# A unified transform method for solving linear and certain nonlinear PDEs

**Dedicated to Peter Lax on the occasion of his 70th birthday**

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A new transform method for solving initial boundary value problems for linear and for integrable nonlinear PDEs in two independent variables is introduced. This unified method is based on the fact that linear and integrable nonlinear equations have the distinguished property that they possess a Lax pair formulation. The implementation of this method involves performing a simultaneous spectral analysis of both parts of the Lax pair and solving a Riemann–Hilbert problem. In addition to a unification in the method of solution, there also exists a unification in the representation of the solution. The sine–Gordon equation in light-cone coordinates, the nonlinear Schrödinger equation and their linearized versions are used as illustrative examples. It is also shown that appropriate deformations of the Lax pairs of linear equations can be used to construct Lax pairs for integrable nonlinear equations. As an example, a new Lax pair of the nonlinear Schrödinger equation is derived.

## 1. Introduction

The aim of this paper is to introduce a new unified method for solving initial boundary-value problems for linear and for integrable nonlinear PDEs in two independent variables. The extension of this method to equations in more than two independent variables will be discussed elsewhere. For simplicity, we will mainly use scalar evolution equations with constant coefficients to illustrate this method. However, non-evolution equations and equations with variable coefficients will also be mentioned.

We will call an equation integrable if it admits a Lax pair (Lax 1968) formulation, i.e. if it can be written as the compatibility condition of two linear eigenvalue equations†. For evolution equations, these two equations will be referred to as the $x$-part and as the $t$-part of the Lax pair.

We first summarize the usual method for solving initial value problems. The Cauchy problem on the infinite line for linear evolution equations can be solved by the Fourier transform. The analogous problem for integrable nonlinear equations

† For equations in two independent variables, it is important that the Lax pair consists of two *eigenvalue* equations. If the pair does not contain a spectral parameter then it cannot be used to solve the associated equation.

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can be solved by the so called inverse spectral method (Gardner *et al.* 1967, 1974; for recent developments see Fokas & Zakharov 1993). This method uses mainly the $x$-part of the Lax pair; the $t$-part plays only an auxiliary role. The spectral analysis of the $x$-part yields a nonlinear Fourier transform, while the $t$-part determines the evolution of the nonlinear Fourier data, which are usually called spectral data. It was shown in Fokas & Gel'fand (1994) that the inverse spectral method for solving an initial value problem (IBV) is conceptually similar to a certain novel approach of the classical Fourier transform method. The main difference is that the associated nonlinear Fourier transform cannot be determined in closed form but is given through the solution of a linear integral equation, or more precisely through the solution of a Riemann–Hilbert problem (Albowitz & Fokas 1996).

We now summarize the usual method for solving IBV problems. Such problems for linear equations are traditionally solved by certain transforms which include the Fourier, the sine, the cosine and the Laplace transforms. The analogous problems for integrable nonlinear equations are usually studied by a method which is similar to the one used for initial value problems: the $x$-part of the Lax pair is used to derive an appropriate nonlinear transform, while the $t$-part of the Lax pair is used to determine the evolution of the inverse spectral data. It is convenient to divide the IBV problems studied so far into the following two classes. (i) In some problems, the equation describing the evolution of the spectral data is explicitly determined by the given initial and boundary data. Examples of such problems are IBV problems associated with the three-wave interaction, with simple harmonic generation, with stimulated Raman scattering and with the sine–Gordon equation in light-cone coordinates. For these problems, the evolution of the spectral data satisfies a well defined linear matrix equation, so although in principle it can be determined, it cannot be given in closed form. (ii) In some problems, the equation describing the evolution of the spectral data is *not* explicitly determined by the given initial and boundary data. Examples of such problems are IBV problems associated with the nonlinear Schrödinger (NLS), the Korteweg–de Vries and the sine–Gordon equation in laboratory coordinates (for example, the IBV problem of the NLS in the quarter plane with $q(x,0)$ and $q(0,t)$ given, involves the unknown function $q_x(0,t)$).

It is the author's opinion that the above methods for studying IBV problems for both linear and integrable nonlinear equations are inadequate. We first discuss linear equations. Given a two dimensional PDE with constant coefficients, separation of variables gives rise to two linear ordinary differential operators; the *independent* spectral analysis of these operators yields two different integral transforms for the solution of certain IBV problems for this PDE. The method presented here is, in a sense, the antithesis of separation of variables; it uses the *simultaneous* spectral analysis of both linear operators defining the Lax pair to construct a *unifying* transform for the solution of certain IBV problems. For concreteness, we will use the particular example of the linearized nonlinear Schrödinger equation to discuss some of the differences between the traditional and the new approaches. Let the complex valued function $q(x,t)$ satisfy

$$\mathrm{i}q_t + q_{xx} = 0, \quad x,t \in [0,\infty), \tag{1.1}$$

$$q(x,0) = q_1(x), \quad q(0,t) = q_2(t), \tag{1.2}$$

where $q_1(x)$ and $q_2(t)$ are given functions decaying sufficiently fast for large $x$ and large $t$, respectively. This problem is traditionally solved by the sine transform in $x$ or the Laplace transform in $t$. Let us consider the $x$-transform. Separation of variables

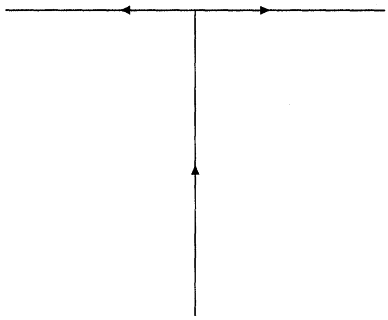

Figure 1. The contour $L$ for the equation $\mathrm{i}q_t + q_{xx} = 0$ in the quarter plane.

and the spectral analysis of $\partial_x^2 + \lambda^2$ imply that the appropriate $x$-transform for the above IBV problem is indeed the sine transform

$$\hat{q}(k,t) = \int_0^\infty q(x,t) \sin kx \, \mathrm{d}x, \quad q(x,t) = \frac{2}{\pi} \int_0^\infty \hat{q}(k,t) \sin kx \, \mathrm{d}k. \tag{1.3}$$

Equation (1.1) and integration by parts imply

$$\hat{q}_t + \mathrm{i}k^2 \hat{q} = \mathrm{i}k q_2(t). \tag{1.4}$$

Thus, the solution of the IBV problem defined by the equations (1.1) and (1.2) is given by

$$q(x,t) = \frac{2}{\pi} \int_0^\infty \sin kx \left[ \mathrm{e}^{-\mathrm{i}k^2 t} \hat{q}_1(k) + \mathrm{i}k \int_0^t \mathrm{e}^{-\mathrm{i}k^2(t-t')} q_2(t') \, \mathrm{d}t' \right] \mathrm{d}k, \tag{1.5}$$

where $\hat{q}_1(k)$ is the sine transform of the initial data. This solution should be contrasted with the following alternative representation obtained by the new method:

$$q(x,t) = \frac{1}{2\pi} \int_L \mathrm{e}^{-\mathrm{i}kx - \mathrm{i}k^2 t} \rho(k) \, \mathrm{d}k, \tag{1.6}$$

where $L$ is the union of the real axis and of the negative imaginary axis of the complex $k$ plane (see figure 1) and the spectral data $\rho(k)$ is explicitly given in terms of certain integral transforms of $q_1(x)$ and of $q_2(t)$ (see theorem 1.1).

Comparing the representations (1.5) and (1.6), it follows that only the latter provides the *complete spectral decomposition* of $q(x,t)$; in equation (1.6), $x$ and $t$ appear parametrically in the form dictated by the underlying dispersion relationship while the spectral data depends only on $k$ (in contrast, the second term of the right-hand side of equation (1.5) is not separable in $t$ and $k$). The main advantage of equation (1.6) is that it provides the solution in a suitable functional class of *any* IBV problem in the quarter plane independent of the particular boundary conditions (see proposition 3.1). The particular form of $\rho(k)$ depends on the particular boundary conditions. The simple cases where $q(0,t)$, or $q_x(0,t)$, or $q_x(0,t) + \alpha q(0,t)$, $\alpha$ constant, are given, are discussed in theorem 1.1.

