Reading notes

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This document contains all notes taken while reading materials (e.g., textbooks, literature) in preparation for the capstone project. Black text are information consolidated from the readings; blue text are notes (proofs, explanations); red text are questions.

Contents

Todo list

1 Theory of Ordinary Differential Equations (Coddington Levinson)

1.1 Chapter 11: Algebraic properties of linear boundary-value problems on a finite interval

1.1.1 Introduction

Definition 1.1.1. Let L be the **linear differential operator of oder** n ($n \geq 1$) defined by

$$Lx = p_0 x^{(n)} + p_1 x^{(n-1)} + \cdots + p_{n-1}x' + p_n x$$

where the p_k are complex-valued functions of class C^{n-k} on a closed bounded interval $[a, b]$ (i.e., derivatives $p_k, p_k', \ldots, p_k^{(n-k)}$ exist on $[a, b]$ and are continuous) and $p_0(t) \neq 0$ on $[a, b]$.

Definition 1.1.2. **Homogeneous boundary conditions** refer to a set of equations/constraints of the type

$$\sum_{k=1}^{n}(M_{jk}x^{(k-1)}(a) + N_{jk}x^{(k-1)}(b)) = 0 \quad (j = 1, \ldots, m) \tag{1.1.1}$$

where M_{jk}, N_{jk} are complex constants.

Definition 1.1.3. A **homogeneous boundary-value problem** concerns finding the solutions of

$$Lx = 0$$

on $[a, b]$ which satisfy some homogeneous boundary conditions defined above.

Definition 1.1.4. For any homogeneous boundary value problem, an **adjoint problem** refers to the problem of finding the solutions of

$$L^+x := (-1)^n(\bar{p}_0 x)^{(n)} + (-1)^{n-1}(\bar{p}_1 x)^{(n-1)} + \cdots + \bar{p}_n x = 0$$

on $[a, b]$ which satisfy some homogeneous boundary conditions "complementary" to the conditions associated with the solutions of $Lx = 0$.

Theorem 1.1.5. *(Green's formula) For $u, v \in C^n$ on $[a, b]$,*

$$\int_{t_1}^{t_2}(Lu)\bar{v}\,dt - \int_{t_1}^{t_2}u(\overline{L^+v})\,dt = [uv](t_2) - [uv](t_1) \tag{1.1.2}$$

where $a \leq t_1 < t_2 \leq b$ and $[uv](t)$ is the form in $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$ given by

$$[uv](t) = \sum_{m=1}^{n}\sum_{j+k=m-1}(-1)^i u^{(k)}(t)(p_{n-m}\bar{v})^{(j)}(t) \tag{1.1.3}$$

Remark 1.1.6. Alternatively, $[uv](t)$ can be written as (checked for $n = 2$)

$$[uv](t) = \sum_{j,k=1}^{n} B_{jk}(t)u^{(k-1)}(t)\bar{v}^{(j-1)}(t) \tag{1.1.4}$$

where B_{jk} are the j, k-entry of the $n \times n$ matrix

$$B(t) = \begin{bmatrix} B_{11} & B_{12} & \cdots & \cdots & p_0(t) \\ \vdots & \vdots & \cdots & -p_0(t) & 0 \\ (-1)^{n-1}p_0(t) & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{1.1.5}$$

Since $B(t)$ is square with $\det B(t) = (p_0(t))^n$ where $p_0(t) \neq 0$ on $[a, b]$ (as in the definition of L), $B(t)$ is nonsingular/invertible for $t \in [a, b]$.

Definition 1.1.7. For vectors $f = (f_1, \dots, f_k)$, $g = (g_1, \dots, g_k)$, define the product

$$f \cdot g := \sum_{i=1}^{k} f_i \bar{g}_i.$$

Note that $f \cdot g = g^* f$.

Definition 1.1.8. A **semibilinear form** is a complex-valued function $\mathcal{S}$ defined for pairs of vectors $f = (f_1, \dots, f_k)$, $g = (g_1, \dots, g_k)$ satisfying

$$\mathcal{S}(\alpha f + \beta g, h) = \alpha \mathcal{S}(f, h) + \beta \mathcal{S}(g, h)$$
$$\mathcal{S}(f, \alpha g + \beta h) = \bar{\alpha} \mathcal{S}(f, g) + \bar{\beta} \mathcal{S}(f, h)$$

for any complex numbers α, β and vectors f, g, h.

Note that $\mathcal{S}$ is linear in the first argument but not the second one. If $\mathcal{S}$ were bilinear, it would be linear in each argument.

Remark 1.1.9. If

$$S = \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \cdots & s_{kk} \end{bmatrix},$$

then $Sf \cdot g$ is a semibilinear form

$$\begin{aligned} \mathcal{S}(f, g) &:= Sf \cdot g \\ &= \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \cdots & s_{kk} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{k} s_{1j} f_j \\ \vdots \\ \sum_{j=1}^{k} s_{kj} f_j \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\ &= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} s_{ij} f_j \right) \bar{g}_i \\ &= \sum_{i,j=1}^{k} s_{ij} f_i \bar{g}_i. \end{aligned} \tag{1.1.6}$$

Indeed:

$$\begin{aligned} \mathcal{S}(\alpha f + \beta g, h) &= \sum_{i,j=1}^{k} s_{ij} (\alpha f_j + \beta g_j) \bar{h}_i \\ &= \alpha \sum_{i,j=1}^{k} s_{ij} f_j \bar{h}_i + \beta \sum_{i,j=1}^{k} g_j \bar{h}_i \\ &= \alpha S f \cdot h + \beta S g \cdot h \\ &= \alpha \mathcal{S}(f, h) + \beta \mathcal{S}(g, h). \end{aligned}$$

Similarly,

$$\mathcal{S}(f, \alpha g + \beta h) = \sum_{i,j=1}^{k} s_{ij} f_j (\alpha g_i \bar{+} \beta h_i)$$

$$
\begin{aligned}
&= \bar{\alpha} \sum_{i,j=1}^{k} s_{ij} f_j \bar{g}_i + \bar{\beta} \sum_{i,j=1}^{k} f_j \bar{h}_i \\
&= \bar{\alpha} S f \cdot g + \bar{\beta} S f \cdot h \\
&= \bar{\alpha} \mathcal{S}(f, g) + \bar{\beta} \mathcal{S}(f, h).
\end{aligned}
$$

Remark 1.1.10. Under a similar matrix framework, we see that $[uv](t)$ is a semibilinear form with matrix $B(t)$: Let $\vec{u} = (u, u', \ldots, u^{(n-1)})$ and $\vec{v} = (v, v', \ldots, v^{(n-1)})$. Then we have

$$
\begin{aligned}
[uv](t) &= \sum_{j,k=1}^{n} B_{jk}(t) u^{(k-1)}(t) \overline{v}^{(j-1)}(t) \quad \text{(by (1.1.4))} \\
&= \sum_{i,j=1}^{n} (B_{ij} u^{(j-1)} \overline{v}^{(i-1)})(t) \qquad (1.1.7) \\
&= (B\vec{u} \cdot \vec{v})(t) \\
&= \mathcal{S}(\vec{u}, \vec{v})(t).
\end{aligned}
$$

