)\,d\xi = f(-x) - 2he^{hx}\int_0^{-x} e^{hy}f(y)\,dy$$

EXAMPLE 5.2.8. *Solve the heat equation $u_t = Ku_{xx}$ for $x > 0, t > 0$, with the boundary condition $u_x(0;t) - hu(0;t) = 0$ and the initial condition $u(x;0) = e^{-ax}$, where $a > 0$.*

Solution. In this case we have $g(x) = -f'(-x) + hf(-x) = ae^{ax} + he^{ax} = (h+a)e^{ax}$. Therefore we have, for $a \neq h$,

$$\begin{aligned}\tilde{f}(x) &= e^{hx} + \int_0^x e^{h(x-\xi)}(h+a)e^{a\xi}\,d\xi \\ &= e^{hx} + \frac{a+h}{a-h}e^{hx}\left(e^{(a-h)x} - 1\right) \\ &= -\frac{2h}{a-h}e^{hx} - \frac{h+a}{h-a}e^{ax}\end{aligned}$$

Therefore the heat equation is solved by

$$\begin{aligned}u(x;t) &= \frac{1}{\sqrt{4\pi Kt}}\int_0^\infty e^{-a\xi}e^{-(x-\xi)^2/4Kt}d\xi \\ &\quad + \frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^0 e^{-(x-\xi)^2/4Kt}\left[-\frac{2h}{a-h}e^{h\xi} - \frac{h+a}{h-a}e^{a\xi}\right]d\xi\end{aligned}$$

If $h = a$, the integral gives $\tilde{f}(x) = e^{hx}[1 + x(h+a)] = e^{ax}(1+2ax)$ and a corresponding formula for $u(x;t)$. •

Figure 5.2.3 displays the method of images in the limiting case $a = 0$, where $f(x) = 1, x > 0$.

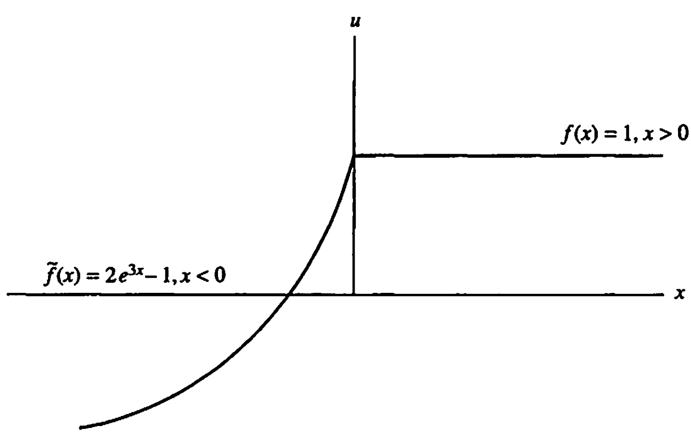


FIGURE 5.2.3 The method of images applied to the boundary condition $u'(0) = 3u(0)$ with the initial condition $f(x) = 1, x > 0$.

EXERCISES 5.2.6

1. Show that the solution formula (5.2.10) can be written in the form
$$u(x;t) = \int_{-\infty}^{\infty} f(x - z\sqrt{2Kt})\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$$
by making the substitution $x - \xi = z\sqrt{2Kt}$.
2. Use the result of Exercise 1 to prove directly that $u(x;t)$ is a solution of the equation. You may assume for this purpose that f', f'' exist.
3. Use the result of Exercise 2 to prove that $|u(x;t) - f(x)| \le M\sqrt{Kt/\pi}$. You may assume for this purpose that f' exists and $|f'(x)| \le M$ everywhere. [*Hint:* Write $u(x;t) - f(x)$ as a definite integral and apply the mean-value theorem for $f(x - z\sqrt{2Kt}) - f(x)$.]
4. Prove the following inequalities, which were used in the proof of Theorem 5.3.
$$|\cos\theta - 1| \le \frac{\theta^2}{2}, \qquad |\sin\theta - \theta| \le \frac{\theta^2}{2}$$
Hint:
$$1 - \cos\theta = \int_0^{\theta}\int_0^{\theta_1} \cos\varphi\, d\varphi\, d\theta_1, \qquad \sin\theta - \theta = -\int_0^{\theta}\int_0^{\theta_1} \sin\varphi\, d\varphi\, d\theta_1$$
5. Complete the details of the proof of formulas (5.2.8) and (5.2.9).
6. Apply the method of images to solve the initial-value problem
$$\begin{aligned} u_t &= Ku_{xx} && t > 0, x > 0 \\ u(0;t) &= 0 && t > 0 \\ u(x;0) &= \begin{cases} 1 & 0 \le x \le L_1 \\ 0 & x > L_1 \end{cases} \end{aligned}$$
Show that $u(x;t) = O(t^{-3/2})$ when $t \to \infty$. (*Hint:* $|e^{-a} - e^{-b}| \le |a - b|$ if $a, b \ge 0$.)
7. Apply the method of images to solve the initial-value problem
$$\begin{aligned} u_t &= Ku_{xx} && t > 0, x > 0 \\ u_x(0;t) &= 0 && t > 0 \\ u(x;0) &= \begin{cases} 1 & 0 \le x \le L_1 \\ 0 & x > L_1 \end{cases} \end{aligned}$$
Show that $u(x;t) = O(t^{-1/2})$ when $t \to \infty$.
8. Consider the following initial-value problem for a heat equation with transport term:
$$\begin{aligned} u_t &= Ku_{xx} + vu_x && t > 0, -\infty < x < \infty \\ u(x;0) &= f(x) \end{aligned}$$

where v is a nonzero constant that represents the mean velocity of the diffusing substance.

(a) Use the method of Fourier transforms to solve the problem in the form

$$u(x;t) = \int_{-\infty}^{\infty} e^{i\mu x} e^{-\mu^2 Kt} e^{iv\mu t} F(\mu)\,d\mu$$

(b) Show that this can be rewritten in the form

$$u(x;t) = \int_{-\infty}^{\infty} \frac{\exp[-(x-y+vt)^2/4Kt]}{\sqrt{4\pi Kt}} f(y)\,dy$$

(c) Show that $u(x;t) = O(e^{-v^2 t/4K})$, $t \to \infty$. You may suppose for this purpose that $f(y) = 0$ for $|y| \geq 5.3 \times 10^6$.

9. Show that (5.2.13) can be written in the form

$$u(x;t) = \int_0^{\infty} F_s(\mu) e^{-\mu^2 Kt} \sin \mu x \, d\mu$$

where $F_s(\mu)$ is the Fourier sine transform of f.

10. Show that (5.2.14) can be written in the form

$$u(x;t) = \int_0^{\infty} F_c(\mu) e^{-\mu^2 Kt} \cos \mu x \, d\mu$$

where $F_c(\mu)$ is the Fourier cosine transform of f.

11. Consider the following initial-value problem for a heat equation with a linear source term:

$$\begin{aligned} u_t &= Ku_{xx} + au \qquad t > 0, -\infty < x < \infty \\ u(x;0) &= f(x) \end{aligned}$$

where a is a positive constant that represents the strength of the source term, per unit of temperature.

(a) Find a Fourier representation of the solution.

(b) Find an explicit representation of the solution, corresponding to the Gauss-Weierstrass integral (5.2.10).

12. Consider the following initial-value problem for a heat equation with a transport term and a linear source term:

$$\begin{aligned} u_t &= Ku_{xx} + vu_x + au \qquad t > 0, -\infty < x < \infty \\ u(x;0) &= f(x) \end{aligned}$$

where a, v are nonzero constants

(a) Find a Fourier representation of the solution.

(b) Find an explicit representation of the solution, corresponding to the Gauss-Weierstrass integral (5.2.10).

13. Consider the following initial-value problem for a heat equation with a linear time-dependent source term:

$$\begin{aligned} u_t &= Ku_{xx} + at^p u \qquad t > 0, -\infty < x < \infty \\ u(x;0) &= f(x) \end{aligned}$$

where a, p are positive constants.

(a) Find a Fourier representation of the solution.

(b) Find an explicit representation of the solution, corresponding to the Gauss-Weierstrass integral (5.2.10).

14. Consider the Gauss-Weierstrass integral (5.2.10) when the function $f(x)$, $-\infty < x < \infty$, satisfies $|f(x)| \le Ae^{bx^2}$ where A, b are positive constants.

(a) Show that the Gauss-Weierstrass integral is convergent if $t < 1/4Kb$.

(b) Show that the initial condition is satisfied if $f(x)$, $-\infty < x < \infty$, is piecewise continuous.

15. Solve the heat equation $u_t = Ku_{xx}$ with the initial conditions $u(x;0) = T_1$ if $x < 0$ and $u(x;0) = 0$ if $x > 0$. Show that the level curves $u(x;t) = C$ are parabolas passing through $(0,0)$ in the (x,t) plane. Plot these level curves if $K = \frac{1}{2}$, $T_1 = 100$ for the values $C = 10$, $C = 30$, $C = 50$.
16. Two materials of the same conductivity K and temperatures T_1, T_2 are brought together at $t = 0$. Find the time τ^* such that $|u(x;\tau^*) - (\frac{1}{2}T_1 + \frac{1}{2}T_2)| < 0.2|T_1 - T_2|$, where $u(x;t)$ is the solution of the heat equation.
17. Solve the heat equation $u_t = Ku_{xx}$ with the initial conditions $u(x;0) = T_1$ if $-L < x < 0$, $u(x;0) = T_2$ if $0 < x < L$, and $u(x;0) = 0$ if $|x| > L$. What is $\lim_{t\to\infty} u(x;t)$?
18. Verify directly that $u(x;t)$, defined by (5.2.6), satisfies the boundary condition $u(0;t) = 0$.
19. Verify directly that $u(x;t)$, defined by (5.2.7), satisfies the boundary condition $u_x(0;t) = 0$.
20. Consider the heat equation $u_t = Ku_{xx}$ with $u(x;0) = x^n$, where $n = 1, 2, \ldots$ Show that a solution can be found in the form $u(x;t) = x^n + a_1(t)x^{n-1} + \cdots + a_{n-1}(t)x + a_n(t)$ for suitable functions $a_1(t), \ldots, a_n(t)$.

5.2.7. The Black-Scholes model. In this subsection we indicate an application of the heat equation to *financial mathematics*—specifically, to the problem of pricing an option on an asset purchased at $t = 0$ and maturing at time $t = T$. At that time it will have an *exercise price* E, which is to be compared to the market value of the asset, denoted S. If $S \le E$, then the option is worthless and the final value is zero. Otherwise $S > E$ and the asset can be sold, realizing a profit of $S - E$. The time-dependent value of the option is denoted $V(S,t)$, defined for $S \ge 0$, $0 \le t \le T$.

In order to formulate the mathematical model, we assume the following financial parameters to be given: σ, the volatility of the asset, and r, the interest rate. The volatility is associated with random fluctuations of the value of assets, whereas the interest rate is associated with deterministic changes in asset values. Using a stochastic model[1] leads to the following partial differential equation for the value function:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad S > 0,\ 0 \le t \le T \tag{5.2.15}$$

with the boundary conditions

$$V(0;t) = 0 \qquad \frac{V(S;t)}{S} \to 1, S \to \infty$$

The first of these indicates that if the stock is worthless, then so is the option. The second indicates that if the price is very large, the value of the option should be accordingly large. The final value of the option is defined by the statement

$$V(S;T) = \max(S - E, 0)$$

as explained above.

Note that the Black-Scholes equation (5.2.15) cannot be solved directly by the previous methods because of two new features: (i) it contains the variable coefficients S and S^2 multiplying two of the terms and (ii) it is written in the backward time direction, with a "final value" instead of an initial value. We will now remedy these two difficulties.

5.2.7.1. *Transformation to the heat equation.* Equation (5.2.15) can be transformed to the standard heat equation by the following series of transformations. First we define new independent variables (x, τ) by

$$x = \log(S/E), \qquad \tau = T - t$$

These satisfy $-\infty < x < \infty$, $0 \le \tau \le T$, $dx/dS = 1/S$, $d\tau/dt = -1$. A new function is defined by setting

$$v(x,\tau) = \frac{V(S,t)}{E}$$

Computing directly, we have

$$\begin{aligned} V_t &= \frac{\partial V}{\partial t} = -E\frac{\partial v}{\partial \tau} = -Ev_\tau \\ V_S &= \frac{\partial V}{\partial S} = E\frac{\partial v}{\partial x}\frac{1}{S} = \frac{E}{S}v_x \\ V_{SS} &= \frac{\partial^2 V}{\partial S^2} = \frac{E}{S^2}\frac{\partial^2 v}{\partial x^2} - \frac{E}{S^2}\frac{\partial v}{\partial x} = \frac{E}{S^2}(v_{xx} - v_x) \end{aligned}$$

resulting in the equation

$$v_\tau = \frac{\sigma^2}{2}v_{xx} + (r - \frac{\sigma^2}{2})v_x - rv \tag{5.2.16}$$

[1] P. Wilmott, S. Howison, and J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge University Press, New York, 1995.

The final condition at $t = T$ is transformed into an initial condition at $\tau = 0$:

$$v(x; 0) = \max(e^x - 1, 0)$$

The above transformations have reduced the Black-Scholes equation (5.2.15) to a second-order partial differential equation with constant coefficients, in the forward time direction. The second step of the reduction is to eliminate the lower-order terms involving v and v_x. To do this, we define a new function $u(x, \tau)$ by setting

$$v(x, \tau) = u(x, \tau)e^{\alpha x + \beta\tau}$$

where the constants α, β are to be determined. Computing directly, we have

$$\begin{aligned} v_\tau &= (\beta u + u_\tau)e^{\alpha x + \beta\tau} \\ v_x &= (\alpha u + u_x)e^{\alpha x + \beta\tau} \\ v_{xx} &= (\alpha^2 u + 2\alpha u_x + u_{xx})e^{\alpha x + \beta\tau} \end{aligned}$$

Substituting these in (5.2.16) and canceling the exponential term results in the equation

$$u_\tau = \frac{\sigma^2}{2}u_{xx} + \left(\alpha\sigma^2 + r - \frac{\sigma^2}{2}\right)u_x + \left(\frac{\alpha(\alpha-1)\sigma^2}{2} + r(\alpha - 1) - \beta\right)u$$

In order to eliminate the terms with u_x and u, we are led to the equations

$$\alpha\sigma^2 + r - \frac{\sigma^2}{2} = 0, \qquad \frac{\alpha(\alpha-1)\sigma^2}{2} + r(\alpha - 1) - \beta = 0$$

Defining a new parameter k through the equation $k = 2r/\sigma^2$, these are solved to yield

$$\begin{aligned} \alpha &= -\frac{r - (\sigma^2/2)}{\sigma^2} = -\frac{1}{2}(k-1) \\ \beta &= \frac{\alpha(\alpha-1)\sigma^2}{2} + r(\alpha - 1) = -\frac{\sigma^2(1+k)^2}{8} \end{aligned}$$

so that the function $u(x; \tau)$ satisfies the familiar heat equation

$$u_\tau = \frac{\sigma^2}{2}u_{xx}$$

The initial conditions are determined by noting that

$$\begin{aligned} v(x; \tau) &= u(x; \tau)e^{\alpha x + \beta\tau} \\ v(x; 0) &= u(x; 0)e^{\alpha x} = \max(e^x - 1, 0) \\ u(x; 0) &= \max(e^{x(1-\alpha)} - e^{-\alpha x}, 0) \\ &= \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \end{aligned}$$

This completes the reduction of the Black-Scholes equation to the standard heat equation. The initial condition $u(x; 0)$ is nonzero precisely when $x > 0$.

5.2.7.2. *Solution of the Black-Scholes equation.* Having reduced the problem to a standard initial-value problem for the heat equation, we can write the solution using (5.2.10) with $K = \sigma^2/2$:

$$\begin{aligned} (5.2.17) \qquad u(x;\tau) &= \int_{-\infty}^{\infty} u_0(\xi)\frac{e^{-(x-\xi)^2/2\sigma^2\tau}}{\sqrt{2\pi\sigma^2\tau}}d\xi \\ &= \int_0^{\infty}[e^{\frac{1}{2}(k+1)\xi} - e^{\frac{1}{2}(k-1)\xi}]\frac{e^{-(x-\xi)^2/2\sigma^2\tau}}{\sqrt{2\pi\sigma^2\tau}}d\xi \end{aligned}$$

This is reduced to a standard integral invoving the normal distribution function $\Phi(x)$ by making the substitution $z = (\xi - x)/(\sigma\sqrt{\tau})$ as follows: we have $dz = d\xi/(\sigma\sqrt{\tau})$ and the new limit of integration $z = -x/(\sigma\sqrt{\tau})$ when $\xi = 0$, so that

$$u(x;\tau) = I_1 - I_2$$

where

$$\begin{aligned} I_1 &= e^{\frac{1}{2}(k+1)x}\int_{-\frac{x}{\sigma\sqrt{\tau}}}^{\infty} e^{\frac{1}{2}(k+1)z\sigma\sqrt{\tau}}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \\ I_2 &= e^{\frac{1}{2}(k-1)x}\int_{-\frac{x}{\sigma\sqrt{\tau}}}^{\infty} e^{\frac{1}{2}(k-1)z\sigma\sqrt{\tau}}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \end{aligned}$$

If we complete the square in the exponent, the result is

$$\begin{aligned} I_1 &= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2}\int_{-\frac{x}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{z^2}{2}+\frac{1}{2}(k+1)z\sigma\sqrt{\tau}-\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2}\frac{dz}{\sqrt{2\pi}} \\ &= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2}\int_{-\frac{x}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}[z-\frac{1}{2}\sigma\sqrt{\tau}(k+1)]^2}\frac{dz}{\sqrt{2\pi}} \\ &= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2}\int_{-\frac{x}{\sigma\sqrt{\tau}}-\frac{\sigma\sqrt{\tau}(k+1)}{2}}^{\infty} e^{-w^2/2}\frac{dw}{\sqrt{2\pi}} \\ &= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2}\left[1 - \Phi\left(-\frac{x+\sigma^2\tau(k+1)/2}{\sigma\sqrt{\tau}}\right)\right] \\ &= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}(\sigma\sqrt{\tau}(k+1))^2}\Phi\left(\frac{x+\sigma^2\tau(k+1)/2}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

where we have used the symmetry of the normal distribution in the form $\Phi(u) + \Phi(-u) = 1$. Similarly, I_2 is obtained by replacing $k+1$ by $k-1$, hence

$$I_2 = e^{\frac{1}{2}(k-1)x}e^{\frac{1}{8}(\sigma\sqrt{\tau}(k-1))^2}\Phi\left(\frac{x+\sigma^2\tau(k-1)/2}{\sigma\sqrt{\tau}}\right)$$

EXERCISES 5.2.7

1. Consider the second-order parabolic equation $v_\tau = (\sigma^2/2)v_{xx} + av_x + bv$, where a, b, and σ are constants. Show that this can be reduced to the standard heat equation for the function u defined by $v = u\,e^{\alpha x+\beta\tau}$ for suitable values of α, β.
2. Consider the second-order parabolic equation $u_\tau = (\sigma(\tau)^2/2)u_{xx}$. Show that this can be reduced to the standard heat equation by defining a new time variable $t = \psi(\tau)$ for a suitable choice of ψ.

5.2.8. Hermite polynomials. The normal density function $e^{-x^2/2}$ is proportional to its Fourier transform. This function is the first in an infinite sequence of functions with that property. To define these, we introduce the generating function

$$e^{tx-t^2/2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x) = H_0(x) + tH_1(x) + \frac{t^2}{2}H_2(x) + \cdots \tag{5.2.18}$$

This power series converges for all t, real and complex; the coefficients $H_k(x)$ are the *Hermite polynomials.* Since the generating function is a Taylor series in the variable t, the coefficients can be obtained by successive differentiation as

$$H_k(x) = \left(\frac{d}{dt}\right)^k (e^{tx-t^2/2})|_{t=0} \qquad k = 0, 1, 2, \ldots$$

The first few are written as follows:

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x \\ H_4(x) &= x^4 - 6x^2 + 3 \end{aligned}$$

Hermite polynomials satisfy the following system of orthogonality relations:

$$\boxed{\int_{-\infty}^{\infty} H_k(x)H_j(x)e^{-x^2/2}\,dx = \begin{cases} k!\sqrt{2\pi} & k = j \\ 0 & k \neq j \end{cases}} \tag{5.2.19}$$

To prove this, multiply the generating function formulas in the variables t, s by the weight factor $e^{-x^2/2}$:

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^k}{k!}\frac{s^j}{j!} H_k(x)H_j(x)e^{-x^2/2} = e^{tx-t^2/2}e^{sx-s^2/2}e^{-x^2/2}$$

We integrate the series term by term to obtain

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{s^j}{j!}\int_{-\infty}^{\infty}H_k(x)H_j(x)e^{-x^2/2}\,dx &= \int_{-\infty}^{\infty}e^{tx-t^2/2}e^{sx-s^2/2}e^{-x^2/2}\,dx \\
&= e^{st}\int_{-\infty}^{\infty}e^{-[x^2-2x(s+t)+(s+t)^2]/2}\,dx \\
&= \sqrt{2\pi}\,e^{st} \\
&= \sqrt{2\pi}\sum_{k=0}^{\infty}\frac{(st)^k}{k!}
\end{aligned}$$

The equality of these two power series implies the equality of the corresponding coefficients—hence the orthogonality relations as stated. In particular, the functions $(2\pi)^{-1/4}H_k(x)/\sqrt{k!}$ are orthonormal with respect to the weight function $e^{-x^2/2}$.

In order to express the orthogonality without a weight function, we introduce the *Hermite functions*

$$h_k(x) = \pi^{-1/4}\frac{H_k(x\sqrt{2})}{\sqrt{k!}}e^{-x^2/2} \qquad k = 0, 1, 2\ldots$$

which satisfy

$$\int_{-\infty}^{\infty}h_k(x)\,h_j(x)\,dx = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

These functions transform simply under the Fourier transform, as follows:

$$\boxed{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h_k(x)\,e^{-i\mu x}\,dx = (-i)^k\,h_k(\mu) \quad k = 0, 1, 2\ldots} \tag{5.2.20}$$

Apart from the constant factor $\sqrt{2\pi}$, the operation of Fourier transformation reproduces the kth Hermite function, multiplied by the factor $(-i)^k$, $k = 0, 1, 2\ldots.$ For example, the function $(2x^2 - 1)e^{-x^2/2}$ is proportional to the negative of its Fourier transform.

To prove (5.2.20), we can use the technique of generating functions, by writing

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{t^k}{k!}H_k(x\sqrt{2})e^{-x^2/2}e^{-i\mu x} &= e^{tx\sqrt{2}-t^2/2-i\mu x-x^2/2} \\
&= e^{-[x^2-2x(t\sqrt{2}-i\mu)]/2}e^{-t^2/2} \\
&= e^{-[x-(t\sqrt{2}-i\mu)]^2/2}e^{t^2/2-i\mu t\sqrt{2}-\mu^2/2}
\end{aligned}$$

Integrating term by term, we find that

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{t^k}{k!}\int_{-\infty}^{\infty}H_k(x\sqrt{2})e^{-x^2/2}e^{-i\mu x}\,dx &= e^{t^2/2-i\mu t\sqrt{2}-\mu^2/2}\int_{-\infty}^{\infty}e^{-[x-(t\sqrt{2}-i\mu)]^2/2}\,dx\\ &= \sqrt{2\pi}\,e^{t^2/2-i\mu t\sqrt{2}-\mu^2/2}\\ &= \sqrt{2\pi}\,e^{-\mu^2/2}\sum_{k=0}^{\infty}\frac{(-it)^k}{k!}H_k(\mu\sqrt{2})\end{aligned}$$

Identifying the coefficients of t^k completes the proof.

One can reinterpret the preceding result as providing a basis of functions in which the Fourier transform has a simple structure. Indeed, for a finite sum,

$$f(x)=\sum_{k=0}^{N}a_kh_k(x)$$

the Fourier transform is

$$F(\mu)=\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{N}(-i)^ka_kh_k(\mu)$$

Hermite polynomials can also be used to solve the heat equation in one dimension with *polynomial* initial conditions. To see this, first note that for any complex number α, the function $(t,x)\to e^{\alpha x}e^{\alpha^2Kt}$ is a solution of the heat equation. On the other hand, from the generating function, we have

$$\begin{aligned}\exp(\alpha x+\alpha^2Kt) &= \exp\left(i\alpha\sqrt{2Kt}\frac{x}{i\sqrt{2Kt}}-\frac{1}{2}(i\alpha\sqrt{2Kt})^2\right)\\ &= \sum_{k=0}^{\infty}\frac{(i\alpha\sqrt{2Kt})^k}{k!}H_k\left(\frac{x}{i\sqrt{2Kt}}\right)\\ &= \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}(i\sqrt{2Kt})^kH_k\left(\frac{x}{i\sqrt{2Kt}}\right)\end{aligned}$$

Since the sum of this power series satisfies the heat equation identically in α, it follows that each term of the series satisfies the heat equation. Formally, we define the *heat polynomials* by

$$H_k(x,t)=(i\sqrt{2Kt})^k\,H_k\left(\frac{x}{i\sqrt{2Kt}}\right)$$

In detail, we have the following for the first few heat polynomials:

k	$H_k(x;t)$
0	1
1	x
2	$x^2 + 2Kt$
3	$x^3 + 6Ktx$
4	$x^4 + 12x^2Kt + 12K^2t^2$

EXAMPLE 5.2.9. *Solve the heat equation $u_t = Ku_{xx}$ for $t > 0, -\infty < x < \infty$, with the initial condition $u(x;0) = 2 + 6x + 3x^4$.*

Solution. Any finite sum of the form $\sum_{k=0}^{N} a_k H_k(x;t)$ is a solution of the heat equation with initial conditions $\sum_{k=0}^{N} a_k x^k$. To satisfy the stated initial conditions, we set $a_0 = 2$, $a_1 = 6$, $a_4 = 3$, and $a_k = 0$ otherwise. The solution is $u(x;t) = 2H_0(x;t) + 6H_1(x;t) + 3H_4(x;t) = 2 + 6x + 3(x^4 + 12x^2Kt + 12K^2t^2)$. •

EXERCISES 5.2.8

1. Use the generating function for Hermite polynomials to prove the equations $H_k'(x) = kH_{k-1}(x), k = 1, 2 \ldots$.
2. Use the generating function for Hermite polynomials to prove the equations $H_{k+1}(x) = xH_k(x) - kH_{k-1}(x), k = 1, 2 \ldots$.
3. Combine the results of the two previous exercises to prove the following differential equation satisfied by the Hermite polynomials: $H_k''(x) - xH_k'(x) = -kH_k(x), k = 0, 1, 2 \ldots$.
4. Consider the partial differential equation $u_t = u_{xx} - xu_x$.

   (a) Show that separated solutions may be obtained in the form $u(x;t) = e^{-kt}H_k(x), k = 0, 1, 2 \ldots$.

   (b) Show directly, or by using the generating function, that for any complex number α, the function $(t, x) \to \exp(\alpha x e^{-t} - \frac{1}{2}\alpha^2 e^{-2t})$ is a solution.

   (c) Find a Fourier representation of the solution of the equation $u_t = u_{xx} - xu_x$ for $t > 0$, $-\infty < x < \infty$ with $u(x;0) = f(x) = \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}$. (*Hint:* Take $\alpha = i\mu$ and form a continuous superposition with respect to a suitable function of μ.)

   (d) Find an explicit representation of $u(x;t)$ by using the convolution theorem and the Fourier transform of the normal density.
5. Use Exercise 3 to show that the functions $\psi_k(x) = e^{-x^2/4}H_k(x)$ satisfy the differential equation $\psi_k''(x) - (x^2/4)\psi_k(x) = -(k + \frac{1}{2})\psi_k(x)$ for $0 = 1, 2, \ldots$.
6. Carry through the computations of Exercise 4 for the partial differential equation $u_t = u_{xx} - (x^2/4)u$.
7. Solve the heat equation $u_t = K\, u_{xx}$ for $t > 0, -\infty < x < \infty$ with the initial conditions $u(x;0) = 2 + 3x + 2x^3 + 6x^4$.

8. Use the definition of the Hermite polynomials to prove

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k!} e^{-x^2/2} H_k(x) = e^{-(t-x)^2/2}$

(b) $\displaystyle e^{-x^2/2} H_k(x) = (-1)^k \left(\frac{d}{dx}\right)^k (e^{-x^2/2}) \qquad k = 0, 1, 2 \ldots$

9. Use part (b) of the previous exercise to give a new proof of the orthogonality relations (5.2.19)
10. Show that the functions $H_k(x)H_l(y)e^{-(k+l)t}$ satisfy the partial differential equation $u_t = u_{xx} + u_{yy} - xu_x - yu_y$. Generalize to three variables.
11. Show that the Fourier transform of $H_k(x)e^{-x^2/2}$ is proportional to the function $(i\mu)^k e^{-\mu^2/2}$ and find the constant of proportionality.
12. Show that the Hermite polynomials are given explicitly by the formulas

$$\begin{aligned} H_{2n}(x) &= \sum_{k=0}^{n} \frac{(-1)^k (2n)! x^{2n-2k}}{2^k k!(2n-2k)!} \\ H_{2n+1}(x) &= \sum_{k=0}^{n} \frac{(-1)^k (2n+1)! x^{2n-2k+1}}{2^k k!(2n-2k+1)!} \end{aligned}$$

13. Use the result of Exercise 12 to show that the Hermite polynomials are bounded, in the form

$$\begin{aligned} |H_{2n}(x)| &\le \frac{(2n)! 2^n (1+|x|)^{2n}}{n!} \\ |H_{2n+1}(x)| &\le \frac{(2n+1)! 2^{n+1} (1+|x|)^{2n+1}}{n!} \end{aligned}$$

5.3. Solutions of the Wave Equation and Laplace's Equation

In this section we apply the Fourier transform to solve the wave equation in one, two, and three spatial dimensions and to solve Laplace's equation in two spatial dimensions.

5.3.1. One-dimensional wave equation and d'Alembert's formula. The Fourier transform can also be applied to solve initial-value problems for the wave equation. The simplest problem of this type is

$$\begin{aligned} y_{tt} &= c^2 y_{xx} && t > 0, -\infty < x < \infty \\ y(x;0) &= f_1(x) && -\infty < x < \infty \\ y_t(x;0) &= f_2(x) && -\infty < x < \infty \end{aligned} \tag{5.3.1}$$

This is a pure initial-value problem, the *Cauchy problem*; $f_1(x)$, $f_2(x)$ are prescribed functions that can be thought of as the initial position and velocity of an infinite vibrating string. We will follow the second method of Sec. 5.2.

To solve (5.3.1), we introduce the Fourier transforms

$$F_k(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mu x} f_k(x)dx \qquad k = 1, 2 \tag{5.3.2}$$

with the inversion formulas

$$f_k(x) = \int_{-\infty}^{\infty} e^{i\mu x} F_k(\mu)d\mu \qquad k = 1, 2 \tag{5.3.3}$$

The desired Fourier representation of the solution y is

$$y(x;t) = \int_{-\infty}^{\infty} Y(\mu, t)e^{i\mu x}d\mu \tag{5.3.4}$$

where $Y(\mu, t)$ is an unknown function, which we will now determine. To do this, we substitute (5.3.4) into the wave equation (5.3.1). Thus

$$0 = \int_{-\infty}^{\infty} [Y_{tt}(\mu;t) + c^2\mu^2 Y(\mu;t)]e^{i\mu x}d\mu$$

This requires that Y be a solution of the ordinary differential equation

$$Y_{tt} + c^2\mu^2 Y = 0$$

that is,

$$Y(\mu;t) = A(\mu)\cos\mu ct + B(\mu)\sin\mu ct \tag{5.3.5}$$

To find $A(\mu)$ and $B(\mu)$, we set $t = 0$ in (5.3.4) and use (5.3.5).

$$y(x;0) = \int_{-\infty}^{\infty} A(\mu)e^{i\mu x}d\mu, \qquad y_t(x;0) = \int_{-\infty}^{\infty} \mu c B(\mu)e^{i\mu x}d\mu$$

Comparing this with (5.3.3) and the initial conditions of (5.3.1), we see that we must have

$$F_1(\mu) = A(\mu), \qquad F_2(\mu) = \mu c B(\mu)$$

Substituting these into (5.3.5) and returning to (5.3.4), we have the *Fourier representation*

$$\boxed{y(x;t) = \int_{-\infty}^{\infty}\left[F_1(\mu)\cos\mu ct + F_2(\mu)\frac{\sin\mu ct}{\mu c}\right]e^{i\mu x}d\mu} \tag{5.3.6}$$

This is the desired representation by Fourier transforms. Using this, we can also obtain an explicit representation in terms of the given functions $f_1(x)$, $f_2(x)$. To do this, recall that

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} F_1(\mu)\cos\mu ct\, e^{i\mu x}d\mu &= \frac{1}{2}\int_{-\infty}^{\infty} F_1(\mu)(e^{i\mu ct}+e^{-i\mu ct})e^{i\mu x}d\mu\\ &= \frac{1}{2}\int_{-\infty}^{\infty} F_1(\mu)(e^{i\mu(x+ct)}+e^{i\mu(x-ct)})d\mu\\ &= \frac{1}{2}[f_1(x+ct)+f_1(x-ct)]\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{-\infty}^{\infty} F_2(\mu)\frac{\sin\mu ct}{\mu c}e^{i\mu x}d\mu &= \frac{1}{2}\int_{-\infty}^{\infty} F_2(\mu)\frac{e^{i\mu ct}-e^{-i\mu ct}}{i\mu c}e^{i\mu x}d\mu\\ &= \frac{1}{2}\int_{-\infty}^{\infty} F_2(\mu)\frac{e^{i\mu(x+ct)}-e^{i\mu(x-ct)}}{i\mu c}d\mu\\ &= \frac{1}{2c}\int_{-\infty}^{\infty} F_2(\mu)\left(\int_{x-ct}^{x+ct} e^{i\mu\xi}d\xi\right)d\mu\\ &= \frac{1}{2c}\int_{x-ct}^{x+ct}\left(\int_{-\infty}^{\infty} e^{i\mu\xi}F_2(\mu)d\mu\right)d\xi\\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi\end{aligned}$$

Adding these together yields *d'Alembert's formula:*

$$\boxed{y(x;t)=\frac{1}{2}[f_1(x+ct)+f_1(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi} \tag{5.3.7}$$

This was derived in Chapter 2 in the context of the vibrating string, where we had the boundary conditions $y(0;t)=0=y(L;t)$. We now see that the formula is, in fact, more general. The first term corresponds to a superposition of two traveling waves, with velocities $\pm c$. The second term has a similar interpretation.

We can show that (5.3.7) solves the initial-value problem (5.3.1) without any recourse to the Fourier transform. For this purpose we assume that f_1 has two continuous derivatives and f_2 has one continuous derivative. (This is in marked contrast with the heat equation, where piecewise smooth initial data are sufficient to ensure that the solution is differentiable to all orders for $t>0$.) The details are left as an exercise.

In many applications of the wave equation we would like to apply d'Alembert's formula in the case where f_1, f_2 are only piecewise smooth. For this purpose we enlarge the concept of solution in the following way. A function $y(x;t)$ is said to be a *weak solution* of the wave equation if there exists a sequence of ordinary solutions $y_n(x;t)$ such that $y(x;t)=\lim_{n\to\infty} y_n(x;t)$. With this extended concept of solution, we can say that (5.3.7) provides a weak solution of the wave equation for *any piecewise smooth* f_1, f_2. The solutions $y_n(x;t)$ may be taken to be the integrals (5.3.6), where the range of integration is restricted to $-n \le \mu \le n$.

The concept of weak solution can be considered at a more abstract, or intrinsic, level. To motivate this, note that if $y(x;t)$ is an ordinary solution of the wave equation, then for any smooth function $\psi(x;t)$ that is zero outside a sphere, we can integrate by parts twice to conclude $\int\int(\psi_{tt}(x;t)-c^2\psi_{xx}(x;t))y(x;t)\,dx\,dt = 0$. We now *define* a weak solution of the wave equation in the extended sense as any function $y(x;t)$ for which this integral is zero for all smooth functions ψ with compact support, i.e., zero outside a sphere. With this intrinsic definition, one can show that any pointwise limit of a sequence of ordinary solutions is a weak solution in the extended sense. It is less obvious, but nonetheless true, that any weak solution in the extended sense is a pointwise limit of ordinary solutions.

The d'Alembert formula can be combined with the method of images to solve initial-value problems for the wave equation in the half-line $x > 0$. This corresponds to an infinitely long vibrating string with boundary conditions imposed at one end. For example, we consider the initial-value problem

$$\begin{aligned} y_{tt} &= c^2 y_{xx} && t > 0, x > 0 \\ y(0;t) &= 0 && t > 0 \\ y(x;0) &= f(x) && x > 0 \\ y_t(x;0) &= 0 && x > 0 \end{aligned}$$

corresponding to an infinite string that is tied down at the end $x = 0$. To solve this problem, we extend f as an odd function to $x < 0$ by defining $\tilde{f}(-x) = -\tilde{f}(x)$ for all x and $\tilde{f}(x) = f(x)$ for $x > 0$. Substituting in d'Alembert's formula, we have

$$y(x;t) = \frac{1}{2}[\tilde{f}(x+ct) + \tilde{f}(x-ct)]$$

This function satisfies the boundary conditions, since $y(0;t) = \frac{1}{2}[\tilde{f}(ct)+\tilde{f}(-ct)] = 0$ and $\tilde{f}$ is an odd function.

If instead of the boundary condition $y(0;t) = 0$ we have the boundary condition $y_x(0;t) = 0$ (meaning that the end of the string at $x = 0$ is free to move), then the solution of the wave equation with the same initial conditions is given by the same formula $y(x;t) = \frac{1}{2}\bar{f}(x+ct)+\frac{1}{2}\bar{f}(x-ct)$, where now $\bar{f}$ is the *even* extension of f, that is, the function $\bar{f}$ defined for all x and satisfying $\bar{f}(-x) = \bar{f}(x)$ for all x while $\bar{f}(x) = f(x)$ for $x > 0$.

5.3.2. General solution of the wave equation. The Fourier method can be used to find the *general solution* of the wave equation $y_{tt} = c^2 y_{xx}$. Writing the solution in the Fourier representation $y(x;t) = \int_{-\infty}^{\infty} Y(\mu;t)e^{i\mu x}d\mu$, we substitute in the wave equation to obtain

$$0 = y_{tt} - c^2 y_{xx} = \int_{-\infty}^{\infty} (Y_{tt} + c^2\mu^2 Y)e^{i\mu x} d\mu$$

The equation $Y_{tt} + c^2\mu^2 Y = 0$ has the general solution $Y(\mu;t) = C(\mu)e^{i\mu ct} + D(\mu)e^{-i\mu ct}$. The solution $y(x;t)$ can be computed as

$$\begin{aligned} y(x;t) &= \int_{-\infty}^{\infty} Y(\mu;t)e^{i\mu x}d\mu \\ &= \int_{-\infty}^{\infty} [C(\mu)e^{i\mu ct} + D(\mu)e^{-i\mu ct}]e^{i\mu x}d\mu \\ &= \int_{-\infty}^{\infty} [C(\mu)e^{i\mu(x+ct)} + D(\mu)e^{i\mu(x-ct)}]d\mu \\ &= f(x+ct) + g(x-ct) \end{aligned}$$

where $f(x) = \int_{-\infty}^{\infty} C(\mu)e^{i\mu x}d\mu$, $g(x) = \int_{-\infty}^{\infty} D(\mu)e^{i\mu x}d\mu$ are the inverse Fourier transforms of $C(\mu)$, $D(\mu)$. This formula, which has been derived by the Fourier transform, is in fact more general, as the following proposition states.

PROPOSITION 5.3.1. *Let f, g be continuous functions for which f', g', f'', g'' exist. Then $y(x;t) = f(x+ct) + g(x-ct)$ is a solution of the wave equation. Furthermore, any solution is of this form.*

Proof. The first part is an immediate verification, since $y_x(x;t) = f'(x+ct)+g'(x-ct)$, $y_{xx}(x;t) = f''(x+ct)+g''(x-ct)$, $y_t(x;t) = cf'(x+ct)-cg'(x-ct)$, $y_{tt}(x;t) = c^2 f''(x+ct) + c^2 g''(x-ct)$. The second part is proved by introducing characteristic coordinates $\xi = x-ct$, $\eta = x+ct$. If $y(x;t)$ is a solution of the wave equation with continuous second-order partial derivatives, let $z(\xi;\eta) = y(x;t)$. By the chain rule for partial derivatives,

$$\begin{aligned} y_t &= \frac{\partial y}{\partial t} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial z}{\partial \xi} + c\frac{\partial z}{\partial \eta} \\ y_{tt} &= \frac{\partial^2 y}{\partial t^2} = -c\left(\frac{\partial^2 z}{\partial \xi^2}\frac{\partial \xi}{\partial t} + \frac{\partial^2 z}{\partial \xi\,\partial \eta}\frac{\partial \eta}{\partial t} + c\frac{\partial^2 z}{\partial \eta\,\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial^2 z}{\partial \eta^2}\frac{\partial \mu}{\partial t}\right) \\ &= c^2\frac{\partial^2 z}{\partial \xi^2} - c^2\frac{\partial^2 z}{\partial \xi\,\partial \eta} - c^2\frac{\partial^2 z}{\partial \eta\,\partial \xi} + c^2\frac{\partial^2 z}{\partial \eta^2} \\ y_x &= \frac{\partial y}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \\ y_{xx} &= \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \xi\,\partial \eta} + \frac{\partial^2 z}{\partial \eta\,\partial \xi} + \frac{\partial^2 z}{\partial \eta^2} \end{aligned}$$

The equation $y_{tt} - c^2 y_{xx} = 0$ translates into $\partial^2 z/\partial\xi\,\partial\eta = 0$. This may be solved by two integrations. We have $(\partial/\partial\eta)(\partial z/\partial\xi) = 0$, which requires $\partial z/\partial\xi = \varphi(\xi)$, independent of η. A further integration gives $z(\xi;\eta) = \psi(\eta) + \int_{\xi_0}^{\xi}\varphi(\xi')d\xi'$. Recalling that $\xi = x - ct$, $\eta = x + ct$, we see that $z(\xi;\eta) = f(x+ct) + g(x-ct)$, where $f(\eta) = \psi(\eta)$, $g'(\xi) = \varphi(\xi)$. The proof is complete. •

The general solution of the one-dimensional wave equation may be used to solve problems with a time-dependent boundary condition, as indicated in the next example.

EXAMPLE 5.3.1. *Find the solution of the wave equation* $y_{tt} - c^2 y_{xx} = 0$ *for* $x > 0$, $t > 0$ *satisfying the initial conditions* $y(x;0) = 0$, $y_t(x;0) = 0$ *for* $x > 0$ *and the boundary condition* $y(0;t) = s(t)$.

Solution. We look for y in the form $y(x;t) = f(x+ct)+g(x-ct)$. Substituting the initial conditions and boundary conditions, we must have

$$\begin{aligned} f(x) + g(x) &= 0 \qquad && x > 0 \\ cf'(x) - cg'(x) &= 0 && x > 0 \\ f(ct) + g(-ct) &= s(t) && t > 0 \end{aligned}$$

Differentiating the first equation and using the second, we see that $f'(x) = 0$ and $g'(x) = 0$ for $x > 0$; hence both $f(x)$ and $g(x)$ are constant for $x > 0$. From the first equation we have $f(x) = C$, $g(x) = -C$ for $x > 0$. The third equation gives $g(-ct) = s(t) - f(ct) = s(t) - C$ for $t > 0$. Thus $g(x) = -C$ for $x > 0$ and $g(x) = s(-x/c) - C$ for $x < 0$. Substituting these in the general solution form for $y(x;t)$, we have $f(x+ct) + g(x-ct) = C - C = 0$ for $x - ct > 0$ and $f(x+ct) + g(x-ct) = C + s((ct-x)/c) - C = s(t - x/c)$ for $x - ct < 0$. Thus,

$$y(x;t) = \begin{cases} 0 & x - ct > 0 \\ s\left(t - (x/c)\right) & x - ct < 0 \end{cases} \bullet$$

If we think of $s(t)$ as a signal emitted from the source $x = 0$, the result of the previous example states that an observer positioned at $x > 0$ does not sense the signal until $t = x/c$ time units have elapsed and, after that, the signal is received verbatim, without distortion or damping. The graphs in Fig. 5.3.1 give the solution for several values of t in the case $s(t) = 3\sin 2t$.

Example 5.3.1 can be combined with d'Alembert's formula to solve a general initial-value problem for the wave equation in the half-space $x > 0$. Consider, for example, the problem

$$\begin{aligned} y_{tt} &= c^2 y_{xx} \qquad && t > 0, x > 0 \\ y(0;t) &= s(t) && t > 0 \\ y(x;0) &= f(x) && x > 0 \\ y_t(x;0) &= 0 && x > 0 \end{aligned}$$

The solution is $y(x;t) = \frac{1}{2}\tilde{f}(x+ct) + \frac{1}{2}\tilde{f}(x-ct)$ for $x > ct$, whereas $y(x;t) = \frac{1}{2}\tilde{f}(x+ct) + \frac{1}{2}\tilde{f}(x-ct) + s(t - x/c)$ for $x < ct$.

5.3.3. Three-dimensional wave equation and Huygens' principle. We now consider the three-dimensional wave equation

$$u_{tt} = c^2 \nabla^2 u = c^2 (u_{xx} + u_{yy} + u_{zz})$$

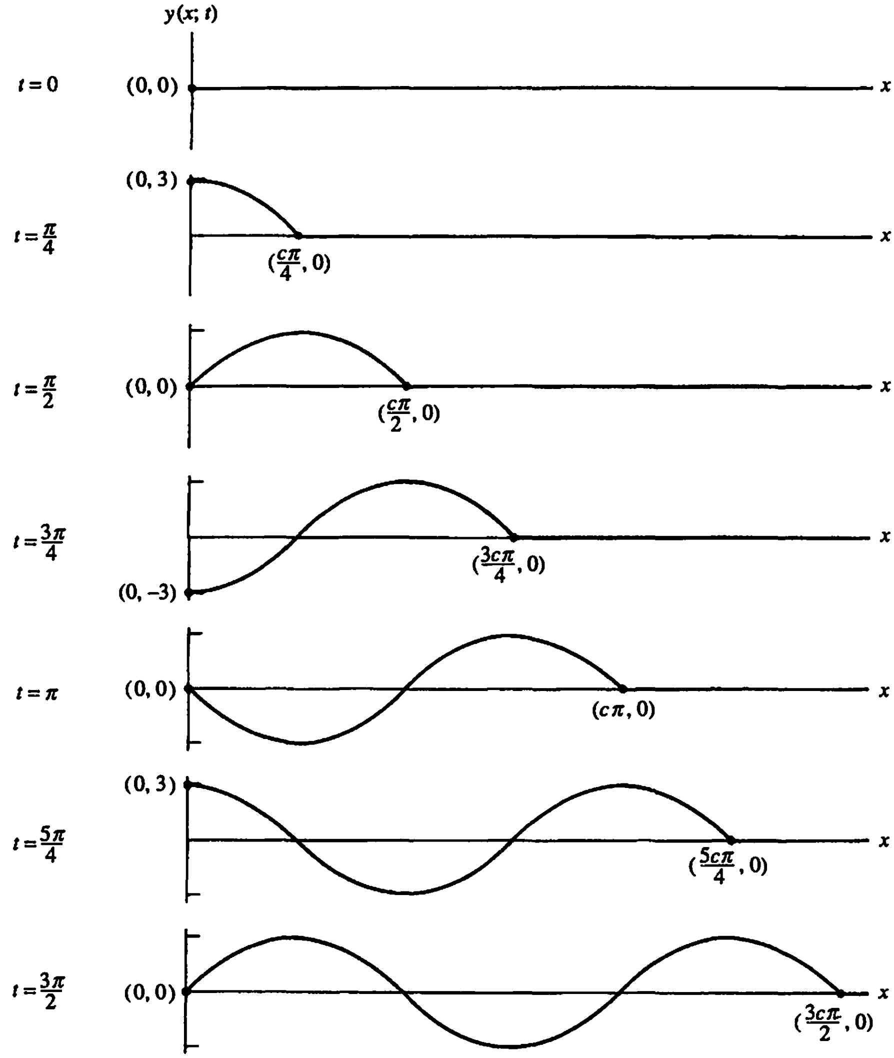


FIGURE 5.3.1 Solution of the wave equation.

which is satisfied by the components of the electric and magnetic fields in a vacuum. The initial conditions are

$$\begin{aligned} u(x, y, z; 0) &= f_1(x, y, z) \\ u_t(x, y, z; 0) &= f_2(x, y, z) \end{aligned}$$

To solve this problem, we introduce the *three-dimensional Fourier transforms* F_1, F_2, U through the formulas

$$f_1(x,y,z) = \iiint F_1(\mu) e^{i\langle \mu, x\rangle} d\mu$$
$$f_2(x,y,z) = \iiint F_2(\mu) e^{i\langle \mu, x\rangle} d\mu$$
$$u(x,y,z;t) = \iiint U(\mu;t) e^{i\langle \mu, x\rangle} d\mu$$

where we have used the inner-product notation $\langle \mu, x\rangle = \mu_1 x + \mu_2 y + \mu_3 z$ and $d\mu = d\mu_1\, d\mu_2\, d\mu_3$. Substituting this form for u in the wave equation, we must have

$$0 = \iiint (U_{tt} + c^2|\mu|^2 U) e^{i\langle \mu, x\rangle} d\mu = 0, \qquad |\mu|^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$$

Thus U is the solution of the second-order equation $U_{tt} + c^2|\mu|^2 U = 0$ with the initial conditions $U(\mu;0) = F_1(\mu)$, $U_t(\mu;0) = F_2(\mu)$. Solving, we have

$$U(\mu;t) = F_1(\mu)\cos ct|\mu| + F_2(\mu)\frac{\sin ct|\mu|}{c|\mu|}$$

Substituting in the formula for $u(x,y,z;t)$, we have the desired Fourier representation

$$\boxed{u(x,y,z;t) = \iiint \left(F_1(\mu)\cos ct|\mu| + F_2(\mu)\frac{\sin ct|\mu|}{c|\mu|}\right) e^{i\langle \mu, x\rangle} d\mu} \tag{5.3.8}$$

EXAMPLE 5.3.2. *Use the Fourier representation (5.3.8) to solve the three-dimensional wave equation $u_{tt} = c^2\nabla^2 u$ if $u(x,y,z;0) \equiv 0$, while $u_t(x,y,z;0) = 1$ for $x^2 + y^2 + z^2 < R^2$ and zero otherwise.*

Solution. We must compute the three-dimensional Fourier transform F_2. Repeating the computations of Example 5.1.10, we have

$$\begin{aligned}
F_2(\mu) &= \left(\frac{1}{2\pi}\right)^3 \iiint_{|x|\le R} \exp[-i\langle \mu, x\rangle] dx \\
&= \left(\frac{1}{2\pi}\right)^3 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \exp[-i|\mu| r\cos\theta] r^2 \sin\theta\, dr\, d\theta\, d\varphi \\
&= \left(\frac{1}{2\pi}\right)^2 \int_0^{R} \left.\frac{\exp[-i|\mu| r\cos\theta]}{i|\mu| r}\right|_{\theta=0}^{\theta=\pi} r^2 dr \\
&= \frac{1}{2\pi^2|\mu|}\int_0^R (\sin|\mu| r) r\, dr \\
&= \frac{1}{2\pi^2|\mu|}\left(\frac{\sin|\mu|R}{|\mu|^2} - \frac{R\cos|\mu|R}{|\mu|}\right)
\end{aligned}$$

Substituting this in (5.3.8) gives the desired Fourier representation. •

We now derive an explicit representation, the three-dimensional generalization of d'Alembert's formula. To do this, we recall, following the discussion after Example 5.1.10, the surface integral formula (5.1.16) on a sphere of radius R:

$$\iint\limits_{|\xi|=R} e^{i\langle\mu,\xi\rangle}dS = 4\pi R^2 \frac{\sin|\mu|R}{|\mu|R} \qquad |\mu| \neq 0$$

If $|\mu| = 0$, the surface integral is $4\pi R^2$, the area of the surface of the sphere. We use this with $R = ct$ to write

$$e^{i\langle\mu,x\rangle}\frac{\sin ct|\mu|}{c|\mu|} = \frac{1}{4\pi tc^2}\iint\limits_{|\xi|=ct} e^{i\langle\mu,x+\xi\rangle}dS$$

We multiply this equation by $F_k(\mu)$, integrate $d\mu$, and formally interchange the order of integration to obtain[2]

$$\begin{aligned}\iiint F_k(\mu)\frac{\sin ct|\mu|}{c|\mu|}e^{i\langle\mu,x\rangle}d\mu &= \frac{1}{4\pi tc^2}\iint\limits_{|\xi|=ct}\left(\iiint e^{i\langle\mu,x+\xi\rangle}F_k(\mu)d\mu\right)dS \\ &= \frac{1}{4\pi tc^2}\iint\limits_{|\xi|=ct} f_k(x+\xi)dS \qquad (5.3.9)\\ &= tM_{ct}f_k \qquad k=1,2\end{aligned}$$

where we have defined the *mean-value operator* by

$$M_Rf(x) = \frac{1}{4\pi R^2}\iint\limits_{|\xi|=R} f(x+\xi)dS$$

Thus we have obtained an explicit representation of the f_2 term of the solution (5.3.8); to handle the f_1 term, we differentiate the identity (5.3.9) with respect to t.

$$\iiint F_k(\mu)\cos|\mu|cte^{i\langle\mu,x\rangle}d\mu = \frac{d}{dt}(tM_{ct}f_k) \qquad k=1,2 \tag{5.3.10}$$

Combining this with the first formula, we have the desired explicit representation:

$$\boxed{u(x,y,z;t) = \frac{d}{dt}(tM_{ct}f_1) + tM_{ct}f_2} \tag{5.3.11}$$

This representation displays an important property of the three-dimensional wave equation, *Huygens' principle*, stated as follows. If f_1 and f_2 are zero for $(x^2+y^2+z^2)^{1/2} \geq R$, then $u(x,y,z;t) = 0$ for $ct > R + (x^2+y^2+z^2)^{1/2}$. This results from the fact that (5.3.11) contains only *surface integrals* over a sphere

[2]This can be made rigorous by inserting the factor $e^{-\epsilon|\mu|^2}$ inside the integral and taking the limit $\epsilon \to 0$ after the interchange.

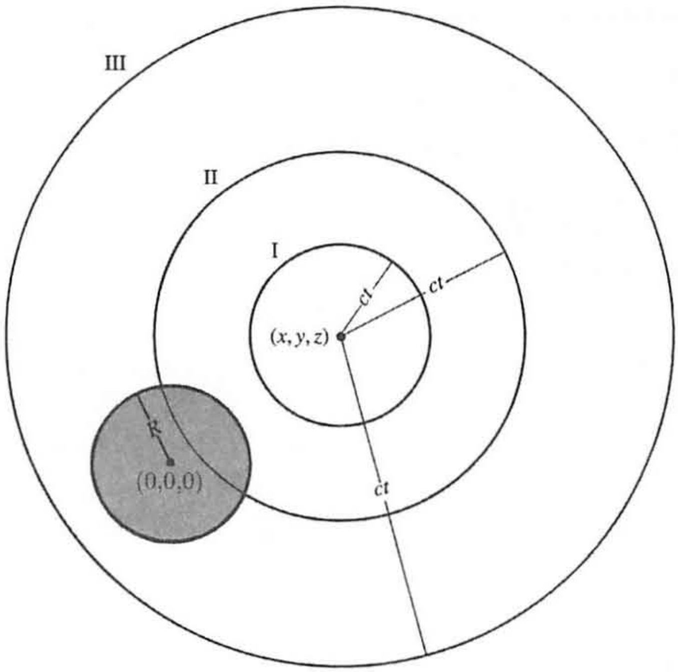


FIGURE 5.3.2 Illustrating Huygens' principle: f_1 and f_2 are zero outside the sphere of radius R about $(0,0,0)$. If $ct > R + \sqrt{x^2+y^2+z^2}$, then f_1 and f_2 are zero on the surface of integration (labeled III).

of a radius ct and that for t sufficiently large, the surface of integration does not intersect the set where $f_1 \neq 0$, $f_2 \neq 0$. This is illustrated in Fig. 5.3.2.

5.3.4. Extended validity of the explicit representation. We have derived the explicit representation formula (5.3.11) for the solution of the wave equation beginning with the Fourier transforms of the initial data f_1, f_2. But the final formula makes no reference to the Fourier transform and can be defined in many cases when the Fourier transform does not exist. For simplicity, the following theorem treats the case $f_1 = 0$. We use the notation $P = (x_1, x_2, x_3)$ to denote a typical point of three-dimensional space.

THEOREM 5.5. *Suppose that $f(P)$ is a real-valued continuous function with two continuous partial derivatives. Then the formula $u(P;t) = tM_{ct}f(P)$ defines a twice-differentiable function that satisfies the wave equation $u_{tt} = c^2\nabla^2 u$ with the initial conditions $u(P;0) = 0, u_t(P;0) = f(P)$.*

Proof. From the definition,

$$u(P;t) = \frac{1}{4\pi c^2 t}\iint_{|\xi|=ct} f(P+\xi)dS = \frac{t}{4\pi}\iint_{|\omega|=1} f(P+ct\omega)dS$$

Clearly $u(P;0) = 0$. The first time derivative is given by the product rule:

$$u_t(P;t) = \frac{1}{4\pi}\iint_{|\omega|=1} f(P+ct\omega)dS + \frac{ct}{4\pi}\iint_{|\omega|=1} (\nabla f)(P+ct\omega)\cdot\omega dS \tag{5.3.12}$$

In particular $u_t(P;0) = f(P)$. To proceed further, we transform the second integral by the divergence theorem as follows:

$$\begin{aligned}
\frac{ct}{4\pi}\iint_{|\omega|=1} (\nabla f)(P+ct\omega)\cdot\omega\, d\omega &= \frac{ct}{4\pi}\iint_{|\omega|=1}\left(\frac{\partial f}{\partial n}\right)(P+ct\omega)d\omega \\
&= \frac{1}{4\pi ct}\iint_{|\xi|=ct}\left(\frac{\partial f}{\partial n}\right)(P+\xi)\,d\xi \\
&= \frac{1}{4\pi ct}\iiint_{|\xi|<ct}(\nabla^2 f)(P+\xi)\,d\xi \\
&= \frac{1}{4\pi ct}\int_0^{ct}\left(\int\int_{|\xi|=r}(\nabla^2 f)(P+\xi)\,dS\right)dr
\end{aligned}$$

We make this substitution in (5.3.12), differentiate with respect to t, and again apply the divergence theorem, to obtain

$$\begin{aligned}
u_{tt}(P;t) &= \frac{c}{4\pi}\iint_{|\omega|=1}\nabla f(P+ct\omega)\cdot\omega\, dS - \frac{1}{4\pi ct^2}\int_0^{ct}\left(\iint_{|\xi|=r}\nabla^2 f(P+\xi)\,dS\right)dr \\
&\quad + \frac{1}{4\pi t}\iint_{|\xi|=ct}(\nabla^2 f)(P+\xi)\,dS \\
&= \frac{1}{4\pi ct^2}\iiint_{|\xi|<ct}(\nabla^2 f)(P+\xi)\,d\xi - \frac{1}{4\pi ct^2}\iiint_{|\xi|<ct}(\nabla^2 f)(P+\xi)\,d\xi \\
&\quad + \frac{1}{4\pi t}\iint_{|\xi|=ct}(\nabla^2 f)(P+\xi)\,dS \\
&= \frac{c^2 t}{4\pi}\iint_{|\omega|=1}\nabla^2 f(P+ct\omega)dS
\end{aligned}$$

On the other hand, we may compute the spatial derivatives of $u(P;t)$ by differentiating under the integral sign, to obtain the formulas

$$\begin{aligned}
u_{x_i}(P;t) &= \frac{t}{4\pi}\iint_{|\omega|=1} f_{x_i}(P+ct\omega)dS \qquad i = 1,2,3 \\
u_{x_i x_j}(P;t) &= \frac{t}{4\pi}\iint_{|\omega|=1} f_{x_i x_j}(P+ct\omega)dS \qquad i,j = 1,2,3
\end{aligned}$$

In particular,

$$\begin{aligned}
\nabla^2 u(P;t) &= u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} \\
&= \frac{t}{4\pi}\iint_{|\omega|=1} \nabla^2 f(P+ct\omega)d\omega \\
&= \frac{1}{c^2}u_{tt}(P;t)
\end{aligned}$$

which was to be proved. •

Now we can use this to treat the case $f_2 = 0$.

THEOREM 5.6. *Suppose that $f(P)$ is a real-valued continuous function with three continuous partial derivatives. Then the formula $v(P;t) = (\partial/\partial t)[t\, M_{ct} f](P)$ defines a twice-differentiable function that satisfies the wave equation $v_{tt} = c^2\nabla^2 v$ with the initial conditions $v(P;0) = f(P)$, $v_t(P;0) = 0$.*

Proof. We have $v = u_t$, where u is the solution of the wave equation obtained in the previous theorem. Application of the requisite derivatives reveals that

$$\begin{aligned}
\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) v &= \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u_t \\
&= \left(\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u\right)_t \\
&= 0
\end{aligned}$$

since u is a solution of the wave equation. To verify the initial conditions, we again appeal to the result of the previous theorem to obtain $v(P;0) = u_t(P;0) = f(P)$ and $v_t(P;0) = u_{tt}(P;0) = c^2\nabla^2 u(P;0) = 0$, completing the proof. •

Adding the results above, we can obtain the solution of the initial-value problem as follows.

THEOREM 5.7. *Suppose that $f_1(P)$ has three continuous partial derivatives and that $f_2(P)$ has two continuous partial derivatives. Then formula (5.3.11) defines a solution of the wave equation satisfying the initial conditions*

$$u(P;0) = f_1(P), \qquad u_t(P;0) = f_2(P)$$

5.3.5. Application to one- and two-dimensional wave equations. Formula (5.3.11) can be used to solve the two-dimensional wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$ with initial conditions $u(x,y;0) = f_1(x,y)$, $u_t(x,y;0) = f_2(x,y)$. Considering these as functions of (x,y,z), we substitute in (5.3.11) and perform the required integrations. If these are written as integrals in the xy plane, we obtain integrals over the *interior* of the circle of radius ct, described in detail below. This has the consequence that Huygens' principle is *not valid* for wave motion in two dimensions. For example, a pebble dropped in a pond of shallow water will create a wave motion on the surface of the water; an observer positioned r units away from the initial disturbance will sense the disturbance at $t = r/c$ time units later. After this time the solution will continue to be nonzero for all later times, according to the wave equation. This is the phenomenon of a *wake* behind the initial disturbance, present in two-dimensional wave motion. Huygens' principle can be restated to say that, in three-dimensional wave motion, no wake is present.

We now present the details of the passage from the three-dimensional wave equation to the two-dimensional wave equation, known as the *method of descent*. We are given two functions $f_1(x,y)$ and $f_2(x,y)$, and we are looking for the solution of the wave equation with $u(x,y;0) = f_1(x,y)$, $u_t(x,y;0) = f_2(x,y)$. To do this, we substitute in the explicit representation formula (5.3.11), which leads us to study the action of the mean-value operator M_{ct} on a function of two variables. Note that this operator involves an integration over the sphere of radius ct; we can parametrize the upper half of the sphere by the formula $\xi_3 = \sqrt{(ct)^2 - \xi_1^2 - \xi_2^2}$, and the element of surface area is computed as

$$\begin{aligned} dS_\xi &= \sqrt{1 + \left(\frac{\partial \xi_3}{\partial \xi_1}\right)^2 + \left(\frac{\partial \xi_3}{\partial \xi_2}\right)^2}\, d\xi_1\, d\xi_2 \\ &= \frac{ct\, d\xi_1\, d\xi_2}{\sqrt{(ct)^2 - \xi_1^2 - \xi_2^2}} \end{aligned}$$

The integral over the lower hemisphere is handled similarly, with the parametrization $\xi_3 = -\sqrt{(ct)^2 - \xi_1^2 - \xi_2^2}$. The mean-value operator M_{ct} is sum of the integrals over the upper and lower hemispheres, which produces a factor of 2 and the formula

$$M_{ct}f(P) = \frac{1}{2\pi ct}\iint_{|\xi|<ct} \frac{f(P+\xi)\, d\xi_1\, d\xi_2}{\sqrt{(ct)^2 - |\xi|^2}} \tag{5.3.13}$$

This is then substituted into (5.3.11) to obtain the solution. We summarize this work as a theorem.

THEOREM 5.8. *Suppose that $f_1(x,y)$ has three continuous partial derivatives and that $f_2(x,y)$ has two continuous partial derivatives. Then the solution of the two-dimensional wave equation satisfying the initial conditions*

$$u(P;0) = f_1(P), \qquad u_t(P;0) = f_2(P)$$

is given by the formula

(5.3.14)

$$\boxed{u(P;t) = \frac{1}{2\pi c}\left[\iint_{|\xi|<ct} \frac{f_2(P+\xi)}{\sqrt{(ct)^2 - |\xi|^2}} d\xi + \frac{d}{dt}\iint_{|\xi|<ct} \frac{f_1(P+\xi)}{\sqrt{(ct)^2 - |\xi|^2}} d\xi\right]}$$

EXAMPLE 5.3.3. *Let $u(x,y;t)$ be the solution of the two-dimensional wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$ with initial conditions $u(x,y;0) \equiv 0$ and $u_t(x,y,0) =$*

1 *for* $x^2 + y^2 \le a^2$ *and zero otherwise. Find* $u(0,0;t)$ *for all* $t > 0$ *and determine the behavior of* $u(0,0;t)$ *when* $t \to \infty$.

Solution. We take $f_1 \equiv 0$ and $f_2(x,y) = 1$ for $x^2 + y^2 \le a^2$ and zero otherwise. We substitute in (5.3.14) to obtain

$$u(x,y;t) = \frac{1}{2\pi c}\iint_{|\xi|<ct, |\xi|<a} \frac{d\xi}{\sqrt{(ct)^2 - |\xi|^2}}$$

The integral can be done by taking polar coordinates with $r = |\xi|$, obtaining

$$\int_0^b \frac{r\,dr}{\sqrt{(ct)^2 - |\xi|^2}} = \frac{1}{2}\left(ct - \sqrt{(ct)^2 - b^2}\right)$$

and the result is

$$u(0,0;t) = \begin{cases} t & ct < a \\ t - \sqrt{t^2 - (a^2/c^2)} & ct > a \end{cases}$$

When $t \to \infty$, we may use the Taylor expansion $\sqrt{1 - (a/ct)^2} = 1 - a^2/2c^2t^2 + O(1/t^4)$ to obtain

$$u(0,0;t) = t - t\sqrt{1 - (a/ct)^2} = a^2/2ct + O(1/t^3) \quad \bullet$$

The final worked example in this subsection illustrates the versatility of formula (5.3.11) by rederiving the solution of the one-dimensional wave equation for the vibrating string.

EXAMPLE 5.3.4. *Show that formula (5.3.11) reduces to d'Alembert's formula in the case where* $f_1 = 0$, $f_2(x,y,z) = f_2(x)$.

Solution. If $f(x,y,z) = f(x)$, then the surface integral defining $M_{ct}f$ can be written as a single integral as follows: for any R,

$$\begin{aligned} M_R f(x,y,z) &= \frac{1}{4\pi R^2}\iint_{|\xi|=R} f(x+\xi_1)\,dS_\xi \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi} f(x + R\cos\theta)\sin\theta\,d\theta\,d\varphi \\ &= \frac{1}{2R}\int_{-R}^{R} f(x+\xi_1)\,d\xi_1 \end{aligned}$$

Thus,

$$tM_{ct}f_2(x,y,z) = \frac{1}{2c}\int_{-ct}^{ct} f(x+\xi_1)\,d\xi_1 = \frac{1}{2c}\int_{x-ct}^{x+ct} f(\eta)\,d\eta$$

which is the appropriate form of d'Alembert's formula in this case. $\bullet$

5.3.6. Laplace's equation in a half-space: Poisson's formula. One can use the method of Fourier transforms to solve Laplace's equation in a half-space and to find the explicit representation of the solution. We now illustrate the solution of Laplace's equation by Fourier transforms. We will solve the following boundary-value problem for Laplace's equation in the half-plane $y > 0$

$$u_{xx} + u_{yy} = 0 \qquad -\infty < x < \infty, y > 0 \tag{5.3.15}$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty \tag{5.3.16}$$

$$|u(x,y)| \leq M \qquad -\infty < x < \infty, y > 0 \tag{5.3.17}$$

where $f(x), -\infty < x < \infty$, is piecewise smooth with $\int_{-\infty}^{\infty} |f(x)| dx < \infty$.

To do this, we follow the second method of Sec. 5.2. Introducing the Fourier transform formulas $f(x) = \int_{-\infty}^{\infty} F(\mu) e^{i\mu x} d\mu$, $F(\mu) = (1/2\pi) \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx$, we look for the solution in the form $u(x,y) = \int_{-\infty}^{\infty} F(\mu, y) e^{i\mu x} d\mu$. Laplace's equation is

$$0 = u_{xx} + u_{yy} = \int_{-\infty}^{\infty} [-\mu^2 F(\mu, y) + F_{yy}(\mu, y)] e^{i\mu x} d\mu$$

Thus F must satisfy the ordinary differential equation $F_{yy}(\mu, y) = \mu^2 F(\mu, y)$ and the initial condition $F(\mu, 0) = F(\mu)$ for each μ. The general solution of the ordinary differential equation is of the form $Ae^{\mu y} + Be^{-\mu y}$. If we impose the initial condition and the boundedness condition, the solution must be

$$F(\mu, y) = \begin{cases} F(\mu) e^{-\mu y} & \text{if } \mu \geq 0 \\ F(\mu) e^{\mu y} & \text{if } \mu < 0 \end{cases}$$

This can be succinctly expressed using the absolute value as $F(\mu) e^{-|\mu| y}$,

$$\boxed{u(x,y) = \int_{-\infty}^{\infty} F(\mu) e^{-|\mu| y} e^{i\mu x} d\mu, \qquad F(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx} \tag{5.3.18}$$

This is the desired Fourier representation of the solution.