It should be noted that the new method provides a *unification in the form of the solution.* Indeed, it can be shown (Fokas 1996) that the same representation (1.6) but with the contour $L$ extended in the upper half complex $k$-plane provides the solution in a suitable functional class of *any* IBV problem of equation (1.1) in the finite $x$-domain. Furthermore, the solution of the initial value problem of equation (1.1) on the infinite line, which is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\hat{q}(k)\,\mathrm{d}k, \tag{1.7}$$

($\hat{q}(k)$ is the Fourier transform of the initial data) is also a particular case of the representation (1.6).

The existence of the representation (1.6) is consistent with the Ehrenpreis principle (Ehrenpreis 1970). Actually, according to this beautiful result, the representation (1.6), with some measure $\rho(k)$ and contour $L$, captures a wide class of solutions of equation (1.1). This new method provides a *constructive implementation of the Ehrenpreis principle.*

Regarding this method we note the following.

(1) Given a PDE in two independent variables $x$ and $y$, one usually asks whether this equation should be solved by an $x$-transform or by a $y$-transform. The new approach suggests that the answer to this question is neither; there exists an alternative transform which is based on the unification, as opposed to separation, of variables. This transform can be used *only* for the solution of this given PDE. This is to be contrasted with the usual approach where, for example, the sine transform or the Laplace transform can be used for the solution of many equations. However, precisely because this transform is 'custom made' it has certain advantages over both the $x$ and the $y$ transforms.

(2) It has been well established that the integral representation of the solution of a given ODE provides a powerful tool for studying many properties of the solution, including its asymptotic behaviour. *The representation presented here, can be thought of as the extension of such integral representations from ODEs to PDEs.*

(3) There exist many physically important linear IBV problems for which the boundary conditions are rather complicated. Prototypical such problems appear in diffraction. Some of these problems have been analysed by the Wiener–Hopf factorization (Noble 1958), which is based on a particular case of a Riemann–Hilbert problem. The method presented here is also based on the Riemann–Hilbert problem, thus one might expect a relationship between these approaches. Indeed, *the method presented here provides a unified approach to solving linear IBV problems for any boundary conditions. This includes problems of Wiener–Hopf type.* We conjecture that the new method, in addition to providing a unified, elegant treatment of IBV problems solvable by other methods, will also yield the solution of many IBV problems which remain unsolved. The solution of several such problems using this method will be presented elsewhere.

(4) Integral representations have been used by some authors for the solution of certain complicated IBV problems such as the problem of waves on sloping beaches (Stoker 1957). However, these approaches are *ad hoc* and they also involve the solution of certain difference equations for the determination of $\rho(k)$. In contrast, the new method provides an algorithmic approach for finding both the contour $L$ and the spectral data $\rho(k)$.

(5) This method yields explicit relations between the boundary values of $q(x,t)$. For example, for the linear equation (1.1) defined in the quarter plane, the following relation among $q(x,0)$, $q(0,t)$ and $q_x(0,t)$ is valid:

$$\int_0^{\infty} \mathrm{e}^{\mathrm{i}k^2t}(\mathrm{i}q_x(0,t)+kq(0,t))\,\mathrm{d}t - \int_0^{\infty} \mathrm{e}^{\mathrm{i}kx}q(x,0)\,\mathrm{d}x = 0, \quad 0 \leqslant \arg k \leqslant \tfrac{1}{2}\pi. \tag{1.8}$$

These type of relations can be used to determine which initial and boundary data make a given IBV problem well posed. They also provide an effective way of finding global constraints needed in the cases that the Shapiro–Lopatinski conditions (Lopatinski 1953; Shapiro 1953) indicate that the problem is ill-posed (see, for example, equation (1.13)).

We now discuss nonlinear IBV problems. For problems of class (ii), the traditional approach fails since the evolution of the spectral data depends on some unknown function $f(t)$. For problems of class (i), the spectral data cannot be given explicitly, thus it is difficult to determine the long time behaviour of the solution. Also, the solution may contain certain coherent structures (solitons) which are not apparent in the representation of the solution obtained by this method†. The main advantage of our method is that *it yields Riemann–Hilbert problems whose $x$ and $t$ dependence is explicit. Furthermore, this dependence can be deduced from the dispersion relationship of the linearized version of the given nonlinear equation.* For example, for the nonlinear Schrödinger equation in the quarter plane, this method yields the Riemann–Hilbert problem

$$\Phi(x,t,k) = \Psi(x,t,k)\mathrm{e}^{(-\mathrm{i}kx-2\mathrm{i}k^2t)\sigma_3}S(k)\mathrm{e}^{(\mathrm{i}kx+2\mathrm{i}k^2t)\sigma_3}, \quad k \in \mathbb{R}\cup\mathrm{i}\mathbb{R},$$

where the $2\times 2$ matrix valued functions $\Phi$ and $\Psi$ are analytic and bounded in certain sectors of the complex $k$-plane, $\sigma_3 = \mathrm{diag}(1,-1)$ and the matrix $S(k)$, which contains the spectral data, is a $2\times 2$ matrix defined for $k$ real and for $k$ purely imaginary. This formulation can be thought of as providing the *extension of the Ehrenpreis principle to nonlinear integrable equations.* For nonlinear equations of class (i), $S(k)$ can be explicitly computed in terms of the given initial and boundary data. For equations of class (ii), one needs to solve an additional inverse problem in order to compute some of the elements of $S(k)$. Unfortunately, this inverse problem appears to be nonlinear.

We note that, for initial value problems, the inverse spectral method can be used to show that the solution exists and is unique and that it decomposes into a number of solitons for large time and for $x/t = O(1)$. Although existence and uniqueness can be obtained by the usual PDE techniques, these techniques cannot be used to obtain the asymptotic behaviour of the solution. For IBV problems, establishing existence by PDE techniques is much harder (see, for example, Bona & Winther 1983). Using the method presented here, it is straightforward to establish existence and uniqueness of problems of class (i) and to derive the large time behaviour of the solution for both problems of class (i) and of class (ii). In particular, since the associated Riemann–Hilbert problems have explicit $x$ and $t$ dependence, it is possible to apply the rigorous technique of Deift & Zhou (1992) for the investigation of the large $t$ asymptotics. Establishing existence for problems of class (ii) involves investigating the additional inverse problem mentioned above; a rigorous analysis of this problem will be presented elsewhere.

As it was mentioned earlier, the new method is based on the existence of Lax pairs. We now briefly discuss the Lax pair formulation of linear and integrable nonlinear equations. A general algorithm for finding one of the Lax pairs for a given evolution linear PDE with constant coefficients is presented in § 2. Using this algorithm, it follows that equation (1.1) possesses the Lax pair

$$\mu_x + \mathrm{i}k\mu = q, \tag{1.9 a}$$

† It turns out that these structures are generated by the discrete spectrum of the $t$-part of the Lax pair.

$$\mu_t + \mathrm{i}k^2\mu = \mathrm{i}q_x + kq, \quad k \in \mathbb{C}, \tag{1.9 b}$$

where $\mu(x,t,k)$ is a scalar function. The interested reader can verify directly that equations (1.9) are compatible iff $q(x,t)$ satisfies equation (1.1). The traditional approach of using an $x$-transform to solve equation (1.1) can be rederived using the Lax pair formulation (1.9): the spectral analysis of equation (1.9 $a$) can be used to derive the Fourier (Fokas & Gel'fand 1994), or the sine, or the cosine transforms, while equation (1.9 $b$) determines the evolution of the relevant transform data. Lax pairs are not unique. For example, equation (1.1), in addition to (1.9), also possesses the Lax pair,

$$\mu_t + \mathrm{i}k\mu = q, \tag{1.10 a}$$

$$\mu_{xx} + k\mu = -\mathrm{i}q. \tag{1.10 b}$$

The traditional approach of using a $t$-transform to solve (1.1) can be rederived using (1.10): the spectral analysis of equation (1.10 $a$) yields the Laplace transform in $t$, while equation (1.10 $b$) determines the $x$-dependence of the Laplace data. These approaches should be contrasted with the new approach which involves performing a simultaneous spectral analysis of *both* equations (1.9), or *both* equations (1.10). The spectral analysis of the former equations is slightly simpler than the spectral analysis of the latter equations†. In general, our experience indicates that for the implementation of the spectral analysis, the most convenient Lax pair is the one involving only first order derivatives of $\mu$.