With this notation, we can rewrite the right hand side of Green's formula as a semibilinear form below:

$$
\begin{aligned}
[uv](t_2) - [uv](t_1) &= \sum_{j,k=1}^{n} B_{jk}(t_2) u^{(k-1)}(t_2)\bar{v}^{(j-1)}(t_2) - \sum_{j,k=1}^{n} B_{jk}(t_2) u^{(k-1)}(t_1)\bar{v}^{(j-1)}(t_1) \\
&= B(t_2)\vec{u}(t_2)\cdot\vec{v}(t_2) - B(t_1)\vec{u}(t_1)\cdot\vec{v}(t_1) \\
&= \begin{bmatrix} B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots \\ B_{n1}(t_2) & \dots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} - \\
&\quad \begin{bmatrix} B_{11}(t_1) & \cdots & B_{1n}(t_1) \\ \vdots & & \vdots \\ B_{n1}(t_1) & \dots & B_{nn}(t_1) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \end{bmatrix} \\
&= \begin{bmatrix} -B_{11}(t_1) & \cdots & -B_{1n}(t_1) \\ \vdots & & \vdots \\ -B_{n1}(t_1) & \dots & -B_{nn}(t_1) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \end{bmatrix} + \\
&\quad \begin{bmatrix} B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots \\ B_{n1}(t_2) & \dots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&= \begin{bmatrix} -B_{11}(t_1) & \cdots & -B_{1n}(t_1) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -B_{n1}(t_1) & \cdots & -B_{nn}(t_1) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & B_{n1}(t_2) & \cdots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \\ u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \\ \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&= \begin{bmatrix} -B(t_1) & 0_n \\ 0_n & B(t_2) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \\ u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \\ \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&=: \hat{B} \begin{bmatrix} \vec{u}(t_1) \\ \vec{u}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \vec{v}(t_1) \\ \vec{v}(t_2) \end{bmatrix}.
\end{aligned}
\tag{1.1.8}
$$

Since $\det \hat{B} = (-1)^n \det B(t_1) \det B(t_2)$, $\hat{B}$ is nonsingular for $t_1, t_2 \in [a,b]$ (since $B(t)$ is nonsingular for $t \in [a,b]$, as shown before).

Why the $(-1)^n$?

1.1.2 Boundary form formula

Definition 1.1.11. Given any set of $2mn$ complex constants M_{ij}, N_{ij} $(i = 1, \dots, m;\ j = 1, \dots, n)$, define m **boundary operators (boundary forms)** $U_1, \dots, U_m$ for functions x on $[a,b]$, for which $x^{(j)}$ $(j = 1, \dots, n-1)$ exists at a and b, by

$$U_i x = \sum_{j=1}^{n} (M_{ij} x^{(j-1)}(a) + N_{ij} x^{(j-1)}(b)) \quad (i = 1, \dots, m) \tag{1.1.9}$$

U_i are **linearly independent** if the only set of complex constants $c_1, \ldots, c_m$ for which

$$\sum_{i=1}^{m} c_i U_i x = 0$$

for all $x \in C^{n-1}$ on $[a, b]$ is $c_1 = c_2 = \cdots = c_m = 0$.

Remark 1.1.12. Note that for $\alpha, \beta \in \mathbb{C}$ and $x_1, x_2 \in C^{n-1}$ on $[a, b]$,

$$\begin{aligned}
U_i(\alpha x_1 + \beta x_2) &= \sum_{j=1}^{n}(M_{ij}(\alpha x_1 + \beta x_2)^{(j-1)}(a) + N_{ij}(\alpha x_1 + \beta x_2)^{(j-1)}(b)) \\
&= \alpha \sum_{j=1}^{n}(M_{ij}x_1^{(j-1)}(a) + N_{ij}x_1^{(j-1)}(b)) + \beta \sum_{j=1}^{n}(M_{ij}x_2^{(j-1)}(a) + N_{ij}x_2^{(j-1)}(b)) \quad \text{(by linearity of derivatives)} \\
&= \alpha U_i x_1 + \beta U_i x_2.
\end{aligned}$$

So U_i are linear operators.

Remark 1.1.13. To describe (1.1.9) with matrices, define

$$\xi := \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix}; \quad U := \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}; \quad M := \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix}; \quad N := \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix}.$$

Then (1.1.9) can be written as

$$Ux = M\xi(a) + N\xi(b).$$

Indeed:

$$\begin{aligned}
M\xi(a) + N\xi(b) &= \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} x(a) \\ x'(a) \\ \vdots \\ x^{(n-1)}(a) \end{bmatrix} + \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} x(b) \\ x'(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n M_{1j}x^{(j-1)}(a) \\ \vdots \\ \sum_{j=1}^n M_{mj}x^{(j-1)}(a) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n N_{1j}x^{(j-1)}(b) \\ \vdots \\ \sum_{j=1}^n N_{mj}x^{(j-1)}(b) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n (M_{1j}x^{(j-1)}(a) + N_{1j}x^{(j-1)}(b)) \\ \vdots \\ \sum_{j=1}^n (M_{mj}x^{(j-1)}(a) + N_{mj}x^{(j-1)}(b)) \end{bmatrix} \\
&= \begin{bmatrix} U_1 x \\ \vdots \\ U_m x \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix} x = Ux.
\end{aligned}$$

Define the $m \times 2n$ matrix

$$(M : N) := \begin{bmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{bmatrix}.$$

Then $U_1, \dots, U_m$ are linearly independent if and only if $\text{rank}(M : N) = m$, or equivalently, $\text{rank}(U) = m$. Recall that the rank of a matrix is the largest number of linearly independent rows or columns in it. For a matrix $A_{m\times n}$, $\text{rank}(A) \le \min\{m, n\}$ and $\text{rank}(A) = \text{rank}(A^T)$.

Ux can be written as

$$\begin{aligned} Ux &= \begin{bmatrix} \sum_{j=1}^{n}(M_{1j}x^{(j-1)}(a) + N_{1j}x^{(j-1)}(b)) \\ \vdots \\ \sum_{j=1}^{n}(M_{mj}x^{(j-1)}(a) + N_{mj}x^{(j-1)}(b)) \end{bmatrix} \\ &= \begin{bmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} x(a) \\ \vdots \\ x^{(n-1)}(a) \\ x(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix} \\ &= (M : N) \begin{bmatrix} \xi(a) \\ \xi(b) \end{bmatrix}. \end{aligned}$$

Definition 1.1.14. If $U = (U_1, \dots, U_m)$ is any boundary form with $\text{rank}(U) = m$ and $U_c = (U_{m+1}, \dots, U_{2n})$ any form with $\text{rank}(U_c) = 2n - m$ such that $(U_1, \dots, U_{2n})$ has rank $2n$, then U and U_c are **complementary boundary forms**. "Adjoining" $U_{m+1}, \dots, U_{2n}$ to $U_1, \dots, U_m$ is equivalent to imbedding the matrix $(M : N)$ in a $2n \times 2n$ nonsingular matrix (recall that for square matrices, nonsingular $\iff$ full rank).

We wish to describe the right hand side of Green's formula (1.1.2) as a linear combination of a boundary form U and a complementary form U_c. To do so, we consider the following results about the semibilinear form (1.1.6).

Definition 1.1.15. For a matrix $A = (a_{ij})$, its **adjoint** is defined as the conjugate transpose $A^* = (\bar{a}_{ij})$.

