SOME APPLICATIONS OF GENERALIZED CALCULUS TO DIFFERENTIAL AND INTEGRO DIFFERENTIAL EQUATIONS*

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1. Introduction. The title of this paper may suggest that the subject under discussion deals with the differential or integro-differential or integral equations of fractional (generalized) order. This is not the case, but in previous work [4], [6], [9], [13], some studies have been made by this writer, in addition to some other authors who have discussed some aspects of this topic [14]. Recently K. Nishimoto and others have made some studies about the integro-differential equations of fractional orders in the complex domain [18], [19].

In this article a study will be made about the use of Fractional (Generalized) Calculus methods in solving some classes of differential and integro-differential equations. This may be accomplished through representations of equations by their equivalent operator equations or, as they are called in some other papers, transform equations. Then the operator equations may be solved by using some operational properties of the integro-differential operators of fractional (generalized) orders (see [1]). Some of this work have been presented in detail by this author, (see [2], [3], [5], [7], [8], [10], [11]). Also, Mambriani's work, [15] and [16], may be considered as being based on related ideas.

So, this article contains two main parts of the subject: the first is dealing with equivalence relations and properties and the second part offers some methods of solving the operator equations which yield solutions to their equivalent differential or integro-differential or integral equations. In

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* Invited talk.

particular the method will be applied to classes of differential or integro--differential equations whose solutions represent some generalized special functions such as hypergeometric functions, Laguerre functions, Legendre functions, Bessel functions, etc.

It may be pointed out that the "H-R transform equations" presented by the author in previous work will be called henceforth "operator equations". If some mathematical symbols or notations are not clearly defined they are found in other stated references of the work of the author.

1. SOME EQUIVALENCE PROPERTIES

1.1 Preliminaries and Definitions. An operator equation is a functional equation which contains the operators of generalized order of an unknown function, given in the equation, which is assumed to be defined and satisfy certain conditions over an interval (a,b). It may be represented by a differential equation or an integro-differential equation or an integral equation. For example the operator equation:

$$\overset{x}{\underset{a}{I}}{}^{-n}\, p(x)\, y(x) = f(x)\,, \quad \left(\overset{x}{\underset{a}{I}}{}^{-n} = D^n = \frac{d^n}{dx^n}\right) \tag{1.1}$$

where n is a positive integer $\geqslant 1$, $p, y \in C^n$ and $f \in C$ on $a \leqslant x \leqslant b$ is the differential equation

$$\sum_{k=0}^{n} \binom{n}{k} p^{(k)}(x)\, y^{(n-k)}(x) = f(x) \tag{1.2}$$

where

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\,\Gamma(k+1)}.$$

The operator equation

$$D^2 y + p_1(x)\, Dy + p_2(x)\, y + p_3(x)\, \overset{x}{\underset{a}{I}}{}^{\alpha} y = 0$$

where

$$R\alpha > 0,\ \overset{x}{\underset{a}{I}}{}^{\alpha} y = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\, y(t)\, dt\,,$$

is an integro-differential equation. Also the operator equation

$$\Gamma(1-\alpha)\, \overset{x}{\underset{a}{I}}{}^{1-\alpha} u(x) = f(x) \tag{1.3}$$

where $0 < \alpha < 1$, $f(x) \in C^1$ on $[a,b]$ and $u(x)$ is the unknown continuous function represents the Abel's integral equation.

The operator equation may assume some other forms different from the ones already mentioned which appear in the forms of successive operations or in the forms of composition of functions such as

$$\overset{x}{\underset{a}{I}}{}^{-w} P_1(x) \overset{x}{\underset{a}{I}}{}^{-1} P_2(x) \overset{x}{\underset{a}{I}}{}^{w-n+1} y(x) = f(x)$$

where P_1 and P_2 are given functions of the variable x and possess as many derivatives as it is required, $y \in C^n$ and $f \in C$ in the interval $a \leqslant \overset{x}{t} \leqslant b$.

This operator equation may be represented by a differential equation or an integro-differential equation or an integral equation. Such representation, however, depends upon the function forms of P_1 and P_2, the values of w and the positive integer n. These types of equations may be called "equations of successive operators".

This work will be confined to the study of some aspects of certain types of operator equations of successive operators and their equivalent equations that may be obtained by applying some operational properties of the operator of generalized order which are in essence the same properties as those possessed by the integral or derivative operator of generalized (fractional) order [1].

Definition. A differential equation or an integro-differential equation is said to be equivalent to an operator equation of successive operators if the first is an expanded form of the second.

For obtaining the expanded forms, the generalized Leibnitz rules of differentiation (see [1], pp.5-9) are used in the expansion process. The main formula of Leibnitz' generalized rule is given by

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f g = \sum_{i=0}^{n} \binom{-\alpha}{i} f^{(i)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha+i} g$$

and

$$\overset{x}{\underset{a}{I}}{}^{-\beta} f g = \sum_{p=0}^{n} \binom{\beta}{p} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{-\beta+p} g$$

where $f(x)$ is a polynomial of degree n, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and

$$\binom{\beta}{p} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-p+1)\Gamma(p+1)} .$$