The above discussion indicates that linear equations possess at least two Lax pairs which correspond to the traditional $x$- and $t$-transforms. By analogy, one might expect the same to be true for integrable nonlinear equations. This is indeed the case. For example, the Lax pair (1.9) is the linear limit, i.e. the limit as $q \to 0$, of the usual Lax pair of the nonlinear Schrödinger equation (NLS). *A new Lax pair for the NLS whose linear limit is the Lax pair (1.10), will be given in § 6.* This result is presented for completeness as well as in order to illustrate that it is possible to construct a Lax pair for some integrable nonlinear equation by 'deforming' one of the Lax pairs of the associated linearized version of this equation. This deformation is based on the so-called dressing method (Fokas & Zakharov 1992).

The rest of this paper is organized as follows: The main theorems are given below. A discussion of how to construct Lax pairs for linear PDEs is presented in § 2; examples include Lax pairs for the Laplace equation, for the reduced wave equation and for a generalization of the linearized Ernst equation. The derivations of theorems 1.1–1.3 are given in §§ 3–5. The new Lax pair of the nonlinear Schrödinger (NLS) equation and an IBV problem for the NLS equation are briefly discussed in § 6. A summary of the steps needed for the implementation of the new method is given in § 7.

We emphasize that here we only study simple IBV problems. Complicated IBV problems for linear PDEs, such as IBV problems with discontinuous boundary conditions and IBV problems with moving boundaries, will be discussed elsewhere. The study of IBV problems in simple domains with discontinuous boundary conditions, involves the formulation of an additional RH problem for the determination of some of the spectral data $\rho(k)$. The study of IBV problems in complicated domains, such

† One might think that for the solution of the IBV (1.1) and (1.2), there is an advantage in using (1.10) instead of (1.9) since $q_x$ does not appear in (1.10). However, this is not the case; in order to perform a simultaneous spectral analysis of (1.10), one needs to study $(\mu_x, \mu)^{\mathrm{T}}$, which then introduces $q_x$.

as problems with moving boundaries, involves an interesting extension of the Ehrenpreis principle: the contour on the complex $k$-plane is not fixed but it depends on $x$ and $t$.

**Notation**. $S(\mathbb{R}^+)$ will denote the space of Schwartz functions on the half line. $H^m(\mathbb{R}^+)$ will denote the space of square integrable functions in $(0,\infty)$, whose first $m$ generalized derivatives are also square integrable. $\dot{q}(x)$ will denote $\mathrm{d}q(x)/\mathrm{d}x$.

**Theorem 1.1.** *Let $q(x,t)$ satisfy the equation $\mathrm{i}q_t + q_{xx} = 0$, $x,t \in \mathbb{R}^+$, the initial condition*

$$q(x,0) = q_1(x) \in H^2(\mathbb{R}^+) \tag{1.11}$$

*and any one of the following boundary conditions:*

$$(a) \quad q(0,t) = q_2(t), \tag{1.12\,a}$$

$$(b) \quad q_x(0,t) = q_2(t), \tag{1.12\,b}$$

$$(c) \quad q_x(0,t) + \mathrm{i}\alpha q(0,t) = q_2(t), \tag{1.12\,c}$$

*where $q_2(t) \in H^1(\mathbb{R}^+)$, $\alpha$ is constant and $\arg\alpha \neq \frac{3}{2}\pi$.*

*The IBV problem satisfying (a), (b) or (c) with $\arg\alpha$ outside the interval $\pi \leqslant \arg\alpha < \frac{3}{2}\pi$ has a unique solution such that $q(x,t) \to 0$ as $x \to \infty$ and $q(x,t) \to 0$ as $t \to \infty$. The IBV problem satisfying (c) with $\pi \leqslant \arg\alpha < \frac{3}{2}\pi$ has a unique solution such that $q(x,t) \to 0$ as $x \to \infty$ and $q(x,t) \to 0$ as $t \to \infty$, provided that $q_1(x)$ and $q_2(t)$ satisfy the global constraint*

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\alpha x} q_1(x)\,\mathrm{d}x = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}\alpha^2 t} q_2(t)\,\mathrm{d}t. \tag{1.13}$$

*The solution of any of the above problems is given by*

$$q(x,t) = \frac{1}{2\pi}\int_\infty^\infty \mathrm{e}^{-\mathrm{i}kx - \mathrm{i}k^2 t}\hat{q}_1(k)\,\mathrm{d}k + \frac{1}{2\pi}\int_{\hat{L}} \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2 t}\nu(k)\,\mathrm{d}k, \tag{1.14}$$

*where $\hat{L}$ is the union of the negative real axis and of the negative imaging axis (see figure 2) and $\hat{q}_1(k)$, $\nu(k)$ are defined as follows:*

$$\hat{q}_1(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx} q_1(x)\,\mathrm{d}x; \tag{1.15}$$

*$\nu(k)$ for (a), (b) or (c), is given by*

$$\nu(k) = 2k\hat{q}_2(k) + \hat{q}_1(-k), \tag{1.16\,a}$$

$$\nu(k) = 2\mathrm{i}\hat{q}_2(k) - \hat{q}_1(-k), \tag{1.16\,b}$$

$$\nu(k) = \frac{2\mathrm{i}k\hat{q}_2(k)}{k-\alpha} - \frac{k+\alpha}{k-\alpha}\hat{q}_1(-k), \tag{1.16\,c}$$

*where*

$$\hat{q}_2(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}k^2 t} q_2(t)\,\mathrm{d}t.$$

*For any of the above problems, the functions $q_1(x)$, $q(0,t)$, $q_x(0,t)$ are related by the equation*

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}kx} q_1(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{\mathrm{i}k^2 t}(\mathrm{i}q_x(0,t) - kq(0,t))\,\mathrm{d}t, \quad \pi \leqslant \arg k \leqslant \tfrac{3}{2}\pi.$$

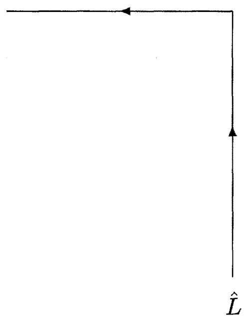


Figure 2. The contour $\hat{L}$.

If $\arg\alpha = \frac{3}{2}\pi$, the solution $q(x,t)$, in addition to the term given by equation (1.14), also contains the term $2\mathrm{i}\alpha\sigma\exp[-\mathrm{i}\alpha x - \mathrm{i}\alpha^2 t]$, $\sigma$ constant (see § 3 for details). This term decays to zero as $x \to \infty$, but it oscillates in $t$. If $\pi \leqslant \arg\alpha < \frac{3}{2}\pi$ and the global constraint is not satisfied then the solution blows up exponentially (like $\exp[-\mathrm{i}\alpha x - \mathrm{i}\alpha^2 t]$) as $t \to \infty$.