Proposition 1.1.16. *In the context of the semibilinear form* (1.1.6), *we have*

$$Sf \cdot g = f \cdot S^* g. \tag{1.1.10}$$

Proof.

$$\begin{aligned} Sf \cdot g &= \sum_{i,j=1}^{k} s_{ij} f_j \bar{g}_i \quad \text{(by (1.1.6))}; \\ f \cdot S^* g &= \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} \bar{s}_{11} & \cdots & \bar{s}_{k1} \\ \vdots & & \vdots \\ \bar{s}_{1k} & \cdots & \bar{s}_{kk} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\ &= \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} \sum_{j=1}^{k} \bar{s}_{j1} g_j \\ \vdots \\ \sum_{j=1}^{k} \bar{s}_{jk} g_j \end{bmatrix} \\ &= \sum_{i=1}^{k} f_i \cdot \overline{\left(\sum_{j=1}^{k} \bar{s}_{ji} g_j \right)} \end{aligned}$$

$$= \sum_{i=1}^{k} f_i \cdot \left(\sum_{j=1}^{k} s_{ji} \bar{g}_j \right)$$
$$= \sum_{i,j=1}^{k} s_{ji} f_i \bar{g}_j = Sf \cdot g.$$

□

Proposition 1.1.17. *Let $\mathcal{S}$ be the semibilinear form associated with a nonsingular matrix S. Suppose $\bar{f} := Ff$ where F is a nonsingular matrix. Then there exists a unique nonsingular matrix G such that if $\bar{g} = Gg$, then $\mathcal{S}(f, g) = \bar{f} \cdot \bar{g}$ for all f, g.*

Proof. Let $G := (SF^{-1})^*$, then

$$\begin{aligned}
\mathcal{S}(f, g) &= Sf \cdot g \\
&= S(F^{-1}F)f \cdot g \\
&= SF^{-1}(Ff) \cdot g \\
&= SF^{-1}\bar{f} \cdot g \\
&= \bar{f} \cdot (SF^{-1})^* g \quad \text{(by (1.1.10))} \\
&= \bar{f} \cdot G * g \\
&= \bar{f} \cdot \bar{g}.
\end{aligned}$$

To see that G is nonsingular, note that $\det G = \det((\overline{SF^{-1}})^T) = \det(\overline{SF^{-1}}) = \overline{\det(SF^{-1})} = \overline{\det(S)\det(F)^{-1}} \neq 0$ since S, F are nonsingular. □

Proposition 1.1.18. *Suppose $\mathcal{S}$ is associated with the unit matrix E, i.e., $\mathcal{S}(f, g) = f \cdot g$. Let F be a nonsingular matrix such that the first j ($1 \leq j < k$) components of $\bar{f} = Ff$ are the same as those of f. Then the unique nonsingular matrix G such that $\bar{g} = Gg$ and $\bar{f} \cdot \bar{g} = f \cdot g$ (as in Proposition 1.1.17) is such that the last $k - j$ components of $\bar{g}$ are linear combinations of the last $k - j$ components of g with nonsingular coefficient matrix.*

Proof. We note that for the condition on F to hold, F must have the form

$$\begin{bmatrix} E_j & 0_+ \\ F_+ & F_{k-j} \end{bmatrix}_{k \times k}$$

where E_j is the $j \times j$ identity matrix, 0_+ is the $j \times (k - j)$ zero matrix, F_+ is a $(k - j) \times j$ matrix, and F_{k-j} a $(k - j) \times (k - j)$ matrix. Let G be the unique nonsingular matrix in Proposition 1.1.17. Write G as

$$\begin{bmatrix} G_j & G_- \\ G_= & G_{k-j} \end{bmatrix}_{k \times k}$$

where $G_j, G_-, G_=, G_{k-j}$ are $j \times j, j \times (k - j), (k - j) \times j, (k - j) \times (k - j)$ matrices, respectively. By the definition of G,

$$f \cdot g = Ff \cdot Gg = \bar{f} \cdot Gg = G^* \bar{f} \cdot g = G^* Ff \cdot g,$$

(where the third equality follows from a reverse application of (1.1.10) with $\bar{f}$ as f, G^* as S) which implies

$$G^* F = E_k.$$

Since

$$\begin{aligned} G^*F &= \begin{bmatrix} G_j^* & G_=^* \\ G_-^* & G_{k-j}^* \end{bmatrix} \begin{bmatrix} E_j & 0_+ \\ F_+ & F_{k-j} \end{bmatrix} \\ &= \begin{bmatrix} G_j^* + G_=^* F_+ & G_=^* F_{k-j} \\ G_-^* + G_{k-j}^* F_+ & G_{k-j}^* F_{k-j} \end{bmatrix} \\ &= \begin{bmatrix} E_j & 0_{j\times(k-j)} \\ 0_{(k-j)\times j} & E_{k-j} \end{bmatrix}. \end{aligned}$$

Thus, $G_=^* F_{k-j} = 0_+$, the $j \times (k-j)$ zero matrix. But $\det F = \det(E_j) \cdot \det(F_{k-j}) \neq 0$, so $\det F_{k-j} \neq 0$ and we must have $G_=^* = 0_+$, i.e., $G_= = 0_{(k-j)\times j}$. Thus, G is upper-triangular, and so $\det G = \det G_j \cdot \det G_{k-j} \neq 0$, which implies $\det G_{k-j} \neq 0$ and G_{k-j} is nonsingular. Hence,

$$\bar{g} = Gg = \begin{bmatrix} G_j & G_- \\ 0_{(k-j)\times j} & G_{k-j} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}$$

where G_{k-j} is the nonsingular coefficient matrix such that

$$\begin{bmatrix} \bar{g}_{j-1} \\ \vdots \\ \bar{g}_k \end{bmatrix} = G_{k-j} \begin{bmatrix} g_{j-1} \\ \vdots \\ g_k \end{bmatrix}.$$

□

Theorem 1.1.19. *(Boundary-form formula) Given any boundary form U of rank m (Definition 1.1.11), and any complementary form U_c (Definition 1.1.14), there exist unique boundary forms U_c^+, U^+ of rank m and $2n-m$, respectively, such that*

$$[xy](b) - [xy](a) = Ux \cdot U_c^+ y + U_{cx} \cdot U^+ y. \tag{1.1.11}$$

If $\tilde{U}_c$ is any other complementary form to U, and $\tilde{U}_c^+, \tilde{U}^+$ the corresponding forms of rank m and $2n-m$, then

$$\tilde{U}^+ y = C^* U^+ y$$

for some nonsingular matrix C.

Does this mean that, given a boundary form, its adjoint boundary forms are related to each other by linear transformation?

Proof. Recall from (1.1.8) that the left hand side of (1.1.11) can be considered as a semibilinear form $\mathcal{S}(f, g) = \hat{B}f \cdot g$ for vectors

$$f = \begin{bmatrix} x(a) \\ \vdots \\ x^{(n-1)}(a) \\ x(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix}, \ g = \begin{bmatrix} y(a) \\ \vdots \\ y^{(n-1)}(a) \\ y(b) \\ \vdots \\ y^{(n-1)}(b) \end{bmatrix}$$

with the nonsingular matrix

$$\hat{B} = \begin{bmatrix} -B(a) & 0_n \\ 0_n & B(b) \end{bmatrix}.$$

Recall from Remark 1.1.13 that

$$Ux = M\xi(a) + N\xi(b) = (M : N)\begin{bmatrix}\xi(a)\\ \xi(b)\end{bmatrix}$$

for M, N, ξ are as defined there. With the definition of f, we have $f = \begin{bmatrix}\xi(a)\\ \xi(b)\end{bmatrix}$ and thus

$$Ux = (M : N)f.$$

By Definition 1.1.14, $U_c x = (\tilde{M} : \tilde{N})f$ for two appropriate matrices $\tilde{M}, \tilde{N}$ for which

$$H = \begin{bmatrix}M & N\\ \tilde{M} & \tilde{N}\end{bmatrix}_{2n\times 2n}$$

has rank $2n$. Thus,

$$\begin{bmatrix}Ux\\ U_c x\end{bmatrix} = \begin{bmatrix}(M : N)f\\ (\tilde{M} : \tilde{N})f\end{bmatrix} = \begin{bmatrix}M & N\\ \tilde{M} & \tilde{N}\end{bmatrix} f = Hf.$$

By Proposition 1.1.17, there exists a unique $2n \times 2n$ nonsingular matrix J such that $\mathcal{S}(f, g) = Hf \cdot Jg$. Let U^+, U_c^+ be such that

Is the direction correct? e.g., from the existence of J, construct U^+ and U_c^+.

$$Jg = \begin{bmatrix}U_c^+ y\\ U^+ y\end{bmatrix},$$

then (1.1.11) holds since

$$[xy](b) - [xy](a) = \mathcal{S}(f, g) = Hf \cdot Jg = \begin{bmatrix}Ux\\ U_c x\end{bmatrix} \cdot \begin{bmatrix}U_c^+ y\\ U^+ y\end{bmatrix} = Ux \cdot U_c^+ y + U_c x \cdot U^+ y.$$

The second statement in the theorem follows from Proposition 1.1.18 with Hf and Jg corresponding to f and g

Proposition 1.1.18 poses condition on F and invokes the existence of G; what are the objects corresponding to F and G here?