According to this definition (1.1) is equivalent to (1.2) and the operator equation

$$\overset{x}{\underset{a}{I}}{}^{-w} e^{-f(x)} \overset{x}{\underset{a}{I}}{}^{-1} e^{f(x)} \overset{x}{\underset{a}{I}}{}^{w} e^{g(x)} y(x) = 0 , \tag{1.4}$$

where w is a number,

$$f(x) = \frac{b_0}{a_0} x ,$$

$$g(x) = \int_a^x \frac{\sum_{i=1}^{n} s_i t^{n-i}}{\sum_{i=0}^{n} a_i t^{n-i}} \, dt \, , \qquad \left[s_i = b_i - \frac{b_0}{a_0} a_i \right],$$

a_k, b_k $(k = 0, 1, 2, \ldots, n)$ are numbers and $a_0 \neq 0$, is equivalent to the linear differential equation of the first order

$$\left(\sum_{i=0}^{n} a_i x^{n-i} \right) y' + \left(\sum_{i=0}^{n} b_i x^{n-i} \right) y = 0 \, . \tag{1.4$_1$}$$

The equivalence can be easily established by expanding (1.4) and writing it in the form

$$\overset{x}{\underset{a}{I}}{}^{-w} e^{-f(x)} \left[f'(x) \overset{x}{\underset{a}{I}}{}^{w} e^{g(x)} y + e^{f(x)} \overset{x}{\underset{a}{I}}{}^{w-1} e^{g(x)} y \right] = 0 \, ,$$

which gives equation $(1.4)_1$ in the form

$$y' + [f'(x) + g'(x)] y = 0 \, .$$

1.2 Some Classes of Equations of Successive Operators.

(1) Let w, k_i, a_i and b_i $(i = 1, 2, \ldots, m)$ be numbers such that not all the b_i's are zeros and $\operatorname{Re} w > 0$. Suppose $y \in C^n$ on the interval (a, b) and m, n are positive integers. Then the operator equation

$$(\mathrm{E}_1) \qquad \overset{x}{\underset{a}{I}}{}^{-w} \prod_{i=1}^{m} \left(a_i + b_i x \right)^{1-k_i} \overset{x}{\underset{a}{I}}{}^{-1} \prod_{i=1}^{m} \left(a_i + b_i x \right)^{k_i} \overset{x}{\underset{a}{I}}{}^{w-n-1} y(x) = 0$$

represents, for $m = n$, a linear differential equation of order n and of the form

$$(\mathrm{E}_1)_1 \qquad \prod_{i=1}^{n} (a_i + b_i x) D^n y + \sum_{r=1}^{n} P_{n-r}(x) D^{n-r} y = 0 \, ,$$

where $D = d/dx$, $D^0 y = y$ and $P_{n-r}(x)$, for fixed r, is a polynomial of degree less than the degree of $\prod_{i=1}^{n} (a_i + b_i x)$ by r.

If the transformation $y(x) = e^{-\mu x} Y(x)$ is applied to (E_1) then the operator equation

$$(\mathrm{E}_2) \qquad \overset{x}{\underset{a}{I}}{}^{-w} \prod_{i=1}^{m} \left(a_i + b_i x \right)^{1-k_i} \overset{x}{\underset{a}{I}}{}^{-1} \prod_{i=1}^{m} \left(a_i + b_i x \right)^{k_i} \overset{x}{\underset{a}{I}}{}^{w-n+1} e^{-\mu x} Y(x) = 0 \, ,$$

for $m = n$, is equivalent to the n^{th} order differential equation of the form

$$(\mathrm{E}_2)_1 \qquad \prod_{i=1}^{n} (a_i + b_i x) D^n Y + \sum_{r=1}^{n} P_r(x) D^{n-r} Y = 0 \, ,$$

where $P_r(x)$, $(r = 1, 2, \ldots, n)$ are polynomials of at most n^{th} degree, i.e. they are of the same degree as that of the polynomial $\prod_{i=1}^{n} (a_i + b_i x)$. The forms of these polynomials can be easily determined.

It may be pointed out that the operator equations (E_1) and (E_2) may no longer represent linear differential equations when $m > n$, (for details see [2] and [3].

It is interesting to note that if the a_i's as well as the b_i's are all equal, i.e., $a_i = A$ and $b_i = B$ for all $(i = 1, 2, \ldots, n)$ and $k = \sum_{i=1}^{n} k_i$, then the operator equation

$$(E_3) \qquad \overset{x}{\underset{a}{I}}{}^{-w} (A+Bx)^{n-k} \overset{x}{\underset{a}{I}}{}^{-1} (A+Bx)^{k} \overset{x}{\underset{a}{I}}{}^{w-n+1} y(x) = 0,$$

is equivalent to the Euler linear differential equation

$$(E_3)_1 \qquad (A+Bx)^n D^n y + \sum_{r=1}^{n} \left[\binom{w}{r} (-n)_r - k \binom{w}{r-1} (-n+1)_{r-1} \right]$$
$$\times (-1)^r B^r (A+Bx)^{n-r} D^{n-r} y = 0,$$

where $(\alpha)_r = \alpha(\alpha+1) \ldots (\alpha+r-1)$, $(\alpha)_0 = 1$, $\alpha \neq 0$.