**Theorem 1.2.** *Let $q(x,t) \in \mathbb{R}$ satisfy*

$$q_{xt} + q = 0, \quad x \in [0,l], \quad t \in [0,\infty), \quad l > 0 \tag{1.17}$$

$$\left.\begin{aligned} q(x,0) &= q_1(x) \in C^1([0,l]), \\ q(0,t) &= q_2(t) \in C^1[0,\infty)] \cup H^1((0,\infty)), \\ q_1(0) &= q_2(0). \end{aligned}\right\} \tag{1.18}$$

*The unique solution of this IBV problem is given by*

$$q = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx-(\mathrm{i}t)/k}\frac{\rho(k)}{k}\,\mathrm{d}k, \tag{1.19}$$

*where this integral is defined as an improper Riemann integral and $\rho(k)$ is given by*

$$\rho(k) = \frac{\mathrm{i}}{k}\int_0^{\infty} \mathrm{e}^{(\mathrm{i}t)/k} q_2(t)\,\mathrm{d}t - \int_0^l \mathrm{e}^{\mathrm{i}kx}\dot{q}_1(x)\,\mathrm{d}x, \quad k \in \mathbb{R}. \tag{1.20}$$

*The functions $q_1(x)$, $q_2(t)$ and $q(l,t)$ are related by the equation*

$$-\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}kl}\int_0^{\infty} \mathrm{e}^{(\mathrm{i}t)/k} q_2(t)\,\mathrm{d}t + \int_0^l \mathrm{e}^{-\mathrm{i}k(l-x)}\dot{q}_1(x)\,\mathrm{d}x + \frac{\mathrm{i}}{k}\int_0^{\infty} \mathrm{e}^{(\mathrm{i}t)/k} q(l,t)\,\mathrm{d}t = 0,$$
$$0 \leqslant \arg k \leqslant \pi. \tag{1.21}$$

**Theorem 1.3.** *Let $q(x,t) \in \mathbb{R}$ satisfy*

$$q_{xt} + \sin q = 0, \quad x \in [0,l], \quad t \in [0,\infty), \quad l > 0 \tag{1.22}$$

$$q(x,0) = q_1(x), \quad q(0,t) = q_2(t). \tag{1.23}$$

*Assume that*

$$q_1(x) \in C^1([0,l]), \quad q_2(t) \in S(\mathbb{R}^+), \quad q_1(0) = q_2(0). \tag{1.24}$$

*The unique solution of this IBV is given by*

$$q_x(x,t) = -\frac{2}{\pi}\int_{-\infty}^{\infty} \overline{M_2}(x,t,k)\frac{\rho_1(k)}{\rho_2(k)}\mathrm{e}^{-2\mathrm{i}kx-(\mathrm{i}t)/2k}\,\mathrm{d}k, \tag{1.25}$$

*where bar denotes complex conjugation and the scalar functions $\rho_1(k)$, $\rho_2(k)$ and $M_2(x,t,k)$, $k \in \mathbb{R}$, are constructed as follows: Let $(\Phi_1(t,k),\, \Phi_2(t,k))^{\mathrm{T}}$ be the unique solution of*

$$\Phi_{1_t} = \frac{\mathrm{i}}{4k}\{(-1+\cos q_2(t))\,\Phi_1 + (\sin q_2(t))\,\Phi_2\}, \tag{1.26 a}$$

$$\Phi_{2_t} - \frac{\mathrm{i}}{2k}\Phi_2 = \frac{\mathrm{i}}{4k}\{(\sin q_2(t))\,\Phi_1 + (1-\cos q_2(t))\,\Phi_2\}, \tag{1.26 b}$$

*which satisfies*

$$\lim_{t\to\infty}(\Phi_1, \mathrm{e}^{-(\mathrm{i}t)/2k}\,\Phi_2)^{\mathrm{T}} = (1,0)^{\mathrm{T}}. \tag{1.26 c}$$

*Let $(\Psi_1(x,k),\, \Psi_2(x,k))^{\mathrm{T}}$ be the unique solution of*

$$\Psi_{1x} = -\tfrac{1}{2}\dot{q}_1(x)\,\Psi_2, \tag{1.27 a}$$

$$\Psi_{2x} - 2\mathrm{i}k\,\Psi_2 = \tfrac{1}{2}\dot{q}_1(x)\,\Psi_1, \tag{1.27 b}$$

*which satisfies*

$$(\Psi_1(0,k),\, \Psi_2(0,k))^{\mathrm{T}} = (\Phi_1(0,k),\, \Phi_2(0,k))^{\mathrm{T}}. \tag{1.27 c}$$

*The functions $\rho_1(k)$ and $\rho_2(k)$ are defined by*

$$\rho_1(k) = -\mathrm{e}^{2\mathrm{i}kl}\,\overline{\Psi_2}(l,k), \quad \rho_2(k) = \overline{\Psi_1}(l,k). \tag{1.28}$$

*Given $\rho_1$ and $\rho_2$, the function $M_2(x,t,k)$ is defined as the unique solution of*

$$M_1(x,t,k) = -\frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}k'x-(\mathrm{i}t)/(2k')}\frac{\rho_1(k')}{\rho_2(k')}\overline{M_2}(x,t,k')\frac{\mathrm{d}k'}{k'-(k+\mathrm{i}0)}, \tag{1.29 a}$$

$$M_2(x,t,k) = 1 + \frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-2\mathrm{i}k'x-(\mathrm{i}t)/(2k')}\frac{\rho_1(k')}{\rho_2(k')}\overline{M_1}(x,t,k')\frac{\mathrm{d}k'}{k'-(k+\mathrm{i}0)}. \tag{1.29 b}$$

*The above formalism is valid if $\rho_2(k) \neq 0$, for $\operatorname{Im} k \leqslant 0$. If $\rho_2(k_j) = 0$, $j = 1,2,\ldots, \operatorname{Im} k_j < 0$, then there exist solitons. In this case, solving equations (1.22) and (1.23) involves solving a system of linear integral equations similar to equations (1.29) and a system of algebraic equations (see § 5).*

The assumption that certain data belong to the Schwartz class is made for simplicity. It is straightforward to replace the Schwartz class by a less restrictive class.

## 2. Lax pairs for linear equations

**Proposition 2.1.** *Let $q(x,y)$ satisfy the linear PDE with constant coefficients*

$$M(\partial_x, \partial_y)q = 0, \tag{2.1}$$

*where $M(\partial_x, \partial_y)$ is a linear operator of $\partial_x$ and $\partial_y$ with constant coefficients. This equation possesses the Lax pair*

$$\mu_x + \mathrm{i}k\mu = q, \quad k \in \mathbb{C}, \tag{2.2 a}$$

$$M(\partial_x, \partial_y)\mu = 0, \tag{2.2 b}$$

*where $\mu(x,y,k)$ is a scalar function.*

*Proof.* Applying the operator $M$ on equation (2.2 $a$), and using the fact that $M$

and $\partial_x + \mathrm{i}k$ commute, it follows that the compatibility condition of equations (2.2) implies equation (2.1). ∎

The Lax pair (2.2) can be rewritten in a form which is more convenient for the implementation of its spectral analysis. We first discuss evolution equations.

**Proposition 2.2.** *Let $q(x,t)$ satisfy the evolution equation*

$$\left(\partial_t + \sum_0^N \alpha_j \partial_x^j\right) q = 0, \quad N \in \mathbb{Z}^+, \tag{2.3}$$

*where $\alpha_j$, $0 \leqslant j \leqslant N$, are constants. This equation possesses the following Lax pair:*

$$\mu_x + \mathrm{i}k\mu = q, \quad k \in \mathbb{C}, \tag{2.4 a}$$

$$\mu_t + \sum_0^N \alpha_j (-\mathrm{i}k)^j \mu = -\sum_{j=1}^N \alpha_j (\partial_x^{j-1} + (-\mathrm{i}k)\partial_x^{j-2} + (-\mathrm{i}k)^2 \partial_x^{j-3} + \cdots + (-\mathrm{i}k)^{j-1}) q. \tag{2.4 b}$$

*Proof.* Equation (2.2 *b*) takes the form

$$\mu_t + \sum_0^N \alpha_j \partial_x^j \mu = 0. \tag{2.5}$$

Using equation (2.2 *a*) to eliminate $\partial_x^j \mu$, $j = 1, \ldots, N$, this equation becomes equation (2.4 *b*). ∎

**Remark 2.3.** Equation (2.3) also possesses the Lax pair

$$\mu_t + \mathrm{i}k\mu = q, \quad k \in \mathbb{C}, \tag{2.6 a}$$

$$\sum_0^N \alpha_j \partial_x^j \mu - \mathrm{i}k\mu = -q. \tag{2.6 b}$$

Indeed, equation (2.3) possesses the Lax pair consisting of equations (2.5) and (2.6 *a*). Replacing $\mu_t$ in equation (2.5) by $-\mathrm{i}k\mu + q$, this equation becomes (2.6 *b*).