. □

1.1.3 Homogeneous Boundary-value Problems and Adjoint Problems

Definition 1.1.20. For any boundary form U of rank m there is associated the **homogeneous boundary condition**

$$Ux = 0 \tag{1.1.12}$$

for functions $x \in C^{n-1}$ on $[a, b]$. If U^+ is any boundary form of rank $2n - m$ determined as in Theorem 1.1.19, then the homogeneous boundary condition

$$U^+ x = 0 \tag{1.1.13}$$

is an **adjoint boundary condition** to 1.1.12.

Proposition 1.1.21. *By Green's formula* (1.1.2) *and the boundary-form formula* (1.1.11)*, for* $(u, v) := \int_a^b u\bar{v}\,dt$*,*

$$(Lu, v) = (u, L^+ v)$$

for all $u \in C^n$ *on* $[a, b]$ *satisfying* (1.1.12) *and all* $v \in C^n$ *on* $[a, b]$ *satisfying* (1.1.13).

Proof.

$$\begin{aligned}(Lu,v)-(u,L^+v) &= \int_a^b Lu\bar{v}\,dt - \int_a^b u(\overline{L^+v})\,dt \\ &= [uv](a)-[uv](b) \quad \text{(by Green's formula (1.1.2))} \\ &= Uu\cdot U_c^+v + U_cu\cdot U^+v \quad \text{(by boundary-form formula (1.1.11))} \\ &= 0\cdot U_c^+v + U_cu\cdot 0 \quad \text{(by (1.1.12) and (1.1.13))} \\ &= 0.\end{aligned} \tag{1.1.14}$$

□

Remark 1.1.22. Let D, D^+ be the set of functions $u \in C^n$ satisfying (1.1.12) and (1.1.13), respectively. Then Theorem 1.1.19 shows that D^+ is uniquely determined by U, although U^+ is not (because U^+ is uniquely determined by U and U_c?).

U^+ is not uniquely determined by U but by U and U_c. But D^+ is the set of functions x satisfying $U^+x = 0$; if U^+ is not uniquely determined by U, how can D^+ be?

Just like how U is associated with two $m\times n$ matrices M, N (Remark 1.1.13), U^+ is associated with two $n\times(2n-m)$ matrices P, Q such that $(P^*:Q^*)$ has rank $2n-m$ and

$$U^+x = P^*\xi(a) + Q^*\xi(b) \tag{1.1.15}$$

Note that imbedding M, N, P^*, Q^* in the same matrix gives

$$\begin{bmatrix}(M:N)_{m\times 2n}\\(P^*:Q^*)_{(2n-m)\times 2n}\end{bmatrix}_{2n\times 2n} = \begin{bmatrix}M & N\\ P^* & Q^*\end{bmatrix}$$

is an $2n\times 2n$ matrix of full rank.

We want to characterize the adjoint condition (1.1.13) in terms of the matrices M, N, P, Q.

Theorem 1.1.23. *The boundary condition $U^+x = 0$ is adjoint to $Ux = 0$ if and only if*

$$MB^{-1}(a)P = NB^{-1}(b)Q \tag{1.1.16}$$

where $B(t)$ is the $n\times n$ matrix associated with the form $[xy](t)$ ((1.1.5)).

Proof. Let $\eta := (y, y', \dots, y^{(n-1)})$, then $[xy](t) = B(t)\xi(t)\cdot\eta(t)$ by (1.1.7).

Suppose $U^+x = 0$ is adjoint to $Ux = 0$. By definition of adjoint boundary condition 1.1.13, U^+ is determined as in Theorem 1.1.19. But by Theorem 1.1.19, in determining U^+, there exist boundary forms U_c, U_c^+ of rank $2n-m$ and m, respectively, such that 1.1.11 holds.

Put

$$\begin{aligned}U_cx &= M_c\xi(a) + N_c\xi(b) \quad \operatorname{rank}(M_c:N_c) = 2n-m\\ U_c^+y &= P_c^*\eta(a) + Q_c^*\eta(b) \quad \operatorname{rank}(P_c^*:Q_c^*) = m.\end{aligned}$$

Then by the boundary-form formula,

$$B(b)\xi(b)\cdot\eta(b) - B(a)\xi(a)\cdot\eta(a) = (M\xi(a)+N\xi(b))\cdot(P_c^*\eta(a)+Q_c^*\eta(b)) + (M_c\xi(a)+N_c\xi(b))\cdot(P^*\eta(a)+Q^*\eta(b)).$$

But M is $m\times n$ and P_c^* is $m\times n$ (P_c is $n\times m$), so considering the matrices' dimensions, the above should be written as

Is this understanding correct?

$$B(b)\xi(b)\cdot\eta(b) - B(a)\xi(a)\cdot\eta(a) = (P_cM + PM_c)\xi(a)\cdot\eta(a) + (Q_cM + QM_c)\xi(a)\cdot\eta(b)$$

$$(P_cN + PN_c)\xi(b) \cdot \eta(a) + (Q_cN + QN_c)\xi(b) \cdot \eta(b).$$

Thus, we have

$$P_cM + PM_c = -B(a) \qquad P_cN + PN_c = 0_n$$
$$Q_cM + QM_c = 0_n \qquad Q_cN + QN_c = B(b).$$

Since $\det B(t) \neq 0$ on $t \in [a, b]$, $B^{-1}(a)$, $B^{-1}(b)$ exist, and thus

$$\begin{bmatrix} -B^{-1}(a)P_c & -B^{-1}(a)P \\ B^{-1}(b)Q_c & B^{-1}(b)Q \end{bmatrix} \begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix} = \begin{bmatrix} E_n & 0_n \\ 0_n & E_n \end{bmatrix}.$$

Recall that $\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix}$ has full rank, which means that it is nonsingular (Definition 1.1.14). Thus, the two matrices on the left are inverses of each other. So we also have

$$\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix} \begin{bmatrix} -B^{-1}(a)P_c & -B^{-1}(a)P \\ B^{-1}(b)Q_c & B^{-1}(b)Q \end{bmatrix} = \begin{bmatrix} E_m & 0_+ \\ 0_- & E_{2n-m} \end{bmatrix}.$$

Therefore,

$$-MB^{-1}(a)P + NB^{-1}(b)Q = 0_+,$$

which is (1.1.16).

Conversely, let U_1^+ is a boundary form of rank $2n - m$ such that

$$U_1^+ y = P_1^* \eta(a) + Q_1^* \eta(b)$$

for appropriate P_1^*, Q_1^* with $\operatorname{rank}(P_1^* : Q_1^*) = 2n - m$. Suppose

$$MB^{-1}(a)P_1 = NB^{-1}(b)Q_1 \tag{1.1.17}$$

holds.