To obtain an explicit representation, we insert the basic defining formula $F(\mu) = (1/2\pi) \int_{-\infty}^{\infty} f(\xi) e^{-i\mu\xi} d\xi$ into the Fourier representation and formally interchange the order of integration. Thus

$$\begin{aligned} u(x,y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\xi) e^{-i\mu\xi} d\xi \right) e^{-|\mu| y} e^{i\mu x} d\mu \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i\mu(x-\xi)} e^{-|\mu| y} d\mu \right) f(\xi) d\xi \end{aligned}$$

The inner integral is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\mu(x-\xi)}e^{-|\mu|y}d\mu &= 2\text{ Re}\int_0^{\infty} e^{i\mu(x-\xi)}e^{-\mu y}d\mu \\ &= 2\text{Re}\,\frac{1}{y-i(x-\xi)} \\ &= \frac{2y}{y^2+(x-\xi)^2}\end{aligned}$$

Therefore the explicit representation is

$$\boxed{u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{y^2+(x-\xi)^2}f(\xi)d\xi} \tag{5.3.19}$$

This is closely related to the Poisson integral formula, obtained in Chapter 2 for the solution of Laplace's equation in the disc.

Having obtained this explicit representation, it may be verified independently that this solution is valid for any piecewise smooth function $f(x)$, $-\infty < x < \infty$, for which $|f(x)| \le K$ for some constant K. It is not necessary to suppose that $\int_{-\infty}^{\infty}|f(x)|dx < \infty$.

THEOREM 5.9. *Suppose that $f(x), -\infty < x < \infty$, is bounded and piecewise continuous. Then the integral (5.3.19) defines a solution of Laplace's equation $u_{xx}+u_{yy}=0$ satisfying the boundary condition that* $\lim_{y\to 0} u(x,y) = \frac{1}{2}f(x+0)+\frac{1}{2}f(x-0)$.

EXAMPLE 5.3.5. *Find the bounded solution of Laplace's equation $u_{xx}+u_{yy}=0$ in the half-plane $y>0$ satisfying the boundary conditions $u(x,0)=1$ if $a<x<b$ and $u(x,0)=0$ otherwise.*

Solution. Referring to (5.3.19), we have

$$u(x,y)=\frac{1}{\pi}\int_a^b \frac{y}{(x-\xi)^2+y^2}\,d\xi$$

Making the substitution $v=(\xi-x)/y$, we have $d\xi = y\,dv$, and the denominator of the integrand is $y^2(1+v^2)$. Changing the limits of integration accordingly, we have

$$\begin{aligned}u(x,y) &= \frac{1}{\pi}\int_{(a-x)/y}^{(b-x)/y}\frac{1}{1+v^2}\,dv \\ &= \frac{1}{\pi}\left(\arctan\frac{b-x}{y}-\arctan\frac{a-x}{y}\right) \\ &= \frac{1}{\pi}(\theta_b-\theta_a)\end{aligned}$$

where the angles θ_a, θ_b are depicted in Fig. 5.3.3. •

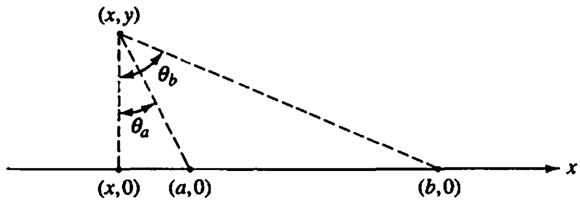


FIGURE 5.3.3 Solution of Laplace's equation.

EXERCISES 5.3

1. Use d'Alembert's formula to solve the wave equation $y_{tt} = c^2 y_{xx}$ with the initial conditions $y(x;0) = 3\sin 2x$, $y_t(x;0) = 0$.
2. Use d'Alembert's formula to solve the wave equation $y_{tt} = c^2 y_{xx}$ with the initial conditions $y(x;0) = 0$, $y_t(x;0) = 4\cos 5x$.
3. Suppose that f_1 has two continuous derivatives and f_2 has one continuous derivative. Show that (5.3.7) is a solution of the initial-value problem (5.3.1).
4. Find the solution of the wave equation $y_{tt} = c^2 y_{xx}$ for $t > 0$, $x > 0$ satisfying the boundary conditions $y(0;t) = 0$ and the initial conditions $y(x;0) = 0$, $y_t(x;0) = g(x)$.
5. Find the solution of the wave equation $y_{tt} = c^2 y_{xx}$ for $t > 0$, $x > 0$ satisfying the boundary conditions $y(0;t) = s(t)$ and the initial conditions $y(x;0) = 0$, $y_t(x;0) = g(x)$.
6. Show that formula (5.3.11) reduces to d'Alembert's formula in the case where $f_1(x,,z) = f_1(x)$, $f_2 = 0$.
7. Use formula (5.3.11) to solve the initial-value problem for the wave equation in three dimensions when the initial data f_1, f_2 depend only on $r = \sqrt{x^2 + y^2 + z^2}$ and $f_1(x,y,z) = f_1(r)$, $f_2(x,y,z) = f_2(r)$.
8. The oscillations of a gas satisfy the three-dimensional wave equation $u_{tt} = c^2\nabla^2 u$ with $u(x,y,z;0) = 0$, $u_t(x,y,z;0) = T$ if $a^2 \geq x^2 + y^2 + z^2$ and zero otherwise. Find the solution of this initial-value problem.
9. Find the bounded solution of Laplace's equation $u_{xx} + u_{yy} = 0$ in the half-plane $y > 0$ satisfying the boundary conditions $u(x,0) = 2$ if $-4 < x < 4$ and $u(x,0) = 0$ otherwise.
10. Consider the problem of solving Laplace's equation $u_{xx} + u_{yy} = 0$ in the quadrant $x > 0$, $y > 0$, with the boundary conditions $u(x,0) = f(x)$, $u(0,y) = 0$. By combining the method of images with Theorem 5.9, find an explicit representation of the solution.
11. Consider the problem of solving Laplace's equation $u_{xx} + y_{yy} = 0$ in the quadrant $x > 0$, $y > 0$, with the boundary conditions $u(x,0) = f(x)$, $u_x(0,y) = 0$. By combining the method of images with Theorem 5.9, find an explicit representation of the solution.

12. Consider the problem of solving Laplace's equation $u_{xx} + u_{yy} = 0$ in the quadrant $x > 0,\ y > 0$ with the boundary conditions $u(x, 0) = f(x)$, $u(0, y) = f(y)$. Find an explicit representation of the solution.
13. Consider the problem of solving Laplace's equation $u_{xx} + u_{yy} = 0$ in the strip $0 < x < L$, $0 < y < \infty$, with the boundary conditions $u(x, 0) = f(x)$, $u(0, y) = 0$, $u(L, y) = 0$. Find the Fourier (series) representation of the bounded solution of this problem.
14. Consider the problem of solving Laplace's equation $u_{xx} + u_{yy} = 0$ in the strip $0 < x < L$, $0 < y < \infty$, with the boundary conditions $u(x, 0) = 0$, $u(0, y) = 0$, $u(L, y) = g(y)$. Find a Fourier representation of the bounded solution of this problem.
15. Find the solution of the wave equation $y_{tt} = c^2 y_{xx}$ for $t > 0, x > 0$, with the boundary condition $y(0; t) = 4\cos t$ and the initial conditions $y(x; 0) = 0$, $y_t(x; 0) = 0$, Sketch the solution for several different values of t.
16. Find the solution of the wave equation $y_{tt} = c^2 y_{xx}$ for $t > 0$, $x > 0$, with the boundary conditions $y(0; t) = 3$ for $0 < t < 5$ and $y(0; t) = 0$ for $t > 5$ and the initial conditions $y(x; 0) = 0$, $y_t(x; 0) = 0$. Sketch the solution for several different values of t.
17. Find the solution of the wave equation $y_{tt} = c^2 y_{xx}$ for $t > 0$, $x > 0$, with the boundary condition $y_x(0; t) = 0$ and the initial conditions $y(x; 0) = 1$ for $0 < x < 3$, $y(x; 0) = 0$ for $x > 3$, and $y_t(x; 0) = 0$ for all $x > 0$. Sketch the solution for several different values of t.

5.4. Solution of the Telegraph Equation

The flow of electricity in a cable is described by the partial differential equation

$$u_{xx} = CLu_{tt} + (RC + GL)u_t + RGu$$

where R is the resistance, L is the inductance, C is the capacitance, and G is the leakage, all measured per unit length of the cable. The unknown function $u(x; t)$ may represent the voltage or the current at the time instant t, at the position x of the cable, where $t > 0$, $-\infty < x < \infty$. The derivation of the telegraph equation is carried out in the appendix to this chapter.

To put this in a more convenient form, let

$$2\beta = \frac{R}{L} + \frac{G}{C}, \quad c^2 = \frac{1}{CL}, \quad \alpha = \frac{RG}{CL}$$

resulting in the equation

$$\boxed{u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx}} \tag{5.4.1}$$

From the defining equations

$$\beta^2 - \alpha = \frac{1}{4}\left(\frac{R}{L} - \frac{G}{C}\right)^2$$

Therefore we need only solve equation (5.4.1) for values of α and β for which $\beta^2-\alpha \geq 0$. In order to illustrate the mathematical methods, we will solve (5.4.1) for arbitrary values of α, β. This will include, for example, the Klein-Gordon equation, which occurs in quantum mechanics, as well as the wave equation, which was treated in Sec. 5.3. In detail, we have three separate cases to consider:

$$\begin{aligned} \text{Case I}: &\quad \beta^2 < \alpha \\ \text{Case II}: &\quad \beta^2 = \alpha \\ \text{Case III}: &\quad \beta^2 > \alpha \end{aligned}$$

Since the telegraph equation is second order in time, it is natural to specify two initial conditions:

$$u(x;0) = f_1(x), \qquad u_t(x;0) = f_2(x)$$

Since the equation is linear and homogeneous, we can first solve with $f_1 = 0$, then solve with $f_2 = 0$, and add the results.

5.4.1. Fourier representation of the solution. To solve the initial-value problem for the telegraph equation, we look for $u(x;t)$ in terms of its Fourier transform,

$$u(x;t) = \int_{-\infty}^{\infty} U(\mu;t)e^{i\mu x}d\mu$$

and formally apply the operations implied by (5.4.1).

$$u_{tt} + 2\beta u_t + \alpha u - c^2 u_{xx} = \int_{-\infty}^{\infty} (U_{tt} + 2\beta U_t + \alpha U + c^2\mu^2 U)e^{i\mu x}d\mu$$

Therefore we solve the ordinary differential equation

$$U_{tt} + 2\beta U_t + (\alpha + c^2\mu^2)U = 0 \tag{5.4.2}$$

with the initial conditions

$$U(\mu;0) = 0, \qquad U_t(\mu;0) = F_2 \tag{5.4.3}$$

To solve this equation with constant coefficients, we look for solutions of the form $e^{\gamma t}$ and obtain the quadratic equation

$$\gamma^2 + 2\beta\gamma + (\alpha + c^2\mu^2) = 0 \tag{5.4.4}$$

We consider separately the three cases.

Case 1: $\alpha - \beta^2 > 0$. In this case both roots of the quadratic equation (5.4.4) are complex numbers, $\gamma = -\beta \pm i\sqrt{(\alpha-\beta^2)+(c\mu)^2}$, and the solution of (5.4.2) is

$$U(\mu;t) = F_2(\mu)e^{-\beta t}\frac{\sin t\sqrt{(\alpha-\beta^2)+(c\mu)^2}}{\sqrt{(\alpha-\beta^2)+(c\mu)^2}}$$

The Fourier representation of the solution is

$$(5.4.5) \qquad \boxed{u(x;t) = e^{-\beta t} \int_{-\infty}^{\infty} F_2(\mu) \frac{\sin t\sqrt{(\alpha-\beta^2)+(c\mu)^2}}{\sqrt{(\alpha-\beta^2)+(c\mu)^2}}\, e^{i\mu x} d\mu}$$

Case 2: $\beta^2 - \alpha = 0$. In this case the roots of the quadratic equation (5.4.4) are the complex numbers $\gamma = -\beta \pm ic\mu$, and the solution of (5.4.2) is

$$U(\mu;t) = F_2(\mu) e^{-\beta t} \frac{\sin c\mu t}{c\mu}$$

The Fourier representation of the solution is

$$(5.4.6) \qquad \boxed{u(x;t) = e^{-\beta t} \int_{-\infty}^{\infty} F_2(\mu) \frac{\sin \mu c t}{\mu c}\, e^{i\mu x} d\mu}$$

Case 3: $\beta^2 - \alpha > 0$. In this case the roots of the quadratic equation may be either real or complex conjugate, depending on the value of μ. This leads to the following two subcases:

Subcase i. If $c|\mu| \le \sqrt{\beta^2 - \alpha}$, then the roots are real and given by $\gamma = -\beta \pm \sqrt{(\beta^2-\alpha)-(c\mu)^2}$, and the Fourier transform is given by

$$U(\mu;t) = F_2(\mu) e^{-\beta t} \frac{\sinh t\sqrt{(\beta^2-\alpha)-(c\mu)^2}}{\sqrt{(\beta^2-\alpha)-(c\mu)^2}}$$

Subcase ii. If $c|\mu| > \sqrt{\beta^2 - \alpha}$, then the roots are conjugate complex and given by $\gamma = -\beta \pm i\sqrt{(c\mu)^2 - (\beta^2-\alpha)}$, and the Fourier transform is given by

$$U(\mu;t) = F_2(\mu) e^{-\beta t} \frac{\sin t\sqrt{(c\mu)^2-(\beta^2-\alpha)}}{\sqrt{(c\mu)^2-(\beta^2-\alpha)}}$$

Combining these two subcases, we find the Fourier representation in case 3:

$$(5.4.7) \quad u(x;t) = e^{-\beta t} \int_{|c\mu| \ge \sqrt{\beta^2-\alpha}} F_2(\mu) \frac{\sin t\sqrt{(c\mu)^2-(\beta^2-\alpha)}}{\sqrt{(c\mu)^2-(\beta^2-\alpha)}}\, e^{i\mu x} d\mu + e^{-\beta t} \int_{|c\mu| < \sqrt{\beta^2-\alpha}} F_2(\mu) \frac{\sinh t\sqrt{(\beta^2-\alpha)-(c\mu)^2}}{\sqrt{(\beta^2-\alpha)-(c\mu)^2]}}\, e^{i\mu x} d\mu$$

We have thus obtained the desired Fourier representation of the solution of the telegraph equation in each of the three cases.

We now verify that these formal solutions are indeed rigorous solutions of the problem (5.4.1) with $f_1 = 0$. For this purpose we assume that the Fourier transform of the initial data satisfies

$$\int_{-\infty}^{\infty} |\mu||F_2(\mu)| d\mu < \infty$$

(This will happen if, for example, f_2 has three continuous derivatives that are absolutely integrable.) With this hypothesis we can follow the arguments of Sec. 5.2 and take the derivatives under the integral sign. For example, in case 1,

$$\begin{aligned}
(e^{\beta t}u)_x &= \int_{-\infty}^{\infty} F_2(\mu)\frac{\sin t[(\alpha-\beta^2)+(c\mu)^2]^{1/2}}{[(\alpha-\beta^2)+(c\mu)^2]^{1/2}}\, i\mu e^{i\mu x}d\mu \\
(e^{\beta t}u)_{xx} &= \int_{-\infty}^{\infty} F_2(\mu)\frac{\sin t[(\alpha-\beta^2)+(c\mu)^2]^{1/2}}{[(\alpha-\beta^2)+(c\mu)^2]^{1/2}}\, (i\mu)^2 e^{i\mu x}d\mu \\
(e^{\beta t}u)_{tt} &= -\int_{-\infty}^{\infty} F_2(\mu)\frac{\sin t[(\alpha-\beta^2)+(c\mu)^2]^{1/2}}{[(\alpha-\beta^2)+(c\mu)^2]^{1/2}}\, [(\alpha-\beta^2)+(c\mu)^2]e^{i\mu x}d\mu
\end{aligned}$$

Clearly, $(e^{\beta t}u)_{tt} - c^2(e^{\beta t}u)_{xx} + (\alpha-\beta^2)(e^{\beta t}u) = 0$, which is equivalent to the telegraph equation.

In case 3 the analysis is the same. In case 2 we can do better, since, apart from the factor $e^{-\beta t}$, this is just the Fourier representation (5.3.6) of the solution of the wave equation, which can be rewritten as

$$u(x;t) = \frac{e^{-\beta t}}{2c}\int_{x-ct}^{x+ct} f_2(\xi)d\xi$$

If f_2' exists, this is a solution of the telegraph equation.

5.4.2. Uniqueness of the solution. We now discuss the *uniqueness* of our solution. We have obtained a solution in the form of a Fourier representation, but it is conceivable that other forms are possible. The property of uniqueness ensures that any other representation must lead to one and the same function $u(x;t)$. For this purpose, let u_1, u_2 be two solutions that have the same initial data and are zero for large x, depending on t. (This is a natural assumption since telegraph signals are expected to move with a finite velocity.) Letting $v = (u_1 - u_2)e^{\beta t}$, we see that v satisfies the equation $v_{tt} + (\alpha-\beta^2)v = c^2 v_{xx}$ with zero initial conditions. We now introduce the *energy functional*

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(v_t^2 + c^2v_x^2 + v^2)dx$$

Differentiating with respect to t, using the equation for v, and integrating by parts, we have

$$\begin{aligned}
E'(t) &= \int_{-\infty}^{\infty}(v_tv_{tt} + c^2v_xv_{xt} + vv_t)dx \\
&= \int_{-\infty}^{\infty}\left(v_t[c^2v_{xx} - (\alpha-\beta^2)v] + c^2v_xv_{xt} + vv_t\right)dx \\
&= -c^2\int_{-\infty}^{\infty} v_xv_{tx}dx + (1+\beta^2-\alpha)\int_{-\infty}^{\infty} vv_tdx + c^2\int_{-\infty}^{\infty} v_xv_{xt}dx \\
&= (1+\beta^2-\alpha)\int_{-\infty}^{\infty} vv_tdx
\end{aligned}$$

But

$$vv_t \leq \frac{1}{2}(v^2 + v_t^2)$$

Therefore,

$$E'(t) \leq bE(t) \qquad b = \frac{1+\beta^2-\alpha}{2}$$

Thus,

$$\frac{d}{dt}[e^{-bt}E(t)] = [E'(t) - bE(t)]e^{-bt} \leq 0$$

But v and v_t are both zero for $t = 0$; hence

$$E(0) = 0$$

Applying the fundamental theorem of calculus,

$$e^{-bt}E(t) \leq 0$$

But $E(t)$ is the integral of a nonnegative function, and hence nonnegative. Therefore we conclude that

$$E(t) \equiv 0$$

This means that all of the terms in the integrand of $E(t)$ are also zero, in particular, v_t. Integrating once more and using the fundamental theorem, we see that v itself is identically zero, and we have proved uniqueness. •

5.4.3. Time-periodic solutions of the telegraph equation. Quite apart from the initial-value problem, it is possible to find solutions of the telegraph equation that are periodic in time, much as we did for the heat equation in Chapter 2.

In order to find time-periodic solutions of the telegraph equation, we look for complex separated solutions of the form

$$u(x;t) = e^{i\omega t}e^{\gamma x}$$

where ω is a given real number and γ is to be found. By taking the real and imaginary parts, we will obtain real-valued solutions of the telegraph equation that are periodic in time. To proceed further, we compute the relevant derivatives:

$$u_x(x;t) = \gamma\, e^{i\omega t}e^{\gamma x} \quad u_{xx}(x;t) = \gamma^2\, e^{i\omega t}e^{\gamma x}$$

$$u_t(x;t) = i\omega\, e^{i\omega t}e^{\gamma x} \quad u_{tt}(x;t) = -\omega^2\, e^{i\omega t}e^{\gamma x}$$

To satisfy the telegraph equation, we must have

$$0 = u_{tt} + 2\beta u_t + \alpha u - c^2 u_{xx} = [-\omega^2 + 2i\omega\beta + \alpha - c^2\gamma^2]u$$

which will be satisfied if and only if γ satisfies

$$c^2\gamma^2 = -\omega^2 + 2i\omega\beta + \alpha$$

This may be solved in terms of complex square roots through the formula

$$\sqrt{A+iB} = \pm\frac{u+iv}{\sqrt{2}}$$

where

$$u = \sqrt{A+\sqrt{A^2+B^2}}, \quad v = \operatorname{sgn}(B)\sqrt{-A+\sqrt{A^2+B^2}}$$

where the *signum* function is defined by $\operatorname{sgn}(B) = +1$ if $B \geq 0$ and $\operatorname{sgn}(B) = -1$ otherwise. Taking $A = \alpha - \omega^2$, $B = 2\omega\beta > 0$, we have

$$c\gamma\sqrt{2} = \pm\left(\sqrt{\alpha-\omega^2+\sqrt{(\alpha-\omega^2)^2+4\beta^2\omega^2}} + i\sqrt{-\alpha+\omega^2+\sqrt{(\alpha-\omega^2)^2+4\beta^2\omega^2}}\right)$$

Real-valued solutions are obtained by taking the real and imaginary parts of the above complex-valued solution.

To do this, we write $\gamma = \pm(\gamma_R + i\gamma_I)$ and obtain

$$\begin{aligned} e^{\gamma x}e^{i\omega t} &= e^{\pm(\gamma_R+i\gamma_I)x}e^{i\omega t} \\ &= e^{\pm\gamma_R x}\left(\cos(\omega t \pm \gamma_I x) + i\sin(\omega t \pm \gamma_I x)\right) \end{aligned}$$

The choice of sign is often dictated by conditions of boundedness, as indicated in the next example.

EXAMPLE 5.4.1. *Find the time-periodic solution of the telegraph equation* $u_{tt} + 2\beta u_t = c^2 u_{xx}$ *defined for* $x > 0$, $-\infty < t < \infty$ *and satisfying the boundary condition that* $u(0;t) = A\cos\omega t$. *Assume further that* $|u(x;t)| \leq M$ *for some constant* M *and that* $\omega > 0$, $\beta > 0$.

Solution. We use the above complex separated solutions with $\alpha = 0$, $\beta > 0$. Thus we have

$$c\gamma\sqrt{2} = \pm\left(\sqrt{-\omega^2+\sqrt{\omega^2+4\beta^2\omega^2}} + i\sqrt{\omega^2+\sqrt{\omega^2+4\beta^2\omega^2}}\right)$$

and the real-valued solutions

$$u(x;t) = \exp\left(\pm\frac{x}{c\sqrt{2}}\sqrt{-\omega^2+\sqrt{\omega^2+4\beta^2\omega^2}}\right)\cos\left(\omega t \pm \frac{x}{c\sqrt{2}}\sqrt{\omega^2+\sqrt{\omega^2+4\beta^2\omega^2)}}\right)$$

$$u(x;t) = \exp\left(\pm\frac{x}{c\sqrt{2}}\sqrt{-\omega^2+\sqrt{\omega^2+4\beta^2\omega^2}}\right)\sin\left(\omega t \pm \frac{x}{c\sqrt{2}}\sqrt{\omega^2+\sqrt{\omega^2+4\beta^2\omega^2)}}\right)$$

In order to satisfy the boundary condition at $x = 0$ we choose the first solution; in order to satisfy the condition of boundedness for $x > 0$, we must choose the solution with a negative real part in the exponent. This leads to the solution

$$u(x;t) = \exp\left(-\frac{x}{c\sqrt{2}}\sqrt{-\omega^2+\sqrt{\omega^2+4\beta^2\omega^2}}\right)\cos\left(\omega t - \frac{x}{c\sqrt{2}}\sqrt{\omega^2+\sqrt{\omega^2+4\beta^2\omega^2)}}\right) \bullet$$

EXERCISES 5.4

1. Let $u(x;t)$ be a solution of the telegraph equation $u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx}$. Show that $v(x;t) = e^{\beta t}u(x;t)$ is a solution of the equation $v_{tt} + (\alpha - \beta^2)v = c^2 v_{xx}$.
2. Let $v(x;t)$ be a solution of the equation $v_{tt} + (\alpha - \beta^2)v = c^2 v_{xx}$. Show that $u(x;t) = e^{-\beta t}v(x;t)$ is a solution of the telegraph equation $u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx}$.
3. Solve the initial-value problem for the telegraph equation

$$\begin{aligned} u_{tt} + 2\beta u_t + \alpha u &= c^2 u_{xx} & & t > 0, -\infty < x < \infty \\ u(x;0) &= f_1(x) & & -\infty < x < \infty \\ u_t(x;0) &= 0 & & -\infty < x < \infty \end{aligned}$$

   Consider separately the three cases $\alpha > \beta^2$, $\alpha = \beta^2$, $\alpha < \beta^2$.
4. Let $f(x)$ be a smooth function whose derivatives $f^{(i)}$ satisfy

$$\int_{-\infty}^{\infty} |f^{(i)}(x)|dx < \infty \qquad i = 0, 1, 2, 3$$

   where $f^{(0)} = f$. Let F be the Fourier transform of f. Show that

$$\int_{-\infty}^{\infty} |\mu||F(\mu)|d\mu < \infty$$

   [*Hint:* Integrate by parts the formula (5.1.6) three times to obtain $F(\mu) = O(|\mu|^{-3})$, $|\mu| \to \infty$.]
5. Use the result of the previous exercise to show that under the stated conditions, the Fourier representation (5.4.5) defines a rigorous solution of the telegraph equation with $f_1 = 0$.
6. Find the bounded time-periodic solution of the telegraph equation $u_{tt} + 2\beta u_t = c^2 u_{xx}$ for $x > 0$, $-\infty < t < \infty$, with $u(0;t) = A\sin\omega t$, where A, ω, β are real and positive.
7. Find the bounded time-periodic solution of the telegraph equation $u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx}$ for $x > 0$, $-\infty < t < \infty$, with $u(0;t) = A\cos\omega t$, where A, ω, α, β are real and positive.
8. Consider the three-dimensional telegraph equation

$$u_{tt} + 2\beta u_t + \alpha u = c^2\nabla^2 u$$

   with initial conditions

$$u(x,y,z;0) = 0 \qquad u_t(x,y,z;0) = f_2(x,y,z)$$

   Find a Fourier representation of the solution.

Appendix 5A: Derivation of the telegraph equation

In this appendix we derive a system of two partial differential equations that lead to the telegraph equation.

Suppose that a conducting cable occupies the x-axis and is endowed with resistance, inductance, capacitance, and leakage—denoted respectively by the letters R, L, C, and G, assumed to be constants, independent of position or time. The unknown functions are the voltage $v(x;t)$ and the current $i(x;t)$ measured at the point x of the cable at time instant t. We suppose that these are continuous functions of both variables, with continuous partial derivatives v_x, v_t, i_x, i_t. The system of equations will be derived by considering the voltage loss and current loss along a section of the cable $x_1 \le x \le x_2$.

The voltage loss comes in two parts. First, there is a voltage loss that is proportional to the current in the cable, where the proportionality factor is (defined to be) the resistance; second, there is a voltage loss that is proportional to the rate of change of current, where the proportionality factor is (defined to be) the inductance. Putting these together, we have

$$v(x_1;t) - v(x_2;t) = R\int_{x_1}^{x_2} i(x;t)\,dx + L\int_{x_1}^{x_2} i_t(x;t)\,dx$$

But the fundamental theorem of calculus gives the alternative expression

$$v(x_1;t) - v(x_2;t) = -\int_{x_1}^{x_2} v_x(x;t)\,dx$$

Equating these two expressions leads to the equation

$$\int_{x_1}^{x_2} [v_x(x;t) + Ri(x,t) + Li_t(x;t)]dx = 0$$

Since the functions in the integrand are assumed to be continuous, we conclude that the integrand is identically zero, leading to the first equation of the system:

$$v_x(x;t) + Ri(x;t) + Li_t(x;t) = 0$$

To deduce the second equation of the system, we consider the change in current in a section of the cable. The current changes based on two factors: first, there is a current loss that is proportional to the voltage level, where the proportionality factor is (defined to be) the leakage constant; second, there is a current loss that is proportional to the rate of change of the voltage, where the constant of proportionality is (defined to be) the capacitance. Putting these together, we have, for the current loss along this section, $G\int_{x_1}^{x_2} v(x;t)\,dx + C\int_{x_1}^{x_2} v_t(x;t)\,dx$. But the basic relation between charge and current is that the above quantity is the difference in the current. Thus

$$i(x_1;t) - i(x_2;t) = G\int_{x_1}^{x_2} v(x;t)\,dx + C\int_{x_1}^{x_2} v_t(x;t)\,dx$$

The fundamental theorem of calculus gives the alternative formula

$$i(x_1;t) - i(x_2t) = -\int_{x_1}^{x_2} i_x(x;t)\,dx$$

Subtracting these two equations, we have, for any segment of the cable,

$$\int_{x_1}^{x_2} [i_x(x;t) + Gv(x;t) + Cv_t(x;t)]\,dx = 0$$

Since the functions are assumed to be continuous, we conclude that the integrand is identically zero, leading to the second equation of the system:

$$i_t(x;t) + Gv(x;t) + Cv_t(x;t) = 0$$

This system of equations may be summarized in the form

$$\boxed{i_x + Cv_t + Gv = 0} \tag{5.4.8}$$

$$\boxed{v_x + Li_t + Ri = 0} \tag{5.4.9}$$

Having established this system of equations, it is straightforward to derive the second-order telegraph equation of the text, assuming that both $v(x;t)$ and $i(x;t)$ have continuous second-order partial derivatives. In detail, take the t-derivative of (5.4.8), multiply by L, and subtract this from the x-derivative of (5.4.9); the term Li_{xt} cancels, and we are left with the equation $LCv_{tt} + GLv_t = v_{xx} + Ri_x$. But Ri_x can be computed from (5.4.8) as $Ri_x = -RCv_t - RGv$, leading to the telegraph equation

$$LCv_{tt} + (GL + RC)v_t + RGv = v_{xx}$$

To obtain the equation satisfied by the current $i(x;t)$, we proceed similarly, taking the x-derivative of (5.4.8) and subrating C times the t-derivative of the (5.4.9). Using (5.4.9) again to solve for v_x and substituting, we obtain the same telegraph equation:

$$LCi_{tt} + (GL + RC)i_t + RGi = i_{xx}$$

CHAPTER 6

ASYMPTOTIC ANALYSIS

INTRODUCTION

Several times in the previous chapters we have encountered notions from asymptotic analysis. This refers to the simplification of a complicated formula that contains a parameter. When the parameter becomes very large, it is often possible to find a simpler formula that gives an extremely accurate approximation to the original formula.

For example, in Chapter 2 we found asymptotic formulas for the solution of the heat equation, by using the first term of the series of separated solutions. However, when the solution is represented by an integral instead of a series, there is no largest term that gives the asymptotic behavior; hence we must develop more systematic methods of analysis.

Section 6.1 illustrates the power of working with an integral representation rather than with an elementary formula in the case of the factorial function, where we derive a form of Stirling's formula by suitable asymptotic analysis. The systematic work begins in Sec. 6.2, where we show that the elementary technique of integration by parts can provide useful asymptotic estimates in many cases of interest. In more refined cases, we can often use *Laplace's method*, developed in Sec. 6.3. There is also a counterpart of Laplace's method for oscillatory integrands represented by purely imaginary exponentials, the *method of stationary phase*, which is developed in Sec. 6.4. These three methods can be consistently applied to the integral representations of solutions of the heat equation and Laplace's equation to deduce one-term asymptotic formulas. If one desires formulas with additional terms, there is the *method of asymptotic expansions*, developed in Sec. 6.5, which extends each of the three methods to obtain additional terms.

The methods of asymptotic analysis can be applied effectively to the solutions of partial differential equations. This is especially natural when we consider the Fourier integral representation, with the heat equation as one of the simplest cases. In Sec. 6.6 we combine the previous methods to obtain the asymptotic analysis of solutions of the telegraph equation.

6.1. Asymptotic Analysis of the Factorial Function

In this section we discuss the asymptotic behavior of the factorial function

$$n! = 1 \cdot 2 \cdots n$$

This formula, although elementary in principle, contains $n-1$ multiplications, and thus its computation can be time-consuming. If we take logarithms to reduce the multiplications to additions, we must then have a table (or program) to compute $n-1$ logarithms and finally compute the exponential. Therefore it is desirable to have a simpler formula, i.e., one that involves fewer computations.

6.1.1. Geometric mean approximation: Analysis by logarithms. As a first attempt, we form the natural logarithm to obtain

$$\log n! = \log 2 + \log 3 + \cdots + \log n$$

The sum of logarithms can be estimated by a definite integral, by noting that

$$\int_1^n \log x\,dx < \log 2 + \cdots + \log n < \int_1^{n+1} \log x\,dx$$

From elementary calculus,

$$\int_1^n \log x\,dx = n\log n - n + 1 \quad \text{and} \quad \int_1^{n+1} \log x\,dx \le \int_1^n \log x\,dx + \log(n+1)$$

so that

$$\begin{aligned} n\log n - n + 1 < \log 2 + \cdots + \log n &< n\log n - n + 1 + \log(n+1) \\ \log n - 1 - \frac{1}{n} < \frac{\log 2 + \cdots + \log n}{n} &< \log n - 1 + O\left(\frac{\log n}{n}\right) \end{aligned}$$

Subtracting $\log n$ from both sides and letting $n \to \infty$, we have

$$\lim_{n\to\infty} \left(\frac{\log 2 + \cdots + \log n}{n} - \log n\right) = -1$$

Taking the exponential of both sides, we obtain

$$\boxed{\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = e^{-1}} \tag{6.1.1}$$

which is called the *geometric mean approximation* and often written in the form $n! \sim (n/e)^n$, $n \to \infty$.

To get a feel for the accuracy of the geometric mean approximation, we compute a few typical values in the table below. The second and third columns are comparable, within a factor of 10, for the tabulated values. By taking nth roots, the accuracy is apparently much improved.

n	$n!$	$(n/e)^n$	$(n!)^{1/n}$	n/e
1	1	0.37	1.00	0.37
10	3.63×10^{6}	4.54×10^{5}	4.53	3.68
100	9.33×10^{157}	3.72×10^{156}	37.99	36.79
1000	4.02×10^{2567}	5.08×10^{2565}	369.49	367.88

In this basic example we have a simple case of asymptotic analysis. Computation of $(n/e)^n$ involves only two multiplications and two logarithmic computations. Hence we have found a simpler formula for $n!$.

6.1.2. Refined method using functional equations. To obtain a finer approximation of $n!$, we write the factorial in the form

$$n! = \frac{n^n}{e^{A(n)}}$$

which defines a new function $A(n)$. From the analysis of the previous subsection, we suspect that $A(n)$ increases like n, when $n \to \infty$. In order to quantify this, we try to find a difference equation for the function $A(n)$. Thus

$$\begin{aligned}\frac{e^{A(n)}}{e^{A(n-1)}} &= \frac{n^n/n!}{(n-1)^{n-1}/(n-1)!}\\ &= \left(1-\frac{1}{n}\right)^{1-n}\end{aligned}$$

Therefore the function $A(n)$ satisfies the difference equation

$$A(n) - A(n-1) = (1-n)\log\left(1-\frac{1}{n}\right)$$

This can be simplified by invoking the two-term Taylor expansion of the logarithm: $\log(1+x) = x - (x^2/2) + O(x^3), x \to 0$; this yields

$$\begin{aligned}A(n) - A(n-1) &= (1-n)\left(-\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right)\\ &= 1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \qquad n \to \infty\end{aligned}$$

so that the function $A(n)$ can be retrieved as the telescoping sum

$$\begin{aligned}A(n) &= A(1) + \sum_{k=1}^{n}(A(k) - A(k-1))\\ &= \sum_{k=1}^{n}\left(1 - \frac{1}{2k} + O\left(\frac{1}{k^2}\right)\right)\end{aligned}$$

But the series $\sum_1^\infty 1/k^2$ converges, and the sum $\sum_1^n 1/k$ is $\log n$ plus a constant plus a term that tends to zero. Combining this, we obtain

$$A(n) = n - \frac{\log n}{2} + C + O\left(\frac{1}{n}\right) \qquad n \to \infty$$

where the constant C is to be determined. By exponentiation, we have

$$(6.1.2)\qquad n! = n^n e^{-A(n)} = n^n \exp\left(-n + \frac{\log n}{2} - C + O\left(\frac{1}{n}\right)\right) = n^{n+(1/2)}e^{-n}e^{-C+O(1/n)}$$

This is closely related to *Stirling's formula*, which states that $e^{-C} = \sqrt{2\pi}$.

6.1.3. Stirling's formula via an integral representation. In order to prove the classical Stirling formula, it suffices to identify the constant C from formula (6.1.2). To do this, we begin with the following integral representation obtained in Chapter 1:

$$(6.1.3)\qquad \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^{2n} x\,dx$$

The left side can be rewritten following (6.1.2):

$$\begin{aligned}\frac{1}{2^{2n}}\binom{2n}{n} &= \frac{(2n)!}{2^{2n}(n!)^2} = \frac{(2n)^{2n+(1/2)}e^{-2n}e^{-C+O(1/n)}}{2^{2n}\left(n^{n+(1/2)}e^{-n}e^{-C+O(1/n)}\right)^2}\\ &= \left(\frac{2}{n}\right)^{1/2} e^{C+O(1/n)}\end{aligned}$$

On the other hand, the integrand on the right side has period π, so that the integral can be written as

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^{2n} x\,dx &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos^{2n} x\,dx\\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(1 - x^2/2 + O(x^4)\right)^{2n} dx\\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(e^{-x^2/2(1+\varepsilon(x))}\right)^{2n} dx\\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-nx^2(1+\varepsilon(x))}\,dx\\ &= \frac{1}{\pi\sqrt{n}}\int_{-\pi\sqrt{n}/2}^{\pi\sqrt{n}/2} e^{-y^2(1+\varepsilon(y/\sqrt{n}))}\,dy\end{aligned}$$

where $\varepsilon(x)$ is a function that satisfies $\lim_{x\to 0}\varepsilon(x) = 0$ and where we have made the substitution $x = y/\sqrt{n}$. When $n \to \infty$, the final integral tends to the value

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}$$

Inserting this in the previous computations shows that

$$\sqrt{\frac{2}{n}}e^{C} = \frac{1}{\sqrt{n\pi}}$$

which is solved to yield $e^{-C} = \sqrt{2\pi}$ and the desired form of Stirling's formula

$$\boxed{n! = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}\,(1 + O(1/n)) \qquad n \to \infty} \tag{6.1.4}$$

Stirling's formula provides a more accurate approximation, with very little additional complexity compared with the geometric mean approximation. In addition to two logarithmic computations, one must also have the value of $\sqrt{2\pi}$. The following table gives a numerical comparison of the two approximations.

n	$n!$	$(n/e)^n$	$(n/e)^n\sqrt{2n\pi}$
1	1	0.37	0.992
10	3.63×10^6	4.54×10^5	3.599×10^6
100	9.33×10^{157}	3.72×10^{156}	9.325×10^{157}
1000	4.02×10^{2567}	5.08×10^{2565}	4.027×10^{2567}

In concluding this section, we emphasize that the derivation of the precise form of Stirling's formula was possible only when we began with a suitable *integral representation.* Quite apart from PDE, this illustrates the power of integral representations in asymptotic analysis. In the remainder of this chapter we will develop systematic methods for obtaining asymptotic formulas for certain functions that are defined by integrals, for example, the Fourier representation of the solution of the heat equation.

EXERCISES 6.1

1. Compute the values of $n!$, $(n/e)^n$, and $(n/e)^n(2\pi n)^{1/2}$ for $n = 5$, $n = 50$, $n = 500$.
2. Show that Stirling's formula can be derived directly from the integral representation

$$n! = \int_0^\infty x^n e^{-x} dx \qquad n = 1, 2, \ldots$$

by the following steps.

(i) Make the change of variable $x = ny$ to obtain

$$\frac{n!}{n^{n+1}e^{-n}} = \int_0^\infty y^n e^{-n(y-1)}\,dy \qquad n = 1, 2, \ldots$$

(ii) Show that $\int_0^{1-\delta} y^n e^{-n(y-1)}\,dy \to 0$ and $\int_{1+\delta}^{\infty} y^n e^{-n(y-1)}\,dy \to 0$ for any $\delta > 0$, when $n \to \infty$.

(iii) Show that $ye^{1-y} = e^{-(y-1)^2/2(1+\varepsilon(y))}$ where $\lim_{y\to 1} \varepsilon(y) = 0$.

(iv) Conclude that $\lim_{n\to\infty} \sqrt{n} \int_{1-\delta}^{1+\delta} \left(ye^{1-y}\right)^n \, dy = \sqrt{2\pi}$.
(v) Derive Stirling's formula in the form

$$\lim_{n\to\infty} \frac{n!}{n^{n+(1/2)}e^{-n}} = \sqrt{2\pi}$$

6.2. Integration by Parts

In the next two sections we will study functions that are defined by an integral of the form

$$f(t) = \int_a^b g(x)e^{th(x)}dx \tag{6.2.1}$$

when $t \to \infty$. Here a and b are fixed limits of integration (independent of t). For technical reasons, we allow one of them to be infinite, but not both; $g(x)$ and $h(x)$ are smooth functions with $h(x)$ real-valued, while $g(x)$ may be either real- or complex-valued.

For background motivation, one should think of a finite sum of powers

$$g_1e^{th_1} + g_2e^{th_2} + \cdots + g_Ne^{th_N}$$

where $h_1 < h_2 < \cdots < h_N$. When t is very large, the sum is well approximated by the largest term, which is the last term in the sum, so that we can write

$$\lim_{t\to\infty} \frac{g_1e^{th_1} + g_2e^{th_2} + \cdots + g_Ne^{th_N}}{e^{th_N}} = g_N$$

In dealing with an integral of exponential terms, there is no "largest term," so the asymptotic formula must be more detailed. As a simple illustration, consider the exact computation

$$\int_0^1 e^{tx}\, dx = \frac{e^t - 1}{t}$$

If we simply divide both sides by e^t, the right side tends to zero when $t \to \infty$. The precise asymptotic formula is e^t/t, where we have an additional factor of $1/t$. This will be mirrored in the general case, discussed below.

The main hypothesis of this section is

$$h'(x) \neq 0 \qquad a \leq x \leq b$$

In particular, $h(x)$ has no interior maximum or minimum for $a < x < b$. If $a = -\infty$ or $b = +\infty$, we also require that $h(x) \to -\infty$ fast enough so that the resulting improper integral converges. More specifically, we can allow $b = \infty$, $a > -\infty$ in case $h'(x) > 0$, whereas we can allow $b < \infty$, $a = -\infty$ in case $h'(x) < 0$.

The main result is

$$\boxed{f(t) = \frac{e^{tH}}{t}\left[C + O\left(\frac{1}{t}\right)\right] \qquad t \to \infty} \tag{6.2.2}$$

where

$$\begin{aligned} H &= \max_{a \le x \le b} h(x) = \max[h(a), h(b)] \\ C &= \begin{cases} g(b)/h'(b) & \text{if } h'(x) > 0 \\ -g(a)/h'(a) & \text{if } h'(x) < 0 \end{cases} \end{aligned}$$

In other words, the integral behaves like the largest exponential in the integrand, softened by a factor of $1/t$. This has already been illustrated by the exact computation above.

Proof. To prove (6.2.2), we multiply and divide the integrand by $th'(x)$ and obtain

$$f(t) = \int_a^b \frac{g(x)}{th'(x)}\, d(e^{th(x)}) = \frac{g(x)e^{th(x)}}{th'(x)}\bigg|_a^b - \frac{1}{t}\int_a^b e^{th(x)}\left(\frac{g}{h'}\right)' dx$$

The second integral is of the same form as the original integral (6.2.1) for $f(t)$, but with g replaced by $(g/h')'$. If we now apply the identical procedure to this integral, we obtain

$$\int_a^b e^{th(x)}\left(\frac{g}{h'}\right)' dx = \frac{1}{th'}\left(\frac{g}{h'}\right)' e^{th(x)}\bigg|_a^b - \frac{1}{t}\int_a^b e^{th(x)}\left[\frac{1}{h'}\left(\frac{g}{h'}\right)'\right]' dx$$

But $e^{th(x)} \le e^{tH}$ for all $a \le x \le b$, and therefore the above expression is $O(e^{tH}/t)$ when $t \to \infty$. Thus we have shown that

$$f(t) = \frac{g(x)e^{th(x)}}{th'(x)}\bigg|_a^b + O\left(\frac{e^{tH}}{t^2}\right) \qquad t \to \infty$$

But $h(x)$ is a continuous function that assumes its maximum at one of the endpoints $x = a$ or $x = b$ but not both. If $h'(x) < 0$, the maximum is assumed at $x = a$; otherwise it is assumed at $x = b$. Thus we have the stated result. •

6.2.1. Two applications. We will now give two important applications of the method of integration by parts. The first of these is the integral that defines the error function, which was used to solve the heat equation in Chapter 5. The second application deals specifically with the initial-value problem for the heat equation.

EXAMPLE 6.2.1. *Find an asymptotic formula for the complementary error function, defined by the integral*

$$1 - \Phi(y) = \frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-x^2/2} dx$$

Solution. The integral is not presented in the form (6.2.1). To transform it to this form, make the change of variable $u = x/y$, with the result

$$1 - \Phi(y) = \frac{y}{\sqrt{2\pi}} \int_1^\infty e^{-u^2y^2/2} du$$

In this integral we make the identifications $a = 1$, $b = \infty$, $t = y^2$, $h(u) = -\frac{1}{2}u^2$, $g(u) = 1$. The maximum of $h(u)$ is $H = h(1) = -\frac{1}{2}$, while $C = -g(1)/h'(1) = 1$. Applying (6.2.2), we have

$$\begin{aligned} 1 - \Phi(y) &= \frac{y}{\sqrt{2\pi}} \frac{e^{-t/2}}{t} \left[1 + O\left(\frac{1}{t}\right)\right] \\ &= \frac{1}{y\sqrt{2\pi}} e^{-y^2/2} \left[1 + O\left(\frac{1}{y^2}\right)\right] \qquad y \to \infty \quad \bullet \end{aligned}$$

The accuracy of this approximation can be inferred from the following table of values:[1]

y	$1 - \Phi(y)$	$e^{-y^2/2}/y\sqrt{2\pi}$
2.25	1.22×10^{-2}	1.41×10^{-2}
3.00	1.30×10^{-3}	1.48×10^{-3}
3.75	8.82×10^{-5}	9.41×10^{-5}
4.50	3.45×10^{-6}	3.56×10^{-6}

We now turn to an example involving the solution of the heat equation.

EXAMPLE 6.2.2. *Two materials of the same conductivity K are initially at temperatures T_1 and T_2. Find asymptotic formulas for the temperature $u(x;t)$ when $t \to 0$, $t \to \infty$.*

Solution. In Example 5.2.3 we found the exact solution

$$u(x;t) = T_2 \Phi\left(\frac{x}{\sqrt{2Kt}}\right) + T_1 \left[1 - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right]$$

To analyze the solution for $t \to 0$, we use Example 6.2.1.

$$\Phi(-z) = 1 - \Phi(z) = \frac{1}{z\sqrt{2\pi}} e^{-z^2/2} \left[1 + O\left(\frac{1}{z^2}\right)\right] \qquad z \to \infty$$

[1]The values for $1 - \Phi(y)$ are from Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1972, p. 316, where $1 - \Phi(y) = \frac{1}{2}\operatorname{erf}_c(y/\sqrt{2})$.

Taking $z = \pm x/\sqrt{2Kt}$, we have

$$\begin{aligned}
1 - \Phi\left(\frac{x}{\sqrt{2Kt}}\right) &= \left(\frac{Kt}{\pi x^2}\right)^{1/2} e^{-x^2/4Kt}[1 + O(t)] \qquad x > 0, t \to 0 \\
\Phi\left(\frac{x}{\sqrt{2Kt}}\right) &= \left(\frac{Kt}{\pi x^2}\right)^{1/2} e^{-x^2/4Kt}[1 + O(t)] \qquad x < 0, t \to 0
\end{aligned}$$

Substituting this, we have

$$\begin{aligned}
u(x;t) &= T_2\left\{1 - \left(\frac{Kt}{\pi x^2}\right)^{1/2} e^{-x^2/4Kt}[1 + O(t)]\right\} \\
&\quad + T_1\left(\frac{Kt}{\pi x^2}\right)^{1/2} e^{-x^2/4Kt}[1 + O(t)] \\
&= T_2 + O(e^{-x^2/4Kt}) \qquad x > 0, t \to 0 \\
u(x;t) &= T_2\left(\frac{Kt}{\pi x^2}\right)^{1/2} e^{-x^2/4Kt}[1 + O(t)] \\
&\quad + T_1\left\{1 - \left(\frac{Kt}{\pi x^2}\right)^{1/2} e^{-x^2/4Kt}[1 + O(t)]\right\} \\
&= T_1 + O(e^{-x^2/4Kt}) \qquad x < 0, t \to 0
\end{aligned}$$

At $x = 0$, $u(x;t) = \frac{1}{2}T_1 + \frac{1}{2}T_2$ for all $t > 0$.

To analyze the solution for $t \to \infty$, we simply use the Taylor expansion of $\Phi(z)$ about $z = 0$:

$$\Phi(z) = \Phi(0) + z\Phi'(0) + O(z^2) \qquad z \to 0$$

Taking $z = x/\sqrt{2Kt}$, we have

$$\Phi\left(\frac{x}{\sqrt{2Kt}}\right) = \frac{1}{2} + \frac{x}{\sqrt{2Kt}}\frac{1}{\sqrt{2\pi}} + O\left(\frac{1}{t}\right) \qquad t \to \infty$$

Substituting this in the exact solution, we have

$$\begin{aligned}
u(x;t) &= T_2\left[\frac{1}{2} + \frac{x}{\sqrt{4\pi Kt}} + O\left(\frac{1}{t}\right)\right] + T_1\left[\frac{1}{2} - \frac{x}{\sqrt{4\pi Kt}} + O\left(\frac{1}{t}\right)\right] \\
&= \frac{1}{2}(T_1 + T_2) + \frac{(T_2 - T_1)x}{\sqrt{4\pi Kt}} + O\left(\frac{1}{\sqrt{t}}\right) \qquad t \to \infty \quad \bullet
\end{aligned}$$

The preceding example shows that when $t \to 0$, $u(x;t)$ tends to $u(x;0)$ faster than any power of t; that is, $u(x;t) - u(x;0) = O(t^n)$, $t \to 0$, for $n = 1, 2, \ldots$. This shows that $\partial^n u/\partial t^n|_{t=0} = 0$, but $u(x;t)$ is not identically zero. We have a "real" example of a function that is not represented by its Taylor series about $t = 0$.

EXERCISES 6.2

Obtain asymptotic expressions for the following integrals when $t \to \infty$.

1. $f(t) = \int_0^1 e^{tx} \sin x \, dx$
2. $f(t) = \int_1^\infty e^{-tx^2} x^4 dx$
3. $f(t) = \int_t^\infty u^{-1} e^{-u} \, du$
4. $f(t) = \int_0^\infty (1+x^2)^{-1} e^{-tx} \, dx$
5. By an appropriate use of integration by parts, find an asymptotic expression for
$$f(t) = \int_0^\infty \frac{\sin x}{x+t} \, dx$$
6. Two materials of the same conductivity K are initially at the temperatures $T_1 = 0$ and $T_2 = 100$. At time $t = 0$ they are brought together. Find an asymptotic formula for $u(at; t)$ when $t \to \infty$, where a is a positive constant.
7. Let $u(x, y)$ be the solution of Laplace's equation $u_{xx} + u_{yy} = 0$ defined in the half-plane $-\infty < x < \infty$, $y > 0$ with $u(x, 0) = f(x)$. Show that when $y \to \infty$ we have the asymptotic formula
$$u(x, y) = \frac{1}{\pi y} \int_{-\infty}^{\infty} f(z) \, dz + O(1/y^2) \quad y \to \infty$$
[*Hint:* Use the Fourier representation (5.3.18) from Chapter 5.]

6.3. Laplace's Method

In this section we continue the asymptotic analysis of integrals of the form

$$f(t) = \int_a^b g(x) e^{th(x)} dx \tag{6.3.1}$$

but now we include the possibility that $h'(x) = 0$ at one or more points. As a first approximation, we can expect that $f(t)$ grows like e^{tH} when $t \to \infty$, where H is the maximum of $h(x)$, $a \le x \le b$. The new feature results from the possibility of points x_i, where $h(x_i) = H$ and $h'(x_i) = 0$.

6.3.1. Statement and proof of the result. We assume that $h(x), a \le x \le b$, is a twice-differentiable function that assumes its maximum H at a finite number of points (x_i) and that $h''(x_i) \neq 0$ at each of these points. [Of course, it follows that $h''(x_i) < 0$ since we are at a maximum of h.] These points fall into two groups: (1) interior global maxima of h and (2) boundary maxima where $h'(x_i) = 0$. The exact contribution of the second type of point is one-half the contribution of the first type of point. We now state the result of Laplace's method.

THEOREM 6.1. *Under the above conditions,*

$$\boxed{f(t) = \frac{e^{tH}}{\sqrt{t}}\left[C + O\left(\frac{1}{\sqrt{t}}\right)\right] \qquad t \to \infty} \tag{6.3.2}$$

where

$$C = \sqrt{2\pi}\left[\sum_{a<x_i<b} \frac{g(x_i)}{[-h''(x_i)]^{1/2}} + \frac{1}{2}\sum_{x_i=a \text{ or } b} \frac{g(x_i)}{[-h''(x_i)]^{1/2}}\right] \tag{6.3.3}$$

where it is understood that the first sum is over those interior points x_i where $h(x_i) = H$, and the second sum contains $x = a$ (resp. $x = b$) if and only if $h(a) = H$, $h'(a) = 0$ [resp. $h(b) = H$, $h'(b) = 0$].

The proof, which is divided into three steps, may be omitted at a first reading.

Proof. For simplicity, let us assume that the maximum is assumed at *exactly one interior point* x_1. (The other cases, which are not essentially different, are dealt with in the exercises.) Now we can write the integral (6.3.1) in the form

$$f(t) = \mathrm{I} + \mathrm{II} + \mathrm{III}$$

where

$$\begin{aligned}
\mathrm{I} &= \int_a^{x_1-\delta} g(x)e^{th(x)}dx \\
\mathrm{II} &= \int_{x_1-\delta}^{x_1+\delta} g(x)e^{th(x)}dx \\
\mathrm{III} &= \int_{x_1+\delta}^{b} g(x)e^{th(x)}dx
\end{aligned}$$

δ is a positive number that will be specified. The first and third integrals are straightforward. Indeed, let

$$h_1 = \max_{a \le x \le x_1-\delta} h(x), \qquad h_2 = \max_{x_1+\delta \le x \le b} h(x)$$

Then

$$\mathrm{I} = O(e^{th_1}), \qquad \mathrm{III} = O(e^{th_2}) \qquad t \to \infty$$

In addition, we must have $h_1 < H$, $h_2 < H$. Otherwise the maximum of the continuous function h would be assumed on the interval $[a, x_1 - \delta]$ or the interval $[x_1 + \delta, b]$, contrary to the hypothesis. Therefore it remains to analyze the middle integral when $t \to \infty$.

To do this, we begin with the Taylor expansions of $g(x)$, $h(x)$ about $x = x_1$, written in the form

$$|g(x) - g(x_1)| \le M_1|x - x_1| \tag{6.3.4}$$

$$\left|h(x) - H - \frac{1}{2}h''(x_1)(x - x_1)^2\right| \le M_2|x - x_1|^3 \tag{6.3.5}$$

The Taylor expansion (6.3.5) implies that

$$h(x) \le H - M_3(x-x_1)^2 \qquad |x-x_1| < \delta \tag{6.3.6}$$

This is proved in detail at the end of this section.

In the following steps we will show that $g(x)$ may be replaced by $g(x_1)$ and that $h(x)$ may be replaced by its second-order Taylor expansion.

Step 1

$$\int_{x_1-\delta}^{x_1+\delta} g(x)e^{th(x)}dx = g(x_1)\int_{x_1-\delta}^{x_1+\delta} e^{th(x)}dx + O\left(\frac{e^{tH}}{t}\right) \qquad t\to\infty \tag{6.3.7}$$

To prove this, we use (6.3.4) to write

$$\left|\int_{x_1-\delta}^{x_1+\delta} g(x)e^{th(x)}dx - g(x_1)\int_{x_1-\delta}^{x_1+\delta} e^{th(x)}dx\right| \le M_1\int_{x_1-\delta}^{x_1+\delta} |x-x_1|e^{th(x)}dx$$

From (6.3.6) the last integral is less than or equal to

$$M_1e^{tH}\int_{x_1-\delta}^{x_1+\delta} |x-x_1|\exp[-tM_3(x-x_1)^2]dx$$

Making the change of variables $y=\sqrt{t}(x-x_1)$, this is

$$M_1\frac{e^{tH}}{t}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} |y|e^{-M_3y^2}dy = O\left(\frac{e^{tH}}{t}\right)$$

This completes the proof of (6.3.7).

Step 2

$$\int_{x_1-\delta}^{x_1+\delta} e^{th(x)}dx = e^{tH}\int_{x_1-\delta}^{x_1+\delta}\exp\left[\frac{th''(x_1)(x-x_1)^2}{2}\right]dx + O\left(\frac{e^{tH}}{t}\right) \tag{6.3.8}$$

To prove this, we use the following inequality, valid for real numbers A, B:

$$|e^A - e^B| \le |A-B|e^C \qquad C = \max(A,B)$$

We apply this with $A = h(x) - H$, $B = \frac{1}{2}h''(x_1)(x-x_1)^2$. We have $\max(A,B) \le -M_3(x-x_1)^2$ for $|x-x_1|<\delta$, $M_3 = \frac{1}{3}|h''(x_1)|$. Thus

$$\begin{aligned}
&\left|\int_{x_1-\delta}^{x_1+\delta} e^{th(x)}dx - e^{tH}\int_{x_1-\delta}^{x_1+\delta}\exp\left[\frac{th''(x_1)(x-x_1)^2}{2}\right]dx\right| \\
&= e^{tH}\left|\int_{x_1-\delta}^{x_1+\delta}(e^{tA}-e^{tB})dx\right| \\
&\le te^{tH}\int_{x_1-\delta}^{x_1+\delta} M_2|x-x_1|^3\exp[-tM_3(x-x_1)^2]dx
\end{aligned}$$

We again make the change of variable $y = \sqrt{t}(x - x_1)$ in this integral and see that the integral is

$$\frac{e^{tH}}{t^2}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} M_2|y|^3 e^{-M_3 y^2}\,dy = O\left(\frac{e^{tH}}{t}\right)$$

This completes the proof of (6.3.8).

Step 3

(6.3.9)

$$e^{tH}\int_{x_1-\delta}^{x_1+\delta} \exp\left[\frac{th''(x_1)(x-x_1)^2}{2}\right] dx = e^{tH}\left(\frac{2\pi}{-th''(x_1)}\right)^{1/2}\left[1 + O\left(\frac{1}{t}\right)\right]$$

To prove this, we again apply the change of variable $y = \sqrt{t}(x-x_1)$ and transform the original integral to

$$\frac{e^{tH}}{\sqrt{t}}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} \exp\left[\frac{y^2 h''(x_1)}{2}\right] dy$$

When $t \to \infty$, the integral is equal to $= \{2\pi/[-h''(x_1)]\}^{1/2} + O(1/t)$, which is of the required form. Combining steps 1, 2, and 3 completes the proof of Laplace's method. •

If we also have a maximum at a boundary point, for example $x = a$, where $h'(a) = 0$, $h''(a) < 0$, then we must modify the analysis by taking the integral

$$\mathrm{II} = \int_a^{a+\delta} g(x)e^{th(x)}dx$$

By repeating the same steps, we find

$$\mathrm{II} \sim \frac{g(a)e^{th(a)}}{-th''(a)}\int_0^{\delta\sqrt{t}} e^{-u^2/2}du$$

When $t \to \infty$, the integral has the limiting value $\frac{1}{2}\sqrt{2\pi}$, which explains the factor of $\frac{1}{2}$ for the boundary terms in (6.3.3).

6.3.2. Three applications to integrals.

EXAMPLE 6.3.1. *Find an asymptotic formula for* $f(t)=\int_0^1 e^{tx(1-x)}dx$, $t \to \infty$.

Solution. In this case $h(x) = x(1-x)$ and the maximum is attained at $x = \frac{1}{2}$, where $h'(\frac{1}{2}) = 0$, $h''(\frac{1}{2}) = -2$, which is nonzero. Thus we may apply Laplace's method to obtain

$$\int_0^1 e^{tx(1-x)}dx = \sqrt{\frac{\pi}{t}}e^{t/4}\left[1 + O\left(\frac{1}{\sqrt{t}}\right)\right] \qquad t \to \infty \quad \bullet$$

EXAMPLE 6.3.2. *Find an asymptotic formula for* $f(t)=\int_0^1 e^{tx(2-x)}dx$, $t \to \infty$.

Solution. In this case $h(x) = 2x - x^2$ and the maximum is assumed at $x = 1$, where $h'(1) = 0$, $h''(1) = -2 < 0$. Thus we may apply Laplace's method at the endpoint $x = 1$, with the result

$$\int_0^1 e^{t(2x-x^2)}dx = e^t\sqrt{\frac{\pi}{4t}}\left[1+O\left(\frac{1}{\sqrt{t}}\right)\right] \qquad t \to \infty \quad \bullet$$

The next example gives an independent treatment of Stirling's formula, discussed in Sec. 6.1.

EXAMPLE 6.3.3. *Find an asymptotic formula for* $f(t)=\int_0^\infty e^{t\ln x-x}dx$, $t \to \infty$.

Solution. This is the integral that defines the factorial function $f(t) = t!$ for $t = 1, 2, 3, \ldots$. We will apply Laplace's method to obtain Stirling's formula. To do this, we might try $h(x) = \ln x$, $g(x) = e^{-x}$; but $\ln x$ has no maximum for $0 < x < \infty$; therefore Laplace's method is not applicable in this form. Nevertheless the function $x \to e^{t\ln x - x}$ has a maximum at $x = t$, so we make the change of variable $y = x/t$. This gives

$$\begin{aligned} f(t) &= t\int_0^\infty \exp[t\ln y + t\ln t - yt]dy \\ &= t^{t+1}\int_0^\infty e^{th(y)}dy \end{aligned}$$

where

$$\begin{aligned} h(y) &= \ln y - y \\ h'(y) &= \frac{1}{y} - 1 \\ h''(y) &= -\frac{1}{y^2} \end{aligned}$$

We can now apply Laplace's method. h has a global maximum at $y = 1$, where $h(1) = -1$, $h''(1) = -1$. Therefore

$$f(t) = t^{t+1}\sqrt{\frac{2\pi}{t}}\,e^{-t}\left[1+O\left(\frac{1}{\sqrt{t}}\right)\right] \qquad t \to \infty$$

This is the required form of Stirling's formula. •

6.3.3. Applications to the heat equation. Laplace's method applies naturally to solutions of the heat equation, written in the Fourier representation

$$u(x;t) = \int_{-\infty}^{\infty} e^{i\mu x}e^{-\mu^2 Kt}F(\mu)\,d\mu \tag{6.3.10}$$

where $F(\mu)$ is the Fourier transform of the initial data,

$$F(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}d\xi \tag{6.3.11}$$

EXAMPLE 6.3.4. *Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ with initial condition $u(x;0) = f(x)$ for $t > 0$, $-\infty < x < \infty$. Use Laplace's method to obtain asymptotic formulas for the temperature and heat flux when $t \to \infty$.*

Solution. We apply Laplace's method to the Fourier representation (6.3.10) with $h(\mu) = -K\mu^2$, $a = -\infty$, $b = \infty$, $g(\mu) = F(\mu)e^{i\mu x}$. The function $h(\mu)$ has a single maximum at $\mu = 0$, where $H = h(0) = 0$, $h'(0) = 0$, $h''(0) = -2K$. Applying (6.3.2), we have

$$u(x;t) = \frac{1}{\sqrt{t}}\left[F(0)\sqrt{\frac{2\pi}{2K}} + O\left(\frac{1}{\sqrt{t}}\right)\right] \qquad t \to \infty$$

Recalling (6.3.11), we have

$$u(x;t) = \frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty} f(\xi)d\xi + O\left(\frac{1}{t}\right) \qquad t \to \infty$$

To study the heat flux, we differentiate and get

$$u_x(x;t) = \int_{-\infty}^{\infty} i\mu e^{i\mu x}F(\mu)e^{-\mu^2 Kt}d\mu$$

Applying Laplace's method again with $h(\mu) = -K\mu^2$, $g(\mu) = i\mu e^{i\mu x}F(\mu)$, we have a single maximum at $\mu = 0$, where $h(0) = 0$, $g(0) = 0$. Applying (6.3.2), we have $C = 0$, and thus

$$u_x(x;t) = O\left(\frac{1}{t}\right) \qquad t \to \infty$$

The heat flux tends to zero faster than the temperature. •

6.3.4. Improved error with gaussian approximation. If $h(x)$ is a quadratic polynomial and $g(x)$ has a three-term Taylor expansion about the maximum of h, then we can obtain a more detailed statement of Laplace's method. Assuming that $h(x) = H - (h_2/2)(x - x_1)^2$ with $h_2 > 0$, we write

$$\begin{aligned} g(x) &= g(x_1) + (x - x_1)g'(x_1) + \frac{1}{2}(x - x_1)^2 g''(x_1) + \cdots \\ &= \alpha \exp[\beta(x - x_1) + \frac{1}{2}\gamma(x - x_1)^2 + \epsilon(x)] \end{aligned}$$

where $\epsilon(x) = O((x - x_1)^3), x \to x_1$. The coefficients α, β, γ can be obtained from

$$\alpha = g(x_1), \quad \alpha\beta = g'(x_1), \quad \alpha(\gamma + \beta^2) = g''(x_1)$$

In terms of these quantities, the amplified statement of Laplace's method is

$$\boxed{\int_a^b g(x)\, e^{th(x)}\, dx = e^{tH}\sqrt{\frac{2\pi}{th_2-\gamma}}\alpha\, e^{\beta^2/2(th_2-\gamma)} + O(t^{-2})} \tag{6.3.12}$$

This can be immediately applied to the heat equation, to obtain a more informative asymptotic formula than the one obtained for $u(x;t)$ in Example 6.3.4, which does not explicitly involve x. In order to obtain a more detailed formula, we can make use of (6.3.12) as shown in the following example.

EXAMPLE 6.3.5. *Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ with initial condition $u(x;0) = f(x)$ for $t > 0$, $-\infty < x < \infty$. Use (6.3.12) to find an asymptotic formula in terms of the following quantities:*

$$\begin{aligned} A &= \int_{-\infty}^{\infty} f(\xi)\, d\xi \neq 0 \\ \bar{x} &= \frac{1}{A}\int_{-\infty}^{\infty} \xi f(\xi) d\xi \\ \tau &= \frac{1}{2A}\int_{-\infty}^{\infty} (\xi-\bar{x})^2 f(\xi) d\xi \end{aligned}$$

Solution. In terms of the Fourier transform, we have

$$\begin{aligned} F(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi = \frac{A}{2\pi} \\ F'(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\xi)f(\xi)d\xi = -i\bar{x}\frac{A}{2\pi} \\ F''(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\xi^2)f(\xi)d\xi = -(2\tau+\bar{x}^2)\frac{A}{2\pi} \end{aligned}$$

so that we can write

$$\begin{aligned} u(x;t) &= \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}e^{-\mu^2 Kt}\, d\mu \\ &= \frac{A}{2\pi}\int_{-\infty}^{\infty} e^{-\mu^2 Kt}e^{i\mu(x-\bar{x})-\mu^2\tau+c(\mu)}\, d\mu \end{aligned}$$

where $c(\mu) = O(\mu^3), \mu \to 0$. Applying (6.3.12), we obtain

$$u(x;t) = \frac{A}{\sqrt{4\pi(Kt+\tau)}}e^{-(x-\bar{x})^2/4(Kt+\tau)} + O(t^{-2}) \qquad t\to\infty \quad \bullet$$

Proof of (6.3.6). By the extended mean-value theorem, we may write

$$h(x) = H + \frac{1}{2}h''(x_1)(x-x_1)^2 + \frac{1}{6}h'''(\xi)(x-x_1)^3$$

for some ξ with $|\xi - x_1| \le |x - x_1|$. If h''' is identically zero, there is nothing to prove. Otherwise let $M = \max |h'''|$, where the maximum is taken over any interval $(x_1 - \delta, x_1 + \delta)$. Write

$$H - h(x) = \frac{1}{2}|h''(x_1)|(x - x_1)^2 \left[1 + \frac{h'''(\xi)(x - x_1)}{3|h''(x_1)|}\right]$$

Let $|x - x_1| < |h''(x_1)|/M$. Then $|h'''(\xi)(x - x_1)/h''(x_1)| < 1$, and we have

$$\begin{aligned} H - h(x) &\ge \frac{1}{2}|h''(x_1)|(x - x_1)^2 \left(1 - \frac{1}{3}\right) \\ &= \frac{1}{3}|h''(x_1)|(x - x_1)^2 \end{aligned}$$

which was to be proved.

EXERCISES 6.3

In Exercises 1 to 3, apply Laplace's method to obtain an asymptotic formula for $f(t)$, $t \to \infty$.