**Examples.** (1) The linearized NLS equation (1.1) possesses the Lax pairs (1.9) and (1.10). Indeed, $\alpha_2 = -\mathrm{i}$ and equations (2.4 *b*), (2.6 *b*) become

$$\mu_t + \mathrm{i}k^2\mu = \mathrm{i}(q_x - \mathrm{i}kq), \quad -\mathrm{i}(\mu_{xx} + k\mu) = -q.$$

(2) The linearized Korteweg–de Vries equation

$$q_t + q_{xxx} = 0 \tag{2.7}$$

possesses the Lax pair

$$\mu_x + \mathrm{i}k\mu = q \tag{2.8 a}$$

$$\mu_t + \mathrm{i}k^3\mu = -q_{xx} + \mathrm{i}kq_x + k^2 q. \tag{2.8 b}$$

**Remark 2.4.** The expression

$$\mathrm{e}^{-\mathrm{i}kx - w(k)y}$$

is a solution of equation (2.1) iff

$$M(-\mathrm{i}k, -w(k)) = 0. \tag{2.9}$$

In the particular case of evolution equations, equation (2.9) has one root $w(k)$, and the left-hand sides of the equations forming its Lax pair are given by

$$\mu_x + \mathrm{i}k\mu, \quad \mu_t + w(k)\mu.$$

In general, if the symbol of the differential operator $M$ can be factorized into $n$ roots, there exist $n$ Lax pairs; the left-hand sides of the equations forming the $l$th Lax pair are given by

$$\mu_x + \mathrm{i}k\mu, \quad \mu_t + w_l(k)\mu, \quad l = 1, \ldots, n.$$

**Examples.** (1) The symbol of the Laplace operator is $w^2 = k^2$. Laplace's equation can be written as

$$(\partial_x^2 + k^2)q = -(\partial_y^2 - w^2)q, \tag{2.10}$$

or

$$(\partial_x + \mathrm{i}k)(\partial_x - \mathrm{i}k)q = -(\partial_y - w)(\partial_y + w)q. \tag{2.11}$$

This implies that the Laplace equation

$$q_{xx} + q_{yy} = 0, \tag{2.12}$$

possesses the two Lax pairs

$$\mu_x + \mathrm{i}k\mu = q_y - wq, \tag{2.13 a}$$

$$\mu_y + w\mu = -q_x + \mathrm{i}kq, \tag{2.13 b}$$

where $w = \pm k$.

(2) The symbol of the operator $\partial_x^2 + \partial_y^2 + \beta^2$, $\beta$ real constant, is $w^2 = k^2 - \beta^2$. The reduced wave equation can also be written in the form (2.11). Thus, the reduced wave equation

$$q_{xx} + q_{yy} + \beta^2 q = 0, \tag{2.14}$$

possesses the two Lax pairs (2.13), where $w = \pm\sqrt{k^2 - \beta^2}$.

For the implementation of the spectral analysis, it is more convenient to 'uniformize' $k$, i.e. to replace $k$ by $\frac{1}{2}\beta(k + (1/k))$. Then equations (2.13) imply that equation (2.14) possesses the Lax pair

$$\mu_x + \tfrac{1}{2}\mathrm{i}\beta\left(k + \frac{1}{k}\right)\mu = q_y \mp \tfrac{1}{2}\beta\left(k - \frac{1}{k}\right)q, \tag{2.15 a}$$

$$\mu_y \pm \tfrac{1}{2}\beta\left(k - \frac{1}{k}\right)\mu = -q_x + \tfrac{1}{2}\mathrm{i}\beta\left(k + \frac{1}{k}\right)q, \quad k \in \mathbb{C}. \tag{2.15 b}$$

**Remark 2.5.** It is also possible to construct Lax pairs for linear PDEs with non-constant coefficients. For example, the equation

$$(x - y)q_{xy} + \beta q_x - \alpha q_y = 0, \quad \alpha, \beta \,\mathrm{const.} \tag{2.16}$$

possesses the Lax pair

$$\mu_x - \frac{\alpha}{x - k}\mu = \frac{q_x}{x - k}, \tag{2.17 a}$$

$$\mu_y - \frac{\beta}{y - k}\mu = \frac{q_y}{y - k}, \quad k \in \mathbb{C}. \tag{2.17 b}$$

The particular case of $\alpha = \beta = \frac{1}{2}$ is the linearized version of the Ernst equation.

## 3. The linearized NLS equation

In order to derive the integral representation (1.6), we first *assume* that there exists a solution $q(x,t)$ which decays to zero as $x \to \infty$ and as $t \to \infty$. This yields an expression for $q(x,t)$ in terms of appropriate spectral data. We then verify directly that this expression solves equation (1.1) and satisfies the given initial and boundary data *without* the *a priori* assumption that $q(x,t)$ exists.

Let

$$\tilde{q}(x,t,k) = \mathrm{i}q_x(x,t) + kq(x,t). \tag{3.1}$$

The general solution of equation (1.9 $a$) is

$$\mu(x,t,k) = \mathrm{e}^{-\mathrm{i}kx}\mu(0,t,k) + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')}q(x',t)\,\mathrm{d}x'.$$

Equations (1.9) are compatible, thus the solution $\mu(x,t,k)$ defined above also solves equation (1.9 $b$) iff $\mu(0,t,k)$ solves

$$\mu_t(0,t,k) + \mathrm{i}k^2\mu(0,t,k) = \tilde{q}(0,t,k).$$

A particular solution of this equation is

$$\mu(0,t,k) = \int_0^t \mathrm{e}^{-\mathrm{i}k^2(t-t')}\tilde{q}(0,t',k)\,\mathrm{d}t'.$$

Thus, the function $\mu^{(4)}(x,t,k)$, defined by

$$\mu^{(4)}(x,t,k) = \mathrm{e}^{-\mathrm{i}kx}\int_0^t \mathrm{e}^{-\mathrm{i}k^2(t-t')}\tilde{q}(0,t',k)\,\mathrm{d}t' + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')}q(x',t)\,\mathrm{d}x', \tag{3.2}$$

solves both equations (1.9). Similarly, the functions $\mu^{(3)}(x,t,k)$ and $\mu^{(12)}(x,t,k)$, defined by

$$\mu^{(3)}(x,t,k) = -\mathrm{e}^{-\mathrm{i}kx}\int_t^\infty \mathrm{e}^{-\mathrm{i}k^2(t-t')}\tilde{q}(0,t',k)\,\mathrm{d}t' + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')}q(x',t)\,\mathrm{d}x' \tag{3.3}$$

and

$$\mu^{(12)}(x,t,k) = -\int_x^\infty \mathrm{e}^{-\mathrm{i}k(x-x')}q(x',t)\,\mathrm{d}x', \tag{3.4}$$

also solve both equations (1.9).

The superscripts indicate that the functions $\mu^{(4)}$, $\mu^{(3)}$, $\mu^{(12)}$ are analytic and bounded in the fourth quadrant, the third quadrant and the upper half of the complex $k$-plane. The boundness of these functions follows from the following elementary facts. The function $\mathrm{e}^{\mathrm{i}k\xi}$ is bounded in the upper half of the complex $k$-plane for $\xi \in \mathbb{R}^+$, while the function $\mathrm{e}^{\mathrm{i}k^2\tau}$ is bounded in the first and the third quadrants of the complex $k$-plane for $\tau \in \mathbb{R}^+$. Let I, ..., IV denote the first, ..., the fourth quadrant of the complex $k$-plane. The exponentials appearing in (3.2) are bounded in the intersection of (III $\cup$ IV) and of (II $\cup$ IV), i.e. in IV. Similarly, the exponentials in (3.3) are bounded in the intersection of (III $\cup$ IV) and of (I $\cup$ III), i.e. in III, while the exponential in (3.4) are bounded in (I $\cup$ II).