Recall that dim(solution space) + rank(matrix) = # of unknown variables. Let u be a $2n \times 1$ vector, then there exist exactly $2n - m$ linearly independent $2n \times 1$ vector solutions of the linear system $(M : N)_{m \times 2n} u = 0$. By (1.1.17),

$$MB^{-1}(a)P_1 - NB^{-1}(b)Q = 0,$$

and thus

$$(M : N)_{m \times 2n} \begin{bmatrix} B^{-1}(a)P_1 \\ -B^{-1}(b)Q_1 \end{bmatrix}_{2n \times (2n-m)} = 0_{m \times (2n-m)}.$$

So the $2n - m$ columns of the matrix

$$H_1 := \begin{bmatrix} B^{-1}(a)P_1 \\ -B^{-1}(b)Q_1 \end{bmatrix}$$

are solutions of this system. Since $\operatorname{rank}(P_1^* : Q_1^*) = 2n - m$,

$$\operatorname{rank} \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} = 2n - m.$$

Since $B(a)$, $B(b)$ are nonsingular, $\operatorname{rank}(H_1) = 2n - m$.

If $U^+ x = P^* \xi(a) + Q^* \xi(b) = 0$ is a boundary condition adjoint to $Ux = 0$, then the matrix

$$\begin{bmatrix} -B^{-1}(a)P_c & -B^{-1}(a)P \\ B^{-1}(b)Q_c & B^{-1}(b)Q \end{bmatrix}_{2n \times 2n}$$

is nonsingular (because it has inverse $\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix}$), i.e., it has full rank. Thus, if

$$H = \begin{bmatrix} -B^{-1}(a)P \\ B^{-1}(b)Q \end{bmatrix}_{n\times(2n-m)},$$

then $\operatorname{rank}(H) = 2n-m$. Therefore, by (1.1.16), the $2n-m$ columns of H also form $2n = m$ linearly independent solutions of $(M : N)u = 0$, as in the case of H_1. Hence, there exists a nonsingular $(2n-m)\times(2n-m)$ matrix A such that $H_1 = HA$ (change of basis in the solution space). Thus we have

$$\begin{bmatrix} B^{-1}(a)P_1 \\ -B^{-1}(b)Q_1 \end{bmatrix} = H_1 = HA = \begin{bmatrix} B^{-1}(a)PA \\ -B^{-1}(b)QA \end{bmatrix},$$

or $P_1 = PA$, $Q_1 = QA$. Thus,

$$U_1^+ y = P_1^*\eta(a) + Q_1^*\eta(b) = A^*P^*\eta(a) + A^*Q^*\eta(b) = A^*U^+y.$$

This implies that $U_1^+ y = 0$ is an adjoint boundary condition to $Ux = 0$.

Is this by Theorem 1.1.19? But it says adjoint boundary forms are related by multiplication by nonsingular matrices, not that the multiplication of an adjoint boundary form by a nonsingular matrix is still an adjoint boundary form.

□

Definition 1.1.24. If U is a boundary form of rank m, the problem of finding solutions of

$$\pi_m : Lx = 0 \quad Ux = 0$$

on $[a,b]$ is a **homogeneous boundary-value problem of rank** m. The problem

$$\pi^+_{2n-m} : L^+x = 0 \quad U^+x = 0$$

on $[a,b]$ is the **adjoint boundary-value problem to** π_m.

Given that U^+ is not uniquely determined by U, is the adjoint problem unique?

In fact, π_m and π^+_{2n-m} are adjoint problems to each other. The zero function on $[a,b]$ is a solution to both π_m and π^+_{2n-m}, known as the **trivial solution**.

Theorem 1.1.25. *If $m = n$, the boundary condition $Ux = 0$ is adjoint to itself if and only if*

$$MB^{-1}(a)M^* = NB^{-1}(b)N^*.$$

Proof. Replace P, Q with M, N in Theorem 1.1.23. □

Theorem 1.1.26. *If $Ux = 0$ is self-adjoint and $L^+ = L$, the boundary-value problem π_m is self-adjoint, i.e., if $u, v \in C^n$ on $[a,b]$ and satisfy $Ux = 0$, then*

$$(Lu, v) = (u, Lv).$$

Proof. The equation follows as a special case of Proposition 1.1.21. □

Definition 1.1.27. Let $\varphi_1, \ldots, \varphi_n$ be a fundamental set (basis of the solution space to $Lx = 0$). Let Φ denote the nonsingular matrix

$$\Phi := \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \vdots & \varphi_n^{(n-1)} \end{bmatrix}.$$

Then Φ is a **fundamental matrix associated with** $Lx = 0$. Similarly, if $\psi_1, \ldots, \psi_n$ is a fundamental set fo $L^+x = 0$, then the corresponding fundamental matrix is

$$\Psi := \begin{bmatrix} \psi_1 & \cdots & \psi_n \\ \psi_1' & \cdots & \psi_n' \\ \vdots & & \vdots \\ \psi_1^{(n-1)} & \vdots & \psi_n^{(n-1)} \end{bmatrix}.$$

The meanings of U, U^+ can be extended from vectors (Remark 1.1.13) to matrices as follows:

$$U\Phi := M\Phi(a) + N\Phi(b)$$
$$U^+\Psi := P^*\Psi(a) + Q^*\Psi(b).$$

Remark 1.1.28. We note that

$$\begin{aligned}
U\Phi &= M\Phi(a) + N\Phi(b) \\
&= \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} \varphi_1(a) & \cdots & \varphi_n(a) \\ \varphi_1'(a) & \cdots & \varphi_n'(a) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(a) & \vdots & \varphi_n^{(n-1)}(a) \end{bmatrix} + \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} \varphi_1(b) & \cdots & \varphi_n(b) \\ \varphi_1'(b) & \cdots & \varphi_n'(b) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(b) & \vdots & \varphi_n^{(n-1)}(b) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n M_{1j}\varphi_1^{(j-1)}(a) & \cdots & \sum_{j=1}^n M_{1j}\varphi_n^{(j-1)}(a) \\ \vdots & & \vdots \\ \sum_{j=1}^n M_{mj}\varphi_1^{(j-1)}(a) & \cdots & \sum_{j=1}^n M_{mj}\varphi_n^{(j-1)}(a) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n N_{1j}\varphi_1^{(j-1)}(b) & \cdots & \sum_{j=1}^n N_{1j}\varphi_n^{(j-1)}(b) \\ \vdots & & \vdots \\ \sum_{j=1}^n N_{mj}\varphi_1^{(j-1)}(b) & \cdots & \sum_{j=1}^n N_{mj}\varphi_n^{(j-1)}(b) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n (M_{1j}\varphi_1^{(j-1)}(a) + N_{1j}\varphi_1^{(j-1)}(b)) & \cdots & (\sum_{j=1}^n M_{1j}\varphi_n^{(j-1)}(a) + N_{1j}\varphi_n^{(j-1)}(b)) \\ \vdots & & \vdots \\ \sum_{j=1}^n (M_{mj}\varphi_1^{(j-1)}(a) + N_{mj}\varphi_1^{(j-1)}(b)) & \cdots & (\sum_{j=1}^n M_{mj}\varphi_n^{(j-1)}(a) + N_{mj}\varphi_n^{(j-1)}(b)) \end{bmatrix} \\
&= \begin{bmatrix} U_1\varphi_1 & \cdots & U_1\varphi_n \\ \vdots & & \vdots \\ U_m\varphi_1 & \cdots & U_m\varphi_n \end{bmatrix}.
\end{aligned}$$

Theorem 1.1.29. *The problem π_m has exactly k $(0 \le k \le n)$ linearly independent solutions if and only if $U\Phi$ has rank $n-k$, where Φ is any fundamental matrix associated with $Lx = 0$.*

Proof. A function φ satisfies $Lx = 0$ if and only if the corresponding vector $\vec{\varphi} = (\varphi, \varphi', \ldots, \varphi^{(n-1)})$ is of the form $\vec{\varphi} = \Phi\vec{c}$, where $\vec{c} = (c_1, \ldots, c_n)$ is a constant vector.