1. $f(t) = \int_{-\pi/2}^{\pi/2} (3x + 2)e^{-t\sin^2 x} dx$
2. $f(t) = \int_{-2}^{2} (3 + 2\cos x)e^{-tx^2} dx$
3. $f(t) = \int_{-1}^{1} e^{-tP_4(x)} dx$, where P_4 is the fourth Legendre polynomial.
4. Let $I_0(t)$ be the modified Bessel function

$$I_0(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{t\sin\theta} d\theta$$

   Apply Laplace's method to find an asymptotic formula for $I_0(t)$, $t \to \infty$.
5. Let $h(x)$, $a < x < b$, be a differentiable function with a single maximum at x_1, $a < x_1 < b$. Assume that $h'(x_1) = 0$, $h''(x_1) = 0$, $h'''(x_1) = 0$, $h^{(4)}(x_1) < 0$. Show that Laplace's method can be modified to obtain an asymptotic formula of the form

$$\int_a^b e^{th(x)} dx \approx C \frac{e^{th(x_1)}}{[-th^{(4)}(x_1)]^{1/4}}$$

   with $C = (24)^{1/4} \int_{-\infty}^{\infty} e^{-u^4} du$.
6. Apply the method of Exercise 5 to obtain an asymptotic formula for the integral

$$\int_{-\pi/2}^{\pi/2} \exp\left[t\left(1 - \cos x - \frac{x^2}{2}\right)\right] dx \qquad t \to \infty$$

7. Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ for $x > 0$, $t > 0$, with the initial condition $u(x;0) = f(x)$ and the boundary condition $u(0;t) = 0$. Use Laplace's method to obtain asymptotic formulas for the temperature and heat flux when $t \to \infty$.
8. Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ for $x > 0$, $t > 0$ with the initial condition $u(x;0) = f(x)$ and the boundary condition $u_x(0;t) = 0$. Use Laplace's method to obtain asymptotic formulas for the temperature and heat flux when $t \to \infty$.
9. Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ for $t > 0$, $-\infty < x < \infty$ with initial condition $u(x;0) = 100/(1+x^2)$. Use Laplace's method to obtain asymptotic formulas for the temperature and heat flux when $t \to \infty$.
10. Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ for $t > 0$, $x > 0$ with initial condition $u(x;0) = xe^{-x^2}$ for $x > 0$ and the boundary condition $u_x(0;t) = 0$ for $t > 0$. Use Laplace's method to find asymptotic formulas for the temperature and heat flux when $t \to \infty$.
11. Let $u(x : t)$ be the solution of the heat equation $u_t = Ku_{xx}$ for $t > 0$, $-\infty < x < \infty$ with initial condition $u(x;0) = f(x)$. Modify the method of Example 6.3.5 to obtain the asymptotic formula

$$u(x;t) = \frac{A}{\sqrt{4\pi(Kt+\tau)}} e^{-(x-\bar{x})^2/4(Kt+\tau)} \left(1 - \frac{m_3}{(Kt+\tau)^{3/2}} H_3\left(\frac{x-\bar{x}}{\sqrt{Kt+\tau}}\right)\right) + O(t^{-3})$$

where $H_3(x) = x^3/8 - 3x/4$.

6.4. The Method of Stationary Phase

In the previous two sections we dealt with integrals containing an exponential factor with a real exponent, where the maximum value of the exponent influences the asymptotic behavior. In this section we turn our attention to integrals that are formally similar, but where the exponent is purely imaginary. Specifically, we consider complex-valued functions of the form

$$f(t) = \int_a^b e^{it\varphi(x)} g(x)\, dx, \tag{6.4.1}$$

where φ is a real-valued function called the *phase function*. The function $g(x)$ may be either real- or complex-valued. If $\varphi'(x) \neq 0$, then we may integrate by parts, as in Sec. 6.2, and conclude that $f(t) = O(1/t), t \to \infty$. However, if $\varphi'(x) = 0$ for some x, then this conclusion is no longer valid. In order to find the correct result, we focus attention upon those points x_j where $\varphi'(x) = 0$, the so-called stationary points.

6.4.1. Statement of the result. The complete statement of the result is given as follows.

THEOREM 6.2. *(The method of stationary phase). Suppose that $g(x), \varphi(x)$ have two continuous derivatives for $a \le x \le b$, that $\varphi(x)$ is real-valued, and that $\varphi'(x) \neq 0$ except for a finite number of stationary points x_j, where $\varphi''(x_j) \neq 0$. Let these be labeled so that $\varphi''(x_j) > 0$ for $1 \le j \le K$ and $\varphi''(x_j) < 0$ for $K+1 \le j \le K+L$. Then when $t \to \infty$,*

$$\boxed{\int_a^b e^{it\varphi(x)} g(x)\, dx = I^+(t) + I^-(t) + O\left(\frac{1}{t}\right)} \tag{6.4.2}$$

where

$$\boxed{I^+(t) = \sum_{j=1}^{K} \left(\frac{2\pi}{t\,\varphi''(x_j)}\right)^{1/2} e^{it\varphi(x_j)} e^{i\pi/4} g(x_j)} \tag{6.4.3}$$

$$\boxed{I^-(t) = \sum_{j=K+1}^{K+L} \left(\frac{2\pi}{-t\,\varphi''(x_j)}\right)^{1/2} e^{it\varphi(x_j)} e^{-i\pi/4} g(x_j)} \tag{6.4.4}$$

If either of the endpoints $x = a$, $x = b$ are also stationary points, then they contribute to (6.4.3)-(6.4.4) with a factor of $\frac{1}{2}$, exactly as in Laplace's method.

Note that, in contrast with Laplace's method, we must sum over *all* points where $\varphi'(x) = 0$, not simply those where $\varphi(x)$ is maximum.

A simple tool to remember this complicated formula is to observe that the result is identical to what is obtained by replacing $\varphi(x)$ by its two-term Taylor expansion and replacing $g(x)$ by its value at each stationary point, then doing the resultant integrals (one for each stationary point), and then summing the results.

We illustrate with a typical example.

EXAMPLE 6.4.1. *Apply the method of stationary phase to find an asymptotic formula for the integral*

$$\int_{-\pi/2}^{\pi/2} (2x+3) e^{-it\cos x}\, dx$$

Solution. In this case we have $g(x) = 2x+3$, $\varphi(x) = -\cos x$, $\varphi'(x) = \sin x$, $\varphi''(x) = \cos x$. The only stationary point is $x = 0$, where $\varphi''(0) = +1$, $g(0) = 3$. Applying (6.4.2), we have

$$\int_{-\pi/2}^{\pi/2} e^{-it\cos x}\, dx = 3\sqrt{\frac{2\pi}{t}} e^{-it} e^{i\pi/4} + O\left(\frac{1}{t}\right) \qquad t \to \infty \quad \bullet$$

6.4.2. Application to Bessel functions. As a primary application of the method of stationary phase, we propose to obtain the asymptotic behavior of the Bessel function $J_m(t)$, $t \to \infty$. Recall that in Chapter 3 we obtained a formula containing constants that could not be identified. We can make an explicit identification of these constants with the method of stationary phase, as follows:

$$\boxed{J_m(t) = \sqrt{\frac{2}{\pi t}} \cos(t - \pi/4 - m\pi/2) + O\left(\frac{1}{t}\right) \qquad t \to \infty} \tag{6.4.5}$$

Proof. To prove this, we begin with the integral representation (3.2.12), where $\varphi(x) = \cos x$, $g(x) = (1/2\pi)e^{-imx}e^{-im\pi/2}$. Since $\varphi'(x) = -\sin x$, $\varphi''(x) = -\cos x$, there are three stationary points: $x = 0$, $x = \pi$, $x = -\pi$, with $\varphi''(0) = -1$, $\varphi''(\pi) = 1 = \varphi''(-\pi)$; also $g(0) = (1/2\pi)e^{-im\pi/2}$, $g(\pi) = (1/2\pi)e^{-3im\pi/2} = g(-\pi)$. We apply the method of stationary phase, noting that the endpoints contribute with a factor of $\frac{1}{2}$. Hence

$$\begin{aligned}
J_m(t) &= \sqrt{\frac{2\pi}{t}} e^{it} e^{-i\pi/4} \frac{e^{-im\pi/2}}{2\pi} + \left(\frac{1}{2} + \frac{1}{2}\right) \sqrt{\frac{2\pi}{t}} e^{-it} e^{i\pi/4} \frac{e^{-3im\pi/2}}{2\pi} + O\left(\frac{1}{t}\right) \\
&= \sqrt{\frac{2\pi}{t}} e^{it} e^{-i\pi/4} \frac{e^{-im\pi/2}}{2\pi} + \sqrt{\frac{2\pi}{t}} e^{-it} e^{i\pi/4} \frac{e^{im\pi/2}}{2\pi} + O\left(\frac{1}{t}\right) \\
&= \sqrt{\frac{1}{2\pi t}} \left(e^{i(t-\pi/4-m\pi/2)} + e^{-i(t-\pi/4-m\pi/2)}\right) + O\left(\frac{1}{t}\right) \\
&= \sqrt{\frac{2}{\pi t}} \cos(t - \pi/4 - m\pi/2) + O\left(\frac{1}{t}\right) \quad \bullet
\end{aligned}$$

6.4.3. Proof of the method of stationary phase. We now outline the steps used to prove (6.4.2). The idea is to reduce the study to each stationary point, where we can approximate using the Taylor expansions with an error of $O(1/t)$.

Step 1. If the interval $c \le x \le d$ does not contain any stationary points, then

$$\int_c^d e^{it\varphi(x)} g(x)\, dx = O\left(\frac{1}{t}\right) \qquad t \to \infty$$

Proof. We multiply and divide by $\varphi'(x)$ and integrate by parts as follows:

$$\begin{aligned}
\int_c^d g(x) e^{it\varphi(x)}\, dx &= \int_c^d \frac{g(x)}{it\varphi'(x)} d\left(e^{it\varphi(x)}\right) dx \\
&= \frac{g(x)}{it\varphi'(x)} e^{it\varphi(x)} \Big|_{x=c}^{x=d} - \frac{1}{it} \int_c^d e^{it\varphi(x)} \frac{d}{dx}\left(\frac{g(x)}{\varphi'(x)}\right) dx
\end{aligned}$$

Both terms are $O(1/t)$, $t \to \infty$, and can therefore be included in the remainder term.

Therefore we can restrict attention to contributions from intervals containing the stationary points. Assume that x_1 is a stationary point for which $\varphi''(x_1) > 0$, and let $\delta > 0$ be chosen so that $\varphi(x) - \varphi(x_1) > 0$ in the interval $x_1 - \delta < x < x_1 + \delta$. We introduce a new variable of integration v through the equation

$$v = (x - x_1)\sqrt{\frac{\varphi(x) - \varphi(x_1)}{(x - x_1)^2}} \qquad x_1 - \delta < x < x - 1 + \delta$$

The function $x \to v(x)$ vanishes at $x = x_1$, with $v'(x_1) = \sqrt{\varphi''(x_1)/2} > 0$. Therefore there exists an inverse function $x = X(v)$ with $X(0) = x_1$, $X'(0) = \sqrt{2/\varphi''(x_1)}$.

Step 2

$$\int_{x_1-\delta}^{x_1+\delta} g(x)e^{it\varphi(x)}\,dx = e^{it\varphi(x_1)}\int_{-\bar{\delta}_1}^{\bar{\delta}_2} G(v)e^{itv^2}\,dv$$

where $\bar{\delta}_2 = v(x_1 + \delta), -\bar{\delta}_1 = v(x_1 - \delta)$ and $G(v) = g(X(v))/v'(X(v))$, $G(0) = g(x_1)\sqrt{2/\varphi''(x_1)}$.

Proof. We make these substitutions in the integral and change variables to obtain the result.

Step 3

$$\int_{-\bar{\delta}_1}^{\bar{\delta}_2} G(v)e^{itv^2}\,dv = G(0)\int_{-\bar{\delta}_1}^{\bar{\delta}_2} e^{itv^2}\,dv + O(1/t) \qquad t \to \infty$$

Proof. We write $G(v) = G(0) + vh(v)$, which defines the differentiable function $h(v)$. The second term contributes to the integral

$$\begin{aligned}\int_{-\bar{\delta}_1}^{\bar{\delta}_2} vh(v)e^{itv^2}dv &= \frac{1}{2it}\int_{-\bar{\delta}_1}^{\bar{\delta}_2} h(v)d\left(e^{itv^2}\right)\\ &= \frac{1}{2it}\left(h(v)e^{itv^2}\Big|_{\bar{\delta}_1}^{\bar{\delta}_2} - \int_{-\bar{\delta}_1}^{\bar{\delta}_2} h'(v)e^{itv^2}dv\right)\\ &= O\left(\frac{1}{t}\right)\end{aligned}$$

as required.

Step 4

$$\int_{-\bar{\delta}_1}^{\bar{\delta}_2} e^{itv^2}\,dv = \sqrt{\frac{\pi}{t}}e^{i\pi/4} + O\left(\frac{1}{t}\right) \qquad t \to \infty$$

Proof. This is the *Fresnel integral.* Readers familiar with complex-variable methods will find this a one-liner. Apply Cauchy's theorem to the function $f(z) = e^{iz^2}$ on the crescent-shaped contour formed by the ray $z = re^{i\pi/4}, 0 \le r \le R$, the arc of the circle $|z| = R$, and the real axis from $(0,0)$ to $(R,0)$, when $R \to \infty$. We now outline a proof[2] that does not use complex-variable methods.

The qualitative fact of convergence of the improper integral is established by the following partial integration:

$$\begin{aligned}\int_M^N e^{ix^2}\,dx &= \int_M^N \frac{1}{2ix} d(e^{ix^2}) \\ &= \frac{e^{iN^2}}{2iN} - \frac{e^{iM^2}}{2iM} + \frac{1}{2i}\int_M^N \frac{e^{ix^2}}{x^2}\,dx\end{aligned}$$

The final integral is less than or equal to $1/N$, so that the right side tends to zero when $M, N \to \infty$. This proves that the improper integral $\int_0^\infty e^{ix^2}\,dx$ is convergent. Letting $N \to \infty$ shows furthermore that

$$\int_M^\infty e^{ix^2}\,dx = -\frac{e^{iM^2}}{2iM} + \frac{1}{2i}\int_M^\infty \frac{e^{ix^2}}{x^2}\,dx$$

Both terms on the right are $O(1/M)$, so that we have the required speed of convergence:

$$\int_0^M e^{ix^2}\,dx = \int_0^\infty e^{ix^2}\,dx + O(1/M) \qquad M \to \infty$$

It remains to compute the numerical value of the improper integral. To do this, we let $p > 0$ and examine the double integral

$$J_p = \int_0^\infty \int_0^\infty e^{-p(x^2+y^2)} e^{i(x^2+y^2)}\,dx\,dy$$

On the one hand, we can take polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and compute

$$\begin{aligned}J_p &= \int_0^\infty \int_0^{\pi/2} e^{-pr^2} e^{ir^2} r\,dr\,d\theta \\ &= \frac{\pi}{2}\int_0^\infty re^{-r^2(p-i)}\,dr \\ &= \frac{\pi}{2}\frac{1}{2(p-i)}\end{aligned}$$

On the other hand, the double integral is the square of a single integral:

$$J_p = \left(\int_0^\infty e^{-px^2} e^{ix^2}\,dx\right)^2$$

[2]For further details, see Harley Flanders, "On the Fresnel Integrals," *American Mathematical Monthly*, vol. 89, 1982, pp. 264–266.

Letting $I = \int_0^\infty e^{ix^2}\,dx$, we conclude that

$$I^2 = \lim_{p\to 0} J_p = \frac{\pi i}{4}$$

But the complex number I has both positive real and imaginary parts, so that the appropriate square root is

$$I = \sqrt{\frac{\pi}{4}}e^{i\pi/4}$$

To apply this to step 4, write

$$\int_0^\delta e^{itv^2}\,dv = \frac{1}{\sqrt{t}}\int_0^{\delta\sqrt{t}} e^{ix^2}\,dx = \frac{1}{\sqrt{t}}\left[I + O\left(\frac{1}{\sqrt{t}}\right)\right] \qquad t\to\infty$$

The equality of the two limits follows from Abel's lemma (see Appendix A.2), which completes the proof.

EXERCISES 6.4

Apply the method of stationary phase to find asymptotic formulas for the following functions when $t \to \infty$:

1. $f(t) = \int_{-2}^{2} e^{it\mu^2}\,d\mu$
2. $f(t) = \int_{-1}^{1} e^{it\cos\theta}\cos^2\theta\,d\theta$
3. $f(t) = \int_0^{\pi/2} e^{it\cos\theta}\cos\theta\,d\theta$
4. $f(t) = \int_{-\pi}^{\pi} e^{it\cos\theta}e^{-im\theta}\,d\theta$
5. Modify the method of stationary phase to find an asymptotic formula for the integral

$$f(t) = \int_{-1}^{1} e^{it\mu^4}\,d\mu$$

6. Suppose that $g(0) = 0$ and $g'(0) \neq 0$. Show how to modify the method of stationary phase to obtain an asymptotic formula for the integral

$$f(t) = \int_{-1}^{1} e^{it\mu^2}g(\mu)\,d\mu$$

7. Suppose that $g(0) = 0$, $g'(0) = 0$ and $g''(0) \neq 0$. Show how to modify the method of stationary phase to obtain an asymptotic formula for the integral

$$f(t) = \int_{-1}^{1} e^{it\mu^2}g(\mu)\,d\mu$$

6.5. Asymptotic Expansions[3]

In the preceding sections we have used integration by parts and Laplace's method to obtain one-term asymptotic formulas of the form

$$\int_a^b g(x)e^{th(x)}\,dx = \frac{e^{tH}}{t^\alpha}\left[C + O\left(\frac{1}{t^\alpha}\right)\right] \tag{6.5.1}$$

In this section we indicate the steps necessary to obtain an *asymptotic expansion*, which is displayed as

$$\int_a^b g(x)e^{th(x)}\,dx = \frac{e^{tH}}{t^\alpha}\left[C_0 + \frac{C_1}{t} + \cdots + \frac{C_N}{t^N} + O\left(\frac{1}{t^{N+1}}\right)\right] \quad t \to \infty$$

We will indicate separately the steps for each of the two methods.

6.5.1. Extension of integration by parts. If $h'(x) \neq 0$ for $a \le x \le b$, we can write

$$\int_a^b g(x)e^{th(x)}\,dx = \frac{g(x)e^{th(x)}}{th'(x)}\bigg|_a^b - \frac{1}{t}\int_a^b e^{th(x)}\,d(g/h')'\,dx$$

The second integral is of the same form as the first, with g replaced by $(g/h')'$. By repeating this procedure N times, we can obtain N terms in the asymptotic expansion of the integral. We illustrate the method with the classical *exponential integral.*

EXAMPLE 6.5.1. *Find an asymptotic expansion when $t \to \infty$ for the function defined by the integral*

$$f(t) = \int_t^\infty \frac{e^{-y}}{y}\,dy$$

Solution. We first put this in standard form by defining a new integration variable $y = tx$. Thus

$$\begin{aligned} f(t) &= \int_1^\infty \frac{e^{-tx}}{x}\,dx \\ &= \frac{e^{-t}}{t} - \frac{1}{t}\int_1^\infty \frac{e^{-tx}}{x^2}\,dx \\ &= \frac{e^{-t}}{t} - \frac{e^{-t}}{t^2} + \frac{2}{t^2}\int_1^\infty \frac{e^{-tx}}{x^3}\,dx \\ &= \frac{e^{-t}}{t} - \frac{e^{-t}}{t^2} + \frac{2e^{-t}}{t^3} - \frac{3!}{t^3}\int_1^\infty \frac{e^{-tx}}{x^4}\,dx \end{aligned}$$

[3]This section is optional. The results are not used in the following sections.

Each partial integration produces a new power of $1/t$ together with a new integral of the same order. Continuing inductively, we obtain

$$f(t) = \frac{e^{-t}}{t}\left[1 - \frac{1}{t} + \frac{2}{t^2} - \frac{6}{t^3} + \cdots + \frac{(-1)^N N!}{t^N} + O\left(\frac{1}{t^{N+1}}\right)\right] \quad \bullet$$

The next example is related to the Gauss-Weierstrass integral of Chapter 5.

EXAMPLE 6.5.2. *Find an asymptotic expansion when $t \to \infty$ for the function defined by the integral*

$$f(t) = \int_t^\infty e^{-y^2/2}\,dy$$

Solution. This can be done by reducing to the standard form using the substitution $y = tx$. The same results can be obtained directly, as follows:

$$\begin{aligned}\int_t^\infty e^{-y^2/2}\,dy &= -\int_t^\infty \frac{1}{y}d(e^{-y^2/2}) \\ &= \frac{1}{t}e^{-t^2/2} - \int_t^\infty \frac{e^{-y^2/2}}{y^2}\,dy\end{aligned}$$

In order to effect the extension, we note that for any $k = 1, 2, \ldots,$

$$\begin{aligned}\int_t^\infty \frac{1}{y^{2k}}e^{-y^2/2}\,dy &= -\int_t^\infty \frac{1}{y^{2k+1}}d\left(e^{-y^2/2}\right) \\ &= \frac{e^{-t^2/2}}{t^{2k+1}} - (2k+1)\int_t^\infty \frac{e^{-y^2/2}}{y^{2k+2}}\,dy\end{aligned}$$

Applying this repeatedly leads to the asymptotic expansion

$$f(t) = \frac{e^{-t^2/2}}{t}\left[1 - \frac{1}{t^2} + \frac{3}{t^4} - \frac{15}{t^6} + \cdots + (-1)^N\frac{1\cdot 3\cdots(2N-1)}{t^{2N}} + O\left(\frac{1}{t^{2N+2}}\right)\right] \quad \bullet$$

6.5.2. Extension of Laplace's method. We now turn to Laplace's method, with the goal of an asymptotic expansion. To simplify the exposition, we assume a single maximum at the point $x = 0$, where $a < 0 < b$, $h(0) = H$, $h''(0) = h_2 < 0$. Given $\epsilon > 0$, we choose $\delta > 0$ so that $h(x) < H - \epsilon$ for $|x| > \delta$. The contribution to the integral from $|x| > \delta$ is $O(e^{t(H-\epsilon)})$ and can therefore be ignored; in detail,

$$\int_a^b g(x)e^{th(x)}\,dx = \int_{-\delta}^{\delta} g(x)e^{th(x)}\,dx + O(e^{t(H-\epsilon)}) \tag{6.5.2}$$

In order to determine the asymptotic expansion, we introduce the Taylor expansions of $g(x)$ and $h(x)$:

$$\begin{aligned}g(x) &= g_0 + xg_1 + \cdots + \frac{x^N}{N!}g_N + O(x^{N+1}) \qquad x \to 0 \\ h(x) &= H + \frac{x^2h_2}{2} + \cdots + \frac{x^N}{N!}h_N + O(x^{N+1}) \qquad x \to 0\end{aligned}$$

We substitute these into (6.5.2) and make the change of variable $y = x\sqrt{t}$, with the result

$$\int_{-\delta}^{\delta} g(x)e^{th(x)}\,dx = \frac{e^{tH}}{\sqrt{t}}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} e^{h_2y^2/2}\left(\sum_{k\geq 0}\frac{g_ky^k}{k!t^{k/2}}\right)\exp\left(\sum_{k\geq 3}\frac{h_ky^k}{k!t^{(k-2)/2}}\right)dy$$

We expand the second exponential in a power series in $1/t$, multiply by the first sum, and collect powers of $1/t$. The first three terms of the resulting expression have the form

$$\frac{e^{tH}}{\sqrt{t}}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} e^{h_2y^2/2}\left(g_0+\frac{yg_1+y^3h_3/6}{\sqrt{t}}+\frac{1}{t}\left[\frac{y^2g_2}{2}+\frac{y^4g_1h_3}{6}+\frac{y^4g_0h_4}{24}+\frac{y^6g_0h_3^2}{72}\right]\right)dy$$

The higher terms are similar, invoving the known integrals

$$\int_{-\infty}^{\infty} y^ne^{-y^2/2}\,dy = \begin{cases} 0 & n \text{ odd} \\ \sqrt{2\pi}\;1\cdot 3\cdot 5\cdots(2m-1) & n = 2m \end{cases}$$

Using these we may compute as many terms as are desired in an asymptotic expansion. The fractional powers of $1/t$ are multiplied by odd powers of y, and hence do not appear in the final result. We illustrate the method with an improved version of Stirling's formula.

EXAMPLE 6.5.3. *Find two terms in an asymptotic expansion of the integral*

$$f(t) = \int_0^{\infty} e^{t(\log x - x)}\,dx$$

Solution. In this case we have $g(x) = 1$, $h(x) = \log x - x$, with a single global maximum at $x = 1$, where $H = h(1) = -1$, $h''(1) = -1$, and the Taylor expansion

$$h(x) = -1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{N-1}\frac{(x-1)^N}{N} + O((x-1)^{N+1})$$

Hence $g_0 = 1$, $g_k = 0$ for $k = 1, 2, \ldots$ and $h_3 = 2$, $h_4 = -6$. Following the steps above, we are led to the integral

$$\frac{e^{-t}}{\sqrt{t}}\int_{-\delta\sqrt{t}}^{\delta\sqrt{t}} e^{-y^2/2}\left(1+\frac{y^3/3}{\sqrt{t}}+\frac{1}{t}\left[-\frac{6y^4}{24}+\frac{4y^6}{72}\right]\right)dy = \sqrt{2\pi}\frac{e^{-t}}{\sqrt{t}}\left[1+\frac{1}{12t}\right] \quad \bullet$$

This result may be applied to Stirling's formula as follows. For $n = 1, 2, \ldots,$

$$n! = \int_0^{\infty} u^ne^{-u}\,du$$

We make the change of variable $u = nx$, $du = n\,dx$ to obtain

$$\begin{aligned} n! &= n^{n+1}\int_0^\infty x^n e^{-nx}\,dx \\ &= n^{n+1}\int_0^\infty e^{n\log x}e^{-nx}\,dx \\ &= n^{n+1}f(n) \end{aligned}$$

Thus we have the following improved version of Stirling's formula:

$$n! = \sqrt{2\pi}\frac{n^{n+(1/2)}}{e^n}\left[1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right] \qquad n\to\infty$$

This shows that $n!$ is larger than the Stirling approximation, for large n.

EXERCISES 6.5

1. Show that the asymptotic expansion of Stirling's integral for $n!$ contains only *integral* powers of n^{-1}.
2. Obtain the coefficient of $1/n^2$ in Stirling's formula.
3. Use integration by parts to show that, for $t > 0$,

$$\left(\frac{1}{t}-\frac{1}{t^2}\right) \le \int_t^\infty e^{-u^2/2}\,du \le \frac{1}{t}e^{-u^2/2}$$

4. Obtain an asymptotic expansion of the exponential integral

$$f(t) = \int_t^\infty \frac{e^{-u}}{u}\,du \qquad \text{when} \quad t\to\infty$$

5. Obtain three terms of the asymptotic expansion of

$$f(t) = \int_0^\infty \frac{e^{-tx}}{1+x^2}\,dx \qquad \text{when} \quad t\to\infty$$

6.6. Asymptotic Analysis of the Telegraph Equation

In this section we will apply the asymptotic methods to the initial-value problem for the telegraph equation. This example illustrates all of the methods discussed in this chapter. The wave equation is included as a special case.

The general initial-value problem for the telegraph equation is written

$$u_{tt} + 2\beta u_t + \alpha u = c^2 u_{xx} \tag{6.6.1}$$

$$u(x;0) = f_1(x) \tag{6.6.2}$$

$$u_t(x;0) = f_2(x) \tag{6.6.3}$$

The initial data $f_1(x)$, $f_2(x)$ are assumed to be integrable functions with integrable Fourier transforms.

In Chapter 5 it was found that the form of the solution depends on whether $\beta^2 < \alpha$, $\beta^2 = \alpha$, or $\beta^2 > \alpha$. Furthermore, it was found that the new function $e^{\beta t}u(x;t)$ satisfies a telegraph equation with β replaced by zero and α replaced by $\alpha - \beta^2$. Therefore it constitutes no loss of generality to suppose that $\beta = 0$ and to consider separately the cases $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$.

6.6.1. Asymptotic behavior in case $\alpha = 0$. The simplest case is $\alpha = 0$, where we have the wave equation $u_{tt} = c^2 u_{xx}$, whose general solution was found in Sec. 5.3:

$$u(x;t) = \frac{f_1(x+ct) + f_1(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(y)\,dy$$

With the given hypotheses on f_1, f_2, it follows that $\lim_{|x|\to\infty} f_1(x) = 0$ and that the improper integral of f_2 exists. Combining these facts, it is possible to determine the following asymptotic behavior of the solution of the wave equation, as we *move along* at various speeds:

$$\lim_{t\to\infty} u(x+vt;t) = \begin{cases} 0 & \text{if } v > c \\ f_1(x)/2 + (1/2c)\int_x^\infty f_2(y)dy & \text{if } v = c \\ (1/2c)\int_{-\infty}^\infty f_2(y)dy & \text{if } -c < v < c \\ f_1(x)/2 + (1/2c)\int_{-\infty}^x f_2(y)dy & \text{if } v = -c \\ 0 & \text{if } v < -c \end{cases}$$

We note that the result is a constant function of x when we move below the wave speed, $|v| < c$, and that the result is zero when we move above the wave speed, $|v| > c$. The most interesting result is obtained when we move at the wave speed, $v = \pm c$, when the limit depends on x as written. This behavior will be used as a guide in discussing the telegraph equation in case $\alpha > 0$.

6.6.2. Asymptotic behavior in case $\alpha > 0$. In Chapter 5 the Fourier representation of the solution of the initial-value problem was found to be

$$\begin{aligned} u(x;t) &= \int_{-\infty}^{\infty} F_1(\mu)\cos t\sqrt{\alpha + (c\mu)^2} e^{i\mu x}\,d\mu \\ &+ \int_{-\infty}^{\infty} F_2(\mu)\frac{\sin t\sqrt{\alpha+(c\mu)^2}}{\sqrt{\alpha+(c\mu)^2}} e^{i\mu x}\,d\mu \end{aligned}$$

Each of these integrals can be analyzed by the method of stationary phase in order to produce an asymptotic result. In detail, we write

$$u(x+vt;t) = \int_{-\infty}^{\infty} e^{it\phi_+(\mu)}F(\mu)e^{i\mu x}\,d\mu + \int_{-\infty}^{\infty} e^{it\phi_-(\mu)}G(\mu)e^{i\mu x}\,d\mu$$

where

$$F(\mu) = \frac{F_1(\mu)}{2} + \frac{F_2(\mu)}{2i\sqrt{\alpha+(c\mu)^2}}, \qquad G(\mu) = \frac{F_1(\mu)}{2} - \frac{F_2(\mu)}{2i\sqrt{\alpha+(c\mu)^2}}$$

and

$$\phi_+(\mu) = v\mu + \sqrt{\alpha + (c\mu)^2}, \qquad \phi_-(\mu) = v\mu - \sqrt{\alpha + (c\mu)^2}$$

To examine the first integral, we must determine the stationary points of the phase function $\phi_+(\mu)$. Computing the first and second derivatives, we have

$$\begin{aligned} \phi'_+(\mu) &= v + \frac{c^2\mu}{\sqrt{\alpha + (c\mu)^2}} \\ \phi''_+(\mu) &= \frac{c^2\alpha}{[\alpha + (c\mu)^2]^{3/2}} \end{aligned}$$

If $|v| \geq c$, then the first term of $\phi'_+(\mu)$ is larger in absolute value than the second term, so there is no stationary point. In this case a simple integration by parts shows that

$$\boxed{u(x+vt;t) = O(1/t) \qquad t \to \infty \qquad |v| \geq c}$$

If $|v| < c$, we may solve the equation $\phi'_+(\mu) = 0$ to obtain the stationary point

$$\mu^+(v) = -\frac{v}{c}\sqrt{\frac{\alpha}{c^2 - v^2}}$$

Computing the phase function at the stationary point, we have

$$\phi_+(\mu^+(v)) = \frac{1}{c}\sqrt{\alpha(c^2 - v^2)}, \qquad \phi''_+(\mu^+(v)) = \frac{1}{c}\sqrt{\frac{(c^2 - v^2)^3}{\alpha}}$$

Applying the method of stationary phase gives the result

$$\int_{-\infty}^{\infty} e^{it\phi_+(\mu)} F(\mu) e^{i\mu x}\, d\mu = \sqrt{\frac{2\pi}{t\phi''_+(\mu^+(v))}} e^{it\phi_+(\mu^+(v))} e^{i\pi/4} F(\mu^+(v)) e^{i\mu^+(v)x} + O\left(\frac{1}{t}\right)$$

To examine the second integral, we must determine the stationary points of the phase function $\phi_-(\mu)$. Computing the first and second derivatives, we have

$$\begin{aligned} \phi'_-(\mu) &= v - \frac{c^2\mu}{\sqrt{\alpha + (c\mu)^2}} \\ \phi''_-(\mu) &= -\frac{c^2\alpha}{[\alpha + (c\mu)^2]^{3/2}} \end{aligned}$$

If $|v| \geq c$, then the first term of $\phi'_-(\mu)$ is larger in absolute value than the second term, so there is no stationary point. If $|v| < c$, we may solve the equation $\phi'_-(\mu) = 0$ to obtain the stationary point

$$\mu^-(v) = \frac{v}{c}\sqrt{\frac{\alpha}{c^2 - v^2}}$$

Computing the phase function at the stationary point, we have

$$\phi_-(\mu^-(v)) = -\frac{1}{c}\sqrt{\alpha(c^2-v^2)}, \qquad \phi''_+(\mu^+(v)) = -\frac{1}{c}\sqrt{\frac{(c^2-v^2)^3}{\alpha}}$$

Applying the method of stationary phase gives the result

$$\int_{-\infty}^{\infty} e^{it\phi_+(\mu)}F(\mu)e^{i\mu x}\,d\mu = \sqrt{\frac{2\pi}{t|\phi''_+(\mu^+(v))|}}e^{it\phi_-(\mu^+(v))}e^{-i\pi/4}G(\mu^+(v))e^{i\mu^+(v)x} + O\left(\frac{1}{t}\right)$$

Adding the two results gives

$$\boxed{u(x+vt;t) = u_+(x;t) + u_-(x;t) + O\left(1/t\right) \qquad |v| < c}$$

where

$$\begin{aligned}
u_+(x;t) &= \sqrt{\frac{2\pi}{t|\phi''_+(\mu^+(v))|}}e^{i(\omega t+\pi/4+\mu_+(v)x)}\left[\frac{F_1(\mu_+(v))}{2} + \frac{F_2(\mu_+(v))}{2i\sqrt{\alpha+(c\mu_+(v))^2}}\right] \\
u_-(x;t) &= \sqrt{\frac{2\pi}{t|\phi''_+(\mu^+(v))|}}e^{-i(\omega t+\pi/4+\mu_+(v)x)}\left[\frac{F_1(\mu_-(v))}{2} - \frac{F_2(\mu_-(v))}{2i\sqrt{\alpha+(c\mu_-(v))^2}}\right] \\
\omega &= \phi_+(\mu^+(v)) = \frac{1}{c}\sqrt{\alpha(c^2-v^2)}
\end{aligned}$$

The physical interpretation of these results is that, although there is no definite wave speed (as for the wave equation), one can say that when $t \to \infty$ the largest values of the solution are within the boundaries $x \pm ct$. Thus *most* of the wave motion is contained within these boundaries. Historically, this resolved a paradoxical situation, since the Fourier representation is a sum of "plane waves" in the form $e^{i(\mu x \pm t\sqrt{\alpha+(c\mu)^2})}$, where the *phase velocity* $v = \sqrt{\alpha+(c\mu)^2}/\mu$ is strictly larger than c, which would violate the principles of special relativity theory. The method of stationary phase shows that we have no contradiction with special relativity theory.

6.6.3. Asymptotic behavior in case $\alpha < 0$. This case is more complex, since the Fourier representation involves two subcases. In detail, we have $u(x;t) = u_I(x;t) + u_{II}(x;t)$, where

$$\begin{aligned}
u_I(x;t) &= \int_{c|\mu|>\sqrt{|\alpha|}} F_1(\mu)\cos t\sqrt{\alpha+(c\mu)^2}e^{i\mu x}\,d\mu \\
&\quad + \int_{c|\mu|>\sqrt{|\alpha|}} F_2(\mu)\,\frac{\sin t\sqrt{\alpha+(c\mu)^2}}{\sqrt{\alpha+(c\mu)^2}}e^{i\mu x}\,d\mu
\end{aligned}$$

$$\begin{aligned} u_{II}(x;t) &= \int_{c|\mu|<\sqrt{|\alpha|}} F_1(\mu)\cosh t\sqrt{|\alpha|-(c\mu)^2}e^{\imath\mu x}\,d\mu \\ &+ \int_{c|\mu|<\sqrt{|\alpha|}} F_2(\mu)\,\frac{\sinh t\sqrt{|\alpha|-(c\mu)^2}}{\sqrt{|\alpha|-(c\mu)^2}}e^{\imath\mu x}\,d\mu \end{aligned}$$

We will now show that u_I is bounded for large t, while u_{II} grows exponentially, according to Laplace's method. To see this, note that

$$|u_I(x;t)| \le \int_{c|\mu|>\sqrt{|\alpha|}} |F_1(\mu)|\,d\mu + \int_{c|\mu|>\sqrt{|\alpha|}} \frac{|F_2(\mu)|}{\sqrt{\alpha+(c\mu)^2}}\,d\mu$$

To analyze u_{II}, write $2\sinh\theta = e^{\theta}-e^{-\theta}$, $2\cosh\theta = e^{\theta}+e^{-\theta}$, so that the negative exponential terms contribute at most $O(e^{-t\sqrt{|\alpha|}})$, $t\to\infty$. The remaining integrals can be written in the form

$$\int_{c|\mu|<\sqrt{|\alpha|}} e^{th(\mu)}e^{\imath\mu x}F(\mu)\,d\mu$$

where

$$\begin{aligned} h(\mu) &= \sqrt{|\alpha|-(c\mu)^2} \\ F(\mu) &= \frac{F_1(\mu)}{2}+\frac{F_2(\mu)}{2\sqrt{|\alpha|-(c\mu)^2}} \end{aligned}$$

We have

$$h'(\mu) = -\frac{c^2\mu}{\sqrt{|\alpha|-(c\mu)^2}},\quad h''(\mu) = -\frac{c^2|\alpha|}{[|\alpha|-(c\mu)^2]^{3/2}}$$

so that $h(\mu)$ has a global maximum at $\mu=0$, where $h''(0)<0$. Laplace's method gives the asymptotic formula

$$\int_{c|\mu|<\sqrt{|\alpha|}} e^{th(\mu)}e^{\imath\mu x}F(\mu)\,d\mu = \sqrt{\frac{2\pi}{-th''(0)}}e^{t\sqrt{|\alpha|}}\left[F(0)+O\left(\frac{1}{t}\right)\right]$$

Therefore we have shown that

$$\boxed{u(x;t) = t^{-1/2}e^{t\sqrt{|\alpha|}}\left[C_2+O\left(t^{-1/2}\right)\right]}$$

where

$$C_2 = \frac{\sqrt{2\pi}|\alpha|^{1/4}F(0)}{c}$$

In case $v\ne 0$, the integral defining $u(x+vt;t)$ comes in two parts. As above, we need only consider the positive exponential terms coming from u_{II}; thus

$$u(x+vt;t) = \int_{c|\mu|<\sqrt{|\alpha|}} e^{t[h(\mu)+iv\mu]}F(\mu)e^{i\mu x}\,dx + O(1) \qquad t\to\infty$$

The new exponent is a complex-valued function whose derivative is never zero and whose real part has a maximum at $\mu = 0$, where $h(0) = \sqrt{|\alpha|}$. Therefore we can extend the method of integration by parts, to conclude that this integal is $O(e^{t\sqrt{|\alpha|}}/t)$, $t \to \infty$. Combining this with the (bounded) contribution from $u_I(x;t)$, we conclude that

$$\boxed{u(x+vt;t) = O\left(t^{-1/2}e^{t\sqrt{|\alpha|}}\right) \quad t \to \infty} \tag{6.6.4}$$

This completes the asymptotic analysis of the telegraph equation.

EXERCISES 6.6

1. Use the asymptotic result for $\alpha = 0$ to find an asymptotic formula for the solution of the *critically damped* telegraph equation $u_{tt}+2\beta u_t+\beta^2 u = c^2 u_{xx}$ with $u(x;0) = f_1(x)$, $u_t(x;0) = f_2(x)$.
2. Use the asymptotic result in case $\alpha > 0$ to show that if $u(x;t)$ is the solution of the equation $u_{tt} + m^2 u = c^2 u_{xx}$ with $u(x;0) = 0$, $u_t(x;0) = f_2(x)$, then
$$u(x;t) = \frac{C_1}{\sqrt{t}}\sin(mt+\pi/4) + O\left(\frac{1}{t}\right) \qquad t \to \infty$$
where $C_1 = 1/\sqrt{2\pi mc^2}\int_{-\infty}^{\infty} f_2(x)\,dx$.
3. Use the asymptotic result in case $\alpha > 0$ to find an asymptotic formula for $u(x;t)$, the solution of the equation $u_{tt}+m^2 u = c^2 u_{xx}$ with $u(x;0) = f_1(x)$, $u_t(x;0) = 0$.
4. Use the asymptotic result in case $\alpha < 0$ to show that if $u(x;t)$ is the solution of the equation $u_{tt} + 2\beta u_t = c^2 u_{xx}$ with $\beta > 0$ and $u(x;0) = 0$, $u_t(x;0) = f_2(x)$, then we have the asymptotic formulas
$$\begin{aligned} u(x;t) &= \frac{C_2}{\sqrt{t}} + O\left(\frac{1}{t}\right) \qquad t \to \infty \\ u(x+vt;t) &= O\left(\frac{1}{t}\right) \qquad t \to \infty, v \neq 0 \end{aligned}$$
for an appropriate constant C_2.
5. Use the asymptotic result in case $\alpha < 0$ to find asymptotic formulas for $u(x;t)$ and $u(x+vt;t)$, where u is the solution of the equation $u_{tt}+2\beta u_t = c^2 u_{xx}$ with $\beta > 0$ and $u(x;0) = f_1(x)$, $u_t(x;0) = 0$.
6. Suppose that $g(\mu), h(\mu)$ are differentiable functions and that $h(\mu)$ is real-valued with a global maximum at $\mu = 0$. Show that if $v \neq 0$, when $t \to \infty$,
$$\int_{-A}^{A} g(\mu)e^{t[h(\mu)+iv\mu]}\,d\mu = O\left(\frac{e^{th(0)}}{t}\right)$$

7. Let $u(x;t)$ be the solution of the equation $u_{tt} + 2\beta u_t = c^2 u_{xx}$ with $\beta > 0$ and $u(x;0) = 0$, $u_t(x;0) = f_2(x)$. Combine the methods of this section with the *gaussian approximation* from Sec. 6.3, Eq. (6.3.12), to show that we have an asymptotic formula of the form

$$u(x;t) = B_0 \frac{e^{-(x-b)^2/4(B_1+B_2 t)}}{\sqrt{4\pi(B_1 + B_2 t)}} + O\left(\frac{1}{t^2}\right) \qquad t \to \infty$$

for suitable constants B_0, B_1, B_2. [*Hint:* Write the integrand in the form $F(\mu)e^{i\mu x} = Ae^{i\mu(x-b)-\sigma^2\mu^2/2+O(\mu^3)}$ and replace the integral by a suitable gaussian integral.]

CHAPTER 7

NUMERICAL ANALYSIS

INTRODUCTION

In this chapter we present methods for obtaining approximate numerical solutions of certain boundary-value problems for partial differential equations. In practice, this may be necessary for a variety of reasons, for example, the geometry of the boundary or the explicit form of the coefficients in the equation. Even if the problem admits an explicit solution by a series or integral, it may be difficult to compute the numerical value of the solution using the series or integral.

The most widely used numerical method is the method of finite differences, which is discussed for both the heat equation and the Laplace equation in Secs. 7.2 and 7.3. Then we turn to methods that are suggested by the calculus of variations, including the methods of Ritz and Kantorovich. These lead naturally to the so-called orthogonality method of Galerkin, which in turn leads to the finite element method.

In order to orient the chapter, we include an initial section on some elementary finite difference methods for ordinary differential equations, including an error analysis.

7.1. Numerical Analysis of Ordinary Differential Equations

Most differential equations do not have explicit solutions. Even if a differential equation has an explicit solution, the solution may be so complicated that it is useless for practical purposes.

On the other hand, we can try to find an approximation to the solution of a differential equation with an initial condition. For example, suppose we are given a first-order initial-value problem

$$y' = f(t, y), \quad y(a) = y_0 \tag{7.1.1}$$

on an interval $a \le t \le b$. Although we may not be able to obtain a formula for the solution of (7.1.1), we can subdivide the interval as

$$a = t_0 < t_1 < \cdots < t_N = b$$

and try to assign approximate values Y_n to t_n for $n = 1, \ldots, N$. Instead of a formula we will have an approximation to the solution of (7.1.1) expressed as a table of the Y_n's in terms of the t_n's. By graphing the table, we can visualize the solution.

We begin by considering simple methods of numerical approximations to solutions to differential equations. The simplest of all is the Euler method; the next simplest is the Heun method, which is derived from the Euler method. Subsection 7.1.3 discusses the error analysis, with a complete proof in the case of Euler's method.

7.1.1. The Euler method. The Euler method is the simplest method used to find numerical solutions to differential equations. The objective is to construct an approximation to the solution of the initial-value problem (7.1.1) for $a \le t \le b$. We divide the interval $a \le t \le b$ into equal subintervals:

$$a = t_0 < t_1 < \cdots < t_N = b$$

where $h = t_{n+1} - t_n$ is called the *step size*. Let us first suppose that (7.1.1) has a solution $y(t)$, which we assume to be twice differentiable. We need the finite Taylor expansion

$$y(t_1) = y(t_0) + (t_1 - t_0)y'(t_0) + \frac{(t_1 - t_0)^2}{2}y''(\xi_0) \tag{7.1.2}$$

where $t_0 < \xi_0 < t_1$. Using (7.1.1), we can rewrite (7.1.2) as

$$y(t_1) = y(t_0) + h\,f(t_0, y_0) + \frac{h^2}{2}y''(\xi_0) \tag{7.1.3}$$

Now if h is a positive number less than 1, the quantity h^2 is even smaller. Therefore, we have the approximate equality

$$y(t_1) \approx y(t_0) + hf(t_0, y_0) \tag{7.1.4}$$

The initial condition of (7.1.1) is $y(t_0) = y_0$. We can use this as the starting point of a sequence of numerical approximations (Y_n) by taking $Y_0 = y_0$. However, there may be roundoff errors or other inaccuracies that suggest a more general approach, beginning with a value Y_0, not necessarily equal to y_0. Let us now *define* Y_1 by

$$Y_1 = Y_0 + hf(t_0, Y_0) \tag{7.1.5}$$

Then Y_1 is an approximation to $y(t_1)$. Furthermore, if we put

$$Y_2 = Y_1 + hf(t_1, Y_1)$$

then we can expect Y_2 to be a reasonable approximation of $y(t_2)$. More generally, we define

$$Y_{n+1} = Y_n + hf(t_n, Y_n) \tag{7.1.6}$$

for $0 \le n \le N - 1$. The *Euler method* consists of approximating the solution to (7.1.1) by means of (7.1.6). From (7.1.1) we know the point (t_0, Y_0) and the slope of the tangent line to the solution curve at (t_0, Y_0), namely, $f(t_0, Y_0)$. Thus the tangent line is the graph of

$$t \longmapsto Y_0 + (t - t_0)f(t_0, Y_0)$$

To see how the Euler method works, we use it to approximate the solution to a simple first-order linear equation.

EXAMPLE 7.1.1. *Use the Euler method to find an approximate solution to the initial-value problem*

$$y' = y + 1, \quad y(0) = 0 \tag{7.1.7}$$

for $0 \le t \le 1$. *Use the step size* $h = 0.1$ *and compare the approximation with the exact solution.*

Solution. The Euler method formula (7.1.6) for the initial-value problem (7.1.7) with step size $h = 0.1$ becomes

$$Y_{n+1} = Y_n + 0.1(Y_n + 1)$$

for $0 \le n \le N - 1$. The interval $0 \le t \le 1$ is divided into 10 equal pieces, so $N = 10$. We are given $Y_0 = 0$; then $Y_1 = Y_0 + 0.1(Y_0 + 1) = 0.1$ and

$$Y_2 = Y_1 + 0.1(Y_1 + 1) = 0.1 + 0.1(1.1) = 0.21$$

Continuing in this way, we get the first three columns in the following table:

n	t_n	Y_n	$y_{\text{exact}}(t_n)$
0	0.0	0.0	0.0
1	0.1	0.1	0.10517
2	0.2	0.21	0.22140
3	0.3	0.331	0.34986
4	0.4	0.4641	0.49182
5	0.5	0.61050	0.64872
6	0.6	0.77156	0.82212
7	0.7	0.94871	1.01375
8	0.8	1.14359	1.22554
9	0.9	1.35795	1.45960
10	1.0	1.59370	1.71828

Since $y' = y + 1$ is a first-order linear differential equation, the exact solution of (7.1.7) is easily found to be

$$y_{\text{exact}}(t) = e^t - 1$$

Then $y_{\text{exact}}(0.1) = e^{0.1} - 1 \approx 0.10517$, $y_{\text{exact}}(0.2) = e^{0.2} - 1 \approx 0.22140$, and so forth. Values for the exact solution are shown in the fourth column. The difference $|Y_{10} - y(t_{10})|$ is approximately 0.12, an error of less than 10 percent. •

Next let us consider a differential equation for which it is impossible to find the solution, at least by elementary techniques.

EXAMPLE 7.1.2. *Use the Euler method to find an approximate solution to*

$$y' = 5 - t^2 y^3, \quad y(0) = 0 \tag{7.1.8}$$

for $0 \le t \le 1$. *Use step sizes* $h = 0.1$ *and* $h = 0.01$ *and compare the results.*

Solution. The Euler method formula (7.1.6) for the initial-value problem (7.1.8) with step size h is

$$Y_{n+1} = Y_n + h\big(5 - t_n^2 Y_n^3\big) \tag{7.1.9}$$

Taking $h = 0.01$ and then $h = 0.001$ in (7.1.9) gives us the following table:

t	Y_n $(h = 0.1)$	Y_n $(h = 0.01)$
0.0	0.0	0.0
0.1	0.50000	0.49998
0.2	0.99987	0.99886
0.3	1.49588	1.48647
0.4	1.96575	1.92540
0.5	2.34422	2.24190
0.6	2.52216	2.36963
0.7	2.44457	2.32246
0.8	2.22875	2.18041
0.9	2.02021	2.01484
1.0	1.85237	1.86018

●

EXERCISES 7.1.1

In the following exercises, perform the indicated computations by hand, retaining only four significant digits at each step of the calculations.

1. Use the Euler method to compute an approximate solution to the initial-value problem $y' = \sqrt{y+1}, y(0) = 0$ with step size $h = 0.1$. Compare with the exact solution $y(t) = t(4+t)/4$.
2. Consider the following initial-value problem:

$$\begin{cases} y' = y - t\,y^2 \\ y(0) = 2 \end{cases}$$

(a) Use the Euler method to compute an approximate solution at $t = 1.0$. Use the values $n = 2$, 4, 8, which correspond to step sizes of $h = 1/2$, 1/4, 1/8.

(b) Determine the actual solution and compare the value of $y(1)$ with the results from part (a).

Repeat Exercise 2 for the following initial-value problems:

3. $\begin{cases} y' + 2y = t \\ y(0) = 1 \end{cases}$
4. $\begin{cases} y' = t + 1 \\ y(0) = 1 \end{cases}$

5. $\begin{cases} y' = t^3 e^{-2y} \\ y(0) = 0 \end{cases}$

6. $\begin{cases} yy' = e^t \\ y(0) = 1 \end{cases}$

7. Consider the following initial value problem:

$$y' = y + t^2, \qquad y(0) = 1$$

(a) Find the solution $y(t)$ and evaluate it for $t = 0.2, 0.4, \ldots, 1.0$.

(b) Using the Euler method, with step size of $h = 0.2$, find approximate values for the solution at the t values in part (a).

(c) Repeat part (b) by using $h = 0.1$.

(d) Compare the results of part (b) with those of part (c) and the exact values.

The initial-value problems in the following exercises cannot be solved by symbolic methods. Use the Euler method with step size $h = 0.1$ to approximate the solution at $t = 1$ to three significant digits (see Exercise 7).

8. $y' = \sin(y) + e^t, y(0) = 0$
9. $y' = y^{3/2} + t, y(0) = 1$
10. $y' = \sin(t) + \cos(y), y(0) = 1$
11. $y' = e^{t^3}, y(0) = 1$
12. Consider the initial-value problem

$$y' = t^2 y, \qquad y(0) = 1$$

(a) Find the exact solution at $t = 1.0$. Express this value to four decimal places.

(b) Use the Euler method with $h = 1/8$ to approximate the solution at $t = 1.0$. Compute the absolute error.

(c) Repeat part (b) with $h = 1/16$, $h = 1/32$. Create a table and a graph showing the absolute errors corresponding to the various step sizes. A theoretical analysis for the Euler method suggests a linear relationship between the absolute error and the step size. Do the numbers agree with the theory?

(d) Observe in part (c) that the error is roughly proportional to the step size. Use these data to estimate the constant of proportionality.

13. Apply the Euler method with successively smaller step sizes on the interval $0 \le t \le 1$ to verify empirically that the solution of the initial-value problem

$$y' = t^2 + y^2, \quad y(0) = 1$$

has a vertical asymptote near $t = 0.97$.

7.1.2. The Heun method. As a first attempt to obtain an improvement of the Euler method formula (7.1.6), let us replace $f(t_n, Y_n)$ with the *average* of $f(t_n, Y_n)$ and $f(t_{n+1}, Y_{n+1})$; this leads to the formula

$$Y_{n+1} = Y_n + \frac{h}{2}\big(f(t_n, Y_n) + f(t_{n+1}, Y_{n+1})\big) \tag{7.1.10}$$

Unfortunately, Y_{n+1} occurs on both sides of (7.1.10), so we must solve the equation for Y_{n+1}, which may be difficult. Fortunately, we already have an approximation for Y_{n+1}, namely, the value

$$Y_n + hf(t_n, Y_n)$$

from the Euler method formula (7.1.6). Let us substitute $Y_n + hf(t_n, Y_n)$ for the term Y_{n+1}, occurring on the right-hand side of (7.1.10). The result is

$$Y_{n+1} = Y_n + \frac{h\big(f(t_n, Y_n) + f(t_{n+1}, Y_n + hf(t_n, Y_n))\big)}{2} \tag{7.1.11}$$

The *Heun method* consists of approximating the solution to

$$y' = f(t, y), \quad y(a) = y_0 \tag{7.1.12}$$

by means of (7.1.11), which we call the *Heun method formula*. The Heun method is an example of a *predictor-corrector method*. The Euler method is used to "predict" a value for Y_{n+1}; this value is then used in (7.1.11) to obtain a better (or "more correct") approximation.

EXAMPLE 7.1.3. *Use the Heun method to find an approximate solution to the initial-value problem*

$$y' = 5 - t^2y^3, \quad y(0) = 0 \tag{7.1.13}$$

for $0 \le t \le 1$. *Use the step size* $h = 0.1$. *Compare the Heun method approximation with the Euler method approximation.*

Solution. Let $f(t, Y) = 5 - t^2Y^3$. Since the Euler method approximation to Y_{n+1} is $Y_n + hf(t_n, Y_n) = Y_n + h(5 - t_n^2Y_n^3)$, we compute

$$\begin{aligned} f\big(t_{n+1}, Y_n + hf(t_n, Y_n)\big) &= f\big(t_n + h, Y_n + h(5 - t_n^2Y_n^3)\big) \\ &= 5 - (t_n + h)^2\big(Y_n + h(5 - t_n^2Y_n^3)\big)^3 \end{aligned}$$

We also have $f(t_n, Y_n) = 5 - t_n^2Y_n^3$. Thus the Heun method formula (7.1.11) becomes

$$Y_{n+1} = Y_n + \frac{h}{2}\Big(10 - t_n^2Y_n^3 - (t_n + h)^2\big(Y_n + h(5 - t_n^2Y_n^3)\big)^3\Big) \tag{7.1.14}$$

We obtain $Y_1 = (h/2)(10 - h^5(125))$, and so forth. Taking $h = 0.1$ in (7.1.14) gives us the last column in the following table. The middle column is computed using the Euler method formula (7.1.6).

t	Y_n (Euler)	Y_n (Heun)
0.0	0.0	0.0
0.1	0.50000	0.49994
0.2	0.99987	0.99788
0.3	1.49588	1.48089
0.4	1.96575	1.90680
0.5	2.34422	2.20007
0.6	2.52216	2.30745
0.7	2.44457	2.26215
0.8	2.22875	2.14016
0.9	2.02021	1.99622
1.0	1.85236	1.85650

●

EXERCISES 7.1.2

In the following exercises, perform the indicated computations by hand, retaining only four significant digits at each step of the calculations.

1. Use the Heun method to compute an approximate solution to the initial-value problem $y' = y + 1$, $y(0) = 0$ with step size $h = 0.1$. Compare this with the exact solution $y(t) = e^t - 1$ and with the Euler method approximation from Example 7.1.1.
2. Consider the following initial-value problem:

$$\begin{cases} y' = y - t\,y^2 \\ y(0) = 2 \end{cases}$$

(a) Use the Heun algorithm to compute an approximate solution at $t = 1.0$. Use the values $n = 2, 4, 8$, which correspond to step sizes of $h = 1/2, 1/4, 1/8$.

(b) Determine the actual solution and compare the value of $y(1)$ with the results from part (a).

Repeat Exercise 2 for the following initial-value problems:

3. $\begin{cases} y' + 2y = t \\ y(0) = 1 \end{cases}$
4. $\begin{cases} y' = t + 1 \\ y(0) = 1 \end{cases}$
5. $\begin{cases} y' = t^3 e^{-2y} \\ y(0) = 0 \end{cases}$

6. $\begin{cases} y' = \dfrac{e^t}{y} \\ y(0) = 1 \end{cases}$

7. Consider the following initial-value problem:

$$\begin{cases} y' = y + t^2 \\ y(0) = 1 \end{cases}$$

(a) Find the solution $y(t)$ and evaluate it for $t = 0.2, 0.4, \dots, 1.0$.

(b) Using the Heun method with step size $h = 0.2$, find approximate values for the solution at the t values in part (a).

(c) Repeat part (b) using $h = 0.1$.

(d) Compare the results of part (b) with those of part (c) and the exact values. (The differences in the results for $h = 0.2$ and $h = 0.1$ tend to indicate whether a smaller step size must be used for the desired range of t values. The general rule of thumb is to use the smaller step size if the two solutions agree to the desired accuracy. If they do not agree, reduce h and repeat the calculations. This gives an indication but not a proof of the accuracy of the result.)

The initial-value problems in the following exercises cannot be solved by exact methods. Use the Heun method with step size $h = 0.1$ to approximate the solution at $t = 1$ to three significant digits (see Exercise 7).

8. $\begin{cases} y' = \sin(y) + e^t \\ y(0) = 0 \end{cases}$

9. $\begin{cases} y' = y^{3/2} + t \\ y(0) = 1 \end{cases}$

10. $\begin{cases} y' = \sin(t) + \cos(y) \\ y(0) = 1 \end{cases}$

11. $\begin{cases} y' = e^{t^3} \\ y(0) = 1 \end{cases}$

12. Consider the initial-value problem

$$\begin{cases} y' = t^2 y \\ y(0) = 1 \end{cases}$$

(a) Find the exact solution at $t = 1.0$. Express this value to four decimal places.

(b) Use the Heun method with $h = 1/8$ to approximate the solution at $t = 1.0$. Compute the absolute error.

(c) Repeat part (b) with $h = 1/16$, $h = 1/32$. Create a table and a graph showing the absolute errors corresponding to the various step sizes. A theoretical analysis for the Heun method suggests a linear relationship between the absolute error and the square of the step size. Do the numbers agree with the theory?

(d) Observe in part (c) that the error is roughly proportional to the square of the step size. Use the data to estimate the constant of proportionality.

13. Apply the Heun method with successively smaller step sizes on the interval $0 \le t \le 1$ to verify empirically that the solution of the initial-value problem

$$\begin{cases} y' = t^2 + y^2 \\ y(0) = 1 \end{cases}$$

has a vertical asymptote near $t = 0.97$.

7.1.3. Error analysis. A common way to classify methods is to give their order of accuracy. This order is associated with truncation error as defined by the particular method and the Taylor expansion of the solution $y(t)$. Taylor's theorem states that if $y(t)$ has $k+1$ continuous derivatives on the interval $t_0-\delta < t < t_0+\delta$, then

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \cdots + \frac{y^{(k)}(t_0)}{k!}(t - t_0)^k + \mathrm{TE}$$

where $t_0 - \delta < t < t_0 + \delta$. Here the Taylor remainder TE is called the *local truncation error*; it is defined by

$$\mathrm{TE} = \frac{y^{(k+1)}(\eta)}{(k+1)!}(t - t_0)^{k+1}$$

where $t_0 - \delta \le \eta \le t_0 + \delta$. If $t_1 = t_0 + h$, then we may write

$$\mathrm{TE} = \frac{y^{(k+1)}(\eta)}{(k+1)!}h^{k+1}$$

and we say that the local truncation error is proportional to h^{k+1}. When this occurs, we say that the method is of order k. The reason for defining it this way is because the *global truncation error*

$$\mathrm{GTE} = |y(t_i) - Y(t_i)|$$

is asymptotically proportional to one lower power of h when h tends to zero. Here we use y for the true solution and Y for the approximate solution. In order to discuss the error analysis, we recall the *big-O notation.*

Definition Suppose that $f(h), g(h)$ are functions defined in some interval $0 < h < a$ with $g(h) > 0$. Then we write

$$f(h) = O(g(h)), \quad t \to 0$$

if there exist constants $M > 0, \delta > 0$ so that $|f(h)| \le Mg(h)$ for $0 < h < \delta$. For example, $\sin(h) = O(h), h \to 0$, but $\sin(h) \ne O(h^2), t \to 0$. In many cases we will omit the quantifier $h \to 0$ when it is obvious from the context.

This may be applied to the discussion of error analysis of the various finite difference schemes. Typically a given scheme (Euler, e.g.) will satisfy a pair of statements of the form

$$\mathrm{TE} = O(h^{k+1}) \qquad \text{and} \qquad \mathrm{GTE} = O(h^k) \qquad h \to 0$$

for a certain value of k.

To see how this analysis is done, let us consider the Euler method. Using Taylor's theorem,

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2} y''(\eta) \tag{7.1.15}$$

for some $\eta \in [t_n, t_{n+1}]$. To analyze the error in the Euler method, we state the method in terms of the approximate solution Y,

$$Y_{n+1} = Y_n + hf(t_n, Y_n) \qquad n = 0, 1, \ldots, N-1$$

and subtract this equation from Eq. (7.1.15) to obtain

$$y(t_{n+1}) - Y_{n+1} = y(t_n) - Y_n + h\big(f(t_n, y(t_n)) - f(t_n, Y_n)\big) + \frac{h^2}{2} y''(\eta) \tag{7.1.16}$$

The error in Y_{n+1} consists of two parts: (1) the local truncation error TE introduced at step t_{n+1} and (2) the *propagated error*

$$y(t_n) - Y_n + h\big(f(t_n, y(t_n)) - f(t_n, Y_n)\big)$$

The propagated error can be simplified by applying the mean value theorem to $f(t, z)$, considered as a function of z:

$$f(t_n, y(t_n)) - f(t_n, Y_n) = \frac{\partial f(t_n, \xi)}{\partial z} \big(y(t_n) - Y_n\big)$$

where ξ is between $y(t_n)$ and Y_n. We let $\epsilon_n = y(t_n) - Y_n$, and use the above to rewrite Eq. (7.1.16), obtaining

$$\epsilon_{n+1} = \left(1 + h \frac{\partial f(t_n, \xi)}{\partial z}\right) \epsilon_n + \frac{h^2 y''(\eta)}{2} \tag{7.1.17}$$

These computations yield a general error analysis for the Euler method for the initial-value problem as stated in the following theorem.

THEOREM 7.1. *Assume*

$$0 < K = \sup \left| \frac{\partial f(t, z)}{\partial z} \right| < \infty \tag{7.1.18}$$

where the supremum is taken over (t, z) of the form $z = y(t)$ with $t_0 \le t \le b$. Then the Euler method solution (Y_n) satisfies the error bound

$$(7.1.19) \qquad |y(t_n) - Y_n| \le e^{(b-t_0)K}|y(t_0) - Y_0| + h\left(\frac{e^{(b-t_0)K} - 1}{2K}\right) \sup_{t_0 \le t \le b} |y''(t)|$$

for all n with $t_0 \le t_n \le b$.

Proof. The proof can be accomplished by mathematical induction on the sequence of numbers $\epsilon_n, n = 0, 1, \ldots, N$. From Eqs. (7.1.17) and (7.1.18) we have

$$(7.1.20) \qquad |\epsilon_{n+1}| \le A\,|\epsilon_n| + B \qquad n = 0, 1, \ldots, N-1$$

where $A = 1 + hK, B = h^2M/2$, and $M = \sup_{t_0 \le t \le b} |y''(t)|$. We propose to show that

$$(7.1.21) \qquad |\epsilon_n| \le |\epsilon_0|\,A^n + \frac{B}{A-1}(A^n - 1) \qquad n = 0, 1, \ldots, N$$

Clearly, (7.1.21) holds for the value $n = 0$. Assuming its truth for the value $n = m$, we have

$$\begin{aligned}
|\epsilon_{m+1}| &\le A|\epsilon_m| + B \\
&\le A\left(|\epsilon_0|A^m + \frac{B}{A-1}(A^m - 1)\right) + B \\
&= |\epsilon_0|A^{m+1} + \frac{B}{A-1}(A^{m+1} - 1)
\end{aligned}$$

which proves (7.1.21) for the value $n = m+1$, and hence for all n by mathematical induction. To obtain the conclusion (7.1.19), we apply this with $A = 1+hK, B = h^2M/2$ and take note of the inequality $1 + x \le e^x$ to write

$$\begin{aligned}
|\epsilon_n| &\le |\epsilon_0|(1 + hK)^n + \frac{h^2M/2}{hK}((1 + hK)^n - 1) \\
&\le |\epsilon_0|e^{hnK} + \frac{hM}{2K}(e^{nhK} - 1) \\
&\le |\epsilon_0|e^{(b-t_0)K} + \frac{hM}{2K}(e^{(b-t_0)K} - 1)
\end{aligned}$$

Making the identification $\epsilon_n = y(t_n) - Y_n$ completes the proof. •

When $Y_0 = y(t_0)$ (as is commonly the case), (7.1.19) can be written

$$|y(t_n) - Y_n| \le c\,h \qquad t_0 \le t_n \le b$$

where c is a constant. Therefore we say that the Euler method is an order 1 or first-order method. When h is halved, the error is halved. Also, the Euler method is said to converge with order 1. In general, if we have

$$|y(t_n) - Y_n| \le c\,h^k \qquad t_0 \le t_n \le b$$

then we say that the method is an order k method or is convergent with order k. To see what this means, let us consider an example.

EXAMPLE 7.1.4. *Illustrate the error bound (7.1.17) for the equation*

$$y' = 4t, \qquad y(0) = 0$$

whose exact solution is $y(t) = 2t^2$.

Solution. The error formula (7.1.17) becomes

$$\epsilon_{n+1} = \epsilon_n + 2h^2, \qquad \epsilon_0 = 0$$

Using induction, we get

$$\epsilon_n = 2n\,h^2 \qquad n \geq 0$$

Since $n\,h = t_n$,

$$\epsilon_n = 2t_n h \quad \bullet$$

In the above example we see that for each t_n, the error of approximation in the Euler method at t_n is proportional to h. The local truncation error TE is proportional to h^2, but the cumulative effect of these errors is a total error proportional to h.

EXERCISES 7.1.3

1. Apply the Euler method to the initial-value problem

$$y' = \sin(10^6 y), \qquad y(0) = 1$$

and show that the local truncation error TE is

$$\text{TE} = \frac{h^2}{4} 10^6 \sin\big(2 \times 10^6 y(\eta)\big)$$

2. Determine an upper bound on the global truncation error GTE in using the Euler method with $N = 10$ steps to solve the initial-value problem

$$\begin{cases} y' = t/y \\ y(0) = 2 \end{cases}$$

on the interval $0 \leq t \leq 1$.

3. Use Euler's method with $h = 0.1$ to approximate the solution to

$$y' = \frac{2}{t} y + t^2 e^t, \qquad y(1) = 0$$

on the interval $1 \leq t \leq 2$. Find the value of h so that $|y(t_i) - Y_i| \leq 0.1$.

4. Consider the initial-value problem

$$y' = -10y, \qquad y(0) = 1$$

on the interval $0 \leq t \leq 2$, which has the solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this violate Theorem 7.1?

5. Apply the second-order Euler method with $h = 1/2$ to the initial-value problem

$$y' = t^2 - y^2, \qquad y(1) = 1$$

Compute $y(2)$. What can be said about the error for this method?

6. This exercise outlines the counterpart of the error estimate (7.1.19) for the Heun method. Assume that the true solution $y(t)$ has a bounded third derivative and that f and its partial derivatives of orders 1 and 2 are continuous and bounded by a constant K on their respective domains. Use the notation $f_1 = \partial f/\partial t$, $f_2 = \partial f/\partial y$.

(a) Beginning with the Heun formula (7.1.10), use the mean value theorem to show that

$$Y_{n+1} - Y_n = hf(t_n, Y_n) + (h^2/2)\big(f_2(t_n, \xi_1)f(\tau_1, \xi_2) + f_1(\tau_2, Y_{n+1})\big)$$

for appropriate values of $\tau_1, \tau_2, \xi_1, \xi_2$.

(b) Use the second-order Taylor theorem to show that the exact solution $y(t)$ satisfies

$$\begin{aligned} y(t_{n+1}) - y(t_n) &= hy'(t_n) + (h^2/2)y''(t_n) + (h^3/6)y'''(\eta) \\ &= hf(t_n, y(t_n)) + \frac{h^2}{2}\big(f_1(t_n, y(t_n)) + (ff_2)(t_n, y(t_n))\big) \\ &\quad + O(h^3) \end{aligned}$$

for an appropriate value of η.

(c) Letting $\epsilon_n = Y_n - y(t_n)$, perform the subtraction to obtain

$$\begin{aligned} \epsilon_{n+1} - \epsilon_n &= h[f(t_n, Y_n)) - f(t_n, y(t_n))] + B \\ &= h\epsilon_n f_2(t_n, \xi_3) + B \end{aligned}$$

where $B = O(h^3), h \to 0$.

(d) Show that

$$|\epsilon_{n+1}| \leq A|\epsilon_n| + B \qquad n = 0, 1, \ldots, N-1$$

where $A = 1 + hK$, $B = O(h^3), h \to 0$.

(e) Use an inductive argument to show that

$$|\epsilon_n| \leq |\epsilon_0|\, A^n + B\frac{A^n - 1}{A - 1}$$

(f) Assuming that we begin with $Y_0 = y_0$, conclude that the Heun method satisfies the numerical error estimate $Y_n - y(t_n) = O(h^2), h \to 0$, where $n = 1, 2, \ldots, N$.