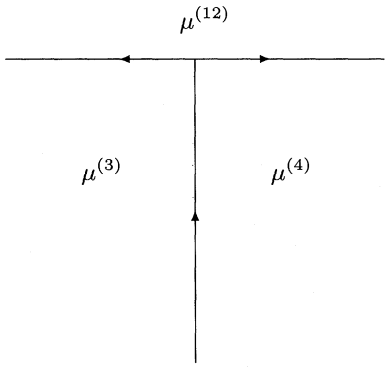


Figure 3. The Riemann–Hilbert problem for the equation $\mathrm{i}q_t + q_{xx} = 0$ in the quarter plane.

Integration by parts implies that if $\mu$ denotes $\mu^{(4)}$, or $\mu^{(3)}$, or $\mu^{(12)}$ then, for $k$ off the contour defined in figure 1,

$$\mu(x,t,k) = \frac{q(x,t)}{\mathrm{i}k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{3.5}$$

In the domain of their overlap, the functions $\mu^{(4)}$, $\mu^{(3)}$, $\mu^{(12)}$ are simply related

$$\mu^{(12)}(x,t,k) - \mu^{(4)}(x,t,k) = -\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}s(k), \quad k \in \mathbb{R}^+, \tag{3.6 a}$$

$$\mu^{(3)}(x,t,k) - \mu^{(12)}(x,t,k) = \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}[s(k) - \nu(k)], \quad k \in \mathbb{R}^-, \tag{3.6 b}$$

$$\mu^{(3)}(x,t,k) - \mu^{(4)}(x,t,k) = -\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\nu(k), \quad k \in \mathrm{i}\mathbb{R}^-, \tag{3.6 c}$$

where

$$s(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx}q(x,0)\,\mathrm{d}x, \quad 0 \leqslant \arg k \leqslant \pi, \tag{3.7}$$

$$\nu(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}k^2t}(\mathrm{i}q_x(0,t) + kq(0,t))\,\mathrm{d}t, \quad \pi \leqslant \arg k \leqslant \tfrac{3}{2}\pi. \tag{3.8}$$

Indeed, let $\Delta\mu$ denote the difference of any two solutions of equations (1.9). Since these solutions satisfy equation (1.9 $a$), it follows that $\Delta\mu = \mathrm{e}^{-\mathrm{i}kx}f(t,k)$. Similarly, since these solutions satisfy equation (1.9 $b$), it follows that $\Delta\mu = \mathrm{e}^{-\mathrm{i}k^2t}g(x,k)$. Thus, $\Delta\mu = \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\rho(k)$, where the function $\rho(k)$ can be determined by evaluating this equation at any suitable $x$, $t$. For example, evaluating the equation $\mu^{(12)} - \mu^{(4)} = \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\rho(k)$ at $x = t = 0$, it follows that $\rho(k) = -s(k)$. Similarly for (3.6 $b$), (3.6 $c$).

The unique solution of this Riemann–Hilbert problem is given by

$$\mu = -\frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathrm{e}^{-\mathrm{i}k'x-\mathrm{i}k'^2t}\rho(k')}{k'-k}\,\mathrm{d}k', \tag{3.9}$$

where $\mu$ is $\mu^{(12)}$ if $k \in \mathrm{I} \cup \mathrm{II}$, $\mu$ is $\mu^{(3)}$ if $k \in \mathrm{III}$, $\mu$ is $\mu^{(4)}$ if $k \in \mathrm{IV}$, the directed contour $L$ is depicted in figure 3 and $\rho(k) = \nu(k) - s(k)$ for $k \in \mathbb{R}^-$, $\rho(k) = s(k)$ for $k \in \mathbb{R}^+$, $\rho(k) = \nu(k)$ for $k \in \mathrm{i}\mathbb{R}^-$.

Equation (3.5) and the large $k$ behaviour of equation (3.9) yield equation (1.6).

We now derive equation (1.8). This equation is a consequence of the fact that each solution of equations (1.9) possesses two different representations. For example,

$\mu^{(12)}$, in addition to equation (3.4), also satisfies

$$\mu^{(12)}(x,t,k) = \mathrm{e}^{-\mathrm{i}k^2 t}\mu^{(12)}(x,0,k) + \int_0^t \mathrm{e}^{-\mathrm{i}k^2(t-t')}\tilde{q}(x,t',k)\,\mathrm{d}t',$$

or

$$\mu^{(12)}(x,t,k) = -\mathrm{e}^{-\mathrm{i}k^2 t}\int_x^\infty \mathrm{e}^{-\mathrm{i}k(x-x')}q(x',0)\,\mathrm{d}x' + \int_0^t \mathrm{e}^{-\mathrm{i}k^2(t-t')}\tilde{q}(x,t',k)\,\mathrm{d}t', \quad (3.10)$$

where equation (3.4) has been used to evaluate $\mu^{(12)}(x,0,k)$. The exponentials appearing in (3.10) are bounded in II. But equation (3.4) implies that $\mu^{(12)}$ is well defined in both I and II. Rewriting the $\int_0^t$ in (3.10) as $\int_0^\infty - \int_t^\infty$, it follows that $\mu^{(12)}$ is analytic and bounded in (I $\cup$ II) iff

$$\int_0^\infty \mathrm{e}^{\mathrm{i}k^2 t}\tilde{q}(x,t,k)\,\mathrm{d}t - \int_x^\infty \mathrm{e}^{-\mathrm{i}k(x-x')}q(x',0)\,\mathrm{d}x' = 0, \quad k \in \mathrm{I}. \quad (3.11)$$

Letting $x = 0$ in equation (3.11), this equation becomes (1.8). Replacing $k$ by $-k$ in equation (1.8), it follows that

$$\int_0^\infty \mathrm{e}^{\mathrm{i}k^2 t}(\mathrm{i}q_x(0,t) - kq(0,t))\,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\mathrm{i}kx}q(x,0)\,\mathrm{d}x = 0, \quad \pi \leqslant \arg k \leqslant \tfrac{3}{2}\pi. \quad (3.12)$$

The above calculations motivate the following proposition.

**Proposition 3.1.** *Let $q(x,t)$ satisfy equation (1.1) and the initial condition $q(x,0) = q_1(x) \in H^2(\mathbb{R}^+)$. Assume that there exists a solution of this problem such that $q(x,t) \to 0$ as $x \to \infty$ and $q(x,t) \to 0$ as $t \to \infty$. This solution is given by (1.14), where $\hat{q}_1(k)$ and $\nu(k)$ are defined by*

$$\hat{q}_1(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx}q_1(x)\,\mathrm{d}x, \quad 0 \leqslant \arg k \leqslant \pi, \quad (3.13)$$

$$\nu(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}k^2 t}(\mathrm{i}q_x(0,t) + kq(0,t))\,\mathrm{d}t, \quad \pi \leqslant \arg k \leqslant \tfrac{3}{2}\pi. \quad (3.14)$$

*The boundary values of $q(x,t)$ satisfy the global constraint*

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}kx}q(x,0)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{\mathrm{i}k^2 t}(\mathrm{i}q_x(0,t) - kq(0,t))\,\mathrm{d}t, \quad \pi \leqslant \arg k \leqslant \tfrac{3}{2}\pi \quad (3.15)$$

*Proof.* For the sake of economy of presentation, the proof is omitted. It involves calculations similar to the ones presented in the proof of theorem 1.1. ■

*Proof of theorem 1.1.* A well posed problem for equation (1.1) is defined by prescribing $q(x,0)$ and some combination of $q_x(0,t)$ and $q(0,t)$. Suppose, for example, that $q(0,t) = q_2(t)$ is given. Then subtracting equations (3.14) and (3.15), we find equation (1.16 *a*). Similarly, linear combinations of equations (3.14) and (3.15) imply equations (1.16 *b*) and (1.16 *c*).