The equivalence of (E_3) and $(E_3)_1$ can be shown easily by using the properties of the differential operator of generalized order, as equation (E_3) can be written in the form

$$Bk \overset{x}{\underset{a}{I}}{}^{-w} (A+Bx)^{n-1} \overset{x}{\underset{a}{I}}{}^{w-n+1} y + \overset{x}{\underset{a}{I}}{}^{-w} (A+Bx)^{n} \overset{x}{\underset{a}{I}}{}^{w-n} y = 0,$$

which, after expanding the terms, becomes

$$k \sum_{r=0}^{n-1} (-1)^r \binom{w}{r} (1-n)_r B^{r+1} (A+Bx)^{n-r-1} D^{n-r-1} y$$
$$+ \sum_{r=0}^{n} (-1)^r \binom{w}{r} (-n)_r B^r (A+Bx)^{n-r} D^{n-r} y = 0.$$

This equation takes the form of $(E_3)_1$ by replacing r, in the first summation, by $r-1$.

The significance of the equivalence between operator equations and their corresponding linear differential equations or integro-differential or integral equations is that once the equation is transformed into an equation of successive operators, the problem of finding solutions of an equation becomes easier to deal with through the use of some operational properties of the integro-differential operator of generalized (fractional) order. This matter will be clarified in the next sections when the problem of solving certain classes of operator equations is studied.

(2) For other cases of (E_1) we may present the following:

Let α_i, a_i be numbers, $R(\lambda-w)>0$ $(\lambda=1,2,\dots)$, $f(x)$ is a given function of Class $C^{(0)}$, $y(x)$ is a function of Class $C^{(n)}$ on $[a,b]$ and m and n are positive integers such that

$$(E_4)\quad \overset{x}{\underset{a}{I}}{}^{-w}\prod_{i=1}^{m}(x-a_i)^{a_i}\ \overset{x}{\underset{a}{I}}{}^{-1}\prod_{i=1}^{m}\left(x-a_i\right)^{1-a_i}\ \overset{x}{\underset{a}{I}}{}^{w-n+1}\ y(x)=f(x)\,.$$

By using the differentiation properties of the operator of generalized order it can be easily verified that equation (E_4) is equivalent to the integro-differential equation

$$(E_4)_1\quad \frac{d^n y}{dx^n}+\sum_{p=-1}^{m-2}\phi_p\sum_{1\leqslant i_1<i_2<\dots<i_{p+2}\leqslant m}\frac{w-\sum\limits_{k=1}^{p+2}\alpha_{i_k}+1}{\sum\limits_{k=1}^{p+2}(x-a_{i_k})}$$

$$\times\overset{x}{\underset{a}{I}}{}^{p-n+2}\ y(x)=f(x)$$

where $\phi_{-1}=1$, $\phi_r=w(w-1)\dots(w-r)$, $(r=0,1,\dots,m-2)$. From the equivalence of these two forms, we may conclude the following:

(i) If $m=n$, then $(E_4)_1$ is a non-homogeneous differential equation of n^{th} order of the Fuchsian class, with singularities at $x=a_i$ $(i=1,2,\dots,n)$.

(ii) If $m>n$ $(m-n=\nu)$, then $(E_4)_1$ is a non-homogeneous (n^{th} order differential $-\nu^{th}$ order integral), integro-differential equation of Fuchs--Volterra type with ν integral terms containing integrals of the operator or integral orders of the forms $\overset{x}{\underset{a}{I}}{}^{q}y$, $(q=1,2,\dots,\nu)$. The equation's singularities are at a_i, $(i=1,2,\dots,m)$.

(iii) If $m<n$ $(n-m=\mu)$ then $(E_4)_1$ is a non-homogeneous differential equation of n^{th} order with the last term containing μ^{th} dirivative of y.

(iv) For the particular case $m=n+1$, equation $(E_4)_1$ is a non-homogeneous, Funchsian type n^{th} order differential equation if and only if

$$w-\sum_{k=1}^{k=n+1}a_{i_k}+1=0\,.$$

Details about equivalence and particular cases may be seen in [2], [3].

1.3 <u>Some Cases of Equivalence Properties</u>

Property 1. The necessary and sufficient condition that the second order linear differential equation

$$(a+bx)(c+dx)Y''+(a_1+b_1x)Y'+d_1Y=0\,, \tag{3.1}$$

is equivalent to the operator equation

$$\underset{x_0}{\overset{x}{I}}{}^{-w} (a+bx)^{1-p}(c+dx)^{1-q} \underset{x_0}{\overset{x}{I}}{}^{-1} (a+bx)^p (c+dx)^q \underset{x_0}{\overset{x}{I}}{}^{w-1} Y = 0, \tag{3.2}$$

where $a, b, c, d, a_1, b_1, d_1, w, p$ and q are numbers such that a and c are not both zero and $bd \neq 0$, is that

$$bcU + adV = a_1 \tag{3.3}$$

$$bd(U+V) = b_1 \tag{3.4}$$

$$bdw(U+V-w-1) = d_1 \tag{3.5}$$

where $U = w + p$ and $V = w + q$.

It may be pointed out that the values of w, in this equivalence case are the roots of the quadratic equation

$$bdw^2 + (bd - b_1)\ w + d_1 = 0.$$

Property 2. If the differential equation (3.1) is equivalent to the operator equation (3.2), then the necessary and sufficient condition that the equation

$$P_0(x)\ y'' + P_1(x)\, y' + P_2(x)\ y = 0, \tag{3.6}$$

where

$$\begin{aligned} P_0(x) &= (a+bx)(c+dx) \\ P_1(x) &= A_1x^2 + B_1x + C_1 \\ P_2(x) &= A_2x^2 + B_2x + C_2 \end{aligned}$$

and A_1, B_1, C_1, A_2, B_2, C_2 are constants, is equivalent to the operator equation (3.2) with $Y = (\exp \mu x)y$, $\mu \neq 0$, is that:

$$bd(\mu^2 + 2\mu) = A_1 + A_2 \tag{3.7}$$

$$B_1 = b_1 + 2\mu S \tag{3.8}$$

$$C_1 = a_1 - 2\mu ac \tag{3.9}$$

$$d_1 = C_2 - C_1\mu + ac\mu^2 \tag{3.10}$$

and

$$A_1^2 = 4bdA_2 \tag{3.11}$$

$$B_1 = \mu S + B_2\mu^{-1} \tag{3.12}$$

where $S = bc + ad$.