7. For an arbitrary smooth function $u(x)$, define the *forward replacement error* by

$$e^+(x, h) = u'(x) - \frac{u(x+h) - u(x)}{h}$$

Compute $e^+(x,h)$ for the following cases:
(a) $u(x) = \cos x$, $x = 0$, $h = 0.1$
(b) $u(x) = x^2$, $x = 1$, $h = 0.1$
(c) $u(x) = e^{3x}$, $x = 0$, $h = 0.1$
(d) $u(x) = x \sin 15x$, $x = 0$, $h = 0.1$

8. Define the *backward replacement error* by
$$e^-(x,h) = u'(x) - \frac{u(x) - u(x-h)}{h}$$
Compute $e^-(x,h)$ for each of the cases (a), (b), (c), (d) in Exercise 7.

9. Define the *symmetric replacement error* by
$$e^0(x,h) = u'(x) - \frac{u(x+h/2) - u(x-h/2)}{h}$$
Compute $e^0(x,h)$ for each of the cases (a), (b), (c), (d) in Exercise 7.

10. With reference to the previous exercises, show that the following bounds hold for an arbitrary smooth function $u(x)$:
(a) $|e^+(x,h)| \le (h/2) \max_{x \le t \le x+h} |u''(t)|$
(b) $|e^-(x,h)| \le (h/2) \max_{x-h \le t \le x} |u''(t)|$
(c) $|e^0(x,h)| \le (h^2/24) \max_{x-h/2 \le t \le x+h/2} |u'''(t)|$
Hint: Begin with the Taylor formula with remainder
$$u(x+h) - u(x) = hu'(x) + \int_x^{x+h} (t-x)u''(t)dt$$

11. Define the *second-order symmetric replacement error* by
$$E^0(x,h) = u''(x) - \frac{u(x+h) + u(x-h) - 2u(x)}{h^2}$$
Compute $E^0(x,h)$ for each of the following cases:
(a) $u(x) = x^2$, $x = 0$, $h = 0.1$
(b) $u(x) = e^{3x}$, $x = 0$, $h = 0.1$
(c) $u(x) = \cos x$, $x = 0$, $h = 0.1$
(d) $u(x) = x^2 \sin 15x$, $x = 0$, $h = 0.1$

12. Show that the second-order symmetric replacement error satisfies the bound
$$|E^0(x,h)| \le \frac{h^2}{12} \max_{|t-x| \le h} |u^{(\mathrm{iv})}(t)|$$
Hint: Use the Taylor formula of order 4:
$$\begin{aligned} u(x+h) - u(x) &= hu'(x) + \frac{h^2}{2} u''(x) \\ &\quad + \frac{h^3}{6} u'''(x) + \int_x^{x+h} \frac{(t-x)^3}{6} u^{(\mathrm{iv})}(t)dt \end{aligned}$$

13. Define the *second-order forward replacement error* by

$$E^+(x,h) = u''(x) - \frac{u(x+2h)+u(x)-2u(x+h)}{h^2}$$

Show that we have the bound $|E^+(x,h)| \le 2h \max_{x\le t\le x+2h} |u'''(t)|$.

14. Let $u(x)$ be the arbitrary smooth function and define the *fourth-order symmetric replacement error* by

$$\begin{aligned} F^0(x,h) &= u^{(\mathrm{iv})}(x) \\ &\quad - \frac{u(x+2h)-4u(x+h)+6u(x)-4u(x-h)+u(x-2h)}{h^4} \end{aligned}$$

Show that we have the following bound:

$$|F^0(x,h)| \le \frac{h^2}{45} \max_{|t-x|\le h} |u^{(\mathrm{iv})}(t)|$$

7.2. The One-Dimensional Heat Equation

In this section we begin the study of numerical solutions of partial differential equations. As our first model, we consider the following problem for the heat equation:

$$\begin{aligned} u_t &= K u_{xx} && t>0, 0<x<L \\ u(0;t) &= 0 = u(L;t) && t>0 \\ u(x;0) &= f(x) && 0<x<L \end{aligned} \tag{7.2.1}$$

Physically this represents heat flow in a slab whose faces are maintained at equal temperatures. In Chapter 2 we solved this problem using Fourier series.

7.2.1. Formulation of a difference equation. To solve this problem by the method of finite differences, we choose a mesh $0 = x_0 < x_1 < x_2 < \cdots < x_{n+1} = L$ with $x_{i+1} - x_i = \Delta x$, independent of i for $i = 0, 1, \ldots, n$. We replace $u(x_i;t)$ by $u_i(t)$ for $i = 0, 1, \ldots, n+1$. Similarly, the t-axis is replaced by a mesh of points (t_i), with $t_{i+1} - t_i = \Delta t$. To employ the method of finite differences, we make the following replacements for the partial derivatives that occur in the heat equation:

$$\begin{aligned} &u_t(x_i;t) && \text{is replaced by} && \frac{u_i(t+\Delta t)-u_i(t)}{\Delta t} \\ &u_{xx}(x_i;t) && \text{is replaced by} && \frac{u_{i+1}(t)+u_{i-1}(t)-2u_i(t)}{(\Delta x)^2} \end{aligned}$$

Thus the partial differential equation is replaced by the following system of linear equations with boundary conditions:

$$\frac{u_i(t+\Delta t) - u_i(t)}{\Delta t} = K\frac{u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)}{(\Delta x)^2} \qquad 1 \le i \le n$$
$$u_0(t) = 0, \qquad u_{n+1}(t) = 0$$

In order to make this more natural, recall the original formulation of the one-dimensional heat equation in Chapter 0, Eq. (0.1.2), where we postulated that the time rate of change of temperature is proportional to the difference between the local temparature and that of its neighbors. That formulation differs from the above finite difference scheme only in the further replacement of the time derivative by the time difference quotient.

We solve these linear equations for $u_i(t+\Delta t)$ to obtain

$$\boxed{u_i(t+\Delta t) = \frac{K(\Delta t)}{(\Delta x)^2}u_{i+1}(t) + \frac{K(\Delta t)}{(\Delta x)^2}u_{i-1}(t) + \left(1 - \frac{2K\,\Delta t}{(\Delta x)^2}\right)u_i(t)} \tag{7.2.2}$$

Thus $u_i(t+\Delta t)$ is a weighted average of the numbers $u_{i+1}(t)$, $u_{i-1}(t)$, and $u_i(t)$ provided that $2K\Delta t/(\Delta x)^2 \le 1$. This is a *stability condition*, meaning that the numerical approximations will be close to the true solution in a suitable sense. When this condition is satisfied, all of the coefficients in (7.2.2) are positive and their sum is unity—thus the term "weighted average." For given values of the diffusivity K and the mesh size Δx, this condition can be realized by a suitable choice of the time step Δt. It is to our advantage to choose Δt as large as possible, consistent with this restriction. In particular, we may we choose $2K\Delta t/(\Delta x)^2 = 1$.

7.2.2. Computational molecule. We can graphically illustrate the numerical algorithm implied by Eq. (7.2.2) in terms of the "computational molecule" depicted in Fig. 7.2.1. The coefficient $r = K\Delta t/(\Delta x)^2$ multiplies the current values $u_{i+1}(t)$, $u_{i-1}(t)$, while the coefficient $(1-2r)$ multiplies the current value $u_i(t)$. Together they produce the new value $u_i(t+\Delta t)$. On a larger scale we may graph the boundary values and unknown values as part of a rectangular grid, as depicted in Fig. 7.2.2. The initial and boundary values are represented by small cubes, while the unknown solution values are represented by black dots.

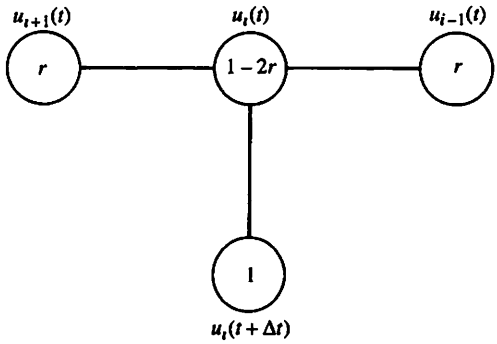


FIGURE 7.2.1 Computational molecule for the heat equation.

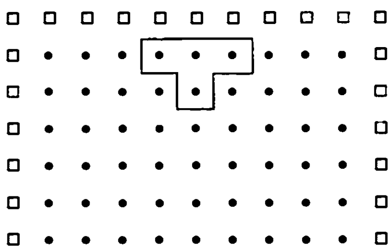


FIGURE 7.2.2 Global rectangular grid.

7.2.3. Examples and comparison with the Fourier method. We now present an example that affords comparison between the numerical solution and the Fourier series solution.

EXAMPLE 7.2.1. *Find the approximate numerical solution of the heat equation* $u_t = Ku_{xx}$ *for* $0 < x < 1$, $t > 0$ *with the boundary conditions* $u(0;t) = 0$, $u(1;t) = 0$ *and the initial condition* $u(x;0) = 100x(1-x)$. *Use the mesh with* $\Delta x = 0.1$, $K\Delta t/(\Delta x)^2 = \frac{1}{3}$, *and compute the solution for* $0 \le t \le 20\Delta t$.

Solution. In this case the basic equations (7.2.2) take the form

$$\begin{aligned} u_i(t+\Delta t) &= \frac{K\,\Delta t}{(\Delta x)^2}\, u_{i+1}(t) + \frac{K\,\Delta t}{(\Delta x)^2}\, u_{i-1}(t) + \left[1 - \frac{2K\,\Delta t}{(\Delta x)^2}\right] u_i(t) \\ &= \frac{1}{3} u_{i-1}(t) + \frac{1}{3} u_{i+1}(t) + \frac{1}{3} u_i(t) \end{aligned}$$

Since the initial conditions are symmetric about $x = \frac{1}{2}$, we may restrict attention to the values $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. The computational grid is depicted in Table 7.2.1.

	$x=0$	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
$t=0$	0	9.00	16.00	21.00	24.00	25.00
$t=1$	0	8.33	15.33	20.33	23.33	24.33
$t=2$	0	7.89	14.66	19.66	22.66	23.66
$t=3$	0	7.52	14.07	18.99	21.99	22.99
$t=4$	0	7.20	13.53	18.35	21.32	22.32
$t=5$	0	6.91	13.03	17.73	20.66	21.65
$t=6$	0	6.65	12.56	17.14	20.01	20.99
$t=7$	0	6.40	12.12	16.57	19.38	20.34
$t=8$	0	6.17	11.70	16.02	18.76	19.70
$t=9$	0	5.96	11.30	15.49	18.16	19.07
$t=10$	0	5.75	10.92	14.98	17.57	18.46
$t=11$	0	5.56	10.55	14.49	17.00	17.87
$t=12$	0	5.37	10.20	14.00	16.45	17.29
$t=13$	0	5.19	9.86	13.55	15.91	16.73
$t=14$	0	5.02	9.53	13.11	15.40	16.18
$t=15$	0	4.85	9.22	12.68	14.90	15.66
$t=16$	0	4.69	8.92	12.27	14.41	15.15
$t=17$	0	4.54	8.63	11.87	13.61	14.66
$t=18$	0	4.39	8.35	11.37	13.38	13.96
$t=19$	0	4.25	8.02	11.03	12.90	13.57
$t=20$	0	4.09	7.77	10.65	12.50	13.12

TABLE 7.2.1 Numerical results from Example 7.2.1

It is instructive to compare the numerical solution with the exact solution obtained by Fourier series. To do this, we recall the Fourier sine series for $0 < x < 1$:

$$100x(1-x) = \frac{800}{\pi^3}\left(\sin \pi x + \frac{1}{27}\sin 3\pi x + \cdots\right)$$

The Fourier series solution of the heat equation with this initial condition is

$$u(x;t) = \frac{800}{\pi 3}\left(\sin \pi x\, e^{-\pi^2 Kt} + \frac{1}{27}\sin 3\pi x e^{-(3\pi)^2 Kt} + \cdots\right)$$

We compute the values of the various terms for $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ and $Kt = K(20\Delta t) = (20/3)(\Delta x)^2 = 1/15$:

x	0	0.1	0.2	0.3	0.4	0.5
$\sin \pi x$	0	0.309	0.588	0.809	0.951	1.000
$(1/27)\sin 3\pi x$	0	0.011	0.035	0.030	0.951	1.000
$(800/\pi^3)\sin \pi x e^{-\pi^2/15}$	0	4.128	7.856	10.808	12.705	13.363
$(800/27\pi^3)\sin 3\pi x e^{-9\pi^2/15}$	0	0.000	0.002	0.002	−0.002	−0.002
$u(x;t)$	0	4.128	7.858	10.810	12.703	13.361

Thus we see that the numerical solution and the Fourier solution agree to within 2 percent for the tabulated values.

This example suggests that we can expect close agreement between the numerical solution and the exact solution of the heat equation. The following theorem[1] affirms this result more generally.

THEOREM 7.2. *Let $u(x;t)$ be the solution of the heat equation $u_t = Ku_{xx}$ for $t > 0$, $0 < x < L$ with the boundary conditions $u(0;t) = \alpha(t)$, $u(L;t) = \beta(t)$ and the initial condition $u(x;0) = f(x)$. Let $u_i(t)$ be the numerical solution obtained with $2K(\Delta t)/(\Delta x)^2 \le 1$. Then*

$$\max_{1\le i\le n} |u(x_i;t) - u_i(t)| \le \frac{MK\Delta t(\Delta x)^4}{135}$$

where M is a bound for $u(x;t)$ and its first six derivatives.

In order to illustrate the role of the stability condition $2K(\Delta t)/(\Delta x)^2 \le 1$ we consider a simple example with two different mesh sizes. In the first case we have stability, and in the second case we have instability.

EXAMPLE 7.2.2. *Find the approximate numerical solution of the heat equation $u_t = u_{xx}$ for $t > 0$, $0 < x < 1$ with the boundary conditions $u(0;t) = 0$, $u(1;t) = 0$ and the initial condition $u(x;0) = 4x$. First use the mesh size $\Delta x = 1/4$, $\Delta t = 1/32$; then use the mesh size $\Delta x = 1/4$, $\Delta t = 1/8$.*

Solution. For the first choice, we have $K(\Delta t)/(\Delta x)^2 = 1/2$ and the form of the solution (7.2.2) is $u_i(t + \Delta t) = \frac{1}{2}u_{i+1}(t) + \frac{1}{2}u_{i-1}(t)$. Applying this to the initial data $u(x;0) = 4x$, we have the following table of values:

	$x = 0$	$x = 1/4$	$x = 1/2$	$x = 3/4$	$x = 1$
$t = 0$	0	1.0	2.0	3.0	4.0
$t = 1/32$	0	1.0	2.0	3.0	0.0
$t = 1/16$	0	1.0	2.0	1.0	0.0
$t = 3/32$	0	1.0	1.0	1.0	0.0
$t = 1/8$	0	0.5	1.0	0.5	0.0

For the second choice, we have $K(\Delta t)/(\Delta x)^2 = 2$. The form of the solution (7.2.2) is $u_i(t + \Delta t) = 2u_{i+1}(t) + 2u_{i-1}(t) - 3u_i(t)$, leading to the following table of values:

[1] W. E. Milne, *Numerical Solution of Differential Equations*, Dover Publications, New York, 1970, p. 122

t	$x=0$	$x=1/4$	$x=1/2$	$x=3/4$	$x=1$
0	0	1.0	2.0	3.0	4.0
1/32	0	1.0	2.0	3.0	0.0
1/16	0	1.0	2.0	−5.0	0.0
3/32	0	1.0	−14.0	19.0	0.0
1/8	0	−31.0	82.0	−85.0	0.0

From this example we see that the second choice of $\Delta x = 1/4$, $\Delta t = 1/8$ leads to an absurd result, since we expect on physical grounds that the temperature will remain positive and tend to zero as the time t becomes large; in fact, this is known from the analysis of Chapter 2, where we solved this initial-boundary value problem using a Fourier sine series. The absurd values in this example are not surprising, in view of the fact that the ratio $2K(\Delta t)/(\Delta x)^2$ is larger than unity. •

7.2.4. Stability analysis. We now present the derivation of the stability condition for the one-dimensional heat equation with zero boundary conditions. The first step is to rewrite the difference scheme (7.2.2) in the form

$$u_i^{n+1} = ru_{i+1}^n + ru_{i-1}^n + (1-2r)u_i^n \tag{7.2.3}$$

where $r = K\Delta t/(\Delta x)^2$ and we have set $u_i^n = u_i(n\Delta t)$.

The system (7.2.3) can be solved by separation of variables, reminiscent of the Fourier solution of the heat equation $u_t = Ku_{xx}$. This is written in the form

$$u_i^n = A(i)B(n)$$

Substituting in (7.2.3) and clearing fractions leads to the equation

$$\frac{B(n+1)}{B(n)} = \frac{r\,A(i-1) + r\,A(i+1) + (1-2r)A(i)}{A(i)}$$

The left side depends only on n, and the right side depends only on i; therefore they must both be constant, denoted θ. Thus

$$B(n+1) = \theta B(n), \qquad r\,A(i-1) + r\,A(i+1) + (1-2r-\theta)A(i) = 0 \tag{7.2.4}$$

From this it follows that $B(n) = \theta^n B(0)$ and, by analogy with the continuous case, we can find $A(i)$ as a sine function:

$$A(i) = \sin\left(\frac{k\pi i}{N}\right) \qquad k = 1, 2, \ldots, N-1 \tag{7.2.5}$$

The stability analysis below will hinge on the dichotomy between $|\theta| \leq 1$ and $|\theta| > 1$. In the first case one can expect all solutions of the difference scheme to remain bounded when $n \to \infty$, whereas in the second case we can expect some solutions to become unbounded when $n \to \infty$. We now present the details.

Substitution of (7.2.5) into (7.2.4) yields

$$2r\cos\left(\frac{k\pi}{N}\right) + (1 - 2r - \theta) = 0 \tag{7.2.6}$$

which is solved explicitly for θ as

$$\boxed{\theta = (1 - 2r) + 2r\cos(k\pi/N)} \tag{7.2.7}$$

We consider separately two cases:

Case 1: $0 < 2r \le 1$. Since the cosine function is strictly between -1 and $+1$, it follows that $(1-2r) - 2r < \theta < (1-2r) + 2r$. But the upper bound is $+1$ and the lower bound is greater than or equal to $-2r$, which is greater than or equal to -1. Combining these yields the result $-1 < \theta < 1$ for $k = 1, 2, \ldots, N-1$.

Case 2: $2r > 1$. In this case the upper bound of case 1 still applies, and we have $\theta < 1$ for all k. However, if we take $k = N - 1$ and note that the cosine function lies below its quadratic Taylor approximation about π, it follows that

$$\begin{aligned}
\theta &= (1-2r) + 2r\cos(\pi - \pi/N) \\
&\le (1-2r) + 2r(-1 + \pi^2/2N^2) \\
&= (1-4r) + r\pi^2/N^2
\end{aligned}$$

Since $2r > 1$, it follows that $1 - 4r < -1$. By taking N sufficiently large, we can still achieve $\theta < -1$. We summarize the above work as follows.

PROPOSITION 7.2.1. *Suppose that the mesh sizes Δt, Δx satisfy the condition that $r := K\Delta t/(\Delta x)^2 \le 1/2$; then all solutions of the difference scheme (7.2.2) remain bounded when $n \to \infty$. If the mesh sizes Δt, Δx are such that $r := K\Delta t/(\Delta x)^2 > 1/2$, then there exist solutions of the difference scheme (7.2.2) that become unbounded when $n \to \infty$.*

7.2.5. Other boundary conditions. The method of finite differences can also be applied to solve the heat equation with boundary conditions involving the derivative u_x. Consider, for example, the problem

$$\begin{aligned}
u_t &= Ku_{xx} & t > 0, 0 < x < L \\
u_x(0;t) &= 0 = u_x(L;t) & t > 0 \\
u(x;0) &= f(x) & 0 < x < L
\end{aligned} \tag{7.2.8}$$

which represents heat flow in a slab whose faces are insulated. In this problem it is natural to replace the boundary condition by the equations $u_0 = u_1$, $u_n = u_{n+1}$. As before, the difference scheme is obtained from the equations (7.2.2).

EXAMPLE 7.2.3. *Find the approximate numerical solution of the heat equation $u_t = u_{xx}$ for $t > 0$, $0 < x < 1$ with the boundary conditions $u_x(0;t) = 0$, $u_x(1;t) = 0$ and the initial condition $u(x;0) = 4x$. Use the mesh size $\Delta x = 1/4$, $\Delta t = 1/32$.*

Solution. For this mesh size, we have $K\Delta t/(\Delta x)^2 = 1/2$ and the form of the solution (7.2.2) is $u_i(t+\Delta t) = \frac{1}{2}u_{i+1}(t) + \frac{1}{2}u_{i-1}(t)$ for $i = 1, 2, 3$. When we impose the conditions $u_0 = u_1$, $u_3 = u_4$, we obtain the following table of values:

t	$x=0$	$x=1/4$	$x=1/2$	$x=3/4$	$x=1$
0	0	1.0	2.0	3.0	4.0
1/32	1.00	1.00	2.00	3.00	3.00
1/16	1.50	1.50	2.00	2.50	2.50
3/32	1.75	1.75	2.00	2.25	2.25
1/8	1.88	1.88	2.00	2.13	2.13

●

We now turn to a problem involving a slab of variable conductivity. To be specific, suppose that the slab is composed of two materials whose diffusivity coefficients are K_1, K_2 and that these are of thickness L_1, L_2, respectively. This leads to the following initial-boundary-value problem:

$$\begin{aligned} u_t &= K_1 u_{xx} && 0 < x < L_1 \\ u_t &= K_2 u_{xx} && L_1 < x < L_1 + L_2 \\ u(0;t) &= 0 && t > 0 \\ u(L_1 + L_2;t) &= 0 && t > 0 \\ u(x;0) &= f(x) && 0 < x < L_1 + L_2 \end{aligned}$$

This problem is not easily solved by separation of variables. To employ the method of finite differences, we make the additional requirement that the temperature and heat flux be continuous at the interface $x = L_1$. This is translated into the following two additional boundary conditions:

$$\begin{aligned} u(L_1 - 0;t) &= u(L_1 + 0;t) \\ K_1 u_x(L_1 - 0;t) &= K_2 u_x(L_1 + 0;t) \end{aligned}$$

To solve this problem by the methods of finite differences, we first select a mesh:

$$0 = x_0 < x_1 < \cdots < x_{n+1} = L_1 < x_{n+2} < \cdots < x_{n+m+1} = L_1 + L_2$$

The various derivatives are replaced by the following difference quotients:

$$
\begin{aligned}
u_t(x_i;t) &\quad \text{is replaced by} \quad \frac{u_i(t+\Delta t)-u_i(t)}{\Delta t}\\
u_{xx}(x_i;t) &\quad \text{is replaced by} \quad \frac{u_{i+1}(t)+u_{i-1}(t)-2u_i(t)}{(\Delta x)^2}\\
u_x(L_1-0) &\quad \text{is replaced by} \quad \frac{u_{n+1}(t)-u_n(t)}{\Delta x}\\
u_x(L_1+0) &\quad \text{is replaced by} \quad \frac{u_{n+2}(t)-u_{n+1}(t)}{\Delta x}
\end{aligned}
$$

Substituting these in the previous equations and solving, we obtain the following system of equations for the solution:

$$u_0(t)=0, \qquad u_{n+m+1}(t)=0$$

(7.2.9)

$$u_i(t+\Delta t)=\frac{K_1\Delta t}{(\Delta x)^2}[u_{i+1}(t)+u_{i-1}(t)]+\left(1-\frac{2K_1\Delta t}{(\Delta x)^2}\right)u_i(t), \quad 1\le i\le n$$

(7.2.10)

$$u_i(t+\Delta t)=\frac{K_2\Delta t}{(\Delta x)^2}[u_{i+1}(t)+u_{i-1}(t)]+\left(1-\frac{2K_2\Delta t}{(\Delta x)^2}\right)u_i(t), \quad n+2\le i\le n+m$$

$$u_{n+1}(t)=\frac{K_2u_{n+2}(t)+K_1u_n(t)}{K_1+K_2} \tag{7.2.11}$$

Thus, to obtain the solution, we first obtain $u_1(t),\dots,u_n(t)$ from (7.2.9). Next we obtain $u_{n+2}(t),\dots,u_{n+m}(t)$ from (7.2.10). Finally, we obtain $u_{n+1}(t)$ from the interface condition (7.2.11).

EXAMPLE 7.2.4. *Find the approximate numerical solution of the heat equation $u_t=u_{xx}$ for $0<x<\frac{1}{2}$ and $u_t=\frac{1}{2}u_{xx}$ for $\frac{1}{2}<x<1$ with the boundary conditions $u(0;t)=0$, $u(1;t)=0$ and the initial condition $u(x;0)=4x$. Use the mesh size $\Delta x=\frac{1}{4}$, $\Delta t=\frac{1}{32}$.*

Solution. We have $K_1=1$, $K_2=\frac{1}{2}$, $K_1\Delta t/(\Delta x)^2=\frac{1}{2}$, $K_2\Delta t/(\Delta x)^2=\frac{1}{4}$, and the previous equations become

$$
\begin{aligned}
u_0(t) &= 0, \qquad u_4(t)=0\\
u_1(t+\Delta t) &= \frac{1}{2}u_0(t)+\frac{1}{2}u_2(t)\\
u_3(t+\Delta t) &= \frac{1}{4}u_2(t)+\frac{1}{4}u_4(t)+\frac{1}{2}u_3(t)\\
u_2(t) &= \frac{2}{3}u_1(t)+\frac{1}{3}u_3(t)
\end{aligned}
$$

Solving these equations yields the following table of values for $t = 0, \frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \frac{1}{8}$.

t	$x=0$	$x=1/4$	$x=1/2$	$x=3/4$	$x=1$
0	0	1.0	2.0	3.0	4.0
1/32	0	1.00	1.67	3.00	0
1/16	0	0.83	1.19	1.92	0
3/32	0	0.60	0.82	1.26	0
1/8	0	0.41	0.55	0.83	0

●

One can also treat the wave equation $u_{tt} = c^2 u_{xx}$ using the method of finite differences. We choose a mesh and replace the derivatives by the appropriate difference quotients, to obtain the system of equations

$$\frac{u_i(t+\Delta t) + u_i(t-\Delta t) - 2u_i(t)}{(\Delta t)^2} = c^2 \frac{u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)}{(\Delta x)^2}$$

If we choose Δt, Δx so that $\Delta x/\Delta t = c$, these equations simplify to

$$u_i(t+\Delta t) + u_i(t-\Delta t) = u_{i+1}(t) + u_{i-1}(t)$$

This system of difference equations has the general solution

$$u_i(t) = f(x_i + ct) + g(x_i - ct)$$

which is the same as the general solution of the wave equation obtained in Chapter 5 in connection with the vibrating string (see Exercise 8 below)

EXERCISES 7.2

1. Find the approximate numerical solution of the initial-boundary-value problem for the heat equation $u_t = \frac{1}{2}u_{xx}$ for $t > 0$, $0 < x < 1$ with the boundary conditions $u(0;t) = 0$, $u(1;t) = 0$ and the initial condition $u(x;0) = \sin \pi x$. Use the mesh size $\Delta x = \frac{1}{4}$, $\Delta t = \frac{1}{64}$ and compute the solution for $t = 0$, $t = \frac{1}{64}$, $t = \frac{1}{32}$, $t = \frac{3}{64}$, $t = \frac{1}{4}$. Compare this with the exact solution.

2. Find the approximate numerical solution of the initial-boundary-value problem for the heat equation $u_t = u_{xx}$ for $t > 0$, $0 < x < 1$ with the boundary conditions $u(0;t) = 0$, $u_x(1;t) = 0$ and the initial condition $u(x;0) = 4x$. Use the mesh size $\Delta x = \frac{1}{4}$, $\Delta t = \frac{1}{32}$ and compute the solution for $t = 0$, $t = \frac{1}{32}$, $t = \frac{1}{16}$, $t = \frac{3}{32}$, $t = \frac{1}{8}$.

3. Find the approximate numerical solution of the initial-boundary-value problem for the heat equation $u_t = u_{xx}$ for $t > 0$, $0 < x < 1$ with the (time-dependent) boundary conditions $u(0;t) = t$, $u(1;t) = t$ and the initial condition $u(x;0) = 0$. Use the mesh size $\Delta x = 0.1$, $\Delta t = 0.01$ and compute the solution for $t = 0.1, 0.2, \ldots, 0.9, 1.0$.

4. For each of the following equations, suppose that the mesh size Δx is given. In each case, derive an appropriate stability condition and find the largest time step Δt that satisfies the stability condition. (You are not required to find the approximate numerical solution.)
   (a) $u_t = \frac{1}{2}u_{xx}$
   (b) $u_t = u_{xxxx}$
   (c) $u_t = u_{xx} + 3u_x$
   (d) $u_t = u_{xx} - 4u$

5. Find the approximate numerical solution of the initial-boundary-value problem for the heat equation $u_t = u_{xx}$ for $t > 0$, $0 < x < 1$ with the boundary conditions $u(0;t) = 0$, $u(1;t) = 0$ and the initial condition $u(x;0) = 4x(1-x)$. Use the mesh size $\Delta x = 0.1$, $\Delta t = 0.005$ and compute the solution for $t = 0.005, 0.010, \ldots, 0.100$.

6. Consider the heat equation $u_t = Ku_{xx}$ with the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$. Recall that the *relaxation time*, from Chapter 2, is given by $T = L^2/(\pi^2 K)$. Assuming that an approximate numerical solution has been found with the largest possible time step Δt, how many time steps N are necessary so that $N\Delta t = T$?

7. Find the approximate numerical solution of the initial-boundary-value problem for the heat equation $u_t = u_{xx}$ for $t > 0$, $0 < x < \frac{1}{2}$ and $u_t = \frac{1}{2}u_{xx}$ for $t > 0$, $\frac{1}{2} < x < 1$ with the boundary conditions $u(0;t) = 0$, $u(1;t) = 0$ and the initial condition $u(x;0) = 4x$. Use the mesh size $\Delta x = \frac{1}{4}$, $\Delta t = \frac{1}{32}$.

8. Show that any function of the form $u_i(t) = f(x_i + ct) + g(x_i - ct)$ is a solution of the system of difference equations $u_i(t + \Delta t) + u_i(t - \Delta t) = u_{i+1}(t) + u_{i-1}(t)$, provided that $\Delta x/\Delta t = c$.

7.3. Equations in Several Dimensions

In this section we formulate and obtain the numerical solution of the heat equation and the Laplace equation in two and three dimensions. The methods are adapted to domains of arbitrary shape, many of which do not lend themselves naturally to a solution by separation of variables. Hence numerical solutions become an indispensable tool in obtaining the solution of the boundary-value problem.

We consider first the two-dimensional case. To study either the heat equation or the Laplace equation, we must obtain a suitable finite difference replacement for the Laplacian $\nabla^2 u = u_{xx} + u_{yy}$. There are many ways of doing this, and we choose the simplest, consistent with our treatment in Sec. 7.2. We choose a mesh of points (x_i, y_j) with $x_{i+1} - x_i = \Delta x$, $y_{i+1} - y_i = \Delta y$ and set $u_{ij} =$

$u(x_i, y_j)$. Making the usual replacements for u_{xx} and u_{yy}, we obtain the following replacement for $\nabla^2 u$:

$$\frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{(\Delta x)^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{(\Delta y)^2}$$

If the mesh sizes have a common value $\Delta x = \Delta y = h$, then the formula simplifies to

$$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

This can be paraphrased as follows. We take the average of u at the four neighboring points $(i+1,j)$, $(i-1,j)$, $(i,j+1)$, $(i,j-1)$ and subtract the value of u at the base point (i,j) (see Fig. 7.3.1).

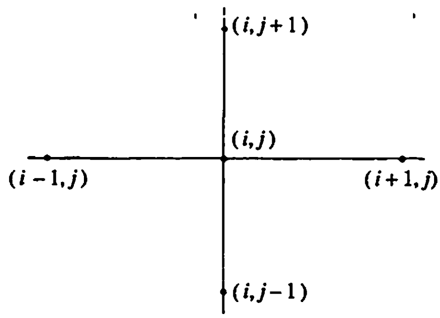


FIGURE 7.3.1 Computational molecule for the Laplacian.

7.3.1. Heat equation in a triangular region. We now consider the heat equation $u_t = K\nabla^2 u$. Replacing u_t by $[u_{i,j}(t+\Delta t) - u_{i,j}(t)]/\Delta t$, we obtain the finite difference equation

$$\frac{u_{i,j}(t+\Delta t) - u_{i,j}(t)}{\Delta t} = K\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}$$

which is solved to yield

$$\begin{aligned} u_{i,j}(t+\Delta t) &= \frac{K\Delta t}{h^2}[u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t)] \\ &\quad + \left(1 - \frac{4K\Delta t}{h^2}\right) u_{i,j}(t) \end{aligned}$$

This is a weighted average of the five displayed values, provided $4K\Delta t/h^2 \le 1$, which is the appropriate stability condition for the two-dimensional heat equation. When this is satisfied, the approximate numerical solutions obtained can be expected to yield a suitable approximation to the true solution when the mesh size is small.

EXAMPLE 7.3.1. *Solve the heat equation $u_t = u_{xx} + u_{yy}$ for $t > 0$ in the triangular region $0 < x < y < 1$ with the boundary conditions that $u = 0$ on the three sides $x = 0$, $y = 1$, and $x = y$. Use the initial condition $u(x, y, 0) = 8x(1-y)$ and the mesh size $h = \Delta x = \Delta y = \frac{1}{4}$, $\Delta t = \frac{1}{64}$.*

Solution. We have $K\Delta t/h^2 = \frac{1}{4}$ and the difference scheme

$$u_{ij}(t + \Delta t) = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1})(t)$$

The numerical values are represented in the triangular tables in Fig. 7.3.2, corresponding to $t = 0$, $t = \frac{1}{64}$, $t = \frac{1}{32}$, $t = \frac{3}{64}$, $t = \frac{1}{16}$. •

$t = 0$

$y \backslash x$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
1	0	0	0	0	0
$\frac{3}{4}$	0	$\frac{1}{2}$	1	0	
$\frac{1}{2}$	0	1	0		
$\frac{1}{4}$	0	0			
0	0				

$t = \frac{1}{64}$

$y \backslash x$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
1	0	0	0	0	0
$\frac{3}{4}$	0	$\frac{1}{2}$	$\frac{1}{8}$	0	
$\frac{1}{2}$	0	$\frac{1}{8}$	0		
$\frac{1}{4}$	0	0			
0	0				

$t = \frac{1}{32}$

$y \backslash x$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
1	0	0	0	0	0
$\frac{3}{4}$	0	$\frac{1}{16}$	$\frac{1}{8}$	0	
$\frac{1}{2}$	0	$\frac{1}{8}$	0		
$\frac{1}{4}$	0	0			
0	0				

$t = \frac{3}{64}$

$y \backslash x$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
1	0	0	0	0	0
$\frac{3}{4}$	0	$\frac{1}{16}$	$\frac{1}{64}$	0	
$\frac{1}{2}$	0	$\frac{1}{64}$	0		
$\frac{1}{4}$	0	0			
0	0				

$t = \frac{1}{16}$

$y \backslash x$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
1	0	0	0	0	0
$\frac{3}{4}$	0	$\frac{1}{128}$	$\frac{1}{64}$	0	
$\frac{1}{2}$	0	$\frac{1}{64}$	0		
$\frac{1}{4}$	0	0			
0	0				

FIGURE 7.3.2 Numerical solution of the heat equation.

7.3.2. Laplace's equation in a triangular region. The numerical solution of the heat equation is characterized by an *explicit procedure*; i.e., the values of $u_{i,j}(t + \Delta t)$ are obtained explicitly as linear combinations of $u_{k,l}(t)$ for certain values of (k, l). This feature is not present in the numerical solution of Laplace's equation, where we must solve a system of linear equations to obtain the approximate numerical solution. Consider, for example, the following boundary-value problem for Laplace's equation in the triangular region $0 < x < y < 1$:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 & 0 < x < y < 1 \\ u(0, y) &= f_1(y) & 0 < y < 1 \\ u(x, 1) &= f_2(x) & 0 < x < 1 \\ u(x, x) &= f_3(x) & 0 < x < 1 \end{aligned}$$

To solve this problem by the method of finite differences, we take a mesh with $h = \Delta x = \Delta y = 1/N$ and replace $u(i/N, j/N)$ by $u_{i,j}$. The Laplace equation is replaced by the difference equation

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \qquad 0 < i < j < N$$

and the boundary conditions are replaced by the equations

$$\begin{aligned} u_{0,j} &= f_1(j/N) \quad 0 \le j \le N \\ u_{i,N} &= f_2(i/N) \quad 0 < i \le N \\ u_{i,i} &= f_3(i/N) \quad 0 < i < N \end{aligned}$$

This is a system of $(N+1)(N+2)/2$ equations for the unknowns $u_{i,j}, 0 \le i \le j \le N$. It can be shown that this system of linear equations has a unique solution for any choice of the functions f_1, f_2, f_3. If N is small, the solution can be found by elementary linear algebra. If N is large, it may be necessary to do extensive machine computation in order to obtain the approximate numerical solution.

EXAMPLE 7.3.2. *Find the approximate numerical solution of Laplace's equation $u_{xx} + u_{yy} = 0$ in the triangle $0 < x < y < 1$ with the boundary conditions $u(0,y) = 0$, $u(x,1) = x(1-x)$, and $u(x,x) = 0$. Use the mesh size $h = \Delta x = \Delta y = \frac{1}{4}$.*

Solution. Replacing $u(i/4, j/4)$ by $u_{i,j}$, we have the equations

$$\begin{aligned} u_{11} + u_{13} + u_{02} + u_{22} - 4u_{12} &= 0 \\ u_{12} + u_{14} + u_{03} + u_{23} - 4u_{13} &= 0 \\ u_{22} + u_{24} + u_{13} + u_{33} - 4u_{23} &= 0 \end{aligned}$$

$$\begin{aligned} u_{00} &= 0, \quad u_{01} = 0, \quad u_{02} = 0, \quad u_{03} = 0, \quad u_{04} = 0 \\ u_{14} &= \frac{3}{16}, \quad u_{24} = \frac{1}{4}, \quad u_{34} = \frac{3}{16}, \quad u_{44} = 0 \\ u_{11} &= 0, \quad u_{22} = 0, \quad u_{33} = 0 \end{aligned}$$

Making the appropriate substitutions, we have

$$u_{13} - 4u_{12} = 0, \qquad u_{12} + \frac{3}{16} + u_{23} - 4u_{13} = 0, \qquad \frac{1}{4} + u_{13} - 4u_{23} = 0$$

The solution of this system of three equations is $u_{12} = \frac{1}{56}, u_{13} = \frac{1}{14}, u_{23} = \frac{9}{112}$. •

For larger values of N, this numerical method for solving Laplace's equation leads to large systems of linear equations that may be difficult to solve. To deal with such cases, we regard the solution of Laplace's equation as the limit of the solution of the heat equation when the time becomes large. In symbols,

$$u(x,y) = \lim_{t \to \infty} u(x,y;t)$$

The function $u(x,y;t)$ is the solution of the heat equation $u_t = K(u_{xx} + u_{yy})$, satisfying the same boundary conditions as $u(x,y)$ and with an arbitrary initial condition—for example, $u(x,y;0) = 0$. By an appropriate choice of the initial conditions we can obtain an effective numerical algorithm for an approximate numerical solution of Laplace's equation.

We illustrate these ideas with the following numerical example.

EXAMPLE 7.3.3. *Find the approximate numerical solution of Laplace's equation in the triangle* $0 < x < y < 1$ *with the boundary conditions* $u(0,y) = 0$, $u(x,1) = 1$, $u(x,x) = 0$. *Use the mesh size* $h = \Delta x = \Delta y = \frac{1}{6}$.

Solution. We solve the heat equation $u_t = u_{xx} + u_{yy}$ numerically, with the initial condition $u(x,y;0) = 0$ and with the same boundary conditions: $u(0,y;t) = 0$, $u(x,1;t) = 1$, $u(x,x;t) = 0$. Choosing $\Delta t = \frac{1}{144}$ gives the difference scheme $u_{ij}(t+\Delta t) = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1})(t)$. Figure 7.3.3 presents numerical tables giving the values of the first five iterations, correct to three decimals, for $t = 0, \frac{1}{72}, \frac{2}{72}, \frac{3}{72}, \frac{4}{72}, \frac{5}{72}, \frac{6}{72}$. The values u_{06}, u_{66}, and u_{00} are not listed, since they are not used in the iterations. •

Initial conditions

$x \backslash y$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
1		1	1	1	1	1	
$\frac{5}{6}$	0	0	0	0	0	0	
$\frac{2}{3}$	0	0	0	0	0		
$\frac{1}{2}$	0	0	0	0			
$\frac{1}{3}$	0	0	0				
$\frac{1}{6}$	0	0					
0							

First iteration

$x \backslash y$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
1		1	1	1	1	1	
$\frac{5}{6}$	0	0.250	0.250	0.250	0.250	0	
$\frac{2}{3}$	0	0	0	0	0		
$\frac{1}{2}$	0	0	0	0			
$\frac{1}{3}$	0	0	0				
$\frac{1}{6}$	0	0					
0							

Second iteration

$x \backslash y$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
1		1	1	1	1	1	
$\frac{5}{6}$	0	0.313	0.375	0.375	0.375	0	
$\frac{2}{3}$	0	0.063	0.063	0.063	0		
$\frac{1}{2}$	0	0	0	0			
$\frac{1}{3}$	0	0	0				
$\frac{1}{6}$	0	0					
0							

Third iteration

$x \backslash y$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
1		1	1	1	1	1	
$\frac{5}{6}$	0	0.359	0.438	0.453	0.343	0	
$\frac{2}{3}$	0	0.094	0.125	0.169	0		
$\frac{1}{2}$	0	0.016	0.016	0			
$\frac{1}{3}$	0	0	0				
$\frac{1}{6}$	0	0					
0							

Fourth iteration

$x \backslash y$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
1		1	1	1	1	1	
$\frac{5}{6}$	0	0.383	0.484	0.473	0.348	0	
$\frac{2}{3}$	0	0.125	0.164	0.145	0		
$\frac{1}{2}$	0	0.027	0.035	0			
$\frac{1}{3}$	0	0.004	0				
$\frac{1}{6}$	0	0					
0							

Fifth iteration

$x \backslash y$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
1		1	1	1	1	1	
$\frac{5}{6}$	0	0.402	0.505	0.494	0.368	0	
$\frac{2}{3}$	0	0.144	0.197	0.159	0		
$\frac{1}{2}$	0	0.041	0.048	0			
$\frac{1}{3}$	0	0.007	0				
$\frac{1}{6}$	0	0					
0							

•

FIGURE 7.3.3 From the heat equation to Laplace's equation.

EXERCISES 7.3

1. Obtain the approximate numerical solution of the heat equation $u_t = u_{xx} + u_{yy}$ in the square $0 < x < 1$, $0 < y < 1$ with the boundary condition $u(x, y; t) = 0$ on all four sides and the initial condition $u(x, y; 0) = 8xy$. Use the mesh size $\Delta x = \Delta y = \frac{1}{4}$, $\Delta t = \frac{1}{64}$ and solve for $0 \leq t \leq \frac{1}{16}$.
2. Obtain the approximate numerical solution of the heat equation $u_t = u_{xx} + u_{yy}$ in the L-shaped region obtained by removing the square $0 < x < 1$, $0 < y < 1$ from the square $0 < x < 2$, $0 < y < 2$. Use the boundary condition $u(x, y; t) = 0$ on all six sides and the initial condition $u(x, y; 0) = 4x + 2y$. Use the mesh size $\Delta x = \Delta y = \frac{1}{4}$, $\Delta t = \frac{1}{64}$ and solve for $0 \leq t \leq \frac{1}{16}$.
3. Obtain the approximate numerical solution of the Laplace equation $u_{xx} + u_{yy} = 0$ in the square $0 < x < 1, 0 < y < 1$ with the boundary conditions $u(x, 0) = 0$, $u(x, 1) = 0$, $u(0, y) = 0$, $u(1, y) = 1$. Use the mesh size $\Delta x = \Delta y = \frac{1}{4}$ and explicitly solve the resulting system of linear equations.
4. Solve Exercise 3 by the iterative method of Example 7.3.3 by solving the heat equation with the initial condition $u(x, y; 0) = 0$. Compare the result of five iterations with the result obtained in Exercise 3.
5. Solve Exercise 3 with the mesh size $\Delta x = \Delta y = \frac{1}{10}$ by using the iterative method of Example 7.3.3, using five iterations.
6. Without reference to numerical solutions, sketch a proof of the following theorem, implicit in Example 7.3.3: *Suppose that $u(x, y; t)$ is a solution of the heat equation $u_t = K(u_{xx} + u_{yy})$ in the square $0 < x < 1$, $0 < y < 1$ with boundary conditions $u(0, y; t) = f_1(y)$, $u(1, y; t) = f_2(y)$, $u(x, 0; t) = f_3(x)$, $u(x, 1; t) = f_4(x)$. Then $\lim_{t\to\infty} u(x, y; t) = u(x, y)$, the solution of Laplace's equation with the same boundary conditions.*
7. In Exercise 6, replace the square by the triangle $0 < x < y < 1$. State and prove a theorem analogous to Exercise 6.
8. Formulate a difference scheme to obtain the approximate numerical solution of the heat equation $u_t = K(u_{xx} + u_{yy} + u_{zz})$ in the three-dimensional cube $0 < x < 1$, $0 < y < 1$, $0 < z < 1$. Formulate an appropriate stability condition for the difference scheme.
9. Obtain the approximate numerical solution of Exercise 8 when the boundary condition is $u(x, y, z; t) = 0$ on all six faces and the initial condition is $u(x, y, z; 0) = 1$. Use the mesh size $\Delta x = \Delta y = \Delta z = \frac{1}{4}$ and the largest possible time step Δt consistent with the stability condition.
10. Formulate the approximate numerical solution of Laplace's equation $u_{xx} = u_{yy} + u_{zz} = 0$ in the three-dimensional cube $0 < x < 1$, $0 < y < 1$, $0 < z < 1$ corresponding to the mesh size $\Delta x = \Delta y = \Delta z = \frac{1}{4}$. How many simultaneous linear equations must one solve to obtain an approximate numerical solution by this method?

7.4. Variational Methods

The method of finite differences is well suited to numerical approximations in domains with a rectangular boundary. When we turn to regions with a more general polygonal or curvilinear boundary, other methods are more efficient for obtaining numerical approximations.

Many solutions of partial differential equations that come from physics can be characterized as solutions of *variational problems*, where one is required to minimize a certain functional with boundary conditions. This immediately suggests many constructive approaches for finding an approximate solution.

7.4.1. Variational formulation of Poisson's equation. Consider the solution of Poisson's equation $\nabla^2 u = -\rho$ in the interior of a bounded three-dimensional region D, with the condition that $u = 0$ on the boundary, denoted ∂D. We claim that this solution can be obtained as the solution of the following minimization problem: minimize

$$\iiint_D \left[\frac{1}{2}|\nabla u|^2 - \rho u\right] dV = \iiint_D \left[\frac{1}{2}(u_x^2 + u_y^2 + u_z^2) - \rho u\right] dx\,dy\,dz$$

subject to $u = 0$ on ∂D. Indeed, suppose that there exists a smooth function $u_0(x, y, z)$ that achieves the stated minimum, in competition with all smooth functions that satisfy the boundary condition. We have to show that u_0 satisfies Poisson's equation.

To do this, consider the function $u_0 + \epsilon v$, where $\epsilon > 0$ and $v(x, y, z)$ is any smooth function that satisfies the boundary condition. By hypothesis

$$\iiint_D \left[\frac{1}{2}|\nabla(u_0 + \epsilon v)|^2) - \rho(u_0 + \epsilon v)\right] dV \geq \iiint_D \left[\frac{1}{2}|\nabla u_0|^2) - \rho u_0\right] dV$$

Expanding the left side and simplifying, we have the inequality

$$\iiint_D \left(\epsilon \nabla u_0 \cdot \nabla v + \frac{1}{2}\epsilon^2 |\nabla v|^2 - \epsilon \rho v\right) dV \geq 0$$

This quadratic function of ϵ has a minimum when $\epsilon = 0$; hence the derivative must be zero, which is written

$$0 = \iiint_D (\nabla u_0 \cdot \nabla v - \rho v)\, dV$$

The first term can be transformed using the boundary condition $v = 0$ and the divergence theorem as follows:

$$\begin{aligned}
0 &= \iint_{\partial D} v(\nabla u_0 \cdot \mathbf{n})\, dS \\
&= \iiint_D \operatorname{div}(v \cdot \nabla u_0)\, dV \\
&= \iiint_D (v\nabla^2 u_0 + \nabla v \cdot \nabla u_0)\, dV \\
&= \iiint_D v(\nabla^2 u_0 + \rho)\, dV
\end{aligned}$$

In order to conclude that $\nabla^2 u_0 + \rho = 0$, we develop the following useful principle.

LEMMA 7.4.1. **(Fundamental lemma of the calculus of variations).** *Suppose that $w(P) = w(x, y, z)$ is a continuous function in a region D such that*

$$\iiint_D \varphi(P) w(P)\, dV = \iiint_D \varphi(x, y, z) w(x, y, z)\, dx\, dy\, dz = 0$$

for all differentiable functions φ for which $\varphi = 0$ on ∂D. Then $w(x, y, z) = 0$ in D.

Proof. Suppose $w(P_0) > 0$. Then there is some $\delta > 0$ such that $\delta < \operatorname{dist}(P_0, \partial D)$ and $w(P) \geq w(P_0)/2$ for $|P - P_0| < \delta$. Let $\varphi(P) = (\delta^2 - |P - P_0|^2)^2$ for $|P - P_0| < \delta$ and $\varphi(P) = 0$ elsewhere. Then φ is differentiable in D and we can apply the hypothesis to obtain

$$\begin{aligned}
0 &= \iiint_D \varphi(P) w(P)\, dV \\
&= \iiint_{|P - P_0| < \delta} \varphi(P) w(P)\, dV \\
&\geq \iiint_{|P - P_0| < \delta/2} \varphi(P) w(P)\, dV \\
&\geq \frac{|w(P_0)|}{2} \iiint_{|P - P_0| < \delta/2} \varphi(P)\, dV
\end{aligned}$$

But $\varphi(P) > 0$ when $|P - P_0| < \delta/2$, which implies that the final integral is positive—a contradiction. If instead we assume that $w(P_0) < 0$, the above argument can be applied to $-w(P)$; in either case we obtain the conclusion that $w(P_0) = 0$, which was to be proved. •

7.4.2. More general variational problems. Going beyond the simple case of Poisson's equation, we consider the problem of minimizing the more general functional

$$I(u) := \iiint_D F(x, y, z, u, u_x, u_y, u_z) dV$$

with the condition that on the boundary ∂D we have $u = f$, a given continuous function.

The corresponding partial differential equation is the *Euler-Lagrange equation* associated with the functional I. In detail, this is written

$$\partial_x F_5 + \partial_y F_6 + \partial_z F_7 = F_4$$

where F_4, F_5, F_6, F_7 refer to the partial derivatives of F with respect to the variables u, u_x, u_y, u_z, respectively. In the case of Poisson's equation, we have $F = \frac{1}{2}(u_x^2 + u_y^2 + u_z^2) - \rho u$ with $F_4 = -\rho$, $F_5 = u_x$, $F_6 = u_y$, $F_7 = u_z$.

EXAMPLE 7.4.1. *Consider the functional*

$$I(u) = \iiint_D \left[\frac{1}{2}(u_x^2 + u_y^2 + u_z^2 + du^2) + auu_x + buu_y + cuu_z - \rho u\right] dV$$

where a, b, c, d are differentiable functions. Find the Euler-Lagrange equation.

Solution. We have $F_4 = du + au_x + bu_y + cu_z - \rho$, while $F_5 = u_x + au$, $F_6 = u_y + bu$, $F_7 = u_z + cu$. The Euler-Lagrange equation is

$$(u_x + au)_x + (u_y + bu)_y + (u_z + cu)_z = du + au_x + bu_y + cu_z - \rho$$

When we expand the derivatives on the left side and simplify, the result is the equation

$$\nabla^2 u + (a_x + b_y + c_z - d)u = -\rho \quad \bullet$$

7.4.3. Variational formulation of eigenvalue probems. As a second illustration of variational methods, consider the problem of determining the vibrating frequencies of a two-dimensional region. These are the eigenvalues (λ_n) obtained by solving the boundary-value problem $\nabla^2 u + \lambda u = 0$ with the condition that $u = 0$ on the boundary.

The variational approach begins with the functional

$$I(u) = \frac{\iint_D (u_x^2 + u_y^2)\, dx\, dy}{\iint_D u^2\, dx\, dy}$$

Suppose that $I(u)$ has a minimum when $u = u_0$, in competition with all smooth functions that are zero on the boundary. We will show that $\nabla^2 u_0 + \lambda_1 u_0 = 0$, where $\lambda_1 = I(u_0)$.

To show this, we write the hypothesis in the form

$$\iint_D |\nabla(u_0 + \epsilon v)|^2 dx\, dy \geq \lambda_1 \iint_D |u_0 + \epsilon v|^2 dx\, dy$$

where v is an arbitrary smooth function with $v = 0$ on ∂D and ϵ is an arbitrary real number. Expanding and simplifying, we have

$$2\epsilon \iint_D (\nabla u_0 \cdot \nabla v - \lambda_1 u_0 v)\,dx\,dy + \epsilon^2 \iint_D (|\nabla v|^2 - \lambda_1 v^2)\,dx\,dy \geq 0$$

This quadratic function has a minimum at $\epsilon = 0$; hence the first integral is zero. Transforming the first term by the divergence theorem as before and using the boundary condition $v = 0$, we obtain the condition

$$0 = \iint_D v(\nabla^2 u_0 + \lambda_1 u_0)\,dx\,dy = 0$$

Since this holds for all smooth functions that satisfy the boundary condition $v = 0$, we may apply the fundamental lemma of the calculus of variations to conclude that the term in parentheses is identically zero. Thus $\nabla^2 u_0 + \lambda_1^2 u_0 = 0$, as required.

We observe in passing that λ_1 must be the *smallest* eigenvalue of the region D with the given boundary conditions. Indeed, suppose λ_2 is another eigenvalue, associated with the equation $\nabla^2 w + \lambda_2 w = 0$. Then by applying the divergence theorem and the boundary conditions, we obtain $\lambda_2 = I(w)$. But the minimum of I is obtained as the value λ_1, from which we conclude that $\lambda_2 \geq \lambda_1$, as required.

7.4.4. Variational problems, minimization, and critical points. The *critical points* of a functional are, by definition, the points u_0 where the first derivative is zero. In detail,

$$\lim_{\epsilon \to 0} \frac{I(u_0 + \epsilon v) - I(u_0)}{\epsilon} = 0$$

for all admissible directions v. As in elementary calculus, it is not true that all critical points of a function(al) correspond to maxima or minima. Nevertheless, it has been found extremely useful to formulate many of the basic principles of mechanics as *variational principles*, namely, to find the critical points of a functional. In static, time-independent problems, these often correspond to maxima or minima of the functional. But in dynamic, time-dependent problems this is often not the case. A prototype example relative to the vibrating string is the Euler-Lagrange equation corresponding to the functional

$$I(u) = \int_0^T \int_0^L (c^2 u_x^2 - u_t^2)\,dx\,dt$$

with the boundary conditions $u(0;t) = 0$, $u(L;t) = 0$, $u(x;0) = A(x)$, $u(T;x) = B(x)$, given functions. The associated Euler-Lagrange equation is the wave equation $u_{tt} = c^2 u_{xx}$, so that any critical point of $I(u)$ must satisfy the wave equation. But there are many solutions of the wave equation that are critical points of $I(u)$ not corresponding to local maxima or minima. The exercises below provide further details.

EXERCISES 7.4

1. Let D be a two-dimensional region, and let u_0 be the solution of Poisson's equation $u_{xx} + u_{yy} = -\rho$ in D with $u = 0$ on ∂D. Show that $I(u) \geq I(u_0)$ where

$$I(u) = \iint_D \left(\frac{1}{2}|\nabla u|^2 - \rho u\right) dx\, dy$$

2. Let D be a three-dimensional region and let

$$I(u) = \iiint_D \left[\frac{1}{2}(Au_x^2 + Bu_y^2 + Cu_z^2 + Du^2) - \rho u\right] dx\, dy\, dz$$

Suppose that u_0 is a smooth function that minimizes $I(u)$ in competition with all smooth functions that are zero on ∂D. Prove that u_0 satisfies the equation $(Au_x)_x + (Bu_y)_y + (Cu_z)_z = -\rho$ in D.

3. Let D be a two-dimensional region and let

$$I(u) = \iint_D \left[\frac{1}{2}(Au_x^2 + 2Bu_xu_y + Cu_y^2 + Du^2) - \rho u\right] dx\, dy$$

Suppose u_0 is a smooth function that minimizes $I(u)$ in competition with all once-differentiable functions that are zero on ∂D. Prove that u_0 satisfies the equation $(Au_x + Bu_y)_y + (Bu_x + Cu_y)_y + Du = -\rho$ in D.

4. Let D be a two-dimensional region and let

$$I(u) = \iint_D \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy$$

(the surface area functional). Suppose that u_0 is a twice-differentiable function that minimizes $I(u)$ in competition with all twice-differentiable functions for which $u = f$ on ∂D. Prove that u_0 satisfies the equation

$$(1 + u_x^2)u_{xx} + (1 + u_y^2)u_{yy} - 2u_xu_yu_{xy} = 0$$

This is called the *minimal surface equation.*

5. Let D be a two-dimensional region and let

$$I(u) = \iint_D \left[\frac{1}{2}(u_{xx} + u_{yy})^2 + \rho u\right] dx\, dy$$

where ρ is a continuous function. Suppose that u_0 is a smooth function that minimizes $I(u)$ in competition with all twice-differentiable functions for which $u = 0$ and $\partial u/\partial n = 0$ on ∂D. Prove that u_0 satisfies the equation $\nabla^2\nabla^2 u_0 = -\rho$ in D.

6. Let D be a two-dimensional region and let

$$I(u) = \iint_D \left[\frac{1}{2}(u_x^2 + u_y^2) - \rho u\right] dx\, dy$$

Suppose that u_0 is a smooth function that minimizes $I(u)$ in competiton with all once-differentiable functions u (assuming no boundary conditions). Prove that
   (a) $\nabla^2 u_0 = -\rho$ in D
   (b) $\partial u/\partial n = 0$ on ∂D
   (c) $\iint_D \rho(x, y)\, dx\, dy = 0$
7. Let D be a two-dimensional region and let

$$I(u) = \iint_D \left[\frac{1}{2}(u_x^2 + u_y^2) - \rho u\right] dx\, dy - \int_{\partial D} f(s)u(s)\, ds$$

where f is a given continuous function on ∂D. Suppose that u_0 is a smooth function that minimizes $I(u)$ in competition with all once-differentiable functions u (assuming no boundary conditions). Prove that
   (a) $\nabla^2 u_0 = -\rho$ in D
   (b) $\partial u_0/\partial n = f$ on ∂D
8. Let D be a two-dimensional region and let

$$I(u) = \iint_D \left[\frac{1}{2}(u_x^2 + u_y^2) - \rho u\right] dx\, dy - \int_{\partial D} (f(s)u(s) - g(s)u(s)^2)\, ds$$

where $f(s), g(s)$ are given continuous functions on ∂D. Suppose that u_0 is a smooth function that minimizes $I(u)$ in competition with all once-differentiable functions. Find the partial differential equation satisfied by u_0.
9. Repeat the previous exercise for the functional

$$I(u) = \iint_D \varphi(x, y, u, u_x, u_y)\, dx\, dy + \int_{\partial D} \psi(s, u, \partial u/\partial s)\, ds$$

to find the partial differential equation and boundary condition satisfied by u_0. Assume the necessary smoothness of φ and ψ.

The following exercises provide examples in which the solution of the Euler-Lagrange equation is not a maximum or minimum of the functional whose stationarity is sought. This clarifies the "principle of least action" in mechanics. Exercises 10 and 11 treat the simple harmonic oscillator, and Exercise 12 treats the wave equation.

10. Consider the functional $I_T(u) = \int_0^T [u'(t)^2 - u(t)^2]dt$ with the boundary conditions $u(0) = A$, $u(T) = B$.
    (a) Show that any minimum u_0 must satisy the equation $u'' + u = 0$.

(b) If $T \neq \pi, 2\pi, \ldots$, show that there is a unique solution v of the equation $u'' + u = 0$ satisfying the boundary conditions.

(c) If $u(t) = v(t) + \sum_{n=1}^{\infty} c_n \sin(n\pi t/T)$ is another competing function, show that

$$I_T(u) = I_T(v) + \frac{T}{2}\sum_{n=1}^{\infty} c_n^2 \left[\left(\frac{n\pi}{T}\right)^2 - 1\right]$$

(d) Conclude that if $T < \pi$, then $I_T(u) \geq I_T(v)$ with equality if and only if $u(t) \equiv v(t)$.

11. With the same choice of $I_T(u)$, show that if $T > \pi$, we have $\sup I_T(u) = +\infty$, $\inf I_T(u) = -\infty$. Hence we have no maximum or minimum in case $T > \pi$.

12. Consider the functional

$$I_T(u) = \int_0^T \int_0^\pi (u_t^2 - u_x^2)\, dx\, dt$$

with the boundary conditions $u(0;t) = 0$, $u(\pi;t) = 0$, $u(x;0) = 0$, $u(x;T) = B \sin x$.

(a) Show that the Euler-Lagrange equation is the wave equation $u_{tt} = u_{xx}$ and solve it if T is not of the form π, 2π, 3π, $\ldots$.

(b) Compute the value of $I_T(u)$ for a function of the form $u(x;t) = (Bt/T)\sin x + a \sin Nx \sin(k\pi t/T)$ where N, k are integers and a is any constant.

(c) By choosing k, N, a appropriately, show that we have $\inf I_T(u) = -\infty$, $\sup I_T(u) = +\infty$ over the indicated class of functions, for any $T > 0$, no matter how small. Therefore there is no maximum or minimum in this variation problem, for any $T > 0$.

13. (Exercise 10 without Parseval's theorem) Let $T < \pi$ and define the function $\varphi(t) = \cot[t + (\pi - T)/2]$ for $0 < t < T$, so that $\varphi' + 1 = -\varphi^2$.

(a) Prove the identity

$$\int_0^T (u' - \varphi u)^2\, dt = I_T(u) - \varphi^2 u|_0^T$$

for any differentiable function $u(t), 0 \leq t \leq T$.

(b) Conclude that if $T < \pi$, the functional $I_T(u)$ is bounded below by a constant, for all differentiable functions $u(t), 0 \leq t \leq T$, satisfying the boundary conditions $u(0) = A, u(T) = B$.

7.5. Approximate Methods of Ritz and Kantorovich

Having established the theoretical connection between certain partial differential equations and related minimization problems, we now turn to some approximate methods of solution. All such methods involve the use of *trial solutions*, which are functions that satisfy the boundary conditions and are determined by additional conditions that make them close to the desired minimum. These additional conditions may be obtained by means such as

- Solution of a minimum problem in a finite number of parameters
- Orthogonality conditions in a finite number of parameters
- Solution of a related ordinary differential equation
- Combinations of the above three methods

In this section we discuss the methods of Ritz and Kantorovich, both of which involve minimization in a finite number of parameters. In Sec. 7.6 we proceed to the Galerkin method and the finite element method, which involve orthogonality relations in a finite number of parameters.

7.5.1. The Ritz method: Rectangular regions. The general idea behind the Ritz method is to look for a trial solution in the form

$$u(x,y,z) = U(x,y,z;c_1,c_2,\dots,c_n)$$

which satisfies the boundary condition identically, where $c_1,\dots,c_n$ are parameters that may be adjusted. If we substitute this function into the variational problem, we obtain the problem of minimizing a function of a finite number of parameters, namely,

$$\Phi(c_1,\dots,c_n) = \iiint_D F(x,y,z;u,u_x,u_y,u_z)\,dx\,dy\,dz$$

We minimize this function of n parameters by setting the partial derivatives to zero:

$$\frac{\partial\Phi}{\partial c_i}(c_1,\dots,c_n) = 0 \qquad 1 \le i \le n$$

We then substitute these values of $c_1,\dots,c_n$ into the function u to find the approximate minimizer. We illustrate the method with a problem in two variables.

EXAMPLE 7.5.1. *Find the approximate solution of Poisson's equation* $\nabla^2 u = -\rho$ *in the square* $|x| < a$, $|y| < a$ *with the boundary condition* $u = 0$, *where* ρ *is a constant. Use the trial solution* $u = c(x^2 - a^2)(y^2 - a^2)$.

Solution. We have $u_x = 2cx(y^2 - a^2)$, $u_y = 2cy(x^2 - a^2)$, $(u_x^2 + u_y^2)/2 = 2c^2(x^4y^2 + x^2y^4 - 4a^2x^2y^2 + a^4x^2 - a^4y^2)$, which leads to $(1/2)\iint (u_x^2 + u_y^2)\,dx\,dy = (128/45)a^8c^2$. Likewise $\iint \rho u\,dx\,dy = (16/9)\rho c a^6$. Thus we are required to minimize the function $\Phi(c) = (128/45)a^8c^2 - (16/9)\rho c a^6$. The required mimimum is attained at $c = (5/16)\rho/a^2$, which gives the required trial solution with $\Phi(c) = -(5/18)\rho^2 a^4 \sim -0.277\rho^2 a^4$. •

EXAMPLE 7.5.2. *Find the approximate solution of Poisson's equation $\nabla^2 u = -\rho$ in the square $|x| < a$, $|y| < a$ with the boundary condition $u = 0$, where ρ is a constant. Use a trial solution of the form $u = c\cos(\pi x/2a)\cos(\pi y/2a)$ and compare with Example 7.5.1.*

Solution. We have

$$u_x = -(c\pi/2a)\sin(\pi x/2a)\cos(\pi y/2a), \quad u_y = -(c\pi/2a)\cos(\pi x/2a)\sin(\pi y/2a)$$

which leads to $(1/2)\iint(u_x^2+u_y^2)\,dx\,dy = \pi^2c^2/4$. Likewise $\iint \rho u\,dx\,dy = \rho c(4a/\pi)^2$. Thus we are required to minimize the function $\Phi(c) = \pi^2c^2/4 - (\rho c)(4a/\pi)^2$. The required minimum is attained at $c = 32\rho a^2/\pi^4$, which gives the required trial solution with $\Phi(c) = -(256/\pi^6)\rho^2a^4 \sim -0.266\rho^2a^4$, which is 4 percent larger than the minimum obtained in Example 7.5.1. •

We note here that the Ritz method may be applied to problems in any number of variables. In particular, it may be applied to variational problems for *ordinary* differential equations (see the exercises).

7.5.2. The Kantorovich method: Rectangular regions. This method begins with a trial solution that contains one or more arbitrary *functions* and that identically satisfies the boundary conditions. The functions are chosen to solve the Euler-Lagrange equation that results from minimizing the composite functional.

EXAMPLE 7.5.3. *Find the approximate solution of Poisson's equation $\nabla^2 u = -\rho$ in the square $|x| < a$, $|y| < a$ with the boundary condition $u = 0$, where ρ is a constant. Use a trial solution of the form $u = (a^2 - y^2)c(x)$ where $c(x)$ is an arbitrary function.*

Solution. We have $u_x = (a^2 - y^2)c'(x)$, $u_y = -2y\,c(x)$ and thus

$$\iint_D \left(\frac{1}{2}(u_x^2 + u_y^2) - \rho u\right) dx\,dy = \int_{-a}^{a} \left(\frac{8a^5}{15}c'(x)^2 + \frac{4a^3}{3}c(x)^2 - \frac{4\rho a^3}{3}c(x)\right) dx$$

The Euler-Lagrange equation for this functional is $c''(x) = (5/2a^2)c(x) - (5\rho/4a^2)$. The solution with the boundary conditions $c(-a) = 0$, $c(a) = 0$ is $c(x) = (\rho/2)(1 - \cosh kx/\cosh ka)$ where $ka = \sqrt{5/2} \sim 1.58$.

In order to compute $I(u)$, we note that if we multiply the Euler-Lagrange equation by $c(x)$ and integrate by parts, we obtain

$$\int_{-a}^{a} \left(c'(x)^2 + (5/2a^2)c(x)^2\right) dx = (5\rho/4a^2)\int_{-a}^{a} c(x)\,dx$$

so that for this solution we have

$$I(u) = -(2/3)\rho a^3 \int_{-a}^{a} c(x)\,dx$$

A direct calculation yields

$$\int_{-a}^{a} c(x)\,dx = \frac{\rho}{2}\left(2a - \frac{2\sinh ka}{k\cosh ka}\right) = (0.42)\rho a$$

so that

$$I(u) = -(2/3)\rho a^3(0.42\rho a) \sim (-0.28)\rho^2 a^4$$

which is a smaller value than the one obtained in Example 7.5.1 or 7.5.2. •

EXAMPLE 7.5.4. *Find the approximate solution of Poisson's equation $u_{xx} + u_{yy} = -\rho$ in the square $|x| < a$, $|y| < a$ with the boundary condition that $u = 0$ on all four sides. Use trial solutions of the form $u = \sum_{j=0}^{N} c_j(x)\cos\left[\left(j+\frac{1}{2}\right)\pi y/a\right]$.*

Solution. For this trial solution, we have

$$u_x = \sum_{j=0}^{N} c_j'(x)\cos\left[\left(j+\frac{1}{2}\right)\frac{\pi y}{a}\right]$$

$$u_y = -\sum_{j=0}^{N} c_j(x)\left(j+\frac{1}{2}\right)\frac{\pi}{a}\sin\left[\left(j+\frac{1}{2}\right)\frac{\pi y}{a}\right]$$

$$\begin{aligned} I(u) &= \iint\left[\frac{u_x^2+u_y^2}{2} - \rho u\right]dx\,dy \\ &= \sum_{j=0}^{N}\int_{-a}^{a}\left[a\,c_j'(x)^2 + \left(j+\frac{1}{2}\right)^2\left(\frac{\pi}{a}\right)^2 - a\rho c_j(x)\frac{(-1)^j}{j+\frac{1}{2}}\left(\frac{a}{\pi}\right)\right]dx \end{aligned}$$

The Euler-Lagrange equation is

$$c_j''(x) = \left(j+\frac{1}{2}\right)^2\left(\frac{\pi}{a}\right)^2 c_j(x) - \frac{\rho(-1)^j}{j+\frac{1}{2}}\left(\frac{a}{\pi}\right)$$

The unique solution that satisfies the boundary conditions $c_j(-a) = 0 = c_j(a)$ is

$$c_j(x) = \left[\frac{\rho(-1)^j}{(j+\frac{1}{2})^3}\left(\frac{a}{\pi}\right)^3\right]\left(1 - \frac{\cosh(j+\frac{1}{2})(\pi x/a)}{\cosh[(j+\frac{1}{2})\pi]}\right)$$

Note that the resulting function $u(x,y)$ is the Nth partial sum of the Fourier representation of the solution of Poisson's equation. •

Unlike the Ritz method, the Kantorovich method can be applied only in two or more variables. It is specifially suited to partial differential equations, and has no counterpart for ordinary differential equations.