It is straightforward to verify directly that, if $q(x,t)$ is defined by equation (1.14), it solves the given IBV problem. Consider, for example, the IBV problem defined by equations (1.1) and (1.2). Evaluating equations (1.14) at $x = 0$, we find

$$q(x,0) = \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}kx}\hat{q}_1(k)\,\mathrm{d}k + \int_{\hat{L}} \mathrm{e}^{-\mathrm{i}kx}\nu(k)\,\mathrm{d}k,$$

where the contour $\hat{L}$ is the union of the negative imaginary axis and of the negative real axis. Since the integrand of $\int_{\hat{L}}$ is analytic and bounded in the third quadrant, this integral vanishes. Thus, the above yields $q_1(x)$. Similarly,

$$q(0,t) = \int_{\hat{L}} \mathrm{e}^{-\mathrm{i}k^2t} 2k\hat{q}_2(k)\,\mathrm{d}k + \int_{\tilde{L}} \mathrm{e}^{-\mathrm{i}k^2t}\hat{q}_1(-k)\,\mathrm{d}k,$$

where the contour $\tilde{L}$ is the union of the negative imaginary axis and of the positive real axis. Since the integrand of $\int_{\tilde{L}}$ is analytic and bounded in the fourth quadrant, the integral vanishes. Thus, the above yields $q_2(t)$. Similar considerations are valid for the cases that $q_x(0,t)$, or $\mathrm{i}q_x(0,t)-\alpha q(0,t)$, $\arg\alpha \neq \frac{3}{2}\pi$, are given. In the latter case, it should be noted that because of the global constraint (3.15), the possible singularity of $\nu(k)$ is removable. This constraint follows from the evaluation of equation (3.15) at $k=-\alpha$. ∎

**Remark 3.2.** For more complicated IBV problems, equation (1.8) can be used to formulate a RH problem for some of the spectral data. For example if $q(0,t)=Q_1(t)$ for $t\in[0,T]$ and $q_x(0,t)=Q_2(t)$ for $t\in[T,\infty)$, then equation (1.8) can be used to formulate a RH problem for the functions

$$\int_0^T \mathrm{e}^{\mathrm{i}k^2(T-t)} q_x(0,t)\,\mathrm{d}t, \quad k\int_T^\infty \mathrm{e}^{\mathrm{i}k^2(t-T)}q(0,t)\,\mathrm{d}t.$$

The jumps of this RH problem occur for $k\in\mathbb{R}\cup\mathrm{i}\mathbb{R}$. This RH problem can be solved in closed form.

We now consider the case $\arg\alpha=\frac{3}{2}\pi$.

**Theorem 3.3.** *The solution of the equation* $\mathrm{i}q_t+q_{xx}=0$, *with*

$$q(x,0)=q_1(x), \quad q_x(0,t)+\mathrm{i}\alpha q(0,t)=q_2(t), \quad \arg\alpha=\tfrac{3}{2}\pi, \tag{3.16}$$

*where* $q_1(x)\in H^2(\mathbb{R}^+)$ *and* $q_2(t)\in H^1(\mathbb{R}^+)$ *is given by*

$$q=\frac{1}{2\pi}\int_\infty^\infty \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\hat{q}_1(k)\,\mathrm{d}k+\frac{1}{2\pi}\int_{\hat{L}}\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\nu(k)\,\mathrm{d}k+2\mathrm{i}\alpha\sigma\mathrm{e}^{-\mathrm{i}\alpha x-\mathrm{i}\alpha^2t}. \tag{3.17 a}$$

*The constant* $\sigma$ *is defined by*

$$\sigma \doteqdot \int_0^\infty \mathrm{e}^{-\mathrm{i}\alpha x}q_1(x)\,\mathrm{d}x-\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}\alpha^2t}q_2(t)\,\mathrm{d}t, \tag{3.17 b}$$

*the directed contour* $\hat{L}$ *is the union of the negative real axis and of the negative imaginary axis, (see figure 2) and* $\hat{q}_1(k)$, $\nu(k)$ *are defined as follows:*

$$\hat{q}_1(k)=\int_0^\infty \mathrm{e}^{ikx}q_1(x)\,\mathrm{d}x+\frac{2\alpha\sigma}{k-\alpha}, \tag{3.18}$$

$$\nu(k)=\frac{1}{k-\alpha}\left[2k\mathrm{i}\int_0^\infty \mathrm{e}^{ik^2t}q_2(t)\,\mathrm{d}t-(\alpha+k)\int_0^\infty \mathrm{e}^{-ikx}q_1(x)\,\mathrm{d}x+2\alpha\sigma\right]. \tag{3.19}$$

*Proof.* In order to verify directly this result, we let

$$\tilde{q}(x,t)=q(x,t)-2\mathrm{i}\alpha\sigma\mathrm{e}^{-\mathrm{i}\alpha x-\mathrm{i}\alpha^2t}. \tag{3.20}$$

Thus, $\tilde{q}(x,t)$ satisfies the same equation as $q(x,t)$ and

$$\tilde{q}(x,0)=q_1(x)-2\mathrm{i}\alpha\sigma\mathrm{e}^{-\mathrm{i}\alpha x}, \quad \tilde{q}_x(0,t)+\mathrm{i}\alpha\tilde{q}(0,t)=q_2(t). \tag{3.21}$$

It can be verified directly that if $\tilde{q}$ is defined by

$$\tilde{q} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\hat{q}_1(k)\,\mathrm{d}k + \frac{1}{2\pi}\int_{\hat{L}} \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\nu(k)\,\mathrm{d}k,$$

then $\tilde{q}$ solves the IBV problem defined by equations (1.1) and (3.21). For this verification, one uses that $\int_0^\infty \mathrm{e}^{\mathrm{i}(k-\alpha)x}\,\mathrm{d}x = \mathrm{i}/(k-\alpha)$ and that the singularity of $\nu(k)$ at $k=\alpha$ is removable. ■

**Remark 3.4.** If $\arg\alpha = \frac{3}{2}\pi$, the term $\mathrm{e}^{-\mathrm{i}\alpha x-\mathrm{i}\alpha^2 t}$ decays in $x$ but it oscillates in $t$. Thus, the solution defined in proposition 3.1, which decays to zero as $t\to\infty$, does not satisfy this condition. However, this case can also be handled by the formalism presented in this paper. Define $\tilde{q}(x,t)$ by equation (3.20). Let $\tilde{q}_1(x)$ and $\tilde{q}_2(t)$ denote the initial and boundary data associated with $\tilde{q}(x,t)$. Since $\tilde{q}(x,t)$ decays in $t$, it satisfies the global constraint

$$\int_0^\infty \mathrm{e}^{-\mathrm{i}\alpha x}\tilde{q}_1(x)\,\mathrm{d}x = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}\alpha^2 t}\tilde{q}_2(t)\,\mathrm{d}t.$$

This equation implies that $\sigma$ is given by equation (3.17 $b$)†.

**Remark 3.5.** The solution of the IBV problem defined by equations (1.1) and (1.2) can also be expressed in the form (1.6), where

$$\left.\begin{aligned} &\rho(k) = \hat{q}_1(k) - \hat{q}_1(-k), \quad \text{for } k\in\mathbb{R}^+,\\ &\rho(k) = \hat{q}_1(-k) - \hat{q}_1(k) + 2k\hat{q}_2(k), \quad \text{for } k\in\mathbb{R}^-,\\ &\rho(k) = 2k\hat{q}_2(k), \quad \text{for } k\in \mathrm{i}\mathbb{R}^-, \end{aligned}\right\} \tag{3.22}$$

and

$$\hat{q}_1(k) \doteqdot \int_0^\infty \mathrm{e}^{\mathrm{i}kx}q_1(x)\,\mathrm{d}x, \quad \hat{q}_2(k) \doteqdot \int_0^\infty \mathrm{e}^{\mathrm{i}k^2t}q_2(t)\,\mathrm{d}t. \tag{3.23}$$

Indeed,

$$\hat{q}_1(k) = \hat{q}_1(k) - \hat{q}_1(-k) + \hat{q}_1(-k), \quad \nu(k) = 2k\hat{q}_2(k) + \hat{q}_1(-k).$$

The last terms of these expressions give a zero contribution to (1.6), thus the above equations yield (3.22) (the contributions along $k\in\mathbb{R}^-$ cancel, while the integral along $k\in\mathbb{R}^+$ and $k\in\mathrm{i}\mathbb{R}^-$ is zero since $\exp[-\mathrm{i}kx - \mathrm{i}k^2t]\hat{q}_1(-k)$ is analytic and bounded for $k\in\mathrm{IV}$).