Indeed: Suppose φ is a solution to $Lx = 0$. Then by definition of fundamental set $\varphi_1, \ldots, \varphi_n$, $\varphi = c_1\varphi_1 +$

$\cdots + c_n\varphi_n$ for some $c_1, \dots, c_n \in \mathbb{C}$. By linearity of derivatives, $\varphi^{(j)} = c_1\varphi_1^{(j)} + \cdots + c_n\varphi_n^{(j)}$. Thus,

$$\vec{\varphi} = \begin{bmatrix} \varphi \\ \varphi' \\ \vdots \\ \varphi^{(n-1)} \end{bmatrix} = \begin{bmatrix} c_1\varphi_1 + \cdots + v_n\varphi_n \\ c_1\varphi_1' + \cdots + v_n\varphi_n' \\ \vdots \\ c_1\varphi_1^{(n-1)} + \cdots + v_n\varphi_n^{(n-1)} \end{bmatrix}$$
$$= \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi\vec{c}.$$

Thus, $U\varphi = 0$

This is the definition of Ux in Remark 1.1.13?

if and only if

$$U(\Phi c) = (U\Phi)c = 0.$$

Since dim(solution space)+rank(matrix) = # of unknown variables, the number of linearly independent vectors $\vec{c}$ satisfying $(U\Phi)c = 0$ is $n - \text{rank}(U\Phi)$. Thus, the number of solutions φ to $Lx = 0$ is $n - \text{rank}(U\Phi)$.

If Φ_1 is any other fundamental matrix associated with $Lx = 0$, then $\Phi_1 = \Phi C$, where C is a nonsingular constant matrix. Therefore

By change of basis?

$$\text{rank}(U\Phi_1) = \text{rank}(U\Phi).$$

□

Theorem 1.1.30. *If π_m has exactly k linearly independent solutions, then π_{2n-m}^+ has exactly $k+m-n$ linearly independent solutions.*

Proof. Let $\varphi_1, \dots, \varphi_k$ be k linearly independent solutions of π_m. Suppose U_c where

$$U_c x = M_c\xi(a) + N_c\xi(b)$$

is a boundary form of rank $2n - m$ complementary to U. We show that the vectors $U_c\varphi_i$ $(i = 1, \dots, k)$ are linearly independent. Suppose not, then for some constants $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ not all zero,

$$\sum_{i=1}^{k} \alpha_i U_c \varphi_i = 0,$$

which implies

$$U_c\left(\sum_{i=1}^{k} \alpha_i \varphi_i\right) = 0.$$

But since each φ_i is a solution to π_m, they each satisfy $Ux = 0$. Thus,

$$U\left(\sum_{i=1}^{k} \alpha_i \varphi_i\right) = 0.$$

Let $\bar{\varphi} = \sum_{i=1}^{k} \alpha_i \varphi_i$. Let $\bar{\xi} = (\bar{\varphi}, \bar{\varphi}', \dots, \bar{\varphi}^{(n-1)})$. Then by Remark 1.1.13, the above equations imply

$$M\bar{\xi}(a) + N\bar{\xi}(b) = U\bar{\xi} = 0$$
$$M_c\bar{\xi}(a) + N_c\bar{\xi}(b) = U_c\bar{\xi} = 0.$$

Or

$$\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix} \begin{bmatrix} \bar{\xi}(a) \\ \bar{\xi}(b) \end{bmatrix} = 0_{2n\times 1}.$$

But rank $\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix} = 2n$, which implies it is nonsingular. Thus $\bar{\xi}(a) = \bar{\xi}(b) = 0_{n\times 1}$. But since $\varphi_1, \ldots, \varphi_k$ are solutions to $Lx = 0$, we have

$$L\bar{\varphi} = L\left(\sum_{i=1}^{k} \alpha_i \varphi_i\right) = \sum_{i=1}^{k} \alpha_i L\varphi_i = 0.$$

We show that this implies $\bar{\varphi} = 0$. Indeed: If not, then L maps a nonzero function to 0, which means if two distinct functions x_1, x_2 are such that $x_1 - x_2 = \bar{\varphi}$, then $Lx_1 - Lx_2 = L(x_1 - x_2) = 0$, i.e., the pre-image of 0 under L is not unique.

This is how I interpreted "uniqueness" in the next line. But why is this a problem / where is the contradiction?

Thus by uniqueness, $\bar{\varphi}(t) = 0$ for $t \in [a, b]$. This contradicts the definition of $\bar{\varphi}$ as a nontrivial linear combination of $\varphi_1, \ldots, \varphi$ (i.e., not all α_i are 0). Hence

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

and $U_c\varphi_i$ are linearly independent.

Let $\psi_1, \ldots, \psi_n$ be n linearly independent solutions of $L^+x = 0$. Suppose Ψ is the corresponding fundamental matrix. Since φ_i, ψ_j are solutions to π_m and $L^+x = 0$,

This is not the same as requiring ψ_j to be solutions to π_{2n-m}, is it? Since there is an extra $U^+\psi_j = 0$ to fulfill?

respectively, by Proposition 1.1.21,

Proposition 1.1.21 requires that $U\varphi_i = 0$ and $U^+\psi_j = 0$; are these conditions fulfilled?

$$(L\varphi_i, \psi_j) = (\varphi_i, L^+\psi_j).$$

By Green's formula (1.1.2),

$$0 = (L\varphi_i, \psi_j) - (\varphi_i, L^+\psi_j) = [\varphi_i\psi_j](b) - [\varphi_i\psi_j](a)$$

for $i = 1, \ldots, k$, $j = 1, \ldots, n$. By the boundary-form formula (1.1.11),

$$[\varphi_i\psi_j](b) - [\varphi_i\psi_j](a) = U_{\varphi_i} \cdot U_c^+\psi_j + U_{c\varphi_i} \cdot U^+\psi_j.$$

Since φ_i are solutions to π_m, we have $U\varphi_i = 0$ for $i = 1, \ldots, k$. Thus,

$$U_{c\varphi_i} \cdot U^+\psi_j = 0.$$

Does this mean we don't know $U^+\psi_j = 0$? If so, how could we use Proposition 1.1.21 above?

By definition of $f \cdot g$1.1.7, $f \cdot g = g^* f$ for any column vectors f, g of the same dimension, so

$$(U^+\psi_j)^* U_{c\varphi_i} = 0 \quad (i = 1, \ldots, k).$$

We have shown before that $U_c\varphi_i$ are linearly independent. So the system $(U^+\psi_j)^* v = 0$ has (at least) the k linearly independent $(2n - m \times 1)$ vectors $U_c\varphi_1, \ldots, U_c\varphi_k$ as solutions. Therefore,

$$\operatorname{rank}(U^+\Psi) = \operatorname{rank}(U^+\Psi)^* \leq (2n - m) - k.$$

Suppose $\operatorname{rank}(U^+\psi) = r < (2n-m) - k$. Then by similar reasoning it can be shown that, if Φ is any fundamental matrix associated with $Lx = 0$, $\operatorname{rank}(U\phi) \leq m - (n-r) < n-k$. By Theorem 1.1.29, this contradicts with the assumption that π_m has exactly k linearly independent solutions. Thus, we must have

$$\operatorname{rank}(U^+\Psi) = 2n - m - k.$$

By Theorem 1.1.29, there exist exactly $k+m-n$ linearly independent solutions of π^+_{2n-m}. □

Corollary 1.1.31. *π_n and $\pi^+ n$ have the same number of independent solutions.*

Proof. Apply Theorem 1.1.30 on $m = n$. □

1.1.4 Nonhomogeneous Boundary-value Problems and Green's Function

Definition 1.1.32. A **nonhomogeneous boundary-value problem** associated with π_m is a problem of the form

$$Lx = f \quad Ux = \gamma \tag{1.1.18}$$

on $t \in [a,b]$, where f is a complex-valued continuous function on $[a,b]$ and γ is a complex constant vector such that either f is not the zero function or $\gamma \neq 0$.