Details of these properties are found in [3].

Property 3. (A) The necessary and sufficient condition that the differential equation

$$(a+bx)Y'' + (a_1 + b_1x)Y' + d_1Y = 0 \tag{3.13}$$

is equivalent to the operator equation

$$\underset{x_0}{\overset{x}{I}}{}^{-\alpha}(a+bx)^{1-k}e^{\mu x}\underset{x_0}{\overset{x}{I}}{}^{-1}(a+bx)^{k}e^{-\mu x}\underset{x_0}{\overset{x}{I}}{}^{\alpha-1}Y=0\,, \tag{3.14}$$

where $a, b, a_1, b_1, d_1, \alpha, k$ and μ are numbers such that $b \neq 0$, $b_1 d_1 \neq 0$, is that

$$\alpha = \frac{f(s_1)}{g'(s_1)} \tag{3.15}$$

$$\mu = -\frac{g'(s_1)}{b} \tag{3.16}$$

$$k = \frac{f(s_2)}{g(s_2)}\,, \tag{3.17}$$

$f(s) = as^2 + a_1 s + d_1$, $g(s) = bs^2 + b_1 s$ and s_1, s_2 are the roots of $g(s) = 0$.

(B) By using the transformation $Y = e^{-\lambda x}y$, $\lambda \neq 0$, and assuming that (3.15), (3.16) and (3.17) hold, then the operator equation:

$$\underset{x_0}{\overset{x}{I}}{}^{-\alpha}(a+bx)^{1-k}e^{\mu x}\underset{x_0}{\overset{x}{I}}{}^{-1}(a+bx)^{k}e^{-\mu x}\underset{x_0}{\overset{x}{I}}{}^{\alpha-1}e^{-\lambda x}y=0 \tag{3.18}$$

is equivalent to the second order linear differential equation:

$$(a+bx)y'' + [(a_1 - 2a\lambda) + (b_1 - 2b\lambda)x]y' + [a\lambda^2 - a_1\lambda + d_1 + (b\lambda^2 - b_1\lambda)x]y = 0\,. \tag{3.19}$$

(C) The differential equation

$$(a+bx)y'' + (A_1 + B_1 x)y' + (A_0 + B_0 x)y = 0\,, \tag{3.20}$$

$b \neq 0$, $|B_1| + |B_0| > 0$, is equivalent to (3.18) if

$$\alpha = \frac{F(\lambda_i)}{G'(\lambda_i)}$$

$$\mu = -\frac{G'(\lambda_i)}{b} \tag{3.21}$$

$$k = \frac{bF'(\lambda_i) - aG'(\lambda_i)}{b^2} - \frac{F(\lambda_i)}{G'(\lambda_i)}$$

$(i = 1, 2)$, λ_1, λ_2 are the roots of the quadractic equation $G(\lambda) = 0$, $G(\lambda) = b\lambda^2 + B_1\lambda + B_0$ and $F(\lambda) = a\lambda^2 + A_1\lambda + A_0$.

Details of these properties are found in [8].

2. SOME METHODS OF SOLUTIONS AND PARTICULAR EQUATIONS OF SPECIAL FUNCTIONS

2.1 Some Methods. The representation of differential or integro-differential equations by operator equations may simplify the problems of obtaining solutions of such equations. This is made possible by the use of the operational properties of the integro-differential operator of generalized order [1]. In applying some of these properties to the operator equations, one may find their solutions. These solutions are satisfied by the equivalent differential or integro-differential equation.

In the process of solving an operator equation a term of the form $\left(\overset{x}{\underset{a}{I}}{}^{\alpha}.0\right)$ appears in the solution. When $\alpha = n$ is a positive integer this term has been defined as follows: If $F \in C^n$ on $[a,b]$ and

$$\overset{x}{\underset{a}{I}}{}^{-n} F = D^n F = 0$$

then

$$\overset{x}{\underset{a}{I}}{}^{n} \cdot 0 = F = \sum_{i=1}^{n} \frac{C_i (x-a)^{n-i}}{\Gamma(n-i+1)} ,$$

where C_i are arbitrary constants. This definition has been extended ([1], p.20) to include all values of α where $\operatorname{Re}\alpha + n > 0$. In this case the definition takes the form

$$\overset{x}{\underset{a}{I}}{}^{-\alpha} \cdot 0 = F = \sum_{i=0}^{n-1} C_{i+1} \frac{(x-a)^{-\alpha-i-1}}{\Gamma(-\alpha-i)} . \tag{4.1}$$