EXERCISES 7.5

1. Use the Ritz method to find an approximate minimum of the functional $I(u) = \int_0^1 [u'(t)^2 - u(t)^2 - 2t\,u(t)]\,dt$ with the boundary conditions $u(0) = u(1) = 0$. Use the following sets of trial functions:
   (a) $U(t) = c_0 t(1-t)$
   (b) $U(t) = t(1-t)(c_0 + c_1 t)$
   (c) $U(t) = t(1-t)(c_0 + c_1 t + c_2 t^2)$
   (d) $U(t) = \sum_{n=1}^{N} a_n \sin n\pi t$, where N is fixed but unspecified.

2. Use the Ritz method to find an approximate minimum of the functional $I(u) = \int_0^2 [u'(t)^2 + u(t)^2 - 2t\,u(t)]\,dt$ with the boundary conditions $u(0) = u(2) = 0$. Use the following sets of trial functions:
   (a) $U(t) = c_0 t(2-t)$
   (b) $U(t) = t(2-t)(c_0 + c_1 t)$
   (c) $U(t) = t(2-t)(c_0 + c_1 t + c_2 t^2)$
   (d) $U(t) = \sum_{n=1}^{N} a_n \sin(n\pi t/2)$, where N is fixed but unspecified.

3. Use the Ritz method to find an approximate minimum for the functional $I(u) = \int\int[\frac{1}{2}(u_x^2 + u_y^2) - xyu]\,dx\,dy$, where the integration is over the square $-1 \le x \le 1$, $-1 \le y \le 1$ and the boundary condition is that $u = 0$ on the four sides. Use the following sets of trial functions:
   (a) $U(x, y) = c(1-x^2)(1-y^2)$
   (b) $U(x, y) = (1-x^2)(1-y^2)(c_0 + c_1 x + c_2 y)$
   (c) $U(x, y) = (1-x^2)(1-y^2)(c_0 + c_1 x^2 + c_2 y^2)$

4. Use the Ritz method to find an approximate minimum for the functional $I(u) = \int\int[\frac{1}{2}(u_\rho^2 + \frac{1}{\rho^2} u_\rho]\rho\,d\rho\,d\varphi$, where the integration is over the disc $0 \le \rho \le 1$, $-\pi \le \varphi \le \pi$ and the boundary condition is that $u = 0$ when $\rho = 1$. Use the following sets of trial functions:
   (a) $U(\rho) = c(1-\rho^2)$
   (b) $U(\rho) = (1-\rho^2)(c_0 + c_1\rho)$
   (c) $U(\rho) = (1-\rho^2)(c_0 + c_1\rho + c_2\rho^2)$

5. Consider the Sturm-Liouville eigenvalue problem $\varphi'' + \lambda\varphi = 0$ on the interval $-1 < x < 1$ with the boundary conditions that $\varphi(-1) = 0$, $\varphi(1) = 0$.
   (a) Find (exactly) the smallest eigenvalue.
   (b) Use the Ritz method with the trial function $U(x) = 1 - x^2$ to find a first approximation to the smallest eigenvalue, and compare it with the exact result obtained in part (a).
   (c) Refine the approximation of part (b) by using the trial function $U(x) = (1-x^2)(1+cx^2)$ and show that the error obtained is less than 1 percent.

6. Use the Kantorovich method to find an approximate minimum for the functional $I(u) = \iint [\frac{1}{2}(u_x^2 + u_y^2) - xyu]\,dx\,dy$, where the integration is over the square $-1 \le x \le 1$, $-1 \le y \le 1$ and the boundary condition is that $u = 0$ on the four sides. Use the following sets of trial functions:
   (a) $U(x,y) = (1-y^2)(c_1(x) + y^2 c_2(x))$
   (b) $U(x,y) = \sum_{n=0}^{N} \cos[(n+\frac{1}{2})\pi y]\, c_n(x)$, where N is fixed.

7. Use the Kantorovich method to find an approximate minimum for the functional $I(u) = \frac{1}{2}\iint [u_\rho^2 + (1/\rho^2)u_\varphi^2]\rho\,d\rho\,d\varphi$, where the integration is over the disc $0 \le \rho \le 1$, $-\pi \le \varphi \le \pi$ and the boundary condition is that $u = 0$ when $\rho = 1$. Use the following sets of trial functions:
   (a) $U(\rho,\varphi) = (1-\rho^2)c(\varphi)$
   (b) $U(\rho,\varphi) = J_0(\rho x_1)c_1(\varphi) + J_0(\rho x_2)c_2(\varphi)$ (x_i is the ith zero of J_0.)

7.6. Orthogonality Methods

The approximation methods described in the previous section are based on minimization problems, which are motivated by the variational approach. On the other hand, we may try to find an approximate solution of an elliptic boundary-value problem in the form

$$u(x,y,z) = \sum_{i=1}^{N} c_i \Phi_i(x,y,z)$$

where $\Phi_i(x,y,z)$ are given functions that satisfy the boundary conditions and c_i are parameters that are to be found from suitable orthogonality relations. In case the functions Φ_i are twice-differentiable, we may apply the method of Galerkin, as discussed in the following subsection.

7.6.1. The Galerkin method: Rectangular regions. In the Galerkin method we look for a trial solution in the form

$$U(x,y,z) = \sum_{i=1}^{N} c_i \Phi_i(x,y,z)$$

where $\{\Phi_i\}_{1 \le i \le N}$ are given twice-differentiable functions that satisfy the boundary conditions and c_i are adjustable parameters.

Assume that we wish to solve the equation $\nabla^2 u = -\rho$ with the boundary condition $u = 0$. If u were the true solution, we could multiply both sides of the equation $\nabla^2 u = -\rho$ by the function Φ_i and apply the divergence theorem (Green's second identity):

$$\iiint_D \Phi_i\, \rho\, dx\,dy\,dz = -\iiint_D \Phi_i\, \nabla^2 u\, dx\,dy\,dz = -\iiint_D u\, \nabla^2 \Phi_i\, dx\,dy\,dz$$

Now we replace u by the approximate solution $U = \sum_j c_j\Phi_j$ and we are led to

$$\iiint_D \Phi_i\,\rho\,dx\,dy\,dz = -\sum_{j=1}^{N} c_j \iiint_D \Phi_j\,\nabla^2\Phi_i\,dx\,dy\,dz \quad 1 \le i \le N \tag{7.6.1}$$

This is a finite system of linear equations that may be solved for the parameters $c_1, \ldots, c_N$. The method is called an *orthogonality method* because (7.6.1) is equivalent to the requirement that the trial solution U be such that $\nabla^2 U + \rho$ is orthogonal to the linear span of $\{\Phi_i\}_{1\le i\le N}$. Note that this method does not require that we first formulate the differential equation as a variational problem.

EXAMPLE 7.6.1. *Find the approximate solution of Poisson's equation $u_{xx} + u_{yy} = -\rho$ in the rectangle $|x| < a, |y| < b$ with the boundary condition that $u = 0$ on all four sides. Use Galerkin's method with trial solutions of the form $u(x,y) = A_{00}(a^2 - x^2)(b^2 - y^2)$.*

Solution. We have

$$Lu = u_{xx} + u_{yy} + \rho = A_{00}[-2(a^2 - x^2) - 2(b^2 - y^2)] + \rho$$

Applying the orthogonality condition gives the equation

$$0 = \int_{-a}^{a}\int_{-b}^{b} \left([-2(a^2 - x^2) - 2(b^2 - y^2)]A_{00} + \rho\right)(a^2 - x^2)(b^2 - y^2)\,dx\,dy$$

When we perform the integrations and solve for A_{00}, we obtain the value $A_{00} = 5\rho/8(a^2 + b^2)$ and the trial solution

$$u(x,y) = \frac{5\rho}{8}\frac{(a^2 - x^2)(b^2 - y^2)}{a^2 + b^2} \quad \bullet$$

One may generalize the above example to trial solutions of the form

$$u(x,y) = (a^2 - x^2)(b^2 - y^2)\sum_{i,j} A_{ij}x^{2i}y^{2j}$$

See the exercises for more details.

EXAMPLE 7.6.2. *Find the approximate solution of Poisson's equation $u_{xx} + u_{yy} = -\rho$ in the rectangle $|x| < a, |y| < b$ with the boundary condition that $u = 0$ on all four sides. Use Galerkin's method with trial solutions of the form*

$$u(x,y) = \sum_{m,n\,\text{odd}} A_{mn}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b}$$

Solution. We have

$$Lu = u_{xx} + u_{yy} + \rho = -\sum_{m,n\,\text{odd}} A_{mn}\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} + \rho$$

When we integrate this against the orthogonal functions $\cos(m\pi x/2a)\cos(n\pi y/2b)$, we obtain the equations

$$\frac{16ab\rho}{\pi^2 mn}(-1)^{(m+n)/2-1} - A_{mn}ab\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right] = 0 \qquad m, n \,\text{odd}$$

This leads to the trial solution

$$u(x,y) = \frac{64a^2b^2}{\pi^4} \sum_{m,n\,\text{odd}} (-1)^{(m+n)/2-1} \frac{\cos(m\pi x/2a)\cos(n\pi y/2b)}{mn(b^2m^2 + a^2n^2)} \quad \bullet$$

We now apply Galerkin's method to find an approximate solution to the fundamental frequency of a circular drumhead. This problem was solved exactly in Sec. 3.4 using Bessel functions.

EXAMPLE 7.6.3. *Find the approximate solution of $\nabla^2 u + \lambda u = 0$ in the circle $0 < \rho < a$ in the form $u = A\cos[\pi\rho/2a]$.*

Solution. In polar coordinates we have $\nabla^2 u = u_{\rho\rho} + (1/\rho)u_\rho + (1/\rho^2)u_{\phi\phi}$. In this case u is independent of ϕ and we have

$$\begin{aligned}
u_\rho &= -A\left(\frac{\pi}{2a}\right)\sin\frac{\pi\rho}{2a}, \quad u_{\rho\rho} = -A\left(\frac{\pi}{2a}\right)^2\cos\frac{\pi\rho}{2a} \\
0 &= \iint_{0<\rho<a} (\lambda u + \nabla^2 u)\cos\frac{\pi\rho}{2a}\,\rho\,d\rho\,d\phi \\
&= 2\pi\int_0^a \left[\lambda A\cos^2\frac{\pi\rho}{2a} - A\left(\frac{\pi}{2a}\right)^2\cos^2\frac{\pi\rho}{2a} - A\frac{\pi}{2a}\sin\frac{\pi\rho}{2a}\cos\frac{\pi\rho}{2a}\right]\rho\,d\rho
\end{aligned}$$

Doing the required integrations and canceling the common factor A, we have

$$\frac{\pi^2}{4}\left(\frac{1}{2} + \frac{2}{\pi^2}\right) - \lambda a^2\left(\frac{1}{2} - \frac{2}{\pi^2}\right) = 0$$

or $\lambda = 5.832/a^2$. This approximate solution is to be compared with the exact solution $u = J_0(\rho x_1/a)$, where $x_1 = 2.404...$ is the first zero of the Bessel function that leads to $\lambda = 5.779/a^2$. •

7.6.2. Nonrectangular regions. We now turn to problems in nonrectangular regions. Consider the region D in the xy plane that is bounded by the vertical lines $x = a$ and $x = b$; the horizontal boundaries are defined by the inequalities $\varphi_1(x) \le y \le \varphi_2(x)$, where φ_1, φ_2 are smooth functions. We look for a trial solution of Poisson's equation in the form

$$u(x,y) = (y - \varphi_1(x))(y - \varphi_2(x))f(x)$$

where $f(x), a < x < b$, is to be determined from a suitable ordinary differential equation, according to the method of Kantorovich. To find this equation, we compute the variational integrand $I = \frac{1}{2}(u_x^2 + u_y^2) - \rho u$ as follows:

$$\begin{aligned}
u_x &= (y-\varphi_1(x))(y-\varphi_2(x))f'(x) \\
&\quad -\varphi_1'(x)(y-\varphi_2(x))f(x) - \varphi_2'(x)(y-\varphi_1(x))f(x) \\
u_y &= (y-\varphi_1(x))f(x) + (y-\varphi_2(x))f(x) \\
I &= \frac{1}{2}(y-\varphi_1(x))^2(y-\varphi_1(x))^2 f'(x)^2 \\
&\quad - f(x)f'(x)[(y-\varphi_2(x))^2(y-\varphi_1(x))\varphi_1'(x) + (y-\varphi_1(x))^2(y-\varphi_2(x))\varphi_2'(x)] \\
&\quad + \frac{f(x)^2}{2}\left(\varphi_1'(x)^2(y-\varphi_2(x))^2 + \varphi_2'(x)^2(y-\varphi_1(x))^2\right) \\
&\quad + f(x)^2\left(\varphi_1'(x)\varphi_2'(x)(y-\varphi_1(x))(y-\varphi_2(x))\right) \\
&\quad + \frac{f(x)^2}{2}\left((y-\varphi_1(x))^2 + (y-\varphi_2(x))^2 + 2(y-\varphi_1(x))(y-\varphi_2(x))\right) \\
&\quad - \rho(y-\varphi_1(x))(y-\varphi_2(x))f(x)
\end{aligned}$$

Now we use the following definite integrals:

$$\begin{aligned}
\int_{\varphi_1}^{\varphi_2} (y-\varphi_1)^2(y-\varphi_2)^2\,dy &= \frac{1}{30}(\varphi_2-\varphi_1)^5 \\
\int_{\varphi_1}^{\varphi_2} (y-\varphi_1)(y-\varphi_2)^2\,dy &= \frac{1}{12}(\varphi_2-\varphi_1)^4 \\
\int_{\varphi_1}^{\varphi_2} (y-\varphi_1)^2(y-\varphi_2)\,dy &= -\frac{1}{12}(\varphi_2-\varphi_1)^4 \\
\int_{\varphi_1}^{\varphi_2} (y-\varphi_1)(y-\varphi_2)\,dy &= -\frac{1}{6}(\varphi_2-\varphi_1)^3 \\
\int_{\varphi_1}^{\varphi_2} (y-\varphi_2)^2\,dy &= \frac{1}{3}(\varphi_2-\varphi_1)^3 \\
\int_{\varphi_1}^{\varphi_2} (y-\varphi_1)^2\,dy &= \frac{1}{3}(\varphi_2-\varphi_1)^3
\end{aligned}$$

Making the necessary substitutions and defining $\varphi(x) = \varphi_2(x) - \varphi_1(x)$, $\Psi(x) = \varphi_1'(x)^2 + \varphi_2'(x)^2 - \varphi_1'(x)\varphi_2'(x)$, we have

$$I(u) = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \left[\frac{1}{2}(u_x^2 + u_y^2) - \rho u\right] dy\,dx = \int_a^b \Phi(x)\,dx$$

where

$$\Phi(x) = \left[\frac{f'^2\varphi^5}{60} + \frac{f^2\varphi^3(1+\Psi)}{6} + \frac{ff'\varphi^4\varphi'}{12} + \frac{\rho\varphi^3 f}{6}\right]$$

The Euler-Lagrange ordinary differential equations for this minimization problem are obtained by forming the partial derivatives

$$\frac{\partial\Phi}{\partial f'} = \frac{f'\varphi^5}{30} + \frac{f\varphi^4\varphi'}{12}$$

$$\frac{\partial\Phi}{\partial f} = \frac{f\varphi^3(1+\Psi)}{3} + \frac{f'\varphi^4\varphi'}{12} + \frac{\rho\varphi^3}{6}$$

to obtain the second-order ordinary differential equation

$$\varphi^2 f'' + 5\varphi\varphi' f' + 5\left[\frac{1}{2}\varphi\varphi'' - 2\varphi_1'\varphi_2' - 2\right] f = 5\rho$$

This equation is to be solved with the boundary conditions $f(a) = 0$, $f(b) = 0$. We illustrate with an example.

EXAMPLE 7.6.4. *Find the approximate solution of Poisson's equation $u_{xx} + u_{yy} = -\rho$ in the trapezoid defined by the inequalities $a < x < b$, $-kx < y < kx$, where ρ, k, a are positive constants with $k \neq 1$.*

Solution. In this case we have $\varphi_1(x) = -kx$, $\varphi_2(x) = kx$, $\varphi(x) = 2kx$, with $\varphi'(x) = 2k$, $\varphi''(x) = 0$. The differential equation for $f(x)$ is $k^2x^2f'' + 5k^2xf' + (5/2)(k^2-1)f = 5\rho/4$. This is a differential equation of the Euler type, whose general solution is of the form $f(x) = C_0 + C_1x^{r_1} + C_2x^{r_2}$, where r_1, r_2 are the roots of the indicial equation $k^2r(r-1) + 5k^2r + (5/2)(k^2-1) = 0$ and the constants C_0, C_1, C_2 are determined from ρ and the boundary conditions. In detail we have $r_{1,2} = -2 \pm (1/2)\sqrt{6 + 10\,k^{-2}}$, $C_0 = \rho/2(k^2-1)$. The constants C_1, C_2 are then determined by solving the simultaneous system of equations $0 = C_0 + C_1a^{r_1} + C_2a^{r_2}$, $0 = C_0 + C_1b^{r_1} + C_2b^{r_2}$. •

We consider the following limiting case when $a = 0$.

EXAMPLE 7.6.5. *Find the approximate solution of Poisson's equation $u_{xx} + u_{yy} = -\rho$ in the isoceles triangle defined by the inequalities $0 < x < b$, $-kx < y < kx$ where ρ, k are positive constants with $k \neq 1$. Discuss the special case of an equilateral triangle when $k = 1/\sqrt{3}$.*

Solution. In this case we may dispense with the boundary condition at $x = a$ and determine $f(x)$ by solving the equation $k^2x^2f'' + 5k^2xf' + (5/2)(k^2-1)f = 5\rho/4$ with the boundary condition $f(b) = 0$, while we require that the resultant solution be finite at $x = 0$. This entails that we use the larger root $r = -2 + (1/2)\sqrt{6 + 10k^{-2}}$. The solution that satisfies the boundary condition is written in the form $f(x) = \rho[1 - (x/b)^r]/[2(k^2-1)]$. Since $r > 2$, we note that the product $f(x)(y^2 - k^2x^2)$ is bounded in the triangle and tends to zero when $(x, y) \to (0, 0)$. In the case of an equilateral triangle we have $k = 1/\sqrt{3}$; thus $r = 1$ and the approximate solution is a polynomial, $u(x, y) = -(\rho/2)[1 - (x/b)][(y^2 - x^2/3)]$. In this case it can be verified that we have the exact solution of the problem. •

7.6.3. The finite element method. In case the functions $\Phi_i(x,y,z)$ are not twice-differentiable, we cannot form the functions $\nabla^2\Phi_i$ to use Galerkin's method as described above. However if the first derivatives of Φ_i are suitably well behaved, we may apply the divergence theorem as follows. Assume that we wish to solve the equation $\nabla^2 u = -\rho$ with the boundary condition $u = 0$ by an approximate formula of the type $u = \sum_j c_j\Phi_j(x,y,z)$. If u were the true solution, we could multiply both sides of the equation $\nabla^2 u = -\rho$ by the function Φ_i and apply the divergence theorem (Green's first identity):

$$\iiint_D \Phi_i\,\rho\,dx\,dy\,dz = -\iiint_D \Phi_i\,\nabla^2 u\,dx\,dy\,dz = \iiint_D \nabla\Phi_i\cdot\nabla u\,dx\,dy\,dz$$

Now we replace u by the approximate solution $\sum_j c_j\Phi_j$, and we are led to

$$\iiint_D \Phi_i\,\rho\,dx\,dy\,dz = \sum_{j=1}^{N} c_j \iiint_D \nabla\Phi_i\cdot\nabla\Phi_j\,dx\,dy\,dz \quad 1\le i\le N \tag{7.6.2}$$

This is a finite system of linear equations that may be solved for the parameters $c_1,\ldots,c_N$. These equations make no reference to the second derivatives, so they can be expected to be well defined for functions whose first partial derivatives exist in a suitable sense. In particular, this is the case when the functions have a polygonal profile, as described below.

In the following paragraph we illustrate the finite element method for the two-dimensional Poisson equation

$$\nabla^2 u = -\rho \text{ in } D, \qquad u = 0 \text{ on } \partial D \tag{7.6.3}$$

This will be reduced to a finite system of linear equations by the following steps:

- Approximate the region D by a region D_N that is the union of a finite number of triangles. Let N be the number of *interior vertices* of this system of triangles.
- For each interior vertex V_i, let $\Phi_i(x,y)$ be the function that is equal to 1 at the vertex V_i and equal to zero at all other vertices (interior and boundary), and is linear inside each triangle.
- Introduce the numerical approximate solution

  $$u_N(x,y) = c_1\Phi_1(x,y) + \cdots + c_N\Phi_N(x,y)$$

  where the coefficients $c_1,\ldots,c_N$ will now be determined.
- Require that the approximate solution $u_N(x,y)$ satisfy (7.6.2) for each choice $\Phi_1,\ldots,\Phi_N$; in other words,

  $$\sum_{j=1}^{N} c_j \left(\iint_D \nabla\Phi_i\cdot\nabla\Phi_j\,dx\,dy\right) = \iint_D \rho\,\Phi_j\,dx\,dy \qquad i=1,\ldots,N$$

which is abbreviated

$$\sum_{j=1}^{N} \Phi_{ij} c_j = \rho_i \quad \text{where } \Phi_{ij} = \iint_D \nabla\Phi_i \cdot \nabla\Phi_j \, dx\, dy, \quad \rho_i = \iint_D \rho\, \Phi_i \, dx\, dy$$

EXAMPLE 7.6.6. *Apply the finite element method to find an approximate numerical solution of the Poisson equation $u_{xx} + u_{yy} = -1$ in the rectangle $0 < x < L_1$, $0 < y < L_2$, where we partition the rectangle into four triangles by drawing the diagonals $y = xL_2/L_1$, $y = L_2(L_1 - x)/L_1$.*

Solution. In this case there is just one interior vertex, $V_1 = (L_1/2, L_2/2)$. The piecewise linear function $\Phi_1(x, y)$ is given by the formula

$$\Phi_1(x,y) = \begin{cases} 2x/L_1 & \text{if } 0 < x < L_1/2, xL_2/L_1 < y < L_2(L_1 - x)/L_1 \\ 2(L_1 - x)/L_1 & \text{if } L_1/2 < x < L_1, L_2(L_1 - x)/L_1 < y < xL_2/L_1 \\ 2y/L_2 & \text{if } 0 < y < L_2/2, L_1 y/L_2 < x < L_1(L_2 - y)/L_2 \\ 2(L_2 - y)/L_2 & \text{if } L_2/2 < y < L_2, L_1(L_1 - y)/L_2 < x < L_1 y/L_2 \end{cases}$$

The gradient is given by

$$|\nabla\Phi_1(x,y)|^2 = \begin{cases} 4/L_1^2 & \text{if } 0 < x < L_1/2, xL_2/L_1 < y < L_2(L_1 - x)/L_1 \\ 4/L_1^2 & \text{if } L_1/2 < x < L_1, L_2(L_1 - x)/L_1 < y < xL_2/L_1 \\ 4/L_2^2 & \text{if } 0 < y < L_2/2, L_1 y/L_2 < x < L_1(L_2 - y)/L_2 \\ 4/L_2^2 & \text{if } L_2/2 < y < L_2, L_1(L_2 - y)/L_2 < x < L_1 y/L_2 \end{cases}$$

Since each of the triangles has an area of $L_1L_2/8$, the integral of $|\nabla\Phi_1(x,y)|^2$ is $2[(4/L_1^2)(L_1L_2)/8] + 2[(4/L_2^2)(L_1L_2/8)] = L_2/L_1 + L_1/L_2$. Meanwhile, the right side of (7.6.2) is the sum of four integrals, of which the first is

$$\int_0^{L_1/2} \int_{xL_2/L_1}^{(L_1-x)L_2/L_1} \frac{2x}{L_1}\, dx\, dy = (2L_2/L_1^2) \int_0^{L_1/2} (L_1 - 2x)\, dx = \frac{L_1L_2}{12}$$

and similarly for the other three triangles. Combining these facts, we find for the approximate numerical value at the center

$$u_1(L_1/2, L_2/2) = \frac{L_1L_2/3}{L_1/L_2 + L_2/L_2} = \frac{L_1^2L_2^2}{3(L_1^2 + L_2^2)} \quad \bullet$$

CHAPTER 8

GREEN'S FUNCTIONS

INTRODUCTION

Many boundary-value problems for linear partial differential equations may be solved in terms of integral transforms. These formulas give explicit representations of the solutions as linear transforms of the initial and boundary conditions. For example, the d'Alembert solution of the wave equation (Sec. 2.4), the Poisson integral representation of the solution of Laplace's equation (Sec. 3.2), and the Gauss-Weierstrass representation of the solution of the heat equation (Sec. 5.2) are all of this type. The purpose of this chapter is to pursue this topic more generally.

8.1. Green's Functions for Ordinary Differential Equations

8.1.1. An example. We begin with the simplest example to illustrate the main ideas. This is the boundary-value problem for the ordinary differential equation

$$y'' = -f(x) \qquad 0 < x < L$$
$$y(0) = 0 = y(L)$$

Here $f(x)$, $0 < x < L$, is a given piecewise smooth function. This equation may be used to model the static transverse deflections of a string that is fixed at both ends and subject to a spatially dependent forcing law.

To solve this problem, we use calculus to write $y'(x) = -\int_0^x f(z)dz + A$, $y(x) = -\int_0^x \int_0^\xi f(z)\,dz\,d\xi + Ax + B$, where A, B are constants to be determined. The boundary conditions yield

$$0 = 0 + A \cdot 0 + B$$
$$0 = -\int_0^L \int_0^\xi f(z)\,dz\,d\xi + AL + B$$

with the conclusions $B = 0$, $A = (1/L)\int_0^L \int_0^\xi f(z)\,dz\,d\xi$. We can reduce the iterated integrals to single integrals by interchanging the order of integration. Thus $\int_0^x \int_0^\xi f(z)\,dz\,d\xi = \int_0^x \int_z^x f(z)\,d\xi\,dz = \int_0^x (x-z)f(z)\,dz$, and the solution is written as

$$\begin{aligned}
y(x) &= -\int_0^x (x-z)f(z)\,dz + \frac{x}{L}\int_0^L (L-z)f(z)\,dz \\
&= \int_0^x \frac{z}{L}(L-x)f(z)\,dz + \int_x^L \frac{x}{L}(L-z)f(z)\,dz \\
&= \int_0^L G(x,z)f(z)\,dz
\end{aligned}$$

where *Green's function* $G(x,z)$ is defined by

$$G(x,z) = \begin{cases} \dfrac{z(L-x)}{L} & 0 \le z \le x \\ \dfrac{x(L-z)}{L} & x \le z \le L \end{cases}$$

We have obtained an explicit representation of the solution in terms of the right member $f(x)$, $0 < x < L$, and a function $G(x,z)$ that depends only on the differential equation and the boundary conditions. This formula defines a linear transform of the right member to a solution of the given problem.

Green's function has the following characteristic properties:

1. For each $z, x \to G(x,z)$ satisfies $G'' = 0$ except when $x = z$.
2. G satisfies the boundary conditions $G(0,z) = 0 = G(L,z)$.
3. $G(z+0,z) - G(z-0,z) = 0$
4. $(\partial G/\partial x)(z+0,z) - (\partial G/\partial x)(z-0,z) = -1$
5. $G(x,z) = G(z,x)$

We leave it as an exercise to show that $G(x,z)$ is uniquely determined from conditions 1 through 4. Note that condition 3 signifies that G is a continuous function, while condition 4 signifies that the first derivatives are discontinuous in a precise manner.

Green's function may also be represented by a Fourier sine series. The nth Fourier sine coefficient of G is $B_n(z) = (2/L)\int_0^L G(x,z)\sin(n\pi x/L)dx$. By what we have shown, this is the solution of $B''(z) = -(2/L)\sin(n\pi z/L)$ with boundary conditions $B(0) = 0 = B(L)$, namely, $B(z) = (2/L)[L/(n\pi)]^2\sin(n\pi z/L)$, leading to the Fourier representation of Green's function in the form

$$G(x,z) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L}\sin\frac{n\pi z}{L}$$

This series converges uniformly for $0 \le x,\ z \le L$.

8.1.2. The generic case. We now turn to the case of a general self-adjoint second-order ordinary differential equation. Consider the boundary-value problem

$$Ly := [p(x)y']' + q(x)y = -f(x) \qquad a < x < b \tag{8.1.1}$$

$$\cos\alpha\, y(a) - L\sin\alpha\, y'(a) = 0 \tag{8.1.2}$$

$$\cos\beta\, y(b) + L\sin\beta\, y'(b) = 0 \tag{8.1.3}$$

Here f, q, p are given continuous functions with $p(x) > 0$ for $a \le x \le b$, and α, β are real constants. In addition, *we assume that $\lambda = 0$ is not an eigenvalue of the associated Sturm-Liouville eigenvalue problem.*

We solve this problem by the method of variation of parameters, familiar for ordinary differential equations. The solution is sought in the form

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1, y_2 are solutions of the homogeneous equation and the functions u_1, u_2 are to be chosen. To be specific, we determine y_1, y_2 up to constant multiples by requiring that y_1 satisfy (8.1.2) and that y_2 satisfy (8.1.3). Since $\lambda = 0$ is not an eigenvalue, we conclude that y_1, y_2 must be linearly independent. We have

$$y' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2$$

The method of variation of parameters further requires that $u_1'y_1 + u_2'y_2 = 0$. With this determination, we see that y satisfies (8.1.2) if $u_2(a) = 0$. Similarly, y satisfies (8.1.3) if $u_1(b) = 0$. To satisfy (8.1.1), we write

$$\begin{aligned}(py')' + qy &= [p(u_1y_1' + u_2y_2')]' + q(u_1y_1 + u_2y_2)\\ &= u_1[(py_1')' + qy_1] + u_2[(py_2')' + qy_2] + p(u_1'y_1' + u_2'y_2')\end{aligned}$$

The first two terms are zero since y_1, y_2 are both solutions to the homogeneous equation. Therefore (8.1.1) can be solved by the method of variation of parameters if we have satisfied the simultaneous system

$$\begin{aligned}p(x)(u_1'y_1' + u_2'y_2') &= -f(x)\\ u_1'y_1 + u_2'y_2 &= 0\end{aligned}$$

These are easily solved to yield

$$u_1'(x) = \frac{f(x)y_2(x)}{p(x)W(x)}, \qquad u_2'(x) = -\frac{f(x)y_1(x)}{p(x)W(x)}$$

where $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the wronskian of the two solutions. We determine u_1, u_2 uniquely by $u_1(b) = 0$, $u_2(a) = 0$ to obtain

$$y(x) = -y_1(x)\int_x^b \frac{f(\xi)y_2(\xi)}{p(\xi)W(\xi)}d\xi - y_2(x)\int_a^x \frac{f(\xi)y_1(\xi)}{p(\xi)W(\xi)}d\xi$$

This can be written as a single integral if we use the definition

$$G(x,\xi) = \begin{cases} -\dfrac{y_2(x)y_1(\xi)}{p(\xi)W(\xi)} & a \le \xi \le x \\ -\dfrac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} & x \le \xi \le b \end{cases} \tag{8.1.4}$$

to obtain the formula

$$y(x) = \int_a^b G(x,\xi)f(\xi)d\xi \tag{8.1.5}$$

We have obtained Green's function of the general second-order self-adjoint equation (8.1.1) with the separable boundary conditions (8.1.2) and (8.1.3). This function depends only on the boundary conditions and the functions $p(x)$, $q(x)$; it makes no reference to the right member $f(x)$. We note the following properties of Green's function in this case:

1. The function $x \to G(x,\xi)$ satisfies the homogeneous equation if $x \neq \xi$.
2. The function $x \to G(x,\xi)$ satisfies the boundary conditions (8.1.2) and (8.1.3).
3. The function $G(x,\xi)$ is continuous for $a \le \xi$, $x \le b$.
4. The partial derivatives are continuous except for $x = \xi$, where we have
$$\left(\frac{\partial G}{\partial x}\right)(\xi+0,\xi) - \left(\frac{\partial G}{\partial x}\right)(\xi-0,\xi) = -\frac{1}{p(\xi)}$$
5. The function $G(x,\xi)$ is symmetric: $G(x,\xi) = G(\xi,x)$ for $a \le \xi$, $x \le b$.

We can summarize the preceding discussion in the form of a theorem.

THEOREM 8.1. *Suppose zero is not an eigenvalue of (8.1.1). Then the unique solution of Eq. (8.1.1) with boundary conditions (8.1.2) and (8.1.3) is given by the integral (8.1.5), where Green's function is defined by (8.1.4).*

Proof. To prove the stated uniqueness, let y, $\tilde{y}$ be solutions of $Ly = -f$ satisfying the boundary conditions. Then $y - \tilde{y}$ satisfies the boundary conditions, and $L(y - \tilde{y}) = 0$. Since $\lambda = 0$ is not an eigenvalue, we conclude $y - \tilde{y} = 0$. •

EXAMPLE 8.1.1. *Find Green's function of the equation $y'' = -f$ with the boundary conditions $y(0) = 0$, $y'(L) = 0$, and solve the equation.*

Solution. The general solution of the homogeneous equation is given by $y(x) = A + Bx$. The solution that satisfies $y(0) = 0$ is $y_1(x) = x$, up to a constant multiple. The solution that satisfies $y'(L) = 0$ is $y_2(x) = 1$, up to a constant multiple. Their wronskian is $W(y) = y_1y_2' - y_1'y_2 = -1$. Green's function is

$$G(x,\xi) = \begin{cases} \xi & 0 \le \xi \le x \\ x & x \le \xi \le L \end{cases}$$

The solution of the nonhomogeneous equation is $y(x) = \int_0^x \xi f(\xi)d\xi + x\int_x^L f(\xi)d\xi$. A direct verification shows that $y' = \int_x^L f(\xi)d\xi$, $y'' = -f$, $y(0) = 0$, $y'(L) = 0$. •

To obtain a Fourier representation of Green's function in the generic case, let $\{\lambda_n\}_{n\geq 1}$ be the eigenvalues and $\{\varphi_n\}_{n\geq 1}$ a set of normalized eigenfunctions for the equation $(py')' + qy + \lambda y = 0$. Then Green's function is written as the series

$$G(x,\xi) = \sum_{n\geq 1} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}$$

which is absolutely and uniformly convergent for $a \leq x \leq b$.

EXAMPLE 8.1.2. *Find a Fourier representation of Green's function of Example 8.1.1.*

Solution. In this example we have the formulas $\lambda_n = (n - \frac{1}{2})^2\pi^2/L^2$, $\varphi_n(x) = \sqrt{2/L}\sin[(n-\frac{1}{2})\pi x/L]$, $n \geq 1$, and Green's function is written $G(x,\xi) = (2L/\pi^2)\sum_{n\geq 1}\sin[(n-\frac{1}{2})\pi x/L]\sin[(n-\frac{1}{2})\pi\xi/L]/(n-\frac{1}{2})^2$. •

8.1.3. The exceptional case: Modified Green's function. If $\lambda = 0$ is an eigenvalue, then we lose existence and uniqueness. For example, if the equation is $y'' = -f$ and the boundary conditions are $y'(a) = 0 = y'(b)$, then we can solve the equation if and only if $\int_a^b f(x)dx = 0$. The homogeneous equation $y'' = 0$ has infinitely many solutions that satisfy the boundary condition, that is, $y(x) =$ constant, which destroys uniqueness. In these cases we may determine a unique solution by requiring in addition that the solution be orthogonal to the eigenfunction of the associated homogeneous problem. In the case just mentioned, this amounts to the requirement that $\int_a^b y(x)dx = 0$, leading to $y(x) \equiv 0$.

To formulate Green's function for this exceptional case, we consider the nonhomogeneous equation

$$[p(x)y']' + q(x)y = -f(x) + \varphi(x)\int_a^b f(\xi)\varphi(\xi)d\xi$$

where $\varphi(x)$ is an eigenfunction that satisfies the homogeneous boundary conditions (8.1.2) and (8.1.3) and satisfies the normalization $\int_a^b \varphi(x)^2dx = 1$. We have written the equation so that the right side is orthogonal to the eigenfunction $\varphi(x)$. We look for a solution that satisfies the orthogonality condition $\int_a^b y(x)\varphi(x)dx = 0$. The solution is sought in the form $y(x) = \int_a^b G(x,\xi)f(\xi)d\xi$, and $G(x,\xi)$ is the *modified Green's function.*

EXAMPLE 8.1.3. *Find the modified Green's function for the differential equation $y'' = -f$ with the boundary conditions $y(0) = 0$, $y'(1) = y(1)$.*

Solution. The general solution of the homogeneous equation is $y(x) = A + Bx$, which satisfies the boundary conditions with $A = 0$ and B arbitrary. A normalized eigenfunction is obtained by choosing $B = \sqrt{3}$. To solve the equation $y'' = -f + 3x\int_0^1 \xi f(\xi)d\xi$, we write

$$y'(x) = -\int_0^x f(\xi)d\xi + \frac{3x^2}{2}\int_0^1 \xi f(\xi)d\xi + C$$

$$y(x) = -\int_0^x (x-\xi)f(\xi)d\xi + \frac{1}{2}x^3\int_0^1 \xi f(\xi)d\xi + Cx + D$$

The first boundary condition gives $D = 0$, while the second leaves C undetermined. To find C, we use the orthogonality condition $\int_0^1 y(x)\varphi(x)dx = 0$ to obtain $C = \int_0^1 (1 - 9\xi/5 + \xi^3/2)f(\xi)d\xi$, and we have the modified Green function

$$G(x,\xi) = \begin{cases} \xi - x + \frac{1}{2}x^3\xi + \left(1 - \frac{9\xi}{5} + \frac{\xi^3}{2}\right)x & \xi < x \\ \frac{x^3\xi}{2} + \left(1 - \frac{9\xi}{5} + \frac{\xi^3}{2}\right)x & \xi \geq x \end{cases} \bullet$$

The modified Green's function can be represented as a series of eigenfunctions in the form

$$G(x,\xi) = \sum_{n\geq 1} \frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}$$

where the sum is over all *nonzero* eigenvalues $\{\lambda_n\}$ of the equation $(py')' + qy + \lambda y = 0$ with the given boundary conditions. For example, with the boundary conditions $y'(0) = 0$, $y'(L) = 0$ for the equation $y'' = -f$, we have $\lambda_n = (n\pi/L)^2$, $\varphi_n(x) = \sqrt{2/L}\cos(n\pi x/L)$, and Green's function is written as $G(x,\xi) = (2L/\pi^2)\sum_{n\geq 1}[\cos(n\pi x/L)\cos(n\pi\xi/L)]/n^2$.

8.1.4. The Fredholm alternative. The above discussion leads us to the following dichotomy regarding the nonhomogeneous equation (8.1.1) with the boundary conditions (8.1.2) and (8.1.3), known as the *Fredholm alternative* and stated as follows:

- *Either* $\lambda = 0$ is not an eigenvalue of the homogeneous equation; then the nonhomogeneous equation (8.1.1) has a unique solution that satisfies the boundary conditions (8.1.2) and (8.1.3).
- *Or* $\lambda = 0$ is an eigenvalue of the homogeneous equation with eigenfunction $\varphi(x)$; then the nonhomogeneous equation (8.1.1) has a solution if and only if the right side $f(x)$ satisfies the orthogonality condition $\int_a^b f(x)\varphi(x)dx = 0$. The solution is unique provided that we require that $\int_a^b y(x)\varphi(x)dx = 0$.

The reader familiar with the theory of linear equations will recognize the principle at work. Let a system of linear equations be written in the form $Ay = -f$, where A is a square matrix. *Either* det A is nonzero, and we have a unique solution y for every vector f; *or* det A is zero, and we can solve the equation only if the right side is orthogonal to the set of solutions of $\{\varphi\colon A^T\varphi = 0\}$. In the case of a square matrix, det A is zero if and only if $\lambda = 0$ is an eigenvalue of A. In the case of differential equations, we cannot use the determinant to distinguish between the two cases. Nevertheless, the existence or nonexistence of eigenfunctions for $\lambda = 0$ still makes sense and is a valid criterion to distinguish the two cases.

EXERCISES 8.1

1. Show that Green's function for $y'' = -f$, $y(0) = 0 = y(L)$ is uniquely determined by properties 1 to 4.
2. Find Green's function for the equation $y'' = -f$ with the boundary conditions $y(0) = 0$, $y'(L) + hy(L) = 0$, where h is a positive constant.
3. Find Green's function for the equation $y'' = -f$ with the boundary conditions $y'(0) = hy(0)$, $y'(L) + hy(L) = 0$, where h is a positive constant.
4. Find Green's function for the equation $y'' - ky = -f$ with the boundary conditions $y(0) = 0$, $y(L) = 0$, where k is a positive constant.
5. Find Green's function for the equation $y'' - ky = -f$ with the boundary conditions $y(0) = 0$, $y(L) = 0$, where k is a negative constant,
$$k \neq -\left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$$
6. Find the modified Green's function for the equation $y'' = -f$ with the boundary conditions $y'(0) = 0$, $y'(L) = 0$.
7. Find the modified Green's function for the equation $y'' = -f$ with the (periodic) boundary conditions $y(0) = y(L)$, $y'(0) = y'(L)$.
8. Show that the modified Green's function obtained in Example 8.1.3 satisfies properties 1–5 preceding Theorem 8.1.

8.2. The Three-Dimensional Poisson Equation

In this section and the following sections, we extend the concept of Green's function to certain second-order partial differential equations. Since the geometry and types of equations differ for each case, it is most efficient to consider separately the elliptic, parabolic, and hyperbolic problems.

The boundary-value problem for Poisson's equation is

$$\begin{aligned} \nabla^2 u &= -h \qquad && P \in D \\ u &= f && P \in \partial D \end{aligned}$$

Here P is a point of two- or three-dimensional space, and D is a region whose boundary is denoted ∂D. For example, if D is the three-dimensional ball $\{P : |P| < R\}$, then ∂D is the sphere $\{P : |P| = R\}$.

By the superposition principle, the solution may be obtained in the form $u = u_f + u_h$, where u_f is the solution of the homogeneous equation $\nabla^2 u = 0$ with

$u = f$ on ∂D and u_h is the solution of the Poisson equation $\nabla^2 u = -h$ with $u = 0$ on ∂D. The latter problem is the analogue of the problem for ordinary differential equations solved in Sec. 8.1 by using Green's function. We now develop this idea for Poisson's equation.

8.2.1. Newtonian potential kernel. We begin with the case of the entire three-dimensional space $D = \mathbb{R}^3 = \{\mathrm{P} = (x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$. We proceed heuristically by Fourier transforms and then verify that we have obtained a rigorous solution. The Fourier representation of the solution and right member are written

$$\begin{aligned}
u(P) = u(x, y, z) &= \iiint e^{i\langle P,K\rangle} U(K)dK \\
&= \iiint e^{i(kx+ly+mz)} U(k, l, m)\, dk\, dl\, dm \\
h(P) = h(x, y, z) &= \iiint e^{i\langle P,K\rangle} H(K)dK \\
&= \iiint e^{i(kx+ly+mz)} H(k, l, m)\, dk\, dl\, dm
\end{aligned}$$

where the integrals are over all $\mathbb{R}^3$. Differentiating formally, we have $\nabla^2 u = \iiint -|K|^2 e^{i\langle P,K\rangle} U(K)dK$. The equation $\nabla^2 u = -h$ requires that $|K|^2 U(K) = H(K)$. If H is zero in a small neighborhood of $K = 0$, this can be solved by $U(K) = H(K)/|K|^2$. We substitute in the integral for u and interchange orders of integration:

$$\begin{aligned}
u(P) &= \iiint \frac{e^{i\langle P,K\rangle} H(K)}{|K|^2} dK \\
&= \iiint \frac{e^{i\langle P,K\rangle}}{|K|^2} dK \iiint \left(\frac{1}{2\pi}\right)^3 e^{-i\langle Q,K\rangle} h(Q)dQ \\
&= \iiint h(Q)dQ \left(\frac{1}{2\pi}\right)^3 \iiint \frac{e^{i\langle K,P-Q\rangle}}{|K|^2} dK
\end{aligned}$$

The inner integral may be evaluated in spherical polar coordinates as

$$\iiint \frac{e^{i\langle K,P-Q\rangle}}{|K|^2} dK = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{i|P-Q|\rho\cos\theta} \sin\theta\, d\rho\, d\theta\, d\varphi$$

where $\rho = |K|$, θ is the (polar) angle between K and $P - Q$, and φ is the corresponding azimuthal angle. The θ integral is elementary, and the φ integral gives a factor of 2π, leading to

$$\iiint \frac{e^{i\langle K,P-Q\rangle}}{|K|^2}dK = 2\pi \int_0^\infty \frac{2\sin\rho|P-Q|}{\rho|P-Q|}d\rho$$
$$= 2\pi \frac{2}{|P-Q|}\frac{\pi}{2}$$

Recalling the factor $[1/(2\pi)]^3$ from the Fourier inversion, we are led to the explicit representation

$$u(P) = \frac{1}{4\pi}\iiint \frac{h(Q)}{|P-Q|}dQ \tag{8.2.1}$$

It remains to verify that this is a (rigorous) solution of $\nabla^2 u = -h$. The function $P \to G(P,Q) := 1/(4\pi|P-Q|)$ is the *newtonian potential kernel.* It has the following basic properties:

1. $\nabla_P G = -\nabla_Q G = (1/4\pi)(P-Q)/|P-Q|^3$ $\quad P \neq Q$
2. $\nabla_P^2 G = \nabla_Q^2 G = 0$ $\quad P \neq Q$
3. $G(P,Q) \to 0$ $\quad P \to \infty$ or $Q \to \infty$

We leave the verification of these as an exercise.

THEOREM 8.2. *Let $u = u(P)$ be the solution of $\nabla^2 u = -h$ in all $\mathbb{R}^3$ where $u(P) \to 0, |P||\nabla_P u| \to 0$ when $P \to \infty$. Then u is represented by (8.2.1) in the sense that $u(P) = [1/(4\pi)]\lim_{\epsilon\downarrow 0, R\uparrow\infty}\iiint_{\{Q:\epsilon<|Q-P|<R\}}[h(Q)/|P-Q|]dQ$.*

Proof. We use Green's identity in the form

$$\iiint\limits_{D_{\epsilon,R}} (u\nabla^2 G_P - G_P\nabla^2 u)dQ = \iint\limits_{\partial D_{\epsilon,R}} \left(u\frac{\partial G_P}{\partial n} - G_P\frac{\partial u}{\partial n}\right)dS_Q$$

where $D_{\epsilon,R} = \{Q \in \mathbb{R}^3 : \epsilon < |Q-P| < R\}$ and $G_P(Q) = G(P,Q)$. The right member consists of two surface integrals, over the spheres of radius ϵ and radius R centered at P; the outward normal derivative is defined by $\partial G_P/\partial n = \langle \nabla_Q G_P, n\rangle$. To analyze the integral on the left, we note that $\nabla^2 G_P = 0$ in $D_{\epsilon,R}$ and $\nabla^2 u = -h$, by hypothesis. The integrals on the right are as follows:

$$\iint\limits_{|Q-P|=\epsilon} u\frac{\partial G_P}{\partial n}dS_Q \tag{8.2.2}$$
$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(x+\epsilon\sin\theta\cos\varphi, y+\epsilon\sin\theta\sin\varphi, z+\epsilon\cos\theta)\sin\theta\, d\theta\, d\varphi$$

$$(8.2.3)\quad \iint\limits_{|Q-P|=R} u\frac{\partial G_P}{\partial n}dS_Q$$
$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(x+R\sin\theta\cos\varphi, y+R\sin\theta\sin\varphi, z+R\cos\theta)\sin\theta\, d\theta\, d\varphi$$

$$(8.2.4)\quad \iint\limits_{|Q-P|=\epsilon} G_P\frac{\partial u}{\partial n}dS_Q$$
$$= \frac{1}{4\pi\epsilon}\int_0^{2\pi}\int_0^{\pi} \frac{\partial u}{\partial n}(x+\epsilon\sin\theta\cos\varphi, y+\epsilon\sin\theta\sin\varphi, z+\epsilon\cos\theta)\epsilon^2\sin\theta\, d\theta\, d\varphi$$

$$(8.2.5)\quad \iint\limits_{|Q-P|=R} G_P\frac{\partial u}{\partial n}dS_Q$$
$$= \frac{1}{4\pi R}\int_0^{2\pi}\int_0^{\pi} \frac{\partial u}{\partial n}(x+R\cos\theta, y+R\sin\theta\cos\varphi, z+R\sin\theta\sin\varphi)R^2\sin\theta\, d\theta\, d\varphi$$

The function u is assumed to be twice-differentiable—in particular, continuous. Therefore the integral (8.2.2) tends to $u(P)$ when $\epsilon \downarrow 0$. Integral (8.2.3) tends to zero when $R \uparrow \infty$ by the assumption on u. Integral (8.2.4) tends to zero by the continuity of $\partial u/\partial n$. Finally, integral (8.2.5) tends to zero by the hypothesis on $|\nabla u|$. Putting these facts together, we see that the right side tends to $u(P)$ when $\epsilon \downarrow 0, R \uparrow \infty$. Therefore the left side tends to a limit, and we have proved that $u(P) = [1/(4\pi)]\lim_{\epsilon\downarrow 0, R\uparrow\infty} \iiint_{\epsilon<|P-Q|<R} h(Q)/|P-Q|dQ$, as required. •

8.2.2. Single- and double-layer potentials. The newtonian potential integral (8.2.1) may be considered in its own right, without reference to Poisson's equation. In the theory of electrostatics, the function $P \to 1/(4\pi|P-Q|)$ is interpreted as the potential energy necessary to bring a particle of unit charge from infinity to point Q. The gradient $\nabla_P G$ is the force felt at point P due to the unit charge at point Q. If, instead of a unit charge, we have charge distributed according to a continuous density, then the resultant potential energy is given by the superposition integral (8.2.1). The force is obtained as the negative gradient of this integral. But we can also consider more general superpositions of charge, e.g., on surfaces, lines, or points.

A surface distribution of charges with surface density $h(Q)$ gives rise to the potential function

$$\iint\limits_S \frac{h(Q)}{4\pi|P-Q|}dS_Q$$

This is called a *single-layer potential.*

A linear distribution of charges with linear density $h(Q)$ gives rise to the potential

$$\int_C \frac{h(Q)}{4\pi|P-Q|} dl_Q$$

where the integration is over the parametrized curve C, for example, a straight-line segment.

It may be shown that each of the above integrals is a solution of Laplace's equation $\nabla^2 u = 0$ on the set of P for which $h(P) = 0$.[1]

The simplest charge distribution is that of a finite aggregate of point charges that gives rise to a potential of the form

$$\sum_{i=1}^{N} \frac{C_i}{4\pi|P-Q_i|}$$

for a finite set of points Q_i with corresponding charges C_i, which may be positive or negative. In particular, we may take two nearby points $Q_0 = (x_0, y_0, z_0)$ and $Q_\epsilon = Q_0 + \epsilon n$, where n is a unit vector and the charges are of strength $1/\epsilon$ and $-1/\epsilon$. The total charge is zero, but in the limit we obtain a nonzero potential, since

$$\begin{aligned}\lim_{\epsilon \downarrow 0} \frac{1}{4\pi\epsilon}\left(\frac{1}{|P-Q_0|} - \frac{1}{|P-Q_\epsilon|}\right) &= \left(\frac{\partial}{\partial n}\right)\left(\frac{1}{4\pi|P-Q|}\right) \\ &= -\frac{(P-Q_0)\cdot n}{4\pi|P-Q_0|^3}\end{aligned}$$

This is the *dipole potential.* It satisfies Laplace's equation for $P \neq Q_0$. A superposition of dipole potentials over a surface is called a *double-layer potential* and is written in the form

$$u(P) = \iint_S \frac{(P-Q)\cdot n}{4\pi|P-Q|^3} h(Q)\, dS_Q$$

where n is the unit normal to surface S. We will see below that double-layer potentials provide an explicit representation for many solutions of Laplace's equation.

We now return to the discussion of Green's function.

8.2.3. Green's function of a bounded region. In the case of a smoothly bounded region D in three dimensions, Green's function is defined as the function

[1] I. G. Petrovsky, *Partial Differential Equations,* Wiley-Interscience, New York, 1964, pp. 219–223.

$G(P, Q)$ with the following properties:

(8.2.6) $$\nabla_Q^2 G(P, Q) = 0 \qquad \text{for } P, Q \in D, P \neq Q$$

(8.2.7) $$G(P, Q) = 0 \qquad \text{for } Q \in \partial D, P \in D$$

(8.2.8) $$G(P, Q) - \frac{1}{4\pi |P - Q|} \text{ is a smooth function in all of } D$$

Let's show that such a function is satisfactory for the solution of Poisson's equation.

THEOREM 8.3. *Suppose that $G(P, Q)$ satisfies (8.2.6) through (8.2.8) in the smoothly bounded three-dimensional region D. Then the solution of Poisson's equation $\nabla^2 u = -h$ in D with $u = 0$ on ∂D has the explicit representation*

(8.2.9) $$u(P) = \iiint_D G(P, Q) h(Q) dQ$$

Proof. We again use Green's identity, with $D_\epsilon = \{Q \in D : |Q - P| > \epsilon\}, G_P(Q) = G(P, Q)$. We have

$$\iiint_{D_\epsilon} (u \nabla^2 G_P - G_P \nabla^2 u) dQ = \iint_{\partial D_\epsilon} \left(u \frac{\partial G_P}{\partial n} - G_P \frac{\partial u}{\partial n} \right) dS_Q$$

In D_ϵ we have $\nabla^2 G_P = 0$ and $\nabla^2 u = -h$; thus the left member is $\iiint_{D_\epsilon} G_P h \, dQ$. The right member consists of two integrals, one on the outer boundary ∂D and one on the inner boundary $\{Q : |Q - P| = \epsilon\}$. On the outer boundary both G_P and u are zero; hence this integral is zero. On the inner boundary we can replace G_P by $1/(4\pi |P - Q|)$, since the difference is a smooth function, that will contribute zero to the integral in the limit $\epsilon \downarrow 0$. But the proof of Theorem 8.2 shows that for this choice of G, the integral $\iint_{|Q-P|=\epsilon} u(\partial G/\partial n) dS_Q$ tends to $u(P)$ and the integral $\iint_{|Q-P|=\epsilon} G_P(\partial u/\partial n) dS_Q$ tends to zero, for any smooth function u. Therefore we may let $\epsilon \downarrow 0$ to obtain the required representation. •

One may note that the integral (8.2.9) is absolutely convergent, for any continuous function $h(Q)$. This may be seen by writing the relevant part of the integral in a polar coordinate system about P, where r_0 is chosen small enough:

$$\iiint_{|Q-P|<r_0} \frac{h(Q)}{|P - Q|} dQ = \int_0^{2\pi} \int_0^{\pi} \int_0^{r_0} h(x + r \sin\theta \cos\varphi,$$
$$y + r \sin\theta \sin\varphi, z + r \cos\theta) r \sin\theta \, dr \, d\theta \, d\varphi$$

This means that $\lim_{\epsilon \downarrow 0} \iiint_{D_\epsilon} G(P, Q) h(Q) dQ = \iiint_D G(P, Q) h(Q) dQ$, an absolutely convergent integral.

We now show, as in the case of ordinary differential equations, that Green's function is symmetric.

THEOREM 8.4. *For any two points P_1, P_2 we have $G(P_1, P_2) = G(P_2, P_1)$.*

Proof. We apply Green's identity with $u(Q) = G_{P_1}(Q) = G(P_1, Q), v(Q) = G_{P_2}(Q) = G(P_2, Q)$ in the region $D_\epsilon = \{Q \in D : |Q - P_1| > \epsilon, |Q - P_2| > \epsilon\}$ where $\epsilon < \frac{1}{2}|P_1 - P_2|$. Since both functions satisfy Laplace's equation in D_ϵ, we have $0 = \iiint_{D_\epsilon}(u\nabla^2 v - v\nabla^2 u)dQ = \iint_{\partial D_\epsilon}(u\,\partial v/\partial n - v\,\partial u/\partial n)dS_Q$. The surface integral on the outer boundary ∂D is zero. It remains to analyze the following four integrals on the inner boundaries:

$$\text{I:} \qquad \iint_{|Q-P_1|=\epsilon} G_{P_1}\frac{\partial G_{P_2}}{\partial n}dS_Q$$

$$\text{II:} \qquad \iint_{|Q-P_1|=\epsilon} G_{P_2}\frac{\partial G_{P_1}}{\partial n}dS_Q$$

$$\text{III:} \qquad \iint_{|Q-P_2|=\epsilon} G_{P_1}\frac{\partial G_{P_2}}{\partial n}dS_Q$$

$$\text{IV:} \qquad \iint_{|Q-P_2|=\epsilon} G_{P_2}\frac{\partial G_{P_1}}{\partial n}dS_Q$$

The first and fourth integrals tend to zero when $\epsilon \downarrow 0$, since each integrand is $O(1/\epsilon)$ and the surface area is $O(\epsilon^2)$ when $\epsilon \downarrow 0$. For the second and third integrals, we repeat the analysis used in the proof of Theorem 8.3 to conclude that II $\to G_{P_2}(P_1)$ and III $\to G_{P_1}(P_2)$ when $\epsilon \downarrow 0$. The proof is complete. •

We now turn to some applications of these ideas.

EXAMPLE 8.2.1. *Find Green's function of the ball $D = \{Q \in \mathbb{R}^3 : |Q| < R\}$.*

Solution. This can be obtained from the case $D = \mathbb{R}^3$ by the *method of images.* We look for Green's function in the form

$$G(P,Q) = \frac{1}{4\pi|P-Q|} - \frac{C}{|P-Q'|}$$

where the image point Q' is suitably chosen with $Q' \notin D$ and C is a constant. The above combination satisfies (8.2.6) and (8.2.8). It remains to satisfy the boundary condition (8.2.7). To do this requires that $|P - Q| = |P - Q'|/C$ for all $P \in \partial D$. Specifically, we choose the image point as $Q' = Q(R^2/|Q|^2)$; this is the point along ray OQ whose distance from O satisfies $|Q||Q'| = R^2$. To choose the constant C, we write for $P \in \partial D$

$$\begin{aligned} |P-Q|^2 &= |P|^2 - 2\langle P,Q\rangle + |Q|^2 \\ &= R^2 - 2\langle P,Q\rangle + |Q|^2 \\ |P-Q'|^2 &= |P|^2 - 2\langle P,Q'\rangle + |Q'|^2 \\ &= R^2 - 2\frac{R^2}{|Q|^2}\langle P,Q\rangle + \frac{R^4}{|Q|^2} \\ &= \frac{R^2}{|Q|^2}(|Q|^2 - 2\langle P,Q\rangle + R^2) \end{aligned}$$

Therefore $|P-Q'|/|P-Q| = R/|Q|$ if $|P| = R$. This leads to the choice $C = R/4\pi|Q|$ and Green's function in the form

$$G(P,Q) = \frac{1}{4\pi}\left(\frac{1}{|P-Q|} - \frac{R}{|Q|}\frac{1}{|P-Q'|}\right) \quad \bullet$$

We now return to the theory.

8.2.4. Solution of the Dirichlet problem. Green's function of a region D can also be used to solve the Dirichlet problem $\nabla^2 u = 0$ in D with $u = f$ on ∂D. For this purpose we again write Green's identity

$$\iiint_{D_\epsilon} (G\nabla^2 u - u\nabla^2 G)\, dQ = \iint_{\partial D_\epsilon} \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right) dS_Q$$

The left side is identically zero. The right member contributes $-\iint f(\partial G/\partial n)$ on the outer boundary ∂D; on the inner boundary we have $\iint u(\partial G/\partial n) \to -u(P)$. We have proved the following result.

THEOREM 8.5. *The solution of the Dirichlet problem $\nabla^2 u = 0$ in D with $u = f$ on ∂D has the representation*

$$u(P) = \iint_{\partial D} \frac{\partial G}{\partial n} f(Q)\, dS_Q$$

where $G(P,Q)$ is Green's function of the region D.

We can combine this with the particular solution of Poisson's equation to find the solution of Poisson's equation with general boundary conditions.

EXAMPLE 8.2.2. *Find the explicit representation of the solution of Poisson's equation $\nabla^2 u = -h$ in D with the boundary condition $u = f$ on ∂D.*

Solution. By the superposition principle, the solution may be obtained in the form

$$u(P) = \iiint_D G(P,Q)h(Q)\,dQ + \iint_{\partial D} \frac{\partial G}{\partial n} f(Q)\,dS_Q \quad \bullet$$

We can use Green's function for the ball to find a suitable form of Poisson's integral formula in three dimensions (cf. Sec. 3.2 for the two-dimensional case). We have $G(P,Q) = [1/(4\pi)][1/|P-Q| - (R/|Q|)(1/|P-Q'|)]$. We take a system of polar coordinates for which

$$\begin{aligned} P &= (\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta) \\ Q &= (\tau\cos\psi\sin\alpha, \tau\sin\psi\sin\alpha, \tau\cos\alpha) \\ Q' &= \frac{R}{\tau}(R\cos\psi\sin\alpha, R\sin\psi\sin\alpha, R\cos\alpha) \end{aligned}$$

A straightforward differentiation yields

$$\left.\frac{\partial G}{\partial \tau}\right|_{\tau=R} = \frac{R^2-\rho^2}{4\pi R(R^2+\rho^2-2R\rho\cos\gamma)^{3/2}}$$

where $\cos\gamma = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\psi-\varphi)$. The solution of Laplace's equation $\nabla^2 u = 0$ in the ball with the boundary condition $u = f$ is written

$$u(P) = \frac{1}{4\pi R}\iint_{|Q|=R} \frac{R^2-\rho^2}{(R^2+\rho^2-2R\rho\cos\gamma)^{3/2}} f(Q)dS_Q$$

EXERCISES 8.2

1. Show that the newtonian potential kernel satisfies properties 1 through 3 of subsection 8.2.1.
2. Let $g = g(r)$ be a radial solution of Laplace's equation $\nabla^2 g = 0$ defined for $r > 0$ with the properties that $g(r) \to 0$ when $r \to \infty$ and

   $$\begin{aligned}\lim_{r\downarrow 0} \int_0^{2\pi}\int_0^{\pi} u(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)(\partial g/\partial r)r^2\sin\theta\,d\theta\,d\varphi \\ = -u(0,0,0)\end{aligned}$$

   for every continuous function $u = u(x,y,z)$. Prove that $g(r) = 1/(4\pi r)$, the newtonian potential kernel.
3. Let $G(P,Q) = 1/(4\pi|P-Q|) - R/(4\pi|Q||P-Q'|)$ be Green's function of the ball $|P| < R$. Prove that $\partial G/\partial r = (R^2-r^2)/[4\pi R(R^2+r^2-2Rr\cos\gamma)^{3/2}]$ is the Poisson kernel of the ball.

Exercises 4 to 8 provide some of the basic properties of solutions of Laplace's equation. They must be done in the order given.

4. Modify the proof of Theorem 8.2 to prove the following result: Any solution of Laplace's equation $\nabla^2 u = 0$ in a three-dimensional region D can be written as
$$u(P) = \iint\limits_{|Q-P|=R} \left(u\frac{\partial G_P}{\partial n} - G_P \frac{\partial u}{\partial n} \right) dS_Q$$
where $G_P(Q)$ is the newtonian potential kernel and the integration is over the surface of the sphere.
5. Use Exercise 4 to prove the mean value theorem for harmonic functions: For every solution of $\nabla^2 u = 0$ in a three-dimensional region, we have
$$u(P) = \frac{1}{4\pi R^2} \iint\limits_{|Q-P|=R} u\, dS_Q$$
[*Hint:* First show that on the surface we have $G_P = 1/(4\pi R)$, $\partial G_P/\partial n = -[1/(4\pi R^2)]$, and $\iint_{|Q-P|=R} (\partial u/\partial n) dS_Q = 0$.]
6. Extend Exercise 5 to show that every solution of $\nabla^2 u = 0$ has the *solid mean value property* in the form
$$u(P) = \frac{3}{4\pi R^3} \iiint\limits_{|Q-P|<R} u\, dQ$$
where the integral is over the solid ball of radius R.
7. Use Exercise 6 to prove the *local maximum principle:* if u is a solution of $\nabla^2 u = 0$ in the ball $\{Q : |Q-P| < \delta\}$ such that $u(P) \ge u(Q)$ for all Q and $|Q-P| < \delta$, then $u(Q) = u(P)$ for all Q such that $|Q-P| < \delta$.
8. Extend Exercise 7 to prove the *global maximum principle:* if u is a solution of $\nabla^2 u = 0$ in a connected and smoothly bounded region, for which $\max_{P\in D} u(P)$ is attained at an interior point P_0, then u is constant throughout P. [*Hint:* If there is an interior global maximum at $P_0 \in D$, then connect an arbitrary point $P \in D$ to P_0 by means of a polygonal path C. Along this path draw a finite sequence of spheres $\{B(x_i, \delta_i)\}_{i=0}^{N}$ such that $x_0 = P_0$, $x_N = P$, and $x_i \in C$, $x_i \in B(x_{i-1}, \delta_{i-1})$ for $i = 1, \ldots, N$. By Exercise 7, u is constant throughout $B(x_0, \delta_0)$, especially at x_1. Now apply Exercise 7 to $B(x_1, \delta_1)$ and continue the process inductively until you reach $P = x_N$, to conclude that $u(x_N) = u(x_0)$.]
9. Formulate and prove a local minimum principle and global minimum principle for solutions of Laplace's equation $\nabla^2 u = 0$.

Exercises 10 and 11 are designed to establish directly the properties of the newtonian potential operator $h \to u$, where $u(P) = [1/(4\pi)] \iiint h(Q)/|P-Q| dQ$ is the newtonian potential of the function $h(Q)$.

10. Suppose that h is a continuous function with $h(Q) = 0$ for $|Q| > r$. Prove that u is a solution of Laplace's equation $\nabla^2 u = 0$ for $|P| > r$.

11. Suppose that h is a differentiable function with $h(Q) = 0$ for $|Q| > r$.

(a) Prove that the first derivatives of u can be computed by the formula

$$4\pi \frac{\partial u}{\partial x_i} P = \iiint \frac{h(Q)(P-Q)_i}{|Q-P|^3} dQ = -\iiint \frac{\partial h}{\partial Q_i} Q \frac{1}{|Q-P|} dQ$$

(b) Prove that the second derivatives of u can be computed by the formula

$$4\pi \frac{\partial^2 u}{\partial x_i, \partial x_j} = \frac{1}{4\pi} \iiint \frac{\partial h}{\partial Q_i} \frac{(Q-P)_j}{|P-Q|^3} dQ$$

(c) Transform the last integral to spherical polar coordinates to prove that

$$4\pi \nabla^2 u(x,y,z) =$$
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\frac{\partial}{\partial r}\right) h(x + r\cos\varphi\sin\theta, y + r\sin\varphi\sin\theta, z + r\cos\theta)\sin\theta\, dr\, d\theta\, d\varphi$$
$$= -4\pi\, h(x,y,z)$$

8.3. Two-Dimensional Problems

One may try to replicate the three-dimensional theory in two dimensions. In the case of a bounded region, the theory is entirely parallel, although the formulas are slightly different. For the basic case of the entire plane, we encounter the fundamental difficulty that a Green's function cannot be uniquely obtained that satisfies all the previous conditions.

8.3.1. The logarithmic potential. To find a suitable Green's function for two dimensions, we abstract the fundamental properties of the newtonian potential kernel in three dimensions. These are that $\nabla_Q^2 G_P = 0$ for $Q \neq P$ and $\iint_{|Q-P|=\epsilon} (\partial G_P/\partial n) u\, dS_Q \to -u(P)$, for every smooth function u. To replicate this in two dimensions, we begin with the radial solutions of Laplace's equation

$$0 = \nabla^2 G = G_{rr} + \frac{1}{r} G_r = \frac{1}{r}(rG_r)_r$$

leading to $rG_r = B, G = A + B\log r$ for suitable constants A, B. We can safely set $A = 0$, since we are looking for a particular solution. To determine B, we compute $\partial G/\partial r = B/r$, whose "surface integral" on the circle $|Q-P| = \epsilon$ is

$$\int_{|Q-P|=\epsilon} \frac{\partial G}{\partial r} u\, dS_Q = \int_{-\pi}^{\pi} \frac{B}{\epsilon} u(x + \epsilon\cos\theta, y + \epsilon\sin\theta)\epsilon\, d\theta \to 2\pi B u(x,y) \qquad \epsilon \to 0$$

Therefore we choose $B = -1/(2\pi)$, and we have the *logarithmic potential kernel*

$$G(P,Q) = \frac{1}{2\pi} \log \frac{1}{|P-Q|}$$

This leads to the following explicit representation of the solution of Poisson's equation:

$$u(P) = \frac{1}{2\pi} \iint \log\left(\frac{1}{|P-Q|}\right) h(Q)\, dQ$$

One can show[2] that this formula provides a solution of Poisson's equation. However, it is difficult to obtain a simple uniqueness criterion for the Poisson equation in the entire plane. For this reason we turn to some problems with unique solutions.

8.3.2. Green's function of a bounded plane region. Green's function of a bounded two-dimensional region is defined by the following requirements:

$$\nabla_Q^2 G = 0 \qquad Q \in D, Q \neq P \tag{8.3.1}$$

$$G(P,Q) = 0 \qquad Q \in D, P \in \partial D \tag{8.3.2}$$

$$G(P,Q) - \frac{1}{2\pi} \log \frac{1}{|P-Q|} \text{ is a smooth function} \tag{8.3.3}$$

We have the following theorem, exactly as in the three-dimensional case.