Remark 1.1.33. If φ and $\bar{\varphi}$ are two solutions of 1.1.18, their difference $\varphi - \bar{\varphi}$ is a solution of π_m. Hence, if π_m has k linearly independent solutions $\varphi_1, \ldots, \varphi_k$, then $\varphi = \bar{\varphi} + \sum_{i=1}^k c_i\varphi_i$ for some constants $c_i \in \mathbb{C}$ (since $\varphi_1, \ldots, \varphi_k$ are a basis for the solution space of π_m).

Proposition 1.1.34. *Let A be a matrix and b a vector. $Ax = b$ has a solution if and only if $b \cdot u = u^*b = 0$ for every solution u of $A^*x = 0$.*

Theorem 1.1.35. *The nonhomogeneous problem 1.1.18 has a solution if and only if*

$$(f,\psi) = \gamma \cdot U_c^+\psi \tag{1.1.19}$$

holds for every solution ψ of the adjoint homogeneous problem π^+_{2n-m}.

Remark 1.1.36. Since (f,ψ) is an inner product, $\gamma = 0$ implies f is orthogonal to all solutions ψ of π^+_{2n-m}.

Proof. Let φ be a solution of 1.1.18. Let ψ be a solution of the adjoint homogeneous problem π^+_{2n-m}. Then by Green's formula (1.1.2) and the boundary-form formula (1.1.11),

$$(L\varphi,\psi) - (\varphi, L^+\psi) = U\varphi \cdot U_c^+\psi + U_c\varphi \cdot U^+\psi.$$

Since $L\varphi = f$, $U\varphi = \gamma$, $L^+\psi = 0$, and $U^+\psi = 0$, the above implies that

$$(f,\psi) - (\varphi, 0) = \gamma \cdot U_c^+\psi + 0,$$

or

$$(f,\psi) = \gamma \cdot U_c^+\psi.$$

Now suppose 1.1.19 holds for all solutions ψ of π^+_{2n-m}. Let $\varphi_1, \ldots, \varphi_n$ be a fundamental set for $Lx = 0$. Let $\bar{\varphi}$ be a solution of $Lx = f$. Then every solution φ of $Lx = f$ is of the form

$$\varphi = \bar{\varphi} + \sum_{i=1}^n c_i\varphi_i$$

for some constants $c_i \in \mathbb{C}$. Applying U to both sides, we have that 1.1.18 has a solution if and only if there exist c_i such that

$$U\varphi = U\bar{\varphi} + \sum_{i=1}^{n} c_i U\varphi_i$$

is equal to γ. Equivalently,

$$(U\Phi)c = \gamma - U\bar{\varphi} \tag{1.1.20}$$

where Φ is the fundamental matrix corresponding to $\varphi_1, \ldots, \varphi_n$, and c a constant vector. Note that by Remark 1.1.28,

$$\begin{aligned}(U\Phi)c &= \begin{bmatrix} U_1\varphi_1 & \cdots & U_1\varphi_n \\ \vdots & & \vdots \\ U_m\varphi_1 & \cdots & U_m\varphi_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{n} c_i U_1\varphi_i \\ \vdots \\ \sum_{i=1}^{n} c_i U_m\varphi_i \end{bmatrix} \\ &= \sum_{i=1}^{n} \begin{bmatrix} c_i U_1\varphi_i \\ \vdots \\ c_i U_m\varphi_i \end{bmatrix} \\ &= \sum_{i=1}^{n} c_i U\varphi_i \end{aligned}$$

By Proposition 1.1.34, the above has a solution c if and only if $\gamma - U\bar{\varphi}$ is orthogonal to every solution u of

$$(U\Phi)^* u = 0, \tag{1.1.21}$$

that is,

$$(\gamma - U\bar{\varphi}) \cdot u = 0 \tag{1.1.22}$$

Suppose π^+_{2n-m} have exactly k^+ linearly independent solutions $\psi_1, \ldots, \psi_{k^+}$. By the same argument in Theorem 1.1.30, the k^+ vectors $U_c^+\psi_1, \ldots, U_c^+\psi_{k^+}$ are linearly independent m-dimensional vectors which are solutions to 1.1.21. Let k be the number of linearly independent solutions of π_m. Then by Theorem 1.1.29, $\text{rank}(U\Phi) = n-k$. Thus, the number of linearly independent solutions of 1.1.21 is $m - \text{rank}(U\Phi^*) = m - \text{rank}(U\Phi) = m-(n-k)$. But by Theorem 1.1.30, $k^+ = m - n + k$. Thus, since $U_c^+\varphi_i$ are basis for the solution space of 1.1.21, 1.1.22 holds for every u satisfying 1.1.21 if and only if

$$(\gamma - U\bar{\varphi}) \cdot U_c^+\psi_i = 0 \quad (i = 1, \ldots, k^+) \tag{1.1.23}$$

By Green's formula (1.1.2) and the boundary-form formula (1.1.11),

$$(L\bar{\varphi}, \psi_i) - (\bar{\varphi}, L^+\psi_i) = U_{\bar{\varphi}} \cdot U_c^+\psi_i + U_c\bar{\varphi} \cdot U^+\psi_i. \tag{1.1.24}$$

But ψ_i is a solution to π^+_{2n-m}, so $L^+\psi_i = 0$ and $U^+\psi_i = 0$. Thus, the above becomes

$$(f, \psi_i) = (L\bar{\varphi}, \psi_i) = (L\bar{\varphi}, \psi_i) - 0 = U_{\bar{\varphi}} \cdot U_c^+\psi_i + 0 = U_{\bar{\varphi}} \cdot U_c^+\psi_i. \tag{1.1.25}$$

But by the hypothesis, 1.1.19 holds. Thus,

$$U_{\bar{\varphi}} \cdot U_c^+\psi_i \overset{1.1.25}{=} (f, \psi_i) \overset{1.1.19}{=} \gamma \cdot U_c^+\psi_i.$$

Hence,

$$(\gamma - U\bar{\varphi}) \cdot U_c^+\psi_i = 0.$$

So 1.1.23 is satisfied. Reversing the direction of the argument from 1.1.23, we have that 1.1.22 holds. This implies that 1.1.20 has a solution c, which then implies that 1.1.18 has a solution. $\square$

Corollary 1.1.37. *1.1.18 has a unique solution if $m = n$ and the only solution of π_n is the trivial one.*

Proof. Suppose $m = n$ and π_n only has the trivial solution (note that π_n is the homogeneous problem). Let φ, $\bar{\varphi}$ be two solutions of 1.1.18. Then $\varphi - \bar{\varphi}$ is a solution of π_n. By the hypothesis, $\varphi - \bar{\varphi} = 0$. Thus, 1.1.18 has a unique solution φ.

This is different from the proof in the book. Actually in the corollary statement, is the sentence after "and" part of the conclusion or condition?