Equation (3.1). If we consider the operator equation (3.2) representing the second order linear differential equation (3.1) and they are equivalent to each other we would find, by operating on and multiplying both sides of (3.2) by the operators and the quantities:

$$\overset{x}{\underset{x_0}{I}}{}^{w} , \quad (a + bx)^{p-1}, \quad (c + dx)^{q-1}, \ldots \text{ etc.},$$

that

$$Y(x) = \overset{x}{\underset{x_0}{I}}{}^{-w+1} (a + bx)^{-p} (c + dx)^{-p} (c + dx)^{-q}$$
$$\left[K + \overset{x}{\underset{x_0}{I}} (c + dx)^{q-1} (a + bx)^{p-1} \left(\overset{x}{\underset{x_0}{I}}{}^{w} \cdot 0\right)\right] , \tag{4.2}$$

where K is an arbitrary constant, w is not a positive integer, and $\overset{x}{\underset{x_0}{I}}{}^{w} \cdot 0$ may be determined as indicated by (4.1) after defining the lower limit of the interval of integration which is usually chosen from among the singular points of the equivalent differential equation (see [2], p.52). It is clear that (4.2) is a linear combination of the two particular solutions

$$Y_1(x) = K \overset{x}{\underset{x_0}{I}}{}^{1-w}(a+bx)^{-p}\,(c+dx)^{-q}$$

$$Y_2(x) = \overset{x}{\underset{x_0}{I}}{}^{1-w}(a+bx)^{-p}\,(c+dx)^{-q}\,\overset{x}{\underset{x_0}{I}}\,(a+bx)^{p-1}\,(c+dx)^{q-1}\left(\overset{x}{\underset{x_0}{I}}{}^{w}\cdot 0\right)$$

each of which may be easily shown to satisfy (3.2) and consequently they satisfy (3.1).

Equation (3.20). The representation of this equation by the operator equation (3.18) may be used to solve the equation by the operational method. By operating on and multiplying both sides of (3.18) by:

$$\overset{x}{\underset{x_0}{I}}{}^{\alpha}, \quad (a+bx)^{k-1}, \quad e^{-\mu x}, \text{ etc.}$$

respectively we have

$$y(x_1,\lambda) = e^{\lambda x}\,\overset{x}{\underset{x_0}{I}}{}^{1-\alpha}(a+bx)^{-k}\,e^{\mu x}\left[K + \int_{x_0}^{x} e^{-\mu t}(a+bt)^{k-1}\left(\overset{t}{\underset{x_0}{I}}\cdot 0\right)dt\right], \tag{4.3}$$

where K is an arbitrary constant and $\left(\overset{x}{\underset{x_0}{I}}\cdot 0\right)$ may be determined after defining the lower limit of integration ([2], p.52, [3], [8]). It may be noted that (4.3) is a linear combination of the two particular solutions

$$y_1(x;\lambda) = Ke^{\lambda x}\,\overset{x}{\underset{x_0}{I}}{}^{1-\alpha}(a+bx)^{-k}\,e^{\mu x} \tag{4.4}$$

$$y_2(x;\lambda) = e^{\lambda x}\,\overset{x}{\underset{x_0}{I}}{}^{1-\alpha}(a+bx)^{-k}\,e^{\mu x}(a+bx)^{k-1}\left(\overset{x}{\underset{x_0}{I}}{}^{\alpha}\cdot 0\right). \tag{4.5}$$

These solutions satisfy (3.18) and consequently (4.3) satisfies (3.20).

Remark. In most of these equations solutions y_1 may be considered as the generalized Rodrigues formulae.

2.2 Particular Cases

(1) Confluent hypergeometric equation. The hypergeometric differential equation together with the more general forms of differential equations such as Riemann – Papperitz and Gauss's types have been discussed in a detailed study in a previous paper [2]. So, it is appropriate to present, as an example, this particular equation. This equation is given by

$$xy'' + (\beta - x)\,y' - \gamma y = 0\,. \tag{4.6}$$

By comparing this equation with (3.20) we notice that $F(\lambda) = \beta\lambda - \gamma$, $G(\lambda) = \lambda^2 - \lambda = 0$ and the roots $\lambda_1 = 0$ and $\lambda_2 = 1$.

(A) The case when $\lambda = 0$. We have $\mu = 1$, $\alpha = \gamma$ and $k = \beta - \gamma$. Thus the equivalent operator equation may be written as

$$\text{(A.1)} \qquad \overset{x}{\underset{x_0}{I}}{}^{-\gamma}\, x^{\gamma-\beta+1}\, e^{x}\, \overset{x}{\underset{x_0}{I}}{}^{-1}\, x^{\beta-\gamma}\, e^{-x}\, \overset{x}{\underset{x_0}{I}}{}^{\gamma-1}\, y = 0 ,$$

whose solution may be given by

$$\text{(A.2)} \qquad y(x;0) = y_1(x;0) + y_2(x;0) ,$$

where

$$y_1(x;0) = K_1\, \overset{x}{\underset{0}{I}}{}^{1-\gamma}\, e^{x}\, x^{\gamma-\beta} = K_1\, x^{1-\beta}\, {}_1F_1(\gamma-\beta+1\,;\, 2-\beta;\, x)$$

$$y_2(x;0) = \overset{x}{\underset{0}{I}}{}^{1-\gamma}\, e^{x}\, x^{\gamma-\beta}\, \overset{x}{\underset{0}{I}}\, e^{-x}\, x^{\beta-\gamma-1} \left(\overset{x}{\underset{0}{I}}{}^{\gamma} \cdot 0 \right) = K_2\, {}_1F_1(\gamma\,;\, \beta\,;\, x) ,$$

where K_1 is an arbitrary constant, the lower limit is determined to be zero and

$$\overset{x}{\underset{0}{I}}{}^{\gamma} \cdot 0 = K_2\, \frac{x^{\gamma-1}(\beta-1)}{\Gamma(\gamma)} ,$$

with K_2 being an arbitrary constant.