THEOREM 8.6. *Let u be a solution of the Poisson equation $\nabla^2 u = -h$ in the region D satisfying the boundary condition that $u = 0$ on ∂D. Then we have the explicit representation*

$$u(P) = \iint_D G(P,Q) h(Q)\, dQ$$

Proof. We follow the proof of the three-dimensional case, beginning with Green's formula applied to $D_\epsilon = \{Q \in D : |Q-P| > \epsilon\}$:

$$\iint_{D_\epsilon} (u\nabla^2 G - G\nabla^2 u) dQ = \int_{\partial D_\epsilon} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS_Q$$

The integral on the left is over the interior of the plane region D_ϵ, and the integral on the right is over the boundary curves that define ∂D_ϵ. In D_ϵ, G satisfies Laplace's equation and $\nabla^2 u = -h$, so that the left side becomes $\iint_{D_\epsilon} Gh\, dQ$. The right side is analyzed as before: the outer boundary ∂D contributes zero, while the inner boundary $\{Q : |Q-P| = \epsilon\}$ contributes $[1/(2\pi)] \int_0^{2\pi} u(x + \epsilon\cos\theta, y + \epsilon\sin\theta) d\theta$. In the limit $\epsilon \downarrow 0$, this gives $u(x,y) = u(P)$, which was to be proved. •

EXAMPLE 8.3.1. *Find Green's function of the circular disc $D = \{(x,y) : x^2 + y^2 < a^2\}$, and use this to solve Poisson's equation $\nabla^2 u = -h$ in D with $u = 0$ on ∂D.*

[2] Ibid.

Solution. This can be solved by the method of images, as in the three-dimensional case. We look for Green's function in the form

$$G(P,Q) = \frac{1}{2\pi}\left(\log\frac{1}{|P-Q|} - C_1\log\frac{C_2}{|P-Q'|}\right)$$

where Q' is the image point $Q' = Q(a^2/|Q|^2)$ and the constants C_1, C_2 are to be determined. As we showed in Sec. 8.2, the image point Q' satisfies $|P-Q|/|P-Q'| = a/|Q|$ when $|P| = a$. Therefore we choose $C_1 = 1, C_2 = a/|Q|$. The Poisson equation is solved by the explicit representation

$$u(P) = \frac{1}{2\pi}\int_{|Q|=a}\left(\log\frac{1}{|P-Q|} - \log\frac{a}{|P-Q'||Q|}\right)h(Q)\,dS_Q \quad \bullet$$

EXAMPLE 8.3.2. *Find Green's function for the region $D = \{(x,y) : y > 0\}$, and solve the Poisson equation $\nabla^2 u = -h$ with $u = 0$ on ∂D.*

Solution. In this case we may again use the method of images. If $Q' = (\xi, \eta)$, the image point is $Q' = (\xi, -\eta)$; we have $|P-Q| = |P-Q'|$ if $P \in \partial D$. Green's function is

$$\begin{aligned} G(P,Q) &= \frac{1}{2\pi}\log\frac{1}{|P-Q|} - \frac{1}{2\pi}\log\frac{1}{|P-Q'|} \\ &= -\frac{1}{4\pi}\log\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} \end{aligned}$$

The solution of Poisson's equation is given by $u(P) = \iint_D G(P,Q)h(Q)dQ$. $\bullet$

8.3.3. Solution of the Dirichlet problem. The two-dimensional Green's function can also be used to solve the Dirichlet problem for Laplace's equation $\nabla^2 u = 0$ in a smoothly bounded plane region D with given boundary data f. As in the three-dimensional case, we begin with Green's identity

$$\iint_{D_\epsilon}(u\nabla^2 G - G\nabla^2 u)dQ = \int_{\partial D_\epsilon}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS_Q$$

where $D_\epsilon = \{Q \in D : |Q-P| > \epsilon\}$. The left side is zero. The right side gives $\int f\,\partial G/\partial n$ on the outer boundary ∂D, while on the inner boundary we have $\int u\,\partial G/\partial n \to u(P)$. Thus we have proved the following result.

THEOREM 8.7. *The solution of the Dirichlet problem $\nabla^2 u = 0$ in the smoothly bounded plane region with $u = f$ on ∂D is given by the double-layer potential*

$$u(P) = -\int_{\partial D}\frac{\partial G}{\partial n}f\,dS_Q$$

EXAMPLE 8.3.3. *Solve the Dirichlet problem in the ball* $|P| < a$.

Solution. In this case we have Green's function

$$G(P,Q) = \frac{1}{2\pi}\left(\log\frac{1}{|P-Q|}\right) - \frac{1}{2\pi}\log\frac{a}{|Q||P-Q'|}$$

with

$$\frac{\partial G}{\partial r} = \frac{a^2 - r^2}{2\pi[a^2 + r^2 - 2ar\cos(\theta - \varphi)]}$$

and we retrieve the Poisson integral formula

$$u(P) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \varphi)}f(\theta)\,d\theta \quad \bullet$$

This formula was first derived in Sec. 3.1 from the Fourier series of separated solutions of Laplace's equation. Now we see that it can be done directly, without any appeal to Fourier series or separated solutions.

8.3.4. Green's functions and separation of variables. If we compare the Fourier representation of the solution with the explicit representation by Green's function, we can obtain a representation of the Green function in terms of the eigenfunctions of the associated homogeneous problem.

Suppose that we want to solve Poisson's equation $\nabla^2 u = -h$ with the condition that $u = 0$ on the boundary of the smoothly bounded region D. Let $\{\varphi_n\}$ be a complete system of eigenfunctions satisfying $\nabla^2\varphi_n + \lambda_n\varphi_n = 0$ in D with $\varphi_n = 0$ on the boundary and normalized so that $\langle\varphi_n, \varphi_n\rangle = \int_D \varphi_n(P)^2 dP = 1$. Here n is a "multi-index," depending on the dimension of the space. The Fourier representation of the solution is

$$u(P) = \sum_n \frac{1}{\lambda_n}\langle\varphi_n, h\rangle\varphi_n(P)$$

However, Green's function satisfies the boundary conditions and can be expanded in a series of eigenfunctions as $G(P,Q) = \sum_n c_n(Q)\varphi_n(P)$, where the Fourier coefficients are obtained as $c_n(Q) = \int_D G(P,Q)\varphi_n(P)dP$. Substituting in the explicit representation of u by Green's function and proceeding formally, we have

$$u(P) = \int_D G(P,Q)h(Q)\,dQ = \sum_n \varphi_n(P)\langle c_n, h\rangle$$

Comparing the two formulas for u leads to the identification $\langle c_n, h\rangle = \langle\varphi_n, h\rangle/\lambda_n$ for every h, or $c_n(Q) = \varphi_n(Q)/\lambda_n$, and we are led to the formula

$$G(P,Q) = \sum_n \frac{\varphi_n(P)\varphi_n(Q)}{\lambda_n}$$

This statement is known as *Mercer's theorem.* Although easily remembered, it may not be an efficient method for computation of Green's function. In particular, we do not expect that the series converges for $P = Q$. Rather than explore this in general, we consider the rigorous validity in each case.

EXAMPLE 8.3.4. *Find the Fourier representation of Green's function of* ∇^2 *in the rectangle* $0 < x < a, 0 < y < b$ *with the boundary condition that* $u = 0$ *on all four sides.*

Solution. The normalized eigenfunctions of this problem are

$$\varphi_{mn}(x,y) = \sqrt{\frac{4}{ab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

with $\lambda_{mn} = (m\pi/a)^2 + (n\pi/b)^2$. The above formula for Green's function is

$$\begin{aligned} G(P,Q) &= G(x,y;\xi,\eta) \\ &= \frac{4}{ab}\sum_{mn} \frac{\sin(m\pi x/a)\sin(n\pi y/b)\sin(m\pi\xi/a)\sin(n\pi\eta/b)}{(m\pi/a)^2 + (n\pi/b)^2} \end{aligned}$$

Although this series is not absolutely convergent, it can be shown to be conditionally convergent for $(x,y) \neq (\xi,\eta)$. •

EXAMPLE 8.3.5. *Find the Fourier representation of Green's function of* ∇^2 *in the circle* $x^2 + y^2 < a^2$ *with zero boundary conditions.*

Solution. In polar coordinates the normalized eigenfunctions in complex form are

$$\varphi_{mn}(\rho,\varphi) = C_{mn} J_m\left(\frac{z_{mn}\rho}{a}\right) e^{im\varphi}$$

where the eigenvalues are $\lambda_{mn} = (z_{mn}/a)^2$ and the normalizing constants are $C_{mn}^2 = 2/J_{m+1}(z_{mn})^2$. Green's function is

$$\begin{aligned} G(P,Q) &= G(r,\varphi;\rho,\theta) \\ &= \sum_{mn} C_{mn}^2 J_m\left(\frac{z_{mn}r}{a}\right) J_m\left(\frac{z_{mn}\rho}{a}\right) e^{im(\varphi-\theta)} \end{aligned}$$

It may be shown, as in Example 8.3.4, that the series is conditionally convergent. •

The above representations by double Fourier series are not absolutely convergent. To obtain more effective series representations, we use a Fourier series in one variable only, where the coefficient functions are obtained as Green's function of a closely related ordinary differential equation. We illustrate this for the Poisson equation $u_{xx} + u_{yy} = -f$ in the rectangle $0 < x < a, 0 < y < b$. We write each term as a single Fourier sine series:

$$f(x,y) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) F_m(y), \qquad F_m(y) = \frac{2}{a}\int_0^a f(\xi,y)\sin\frac{m\pi\xi}{a}d\xi$$

$$u(x,y) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right) U_m(y), \qquad U_m(y) = \frac{2}{a}\int_0^a u(\xi,y)\sin\frac{m\pi\xi}{a}d\xi$$

where U_m satisfies the equation $U_m'' - (m\pi/a)^2 U_m = -F_m$ with the boundary condition $U_m(0) = 0 = U_m(b)$. This ordinary differential equation is solved by the one-dimensional Green function

$$U_m(y) = \int_0^b G_m(y,\eta) F_m(\eta) d\eta$$

where

$$G_m(y,\eta) = \frac{\sinh(m\pi y/b)\sinh[m\pi(b-\eta)/b]}{W_m(y)} \quad 0 \le y \le \eta$$

$$G_m(y,\eta) = \frac{\sinh(m\pi(b-y)[b\sinh(m\pi\eta/b)]}{W_m(y)} \quad \eta \le y \le b$$

and the wronskian is

$$W_m(y) = \frac{m\pi}{b}\left[\sinh\frac{m\pi y}{b}\cosh\frac{m\pi(b-y)}{b} + \cosh\frac{m\pi y}{b}\sinh\frac{m\pi(b-y)}{b}\right]$$

Green's function for the problem is written as

$$G(P,Q) = G(x,y;\xi,\eta) = \frac{2}{L}\sum_{m=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{m\pi\xi}{a} G_m(y,\eta)$$

If $y \neq \eta$, this series is absolutely convergent together with all its derivatives. This can be seen easily from the form of the coefficient $G_m(y,\eta)$: when $m \to \infty$, the hyperbolic sine function satisfies $\sinh m\theta \simeq \frac{1}{2}e^{m\theta}$. If we make this substitution everywhere, we have for $y < \eta$

$$\begin{aligned} G_m(y,\eta) &\simeq \frac{e^{m\pi y/b} e^{m\pi(b-\eta)/b}}{2(m\pi/b)e^{m\pi}} \\ &= \frac{e^{(m\pi/b)(y-\eta)}}{2(m\pi/b)} \end{aligned}$$

This tends to zero exponentially when $m \to \infty$. A similar computation applies to the case $y > \eta$.

EXERCISES 8.3

1. Use the method of images to find Green's function and to solve Poisson's equation $\nabla^2 u = -h$ in the quarter-plane $D = \{(x,y) : x > 0, y > 0\}$ with the boundary condition that $u = 0$ on both axes. Use three image points.
2. Use the method of images to find Green's function and to solve Poisson's equation $\nabla^2 u = h$ in the quarter-plane $D = \{(x,y) : x > 0, y > 0\}$ with the boundary conditions $u(x,0) = 0$, $(\partial u/\partial x)(0,y) = 0$.

3. Show that the logarithmic potential can be obtained directly from the newtonian potential kernel by the following "renormalization procedure": let $u_M(x,y) = \int_{-M}^{M} dz/(4\pi\sqrt{x^2+y^2+z^2})$ be the newtonian potential of a uniform line charge on the segment $[-M, M]$ of the z-axis.

   (a) Show that this integral can be evaluated directly as $[1/(2\pi)]\sinh^{-1} \times (M/r)$ where $r = \sqrt{x^2+y^2}$.

   (b) Use the behavior of $\sinh^{-1} x$ when $x \to \infty$ to find a constant c_M such that $\lim_{M\uparrow\infty}[u_M(x,y) - c_M] = [1/(2\pi)]\log(1/\sqrt{x^2+y^2})$.

   (c) Show that the "potential difference" can be expressed directly as

$$\lim_{M\uparrow\infty}[u_M(x_1,y_1) - u_M(x_2,y_2)] = \frac{1}{2\pi}\left(\log\frac{1}{\sqrt{x_1^2+y_1^2}} - \log\frac{1}{\sqrt{x_2^2+y_2^2}}\right)$$

4. Find a Fourier representation for Green's function of ∇^2 in the rectangle $0 < x < a, 0 < y < b$ with the boundary conditions that $u = 0$ on the bottom and two vertical sides while $u_y = 0$ on the top side $y = b, 0 < x < a$.
5. Find a Fourier representation of Green's function of ∇^2 of the disc $x^2+y^2 < a^2$ in the form $G(r,\theta;\rho,\varphi) = \sum G_m(r,\rho)e^{im(\theta-\varphi)}$, where G_m is a suitable one-dimensional Green's function for the ordinary differential operator $u_{rr} + (1/r)u_r - (m^2/r^2)u$.
6. Show that Green's function of ∇^2 for the infinite strip $-\infty < x < \infty, 0 < y < a$ can be written in the form

$$G(x,y;\xi,\eta) = \frac{2}{a}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{m\pi\eta}{a}\right)g_n(x,\xi)$$

   where g_n is a suitably defined Green's function for the ordinary differential equation $g_n'' - (n\pi/a)^2 g_n = 0, -\infty < x < \infty$.
7. Use the method of images to show that Green's function of ∇^2 for the strip $-\infty < x < \infty, 0 < y < a$ can be written in the form

$$G(x,y;\xi,\eta) =$$
$$\frac{1}{2\pi}\sum_n\left[\log\frac{1}{(x-\xi)^2+(y-\eta-2na)^2} - \log\frac{1}{(x-\xi)^2+(y+\eta-2na)^2}\right]$$

8. Let D be the region described in polar coordinates by $\{(r,\theta) : a < r < b, \alpha < \theta < \beta\}$ with $a > 0, 0 < \alpha < \beta < 2\pi$. Show that Green's function can be written in the form

$$G(r,\theta;\rho,\varphi) = \sum_{n\geq 1}\sin\left[\frac{n\pi(\theta-\alpha)}{(\beta-\alpha)}\right]\sin\left[\frac{n\pi(\varphi-\alpha)}{(\beta-\alpha)}\right]f_n(r;\rho)$$

   where f_n is a suitable Green's function for the ordinary differential equation $f'' + (1/r)f' - [n\pi/(\beta-\alpha)^2]f = 0$ on the interval $a < r < b$.

9. Let D be the region described in polar coordinates by $\{(r,\theta) : a < r < b, \alpha < \theta < \beta\}$ with $a > 0, 0 < \alpha < \beta < 2\pi$. Show that Green's function of ∇^2 can be written in the form

$$G(r,\theta;\rho,\varphi) = \sum_{n\geq 1} \sin[n\pi \log(r/a)/\log(b/a)] \sin[n\pi \log(\rho/a)/\log(b/a)] g_n(\theta,\varphi)$$

where g_n is a suitable Green's function for the ordinary differential equation $g_n'' - [n\pi/\log(b/a)]^2 g_n = 0$ on the interval $\alpha < \varphi < \beta$.

8.4. Green's Function for the Heat Equation

In this section we develop the appropriate form of Green's function for the heat equation, which is closely related to the Gauss-Weierstrass integral representation, which was discussed in Chapter 5.

8.4.1. Nonhomogeneous heat equation. We can obtain a particular solution of the nonhomogeneous heat equation

$$u_t - K u_{xx} = h \qquad 0 < t < T, -\infty < x < \infty \tag{8.4.1}$$

by suitably transforming the Fourier representation. To do this, we write

$$u(x;t) = \int_{-\infty}^{\infty} U(\xi;t) e^{i\xi x}\, d\xi, \qquad h(x;t) = \int_{-\infty}^{\infty} H(\xi;t) e^{i\xi x}\, d\xi$$
$$U(\xi;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x;t) e^{-i\xi x}\, dx, \qquad H(\xi;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x;t) e^{-i\xi x}\, dx$$

and transform Eq. (8.4.1) to the ordinary differential equation $U_t + K\xi^2 U = H$. A particular solution with $U(\xi;0) = 0$ is found by means of the integrating factor $e^{K\xi^2 t}$, and we obtain

$$U(\xi;t) = \int_0^t e^{-K\xi^2(t-s)} H(\xi;s)\, ds$$

Substituting the above Fourier integral of $H(\xi;t)$, we obtain

$$\begin{aligned} u(x;t) &= \int_{-\infty}^{\infty} e^{i\xi x}\, d\xi \int_0^t e^{-K\xi^2(t-s)} \left[\frac{1}{2\pi}\int h(y;s) e^{-iy\xi}\, dy\right] ds \\ &= \int_0^t ds \int_{-\infty}^{\infty} h(y;s)\, dy \int e^{i\xi(x-y)} e^{-K\xi^2(t-s)} d\xi \end{aligned}$$

The final integral is recognized from Sec. 5.2 as the heat kernel

$$G(x,y;\tau) = (4\pi K\tau)^{-1/2} e^{-(x-y)^2/(4K\tau)}$$

and we have the explicit representation

$$u(x;t) = \int_0^t \int_{-\infty}^{\infty} G(x,y;t-s)h(y;s)\,dy\,ds \tag{8.4.2}$$

THEOREM 8.8. *Let $h(y;s)$ be a bounded continuous function in the strip $0 \le t \le T, -\infty < x < \infty$. Then integral (8.4.2) defines a solution of the nonhomogeneous heat equation (8.4.1) with $u(x;0) = 0$.*

Proof. We recall from Sec. 5.2 that $G(x,y;t)$ satisfies the heat equation $G_t - KG_{xx} = 0$ and that for any arbitrary bounded continuous function h, $\lim_{\tau\downarrow 0} \int_{-\infty}^{\infty} G(x,y;\tau)h(y)dy = h(x)$. We use this to compute the derivative as follows:

$$\begin{aligned}
u(x;t+\Delta t) - u(x;t) &= \int_0^{t+\Delta t} \int_{-\infty}^{\infty} G(x,y;t+\Delta t - s)h(y;s)\,dy\,ds \\
&\quad - \int_0^t \int_{-\infty}^{\infty} G(x,y;t-s)h(y;s)\,dy\,ds \\
&= \int_0^t \int_{-\infty}^{\infty} [G(x,y;t+\Delta t-s) - G(x,y;t-s)]h(y;s)\,dy\,ds \\
&\quad + \int_t^{t+\Delta t} \int_{-\infty}^{\infty} G(x,y;t+\Delta t-s)h(y;s)\,dy\,ds
\end{aligned}$$

When we divide by Δt and take the limit, the first integral tends to the limit $\int_0^t \int_{-\infty}^{\infty} G_t(x,y;t-s)h(y;s)dy\,ds$ and the second integral tends to $h(x,t)$. Similarly, when we compute u_{xx}, the derivatives can be put directly onto G to obtain the corresponding integral with G replaced by G_{xx}. But G satisfies the heat equation $G_t = KG_{xx}$, from which we conclude $u_t - Ku_{xx} = h$, as required. •

To solve the nonhomogeneous heat equation with general initial data, we apply the superposition principle. The function

$$u(x;t) = \int_{-\infty}^{\infty} G(x,y;t)f(y)\,dy + \int_0^t \int_{-\infty}^{\infty} G(x,y;t-s)h(y;s)\,dy\,ds$$

satisfies the equation $u_t - Ku_{xx} = h$ with the initial condition $u(x;0) = f(x)$.

To solve nonhomogeneous heat equations with boundary conditions, we can use the method of images to suitably modify $G(x,y;t)$.

EXAMPLE 8.4.1. *Find an explicit representation of the solution of the equation $u_t - Ku_{xx} = h$ in the half-space $0 < x < \infty, 0 < t < T$ satisfying the boundary condition $u(0;t) = 0$ and the initial condition $u(x;0) = 0$.*

Solution. The corresponding Green's function for this case is obtained by using the image point $Q' = -y$, which leads to

$$G(P,Q;t) = (4\pi Kt)^{-1/2}(e^{-(x-y)^2/(4Kt)} - e^{-(x+y)^2/(4Kt)})$$

and the solution

$$u(x;t) = \int_0^t \int_{-\infty}^{\infty} G(x,y;t-s)h(y;s)\,dy\,ds \quad \bullet$$

EXAMPLE 8.4.2. *Find the solution of the nonhomogeneous heat equation* $u_t - K\nabla^2 u = h$ *in the region* $\mathbb{R}^3 \times [0,T]$ *satisfying the initial condition* $u(P;0) = 0$.

Solution. We simply modify the previous one-dimensional construction to the case of three dimensions. Green's function for this case is

$$G(P,Q;t) = \iiint e^{-K|\xi|^2 t} e^{i\langle \xi, P-Q\rangle}\,d\xi$$
$$= (4\pi K t)^{-1/2} e^{-|P-Q|^2/(4Kt)}$$

and the solution of Poisson's equation is

$$u(P;t) = \int_0^t \iiint G(P,Q;t-s)h(Q;s)\,dQ\,ds \quad \bullet$$

To solve nonhomogeneous heat equations in higher dimensions, we can suitably modify $G(x,y;t)$. For example, the three-dimensional equation

$$u_t - K\nabla^2 u = h \qquad P \in \mathbb{R}^3, 0 \le t \le T$$

has the particular solution

$$u(P;t) = \frac{1}{[4\pi K(t-s)]^{3/2}} \int_0^t \iiint e^{-|P-Q|^2/[4K(t-s)]} h(Q;s)\,dQ\,ds$$

8.4.2. The one-dimensional heat kernel and the method of images. A closely related problem is the initial-value problem for the homogeneous heat equation

$$u_t = Ku_{xx} \qquad (t > 0, -\infty < x < \infty) \qquad u(x;0) = f(x)$$

that was solved in Sec. 5.2. There we obtained the Gauss-Weierstrass explicit representation

$$u(x;t) = \frac{1}{\sqrt{4\pi K t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4Kt)} f(y)\,dy$$

This can be used to find explicit representations of the solution of initial-boundary-value problems on a finite interval with homogeneous boundary conditions, leading to some interesting identities involving infinite series. To do this, we make a suitable application of the method of images. We illustrate with an example.

EXAMPLE 8.4.3. *Find an explicit representation of the solution of the heat equation* $u_t = Ku_{xx}$ *on the interval* $0 < x < L$ *with the boundary conditions* $u(0;t) = 0, u(L;t) = 0$ *and the initial conditions* $u(x;0) = f(x)$, *a piecewise smooth function.*

Solution. We extend $f(x)$ as an odd $2L$-periodic function by setting $f_{\text{odd}}(x) = -f(-x)$ for $-L < x < 0$ and $f_{\text{odd}}(x+2L) = f_{\text{odd}}(x)$ for all x. We apply the Gauss-Weierstrass formula to $f_{\text{odd}}(x)$ and transform as follows:

$$\begin{aligned}\sqrt{4\pi Kt}\,u(x;t) &= \int_{-\infty}^{\infty} e^{-(x-y)^2/(4Kt)} f_{\text{odd}}(y)\,dy \\ &= \sum_{-\infty<m<\infty} \left\{ \int_{2mL}^{(2m+1)L} + \int_{(2m+1)L}^{(2m+2)L} \right\} e^{-(x-y)^2/(4Kt)} f_{\text{odd}}(y)\,dy\end{aligned}$$

But for each $m = 0, +1, -1, \ldots$ we can write

$$\int_{2mL}^{(2m+1)L} e^{-(x-y)^2/(4Kt)} f_{\text{odd}}(y)\,dy = \int_0^L e^{-(x-y-2mL)^2/(4Kt)} f(y)\,dy$$
$$\int_{(2m+1)L}^{(2m+2)L} e^{-(x-y)^2/(4Kt)} f_{\text{odd}}(y)\,dy = -\int_0^L e^{-[x+y-(2m+2)L]^2/(4Kt)} f(y)\,dy$$

Thus

$$u(x;t) = \sum_{-\infty<m<\infty} \int_0^L \left(e^{-(x-y-2mL)^2/(4Kt)} - e^{-[x+y-(2m+2)L]^2/(4Kt)}\right) \frac{f(y)\,dy}{\sqrt{4\pi Kt}}$$

which is the required representation. •

It is instructive to compare the result of Example 8.4.3 with the Fourier representation obtained in Sec. 2.2. According to that method, the solution is written in the form

$$u(x;t) = \sum_{n\geq 1} A_n \left(\sin\frac{n\pi x}{L}\right) e^{-(n\pi/L)^2 Kt}$$

where the Fourier coefficient is

$$A_n = \frac{2}{L}\int_0^L f(y)\sin\left(\frac{n\pi y}{L}\right)\,dy$$

Interchanging the orders of integration and summation, we have

$$u(x;t) = \frac{2}{L}\int_0^L \left[\sum_{n\geq 1} \sin\frac{n\pi x}{L}\left(\sin\frac{n\pi y}{L}\right) e^{-(n\pi/L)^2 Kt}\right] f(y)\,dy$$

We have represented the function $u(x;t)$ in two different ways. Since this holds for all piecewise smooth functions $f(x), 0 < x < L$, we infer that the integrands are equal. Thus

$$\frac{2}{L}\sum_{n\geq 1}\left(\sin\frac{n\pi x}{L}\right)\left(\sin\frac{n\pi y}{L}\right)e^{-(n\pi/L)^2Kt}$$
$$=\frac{1}{\sqrt{4\pi Kt}}\sum_{-\infty<m<\infty}\left(e^{-(x-y-2mL)^2/(4Kt)}-e^{-[x+y-(2m+2)L]^2/(4Kt)}\right)$$

EXERCISES 8.4

1. Find the solution of the heat equation $u_t - Ku_{xx} = h$ for $0 < x < \infty$ satisfying the boundary conditions $u_x(0;t) = 0, u(x;0) = 0$.
2. Find the solution of the heat equation $u_t - Ku_{xx} = h$ for $0 < x < L$ satisfying the boundary conditions $u(0;t) = 0, u(L;t) = 0, u(x;0) = 0$.
3. Find a particular solution of the two-dimensional heat equation $u_t - K(u_{xx} + u_{yy}) = h$ in the entire plane $-\infty < x < \infty, -\infty < y < \infty, 0 \leq t \leq T$.
4. Find an explicit representation of the solution of the heat equation $u_t = Ku_{xx}$ on the interval $0 < x < L$ with the boundary conditions $u_x(0;t) = 0, u_x(L;t) = 0$ and the initial condition $u(x;0) = f(x)$, a piecewise smooth function.
5. Find an explicit representation of the solution of the heat equation $u_t = Ku_{xx}$ on the interval $0 < x < L$ with the (periodic) boundary conditions $u(0;t) = u(L;t), u_x(0;t) = u_x(L;t)$ and the initial condition $u(x;0) = f(x)$, a piecewise smooth function.

8.5. Green's Function for the Wave Equation

We now seek a Green's function representation for the solution of the three-dimensional wave equation problem

$$u_{tt} - c^2\nabla^2 u = h(P;t) \qquad P\in\mathbb{R}^3, t>0$$
$$u(P;0)=0 \qquad u_t(P;0)=0 \qquad P\in\mathbb{R}^3$$

This will generalize the newtonian potential kernel of Sec. 8.2, which is the case in which $u = u(P), h = h(P)$, independent of time.

8.5.1. Derivation of the retarded potential. To find the explicit representation by Green's function, we begin with the Fourier-transformed equation

$$U_{tt} + c^2|\mu|^2 U = H(\mu;t) \qquad \mu\in\mathbb{R}^3, t>0$$
$$u(\mu;0)=0 \qquad u_t(\mu;0)=0 \qquad \mu\in\mathbb{R}^3$$

where

$$
\begin{aligned}
u(P;t) &= \iiint e^{i\langle\mu,P\rangle} U(\mu;t)\, d\mu \\
U(\mu;t) &= \left(\frac{1}{2\pi}\right)^3 \iiint e^{-i\langle\mu,P\rangle} u(P;t)\, dP \\
h(P;t) &= \iiint e^{i\langle\mu,P\rangle} H(\mu;t)\, d\mu \\
H(\mu;t) &= \left(\frac{1}{2\pi}\right)^3 \iiint e^{-i\langle\mu,P\rangle} h(P;t)\, dP
\end{aligned}
$$

The Fourier-transformed equation is a second-order ordinary differential equation that can be solved by the method of variation of parameters:

$$
U(\mu;t) = \int_0^t \left[\frac{\sin|\mu|cs}{|\mu|c}\right] H(\mu;t-s)\, ds
$$

It remains to invert the Fourier transform.

For this purpose we recall from Sec. 5.3 the surface integral representation of the sine function as a mean value:

$$
\frac{\sin|\mu|R}{|\mu|R} = \frac{1}{4\pi R^2} \iint\limits_{|\xi|=R} e^{i\langle\mu,\xi\rangle} d\xi
$$

Inserting this into the Fourier representation of $u(P;t)$ and interchanging the order of integration, we have

$$
\begin{aligned}
(8.5.1) \qquad u(P;t) &= \iiint e^{i\langle\mu,P\rangle}\, d\mu \int_0^t H(\mu;t-s)\frac{1}{4\pi c^2 s}\, ds \iint\limits_{|\xi|=cs} e^{i\langle\mu,\xi\rangle}\, d\xi \\
&= \frac{1}{4\pi c^2} \int_0^t \frac{1}{s}\, ds \iint\limits_{|\xi|=cs} d\xi \iiint e^{i\langle\mu,P+\xi\rangle} H(\mu;t-s)\, d\mu \\
&= \frac{1}{4\pi c^2} \int_0^t \frac{1}{s}\, ds \iint\limits_{|\xi|=cs} h(P+\xi;t-s)\, d\xi
\end{aligned}
$$

This is the *retarded potential representation* of the solution $u(P;t)$. It can be written as a solid integral purely in terms of the spatial variables by writing

$Q = P + \xi$ and

$$u(P;t) = \frac{1}{4\pi c^2}\int_0^t ds \iint\limits_{|P-Q|=cs} \frac{h(Q, t-(1/c)|P-Q|)\, dQ}{|P-Q|}$$
$$= \frac{1}{4\pi c^2}\iiint\limits_{|P-Q|<ct} \frac{h(Q, t-(1/c)|P-Q|)\, dQ}{|P-Q|}$$

This formula reduces to the newtonian potential in the special case of time-independent problems. To see this, let $h(P;t) = c^2 h(P)$ and let $c \to \infty$ in the above representation. We obtain

$$\lim_{c\uparrow\infty} h(P;t) = \frac{1}{4\pi}\iiint \frac{h(Q)}{|P-Q|} dQ$$

The representation formula (8.5.1) can be expressed in terms of the mean-value operator that was used to solve the Cauchy problem for the homogeneous wave equation in Sec. 5.3, $M_R f(P) = [1/(4\pi R^2)] \iint_{|\xi|=R} f(x+\xi) d\xi$. This can be written as an integral over the unit sphere in the form

$$M_R f(P) = \frac{1}{4\pi}\iint\limits_{|\omega|=1} f(P + R\omega)\, d\omega$$

This form has the advantage that the dependence on radius R appears inside the integral and not in the region of integration. (The first form is preferable when we apply the divergence theorem.) To use this above, we write $h^{(t)}(P) = h(P;t)$, and the inner integral in (8.5.1) is written as $M_{cs}(sh^{(t-s)})(P)$ so that the solution takes the form

$$u(P;t) = \int_0^t M_{cs}(sh^{(t-s)})(P)\, ds = \int_0^t M_{c(t-s)}((t-s)h^{(s)})(P)\, ds$$

From Sec. 5.3, the integrand is the solution of the Cauchy problem $u_{tt} - c^2\nabla^2 u = 0$ with $u(P;0) = 0, u_t(P;0) = h^{(s)}(P)$. This connection makes it possible to first study the homogeneous problem and then apply the results obtained to the nonhomogeneous problem.

First we turn to the proof of the representation formula (5.3.11) for the homogeneous equation, which was left open in Chapter 5.

THEOREM 8.9. *Suppose that $f(P), P \in \mathbb{R}^3$, is a real-valued continuous function with two continuous partial derivatives. Then the formula $u(P;t) = tM_{ct}f(P)$ defines a twice-differentiable function that satisfies the conditions $u_{tt} = c^2\nabla^2 u$, $u(P;0) = 0, u_t(P;0) = f(P)$.*

Proof. We have

$$u(P;t) = \frac{1}{4\pi c^2 t} \iint\limits_{|\xi|=ct} f(P+\xi)\,d\xi = \frac{t}{4\pi} \iint\limits_{|\omega|=1} f(P+ct\omega)\,d\omega$$

Clearly, $u(P;0) = 0$. The first derivative is given by

$$u(P;t) = \frac{1}{4\pi} \iint\limits_{|\omega|=1} f(P+ct\omega)\,d\Omega + \frac{t}{4\pi} \iint\limits_{|\omega|=1} \nabla f(P+ct\omega)\cdot c\omega\,d\omega$$

In particular, $u_t(P;0) = f(P)$. To proceed further, we transform the second integral by the divergence theorem:

$$\begin{aligned}\frac{t}{4\pi} \iint\limits_{|\omega|=1} \nabla f(P+ct\omega)\cdot c\omega\,d\omega &= \frac{1}{4\pi ct} \iint\limits_{|\xi|=ct} \left(\frac{\partial f}{\partial n}\right)(P+\xi)\,d\xi \\ &= \frac{1}{4\pi ct} \iiint\limits_{|\xi|<ct} (\nabla^2 f)(P+\xi)\,d\xi\end{aligned}$$

so that we may write

$$u_t(P;t) = \frac{1}{4\pi} \iint\limits_{|\omega|=1} f(P+ct\omega)\,d\omega + \frac{1}{4\pi ct} \int_0^{ct} ds \iint\limits_{|\xi|=cs} (\nabla^2 f)(P+\xi)\,d\xi$$

Differentiating again with respect to t and applying the divergence theorem, we have

$$\begin{aligned}u_{tt}(P;t) &= \frac{c}{4\pi} \iint\limits_{|\omega|=1} \nabla f(P+ct\omega)\cdot\omega - \frac{1}{4\pi ct^2} \int_0^{ct} ds \iint\limits_{|\xi|=cs} (\nabla^2 f)(P+\xi)\,d\xi \\ &\quad + \frac{1}{4\pi t} \iint\limits_{|\xi|=ct} (\nabla^2 f)(P+\xi)\,d\xi \\ &= \frac{1}{4\pi ct^2} \iiint\limits_{|\xi|<ct} (\nabla^2 f)(P+\xi)\,d\xi - \frac{1}{4\pi ct^2} \iiint\limits_{|\xi|<ct} (\nabla^2 f)(P+\xi)\,d\xi \\ &\quad + \frac{1}{4\pi t} \iint\limits_{|\xi|=ct} (\nabla^2 f)(P+\xi)\,d\xi \\ &= \frac{c^2 t}{4\pi} \iint\limits_{|\omega|=1} \nabla^2 f(P+ct\omega)\,d\omega\end{aligned}$$

But if we compute the spatial derivatives of u by differentiating the integral defining $u(P;t)$, we obtain the formulas

$$u_{x_i}(P;t) = \frac{t}{4\pi} \iint\limits_{|\omega|=1} f_{x_i}(P + ct\omega)\, d\omega \qquad 1 \le i \le 3$$

$$u_{x_i x_j}(P;t) = \frac{t}{4\pi} \iint\limits_{|\omega|=1} f_{x_i x_j}(P + ct\omega)\, d\omega \qquad 1 \le i, j \le 3$$

In particular,

$$\nabla^2 u(P;t) = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = \frac{t}{4\pi} \iint\limits_{|\omega|=1} (\nabla^2 f)(P + ct\omega)\, d\omega = \left(\frac{1}{c^2}\right) u_{tt}$$

which was to be proved. •

We can now use this theorem to solve the general Cauchy problem for the homogeneous wave equation.

THEOREM 8.10. *Suppose that $f(P), P \in \mathbb{R}^3$, is a real-valued function with three continuous partial derivatives. Then the formula $v(P;t) = (\partial/\partial t)[tM_{ct}f(P)]$ defines a twice-differentiable function that satisfies $v_{tt}(P;t) = c^2\nabla^2 v$ with the initial conditions $v(P;0) = f(P), v_t(P;0) = 0$.*

Proof. Both f and ∇f are twice differentiable, so $v = u_t = (\partial/\partial t)(tM_{ct}g)$, given by the first line in the proof of Theorem 8.9, is twice differentiable. Since u satisfies the wave equation (also by Theorem 8.9), we can write

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u_t = \frac{\partial}{\partial t}(u_{tt} - c^2\nabla^2 u) = 0$$

so that v also satisfies the wave equation. From the proof of Theorem 8.9 we have $v(P;0) = u_t(P;0) = f(P), v_t(P;0) = u_{tt}(P;0) = 0$.

Combining the results of the above two theorems by the superposition principle, we have obtained the rigorous proof of the *Poisson formula*

$$u(P;t) = \frac{\partial}{\partial t}(tM_{ct}f_1)(P) + tM_{ct}f_2(P)$$

for the solution of the wave equation $u_{tt} = c^2\nabla^2 u$ with the initial conditions $u(P;0) = f_1(P), u_t(P;0) = f_2(P)$. •

We can now return to the nonhomogeneous equation and prove that formula (8.5.1) gives the rigorous solution of the Cauchy problem for the nonhomogeneous wave equation. To do this, we write

$$u(P;t) = \int_0^t v^{(s)}(P; t-s)\, ds$$

where $v^{(s)}$ is the solution of the homogeneous wave equation $v_{tt}=c^2\nabla^2 v$, $v^{(s)}(P;0) = 0$, $v_t^{(s)}(P;0) = h^{(s)}$. By Theorem 8.9 the solution is $v^{(s)}(P;t) = tM_{ct}h^{(s)}$. Clearly $u(P;0) = 0$. Now we can differentiate under the integral to obtain

$$u_t(P;t) = \left(\frac{\partial}{\partial t}\right)\int_0^t v^{(s)}(P;t-s)\,ds = v^{(s)}(P;0) + \int_0^t v_t^{(s)}(P;t-s)\,ds$$

$$u_{tt}(P;t) = \left(\frac{\partial^2}{\partial t^2}\right)\int_0^t v^{(s)}(P;t-s)\,ds = v_t^{(s)}(P;0) + \int_0^t v_{tt}^{(s)}(P;t-s)\,ds$$

Immediately $u_t(P;0) = v^{(s)}(P;0) = 0$, by construction. From the preceding calculations we know that $v_{tt} = c^2\nabla^2 v$, and hence the final integral in u_{tt} equals $c^2\nabla^2 u$ while the first term of u_{tt} is $h(P;t)$ by construction. We have proved that u satisfies the nonhomogeneous wave equation $u_{tt} - c^2\nabla^2 u = h$.

This is summarized as follows.

THEOREM 8.11. *Suppose that $h(P;t)$ is a continuous function for which $P \to H(P;t)$ has two continuous partial derivatives. Then the solution of the wave equation $u_{tt} - c^2\nabla^2 u = h(P;t)$ with $u(P;0) = 0, u_t(P;0) = 0$ is given by the retarded potential*

$$\begin{aligned} u(P;t) &= \int_0^t M_{c(t-s)}(sh^{(s)})(P)\,ds \\ &= \frac{1}{4\pi c^2}\iiint\limits_{|P-Q|<ct} \frac{h(Q,t-(1/c)|P-Q|)}{|P-Q|}\,dQ \end{aligned} \quad \bullet$$

8.5.2. Green's function for the Helmholtz equation. In case the right side has a harmonic time dependence, we can give a direct treatment of the wave equation in terms of the *Helmholtz equation.* Specifically, we assume that the right side of the wave equation is written in the complex form $h(P;t) = h(P)e^{\iota\omega t}$. We look for a solution in the form $u(P;t) = u(P)e^{\iota\omega t}$. This leads us to the nonhomogeneous Helmholtz equation

$$\nabla^2 u + k^2 u = -h \qquad k = \frac{\omega}{c}$$

We may look for solutions in the entire three-dimensional space or in a region with a smooth boundary.

To formulate Green's function for the entire three-dimensional space, we begin by looking for radial solutions with the characteristic singularity at $r = 0$. In polar coordinates we have $0 = \nabla^2 u + k^2 u = u_{rr} + (2/r)u_r + k^2 u = (1/r)[(ru)_{rr} + k^2(ru)]$ with the general solution $u = (1/r)(A\cos kr + B\sin kr)$. To achieve the proper singularity at $r = 0$, we choose $A = 1/(4\pi)$; the value of B is undetermined, since that part of the solution is smooth at $r = 0$ and is not determined by the singularity of Green's function or by requiring that $u \to 0$ at infinity since $u \to 0$ for any choice of B.

To determine a unique Green's function for the Helmholtz equation, we write the solution in complex form:

$$u = \frac{1}{r}(Ae^{ikr} + Be^{-ikr})$$

which redefines the constants A, B. To have the proper singularity at $r = 0$, we must have $A + B = 1/(4\pi)$. To determine both constants, we look at the corresponding complex separated solutions of the wave equation:

$$u(P;t) = \frac{1}{r}(Ae^{i(kr+\omega t)} + Be^{i(-kr+\omega t)})$$

This describes an *outgoing spherical wave* if $A = 0$. It describes an *incoming spherical wave* if $B = 0$. We use these notions to determine a unique Green's function.

The function $u = e^{-ikr}/r$ satisfies the first-order differential equation $(ru)_r = -ike^{-ikr} = -ikru$, which entails that $ru_r + u = -ikru$ or $u_r + iku = -e^{-ikr}/r^2$. This equation motivates the concept of the *outgoing radiation condition*, which is the statement that

$$u_r + iku = O\left(\frac{1}{r^2}\right) \qquad r \to \infty$$

It is satisfied, in particular, for the function $u = e^{ikr}/r$. Similarly, the incoming *radiation condition* is the statement that $u_r - iku = O(1/r^2), r \to \infty$. It is satisfied, in particular, for the function $u = e^{-ikr}/r$. Either may be used to determine a unique Green's function for the Helmholtz equation. We formulate the following theorem.

THEOREM 8.12. *Let u be a solution of the Helmholtz equation $\nabla^2 u + k^2 u = -h$ in the entire three-dimensional space satisfying the outgoing radiation condition and the condition that $u \to 0$ when $r \to \infty$. Then we have the representation*

$$u(P) = \iiint \frac{e^{-ik|P-Q|}}{4\pi|P-Q|} h(Q)\, dQ$$

Proof. We begin with Green's identity, written in the form

$$\iiint\limits_{D_{\epsilon,R}} [u(\nabla^2 + k^2)G - G(\nabla^2 + k^2)u] = \iint\limits_{\partial D_{\epsilon,R}} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS_Q$$

where we have added and subtracted the term k^2uG on the left side. This integral is analyzed exactly as for the case of Poisson's equation in Sec. 8.2. For the right side, the integral on $|Q - P| = \epsilon$ is analyzed as before. For the integral on $|Q - P| = R$, we note that both u and G satisfy the outgoing radiation condition, and hence we can replace $\partial u/\partial n$ and $\partial G/\partial n$ by $-iku + O(1/r^2)$ and (respectively) $-ikG + O(1/r^2)$. Two of the terms cancel, and the resulting integrals tend to zero when $R \to \infty$, giving the required result. •

8.5.3. Application to the telegraph equation. In Sec. 5.4 we found the Fourier representation of the solution of the telegraph equation $u_{tt}+2\beta u_t+\alpha u = c^2 u_{xx}$. The form of the solution depends on whether $\alpha < \beta^2, \alpha = \beta^2$, or $\alpha > \beta^2$. In the second case, the explicit representation is available from d'Alembert's formula, and there is nothing further to do. We now treat the other two cases, using a *method of descent* beginning with the two-dimensional wave equation.

As a preliminary simplification, we recall that the definition $v(x;t) = u(x;t)e^{\beta t}$ transforms the telegraph equation to a special form with $\beta = 0$, that is, the equation $v_{tt} = c^2 v_{xx} \pm k^2 v$, where either $k^2 = \beta^2 - \alpha$ or $k^2 = \alpha - \beta^2$, whichever is positive, corresponding to case 1 or case 3. First we examine the case where $\alpha < \beta^2$, which is the initial-value problem for

$$v_{tt} - c^2 v_{xx} = k^2 v$$

with

$$v(x;0) = f_1(x), \qquad v_t(x;0) = f_2(x)$$

To relate this system to the two-dimensional wave equation, we introduce a new independent variable y and consider the function

$$w(x,y;t) = v(x;t)e^{(ky/c)}$$

For this function, we have $w_{tt} = v_{tt}e^{(ky/c)}, w_{xx} = v_{xx}e^{(ky/c)}, w_{yy} = (k/c)^2 v e^{(ky/c)}$, and the equation

$$w_{tt} - c^2(w_{xx} + w_{yy}) = 0$$

with

$$w(x,y;0) = f_1(x)e^{(ky/c)}, \qquad w_t(x,y;0) = f_2(x)e^{(ky/c)}$$

This two-dimensional wave equation is solved by formula (5.3.11):

$$w(x,y;t) = \frac{d}{dt}(tM_{ct}F_1) + tM_{ct}F_2$$

where $F_1(x,y) = f_1(x)e^{(ky/c)}, F_2(x,y) = f_2(x)e^{(ky/c)}$.

The mean-value operator is expressed from (5.3.13) as

$$\begin{aligned} M_{ct}F_i(x,y) &= \frac{1}{2\pi ct}\iint_{|\xi|<ct} \frac{F_i(x+\xi_1, y+\xi_2)\,d\xi_1\,d\xi_2}{\sqrt{(ct)^2-\xi_1^2-\xi_2^2}} \qquad i = 1,2 \\ &= \frac{1}{2\pi ct}\iint_{|\xi|<ct} \frac{f_i(x+\xi_1)e^{(k/c)(y+\xi_2)}}{\sqrt{(ct)^2-\xi_1^2-\xi_2^2}}\,d\xi_1\,d\xi_2 \end{aligned}$$

To evaluate the ξ_2 integral, we define a new variable of integration by the formula $\xi_2 = \sqrt{(ct)^2-\xi_1^2}\cos\theta, 0<\theta<\pi$, to find

$$\int_{|\xi_2|<\sqrt{(ct)^2-\xi_1^2}} \frac{e^{(k/c)(y+\xi_2)}}{\sqrt{(ct)^2-\xi_1^2-\xi_2^2}}\,d\xi_2 = e^{(ky/c)}\int_0^\pi e^{(k/c)\sqrt{(ct)^2-\xi_1^2}\cos\theta}\,d\theta$$

The final integral was seen in Sec. 3.2 as the integral representation of the Bessel function with imaginary argument, that is, $I_0((k/c)\sqrt{(ct)^2 - \xi_1^2})$. Therefore we have

$$M_{ct}F_i(x,y) = e^{(ky/c)}\frac{1}{2ct}\int_{-ct}^{ct} f_i(x+\xi_1)I_0\left(\frac{k}{c}\sqrt{(ct)^2-\xi_1^2}\right)\,d\xi_1$$

Canceling the factor $e^{(ky/c)}$, we have found the solution of the telegraph equation with $f_1 = 0$. To find the solution in general, we need to differentiate this integral with respect to t:

$$\begin{aligned}\frac{d}{dt}(tM_{ct}f_i) &= \frac{1}{2c}[f_i(x+ct) + f_i(x-ct)] \\ &\quad + \frac{1}{2c}\int_{-ct}^{ct} f_i(x+\xi_1)\frac{d}{dt}\left\{I_0\left[\frac{k}{c}\sqrt{(ct)^2-\xi_1^2}\right]\right\}\,d\xi_1\end{aligned}$$

But the derivative of the Bessel function I_0 is the Bessel function $I_1 : I_0' = I_1$. Therefore we conclude that

$$\frac{d}{dt}(tM_{ct}f_i) = \frac{1}{2c}[f_i(x+ct) + f_i(x-ct)] + \frac{k}{2}\int_{-ct}^{ct} f_i(x+\xi)I_1\left[\frac{k}{c}\sqrt{(ct)^2-\xi^2}\right]\,d\xi$$

We can summarize these calculations in the following form.

THEOREM 8.13. *The explicit representation of the solution of the telegraph equation* $v_{tt} - c^2v_{xx} = k^2v$ *with* $v(x;0) = f_1(x), v_t(x;0) = f_2(x)$ *is given by*

$$\begin{aligned}v(x;t) &= \frac{1}{2c}\int_{-ct}^{ct} f_2(x+\xi)I_0\left[\frac{k}{c}\sqrt{(ct)^2-\xi^2}\right]\,d\xi \\ &\quad + \frac{1}{2}[f_1(x+ct) + f_1(x-ct)] + \frac{k}{2c}\int_{-ct}^{ct} f_1(x+\xi)I_1\left[\frac{k}{c}\sqrt{(ct)^2-\xi^2}\right]\,d\xi\end{aligned}$$

This concludes the derivation of the solution.

In the exercises we solve the telegraph equation in case 3, for $v_{tt} - c^2v_{xx} = -k^2v$, leading to the replacement of the modified Bessel function by the usual Bessel function J_0.

EXERCISES 8.5

1. Find the explicit representation of the solution of the two-dimensional wave equation $u_{tt} - c^2(u_{xx} + u_{yy}) = h(x,y;t)$ with $u(x,y;0) = 0, u_t(x,y;0) = 0$. [*Hint:* Apply the method of Theorem 8.11 with the two-dimensional mean-value formula (5.3.12).]
2. Find the explicit representation of the solution of the one-dimensional wave equation $u_{tt} - c^2u_{xx} = h(x;t)$ with $u(x;0) = 0, u_t(x;0) = 0$. [*Hint:* Apply the method used in Theorem 8.11 with the one-dimensional mean-value operator $M_{ct}f = [1/(2ct)]\int_{x-ct}^{x+ct} f(z)dz$.]

3. Consider the one-dimensional Helmholtz equation $u'' + k^2 u = -h(x)$.
   (a) Find a Fourier representation of the solution.
   (b) Show that this can be rewritten as the explicit representation
$$u(x) = -\int_{-\infty}^{\infty} \frac{e^{ik|x-y|}}{2ik} h(y)\, dy$$
   (c) Suppose that h is continuous and satisfies $h(x) = O(1/|x|^2)$ when $|x| \to \infty$. Prove that $u'(x), u''(x)$ [defined in (b)] exist and that u satisfies the one-dimensional Helmholtz equation.
4. Find the explicit representation of the solution of the telegraph equation $v_{tt} - c^2 v_{xx} = -k^2 v$ with $u(x;0) = f_1(x), v_t(x;0) = f_2(x)$. [*Hint:* The function $w(x,y;t) := e^{i(ky/c)} v(x;t)$ satisfies the two-dimensional wave equation $w_{tt} = c^2(w_{xx} + w_{yy})$.]
5. Find the explicit representation of the solution of the two-dim. telegraph equation $v_{tt} - c^2(v_{xx} + v_{yy}) = k^2 v$ with $v(x,y;0) = f_1(x,y), v_t(x,y;0) = f_2(x,y)$. [*Hint:* The function $w(x,y,z;t) := e^{(kz/c)} v(x,y;t)$ satisfies the three-dimensional wave equation $w_{tt} = c^2(w_{xx} + w_{yy} + w_{zz})$.]

APPENDIXES

A.1. Review of Ordinary Differential Equations

In this section we give a review of the necessary techniques and theoretical background to deal with the ordinary differential equations that arise in the text. The reader has surely encountered examples of ordinary differential equations in calculus courses. For example, the equation $y' = ky$, which has the general solution $y(t) = Ce^{kt}$, governs exponential growth and decay. The equation $y'' + \omega^2 y = 0$, which has the general solution $y(t) = A\cos\omega t + B\sin\omega t$, governs simple harmonic motion with angular frequency ω. In this section we shall give a brief account of the necessary facts and techniques of ordinary differential equations, which are used throughout the book.

A.1.1. First-order linear equations. The simplest class of explicitly solvable differential equations are the first-order linear equations, written

$$b(t)y'(t) + c(t)y(t) = d(t)$$

where $b(t)$, $c(t)$, $d(t)$ are known functions and $y(t)$ is the unknown function. Equations of this type may be solved by the method of *integrating factors*. We may suppose that $b(t)$ is nonzero in an interval $a < t < b$. The first step is to divide by the nonzero function $b(t)$ and rewrite the equation in the equivalent form

$$y'(t) + p(t)y(t) = q(t) \tag{A.1.1}$$

where $p(t) = c(t)/b(t)$ and $q(t) = d(t)/b(t)$. The method of integrating factors requires that one find a function $\mu(t)$ such that $\mu(t)[y'(t) + p(t)y(t)] = [\mu(t)y(t)]'$ for every possible function $y(t)$. This requires that $\mu(t)$ satisfy the auxiliary equation $\mu'(t) = \mu(t)p(t)$. This may be solved by writing $[\log\mu(t)]' = \mu'(t)/\mu(t) = p(t)$; we choose a fixed reference point t_0 and write

$$\mu(t) = \exp\int_{t_0}^{t} p(s)\,ds$$

Having obtained $\mu(t)$, the solution $y(t)$ may be obtained by multiplying (A.1.1) by $\mu(t)$ to obtain $[\mu(t)y(t)]' = \mu(t)q(t)$, which can be integrated to obtain the solution formula

$$y(t) = \boxed{\frac{C + \int_{t_0}^{t} \mu(s)q(s)\,ds}{\mu(t)}} \tag{A.1.2}$$

where we have incorporated the arbitrary constant C before the definite integral.

One may summarize this method in the statement that a first-order linear equation is solved by means of *two integrations:* the first to find the integrating factor, and the second to integrate the right-hand side and find the solution. We illustrate using the equation with constant coefficients.

EXAMPLE A.1.1. *Find the general solution of the equation $y' + cy = d$ where c, d are constants with $c \neq 0$.*

Solution. An integrating factor is obtained by solving $\mu'(t) = c\mu(t)$ with the base point $t_0 = 0$; thus $\mu(t) = e^{ct}$. Referring to (A.1.2), we have

$$y(t) = \frac{C + \int_0^t de^{cs}\,ds}{e^{ct}} = Ce^{-ct} + (d/c)(1 - e^{-ct})$$

This may also be written in the equivalent form $y(t) = d/c + C_1 e^{-ct}$ by a different choice of the arbitrary constant $C_1 = C - d/c$. •

The next example involves division by the top coefficient.

EXAMPLE A.1.2. *Find the general solution of the equation $ty' + cy = d$, where c, d are constants with $c \neq 0$.*

Solution. We write the equation in standard form: $y' + cy/t = d/t$. Using $t_0 = 1$ as a base point, the integrating factor is $\mu(t) = \exp(\int_1^t (c/s)\,ds) = t^c$, so that the general solution of the equation is

$$y(t) = \frac{C + \int_1^t s^c (d/s)\,ds}{t^c} = \frac{C + d(t^c - 1)/c}{t^c} \quad \bullet$$

A.1.2. Second-order linear equations. When we turn to higher-order linear equations, it is no longer possible to write down the general solution by means of integrals, as we have done for the first-order linear equation. This is possible for the equation of constant coefficients, which will be discussed in detail below. First we review some of the general theory of second-order linear differential equations.

A *second-order linear ordinary differential equation* is written in the form

$$a(t)y'' + b(t)y' + c(t)y = d(t) \tag{A.1.3}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$ are given functions. If the coefficient functions on the left side are constant, $a(t) = a$, $b(t) = b$, $c(t) = c$, then we speak of an *equation* *with constant coefficients.* The equation is said to be *homogeneous* if $d(t) = 0$. Assuming that the top coefficient $a(t)$ is nonzero in an interval $a < t < b$, we may divide by $a(t)$ to obtain the standard form

$$y'' + p(t)y' + q(t)y = r(t) \tag{A.1.4}$$

where $p(t) = b(t)/a(t)$, $q(t) = c(t)/a(t)$, $r(t) = d(t)/a(t)$.

The following fact is of great theoretical importance.

THEOREM A.1. *(Existence-uniqueness theorem) Let $p(t)$, $q(t)$, $r(t)$ be continuous functions on the interval $a \le t \le b$ and t_0 a point in the interval. Let y_0, y_1 be given real numbers. Then there exists a unique function $y(t)$ that satisfies the differential equation (A.1.4) and satisfies the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_1$.*

The proof of this theorem may be found in textbooks on ordinary differential equations. Our interest is in finding the solutions when possible and in using the theorem to obtain other important information

When we specialize to the case $r(t) \equiv 0$, we have the *homogeneous equation*, whose solutions may be studied as follows. Let $y_1(t)$ be the solution of the homogeneous equation with the initial conditions $y_1(t_0) = 1$, $y_1'(t_0) = 0$ and let $y_2(t)$ be the solution of the homogeneous equation with the initial conditions $y_2(t_0) = 0$, $y_2'(t_0) = 1$. Applying the existence-uniqueness theorem, we obtain the following useful corollary.

PROPOSITION A.1.1. *The general solution of the homogeneous equation $y'' + p(t)y' + q(t) = 0$ can be written uniquely in the form*

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

where C_1, C_2 are constants and $y_1(t)$, $y_2(t)$ are the particular solutions defined above.

Proof. The existence of the functions $y(t)$, $y_1(t)$, $y_2(t)$ follows from the existence-uniqueness theorem. If we set $C_1 = y(t_0)$, $C_2 = y'(t_0)$, then both $y(t)$ and $C_1y_1(t) + C_2y_2(t)$ satisfy the same second-order linear differential equation and have the same initial conditions at the point $t = t_0$. Hence they are the same function for all t, $a < t < b$. •

The functions $y_1(t)$, $y_2(t)$ are called a *fundamental set* of solutions of the homogeneous equation. Although it is not possible to explicitly construct a fundamental set for every second-order linear equation, this is always possible for the equation with constant coefficients, to be discusssed in the next subsection.

First, we briefly discuss the nonhomogeneous equation $y'' + p(t)y' + q(t)y = r(t)$. The general solution is obtained by the *method of particular solutions*, as follows.

PROPOSITION A.1.2. *Suppose that $y_0(t)$ is a known solution of the nonhomogeneous equation $y'' + p(t)y' + q(t)y = r(t)$. Then the general solution is obtained in the form $y(t) = y_0(t) + C_1y_1(t) + C_2y_2(t)$, where C_1, C_2 are arbitrary constants and $y_1(t)$, $y_2(t)$ are the solutions of the homogeneous equations discussed above.*

Proof. This also follows from the existence-uniqueness theorem, when we note that both $y(t)$ and $y_0(t) + C_1y_1(t) + C_2y_2(t)$ satisfy the same second-order linear equation and have the same initial conditions. •

A.1.3. Second-order linear equations with constant coefficients. We now outline the procedure for finding the general solution of a second-order homogeneous linear differential equation with constant coefficients, $ay'' + by' + cy = 0$, where a, b, c are real numbers with $a \neq 0$. We try an exponential solution $y(t) = e^{rt}$ and substitute in the equation with $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$; thus $0 = ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt}$. We consider three cases.

Case 1: The characteristic equation $ar^2 + br + c = 0$ has real roots $r_1 \neq r_2$.

Case 2: The characteristic equation $ar^2 + br + c = 0$ has equal real roots $r_1 = r_2$.

Case 3: The characteristic equation $ar^2 + br + c = 0$ has complex roots $r = \lambda \pm i\mu$ with $\mu \neq 0$.

From elementary algebra it is clear that this exhausts all possible cases. Indeed, the quadratic formula yields $r = (-b \pm \sqrt{b^2 - 4ac})/2a$, so that case 1 corresponds to $b^2 - 4ac > 0$, case 2 to $b^2 - 4ac = 0$, and case 3 to $b^2 - 4ac < 0$. We now give the general solution of the differential equation in each of the three cases.

Case 1: The general solution is $y(t) = C_1e^{r_1t} + C_2e^{r_2t}$, where C_1, C_2 are arbitrary constants and r_1, r_2 are the roots of the characteristic equation $ar^2 + br + c = 0$.

Case 2: The general solution is $y(t) = C_1e^{rt} + C_2te^{rt}$, where C_1, C_2 are arbitrary constants and r is the root of the characteristic equation $ar^2 + br + c = 0$.

Case 3: The general solution is $y(t) = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$, where C_1, C_2 are arbitrary constants and $\lambda + i\mu$, $\lambda - i\mu$ are the complex roots of the characteristic equation $ar^2 + br + c = 0$.

EXAMPLE A.1.3. *Find the general solution of the differential equation $y'' + 3y' + 2y = 0$.*

Solution. The characteristic equation is $r^2 + 3r + 2 = 0$, with roots $r_1 = -1$, $r_2 = -2$. Therefore this represents case 1 and the general solution is $y(t) = C_1e^{-t} + C_2e^{-2t}$. •

Often, when working with case 1, we may find it convenient to express the solution in terms of hyperbolic functions. Recall that the defining equations for these functions are $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$. From this it follows that $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$. Thus any linear combination of exponential functions may be written in terms of hyperbolic functions. The solution of case 1 is written as $y(t) = C_1(\cosh r_1 t + \sinh r_1 t) + C_2(\cosh r_2 t - \sinh r_2 t)$. To verify the solution, we may use the differentiation formulas $(d/dx)(\sinh x) = \cosh x$, $(d/dx)(\cosh x) = \sinh x$.

EXAMPLE A.1.4. *Solve the initial-value problem* $y'' - 4y = 0$, $y(0) = 0$, $y'(0) = 6$.

Solution. The characteristic equation is $r^2 - 4 = 0$, with roots $r_1 = 2$, $r_2 = -2$. The general solution is $y(t) = C_1 e^{2t} + C_2 e^{-2t} = (C_1 + C_2)\cosh 2t + (C_1 - C_2)\sinh 2t$. Setting $t = 0$ and using the initial conditions, we must have $0 = (C_1 + C_2)$, $6 = 2(C_1 - C_2)$. The required solution is $y(t) = 3\sinh 2t$. •

We now illustrate a typical example from case 3.

EXAMPLE A.1.5. *Solve the initial-value problem* $y'' + 2y' + 2y = 0$, $y(0) = 3$, $y'(0) = -1$.

Solution. The characteristic equation is $r^2 + 2r + 2 = 0$, which has the complex roots $r = -1 \pm i$. Therefore we have case 3 and the general solution is $y(t) = e^{-t}(C_1 \cos t + C_2 \sin t)$. The initial conditions require $3 = C_1$, $-1 = C_2 - C_1$. The solution is $y(t) = e^{-t}(3\cos t + 2\sin t)$. •

We illustrate the method of particular solutions in the case of an equation with constant coefficients.

EXAMPLE A.1.6. *Find the general solution of the equation* $ay'' + by' + cy = d$, *where* a, b, c, d, *are constants with* $a \neq 0$, $c \neq 0$.

Solution. A particular solution is $y_0(t) = d/c$, since $y_0'(t) = 0$, $y_0''(t) = 0$ for this choice of $y_0(t)$. Thus the general solution of the equation is $y(t) = d/c + C_1 y_1(t) + C_2 y_2(t)$, where C_1, C_2 are arbitrary constants. It will depend upon the exact values of a, b, c whether case 1, 2, or 3 applies. •

EXAMPLE A.1.7. *Find the general solution of the equation* $ay'' + by' = d$, *where* a, b, d *are constants with* $a \neq 0$, $b \neq 0$.

Solution. A particular solution is $y_0(t) = dt/b$, since $y_0'(t) = d/b$, $y_0''(t) = 0$ for this choice of $y_0(t)$. To obtain the general solution, we have the characteristic equation $ar^2 + br = 0$, with roots $r = 0$, $-b/a$. The general solution of the equation is $y(t) = dt/b + C_1 + C_2 e^{-(bt/a)}$. •

A.1.4. Euler's equidimensional equation. Having solved the second-order linear equation with constant coefficients, it is possible to solve other second-order linear equations by a transformation of the independent variable. This is defined by an increasing function $s = \phi(t)$ that is smooth and has a smooth inverse function. A new function Y is defined by writing $Y(s) = y(t)$. The derivatives are computed using the chain rule for composite functions; thus $dy/dt = (dY/ds)(ds/dt)$, $d^2y/dt^2 = (dY/ds)(d^2s/dt^2)+(d^2Y/ds^2)(ds/dt)^2$. When these are expressed in terms of s and substituted in the second-order equation, we obtain a new equation for the transformed function Y.

The Euler equidimensional equation arises in the important case of the transformation $s = e^t$. Applying the chain rule as above, we first note that

$$\frac{ds}{dt} = e^t, \qquad \frac{d^2s}{dt^2} = e^t$$

so that

$$\begin{aligned} y' = \frac{dy}{dt} &= e^t\frac{dY}{ds} = s\frac{dY}{ds} = sY'(s) \\ y'' = \frac{d^2y}{dt^2} &= e^t\frac{dY}{ds} + e^{2t}\frac{d^2Y}{ds^2} = s\frac{dY}{ds} + s^2\frac{d^2Y}{ds^2} = sY' + sY'' \end{aligned}$$

and the constant-coefficient equation is transformed as follows:

$$\begin{aligned} 0 &= ay''(t) + by'(t) + cy(t) \\ &= a(sY'(s) + s^2Y''(s)) + b(sY'(s)) + cY(s) \\ &= as^2Y''(s) + (a+b)sY'(s) + cY(s) \end{aligned}$$

By renaming the coefficients, the Euler equidimensional equation is written in the form

$$\boxed{\alpha s^2Y'' + \beta sY' + \gamma Y = 0} \tag{A.1.5}$$

where we have set $\alpha = a$, $\beta = a + b$, $\gamma = c$ and the new independent variable s is assumed to be positive: $s > 0$.

To find the general solution of (A.1.5), we transform the exponential trial solution $y(t) = e^{rt} = (e^t)^r = s^r = Y(s)$. Substitution of the trial solution $Y(s) = s^r$ into (A.1.5) produces the quadratic equation $\alpha r(r-1) + \beta r + \gamma = 0$. Exactly as for the constant-coefficient equation, we have the following threefold classification.

PROPOSITION A.1.3. *For the Euler equidimensional equation (A.1.5) exactly one of the following three situations arises.*

(*a*) *The quadratic equation $\alpha r(r-1) + \beta r + \gamma = 0$ has two distinct real roots $r_1 \neq r_2$. In this case the general solution is given by*

$$Y(s) = C_1 s^{r_1} + C_2 s^{r_2}$$

where C_1, C_2 are arbitrary constants.

(*b*) *The quadratic equation* $\alpha r(r-1)+\beta r+\gamma=0$ *has a repeated real root* $r_1=r_2$. *In this case the general solution is given by*

$$Y(s)=C_1 s^{r_1}+C_2 s^{r_1}\log s$$

where C_1, C_2 *are arbitrary constants.*

(*c*) *The quadratic equation* $\alpha r(r-1)+\beta r+\gamma=0$ *has two complex conjugate roots* $r_1=\lambda+i\mu, r_2=\lambda-i\mu$. *In this case the general solution is given by*

$$Y(s)=C_1 s^{\lambda}\cos(\mu\log s)+C_2 s^{\lambda}\sin(\mu\log s)$$

where C_1, C_2 *are arbitrary constants.*

EXAMPLE A.1.8. *Find all solutions of the equation* $s^2Y''-3sY'+4Y=0$.

Solution. The quadratic equation is $r(r-1)-3r+4=0$, which simplifies to $0=r^2-4r+4=(r-2)^2$. This has the repeated root $r=2$, so that the general solution is $Y(s)=C_1s^2+C_2s^2\log s$. •

A.1.5. Power series solutions. If the homogeneous linear equation $a(t)y''+b(t)y'+c(t)y=0$ does not have constant coefficients, the solutions $y_1(t)$, $y_2(t)$ cannot be written as elementary functions, in general. The *method of power series* provides a useful algorithm for obtaining solutions for a wide class of second-order equations with nonconstant coefficients. We assume that the functions $a(t)$, $b(t)$, $c(t)$ have convergent power series expansions about $t=t_0$.

$$\begin{aligned} a(t) &= a_0+\sum_{n=1}^{\infty}a_n(t-t_0)^n \\ b(t) &= b_0+\sum_{n=1}^{\infty}b_n(t-t_0)^n \\ c(t) &= c_0+\sum_{n=1}^{\infty}c_n(t-t_0)^n \end{aligned}$$

The point $t=t_0$ is called an *ordinary point* if $a_0=a(t_0)\neq 0$. Under these hypotheses, we can find power series solutions $y(t)$ of the equation $a(t)y''+b(t)y'+c(t)y=0$, convergent for $|t-t_0|<\delta$, an interval on which $a(t)\neq 0$. To find these solutions, we assume a solution of the form

$$y(t)=y_0+\sum_{n=1}^{\infty}y_n(t-t_0)^n$$

Differentiating formally, we have

$$
\begin{aligned}
y'(t) &= \sum_{n=1}^{\infty} n y_n (t - t_0)^{n-1} \\
y''(t) &= \sum_{n=1}^{\infty} n(n-1) y_n (t - t_0)^{n-2}
\end{aligned}
$$

We form the power series for $a(t)y''$, $b(t)y'$, and $c(t)y$. Summing these and equating the coefficients of 1, $(t - t_0), \ldots$ to zero, we obtain linear equations for the coefficients $y_2, y_3, \ldots$. The coefficients y_0, y_1 are determined by the initial conditions. In this way we obtain the power series solution of the equation.

EXAMPLE A.1.9. *Find the power series solution of the equation* $y'' + ty = 0$ *with the initial conditions* $y(0) = 1$, $y'(0) = 0$.