□

Definition 1.1.38. The **Dirac's delta function** can be loosely thought of as a function on $\mathbb{R}$ which is zero everywhere except at the origin, where it is infinite. It is given by

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0. \end{cases}$$

It is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Definition 1.1.39. [1] Given a linear differential operator $L = L(x)$, a **Green's function** $G = G(x, s)$ at the point $s \in \Omega \in \mathbb{R}^n$ corresponding to L is any solution of

$$LG(x, s) = \delta(x - s) \tag{1.1.26}$$

where δ denotes the Dirac's delta function.

Remark 1.1.40. A Green's function is an **integral kernel** that can be used to solve differential equations, such as ordinary differential equations with initial or boundary value conditions. Recall the Fourier kernel $e^{i\lambda x}$ in the Fourier transform

$$\mathcal{F}[f] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx.$$

To see the motivation for defining Green's function, we note that multiplying both sides of 1.1.26 by a function $f(s)$ and integrating with respect to s gives

$$\int LG(x, s) f(s)\, ds = \int \delta(x - s) f(s)\, ds.$$

The right-hand side reduces to $f(x)$ due to properties of δ, and because L is a linear operator acting only on x and not on s, the left-hand side can be rewritten as

$$L\left(\int G(x, s) f(s)\, ds\right).$$

This reduction is particularly useful when solving for $u = u(x)$ in differential equations of the form

$$Lu(x) = f(x),$$

since we have

$$Lu(x) = L\left(\int G(x, s) f(s)\, ds\right),$$

which implies

$$u(x) = \int G(x, s) f(s)\, ds.$$

[1] http://mathworld.wolfram.com/GreensFunction.html

Suppose $m = n$. By Theorem 1.1.30, π_n and π_n^+ have the same number k of linearly independent solutions. If $k = 0$, then π_n has only the trivial solution. In this case, it is possible to solve the nonhomogeneous problem 1.1.18 explicitly in terms of the Green's function:

Proposition 1.1.41. *The nonhomogeneous problem*

$$Lx = f \quad Ux = 0$$

has a unique solution given by $x(t) = \int_a^b f(s)G(s,t)\,ds$ where $G(s,t)$ is a Green's function satisfying some properties.

Definition 1.1.42. The problem

$$\pi : \quad Lx = \ell x \quad Ux = 0$$

is an **eigenvalue problem**. If ℓ is such that π has a nontrivial solution, then ℓ is an **eigenvalue** of π, and the nontrivial solutions of π are the **eigenfunctions** of π.

We introduce the following results.

Proposition 1.1.43. *Let π denote the problem*

$$Lx - \ell x = 0 \quad Ux = 0.$$

If π has no (nontrivial) solution for at least one value of ℓ, then there exists unique $G = G(t,\tau,\ell)$ defined for (t,τ) on the square $a \le t, \tau \le b$ and for all complex ℓ except the eigenvalues of π and having the following properties:

(i) $\frac{\partial^k G}{\partial t^k}$ *$(k = 0, 1, \ldots, n-2)$ exist and are continuous in (t,τ,ℓ) for (t,τ) on the square $a \le t$, $\tau \le b$ and ℓ not at an eigenvalue of π. Moreover,* $\frac{\partial^k G}{\partial t^k}$ *for $k = n-1, n$ are continuous in (t,τ,ℓ) for (t,τ) on the triangles $a \le t \le \tau \le b$ and $a \le \tau \le t \le b$ and ℓ not an eigenvalue of π.*

(ii)
$$\frac{\partial^{n-1} G}{\partial t^{n-1}}(r+0, r, \ell) - \frac{\partial^{n-1} G}{\partial t^{n-1}}(r-0, r, \ell) = \frac{1}{p_0(\tau)}.$$

(iii) As a function of t, G satisfies $Lx = \ell x$ if $t \neq \tau$.

(iv) As a function of t, G satisfies the boundary conditions $Ux = 0$ for $a \le \tau \le b$.

If for one value of ℓ the homogeneous problem π has no solution, then it would have solutions only for a set of ℓ which are the zeroes of an entire function.

Here, $\ell = 0$ and it is assumed that π_n has only the trivial solution. Denote $G(t,\tau,0)$ as $G(t,\tau)$. The unique solution of 1.1.18 with $\gamma = 0$ is given by

$$\mathcal{G}f(t) = \int_a^b G(t,\tau) f(\tau)\,d\tau.$$

If π_n has only the trivial solution, then by Theorem 1.1.30, π_n^+ has only the trivial solution (with $k = 1$ and $m = n$). By Proposition 1.1.43, the Green's function G^+ for π_n^+ exists and is unique.

Theorem 1.1.44. *If π_n has only the trivial solution, Green's function G^+ for π_n^+ is given by*

$$G^+(t,\tau) = \bar{G}(\tau, t). \tag{1.1.27}$$

Proof. Let $a < \tau_1 < \tau_2 < b$. Consider the functions G_1 and G_2^+ given by $G_1(t) = G(t, \tau_1)$, $G_2^+(t) = G^+(t, \tau_2)$. Then applying Green's formula 1.1.2 to the intervals $[a, \tau_1 - 0]$, $[\tau_1 + 0, \tau_2 - 0]$, $[\tau_2 + 0, b]$, we have

$$[G_1G_2^+](\tau_1 - 0) - [G_1G_2^+](a) + [G_1G_2^+](\tau_2 - 0) - [G_1G_2^+](\tau_1 + 0) + [G_1G_2^+](b) - [G_1G_2^+](\tau_2 + 0) = 0 \quad (1.1.28)$$

Why is the equation 0?

By the boundary-form formula 1.1.11,

$$[G_1G_2^+](b) - [G_1G_2^+](a) = 0. \quad (1.1.29)$$

Doesn't this follow from 1.1.28?

From the form of $[xy](t)$ 1.1.3 it follows that the only terms of interest in 1.1.28 are those involving the $(n-1)$st derivatives, and these are

Why?

$$p_0(t)[(-1)^{n-1}x(t)\bar{y}^{(n-1)}(t) + x^{(n-1)}(t)\bar{y}(t)]. \quad (1.1.30)$$

Now by Proposition 1.1.43(ii), G satisfies

$$\frac{\partial^{n-1}G}{\partial t^{n-1}}(r + 0, r, \ell) - \frac{\partial^{n-1}G}{\partial t^{n-1}}(r - 0, r, \ell) = \frac{1}{p_0(\tau)}. \quad (1.1.31)$$

Similarly for G^+,

$$\frac{\partial^{n-1}G^+}{\partial t^{n-1}}(r + 0, r, \ell) - \frac{\partial^{n-1}G^+}{\partial t^{n-1}}(r - 0, r, \ell) = \frac{1}{(-1)^n\bar{p}_0(\tau)}. \quad (1.1.32)$$

Thus, $\bar{G}^+(\tau_1, \tau_2) - G(\tau_2, \tau_1) = 0.$ □

How?

Remark 1.1.45. To consider $G(t, \tau, \ell)$, the differential operation $(L - l)$ is considered instead of L. Let $L_1 = L - \ell$. Consider the problem

$$L_1x = 0 \quad Ux = 0 \quad (1.1.33)$$

The adjoint problem is given in $L_1^+ = L^+ - \bar{\ell}$ and U^+. Applying Theorem 1.1.44 to 1.1.33,

We assume π_n only has the trivial solution?

we have

$$G^+(t, \tau, \ell) = \bar{G}(\tau, t, \bar{\ell}).$$

For the self-adjoint problem where $L^+ = L$ and U equivalent to U^+, it follows that

$$G(t, \tau, \ell) = \bar{G}(\tau, t, \ell).$$