(B) The case when $\lambda = 1$. In this case we have $\mu = -1$, $\alpha = \beta - \gamma$, $k = \gamma$ and the equivalent operator equation may be written as

$$\text{(B.1)} \qquad \overset{x}{\underset{0}{I}}{}^{\gamma-\beta}\, x^{1-\gamma}\, e^{-x}\, \overset{x}{\underset{0}{I}}{}^{-1}\, x^{\gamma}\, e^{x}\, \overset{x}{\underset{0}{I}}{}^{\beta-\gamma-1}\, e^{-x}\, y = 0$$

which has a solution of the form

$$\text{(B.2)} \qquad y(x;1) = y_1(x;1) + y_2(x;1)$$

where

$$y_1(x;1) = C_1\, e^{x}\, \overset{x}{\underset{0}{I}}{}^{\gamma-\beta+1} x^{-\gamma}\, e^{-x} = C\, x^{1-\beta}\, {}_1F_1(\gamma-\beta+1\,;\, 2-\beta; x)$$

$$y_2(x;1) = e^{x}\, \overset{x}{\underset{0}{I}}{}^{\gamma-\beta+1}\, x^{-\gamma}\, e^{-x}\, \overset{x}{\underset{0}{I}}\, e^{x}\, x^{\gamma-1} \left(\overset{x}{\underset{0}{I}}{}^{\beta-\gamma} \cdot 0 \right)$$

$$= D\, {}_1F_1(\gamma;\beta;x)$$

where C_1 is an arbitrary constant, β, γ are assumed to be parameters with ranges determined by the validity of the operations and the properties of function involved,

$$\overset{x}{\underset{0}{I}}{}^{\beta-\gamma} \cdot 0 = (\beta - 1)\, D\, \frac{x^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} , \quad \text{and} \quad C = C_1\, \frac{\Gamma(1-\gamma)}{\Gamma(2-\beta)} ,$$

D is an arbitrary constant.

Details of this work may be found in [8].

(2) Legendre Differential Equation. The equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{4.7}$$

is a particular case of (3.1) where $a=c=d=1$, $b=-1$, $a_1=0$, $b_1=-2$ and $d_1=\alpha(\alpha+1)$. The roots of the quadratic equation

$$w^2 - w - \alpha(\alpha+1) = 0$$

are $w_1=\alpha+1$, $w_2=-\alpha$, and $p_1=q_1=-\alpha$, $p_2=q_2=\alpha+1$. Therefore the two operator equations

$$\underset{x_0}{\overset{x}{I}}{}^{-\alpha-1}(x^2-1)^{\alpha+1}\underset{x_0}{\overset{x}{I}}{}^{-1}(x^2-1)^{-\alpha}\underset{x_0}{\overset{x}{I}}{}^{\alpha}\, y = 0, \qquad (\mathrm{Re}\,\alpha > -1) \tag{4.7$_1$}$$

$$\underset{x_0}{\overset{x}{I}}{}^{\alpha}(x^2-1)^{-\alpha}\underset{x_0}{\overset{x}{I}}{}^{-1}(x^2-1)^{\alpha+1}\underset{x_0}{\overset{x}{I}}{}^{-\alpha-1} y = 0, \qquad (\mathrm{Re}\,\alpha < 0) \tag{4.7$_2$}$$

are equivalent to (4.7) whose solutions are solutions of $(4.7)_1$ or $(4.7)_2$. If $(4.7)_1$ is taken then its solutions may be given by

$$y_1(x) = K_1 \underset{x_0}{\overset{x}{I}}{}^{-\alpha}(x^2-1)^{\alpha} \tag{4.8}$$

$$y_2(x) = K_2 \underset{x_0}{\overset{x}{I}}{}^{-\alpha}(x^2-1)^{\alpha}\underset{x_0}{\overset{x}{I}}(x^2-1)^{-\alpha-1}, \tag{4.9}$$

where K_1 and K_2 are arbitrary constants. It has been shown ([3], p.96), that $y_1(x)$ and $y_2(x)$ are the Legendre functions $P_\alpha(x)$ and $Q_\alpha(x)$ respectively. Equation (4.8) may represent the generalized Rodrigues formula for the solution of Legendre equation which can be written as:

$$y_1 = \frac{1}{2^\alpha\Gamma(\alpha+1)}\underset{\pm\infty}{\overset{x}{I}}{}^{-\alpha}(x^2-1)^{\alpha}. \tag{4.8$_1$}$$

We find that

$$y_1 = P_\alpha(x) = \frac{\Gamma(2\alpha+1)x^\alpha}{2^\alpha[\Gamma(\alpha+1)]^2}\,{}_2F_1\left[\begin{matrix}-\tfrac12\alpha,\ -\tfrac12\alpha+\tfrac12\ ; & \dfrac{1}{x^2}\\ \tfrac12-\alpha\ ; & \end{matrix}\right] \tag{4.8$_2$}$$

which for $\alpha=n$ (positive integer) is reduced to $P_n(x)$. Also we have

$$y_2 = Q_\alpha(x) = \frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{2^{\alpha+1}x^{\alpha+1}\Gamma(\alpha+\frac32)}\,{}_2F_1\left[\begin{matrix}\tfrac12(\alpha+1),\ \tfrac12(\alpha+2) & \dfrac{1}{x^2}\\ \alpha+\tfrac32 & \end{matrix}\right].$$

For further details see [5].