Solution. We assume a power series of the form

$$y(t) = 1 + \sum_{n=2}^{\infty} y_n t^n$$

Differentiating, we have

$$y''(t) = \sum_{n=2}^{\infty} n(n-1) y_n t^{n-2}$$

Thus

$$
\begin{aligned}
y'' + ty &= 2y_2 + t(1 + 6y_3) + t^2(12y_4) + \cdots \\
&\quad + t^n[y_{n-1} + (n+2)(n+1)y_{n+2}] + \cdots
\end{aligned}
$$

Equating each coefficient to zero, we have $y_2 = 0$, $y_3 = -\frac{1}{6}$, $y_4 = 0$, $y_5 = 0$, $y_6 = \frac{1}{180}$, and, in general,

$$
\begin{aligned}
y_{3k} &= \frac{(-1)^k}{3k(3k-1)(3k-3)\cdots 6\cdot 5\cdot 3\cdot 2} \\
y_{3k+1} &= 0 \\
y_{3k+2} &= 0
\end{aligned}
$$

The solution is

$$y(t) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k t^{3k}}{3k(3k-1)(3k-3)\cdots 6\cdot 5\cdot 3\cdot 2} \quad \bullet$$

We now discuss power series solutions of the equation $a(t)y'' + b(t)y' + c(t)y = 0$, with $a(t_0) = 0$. This will be modeled on the Euler equidimensional equation, just as the preceding discussion was modeled on the constant-coefficient equation.

We say that $t = t_0$ is a *regular singular point* if the functions $(t-t_0)b(t)/a(t)$ and $(t-t_0)^2c(t)/a(t)$ both have convergent power series expansions about $t = t_0$.

$$\frac{(t-t_0)b(t)}{a(t)} = \beta(t) = \beta_0 + \sum_{n=1}^{\infty}\beta_n(t-t_0)^n$$

$$\frac{(t-t_0)^2c(t)}{a(t)} = \gamma(t) = \gamma_0 + \sum_{n=1}^{\infty}\gamma_n(t-t_0)^n$$

Thus the equation is written in the form

$$(t-t_0)^2y'' + (t-t_0)\beta(t)y' + \gamma(t)y = 0$$

The *method of Frobenius* states that we can find a power series solution in the form

$$\begin{aligned} y &= (t-t_0)^r\left[1+\sum_{n=1}^{\infty}y_n(t-t_0)^n\right] \\ &= (t-t_0)^r + \sum_{n=1}^{\infty}y_n(t-t_0)^{n+r} \end{aligned} \tag{A.1.6}$$

The exponent r and the coefficients $y_2, y_2, \ldots$ are found by substituting in the differential equation and equating the coefficients of $(t-t_0)^r$, $(t-t_0)^{r+1}, \ldots$. Thus we have the formal series

$$y' = r(t-t_0)^{r-1} + \sum_{n=1}^{\infty}(n+r)y_n(t-t_0)^{n+r-1}$$

$$y'' = r(r-1)(t-t_0)^{r-2} + \sum_{n=1}^{\infty}(n+r)(n+r-1)y_n(t-t_0)^{n+r-2}$$

Equating the coefficient of $(t-t_0)^r$ to zero gives the *indicial equation*

$$r(r-1) + \beta_0 r + \gamma_0 = 0$$

This equation has two roots, r_1, r_2. If they are real, we take the larger root $r = \frac{1}{2}[1-\beta_0+\sqrt{(1-\beta_0)^2-4\gamma_0}]$. With this value of r we equate the coefficients of $(t-t_0)^{r+1}$, $(t-t_0)^{r+2}, \ldots$ to zero to obtain the coefficients $y_1, y_2, \ldots$ and to obtain the Frobenius solution (A.1.6).

EXAMPLE A.1.10. *Find the Frobenius solution of the equation $t^2y'' + ty = 0$.*

Solution. We have $t_0 = 0$, $\beta(t) = 0$, $\gamma(t) = t$. The indicial equation is $r(r-1) = 0$, with roots $r = 0, 1$. We look for the power series solution in the form $y = t + \sum_{n=1}^{\infty} y_n t^{n+1}$. Thus $y' = 1 + \sum_{n=1}^{\infty}(n+1)y_n t^n$, $y'' = \sum_{n=1}^{\infty}(n+1)ny_n t^{n-1}$, $t^2y'' + ty = t^2(2y_1+1) + t^3(6y_2+y_1) + \cdots + t^n[n(n-1)y_{n-1} + y_{n-2}] + \cdots$. Matching each coefficient to zero, we have $y_1 = -\frac{1}{2}$, $y_2 = 1/(3\cdot2\cdot2)$, $y_3 = -1/(4\cdot3\cdot3\cdot2\cdot2)$, $\ldots$, $y_n = (-1)^n/(n+1)(n!)^2$.

The solution is

$$\begin{aligned} y &= t - \frac{t^2}{2} + \frac{t^3}{3 \cdot 2 \cdot 2} - \frac{t^4}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 2} + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nt^n}{(n!)^2} \quad \bullet \end{aligned}$$

A.1.6. Steady state and relaxation time. Many differential equations that occur in applications have solutions $y(t)$ that tend to a limit y_∞ when the time t tends to infinity. This represents an equilibrium state, or *steady state*, of the system. When this happens, it is useful to have a quantitative measure of the time necessary for the system to come within a fixed fraction of the steady state. The *relaxation time* τ is defined by the following limit.

$$\frac{1}{\tau} = - \lim_{t \to \infty} \frac{1}{t} \ln |y(t) - y_\infty|$$

EXAMPLE A.1.11. *For the differential equation $y' + 3y = 6$ with the initial condition $y(0) = 1$, find the steady state and the relaxation time.*

Solution. The solution of the differential equation can be obtained with the integrating factor e^{3t}. Thus $(ye^{3t})' = 6e^{3t}$, $ye^{3t} = 1 + 2(e^{3t} - 1) = 2e^{3t} - 1$. Thus $y(t) = 2 - e^{-3t}$ and the steady state is $y_\infty = 2 = \lim_{t\to\infty} y(t)$. To find the relaxation time, we write $|y(t)-2| = e^{-3t}$, $\ln|y(t)-2| = -3t$, $(-1/t)\ln|y(t)-2| = 3$. Therefore $\tau = \frac{1}{3}$, and we have found the relaxation time. •

The relaxation time is closely related to the *half-life* T, defined by the equation $|y(T) - y_\infty|/|y(0) - y_\infty| = \frac{1}{2}$. In the previous example we found the result that $|y(t) - y_\infty|/|y(0) - y_\infty| = e^{-3t}$; therefore the half-life is obtained by solving $e^{-3T} = \frac{1}{2}$ or $T = \frac{1}{3}\ln 2 = 0.23$, to two decimals. This is typical for first-order equations, where the relaxation time and half-life differ by the factor $\ln 2$. We choose to work with the relaxation time because of the absence of the factor $\ln 2$ in the formulas.

It should be noted that the relaxation time depends on the solution, as well as the equation. Consider, for example, the differential equation $y'' - 5y' + 6y = 0$ with the two solutions $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-2t}$. Both solutions have the same steady state $y_\infty = 0$. For $y_1(t)$ we have the relaxation time $\tau_1 = \frac{1}{3}$, whereas for the solution $y_2(t)$ we have the relaxation time $\tau_2 = \frac{1}{2}$.

In general, the relaxation time furnishes a practical estimate of the time necessary to come "close" to the steady state. To see this, we may solve the equation $[y(t) - y_\infty]/[y(0) - y_\infty] = q$, a fixed fraction. For $q = 0.01$, we get $e^{-(t/\tau)} = 0.01$, $t/\tau = \ln 100 = 4.6$. Thus *after five units of relaxation time the system comes to within 1 percent of the steady state.*

EXERCISES A.1

In Exercises 1 to 5, find the general solution of each of the indicated first-order equations.

1. $y' + 2ty = e^{-t^2}$
2. $ty' + y = 1$
3. $y' + 3y = e^{2t}$
4. $ty' + 4y = t^2$
5. $(\sin t)y' + (\cos t)y = \cos t$

In Exercises 6 to 10, find the general solution of each of the indicated second-order equations.

6. $y'' + 4y = 0$
7. $y'' + 4y' + 4y = 0$
8. $y'' + 2y' - 15y = 0$
9. $y'' + 3y' = 0$
10. $3y'' - 5y' - 2y = 0$

In Exercises 11 to 15, solve the indicated initial-value problems.

11. $y'' + 4y = 0$; $y(0) = 0$, $y'(0) = 2$
12. $y'' + 4y' + 4y = 0$; $y(0) = 1$, $y'(0) = 0$
13. $y'' + 2y' - 15y = 0$; $y(0) = 2$, $y'(0) = 3$
14. $y'' + 3y' = 0$; $y(1) = 1$, $y'(1) = 4$
15. $3y'' - 5y' - 2y = 0$; $y(0) = 1$, $y'(0) = 1$

In Exercises 16 to 20, find the general solution of each of the indicated equations and solve the indicated initial-value problem.

16. $y'' + 4y' + 6y = 2$; $y(0) = 0$, $y'(0) = 2$
17. $y'' - 4y = 2$; $y(0) = 0$, $y'(0) = 0$
18. $y'' + 4y' = 2$; $y(0) = 0$, $y'(0) = 4$
19. $y'' = 0$; $y(0) = 3$, $y'(0) = 4$
20. $ty'' + y' = -1$; $y(1) = 0$, $y'(1) = 0$

The following problems deal with power series solutions of second-order equations.

21. Find the power series solution of the equation $y'' + 4ty = 0$ with the initial conditions $y(0) = 0$, $y'(0) = 1$.
22. Find the power series solution of the equation $y'' + 4ty = 0$ with the initial condition $y(0) = 1$, $y'(0) = 0$.
23. Find the power series solution of the equation $y'' + t^2y = 0$ with the initial condition $y(0) = 0$, $y'(0) = 1$.
24. Find the power series solution of the equation $y'' + t^2y = 0$ with the initial condition $y(0) = 1$, $y'(0) = 0$.
25. Find the power series solution of the equation $y'' + (1 + t)y = 0$ with the initial condition $y(0) = 0$, $y'(0) = 1$.

26. Which of the following second-order differential equations has a regular singular point at $t = 0$?

    (a) $t^2y'' + 3ty + y = 0$ (b) $t^2y'' + y' + 3ty = 0$
    (c) $t^2y'' + t^2y' + y = 0$ (d) $ty'' + y' = 0$
    (e) $ty'' + y = 0$ (f) $y'' + ty = 0$

In Exercises 27 to 30, find the indicial equation and the Frobenius solution for the indicated equation, which has a regular singular point at $t = 0$.

27. $t^2y'' - y - 0$
28. $t^2y'' + ty' - y = 0$
29. $t^2y'' + ty' + 3t^2y = 0$
30. $t^2y'' - (1 + 3t^2)y = 0$

In Exercises 31 to 35, find the steady-state solution and the relaxation time for each of the following solutions of differential equations.

31. $y(t) = 3 + 4e^{-2t}$ (solution of $y' + 2y = 6$)
32. $y(t) = 5 + 3e^{-t} + e^{-3t}$ (solution of $y'' + 4y' + 4y = 15$)
33. $y(t) = e^{-t}\cosh t$ (solution of $y'' + 4y' + 4y = 2$)
34. $y(t) = 1 + e^{-2t}\sinh t$ (solution of $y'' + 4y' + 3y = 3$)
35. $y(t) = 4 + e^{-t} + 3te^{-t}$ (solution of $y'' + 2y' + y = 4$)

A.2. Review of Infinite Series

In this section we collect the necessary facts about infinite series and the related concepts of convergence that are frequently used throughout the text. In the calculus course sequence almost all students have worked with *numerical series*, which are discussed in the first subsection. The remaining subsections deal with issues that arise when we consider the series of *functions* that arise from separated solutions of boundary-value problems for PDEs.

A.2.1. Numerical series. An *infinite series* is, by definition, an expression of the form

$$\sum_{n=1}^{\infty} a_n$$

where a_n is a sequence of numbers. This is a shorthand for the infinite sum

$$a_1 + a_2 + \cdots + a_n + \cdots$$

For example, $\sum_{n=1}^{\infty} 1/n^2$, $\sum_{n=1}^{\infty}(-1)^n/n$, and $\sum_{n=1}^{\infty} 1/3^n$ are familiar examples of infinite series. The summation index n is a dummy variable, so that we make no distinction between $\sum_{n=1}^{\infty} a_n$, $\sum_{m=1}^{\infty} a_m$, and $\sum_{p=1}^{\infty} a_p$.

The *convergence* of the infinite series $\sum_{n=1}^{\infty} a_n$ is formulated in terms of the *partial sums* s_n, defined by $s_n = a_1 + \cdots + a_n$. If $\lim_{n\to\infty} s_n = a$, then we say that the series $\sum_{n=1}^{\infty} a_n$ *converges*, and we write

$$a = \sum_{n=1}^{\infty} a_n$$

If $\lim_{n\to\infty} s_n$ does not exist, then we say that the series *diverges*.

Examples of infinite series that converge are

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty} e^{-n^2}$$

If the terms a_n decrease too slowly or oscillate too erratically, the infinite series will diverge. Examples are

$$\sum_{n=1}^{\infty} \frac{1}{n}, \qquad \sum_{n=1}^{\infty} (-1)^n, \qquad \sum_{n=1}^{\infty} \frac{1}{n} \log(1+n)$$

Convergent series obey the normal laws of arithmetic: if two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then the series $\sum_{n=1}^{\infty} (a_n+b_n)$ converges and $\sum_{n=1}^{\infty} (a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. If the series $\sum_{n=1}^{\infty} a_n$ converges and c is any constant, then the series $\sum_{n=1}^{\infty} c a_n$ converges and $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$.

Tests for convergence are especially simple for series of positive terms: $a_n > 0$. In this case the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sums remain bounded, $s_n \le M$ for some constant M, and all $n = 1, 2, \ldots$. A useful criterion for the convergence of series of positive terms is the *integral test.*

PROPOSITION A.2.1. *(Integral test). Let $a_n = \varphi(n)$, where $\varphi(x)$ is a positive function of $x > 0$, such that $\varphi(x)$ decreases to zero when $x \to \infty$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} \varphi(x)\, dx$ converges.*

EXAMPLE A.2.1. *Determine for which values of $p > 0$ the series $\sum_{n=1}^{\infty} 1/n^p$ converges.*

Solution. We take $\varphi(x) = 1/x^p$, a positive function of $x > 0$, which decreases to zero when $x \to \infty$. The associated improper integral is $\int_1^{\infty} dx/x^p = \lim_{M\to\infty} \int_1^M dx/x^p$. If $p \ne 1$, we have $\int_1^M dx/x^p = (1 - M^{1-p})/(p-1)$, and thus the limit exists if $p > 1$; the limit does not exist if $p < 1$. If $p = 1$, we have $\int_1^M dx/x = \log M$, and the limit does not exist. Therefore the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$. •

When dealing with series whose terms change in sign, we may often conclude convergence from the *ratio test* or the *root test*, stated as follows.

PROPOSITION A.2.2. *(Ratio test). Suppose there are numbers r, N so that $0 < r < 1$ and $N > 1$ such that $|a_{n+1}/a_n| \le r$ for all $n > N$. Then the series $\sum_{n=1}^{\infty} a_n$ converges.*

PROPOSITION A.2.3. *(Root test). Suppose there are numbers r, N so that $0 < r < 1$ and $N > 1$ such that $|a_n|^{1/n} \le r$ for all $n > N$. Then the series $\sum_{n=1}^{\infty} a_n$ converges.*

The ratio test may be applied to conclude convergence of the series $\sum_{n=1}^{\infty}(n/2^n)$. The ratio of two consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)/(2^{n+1})}{n/2^n} = \frac{n+1}{2n}$$

We can take $N = 3$, $r = 2/3$ in the ratio test to conclude that the series converges.

The integral test, ratio test, and root test can be used to deduce the convergence of many of the series that arise in practice. However, none of these can be applied to the harmonic series, $\sum_{n=1}^{\infty}(-1)^n/n$ which arises in many applications. For this purpose, we state the test for *alternating series.*

PROPOSITION A.2.4. *(Alternating series test). Consider a series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$, where $b_n > 0$ and satisfies the conditions that $b_1 \ge b_2 \ge \cdots \ge b_n \ge b_{n+1} \ge \cdots$ and $\lim_{n\to\infty} b_n = 0$. Then the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.*

EXAMPLE A.2.2. *Show that $\sum_{n=1}^{\infty}(-1)^n/n^p$ converges for any $p > 0$.*

Solution. The sequence $b_n = 1/n^p$ satisfies the conditions of the alternating series test for any $p > 0$. •

A.2.2. Taylor's theorem. Another source of convergent series is *Taylor's theorem with remainder:*

$$f(x) - f(x_0) = \sum_{n=1}^{N} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{1}{N!}\int_{x_0}^{x} (x-t)^N f^{(N+1)}(t)\,dt$$

This is not an infinite series, but simply an equation that holds for all functions $f(x)$ for which the derivatives of orders $N+1$ exist and are continuous functions. In particular, for $N = 0$ we obtain the *fundamental theorem of calculus* in the form

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\,dt$$

We can produce convergent infinite series from Taylor's theorem with remainder if we can prove that the integral that defines the "remainder term" tends to zero when $N \to \infty$. This will happen if, for example, all of the derivatives $f^{(n)}$ exist and satisfy the condition that $|f^{(n)}(x)| \le n!c^n$, where $|x - x_0| < 1/c$ for some constant c and $n = 1, 2, \ldots$. Furthermore, we can identify the limit from the left side of Taylor's theorem and conclude the validity of the *Taylor series expansion*

$$f(x) - f(x_0) = \sum_{n=1}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \qquad |x - x_0| < \frac{1}{c}$$

EXAMPLE A.2.3. *Apply Taylor's theorem with remainder to $f(x) = \log x$ with $x_0 = 1$ and find the related Taylor series expansion.*

Solution. The successive derivatives are computed as

$$f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad \ldots, \quad f^{(n+1)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

For any $c > 1$, the ratio $|f^{(N+1)}(t)|/N!$ is less than c^{N+1} if t is restricted to the interval $t \geq 1/c$, so that the series converges provided that $|x - 1| < 1/c$. Since c was arbitrary, we deduce convergence of the series in the interval $|x - 1| < 1$. It is also possible to show that the remainder term tends to zero at the endpoint $x = 2$. The sum of the series is $\log x$, and thus we have the convergent Taylor series

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} \qquad 0 < x < 2 \quad \bullet$$

Another useful series with a finite interval of convergence is the geometric series $\sum_{n=1}^{\infty} x^n$. This arises as the Taylor series of the function $f(x) = 1/(1-x)$ as documented in the next example, which we treat by elementary algebra.

EXAMPLE A.2.4. *Discuss the Taylor expansion of $f(x) = 1/(1-x)$ with respect to the point $x_0 = 0$.*

Solution. By long division, we have for any $x \neq 1$ and any $N = 1, 2, \ldots$,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^N + \frac{x^{N+1}}{1-x}$$

If $|x| < 1$, the last term tends to zero, which proves that the series $\sum_{n=1}^{\infty} x^n$ converges with the explicit sum

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x_N + \cdots = 1 + \sum_{n=1}^{\infty} x^n \qquad -1 < x < 1 \quad \bullet$$

The above series for $\log x$ and $1/(1-x)$ converge in a finite interval. On the other hand, the Taylor series for the exponential and trigonometric functions are convergent for all x.

EXAMPLE A.2.5. *Apply Taylor's theorem with remainder to $f(x) = e^x$ with $x_0 = 0$ and find the related Taylor series expansion.*

Solution. The successive derivatives are computed as

$$f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x, \ldots, f^{(n+1)}(x) = e^x$$

For any $c > 0$, the ratio $f^{(N+1)}(t)/N!$ is less than $e^{1/c}/N!$ if t is restricted to the interval $|t| \le 1/c$, so that the integral remainder term tends to zero and the series converges provided that $|x| < 1/c$. The sum of the series is $e^x - 1$, and thus we have the convergent Taylor series

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty \quad \bullet$$

A.2.3. Series of functions: Uniform convergence. We now consider more general series of functions, of the form

$$\sum_{n=1}^{\infty} u_n(x)$$

All of the Taylor series considered above are examples of these. The functions $u_n(x)$ are assumed to be defined on a common interval $a \le x \le b$. Let $f_n(x)$ be the partial sum $f_n(x) = u_1(x) + \cdots + u_n(x)$.

Definition The series $\sum_{n=1}^{\infty} u_n(x)$ *converges pointwise* if, for each x, $a \le x \le b$, the limit $\lim_{n\to\infty} f_n(x)$ exists. We denote the limit by $f(x) = \sum_{n=1}^{\infty} u_n(x)$.

Definition The series $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly*, $a \le x \le b$, if the series converges pointwise and there exists a sequence of constants ϵ_n such that

$$\begin{aligned} |f_n(x) - f(x)| &\le \epsilon_n \qquad a \le x \le b \\ \lim_{n\to\infty} \epsilon_n &= 0 \end{aligned}$$

If the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly, then it also converges pointwise, but the converse statement is not true. It may happen that, for a judicious choice of $x = x_n$, depending upon n, we have $f_n(x_n) - f(x_n)$ tending to a nonzero limit, for example. To see this in detail, consider the sequence of functions $u_n(x) = nxe^{-nx} - (n+1)xe^{-(n+1)x}$. For any fixed n, we have $f_{n-1}(x) = xe^{-x} - nxe^{-nx}$ for all x, $0 \le x \le 1$. The last term tends to zero, so that the series $\sum_{n=1}^{\infty} u_n(x)$ converges pointwise to $f(x) = xe^{-x}$ for $0 \le x \le 1$. But the convergence is not uniform. Indeed, by taking $x = 1/n$, we have $f_{n-1}(1/n) = (1/n)e^{-(1/n)} - e^{-1}$. Thus $|f_{n-1}(x_n) - f(x_n)|$ has the value $e^{-1} = 0.368\ldots$ for $x_n = 1/n$. This contradicts the possibility of a sequence of constants ϵ_n with $\lim_{n\to\infty} \epsilon_n = 0$ such that $|f_n(x) - f(x)| \le \epsilon_n$ for *all* x, $0 \le x \le 1$.

In order to prove the uniform convergence of a series of functions, we may use the following important criterion.

PROPOSITION A.2.5. *(Weierstrass M-test). Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of functions defined for* $a \le x \le b$. *Suppose there exists a sequence of constants* M_n *such that*

$$|u_n(x)| \le M_n \qquad a \le x \le b$$

and $\sum_{n=1}^{\infty} M_n$ *converges. Then the series* $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly for* $a \le x \le b$.

EXAMPLE A.2.6. *Show that the Taylor series* $\sum_{n=1}^{\infty} x^n/n!$ *converges uniformly for* $-1 \le x \le 1$.

Solution. We have $u_n(x) = x^n/n!$ and $|u_n(x)| \le 1/n!$ for $-1 \le x \le 1$. But the series $\sum 1/n!$ converges. Therefore the Taylor series $\sum_{n=1}^{\infty} x^n/n!$ is uniformly convergent for $-1 \le x \le 1$. •

The Weierstrass M-test only gives *sufficient* conditions for the uniform convergence of a series of functions. For example, the series $\sum_{n=1}^{\infty} (-1)^n x^n/n$ does not satisfy these conditions, but it is uniformly convergent for $0 \le x \le 1$. (See Exercise 22.)

Uniform convergence can be used to justify many operations with infinite series. We restrict attention to series of continuous functions $\sum_{n=1}^{\infty} u_n(x)$. We have the following propositions, which give conditions for continuity, integration, and differentiation of a uniformly convergent series of continuous functions. These results are used in Chapters 2, 3, and 4 when we prove that the sum of a series of separated solutions satisfies a PDE.

PROPOSITION A.2.6. *Suppose* $f(x) = \sum_{n=1}^{\infty} u_n(x)$ *is a uniformly convergent series for* $a \le x \le b$, *where* $u_n(x)$ *is a continuous function for each* $n = 1, 2, \ldots$. *Then* $f(x)$ *is a continuous function,* $a \le x \le b$.

PROPOSITION A.2.7. *Suppose* $f(x) = \sum_{n=1}^{\infty} u_n(x)$ *is a uniformly convergent series for* $a \le x \le b$, *where* $u_n(x)$ *is a continuous function. Then the numerical series* $\sum_{n=1}^{\infty} \int_a^b u_n(x)dx$ *converges and* $\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$.

PROPOSITION A.2.8. *Suppose* $f(x) = \sum_{n=1}^{\infty} u_n(x)$ *is a uniformly convergent series for* $a \le x \le b$, *where* $u_n(x)$ *is a continuous function with a continuous derivative* $u_n(x)$, *and the series* $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly for* $a \le x \le b$. *Then the function* $f(x)$ *has a continuous derivative, which is given by* $f'(x) = \sum_{n=1}^{\infty} u_n'(x)$.

EXAMPLE A.2.7. *Show that the function* $f(x) = \sum_{n=1}^{\infty}(\cos nx)/2^n$ *is continuous for* $-\pi \le x \le \pi$ *and that* $f'(x)$ *and* $f''(x)$ *are continuous for* $-\pi \le x \le \pi$.

Solution. We apply the Weierstrass M-test with $M_n = 1/2^n$, to see that the given series is uniformly convergent, $-\pi \le x \le \pi$. Each function $(\cos nx)/2^n$ is continuous; hence $f(x)$ is continuous by Proposition A.2.6. The differentiated series is $\sum_{n=1}^{\infty}(-n \sin nx)/2^n$, which is also uniformly convergent by the Weierstrass M-test with $M_n = n/2^n$, the general term of a convergent series of constants. Therefore, by Proposition A.2.8, $f(x)$ has a continuous derivative $f'(x)$, which is given by $f'(x) = \sum_{n=1}^{\infty}(-n \sin nx)/2^n$. To study $f''(x)$, we note that the differentiated series for f' is $\sum_{n=1}^{\infty}(-n^2 \cos nx)/2^n$, which is uniformly convergent by the Weierstrass M-test with $M_N = n^2/2^n$, the general term of a convergent series of constants. Therefore, by Proposition A.2.8, $f'(x)$ has a continuous derivative $f''(x)$, which is given by the differentiated series $f''(x) = \sum_{n=1}^{\infty}(-n^2 \cos nx)/2^n$. •

EXAMPLE A.2.8. *Beginning with the geometric series* $x/(1-x) = \sum_{n=1}^{\infty}x^n$, $-1 < x < 1$, *show that*

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \qquad -1 < x < 1$$

$$\sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3} \qquad -1 < x < 1$$

Solution. Let $u_n(x) = x^n$ and let r be any number with $0 < r < 1$. By the Weierstrass M-test, the series $\sum_{n=1}^{\infty}u_n(x)$ is uniformly convergent for $-r \le x \le r$ with $M_n = r^n$, the general term of a convergent series of constants. The differentiated series is $\sum_{n=1}^{\infty}nx^{n-1}$, which is also uniformly convergent for $-r \le x \le r$ by the Weierstrass M-test with $M_n = nr^{n-1}$, the general term of a convergent series. On the other hand, from calculus, $(d/dx)[x/(1-x)] = 1/(1-x)^2$. Therefore from Proposition, A.2.8 we have $1/(1-x)^2 = \sum_{n=1}^{\infty}nx^{n-1}$ for $-r \le x \le r$. But r was any number with $0 < r < 1$, and hence this equation is true for all x with $-1 < x < 1$. Multiplying by x gives the required result $x/(1-x)^2 = \sum_{n=1}^{\infty}nx^n$ for $-1 < x < 1$.

We now apply Proposition A.2.8 to this series, noting that the differentiated series $\sum_{n=1}^{\infty}n^2x^{n-1}$ is again uniformly convergent by the Weierstrass M-test for $-r \le x \le r$ with $M_n = n^2r^{n-1}$, the general term of a convergent series of constants. On the other hand, the derivative can be computed by calculus as $(d/dx)[x/(1-x)^2] = (1+x)/(1-x)^3$. Multiplication by x gives the required result, $\sum_{n=1}^{\infty}n^2x^n = x(1+x)/(1-x)^3$. •

Often we need to consider functions that are defined by integrals. The following statement is used in Chapter 5.

PROPOSITION A.2.9. *Suppose that $f(x) = \int_c^d g(x,y)\,dy$, where the function $g(x,y)$ is continuous for $a \le x \le b, c \le y \le d$. Then $f(x), a \le x \le b$, is a continuous function. If in addition, the partial derivative $\partial g/\partial x$ exists and is a continuous function for $a \le x \le b, c \le y \le d$, then $f'(x)$ exists and is given by the integral formula $f'(x) = \int_c^d (\partial g/\partial x)(x,y)\,dy$.*

A.2.4. Abel's lemma. Abel's lemma states that if $f(t), t > 0$, is a bounded function and $\lim_{t\to\infty} f(t) = L$, then $\lim_{p\downarrow 0} \int_0^\infty f(t)pe^{-pt}\,dt = L$.

The proof consists of writing

$$\int_0^\infty f(t)pe^{-pt}\,dt - L = \int_0^\infty (f(t) - L)pe^{-pt}\,dt$$

Given $\epsilon > 0$, we split the region of integration into the two regions $0 < t < T$ and $T < t < \infty$, where T is chosen such that $|f(t) - L| < \epsilon/2$ for $t > T$, so that the second integral is less than $\epsilon/2$. If $|f(t)| \le M$ for all $t > 0$, then the first integral is less than $2MpT$, so that by choosing $p < \epsilon/4MT$ we render the second integral less than $\epsilon/2$, and the result follows.

Abel's lemma can also be applied to analyze improper integrals, if we let $f(t) = \int_0^t g(s)\,ds$. Interchanging the orders of integration yields

$$\begin{aligned}
\int_0^\infty f(t)pe^{-pt}\,dt &= \int_0^\infty pe^{-pt}\left(\int_0^t g(s)\,ds\right)dt \\
&= \int_0^\infty g(s)\left(\int_s^\infty pe^{-pt}\,dt\right)ds \\
&= \int_0^\infty g(s)e^{-ps}\,ds
\end{aligned}$$

Applying Abel's lemma gives the result that if $\lim_{t\to\infty} \int_0^t g(s)\,ds = L$, then $\lim_{p\downarrow 0} \int_0^\infty g(s)e^{-ps}\,ds = L$. This is used in Chapter 5 to treat the initial-value problem for the heat equation.

There is also a version of Abel's lemma that applies to infinite series and is useful for discussion of the initial-value problem for the heat equation in bounded regions in Chapter 2. We give the combined statement and proof as follows.

THEOREM A.2. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.*

(a) Suppose that $\lim_{n\to\infty} b_n = L$. Then $\lim_{x\uparrow 1}(1-x)\sum_{n=0}^\infty b_n x^n = L$.

(b) Suppose the infinite series $\sum_{n=1}^\infty a_n$ converges to the sum S. Then the power series $\sum_{n=1}^\infty a_n x^n$ has a limit when $x \uparrow 1$, and $\lim_{x\uparrow 1} \sum_{n=1}^\infty a_n x^n = S$.

Proof

(a) As above, we write for $0 < x < 1$,

$$(1-x)\sum_{n=0}^\infty b_n x^n - L = (1-x)\sum_{n=0}^\infty (b_n - L)x^n$$

Given $\epsilon > 0$, choose N so that $|b_n - L| < \epsilon/2$ for $n > N$. Assuming that all of the terms of the sequence are bounded by M,

$$\left|(1-x)\sum_{n=0}^{\infty}(b_n - L)x^n\right| < (1-x^{N+1})2M + \epsilon/2$$

If $1 - x^{N+1} < \epsilon/4M$, then the sum is less than ϵ, as required.
(b) Define $b_0 = 0$ and for $n \geq 1$, $b_n = a_1 + \cdots + a_n$. Then

$$\begin{aligned}(1-x)\sum_{n=0}^{\infty} b_n x^n &= (1-x)\sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} a_k \\ &= (1-x)\sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} x^n \\ &= \sum_{k=1}^{\infty} a_k x^k\end{aligned}$$

By hypothesis, $\lim_{n\to\infty} b_n = S$, so the desired conclusion follows from part (a). •

A.2.5. Double series. We now consider *double series*. These are of the form

$$\sum_{m,n=1}^{\infty} a_{mn}$$

where a_{mn} are real numbers defined for $m = 1, 2, \ldots$ and $n = 1, 2, \ldots$. The *convergence* of a double series is defined in terms of the partial sums

$$s_{mn} = \sum_{1\leq i\leq m, 1\leq j\leq n} a_{ij}$$

If

$$\lim_{m\to\infty, n\to\infty} s_{mn} = a$$

when the indices m, $n \to \infty$ in any order whatever, then we write

$$a = \sum_{m,n=1}^{\infty} a_{mn}$$

as the sum of the series.

A form of the integral test is applicable to double series of positive terms. Let $a_{mn} = \varphi(m, n)$, where the function $\varphi(x, y)$ is positive and decreasing in each of the variables x, y. Then the convergence or divergence of the double series $\sum_{m,n=1}^{\infty} a_{mn}$ is equivalent to the convergence or divergence of the double integral $\int_1^{\infty}\int_1^{\infty} \varphi(x, y)\,dx\,dy$.

EXAMPLE A.2.9. *For which values of $p > 0$ is the double series*

$$\sum_{m,n=1}^{\infty} \frac{1}{(m^2+n^2)^{p/2}}$$

convergent?

Solution. We have $\varphi(x,y) = [1/(x^2+y^2)]^{p/2}$. To study the associated double integral, we take polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, with $dx\,dy = r\,dr\,d\theta$. Thus we examine $\int_0^{\pi/2}\int_1^{\infty}(1/r^p)r\,dr\,d\theta$. This double integral is convergent if and only if $p - 1 > 1$, that is, $p > 2$. We conclude that the double series $\sum_{m,n=1}^{\infty} 1/(m^2+n^2)^{p/2}$ is convergent if and only if $p > 2$. •

The Weierstrass M-test and the properties of uniform convergence may also be generalized to double series of functions of one or more variables, for example, the series $f(x,y) = \sum_{m,n=1}^{\infty} u_{m,n}(x,y)$. We may use the properties of uniform convergence to show that $f(x,y)$ has partial derivatives f_x, f_y, which are continuous functions of (x,y) in a rectangular region $a \le x \le b$, $c \le y \le d$.

A.2.6. Big-O notation. In several places throughout the text we have used the big-O notation to indicate the order of magnitude of a function, in the limiting case when a parameter becomes large. In this subsection we systematically collect the various definitions and properties of this notation.

Definition Let $f(t)$, $g(t)$ be two functions defined for $t > 0$. We write

$$f(t) = O(g(t)) \qquad t \to \infty$$

if there exist constants $M > 0$ and $T > 0$ such that $|f(t)| \le Mg(t)$ for $t > T$. Similarly, if $\{a_n\}$ and $\{b_n\}$ are sequences defined for $n = 1, 2, \ldots$, we say that $a_n = O(b_n)$ if there exist constants $M > 0$ and $N > 0$ such that $|a_n| \le Mb_n$ for $n > N$.

For example, we have

$$\frac{2t^2}{1+t^3} = O\left(\frac{1}{t}\right) \quad t \to \infty, \quad \frac{\sin t}{1+t^2} = O\left(\frac{1}{t^2}\right) \quad t \to \infty$$

$$e^{-t} = O\left(\frac{1}{t^5}\right) \quad t \to \infty, \quad \frac{t^2}{1+t^2} = O(1) \quad t \to \infty$$

Each of these can be proved using facts about the specific functions. In case 1 we have $1+t^3 \ge t^3$ for $t > 0$, so that $2t^2/(1+t^3) \le 2t^2/t^3 = 2/t$; hence the definition is satisfied with $M = 2$, $T = 1$. Likewise in case 2, $|\sin t| \le 1$, $(1+t^2) \ge t^2$, so that the definition is satisfied with $M = 1$, $T = 1$. In case 3 we may use the power series $e^t = 1 + \sum_1^{\infty} t^n/n!$. All of the terms are positive when $t > 0$; hence $e^t > t^5/5!$ or $e^{-t} < 5!/t^5$. Therefore we may take $M = 5!$, $T = 1$. Finally, we note that $f(t) = O(1)$, as in case 4, is equivalent to the statement that $|f(t)| \le M$ for $t > T$. In this case we say that $f(t)$ remains *bounded* when $t \to 0$.

If $f(t) = O(g(t))$, $t \to \infty$, it does not follow that $g(t) = O(f(t))$, $t \to \infty$. For example, $e^{-t} = O(1/t^5)$, $t \to \infty$, but $1/t^5 \ne O(e^{-t})$, $t \to \infty$. Similarly, we write

$$f(t) = O(g(t)) \qquad t \to t_0$$

if there exist constants $M > 0$ and $\delta > 0$ such that $|f(t)| \le Mg(t)$ for $0 < |t - t_0| < \delta$. For example, $\sin t = O(t)$, $t \to 0$, whereas $\sec t = O(1/|t - \pi/2|)$, $t \to \pi/2$.

Often we encounter series that depend on a parameter t, for example, the series $\sum_{n=1}^{\infty} e^{-nt}$. This series converges for each $t > 0$ by the ratio test. If $t > 1$, $e^{-t} < e^{-1} < 0.38$, and thus $\sum_{n=1}^{\infty} e^{-nt} < [1/(1 - 0.38)]e^{-t} < 1.62e^{-t}$. We have proved that

$$\sum_{n=1}^{\infty} e^{-nt} = O(e^{-t}) \qquad t \to \infty$$

At first this may seem somewhat surprising, since we have estimated the *sum* of a series by the first term. This result can be applied to the series $\sum_{n=1}^{\infty} a_n e^{-nt}$, where $\{a_n\}$ is any sequence of constants with $a_n = O(1)$, $n \to \infty$. Thus we have $\sum_{n=1}^{\infty} a_n e^{-nt} = O(e^{-t})$, $t \to \infty$. This may also be true for certain sequences $\{a_n\}$ with a_n tending to infinity. For example, if $a_n = n^2$, we use the result of Example A.2.8:

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n \qquad -1 < x < 1$$

Taking $x = e^{-t}$, we have

$$\sum_{n=1}^{\infty} n^2 e^{-nt} = e^{-t} \frac{1 + e^{-t}}{(1 - e^{-t})^3} = O(e^{-t}) \qquad t \to \infty$$

We have proved that

$$\sum_{n=1}^{\infty} n^2 e^{-nt} = O(e^{-t}) \qquad t \to \infty$$

This is used in Chapter 2.

EXERCISES A.2

1. Apply the integral test to determine the convergence or divergence of each of the following series of positive terms:
   (a) $\sum_{n=1}^{\infty} 1/(n+1)\log(n+1)$ (b) $\sum_{n=1}^{\infty} 1/n(\log 2n)^2$
   (c) $\sum_{n=1}^{\infty} (\log n)/n$ (d) $\sum_{n=1}^{\infty} n e^{-n^2}$

2. Find the Taylor series of the following functions:
   (a) $f(x) = \sin x$, $x_0 = 0$
   (b) $f(x) = \log[(1+x)/(1-x)]$, $x_0 = 0$
   (c) $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$, $x_0 = 0$
   (d) $f(x) = xe^{x^2}$, $x_0 = 0$
3. Let $f(x) = 0$ for $x \le 0$ and $f(x) = e^{-1/x}$ for $x > 0$.
   (a) Compute $f'(x)$, $f''(x)$.
   (b) Show by induction that, for $x > 0$, the nth derivative is of the form $f^{(n)}(x) = e^{-1/x}P_{2n}(1/x)$, where P_{2n} is a polynomial of degree $2n$.
   (c) Deduce that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.
   (d) Discuss the validity of the Taylor expansion

$$f(x) - f(0) = \sum_{n=1}^{\infty} \frac{x^n f^{(n)}(0)}{n!}$$

4. Apply the Weierstrass M-test to verify the uniform convergence of the following series:
   (a) $\sum_{n=1}^{\infty} x^n/2^n, -1 \le x \le 1$ (b) $\sum_{n=1}^{\infty} (\sin nx)/n^2, -\pi \le x \le \pi$
   (c) $\sum_{n=1}^{\infty} e^{-n^2} \cos nx, -\pi \le x \le \pi$ (d) $\sum_{n=1}^{\infty} n^4 (\cos n^2 x)/2^n, 0 \le x \le \pi$
5. Which of the series in Exercise 4 can be differentiated term by term according to Proposition A.2.8?
6. Generalize Example A.2.8 to find the sum of the series

$$\sum_{n=1}^{\infty} n^3 x^n \qquad \text{and} \qquad \sum_{n=1}^{\infty} n^4 x^n \qquad \text{for } -1 < x < 1$$

7. (a) Show that $1/(1+x^2) = 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n}$ for $-1 < x < 1$.
   (b) Show that this series is uniformly convergent for $0 \le x \le r$, where $0 < r < 1$.
   (c) Use Proposition A.2.7 to show that

$$\tan^{-1} x = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad \text{for } 0 < x < 1$$

8. (a) Prove the finite identity

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}$$

   (b) Show that

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1} + (-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2}\, dt$$

   (c) Show that $\int_0^1 [t^{2n+2}/(1+t^2)]dt \le 1/(2n+3)$.
   (d) Conclude that $\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}/(2n-1)$.

9. Which of the following double series are convergent, according to the integral test?
   (a) $\sum_{m,n=1}^{\infty} 1/(m^2+n^2)$ (b) $\sum_{m,n=1}^{\infty} e^{-(m^2+n^2)}$
   (c) $\sum_{m,n=1}^{\infty}(m^2+n^2)e^{-(m+n)}$ (d) $\sum_{m,n=1}^{\infty} 1/(m^2n^2)$
10. Find constants $M > 0$, $T > 0$ such that $|f(t)| \le Mg(t)$ for $t > T$ if
   (a) $f(t) = t^2$, $g(t) = e^t$
   (b) $f(t) = t^{10}$, $g(t) = e^{t/2}$
   (c) $f(t) = (\sin t)(\log t)$, $g(t) = t^{1/2}$
   (d) $g(t) = 100/t$, $f(t) = e^{-t/50}$
11. For each of the following functions, is it true that $f(t) = O(g(t))$, $t \to \infty$? Find suitable constants $M > 0$ and $T > 0$ in each case.
   (a) $f(t) = 3t^3 + 3t^2 + 5$, $g(t) = t^3$
   (b) $f(t) = t^2 + 4t - 2$, $g(t) = t^4$
   (c) $f(t) = (t^3 + 4t)/(t^2 + 2)$, $g(t) = 1$
   (d) $f(t) = t^{100}$, $g(t) = e^{1/4}$
   (e) $f(t) = \sin^2 t$, $g(t) = |\sin t|$
   (f) $f(t) = e^{-0.01t}$, $g(t) = t^{-25}$
12. (a) Show that $\lim_{n\to\infty} n^2xe^{-nx} = 0$ for $0 \le x \le 1$.
   (b) Show that $\lim_{n\to\infty} \int_0^1 n^2xe^{-nx}dx = 1$.
   (c) Find a series of functions $u_n(x)$ for which

$$\int_0^1 \left(\sum_{n=1}^{\infty} u_n(x)\right) dx \neq \sum_{n=1}^{\infty}\left(\int_0^1 u_n(x)dx\right)$$

13. If $f_1(t) = O(g(t))$, $f_2(t) = O(g(t))$, $t \to \infty$, show that $f_1(t) + f_2(t) = O(g(t))$, $t \to \infty$.
14. If $f_1(t) = O(g(t))$, $f_2(t) = O(g(t))$, $t \to \infty$, show that $f_1(t)f_2(t) = O(g(t)^2)$, $t \to \infty$.
15. If $f_1(t) = O(g(t))$, $f_2(t) = O(g(t))$, $t \to \infty$, is it true that $f_1(t)/f_2(t) = O(1)$, $t \to \infty$?
16. Show that $\log(1+t) = \log t + O(1/t)$, $t \to \infty$.
17. Show that $(1+t)^5 = t^5 + O(t^4)$, $t \to \infty$.
18. Show that $\sum_{n=1}^{\infty} ne^{-nt} = O(e^{-t})$, $t \to \infty$.
19. Use mathematical induction to prove that for any $p = 1, 2, \ldots$, $-1 < x < 1$,

$$\sum_{n=1}^{\infty} n^p x^n = \frac{xQ_p(x)}{(1-x)^{p+1}}$$

   where Q_p is a polynomial of degree $p - 1$. For example, $Q_1 = 1, Q_2 = 1 + x, Q_3 = x^2 + 4x + 1$.
20. Let $x = e^{-t}$ in Exercise 19 and show that, for any $p = 1, 2, \ldots$, $\sum_{n=1}^{\infty} n^p e^{-nt} = O(e^{-t})$, $t \to \infty$.

21. Let $\{a_n\}$ be a sequence with $a_n = O(n^p)$ for some $p = 1, 2, \ldots$. Show that $\sum_{n=1}^{\infty} a_n e^{-nt} = O(e^{-t})$, $t \to \infty$.
22. (a) Prove the finite identity $1/(1+t) = 1-t+t^2-\cdots+(-t)^{n-1}+(-t)^n/(1+t)$.
    (b) Integrate this to obtain the identity

$$\log(1+x) = x - \frac{x^2}{x} + \cdots + \frac{(-x)^{n-1}}{n} + (-1)^n \int_0^x \frac{t^n}{1+t} dt$$

for any $x \geq 0$.
    (c) Show that $\int_0^x [t^n/(1+t)]dt \leq 1/(n+1)$ for $0 \leq x \leq 1$.
    (d) Conclude that $\sum_{n=1}^{\infty} (-x)^n/n$ converges uniformly for $0 \leq x \leq 1$.
23. (Cesaro implies Abel summability) Suppose that $g(t), t > 0$ satisfies the relation $\lim_{T\to\infty} T^{-1} \int_0^T g(s)\, ds = L$. Show that $\lim_{p\downarrow 0} \int_0^{\infty} pe^{-pt} g(t) dt = L$. [*Hint:* Letting $f(t) = t^{-1} \int_0^t g(s)\, ds$, we have $g = (tf)'$, so that we have $\int_0^{\infty} pe^{-pt} g(t) dt = p^2 \int_0^{\infty} te^{-pt} f(t)\, dt$. Now imitate the proof of Abel's lemma to obtain the conclusion.]

A.3. Review of Vector Integral Calculus

The starting point for the theory of vector integral calculus is the fundamental theorem of calculus of one variable, written in the form

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

In vector integral calculus we replace the interval $a < x < b$ by a two- or three-dimensional region and replace the scalar function f by a vector field.

In detail, we consider a three-dimensional region D whose boundary consists of a finite number of smooth surfaces. For example, a cube has a boundary consisting of its six faces, whereas a solid sphere has a single smooth boundary surface. A *vector field* consists of an ordered set of three functions:

$$\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$$

The *divergence* of the vector field $\mathbf{F}$ is defined by

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The *divergence theorem* states that

$$\iiint_D \operatorname{div} \mathbf{F}\, dx\, dy\, dz = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS$$

Here $\mathbf{n}$ is the outward-pointing normal vector field on the boundary of D (denoted ∂D) and dS denotes the element of surface area.

EXAMPLE A.3.1. *Verify the divergence theorem in the case where D is the cube $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ and $\mathbf{F}(x, y, z) = (F_1(x), 0, 0)$.*

Solution. In this case $\operatorname{div} \mathbf{F} = F_1'(x)$, so that the left side of the divergence theorem is

$$\int_0^a \int_0^b \int_0^c F_1'(x) dx\, dy\, dz = [F_1(a) - F_1(0)] \times b\, c$$

To compute the right side of the divergence theorem, we must calculate separately the integrand on each of the six faces:

$$\begin{array}{ll} \text{On the face } x = 0, & \mathbf{F} \cdot n = -F_1(0) \\ \text{On the face } x = a, & \mathbf{F} \cdot n = +F_1(a) \\ \text{On the face } y = 0, & \mathbf{F} \cdot n = 0 \\ \text{On the face } y = b, & \mathbf{F} \cdot n = 0 \\ \text{On the face } z = 0, & \mathbf{F} \cdot n = 0 \\ \text{On the face } z = c, & \mathbf{F} \cdot n = 0 \end{array}$$

Therefore the total surface integral reduces to two nonzero terms, each of which is the area of the bounding surface multiplied by the constant value of the integrand; in detail,

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS = -bc\, F_1(0) + bc\, F_1(a) \quad \bullet$$

In the case of a two-dimensional region, the triple integrals are replaced by double integrals and the divergence theorem is written

$$\iint_D \operatorname{div} \mathbf{F}\, dx\, dy = \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS.$$

where $\operatorname{div} \mathbf{F} = \partial F_1/\partial x + \partial F_2/\partial y$. The integral of the normal component of $\mathbf{F}$ can also be written in terms of the tangential component of a vector field, if we write $N = F_1, M = -F_2$ so that we obtain the statement of *Green's theorem*

$$\int_{\partial D} (M\, dx + N\, dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

where the sign is determined by the convention that the boundary curve ∂D is oriented so that when we traverse the boundary, the region is on the left side of the curve.

In order to make effective use of the three-dimensional divergence theorem, we first note that if u is a differentiable function, then the usual product rule for derivatives can be applied to a vector field of the form $\mathbf{W} = u\mathbf{F}$ as follows:

$$\begin{aligned}\operatorname{div}\mathbf{W} &= \frac{\partial(uF_1)}{\partial x} + \frac{\partial(uF_2)}{\partial y} + \frac{\partial(uF_3)}{\partial z} \\ &= u\frac{\partial F_1}{\partial x} + \frac{\partial u}{\partial x}F_1 + u\frac{\partial F_2}{\partial y} + \frac{\partial u}{\partial y}F_2 + u\frac{\partial F_3}{\partial z} + \frac{\partial u}{\partial z}F_3 \\ &= u\operatorname{div}\mathbf{F} + \nabla u \cdot \mathbf{F}\end{aligned}$$

Substituting in the divergence theorem yields

$$\iiint_D (u\operatorname{div}\mathbf{F} + \nabla u \cdot \mathbf{F})\, dx\, dy\, dz = \iint_{\partial D} u\,\mathbf{F}\cdot\mathbf{n}\, dS$$

Now let v be a twice-differentiable function and apply this to the vector field $F = \nabla v$ to obtain *Green's first identity:*

$$\iiint_D \left(u\,\nabla^2 v + \nabla u \cdot \nabla v\right)\, dx\, dy\, dz = \iint_{\partial D} u\,\nabla v\cdot\mathbf{n}\, dS$$

When we interchange the roles of u and v and subtract the two equations, we obtain *Green's second identity:*

$$\iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\, dx\, dy\, dz = \iint_{\partial D} \left(u\nabla v\cdot\mathbf{n} - v\nabla u\cdot\mathbf{n}\right)\, dS$$

which is valid for any pair of twice-differentiable functions u, v.

A.3.1. Implementation with Mathematica. It is possible to illustrate the divergence theorem with Mathematica. This requires Mathematica's **Vector Analysis** package, which is called by typing

```
In[1]:=<<Calculus'VectorAnalysis'
```

Consider the vector field $\mathbf{F}(x,y,z) = x^2(2y+1)z\,\mathbf{j}$ in the triangular prism defined by the planes $x = 0$, $x = 3$, $y = 0$, and $y + z = 1$. Clearly, $\operatorname{div}\mathbf{F} = (\partial/\partial y)(x^2(2y+1)z) = 2x^2z$.

The divergence can be computed by the following Mathematica session

```
In[2]:=F={0,x^2(2y+1)z,0}

          2
Out[2]={0,x  (2 y + 1) z , 0}

In[3]:=divF = Div[F]

            2
Out[3] = 2 x  z
```

The volume integral in the divergence theorem is

$$\begin{aligned}\int_0^3 \left(\int_0^1 \left(\int_0^{1-z} 2x^2 z\, dy \right) dz \right) dx &= \int_0^3 \left(\int_0^1 2x^2 z(1-z)\, dz \right) dx \\ &= \int_0^3 2x^2 \left(2 - \frac{2}{3} \right) dx \\ &= \frac{8}{3} \cdot \frac{1}{3} = \frac{8}{9}\end{aligned}$$

To do this in Mathematica, we have

```
In[4]:-Integrate[
        Integrate[
          Integrate[divF, {x,0,3}],
               {y,0,1-z}],
             {z,0,1}]

Out[4]=8/9
```

A.4. Using Mathematica

A.4.1. Introduction. Mathematica is a very powerful tool for studying partial differential equations. It can be used to find both symbolic and numerical solutions and to graph the solutions. Since no previous knowledge of Mathematica is presumed, in this section we give a brief introduction to its use. Details are given in relevant chapters as they are needed.

Mathematica is a *symbolic manipulation program;* this means that not only numbers but also symbols can be used in calculations. A Mathematica session consists of a series of questions and answers. Although Mathematica notation differs in several crucial ways from ordinary mathematical notation, the translation between the two is straightforward. In addition to symbolic manipulation, Mathematica has powerful graphing capabilities. This important feature will allow us to graph Fourier series and solutions to boundary-value problems with ease.

There are two types of implementations of Mathematica: (i) the notebook (or graphical) interface and (ii) the textual (or command line) interface.

A.4.2. The notebook front end. Notebook versions of Mathematica are available for PCs equipped with Microsoft Windows or NeXTstep, Macintoshes, NeXT workstations, and X Windows running on UNIX workstations. In this section we explain the NeXT, Macintosh, and PC notebook interfaces; notebook interfaces on other computers are similar and can be found by clicking on the Mathematica "Help" menu.

To bring up Mathematica, double click on the Mathematica icon, which is a small polyhedron. This brings up a window called "Untitled"; it is actually a notebook. To enter a formula, just start typing. For example, the polynomial $x^4 - 1$ can be factored by typing

```
Factor[x^4 - 1]
```

The result is

$$(-1+x)(1+x)(1+x^2)$$

Notebooks are divided into cells. Each formula is entered into a "cell," recognizable as a bracket on the right side of the window. After typing a formula into a cell, the formula can be evaluated by hitting the key ENTER. (Notice that ENTER is different from RETURN; the latter is used to go to the next line of a cell. However, SHIFT/RETURN has the same effect as ENTER.) Cells can be used to display either text or graphics and can be printed individually.

A.4.2.1. *Help.* To obtain help on any command, type ? in front of the command and hit ENTER. For example, **?Plot** gives a short description of how to plot a function. Wild cards can also be used. Thus to see all commands that start with **D**, type **?D***. To get more extensive help use ??.

A.4.2.2. *The function browser.* The function browser in the Help menu explains Mathematica functions and commands; it allows functions and commands to be pasted into cells, usually as templates. Although the function browser has been a feature on the NeXT and Macintosh notebook versions of Mathematica for some time, it is a standard feature on all notebook versions of Mathematica beginning with version 3.0. On a NeXT or Macintosh it is also possible to access the function browser with the key combination COMMAND-SHIFT-F. Furthermore, if the name of a function or command in a cell is selected with the mouse, the key combination COMMAND-SHIFT-F will open the function browser and give a short explanation.

A.4.2.3. *Key equivalences between platforms.* On notebook versions of Mathematica, many frequently used menu options can also be performed with specific key combinations. For example, on the NeXT computer, all cells in a notebook can be selected with the key combination COMMAND-SHIFT-A. On the Macintosh it is APPLE-SHIFT-A, and on the PC it is CONTROL-SHIFT-A. In general, whenever a reference is made to a key sequence on the NeXT using the COMMAND key, you may substitute the APPLE key for the Macintosh or the CONTROL key for the Microsoft Windows version on a PC.

A.4.2.4. *Copying input and output from above.* A series of calculations frequently requires evaluating similar expressions, one after the other. Two key combinations greatly facilitate this process. The precise keystrokes can be obtained from the Mathematica Help menu. In the case of NeXT workstations, the key combination COMMAND-L copies the input from the cell above. This new input can be edited and reevaluated. Sometimes the key combination COMMAND-SHIFT-L, which copies the output from the cell above, is also useful.

A.4.2.5. *Command completion.* A partially written command or function can be completed in the NeXT workstation with the key combination COMMAND-K. For other computers, consult the Mathematica Help menu. If there is only one possible completion, Mathematica pastes it in at the insertion point; otherwise a pop-up menu listing the possible completions is activated. Clicking on the selection completes the command or function.

A.4.2.6. *Copying, cutting, and pasting.* Highlighted text or cells can be copied to a clipboard in the NeXT workstation with the key combination COMMAND-C. For other computers, consult the Mathematica Help Menu. (The clipboard is a part of the computer memory that temporarily holds text and graphics. It is not usually possible to view the clipboard directly.) The contents of the clipboard can be copied to another location in a Mathematica notebook with the key combination COMMAND-V. The key combination COMMAND-X is similar to COMMAND-C; it also transfers text or cells to a clipboard, but COMMAND-X deletes the original material.

A.4.2.7. *Printing.* To print a whole notebook, click the mouse on the Print icon. To print an individual cell, use the mouse to move the cursor to a cell bracket and click on it. Then move the cursor to the Print Selection icon and click the mouse button. This brings up a dialog box that allows the graphics to be sent to the printer.

A.4.2.8. *Saving.* The key combination COMMAND-S saves a notebook to disk; to save a notebook under a different name, use COMMAND-SHIFT-S. On all notebook versions of Mathematica except the Macintosh, a saved notebook consists of two files, one with the extension *.ma* and the other with the extension *.mb*. A *.ma* file is an ASCII file, which means that it can be accessed independently of Mathematica with a text editor (although a large number of formatting commands will be visible). On the other hand, a *.mb* file contains binary information. In most situations the information contained in a *.mb* file duplicates that of the corresponding *.ma* file, but the *.mb* information loads more quickly into a Mathematica session. To save disk space, *.mb* files can be deleted. Usually *.mb* files created by Mathematica on different computers are incompatible with one another.

A.4.2.9. *Graphics.* A notebook contains several different kinds of information. In addition to Mathematica definitions and textual information, a notebook can contain graphics. For example, Mathematica has many plotting commands, each of which produces a graphic cell that contains the plot. The key combinations COMMAND-C, COMMAND-X, and COMMAND-V work with graphics cells exactly the same way that they work with cells containing text.

A.4.3. Textual interface: Direct access through a terminal window. Except for older Macintosh versions, the notebook implementation of Mathematica is split into two parts: a kernel and a front end. The user communicates with the front end, which in turn sends messages back and forth to the kernel. It is also possible to access the Mathematica kernel directly through a terminal window. Moreover, for some implementations of Mathematica this is the only way to use the program. To use Mathematica in a terminal window, simply type **math**. Then all of the symbolic commands of Mathematica will be available to the user. There is also a primitive form of graphics available called terminal graphics. On most workstations graphics can also be displayed in a separate window. In the non-Windows version of Mathematica for PCs the display of graphics is full-screen.

The use of Mathematica in a terminal window is especially useful with modems. Typically a user works on a small computer but uses a modem to connect via a telephone line to a more powerful computer. When Mathematica is used in a terminal window that communicates over a telephone line to a large computer equipped with Mathematica, all of Mathematica's symbolic capabilities are available. Although the transfer of data via telephone lines is usually too slow for high-quality graphics, primitive graphics display is possible in a terminal window.

A.4.4. Mathematica notation versus ordinary mathematical notation.

A.4.4.1. *Parentheses.* Parentheses, as used in ordinary mathematics, have at least four distinct meanings according to the context. One of these is the notation for an open interval on the real number line:

$$(a, b) = \{\, t \in \mathbb{R} \mid a < t < b \,\}$$

This formalism is not needed by Mathematica. Different notation is employed by Mathematica for two of the other three uses of parentheses.

A.4.4.2. *Defining functions.* Mathematica usually uses $:=$ instead of $=$ in function definitions. Thus to define in Mathematica a function y that assigns a real number or some other expression, we use

$$y[t\_] := \textit{some expression in } t$$

Here the underscore _ after the t is an important part of the syntax. Whereas t denotes the symbol t, $t\_$ means generic t. Note that square brackets are used instead of parentheses. For example, if we define

```
y[t_]:= t^2 + 3t + 6
```

then we can use this form numerically. Thus

```
y[2]
```

elicits the response

```
16
```

On the other hand, if we evaluate y on a symbol, the result is a symbol; thus

```
y[s]
```

yields

```
6 + 3 s + s^2
```

A.4.4.3. *Multiplication and division.* Most of the mathematical operations, for example, addition (+), subtraction (-), division (/), and exponentiation (^), use the symbols that have become standard in modern programming languages, but multiplication in Mathematica is represented by either an asterisk or a space. For aesthetic reasons we usually prefer to denote multiplication by a space. However, it is sometimes necessary to use an asterisk for multiplication at the end of an intermediate line in the middle of a multiline expression.

Notice that xy represents a single expression in Mathematica; it is never the same as $x\ y$, when both x and y are symbols. But the following expressions are all the same in Mathematica:

```
2y        2 y        2*y
```

They all indicate multiplication of a symbol by a number. Note also that $x2$ denotes a single expression, whereas $x\ x$ denotes x times x.

A.4.4.4. *Universal constants and numerical values.* The numbers $e \approx 2.71828$, $\pi \approx 3.14159$, and $i = \sqrt{-1}$ are represented in Mathematica by **E**, **Pi**, and **I**. Mathematica distinguishes between symbolic values and numerical values. A numerical approximation to π can be found with the command N[**Pi**]:

```
3.14159
```

Mathematica gives exact results, whenever possible. Although Mathematica responds to **Sin**[1] by repeating the expression, Mathematica's answer to N[**Sin**[1]] is the six-digit decimal approximation

```
0.841471
```

Nevertheless, some expressions, such as **Sin**[0] and **Sin**[**Pi**/4], are automatically simplified, since their values are well known. Anytime a numerical value is required, apply the operator **N**.

A.4.4.5. *Lists.* A list consisting of three elements $(a1, a2, a3)$ is denoted in Mathematica by $\{a1, a2, a3\}$. Note that braces are used instead of parentheses. Thus a list of n numbers can be thought of as a point or vector in euclidean space $\mathbb{R}^n$.

A.4.4.6. *Internal functions.* All internal Mathematica functions begin with a capital letter. Thus the Mathematica notation for $\sin(x)$ is **Sin[x]**. A user-defined function can begin with either a lower- or an upper-case letter, but if it coincides with an internal function, Mathematica probably will not work properly. For this reason, it is a good idea to use lower-case letters to define new functions to avoid collision with Mathematica's internally defined functions.

A.4.4.7. *Differentiation.* In ordinary mathematics there are two symbols for differentiation. We write

$$\frac{d}{dt}$$

for the total derivative with respect to t, when t is the only independent variable. On the other hand,

$$\frac{\partial}{\partial t}$$

is used for the partial derivative with respect to t, when t is one of several independent variables. If we assume that $y = y(t)$ is differentiable with respect to t, then the total derivative of y with respect to t can be computed by applying the operator d/dt to y, usually written as

$$y' = \frac{dy}{dt}, \qquad y'' = \frac{d^2y}{dt^2}$$

and so forth. If y is a function of only one variable, then the partial derivative and the total derivative are the same.

Mathematica also has two symbols for differentiation: **Dt**, read "derivative total," and **D** for partial differentiation. In operator notation,

$$\frac{d}{dt} \longleftrightarrow \mathbf{Dt[\ ,t]}, \qquad \frac{d^n}{dt^n} \longleftrightarrow \mathbf{Dt[\ ,\text{-}t,n'']}$$
$$\frac{\partial}{\partial t} \longleftrightarrow \mathbf{D[\ ,t]}, \qquad \frac{\partial^n}{\partial t^n} \longleftrightarrow \mathbf{D[\ ,\text{-}t,n'']}$$

D is used much more frequently in Mathematica than **Dt**. To see the difference between **Dt[,t]** and **D[,t]**, we define a function $y = y(t)$ in Mathematica by

```
y[t_]:= t^2 + 3t + a
```

Then

```
Dt[y[t],t]
```

yields

```
3 + 2 t + Dt[a, t]
```

because **a** is considered a function of **t**. On the other hand,

```
D[y[t],t]
```

results in

```
3 + 2 t
```

because the use of **D[,t]** tells Mathematica to treat **t** as the only variable for the purposes of differentiation.

A.4.4.8. *The prime notation.* In the case of a function f of a single variable t, Mathematica has another notation for the derivative of **f** with respect to **t**, namely, **f'[t]**. This notation is useful since it imitates ordinary mathematical notation closely. For example, if we define

```
f[t_]:= a t^2 + b t + c
```

then Mathematica's response to **f'[t]** is

```
b + 2 a t
```

A.4.4.9. *Integration.* Just as in ordinary mathematics, there are two kinds of integration in Mathematica: indefinite integration and definite integration. To compute

$$\int t\, e^t \sin(t) dt$$

we use

```
Integrate[t E^t Sin[t],t]
```

The answer is

```
 t            t             t
E  Cos[t] - E  t Cos[t] + E  t Sin[t]
-------------------------------------
                  2
```

Mathematica suppresses the constant of integration. For definite integration Mathematica also uses the command **Integrate** but with a different syntax. To find

$$\int_0^1 t\sqrt{1-t}\, dt$$

we use

```
Integrate[t Sqrt[1-t],{t,0,1}]
```

to obtain

```
4
--
15
```

Many definite integrals are too complicated to compute symbolically. In such cases **NIntegrate** can be used. To find

$$\int_0^1 (\sin t)^{1/3} dt$$

we enter the command

```
NIntegrate[Sin[t]^(1/3),{t,0,1}]
```

The approximate value of the integral is

```
0.733269
```

The command **NIntegrate** can be quite slow.

A.4.4.10. *Simplifying expressions.* Mathematica's command **Simplify** attempts to simplify expressions. For example

```
D[(t^4 + 1)/(t^4 - 1),t]
```

gives the complicated answer

```
   3          3       4
 4 t       4 t  (1 + t )
------- - -------------
      4             4 2
-1 + t       (-1 + t )
```

We get a much nicer result when we apply **Simplify**:

```
Simplify[D[(t^4 + 1)/(t^4 - 1),t]]
```

```
      3
  -8 t
----------
       4 2
(-1 + t )
```

An equivalent form of **Simplify**[*expression*] is *expression*//**Simplify**, which is useful when one forgets to write **Simplify** before writing *expression*. Also useful is **Simplify**[%], which simplifies the previous input.

Mathematica contains several other commands that simplify in different ways. For example, **Together** combines fractions, and **Apart** rewrites a rational expression as a sum of terms with minimal denominators.

A.4.4.11. *Clearing values.* Mathematica retains a value assigned to a variable unless it is specifically cleared.

```
Clear[symbol1, symbol2,...]
```

clears values for the specified symbols. A useful shorthand for **Clear** is "=.". On the other hand,

```
Remove[symbol1, symbol2,...]
```

removes symbols completely, so that their names are no longer recognized by Mathematica. To clear or remove all variables starting with a lower-case letter use **Clear["Global"]** or **Remove["Global"]**.

A.4.4.12. *Solving equations.* Linear and other simple algebraic equations can be solved with **Solve**. Here are two examples:

```
Solve[{x + y == 1, x - y == 2},{x,y}]

        3        1
{{x -> -, y -> -(-)}}
        2        2

Solve[x^2 - 5 x + 6 == 0,x]

{{x -> 2}, {x -> 3}}
```

A.4.5. Functional notation in Mathematica. Mathematica, like ordinary mathematics, distinguishes between functions and the values of functions. **Exp** is a function whose value on **z** (which can be either a number or a symbol) is **Exp[z]**. (Note that **Exp[z]** is the same as **E^z**.) One way to generate functions is to use Mathematica's command **Function**. For example, the function that assigns z^2 to z is **Function[z,z^2]**.

EXERCISES A.4

Work the following exercises with Mathematica, using **Simplify** when it is appropriate.

1. Express the differential equation $y''(t) + t\,y'(t) + (1 + t^2)y(t) = t/(1 + t)$ using Mathematica's "prime" notation. Do not evaluate the expression.
2. Express the differential equation in Exercise 1 using the operator **D**. Do not evaluate the expression.
3. Define the function $f(t) = 1 + t + \sin(t) + e^t - \sqrt{5 - t^2}$ and compute the exact and numerical values of $f(2)$.
4. Define the function $f(t, c) = \cos(t) + t^{3/2} + \frac{2}{3}c$.

5. Compute the exact value of

$$\frac{1}{3 \cdot 5 \cdot 7 \cdot 9}$$

Then use N to find a numerical approximation.

6. Compute the following derivatives:
   (a) $(d/dt)[\log(\sin t)/\tan(t)]$ (b) $(d^2/dt^2)[(t^3+1)/(t \sin t)]$

7. Compute the following integrals:
   (a) $\int (t^3 e^{5t} \cos(t))dt$ (b) $\int_0^{\pi} (\sin t)^7 dt$

8. Verify the fundamental theorem of calculus for $g(t) = t e^t + \sec(t)$. [In other words, show that the derivative of the integral of $g(t)$ equals $g(t)$.]

9. Write the function f defined by $f(t) = t^3 + \tan(t)$ as a Mathematica function and compute the symbolic and numerical values of $f(\pi/4)$.

ANSWERS TO SELECTED EXERCISES

CHAPTER 0

Section 0.1.1

1. The most general first-order linear PDE in two variables is written in the form $a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)$ and contains four functions.
2. The most general first-order linear PDE in three variables is written in the form $a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z + d(x,y,z)u = e(x,y,z)$ and contains five functions.
3. The most general first-order linear homogeneous PDE in two variables is written $a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0$ and contains three functions.
4. The most general first-order linear homogeneous PDE in three variables is written $a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z + d(x,y,z)u = 0$ and contains four functions.

Section 0.1.4

1. $u(x;t) = T_1$
2. $u(x;t) = T_0 + \Phi(x - 1/h)$
3. $u(x;t) = \frac{T_0}{2+hL}[1 + h(L-x)] + \frac{T_1}{2+hL}[1 + hx]$

Section 0.1.5

1. Hyperbolic.
2. Elliptic.
3. Parabolic.
4. Elliptic if $x > 0$, hyperbolic if $x < 0$.

Section 0.2.3

1. (a)$X'' - 2\lambda X = 0$, $Y'' - \lambda Y = 0$
   (b) $X'' + 2X' + \lambda X = 0$, $Y'' - \lambda Y = 0$
   (c) $x^2 X'' - \lambda X = 0$, $2yY' - \lambda Y = 0$
   (d) $X'' + X' - \lambda X = 0$, $Y' + (\lambda - 1)Y = 0$
2. (b) is a solution of Laplace's equation.