(3) Bessel's differential equation. By using the transformation $y=x^\nu z$ the equation

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \tag{5}$$

can be written in the form

$$x z'' + (1 + 2\nu) z' + x z = 0. \tag{5.1}$$

Comparing this equation with (3.20) we find that $a = 0$, $b = 1$, $A_1 = 2\nu + 1$, $B_1 = 0$, $A_0 = 0$ and $B_0 = 1$. Also we observe that $G(\lambda) = \lambda^2 + 1$, $F(\lambda) = (2\nu + 1)\lambda$ and the roots of $G(\lambda) = 0$ are $\lambda_1 = i$ and $\lambda_2 = -i$ $(i^2 = -1)$.

(A) The case when $\lambda = i$. In this case, by (3.21) we find that $\alpha = \nu + \frac{1}{2}$, $\mu = -2i$ and $k = \nu + \frac{1}{2}$. Thus the equivalent operator equation may be written as

(A.1) $$\overset{x}{\underset{0}{I}}{}^{-(\nu+\frac{1}{2})}\, x^{\frac{1}{2}-\nu}\, e^{-2ix}\, \overset{x}{\underset{0}{I}}{}^{-1}\, x^{\nu+\frac{1}{2}}\, e^{2ix}\, \overset{x}{\underset{0}{I}}{}^{\nu-\frac{1}{2}}\, e^{-ix}\, z = 0,$$

which has the solutions of the form

(A.2) $$z(x;i) = z_1(x;i) + z_2(x;i),$$

where

$$z_1(x;i) = K_1\, e^{ix}\, \overset{x}{\underset{0}{I}}{}^{\frac{1}{2}-\nu}\, e^{-2ix}\, x^{-(\nu+\frac{1}{2})}$$

$$z_2(x;i) = e^{ix}\, \overset{x}{\underset{0}{I}}{}^{\frac{1}{2}-\nu}\, e^{-2ix}\, x^{-(\nu+\frac{1}{2})}\, \overset{x}{\underset{0}{I}}{}^{\nu-\frac{1}{2}}\, e^{2ix} \left(\overset{x}{\underset{0}{I}}{}^{\nu+\frac{1}{2}} \cdot 0\right)$$

(K_1 is an arbitrary constant).

If $\nu = n + \frac{1}{2}$, (n is a positive integer), we have

(A.3) $$z_1(x;i) = K_1\, (-2)^n\, x^{-n-\frac{1}{2}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left[-iJ_{n+\frac{1}{2}}(x) + (-1)^n J_{-n-\frac{1}{2}}(x)\right]$$

$$z_2(x;i) = M_1\, 2^{-n-\frac{3}{2}}\, \pi^{\frac{1}{2}}\, x^{-n-\frac{1}{2}} \left[J_{n+\frac{1}{2}}(x) - (-1)^n\, iJ_{-n-\frac{1}{2}}(x)\right],$$

from which we obtain the solution of (5) for this case which may be given by

(A.4) $$y(x;i) = L_1\, J_{n+\frac{1}{2}}(x) + M_1\, J_{-n-\frac{1}{2}}(x),$$

where L_1 and M_1 are arbitrary constants.

(B) When $\lambda = -i$, we find that $\alpha = \nu + \frac{1}{2}$, $k = \nu + \frac{1}{2}$, $\mu = -2i$ and the equivalent operator equation takes the form

(B.1) $$\overset{x}{\underset{0}{I}}{}^{-(\nu+\frac{1}{2})}\, x^{\frac{1}{2}-\nu}\, e^{2ix}\, \overset{x}{\underset{0}{I}}{}^{-1}\, x^{\nu+\frac{1}{2}}\, e^{-2ix}\, \overset{x}{\underset{0}{I}}{}^{\nu-\frac{1}{2}}\, e^{ix}\, z = 0,$$

whose solutions may be given by

(B.2) $$z(x;-i) = z_1(x;-i) + z_2(x;-i)$$

where

$$z_1(x;-i) = C_1\, e^{-ix}\, \overset{x}{\underset{0}{I}}{}^{\frac{1}{2}-\nu}\, e^{2ix}\, x^{-(\nu+\frac{1}{2})}$$

$$z_2(x;-i) = e^{ix}\, \overset{x}{\underset{0}{I}}{}^{\frac{1}{2}-\nu}\, e^{2ix}\, x^{-(\nu+\frac{1}{2})}\, \overset{x}{\underset{0}{I}}\, e^{-2ix}\, x^{\nu-\frac{1}{2}} \left(\overset{x}{\underset{0}{I}}{}^{\nu+\frac{1}{2}} \cdot 0\right)$$

where C_1 is an arbitrary constant. Also when $\nu = n + \frac{1}{2}$, then we find that the solution of (5) for this case is given by

$$y(x;-i) = L_2\, J_{n+\frac{1}{2}}(x) + M_2\, J_{-n-\frac{1}{2}}(x)$$

where L_2 and M_2 are arbitrary constants.

For details see [8].

(4) <u>Equations of Laguerre type</u>. In this case certain types of Laguerre differential equations are discussed and solutions are obtained in the form of Laguerre function of generalized type [7] which may be called as Rodrigues generalized formulae for Laguerre functions. These solutions are obtained by dealing with the equivalent operator equations as indicated above.

Consider the second order differential equation

$$xu'' + \left[(2\lambda + \mu)x + \beta + 1\right]u'$$
$$+ \left[\lambda(\lambda+\mu)x + \beta(\lambda+\mu) + \mu(\alpha+1) + \lambda\right]u = 0\,, \qquad (6.1)$$

where λ, μ, α and β are parameters.

This equation can be easily shown to be equivalent to the operator equation

$$\overset{x}{\underset{0}{I}}{}^{-\alpha-1}\, x^{\alpha+\beta+1}\, e^{-\mu x}\, \overset{x}{\underset{0}{I}}{}^{-1}\, x^{-\alpha-\beta}\, e^{\mu x}\, \overset{x}{\underset{0}{I}}{}^{\alpha}\, x^{\beta}\, e^{\lambda x}\, u = 0\,, \qquad (6.2)$$

whose solution may be given by

$$u(x) = u_1(x) + u_2(x)\,,$$

where

$$u_1(x) = K_1\, x^{-\beta} e^{-\lambda x}\, \overset{x}{\underset{0}{I}}{}^{-\alpha}\, e^{-\mu x}\, x^{\alpha+\beta} \qquad (6.2)_1$$

$$\mu_2(x) = K_2\, x^{-\beta} e^{-\lambda x}\, \overset{x}{\underset{0}{I}}{}^{-\alpha}\, e^{-\mu x}\, x^{\alpha+\beta}\, \overset{x}{\underset{0}{I}}\, x^{-\alpha-\beta-1}\, e^{\mu x}\,, \qquad (6.2)_2$$

K_1 and K_2 are arbitrary constants.

Equation $(6.2)_1$ represents the generalized Rodrigues formula for the solution of (6.1) which may be denoted by $L_\alpha^{(\beta)}(\lambda;\mu;x)$, for if we put $K_1 = 1/\Gamma(\alpha+1)$ we would get

$$\mu_1(x) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\, e^{-(\lambda+\mu)x} \sum_{k=0}^{\infty} \frac{(-\alpha)_k\,(\mu x)^k}{(\beta+1)_k\,\Gamma(k+1)}\,,$$

where $(\alpha)_r = \alpha(\alpha+1)\ldots(\alpha+r-1)$, $(\alpha)_0 = 1$. Thus we find that

$$L_\alpha^{(\beta)}(\lambda;\mu;x) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\, e^{-(\mu+\lambda)x}\, {}_1F_1(-\alpha;\beta+1;\mu x)\,.$$

Particular cases of (6.1)

(A) Let $\beta = 0$ in (6.1). Then we have the differential equation

(A.1) $$xu'' + [(2\lambda+\mu)x + 1]u' + [\lambda(\lambda+\mu)x + \mu(\alpha+1) + \lambda]u = 0$$

with the equivalent operator equation

(A.2) $$\overset{x}{\underset{0}{I}}{}^{-\alpha-1} x^{\alpha+1} e^{-\mu x} \overset{x}{\underset{0}{I}}{}^{-1} x^{-\alpha} e^{\mu x} \overset{x}{\underset{0}{I}}{}^{\alpha} e^{x} u = 0 .$$

We notice that the solutions $(6.2)_1$ and $(6.2)_2$ represent solutions of (A.1) when $\beta = 0$. Furthermore we observe that the Rodrigues formula for the solution of (A.1) is $L_\alpha^{(0)}(\lambda;\mu;x)$.

(B) If $\lambda = -1$, $\mu = 1$, then we would have Laguerre differential equation

(B.1) $$xu'' + (1+\beta-x)u' + \alpha u = 0$$

with its equivalent operator equation

(B.2) $$\overset{x}{\underset{0}{I}}{}^{-\alpha-1} x^{\alpha+\beta+1} e^{-x} \overset{x}{\underset{0}{I}}{}^{-1} x^{-\alpha-\beta} e^{x} \overset{x}{\underset{0}{I}}{}^{\alpha} x^{\beta} e^{-x} u = 0 .$$

Equation (B.1) is satisfied by the solutions $(6.2)_1$ and $(6.2)_2$ with $\lambda = -1$ and $\mu = 1$. In particular, the Rodrigues formula for the solution of (B.1) has the form:

$$L_\alpha^{(\beta)}(-1;1;x) = \frac{1}{\Gamma(\alpha+1)} x^{-\beta} e^{x} \overset{x}{\underset{0}{I}}{}^{-\alpha} e^{-x} x^{\alpha+\beta} .$$

It may be pointed out that the Rodrigues formula presented here is in its generalized form, for if $\alpha = n$, a positive integer, we have $L_n^{(\beta)}(-1;1;x)$ which is of the same form as mentioned in references of Special Functions.

Simple Laguerre differential equations and their equivalent operator equations together with their solutions follow in the same way from the above.

Remarks. (1) In this article, many cases of integro-differential equations have not been discussed. For further study it may be of interest to see [10] in the bibliography.

(2) The method applied here may be applied to all types of equations whose solutions are some types of special functions. In fact the writer is studying some operator equations which are equivalent to a generalized form of integro-differential equation of which most of the equations of special functions are particular cases.

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