3. The appropriate separated solutions are

$\lambda < 0 : u(x,y)=(A_1 e^{x(-1+\sqrt{1-\lambda})} + A_2 e^{x(-1-\sqrt{1-\lambda})})(A_3 \cos y\sqrt{-\lambda} + A_4 \sin y\sqrt{-\lambda})$

$\lambda = 0 : u(x,y)=(A_1 + A_2 e^{-2x})(A_3 + A_4 y)$

$0 < \lambda < 1 : u(x,y)=(A_1 e^{x(-1+\sqrt{1-\lambda})} + A_2 e^{x(-1-\sqrt{1-\lambda})})(A_3 e^{y\sqrt{\lambda}} + A_4 e^{-y\sqrt{\lambda}})$

$\lambda = 1 : u(x,y)=(A_1 e^{-x} + A_2 x e^{-x})(A_3 e^{y} + A_4 e^{-y})$

$\lambda > 1 : u(x,y)=(A_1 e^{-x} \cos x\sqrt{\lambda-1} + A_2 e^{-x} \sin x\sqrt{\lambda-1})(A_3 e^{y\sqrt{\lambda}} + A_4 e^{-y\sqrt{\lambda}})$

5. The appropriate separated solutions are

$\lambda < 0 : u(x,y) = [A_1 \cos(\sqrt{-\lambda} \ln|x|) + A_2 \sin(\sqrt{-\lambda} \ln|x|)](A_3 e^{y\sqrt{-\lambda}} + A_4 e^{-y\sqrt{-\lambda}})$

$\lambda = 0 : u(x,y) = (A_1 + A_2 \log|x|)(A_3 + A_4 y)$

$\lambda > 0 : u(x,y) = (A_1 |x|^{\sqrt{\lambda}}) + A_2 |x|^{-\sqrt{\lambda}})(A_3 \cos y\sqrt{\lambda} + A_4 \sin y\sqrt{\lambda})$

7. The appropriate separated solutions are

$\lambda > 0 : u(x,y) = (A_1 e^{x\sqrt{\lambda}} + A_2 e^{-x\sqrt{\lambda}})(1/|y|^{1+\lambda})$

$\lambda = 0 : u(x,y) = (A_1 + A_2 x)(1/y)$

$\lambda < 0 : u(x,y) = (A_1 \cos x\sqrt{-\lambda} + A_2 \sin x\sqrt{-\lambda})(1/|y|^{1+\lambda})$

Section 0.2.4

1. $u_n(x,y) = A_n \cos(n\pi x/L)\sinh(n\pi y/L),\ n = 1,2,\ldots, u_0(x,y) = Cy$
2. $u_n(x,y) = A_n \sin(n\pi x/L)e^{-(n\pi y/L)}, n = 1,2,\ldots$
3. $u_n(x,y) = A_n \sin(n\pi x/L)e^{-(n\pi/L)^2 t}, n = 1,2,\ldots$
4. $u_n(x,t) = A_n \cos(n\pi x/L)e^{-(n\pi/L)^2 t}, n = 0,1,2,\ldots$
5. $u_n(x,t) = A_n \sin((n-1/2)\pi x/L)e^{-((n-1/2)\pi/L)^2 t}, n = 1,2,\ldots$

Section 0.3

1. (a) $\langle\varphi_1,\varphi_2\rangle = \frac{1}{2}$ (b) $\langle\varphi_1,\varphi_3\rangle = \frac{1}{3}$
   (c) $||\varphi_1 - \varphi_2||^2 = \frac{1}{3}$ (d) $||\varphi_1 + 3\varphi_2||^2 = 7$
2. $\langle\varphi_1,\varphi_3\rangle = 0$, $\langle\varphi_1,\varphi_4\rangle = 0$, $\langle\varphi_2,\varphi_3\rangle = 0$, $\langle\varphi_3,\varphi_4\rangle = 0$. Therefore (φ_1,φ_3) are orthogonal, (φ_1,φ_4) are orthogonal, (φ_2,φ_3) are orthogonal, and (φ_3,φ_4) are orthogonal. All others are nonzero.
4. $2/\pi$; $d^2_{\min} = \frac{1}{2} - 4/\pi^2 = 0.0947$, $d_{\min} = 0.3078$
5. $\frac{1}{2} + \frac{1}{2}\cos 2x$
6. (b) $x/2|x|$; $d^2_{\min} = \frac{1}{6}, d_{\min} = 0.4082$
9. (d) $\cos\theta = \sqrt{3}/2, \theta = \pi/6$

10. (a) $\psi_1(x) = 1, \psi_2(x) = x, \psi_3(x) = x^2 - \frac{1}{3}$
(b) $\sqrt{\frac{1}{2}},\ x\sqrt{\frac{3}{2}},\ (x^2 - \frac{1}{3})\sqrt{\frac{45}{8}}$

CHAPTER 1

Section 1.1

1. $\frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2} (-1)^n \cos\frac{n\pi x}{L}$
2. $2L^3 \sum_{n=1}^{\infty} (-1)^n \left[\frac{6}{(n\pi)^3} - \frac{1}{n\pi}\right] \sin\frac{n\pi x}{L}$
3. $\frac{L^3}{4} + 2L^3 \sum_{n=1}^{\infty} \left[\frac{3(-1)^n}{(n\pi)^2} + \frac{6[1-(-1)^n]}{(n\pi)^4}\right] \cos\frac{n\pi x}{L}$
4. $\frac{\sinh L}{L} \left[1 + 2\sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\pi x/L) - (n\pi/L)\sin(n\pi x/L)}{1+(n\pi/L)^2}\right]$
5. $\frac{1}{2} - \frac{1}{2}\cos 4x$
6. $\frac{1}{4}\cos 3x + \frac{3}{4}\cos x$
7. $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin\frac{n\pi x}{L}$
8. $\frac{L}{4} + \sum_{n=1}^{\infty} \left[\frac{L(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{L} - \frac{L}{(n\pi)^2}[1 - (-1)^n] \cos\frac{n\pi x}{L}\right]$
9. $\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1}[1 + (-1)^n]$
10. $\frac{2\sinh\pi}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n\sin nx}{1+n^2}$

14. $f_1(x) = \frac{f(x)-f(-x)}{2}, \quad f_2(x) = \frac{f(x)+f(-x)}{2}$
15. (b),(c),(g),(h) are even. (a),(e),(f) are odd. (d) is neither.
16. (a) $\frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{L}$
(b) $2L^2 \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}}{n\pi} - \frac{2[1-(-1)^n]}{(n\pi)^3}\right] \sin\frac{n\pi x}{L}$
(c) $\frac{2\pi}{L^2}\sum_{n=1}^{\infty} n \left[\frac{1-e^L(-1)^n}{1+(n\pi/L)^2}\right] \sin\frac{n\pi x}{L}$
(d) Same as Exercise 2.
17. (a) $\frac{L}{2} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\frac{n\pi x}{L}$
(b) Same as Exercise 1.
(c) $\frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \left[\frac{(-1)^n e^L - 1}{1+(n\pi/L)^2}\right] \cos\frac{n\pi x}{L}$
(d) Same as Exercise 3.

21. $1 = 1; 1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right);$
$1 = \frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right);$
$1 = \frac{8}{\pi}\left(\frac{\sin 2x}{2} + \frac{\sin 6x}{6} + \frac{\sin 10x}{10} + \cdots\right) \qquad 0 < x < \frac{\pi}{2}$
22. Yes, period is 2.
23. Yes, period is 2π.
24. Yes, period is π.
25. Not periodic.
26. Yes, period is 1.
27. Yes, period is π.
28. Yes, period is 2π.

29. Not periodic.
30. $-\frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin\frac{n\pi x}{L} = x^2 - Lx, \qquad 0 \le x \le L$
31. $\frac{48L^4}{\pi^5}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^5}\sin\frac{n\pi x}{L} = x^4 - 2Lx^3 + L^3x, \qquad 0 \le x \le L$

Section 1.2

1. (b), (d) are piecewise smooth.
9. (a) 1 (b) $\frac{1}{2}$ (c) 1 (d) 0
10. $(1/\pi)(N + \frac{1}{2})$
11. $u = n\pi/(N + \frac{1}{2})$ for $n = \pm 1, \pm 2, \ldots$
12. For $N = 1$, $D_N'(u) = 0$ at $u = 0, \pm\pi$
For $N = 2$, $D_N'(u) = 0$ at $u = 0, \pm\cos^{-1}(-\frac{1}{4}), \pm\pi$
16. (b) $\pi^2/12$ (c) $\pi^2/6$ (d) $\pi^2/8$
17. Zero.
18. $\cosh\pi = \frac{1}{2}(e^{\pi} + e^{-\pi})$
22. $n > 4$
23. $A_n = -\frac{1}{n\pi}\sum_{-\pi<x_i\le\pi}\sin(nx_i)[f(x_i+0) - f(x_i-0)] + O\left(\frac{1}{n^2}\right), \quad n \to \infty$
$B_n = \frac{1}{n\pi}\sum_{-\pi<x_i\le\pi}\cos(nx_i)[\bar{f}(x_i+0) - \bar{f}(x_i-0)] + O\left(\frac{1}{n^2}\right), \quad n \to \infty$

Section 1.3

2. $k = 2, 1.42$; $k = 3, 1.67$; $k = 4, 1.49$
11. $x^2 = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left[1 - \cos\frac{n\pi x}{L}\right] = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{L}$
12. $\frac{x^3}{3} - \frac{L^2x}{3} = \frac{4L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L}, \qquad 0 < x < L$
13. The series for x^2 and $x^3 - L^2x$ are uniformly convergent. The series for x is not uniformly convergent, since the sum of the series is discontinuous at $x = \pm L$.

Section 1.4

1. $\sigma_N^2 = \frac{2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{[(-1)^n-1]^2}{n^2}$
2. $\sigma_N^2 = 8\sum_{n=N+1}^{\infty}\frac{1}{n^4}$
3. $\sigma_N^2 = 0 \qquad$ for $N \ge 10$
4. $\pi^2/8 = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$
5. $\pi^4/90 = 1 + \frac{1}{16} + \frac{1}{81} + \cdots$
8. (a) $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin nx$ (b) $\frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2}\cos nx$
(c) σ_N^2 (sine series) $= \frac{8}{\pi^2}\sum_{n=N+1}^{\infty}\frac{[1-(-1)^n]^2}{n^6} = O(N^{-5})$,
σ_N^2 (cosine series) $= 2\sum_{n=N+1}^{\infty}\frac{[1+(-1)^n]^2}{n^4} = O(N^{-3})$
10. (a) $\frac{1}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2-n^2}\cos nx$ (b) $\sigma_N^2 = O(N^{-3})$
16. $\sum_{n=N+1}^{\infty}e^{-n} = \frac{e^{-N}}{e-1}$ whereas $\int_N^{\infty}e^{-x}dx = e^{-N}$, so that the series is asymptotic to the integral multiplied by the constant factor $1/(e-1)$, in contrast

to the case of a power $f(x) = x^{-s}, s > 1$, when the integral and the series are asymptotic to one another, with the constant $= 1$.

17. $\frac{P^2}{A} = 12\sqrt{3}$
18. $\frac{P^2}{A} = 16$
19. $\frac{P^2}{A} = 4n \tan \frac{\pi}{n}$

Section 1.5

3. $e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{L+in\pi}{L^2+n^2\pi^2} (\sinh L) e^{\frac{in\pi}{L} x}, \qquad -L < x < L$
4. $\frac{1}{1-re^{ix}} = \sum_{n=0}^{\infty} r^n e^{inx}, \qquad -1 < r < 1, -\infty < x < \infty$

Section 1.6

1. $\varphi_n(x) = A \sin((n - \frac{1}{2})\pi x/L)),\ \lambda_n = ((n - \frac{1}{2})\pi/L)^2, n = 1, 2, \ldots$ and A is an arbitrary constant.
2. $\varphi_n(x) = A(\sqrt{\lambda_n} \cos(x\sqrt{\lambda_n}) + h \sin(x\sqrt{\lambda_n}))$ where λ_n is a root of the transcendental equation $2h\sqrt{\lambda} \cos(L\sqrt{\lambda}) + (h^2 - \lambda) \sin(L\sqrt{\lambda}) = 0,\ n = 1, 2, \ldots$ and A is an arbitrary constant.
3. $\varphi_n(x) = A \cos((n - \frac{1}{2})\pi x/L),\ \lambda_n = ((n - \frac{1}{2})\pi/L)^2, n = 1, 2, \ldots$ and A is an arbitrary constant.
4. $\varphi_n(x) = A \cos(2n\pi x/L) + B \sin(2n\pi x/L),\ \lambda_n = (2n\pi/L)^2, n = 1, 2, \ldots,$ where A, B are arbitrary constants.
5. $\varphi_n(x) = A \sin(x\sqrt{\lambda_n})$, where λ_n is a root of the transcendental equation $\tan(L\sqrt{\lambda}) = \sqrt{\lambda},\ n = 1, 2, \ldots$ and A is an arbitrary constant.
6. $\varphi_n(x) = A\left(\sin(x\sqrt{\lambda}) + \sqrt{\lambda} \cos(x\sqrt{\lambda})\right)$, where λ_n is a root of the transcendental equation $\cos(L\sqrt{\lambda}) - \sqrt{\lambda} \sin(L\sqrt{\lambda}) = 0\ n = 1, 2, \ldots$ and A is an arbitrary constant.
9. (a) The point $(\pi/4, \pi/2)$ is in the region $\lambda_1 > 0$ of Fig. 1.6.2; therefore there are no negative eigenvalues in this case.
(b) The point $(\pi/4, 3\pi/4)$ is in the region $\lambda_1 < 0 < \lambda_2$ of Fig. 1.6.2; therefore there is one negative eigenvalue in this case.
(c) The point $(7\pi/8, 7\pi/8)$ is in the region $\lambda_1 < \lambda_2 < 0$ of Fig. 1.6.2; therefore there are two negative eigenvalues in this case.

CHAPTER 2

Section 2.1

1. $U(z) = T_1 + (z/L)(T_2 - T_1)$
2. $\Phi = -(k/L)(T_2 - T_1)$
3. $U(z) = \Phi(z - L) + T_0$
4. $U(z) = \{T_1(k + hz) + T_0[k + h(L - z)]\}/(2k + hL)$

5. $U(z) = T_3 + (T_2 - T_3)\frac{\sinh z\sqrt{\beta/K}}{\sinh L\sqrt{\beta/K}} + (T_1 - T_3)\frac{\sinh(L-z)\sqrt{\beta/K}}{\sinh L\sqrt{\beta/K}}$
6. $U(z) = T_1 + (r/2K)(L^2 - z^2);\qquad \Phi\,|_{z=L} = krL/K$
7. $U(z) = \frac{-rz^2}{2K} + \left(\frac{T_2-T_1}{L} + \frac{rL}{2K}\right) z + T_1$
8. $U(z) = \frac{r_0Lz}{6K},\quad 0 < z < \frac{L}{3};\ U(z) = \frac{r_0z(L-z)}{2K} - \frac{r_0L^2}{18K},\ \frac{L}{3} < z < \frac{2L}{3};$
   $U(z) = \frac{r_0L(L-z)}{6K},\quad \frac{2L}{3} < z < L$
9. 0.001792 cal/s-cm^2
10. The solution is

$$u(z;t) = A_0 + A_1 \exp\left[-z\sqrt{\frac{\pi}{K\tau_1}}\right]\cos\left(\frac{2\pi t}{\tau_1} - z\sqrt{\frac{\pi}{K\tau_1}}\right)$$
$$+ A_2 \exp\left[-z\sqrt{\frac{\pi}{K\tau_2}}\right]\cos\left(\frac{2\pi t}{\tau_2} - \sqrt{\frac{\pi}{K\tau_2}}\,z\right)$$

12. $u(z;t) = e^{\pm(1+i)\sqrt{\beta/2K}\,z}e^{i\beta t}$
14. $u(z;t) = A_0\left(1 - \frac{z}{L}\right) + A_1\mathrm{Re}\left[\frac{e^{-c(z-L)}e^{i[\beta t - c(z-L)]} - e^{c(z-L)}e^{i[\beta t + c(z-L)]}}{e^{cL}e^{icL} - e^{-cL}e^{-icL}}\right],$
   $\beta = \frac{2\pi}{\tau},\quad c = \sqrt{\frac{\beta}{2K}}$
15. $\sqrt{\pi K\tau} = 23.3$ cm
16. $u(z;t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\exp\left[-z\sqrt{\frac{n\pi}{K\tau}}\right]\left[\frac{1-(-1)^n}{n}\right]\sin\left(\frac{2n\pi t}{\tau} - z\sqrt{\frac{n\pi}{K\tau}}\right)$
17. $u(z;t) = -(A_1/2c)e^{-cz}[\cos(\beta t - cz) + \sin(\beta t - cz)],\ c = \sqrt{\frac{\beta}{2K}}$
18. $u(z;t) = -A_1e^{-cz}\frac{(c+h)\cos(\beta t - cz) + \sin(\beta t - cz)}{c^2 + (c+h)^2},\ c = \sqrt{\frac{\beta}{2K}}$
19. $u(z;t) = A_0 + A_1e^{-cz}\cos\left(\frac{2\pi t}{\tau} - z\sqrt{\frac{\pi}{K\tau}}\right)$
20. $U(z) = \frac{rL}{2Kh} + \frac{rz}{2K}(L - z) + \frac{T_1[1 + h(L-z)]}{2+Lh} + \frac{T_2[1+hz]}{2+Lh}$
21. The constants must satisfy $K(\Phi_2 - \Phi_1) + rL = 0$.

Section 2.2

1. $u(z;t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$
2. $u(z;t) = \frac{2T}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right)\sin\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$
3. $u(z;t) = 3\sin\frac{\pi z}{2L}\exp\left[-\left(\frac{\pi}{2L}\right)^2 Kt\right] + 5\sin\frac{3\pi z}{2L}\exp\left[-\left(\frac{3\pi}{2L}\right)^2 Kt\right]$
4. $u_n(z;t) = \cos\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right],\qquad n = 0, 1, 2, \ldots$
5. $u(z;t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)\pi z/L]}{(2n-1)^2}\exp\left[-\frac{(2n-1)^2\pi^2 Kt}{L^2}\right]$
6. $u(z;t) = 3 + 4\cos\frac{\pi z}{L}\exp\left[-\left(\frac{\pi}{L}\right)^2 Kt\right] + 7\cos\frac{3\pi z}{L}\exp\left[-\left(\frac{3\pi}{L}\right)^2 Kt\right]$
9. $u(z;t) = \sum_{n=1}^{\infty}B_n\sin z\sqrt{\lambda_n}\,e^{-\lambda_n Kt},\quad \sqrt{\lambda_n} = -h\tan L\sqrt{\lambda_n},$
   $B_n = \frac{4(1-\cos L\sqrt{\lambda_n})}{2L\sqrt{\lambda_n} - \sin(2L\sqrt{\lambda_n})}$

10. $u_n(z;t) = [h\sin(z\sqrt{\lambda_n}) + \sqrt{\lambda_n}\cos(z\sqrt{\lambda_n})]e^{-\lambda_n Kt}$,
 $\tan(L\sqrt{\lambda_n}) = 2h\sqrt{\lambda_n}/(\lambda_n - h^2)$
11. $u(z;t) = \sum_{n=1}^{\infty} A_n u_n(z;t), \quad A_n = \int_0^L u_n(z;0)dz / \int_0^L u_n(z;0)^2 dz$
12. $\tau = L^2/\pi^2 K$
13. $\tau = L^2/\pi^2 K$
14. $\tau = 4L^2/\pi^2 K$
15. $\tau = (1/\lambda_1 K), \quad \lambda_1 =$ smallest positive root of the equation $\sqrt{\lambda} = -h\tan L\sqrt{\lambda}$
16. $\tau = 1080$ s
18. $u_n(z;t) = \left(A_n \cos\frac{2n\pi z}{L} + B_n \sin\frac{2n\pi z}{L}\right)\exp\left[-\left(\frac{2n\pi}{L}\right)^2 Kt\right]; n = 0,1,2,\ldots$
19. $u(z;t) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin\frac{2n\pi z}{L}\exp\left[-\left(\frac{2n\pi}{L}\right)^2 Kt\right]$
20. $\tau_{\text{ring}} = \frac{L^2}{4\pi^2 K} = \frac{1}{4}\tau_{\text{slab}}$

Section 2.3

1. $u(z;t) = T_1 + \Phi_2 z + \sum_{n=1}^{\infty} A_n \sin\frac{(n-\frac{1}{2})\pi z}{L}\exp\left\{-\left[\frac{(n-\frac{1}{2})\pi}{L}\right]^2 Kt\right\}$
 $A_n = \frac{2(T_3-T_1)}{(n-\frac{1}{2})\pi} - \frac{2L\Phi_2(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2}, \qquad \tau = \frac{4L^2}{\pi^2 K}$
2. $u(z;t) = T_2 + \sum_{n=1}^{\infty} A_n \cos(z\sqrt{\lambda_n})e^{-\lambda_n Kt}, \qquad \sqrt{\lambda_n} = h\cot L\sqrt{\lambda_n}$
 $A_n = \frac{2(T_3-T_2)\sin(L\sqrt{\lambda_n})}{L\sqrt{\lambda_n}+\sin(L\sqrt{\lambda_n})\cos(L\sqrt{\lambda_n})}$
3. $u(z;t) = \frac{rz}{2K}(L-z) + T_1 + \frac{(T_2-T_1)z}{L} + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$
 $A_n = \frac{2(T_3-T_1)}{\pi}\frac{1-(-1)^n}{n} - \frac{2}{\pi}(T_2-T_1)\frac{(-1)^{n+1}}{n} - \frac{2L^2 r}{K\pi^3}\frac{1-(-1)^n}{n^3}$
4. $u(z;t) = 273 + \frac{768L}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$
5. $u(z;t) = T_3 + \left(z - \frac{1}{2}L\right)\Phi + \frac{2L\Phi}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$
6. $U(z) = (a/6)(L^2 z - z^3) \qquad \tau = L^2/\pi^2 K$
 $u(z;t) = U(z) + \frac{2aL^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(\sin\frac{n\pi z}{L}\right)e^{-(n\pi/L)^2 Kt}$
7. $u(z;t) = \frac{Azt}{L} - \frac{A}{6L}(L^2 z - z^3) - \frac{2AL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(n\pi z/L)e^{-(n\pi/L)^2 Kt}$
11. $u(z;t) = v\sin(\pi z/L)\frac{e^{-at}-e^{-\pi^2 Kt/L^2}}{a-(\pi^2 K/L^2)} \qquad a \neq \pi^2 K/L^2$
 $u(z;t) = v\sin(\pi z/L)\, t\, e^{-\pi^2 Kt/L^2} \qquad a = \pi^2 K/L^2$
12. $u(z;t) = \frac{8v}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m \sin(\sqrt{\lambda_m}z)}{(2m-3)(2m+1)}\frac{e^{-\lambda_m Kt}-e^{-at}}{a-\lambda_m K}$ if $a \neq \lambda_m K$,
 where $\lambda_m = [(m-\frac{1}{2})\pi/L]^2$. If $a = \lambda_m K$ for some m, then the corresponding second factor in the series is replaced by $te^{-\lambda_m Kt}$.

Section 2.4

1. $y(s;L/2c) = 0$ for $0 < s < L$
2. $B_{2n+1} = 0$ for $n = 0,1,2,\ldots$
3. $B_{3n+1} = 0$, $B_{3n+2} = 0$ for $n = 0,1,2,\ldots$

4. $E = \frac{L}{4}\sum_{n=1}^{\infty}\left[\rho\omega_n^2\tilde{B}_n^2 + T_0\left(\frac{n\pi}{L}\right)^2\tilde{A}_n^2\right]$
6. $E = \frac{4T_0L}{\pi^2}\left(1+\frac{1}{9}+\frac{1}{25}+\cdots\right) = \frac{T_0L}{2} := \sum_{n=1}^{\infty}E_n,\ \sum_{n>1}E_n = 0.189E$
13. (a) $y(s;t) = \frac{2L}{\pi^2c}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin\frac{n\pi s}{L}\sin\frac{n\pi ct}{L}$
18. $u(x;t) = \frac{e^{-at}}{2}\left[g_1(x+ct)+g_1(x-ct)\right] + \frac{ae^{-at}}{2c}\int_{x-ct}^{x+ct}g_1(s)\,ds$
19. $u(x;t) = \frac{e^{-at}}{2c}\int_{x-ct}^{x+ct}g_2(s)\,ds$
20. $y(s;t) = \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{[(n\pi c/L)^2-\omega^2]\cos\omega t+(2a\omega)\sin\omega t}{[(n\pi c/L)^2-\omega^2]^2+(2a\omega)^2}\sin\left(\frac{n\pi s}{L}\right)\left[1-(-1)^n\right]/n$

Section 2.5

1. $u(x,y;t) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{\sin[(m-\frac{1}{2})(\pi x/L_1)]}{m-\frac{1}{2}}\frac{\sin[(n-\frac{1}{2})(\pi y/L_2)]}{n-\frac{1}{2}}e^{-\lambda_{mn}Kt}$
   $\lambda_{mn} = (m-\frac{1}{2})^2(\pi/L_1)^2 + (n-\frac{1}{2})^2(\pi/L_1)^2$,
   $\tau = (4/\pi^2K)[L_1^2L_2^2/(L_1^2+L_2^2)]$
2. $u(x,y;t) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1-(-1)^m}{m}\sin\frac{m\pi x}{L_1}\exp\left[-\left(\frac{m\pi}{L_1}\right)^2Kt\right],\ \tau = \frac{L_1^2}{\pi^2K}$
3. $u_0(x,y) = Ay+B$, $u_n(x,y) = \cos(n\pi x/L_1)[A\cosh(n\pi y/L_1)+B\sinh(n\pi y/L_1)]$, $n = 1,2,\ldots$
4. $u(x,y) = \frac{yL_1}{2L_2} - \frac{2L_1}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]\cos(n\pi x/L_1)\sinh(n\pi y/L_1)}{n^2\sinh(n\pi L_2/L_1)}$
5. $u(x,y) = y/L_2$
6. $u(x,y) = yT_2/L_2 + (L_2-y)T_1/L_2$
7. $u_{mn}(x,y,z) = \sin[(m-\frac{1}{2})(\pi x/L)]\sin[(n-\frac{1}{2})(\pi y/L)]$
   $\times\{A\cosh(\pi z/L)\sqrt{(m-\frac{1}{2})^2+(n-\frac{1}{2})^2}+B\sinh(\pi z/L)\sqrt{(m-\frac{1}{2})^2+(n-\frac{1}{2})^2}\}$
8. $u_{00}(x,y,z) = Az+B$, $u_{mn}(x,y) = \cos\frac{m\pi x}{L}\cos\frac{n\pi y}{L}[A\cosh\left(\frac{\pi z}{L}\sqrt{m^2+n^2}\right)$
   $+B\sinh\left(\frac{\pi z}{L}\sqrt{m^2+n^2}\right)]$, $m,n = 0,1,2,\ldots$ with $m^2+n^2\neq 0$
9. $u(x,y) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{\sin[(m-\frac{1}{2})(\pi x/L)]\sin[(n-\frac{1}{2})(\pi y/L)]\sinh[(\pi z/L)\sqrt{(m-\frac{1}{2})^2+(n-\frac{1}{2})^2}]}{(m-\frac{1}{2})(n-\frac{1}{2})\sinh(\pi\sqrt{(m-\frac{1}{2})^2+(n-\frac{1}{2})^2})}$
10. $u(x,y,z) = 1$
11. $u(x,y;t) = \frac{T_2y}{L_2} + \frac{(L_2-y)T_1}{L_2} + \sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{L_2}\exp\left[-\left(\frac{n\pi}{L_2}\right)^2Kt\right]$
    $A_n = \frac{2(T_3-T_1)[1-(-1)^n]}{n\pi} + \frac{2(T_1-T_2)(-1)^{n+1}}{n\pi}$
12. $u(x,y;t) = U(x,y) - \frac{4T_1}{\pi^2}\sum_{m,n=1}^{\infty}\frac{1-(-1)^m}{m}\frac{n(-1)^{n+1}}{m^2+n^2}\sin\frac{m\pi x}{L}\sin\frac{n\pi y}{L}e^{-\lambda_{mn}Kt}$,
    $\lambda_{mn} = \frac{(m^2+n^2)\pi^2}{L^2}$, $\tau = \frac{L^2}{2K\pi^2}$
13. $u(x,y;t) = 3\sin(\pi x/L)\sin(2\pi y/L)\cos[\pi ct\sqrt{5}/L]$
    $+4\sin(3\pi x/L)\sin(5\pi y/L)\cos(\pi ct\sqrt{34}/L)$
14. $u_{mn}(x,y;t) = \cos(m\pi x/L)\cos(n\pi y/L)\sin[(\pi ct/L)\sqrt{m^2+n^2}]$, $(m,n)\neq(0,0)$
    $u_{00}(x,y;t) = t$
15. $\frac{\pi c}{L}, \frac{\pi c}{L}, \frac{\pi c}{L}\sqrt{2}, \frac{\pi c}{L}\sqrt{4}, \frac{\pi c}{L}\sqrt{4}, \frac{\pi c}{L}\sqrt{5}, \frac{\pi c}{L}\sqrt{5}, \frac{\pi c}{L}\sqrt{8}, \frac{\pi c}{L}\sqrt{9}, \frac{\pi c}{L}\sqrt{9}$
17. $\frac{\pi c}{L}\sqrt{5}, \frac{\pi c}{L}\sqrt{10}, \frac{\pi c}{L}\sqrt{13}, \frac{\pi c}{L}\sqrt{17}, \frac{\pi c}{L}\sqrt{20}, \frac{\pi c}{L}\sqrt{25}, \frac{\pi c}{L}\sqrt{26}, \frac{\pi c}{L}\sqrt{29}, \frac{\pi c}{L}\sqrt{34}, \frac{\pi c}{L}\sqrt{41}$

CHAPTER 3

Section 3.1

1. $12\rho^2 \cos 2\varphi$
2. 0
3. $n^2 \rho^{n-2}$
4. $(n^2 - m^2)\rho^{n-2} \cos m\varphi$
5. $e^\rho \cos\varphi + (1/\rho)e^\rho \cos\varphi - (1/\rho^2)e^\rho \cos\varphi$
6. 1, 2, 3 if n is even; 4 if $n > m$ and $n - m$ is even.
9. $f(\rho) = A \ln \rho + B, \rho \neq 0$
10. $f(\rho) = -\frac{1}{4}\rho^2 + A \ln \rho + B, \rho \neq 0$
11. $f(\rho) = \frac{2}{\ln 2} \ln \rho + 3$
12. $f(\rho) = -\frac{1}{4}\rho^2 + \frac{7 \ln \rho}{4 \ln 2} + \frac{1}{4}$
13. $u(\rho, \varphi) = 1 + \left(\frac{\rho}{R}\right)^2 \cos 2\varphi + 3\left(\frac{\rho}{R}\right)^3 \sin 3\varphi$
14. $u(\rho, \varphi) = \frac{\ln \rho}{\ln 2} - \frac{\rho^2}{15} \cos 2\varphi + \frac{16 \cos 2\varphi}{15\rho^2}$
15. $u(\rho, \varphi) = \frac{\ln \rho}{2 \ln 2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{\rho^n - \rho^{-n}}{2^n - 2^{-n}}\right] \left[\frac{1-(-1)^n}{n}\right] \sin n\varphi$
16. $u(\rho, \varphi) = 3 + 4\left(\frac{R}{\rho}\right)^2 \cos 2\varphi + 5\left(\frac{R}{\rho}\right)^3 \sin 3\varphi$
18. $u_n(\rho, \varphi) = \rho^{2n} \sin 2n\varphi, \quad n = 1, 2, 3, \ldots$
19. $u(\rho, \varphi) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \rho^{2n} \sin 2n\varphi$
20. $u_n(\rho, \varphi) = \rho^n \cos n\varphi, \quad n = 0, 1, 2, 3, \ldots$
21. $u(\rho, \varphi) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n^2} \rho^n \cos n\varphi$

Section 3.2

8. $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots, \qquad J_1(x) = +\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18{,}432} + \cdots$
28. $A_n = \frac{128/x_n^5 - 16/x_n^3}{J_1(x_n)}$
29. $P_6(x) = \frac{19 - 27x^2 + 9x^4 - x^6}{4608}$
30. $(1 - x^2)^3 = 4608 P_6(x) - 768 P_4(x), \qquad A_n = \frac{4608/x_n^7 - 768/x_n^5}{J_1(x_n)}$
31. $P_8(x) = \frac{211 - 304x^2 + 108x^4 - 16x^6 + x^8}{294{,}912}, \; P_{10}(x) = \frac{3651 - 5275x^2 + 1900x^4 - 300x^6 + 25x^8 - x^{10}}{29{,}491{,}200}$
32. $A_n = 2/x_n J_2(x_n)$
33. $F_3(\rho) = \rho(1 - \rho^2)/16$
34. $F_5(\rho) = \rho(1 - \rho^2)(2 - \rho^2)/384$

Section 3.3

1. $U(\rho) = (g/4c^2)(\rho^2 - a^2)$
4. $u(\rho, \varphi; t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{a}\right) \cos \frac{ctx_n}{a}, \qquad J_0(x_n) = 0,$
   $A_n = \frac{2}{a^2 J_1(x_n)^2} \int_0^a F_1(\rho) J_0\left(\frac{\rho x_n}{a}\right) \rho \, d\rho$
5. $u(\rho, \varphi; t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{a}\right) \sin \frac{ctx_n}{a}, \quad A_n = \frac{2}{acx_n J_1(x_n)^2} \int_0^a F_2(\rho) J_0\left(\frac{\rho x_n}{a}\right) \rho \, d\rho$

6. $u(\rho,\varphi;t) = \frac{2a}{c}\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/a)}{x_n^2 J_1(x_n)} \sin\frac{ctx_n}{a}, \qquad J_0(x_n)=0$
7. $u(\rho,\varphi;t) = \frac{8a^3}{c}\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/a)}{x_n^4 J_1(x_n)} \sin\frac{ctx_n}{a}, \qquad J_0(x_n)=0$
8. $u(\rho,\varphi;t) = (a/cx_1^{(3)})J_3(\rho x_1^{(3)}/a)\sin(ctx_1^{(3)}/a)\cos 3\varphi$

Section 3.4 In the following answers, one may set $R = \rho_{\max}$.

1. $u(\rho,\varphi;t) = 8R^2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/R)}{x_n^3 J_1(x_n)} \exp\left[-\frac{x_n^2 Kt}{R^2}\right]$, where $J_0(x_n)=0$
2. $u(\rho,\varphi;t) = 1 - 2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/R)}{x_n J_1(x_n)} \exp\left[-\frac{x_n^2 Kt}{R^2}\right]$, where
   $J_0(x_n)=0$; $\tau = (0.1736)\frac{R^2}{K}$
3. $u(\rho,\varphi;t) = 1 + \frac{\rho}{2R}\cos\varphi - 2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n^{(0)}/R)}{x_n^{(0)} J_1(x_n^{(0)})} \exp\left[-\frac{[x_n^{(0)}]^2 Kt}{R^2}\right]$
   $-\cos\varphi\sum_{n=1}^{\infty} \frac{J_1(\rho x_n^{(1)}/R)}{x_n^{(1)} J_2(x_n^{(1)})} \exp\left[-[x_n^{(1)}]^2\frac{Kt}{R^2}\right]$
   where $J_0(x_n^{(0)})=0$, $J_1(x_n^{(1)})=0$
4. $u(\rho,\varphi;t) = T_1 + \frac{\sigma(R^2-\rho^2)}{4K} + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{R}\right)\exp\left[-\frac{x_n^2 Kt}{R^2}\right]$,
   where $J_0(x_n)=0$, $A_n = \frac{8[T_2-\sigma R^2/4K]}{x_n^3 J_1(x_n)} - \frac{2T_1}{x_n J_1(x_n)}$
5. $u_{mn}(\rho,\varphi;t) = J_m(\rho x_n^{(m)}/R)\sin m\varphi\exp[-(x_n^{(m)})^2 Kt/R^2]$,
   where $m=1,2,\ldots,J_m(x_n^{(m)})=0$
6. $u(\rho,\varphi;t) = \sum_{m,n=1}^{\infty} A_{mn} J_m\left(\frac{\rho x_n^{(m)}}{R}\right)\sin m\varphi\exp\left[-(x_n^{(m)})^2\frac{Kt}{R^2}\right]$,
   $A_{mn} = \frac{4[1-(-1)^m]}{mR^2\pi J_{m+1}(x_n^{(m)})^2}\int_0^R J_m\left(\frac{\rho x_n^{(m)}}{R}\right) f(\rho)\rho\, d\rho$
7. $U(\rho) = 100\frac{\ln(\rho/3)}{\ln(5)}, \qquad \tau = \frac{15.6}{K}$
8. $u(\rho;t) = 100 - \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{2}\right)\exp\left[-(x_n)^2\frac{Kt}{4}\right]$,
   $A_n = \frac{100}{x_n J_1(x_n)} + \frac{50 J_1(x_n/2)}{x_n J_1(x_n)^2}$, $J_0(x_n)=0$
9. $u_{mn}(\rho,\varphi;t) = J_m\left(\frac{\rho x_n^{(m)}}{R}\right)(A\cos m\varphi + B\sin m\varphi)\exp\left[-(x_n^{(m)})^2\frac{Kt}{R^2}\right]$,
   $J_m'(x_n^{(m)})=0$, $m=0,1,2,\ldots;\quad n=1,2,\ldots$
10. $u(\rho,\varphi;t) = 2R\sin\varphi\sum_{n=1}^{\infty} \frac{J_1(\rho x_n^{(1)}/R)}{x_n J_2(x_n)}\exp\left[-(x_n)^2\frac{Kt}{R^2}\right]$,
    where $J_1(x_n)=0$
11. $u(\rho,\varphi;t) = 2R^2\sum_{n=1}^{\infty} \frac{J_2(\rho x_n/R)}{x_n J_3(x_n)}\cos 2\varphi\exp\left[-x_n^2\frac{Kt}{R^2}\right]$, where $J_2(x_n)=0$
12. $u(\rho,\varphi;t) = T_1 + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{R}\right)\exp\left[-x_n^2\frac{Kt}{R^2}\right]$,
    where $A_n = \frac{1}{[1+(x_n k/hR)^2]J_1(x_n)}\left(\frac{8T_2}{x_n^3} - \frac{2T_1}{x_n}\right)$
    and $kx_n^{(m)}J_m'(x_n^{(m)}) + hRJ_m(x_n^{(m)}) = 0$

Section 3.5

1. $u(\rho,\varphi,z) = 2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/\rho_{\max})\sinh(x_n(L-z)/\rho_{\max})}{x_n J_1(x_n)\sinh(x_n L/\rho_{\max})}$

2. $u(\rho,\varphi,z) = 2T_2\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/\rho_{\max})\sinh(z x_n/\rho_{\max})}{x_n J_1(x_n)\sinh(L x_n/\rho_{\max})}$ $+ 2T_1\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/\rho_{\max})\sinh[(L-z)x_n/\rho_{\max}]}{x_n J_1(x_n)\sinh(L x_n/\rho_{\max})}$, where $J_0(x_n) = 0$
3. $u(\rho,\varphi,z) = I_m((k-\frac{1}{2})\pi\rho/L)(A_m\cos m\varphi + B_m \sin m\varphi)\sin(k-\frac{1}{2})\pi z/L)$
4. $u(\rho,z) = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{I_0((k-\frac{1}{2})\pi\rho/L)}{I_0((k-\frac{1}{2})\pi\rho_{\max}/L)} \frac{\sin(k-\frac{1}{2})\pi z/L}{2k-1}$
5. $u(\rho,z) = \frac{2L}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{I_0(k\pi\rho/L)}{I_0(k\pi\rho_{\max}/L)} \frac{\sin(k\pi z/L)}{k}$
6. (a) $u(\rho,\varphi,z;t) = \cos(k\pi z/L)J_m(\rho x_n^{(m)}/\rho_{\max})(A_m\cos m\varphi + B_m\sin m\varphi)e^{-\lambda_{kmn}Kt}$, where $J_m'(x_n^{(m)}) = 0$ and $k,m = 0,1,2\ldots$
   (b) $u(\rho,\varphi,z;t) = \sum_{k,m,n}^{\infty} A_{kmn}\cos(k\pi z/L)J_m(\rho x_n^{(m)}/\rho_{\max})$ $(A_m\cos m\varphi + B_m\sin m\varphi)e^{-\lambda_{kmn}kt}$, where $J_m'(x_n^{(m)} = 0)$
7. (a) $u(\rho,\varphi,z;t) = \sin(k\pi z/L)J_m(\rho x_n^{(m)}/\rho_{\max})(A_m\cos m\varphi + B_m\sin m\varphi)e^{-\lambda_{kmn}Kt}$, where $J_m'(x_n^{(m)}) = 0$ and $\lambda_{kmn} = (\frac{k\pi}{L})^2 + (\frac{x_n^{(m)}}{\rho_{\max}})^2$
   (b) $u(\rho,\varphi,z) = \sum_{k,m,n}\sin(k\pi z/L)J_m(\rho x_n^{(m)}/\rho_{\max})(A_{kmn}\cos m\varphi + B_{kmn}\sin m\varphi)$, where $J_m'(x_n^{(m)}) = 0$ and $m = 0,1,2\ldots$, $k = 1,2,\ldots$, $n = 1,2,\ldots$

CHAPTER 4

Section 4.1

1. $12r$
2. $3\sin^3\theta + 9\sin\theta\cos^2\theta$
3. $2/r$
4. $(\cot\theta)/r^2$
5. 0
6. $(9+6/r)e^{3r}$
7. $n(n+1)r^{n-2}$
11. $f(r) = (a^2 - r^2)/6$
12. $f(r) = (a^4 - r^4)/20$
13. $f(r) = (a^6 - r^6)/42$
14. $u(r;t) = \frac{3a}{r}\ \mathrm{Re}\ \left\{\exp[c_1(r-a)(1+i)]e^{2it}\frac{1-\exp[-2c_1 r(1+i)]}{1-\exp[-2c_1 a(1+i)]}\right\}$, where $c_1 = \sqrt{\frac{1}{K}}$
17. $u(r;t) = \frac{\sigma(a^2-r^2)}{6K} + T_1 + \frac{1}{r}\sum_{n=1}^{\infty} A_n \sin\frac{n\pi r}{a}\ \exp\left[-\left(\frac{n\pi}{a}\right)^2 Kt\right]$, where $A_n = \frac{2a(T_2-T_1)}{n\pi}(-1)^{n+1} - \frac{2\sigma a^3(-1)^{n+1}}{\pi^3 K n^3}$; $\tau = \frac{a^2}{\pi^2 K}$
18. $u(r;t) = \frac{1}{r}\sum_{n=1}^{\infty} A_n\sin\frac{n\pi r}{2a}\ \exp\left[-\left(\frac{n\pi}{2a}\right)^2 Kt\right] + T_1$, where $A_n = T_2\left[\frac{4a}{(n\pi)^2}\sin\frac{n\pi}{2} - \frac{2a}{n\pi}\cos\frac{n\pi}{2}\right] + T_1\left[\frac{4a(-1)^n}{n\pi}\right]$
20. $u_n(r;t) = \frac{1}{r}\ \sin(r\sqrt{\lambda_n})e^{-\lambda_n K t}$, where $a\sqrt{\lambda_n}\cot(a\sqrt{\lambda_n}) = 1$

22. $u(r;t) = T_1 + \frac{1}{r}\sum_{n=1}^{\infty} A_n \sin\frac{(n-\frac{1}{2})\pi r}{a}\exp\left[-\left(\frac{(n-\frac{1}{2})\pi}{a}\right)^2 Kt\right]$,
where $A_n = -\frac{2(T_2-T_1)(-1)^n a}{\pi^2(n-\frac{1}{2})^2}$
23. $\tau = \frac{4a^2}{\pi^2 K} \cong 2\,\mathrm{min}$
26. $u(r;t) = \frac{Aa}{nr\pi c}\sin\frac{n\pi r}{a}\sin\frac{n\pi ct}{a}$

Section 4.2

1. $0, -\frac{1}{2}, 0, \frac{3}{8}$
2. $1, 0, -\frac{3}{2}, 0$
3. $P_5(s) = \frac{63s^5-70s^3+15s}{8}$, $\quad P_6(s) = \frac{231s^6-315s^4+105s^2-5}{16}$
10. $P_1(0) = 0$, $\quad P_2\left(\pm\frac{1}{\sqrt{3}}\right) = 0$, $P_3(0) = 0$, $P_3\left(\pm\sqrt{\frac{3}{5}}\right) = 0$,
$P_4\left(\pm\sqrt{\frac{15\pm 2\sqrt{30}}{35}}\right) = 0$
12. $\frac{1}{2} + \sum_{k=1}^{\infty}\frac{2k+1}{2k(k+1)}P_k'(0)P_k(s) = \frac{1}{2}[f(s-0)+f(s+0)]$, $\qquad -1 < s < 1$
13. $\sum_{k=1}^{\infty}\frac{2k+1}{k(k+1)}P_k'(0)P_k(s) = \frac{1}{2}[f(s-0)+f(s+0)]$, $\qquad -1 < s < 1$
14. $\frac{1}{2} - \frac{15}{16}P_2(s) + \frac{135}{256}P_4(s) - \frac{273}{2048}P_6(s)$
15. $P_{4,1}(s) = \sqrt{1-s^2}(35s^3-15s)/2 = 5\sqrt{1-s^2}(7s^3-3s)/2$,
$P_{4,2}(s) = (1-s^2)(105s^2-15) = 15(1-s^2)(7s^2-1)/2$,
$P_{4,3}(s) = (1-s^2)^{3/2}(105s)$, $\qquad P_{4,4}(s) = 105(1-s^2)^2$
19. $s^2 = \frac{1}{3}P_0(s) + \frac{2}{3}P_2(s)$, $\qquad s^3 = \frac{3}{5}P_1(s) + \frac{2}{5}P_3(s)$,
$s^4 = \frac{7}{35}P_0(s) + \frac{20}{35}P_2(s) + \frac{8}{35}P_4(s)$

Section 4.3

1. $u(r,\theta) = \frac{11}{3}P_0(\cos\theta) + 4(r/a)P_1(\cos\theta) + \frac{4}{3}(r/a)^2P_2(\cos\theta)$
2. $u(r,\theta) = \frac{2}{3}P_0(\cos\theta) + \frac{4}{3}(r/a)^2P_2(\cos\theta)$
3. $u(r,\theta) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\left(\frac{r}{a}\right)^n P_n(\cos\theta)$
4. $u(r,\theta) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{(2k+1)P_k'(0)}{2k(k+1)}\left(\frac{r}{a}\right)^k P_k(\cos\theta)$
5. $u(r,\theta) = \sum_{k=1}^{\infty}\frac{(2k+1)P_k'(0)}{k(k+1)}\left(\frac{r}{a}\right)^k P_k(\cos\theta)$, $\qquad u\left(r,\frac{\pi}{2}\right) = 0$
7. $u(r,\theta) = \frac{26}{5}(r/a)P_1(\cos\theta) + \frac{4}{5}(r/a)^3P_3(\cos\theta)$
8. See Exercise 5.
9. $u(r,\theta) = \frac{6}{5}(a/r)P_0(\cos\theta) + 2(a/r)^2P_1(\cos\theta) + \frac{4}{7}(a/r)^3P_2(\cos\theta)$
$+ \frac{8}{35}(a/r)^5P_4(\cos\theta)$
10. $u(r,\theta) = \sum_{k=1}^{\infty}\frac{(2k+1)P_k'(0)}{k(k+1)}\left(\frac{a}{r}\right)^{k+1}P_k(\cos\theta)$
11. $u(r,\theta) = -\frac{7}{5}(a^3/r^2)P_1(\cos\theta) - \frac{3}{10}(a^5/r^4)P_3(\cos\theta)$
12. $u(r,\theta) = \frac{3a^3}{a-2}\frac{\cos\theta}{r^2} \qquad (a \neq 2)$
13. $u(r,\theta) = (r/a)\sin\theta\cos\varphi + (r/a)^2\sin^2\theta\sin 2\varphi$
14. $u(r,\theta) = (a/r)^2\sin\theta\cos\varphi + (a/r)^3\sin^2\theta\sin 2\varphi$

CHAPTER 5

Section 5.1

1. $F(\mu) = \frac{\sin 2\mu}{\pi\mu}$
2. $F(\mu) = \frac{4}{i\pi\mu}(1 - \cos\mu)$
3. $F(\mu) = \frac{1}{2\pi}\left(\frac{1}{2-i\mu} + \frac{1}{3+i\mu}\right)$
4. $F(\mu) = \frac{2\mu}{i\pi(1+\mu^2)^2}$
5. $F(\mu) = \frac{1}{2\pi}\left[\frac{1}{1+(1+\mu)^2} + \frac{1}{1+(1-\mu)^2}\right]$
6. $F(\mu) = \frac{1}{4\pi}\left[\frac{2}{1+\mu^2} + \frac{1}{1+(\mu-2)^2} + \frac{1}{1+(\mu+2)^2}\right]$
7. $F(\mu) = \frac{-i\mu}{2}e^{-|\mu|}$
8. $F(\mu) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\mu - \frac{3i}{2}\right)^2\right]$
9. $F(\mu) = \frac{1}{2\sqrt{2\pi}}\left\{\exp\left[-\frac{1}{2}(1+\mu)^2\right] + \exp\left[-\frac{1}{2}(1-\mu)^2\right]\right\}$
10. $F(\mu) = -\frac{i\mu e^{-\mu^2/2}}{\sqrt{2\pi}}$
13. $F_c(\mu) = \frac{2}{\pi}\frac{1-\mu^2}{(1+\mu^2)^2}$ $\quad F_s(\mu) = \frac{4\mu}{\pi(1+\mu^2)^2}$
17. $F(\mu) = \frac{1}{2}e^{-3i\mu}e^{-|\mu|}$
18. $F(\mu) = \frac{1}{\sqrt{2\pi}}e^{-2i\mu}e^{-\mu^2/2}$
19. $F(\mu) = 3e^{-2i\mu}/\pi(9+\mu^2)$
20. $F(\mu) = 1/[\pi(4\mu i + 8 - \mu^2)]$
21. $F(\mu) = (2 - i\mu)/2\pi(4+\mu^2)$

Section 5.2

6. $u(x;t) = \frac{1}{\sqrt{4\pi Kt}}\int_0^{L_1}\left\{\exp\left[-\frac{(x-\xi)^2}{4Kt}\right] - \exp\left[-\frac{(x+\xi)^2}{4Kt}\right]\right\}d\xi$
   $|u(x;t)| \le \frac{xL_1^2}{4K\sqrt{\pi K}}t^{-3/2}$
7. $u(x;t) = \frac{1}{\sqrt{4\pi Kt}}\int_0^{L_1}\left\{\exp\left[-\frac{(x-\xi)^2}{4Kt}\right] + \exp\left[-\frac{(x+\xi)^2}{4Kt}\right]\right\}d\xi$
   $|u(x;t)| \le \frac{2L_1}{\sqrt{4\pi Kt}}$
15. $u(x;t) = T_1\left[1 - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right]$
    If $C = 10$, then $t = (0.61)x^2$. If $C = 30$, then $t = (3.6)x^2$. If $C = 50$, then $x = 0$.
16. $\tau^* = (1.81)x^2/K$
17. $u(x;t) = T_1\left[\Phi\left(\frac{x+L}{\sqrt{2Kt}}\right) - \Phi\left(\frac{x}{\sqrt{2Kt}}\right)\right]$
    $+T_2\left[\Phi\left(\frac{x}{\sqrt{2Kt}}\right) - \Phi\left(\frac{x-L}{\sqrt{2Kt}}\right)\right]$; $\lim_{t\to\infty} u(x;t) = 0$

Section 5.3

1. $y(x;t) = 3\sin 2x\cos 2ct$

2. $y(x;t) = (4/5c)\cos 5x \sin 5ct$
4. $y(x;t) = \frac{1}{2c}\int_{ct-x}^{ct+x} g(\xi)d\xi$ for $0 < x < ct$;
 $y(x;t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$ for $x > ct$
5. $y(x;t) = \frac{1}{2c}\int_{ct-x}^{x+ct} g(\xi)d\xi + s\left(t - \frac{x}{c}\right)$ for $0 < x < ct$
 $y(x;t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi$ for $x > ct$
7. $u(r;t) = \frac{1}{2r}[(r+ct)f_1(r+ct) + (r-ct)f_1(r-ct)] + \frac{1}{2cr}\int_{r-ct}^{r+ct} \xi f_2(\xi)d\xi$
8. $u(r;t) = \begin{cases} 0 & 0 < ct < r-a \\ (T/4cr)[a^2-(r-ct)^2] & 0 < r-a < ct < a+ra \\ Tt & 0 < ct < a-r \\ (T/4cr)[a^2-(r-ct)^2] & 0 < a-r < ct < a+r \\ 0 & ct > a+r \end{cases}$
9. $u(x,y) = \frac{2}{\pi}\left(\tan^{-1}\frac{4-x}{y} + \tan^{-1}\frac{4+x}{y}\right)$
10. $u(x,y) = \frac{1}{\pi}\int_0^\infty \left[\frac{y}{y^2+(x-\xi)^2} - \frac{y}{y^2+(x+\xi)^2}\right] f(\xi)d\xi$
11. $u(x,y) = \frac{1}{\pi}\int_0^\infty \left[\frac{y}{y^2+(x-\xi)^2} + \frac{y}{y^2+(x+\xi)^2}\right] f(\xi)d\xi$
12. $u(x,y) = \frac{1}{\pi}\int_0^\infty \left[\frac{y}{y^2+(x-\xi)^2} + \frac{x}{x^2+(y-\xi)^2} - \frac{y}{y^2+(x+\xi)^2} - \frac{x}{x^2+(y+\xi)^2}\right] f(\xi)d\xi$
13. $u(x,y) = \sum_{n=1}^\infty B_n \sin\frac{n\pi x}{L}\, e^{-(n\pi y/L)}$, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$
14. $u(x,y) = \int_{-\infty}^\infty B(\lambda)\sinh\lambda x e^{i\lambda y}d\lambda$, $B(\lambda)\sinh\lambda L = \frac{1}{2\pi}\int_{-\infty}^\infty g_{\text{odd}} e^{-i\lambda y}dy$

CHAPTER 6

Section 6.1

1. $5! = 120, (5/e)^5 \sim 21.0561, (5/e)^5\sqrt{10\pi} \sim 118.02$;
 $50! \sim (3.0414)10^{64}, (50/e)^{50} \sim (1.7131)10^{63}, (50/e)^{50}\sqrt{100\pi} \sim (3.0363)10^{64}$;
 $500! \sim (1.2201)10^{1134}, (500/e)^{500} \sim (2.1765)10^{1132}$,
 $(500/e)^{500}\sqrt{1000\pi} \sim (1.2199)10^{1134}$

Section 6.2

1. $f(t) = (e^t/t)[(\sin 1) + O(1/t)]$, $t \to \infty$
2. $f(t) = (e^{-t}/t)[\frac{1}{2} + O(1/t)]$, $t \to \infty$
3. $f(t) = (e^{-t}/t)[1 + O(1/t)]$, $t \to \infty$
4. $f(t) = (1/t)[1 + O(1/t)]$, $t \to \infty$
5. $f(t) = (1/t)[1 + O(1/t)]$, $t \to \infty$
6. $u(at;t) = 100 + O(e^{-(a^2t/4K)})$ $t \to \infty, a > 0$

Section 6.3

1. $f(t) = 2\sqrt{\pi/t}[1 + O(1/\sqrt{t})]$, $t \to \infty$

2. $f(t) = 5\sqrt{\pi/t}[1 + O(1/\sqrt{t})], \quad t \to \infty$
3. $f(t) = e^{3t/7}\sqrt{8\pi/15t}[1 + O(1/\sqrt{t})], \quad t \to \infty$
4. $I_0(t) = e^t\sqrt{1/2\pi t}[1 + O(1/\sqrt{t})], \quad t \to \infty$
7. $u(x;t) = O(1/t),\ u_x(x;t) = O(1/t), \quad t \to \infty$
8. $u(x;t) = \frac{1}{\sqrt{\pi K t}}\left[\int_0^\infty f(x)dx + O\left(\frac{1}{\sqrt{t}}\right)\right],\ u_x(x;t) = O\left(\frac{1}{t}\right), \quad t \to \infty$
9. $u(x;t) = 50\sqrt{\frac{\pi}{Kt}}\left[1 + O\left(\frac{1}{\sqrt{t}}\right)\right],\ u_x(x;t) = O\left(\frac{1}{t}\right), \quad t \to \infty$
10. $u(x;t) = \frac{1}{\sqrt{\pi K t}}\left[\frac{1}{2} + O\left(\frac{1}{t}\right)\right],\ u_x(x;t) = O(1/t), \quad t \to \infty$

Section 6.4

1. $f(t) = \sqrt{\pi/t}e^{i\pi/4}[1 + O(1/\sqrt{t})], \quad t \to \infty$
2. $f(t) = \sqrt{2\pi/t}e^{it}e^{-(i\pi/4)}[1 + O(1/\sqrt{t})], \quad t \to \infty$
3. $f(t) = \sqrt{\pi/2t}e^{it}e^{-(i\pi/4)}[1 + O(1/\sqrt{t})], \quad t \to \infty$
4. $f(t) = 2\sqrt{2\pi/t}\cos(t - \pi/4) + O(1/t), \quad t \to \infty \quad m$ even
   $f(t) = 2i\sqrt{2\pi/t}\sin(t - \pi/4) + O(1/t), \quad t \to \infty \quad m$ odd

Section 6.5

4. $f(t) = e^{-t}\left[\frac{1}{t} - \frac{1}{t^2} + \frac{2}{t^3} - \frac{6}{t^4} + \cdots + (-1)^n\frac{n!}{t^{n+1}} + O\left(\frac{1}{t^{n+2}}\right)\right], \quad t \to \infty$

CHAPTER 7

Section 7.1.3

7. (a) 0.04996 (b) −0.1000 (c) −0.4986 (d) −0.9975
8. (a) −0.04996 (b) 0.1000 (c) 0.4082 (d) 0.9975
9. (a) Zero (b) Zero (c) −0.01126 (d) Zero
11. (a) Zero (b) −0.0677 (c) −0.0008 (d) Zero

CHAPTER 8

Section 8.1

2. $G(x,z) = \begin{cases} \frac{1+h(L-z)}{1+hL}x & \text{if } 0 \le x \le z \\ \frac{1+h(L-x)}{1+hL}z & \text{if } z \le x \le L \end{cases}$
3. $G(x,z) = \begin{cases} \frac{1+h(L-z)}{h(2+hL)}(1 + hx) & \text{if } 0 \le x \le z \\ \frac{(1+hz)}{h(2+hL)}[1 + h(L - x)] & \text{if } z \le x \le L \end{cases}$
4. $G(x,z) = \begin{cases} \frac{\sinh[(L-z)\sqrt{k}]\sinh(x\sqrt{k})}{\sinh(L\sqrt{k})} & \text{if } 0 \le x \le z \\ \frac{\sinh[z\sqrt{k}]\sinh((L-x)\sqrt{k})}{\sinh(L\sqrt{k})} & \text{if } z \le x \le L \end{cases}$
5. $G(x,z) = \begin{cases} \frac{\sin[(L-z)\sqrt{-k}]\sin(x\sqrt{-k})}{\sin(L\sqrt{-k})} & \text{if } 0 \le x \le z \\ \frac{\sin[z\sqrt{-k}]\sin((L-x)\sqrt{-k})}{\sin(L\sqrt{-k})} & \text{if } z \le x \le L \end{cases}$

6. $G(x,z) = \begin{cases} -(z-L)^2/2L & \text{if } 0 \le x \le z \\ (x-z) - (z-L)^2/2L & \text{if } z \le x \le L \end{cases}$

7. $G(x,z) = \begin{cases} (L-z)(z-2x)/2L & \text{if } 0 \le x \le z \\ (x-z) + (L-z)(z-2x)/2L & \text{if } z \le x \le L \end{cases}$

Section 8.3

In each case we make the notational convention that $P = (x,y), Q = (\xi,\eta)$.

1. $G(P,Q) = -\frac{1}{4\pi}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} - \frac{1}{4\pi}\log\frac{(x+\xi)^2+(y-\eta)^2}{(x+\xi)^2+(y+\eta)^2}$
2. $G(P,Q) = -\frac{1}{4\pi}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} + \frac{1}{4\pi}\log\frac{(x+\xi)^2+(y-\eta)^2}{(x+\xi)^2+(y+\eta)^2}$
4. $G(P,Q) = \frac{4}{ab}\sum_{m,n=1}^{\infty}\frac{\sin(m\pi x/a)\sin[(n-\frac{1}{2})\pi y/b]\sin(m\pi\xi/a)\sin[(n-\frac{1}{2})\pi\eta/b]}{(m\pi/a)^2+[(n-\frac{1}{2})\pi/b]^2}$

Section 8.4

1. $u(x;t) = \int_0^t\int_0^\infty[4\pi K(t-s)]^{-1/2}\left[e^{-(x-\xi)^2/4K(t-s)} + e^{-(x+\xi)^2/4K(t-s)}\right] h(\xi,s)\,d\xi\,ds$
2. $u(x;t) = \sum_{m=-\infty}^{\infty}\int_0^t\int_0^L[4\pi K(t-s)]^{-1/2}$
   $\left[e^{-(x-\xi-2mL)^2/4K(t-s)} - e^{-(x-\xi-(2m+2)L)^2/4K(t-s)}\right] h(\xi,s)\,d\xi\,ds$
3. $u(x,y;t) = \int_0^t\int_0^\infty\int_0^\infty[4\pi K(t-s)]^{-1/2}e^{-[(x-\xi)^2+(y-\eta)^2]/4K(t-s)}\,h(\xi,\eta,s)\,d\xi\,d\eta\,ds$
4. $u(x;t) = [4\pi Kt]^{-1/2}\sum_{m=-\infty}^{\infty}$
   $\int_0^L\left[e^{-(x-\xi-2mL)^2/4Kt} - e^{-(x-\xi-(2m+2)L)^2/4Kt}\right] f(\xi)\,d\xi$
5. $u(x;t) = [4\pi Kt]^{-1/2}\sum_{m=-\infty}^{\infty}\int_0^L e^{-(x-\xi-mL)^2/4Kt}f(\xi)\,d\xi$

Appendix A.1

1. $y(t) = e^{-t^2}(t+c)$
2. $y(t) = 1 + c/t$
3. $y(t) = e^{2t}/5 + Ce^{-3t}$
4. $y(t) = t^2/6 + C/t^4$
5. $y(t) = 1 + C/\sin t$
6. $y(t) = c_1\cos 2t + c_2\sin 2t$
7. $y(t) = c_1e^{-2t} + c_2te^{-2t}$
8. $y(t) = c_1e^{3t} + c_2e^{-5t}$
9. $y(t) = c_1 + c_2e^{-3t}$
10. $y(t) = c_1e^{-(t/3)} + c_2e^{2t}$
11. $y(t) = \sin 2t$
12. $y(t) = e^{-2t} + 2te^{-2t}$
13. $y(t) = \frac{13}{8}e^{3t} + \frac{3}{8}e^{-5t}$
14. $y(t) = \frac{7}{3} - \frac{4}{3}e^{3-3t}$
15. $y(t) = \frac{3}{7}e^{-(t/3)} + \frac{4}{7}e^{2t}$

16. $y(t) = e^{-2t}\left(-\frac{1}{3}\cos\sqrt{2}t + \frac{2\sqrt{2}}{3}\sin\sqrt{2}t\right) + \frac{1}{3}$
17. $y(t) = \frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} - \frac{1}{2}$
18. $y(t) = \frac{7}{8} - \frac{7}{8}e^{-4t} + \frac{1}{2}t$
19. $y(t) = 3 + 4t$
20. $y(t) = -t + \ln t + 1$
21. $y(t) = t + \sum_{n=1}^{\infty} \frac{(-4)^n \prod_{m=1}^{n}(3m-1)}{(3n+1)!}\, t^{3n+1}$
22. $y(t) = 1 + \sum_{n=1}^{\infty} \frac{(-4)^n \prod_{m=1}^{n}(3m-2)}{(3n)!}\, t^{3n}$
26. (a), (c), (d), (e) have regular singular points.
27. $r(r-1) - 1 = 0, y(t) = t^{\frac{1+\sqrt{5}}{2}}$
28. $r^2 = 1, y(t) = t$
29. $r^2 = 0, y(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}(n!)^2}\, t^{2n}$
30. $r(r-1) - 1 = 0, y(t) = t^r\left[1 + \sum_{n=1}^{\infty} \frac{(-3)^n t^{2n}}{\prod_{j=1}^{n} r(4j-2) + (4j^2 - 2j + 2)}\right], r = \frac{1+\sqrt{5}}{2}$
31. $y_\infty = 3, \quad \tau = \frac{1}{2}$
32. $y_\infty = 5, \quad \tau = 1$
33. $y_\infty = \frac{1}{2}, \quad \tau = \frac{1}{2}$
34. $y_\infty = 1, \quad \tau = 1$
35. $y_\infty = 4, \quad \tau = 1$

Appendix A.2

1. (a) Diverges. (b) Converges. (c) Diverges. (d) Converges.
2. (a)$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$ (b) $2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}$
   (c) $1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}$ (d)$\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(n-1)!}$ $(0! = 1)$
3. (a) $f'(x) = \frac{1}{x^2}e^{-(1/x)}$ and $f''(x) = \left(\frac{1}{x^4} - \frac{2}{x^3}\right)e^{-(1/x)}$ for $x > 0$.
   (d) The Taylor series converges for all x and its sum is zero.
   This equals $f(x) - f(0)$ only when $x \le 0$.
5. (a), (c), (d) can be differentiated term by term according to Proposition A.2.8.
6. $\sum_{n=1}^{\infty} n^3x^n = \frac{x(1+4x+x^2)}{(1-x)^4}$, $\sum_{n=1}^{\infty} n^4x^n = \frac{x(1+11x+11x^2+x^3)}{(1-x)^5}$, $-1 < x < 1$
9. (a) Diverges (b) Converges (c) Converges (d) Converges
10. (a) Can choose any $T > 0, M = 2$. (b)Can choose any $T > 0, M = 2^{10}(10)!$
    (c) Can choose $T = 1, M = 2$ (d) Can choose any $T > 0, M = 2$
12. (a) Use L'Hospital's rule.
    (b) $\int_0^1 n^2xe^{-nx}dx = \int_0^n ye^{-y}dy = -(1+y)e^{-y}\,|_0^n \to 1$ when $n \to \infty$.
    (c) Choose $u_n(x) = n^2xe^{-n^2x} - (n-1)^2xe^{-(n-1)^2x}$ for $n = 1, 2, \ldots$.
15. Not necessarily; for example, let $f_1(t) = \sin t$, $f_2(t) = \cos t$, $g(t) = 1$.

Appendix A.4

1. `y''[t] + t y'[t] + (1 + t^2)y[t] == t/(1 + t)`
2. `D[y[t],t,t] + t D[y[t],t] + (1 + t^2)y[t] == t/(1 + t)`

```
D[y[t],{t,2}] + t D[y[t],t] + (1+t^2)y[t] == t/(1 + t)
```

3.
```
f[t_]:= 1 + t + Sin[t] + E^t - Sqrt[5 - t^2]
f[2]
f[2]//N
```
4.
```
f[t_,c_]:= Cos[t] + t^(3/2) + (2/3) c
```
5.
```
1/(3 5 7 9)
1/(3 5 7 9)//N
```
6.
```
D[Log[Sin[t]]/Tan[t],t]
D[(t^3 + 1)/(t Sin[t]),{t,2}]//Simplify
```
7.
```
(a)Integrate[t^3 E^(5 t)Cos[t],t]
(b)Integrate[(Sin[t])^7,{t,0,Pi}]
```
8.
```
D[Integrate[t E^t + Sec[t],t],t]//Simplify
```
9.
```
Clear["Global`@*"]
f = Function[t,t^3 + Tan[t]]
f = (#^3 + Tan[#])&
f[Pi/4]
f[Pi/4] // N
```

INDEX

Titles in This Series

Volume

15 **Mark A. Pinsky**
Partial Differential Equations and Boundary-Value Problems with Applications, Third Edition

14 **Michael E. Taylor**
Introduction to Differential Equations

13 **Randall Pruim**
Foundations and Applications of Statistics: An Introduction Using R

12 **John P. D'Angelo**
An Introduction to Complex Analysis and Geometry

11 **Mark R. Sepanski**
Algebra

10 **Sue E. Goodman**
Beginning Topology

9 **Ronald Solomon**
Abstract Algebra

8 **I. Martin Isaacs**
Geometry for College Students

7 **Victor Goodman and Joseph Stampfli**
The Mathematics of Finance: Modeling and Hedging

6 **Michael A. Bean**
Probability: The Science of Uncertainty with Applications to Investments, Insurance, and Engineering

5 **Patrick M. Fitzpatrick**
Advanced Calculus, Second Edition

4 **Gerald B. Folland**
Fourier Analysis and Its Applications

3 **Bettina Richmond and Thomas Richmond**
A Discrete Transition to Advanced Mathematics

2 **David Kincaid and Ward Cheney**
Numerical Analysis: Mathematics of Scientific Computing, Third Edition

1 **Edward D. Gaughan**
Introduction to Analysis, Fifth Edition

Building on the basic techniques of separation of variables and Fourier series, the book presents the solution of boundary-value problems for basic partial differential equations: the heat equation, wave equation, and Laplace equation, considered in various standard coordinate systems—rectangular, cylindrical, and spherical. Each of the equations is derived in the three-dimensional context; the solutions are organized according to the geometry of the coordinate system, which makes the mathematics especially transparent. Bessel and Legendre functions are studied and used whenever appropriate throughout the text. The notions of steady-state solution of closely related stationary solutions are developed for the heat equation; applications to the study of heat flow in the earth are presented. The problem of the vibrating string is studied in detail both in the Fourier transform setting and from the viewpoint of the explicit representation (d'Alembert formula). Additional chapters include the numerical analysis of solutions and the method of Green's functions for solutions of partial differential equations. The exposition also includes asymptotic methods (Laplace transform and stationary phase).

With more than 200 working examples and 700 exercises (more than 450 with answers), the book is suitable for an undergraduate course in partial differential equations.

ISBN 978-0-8218-6889-8

9 780821 868898

AMSTEXT/15

For additional information and updates on this book, visit
www.ams.org/bookpages/amstext-15

This series was founded by the highly respected mathematician and educator, Paul J. Sally, Jr.