Similarly, there exist other equivalent representations for the other IBV problems studied in theorem 1.1.

## 4. The linearized sine–Gordon equation in light-cone coordinates

The linearized sine–Gordon equation (1.17) possess the Lax pair

$$\mu_x + \mathrm{i}k\mu = q_x \tag{4.1 a}$$

$$\mu_t + \frac{\mathrm{i}}{k}\mu = \frac{\mathrm{i}}{k}q, \quad k\in\mathbb{C}. \tag{4.1 b}$$

† A similar observation for the problem of forced surface water waves was made by P. Bressloff.

Let $\mu^+(x,t,k)$ and $\mu^-(x,t,k)$ be defined by

$$\mu^+(x,t,k) = \frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}k(x-l)} \int_0^t \mathrm{e}^{-(\mathrm{i}/k)(t-t')} q(l,t')\,\mathrm{d}t' - \int_x^l \mathrm{e}^{-\mathrm{i}k(x-x')} q_x(x',t)\,\mathrm{d}x', \quad k \in \mathrm{I} \cup \mathrm{II}, \tag{4.2}$$

$$\mu^-(x,t,k) = -\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}kx} \int_t^\infty \mathrm{e}^{-(\mathrm{i}/k)(t-t')} q(0,t')\,\mathrm{d}t' + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')} q_x(x',t)\,\mathrm{d}x', \quad k \in \mathrm{III} \cup \mathrm{IV}. \tag{4.3}$$

These functions satisfy (4.1) and they exhibit the following asymptotic behaviour:

$$\mu = \frac{q_x}{\mathrm{i}k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \quad \operatorname{Im} k \neq 0, \tag{4.4}$$

$$\mu = q + \mathrm{i}kq_t + O(k^2), \quad k \to 0. \tag{4.5}$$

Furthermore, they are related by

$$\mu^+(x,t,k) - \mu^-(x,t,k) = \mathrm{e}^{-\mathrm{i}kx-(\mathrm{i}t/k)}\rho(k), \quad k \in \mathbb{R}, \tag{4.6}$$

where $\rho(k)$ is defined in (1.20). Equations (4.4)–(4.6) define a Riemann–Hilbert problem whose unique solution is

$$\mu(x,t,k) = \frac{1}{2\mathrm{i}\pi} \int_{-\infty}^\infty \frac{\mathrm{e}^{-\mathrm{i}k'x-(\mathrm{i}t/k')}\rho(k')}{k'-k}\,\mathrm{d}k', \qquad k \in \mathbb{C}. \tag{4.7}$$

The definition of $\rho(k)$ implies

$$\rho(k) = \frac{\mathrm{i}}{k}\left[\int_0^\infty q_2(t)\,\mathrm{d}t - \dot{q}_1(0) + \dot{q}_1(l)\mathrm{e}^{\mathrm{i}kl}\right] + o\left(\frac{1}{|k|}\right), \quad k \to \infty.$$

Because of the slow decay of $\rho(k)$, some of the integrals defining $\mu$ must be interpreted as improper Riemann integrals. For example, the large $k$ behaviour of $\mu$ is given by

$$\mu = \frac{\mathrm{i}}{2\pi k}[I_1 + I_2 + I_3] + o\left(\frac{1}{|k|}\right), \quad k \to \infty, \tag{4.8}$$

where

$$I_1 = \int_{-1}^1 \mathrm{e}^{-\mathrm{i}(kx+(t/k))}\rho(k)\,\mathrm{d}k, \quad I_2 = \int_{|k|>1} \mathrm{e}^{-\mathrm{i}kx}(\mathrm{e}^{-\mathrm{i}t/k} - 1)\rho(k)\,\mathrm{d}k,$$

$$I_3 = \int_{|k|>1} \mathrm{e}^{-\mathrm{i}kx}\rho(k)\,\mathrm{d}k,$$

and $I_3$ is an improper Riemann integral. Equations (4.4) and (4.8) imply equation (1.19).

The functions $q(0,t)$, $q(l,t)$ and $q(x,0)$ are related by equation (1.21). Indeed, the function $\mu^-$, in addition to equation (4.3), also satisfies

$$\mu^-(x,t,k) = \mathrm{e}^{-\mathrm{i}t/k}\left[-\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}kx}\int_0^\infty \mathrm{e}^{\mathrm{i}t/k}q(0,t)\,\mathrm{d}t + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')}q_{x'}(x',0)\,\mathrm{d}x'\right] + \frac{\mathrm{i}}{k}\int_0^t \mathrm{e}^{-(\mathrm{i}/k)(t-t')}q(x,t')\,\mathrm{d}t'.$$

Writing $\int_0^t$ as $\int_0^\infty - \int_t^\infty$, it follows that $\mu^-$ is analytic in the lower half complex $k$-plane iff

$$-\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}kx}\int_0^\infty \mathrm{e}^{\mathrm{i}t/k}q(0,t)\,\mathrm{d}t \\ +\int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')}q_{x'}(x',0)\,\mathrm{d}x' + \frac{\mathrm{i}}{k}\int_0^\infty \mathrm{e}^{\mathrm{i}t/k}q(x,t)\,\mathrm{d}t = 0, \quad k\in \mathrm{III}\cup\mathrm{IV}.$$

The evaluation of this equation at $x=0$ yields an identity, while its evaluation at $x=l$ yields equation (1.21).

*Proof of theorem 1.2.* It is straightforward to verify directly that if $q$ is defined by equations (1.19) and (1.20), then it solves the IBV problem defined by (1.17), (1.18). We first note that the definition of $q(x,t)$ (equation (1.19)) yields

$$q(x,0) = -\tfrac{1}{2}\rho(0) + \frac{1}{2\pi\mathrm{i}}\,⨍_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\frac{\rho(k)}{k}\,\mathrm{d}k, \tag{4.9}$$

where

$$⨍_{-\infty}^{\infty} = \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}.$$

Indeed, $q(x,t)$ can be written in the form

$$q(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\mathrm{e}^{-\mathrm{i}(kx+(t/k))}\left(\frac{\rho(k)-\rho(0)}{k}\right)\mathrm{d}k + \frac{1}{2\pi\mathrm{i}}\int_{|k|>1}\mathrm{e}^{-\mathrm{i}(kx+(t/k))}\frac{\rho(k)}{k}\,\mathrm{d}k \\ +\frac{1}{2\pi\mathrm{i}}\rho(0)\int_{-1}^{1}\mathrm{e}^{-\mathrm{i}(kx+(t/k))}(1-\mathrm{e}^{-\mathrm{i}t/k})\frac{\mathrm{d}k}{k} + \frac{1}{2\pi\mathrm{i}}\rho(0)\int_{-1}^{1}\mathrm{e}^{-\mathrm{i}t/k}\frac{\mathrm{d}k}{k}. \tag{4.10}$$

The first three integrals of the right-hand side of equation (4.10) are continuous at $t=0$, while the fourth integral equals $-\frac{1}{2}\rho(0)$ as $t\to 0$ and $t>0$. Thus,

$$q(x,0) = -\tfrac{1}{2}\rho(0) + \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\mathrm{e}^{-\mathrm{i}kx}\left(\frac{\rho(k)-\rho(0)}{k}\right)\mathrm{d}k + \frac{1}{2\pi\mathrm{i}}\int_{|k|>1}\mathrm{e}^{-\mathrm{i}kx}\frac{\rho(k)}{k}\,\mathrm{d}k.$$

Using the fact that

$$⨍_{-1}^{1}\mathrm{d}k/k = 0,$$

this equation can be written in the more convenient form of equation (4.9).

We now prove that if $q(x,0)$ is defined by equation (4.9), and if $\rho(k)$ is defined by equation (1.20), then $q(x,0) = q_1(x)$. The right-hand side of equation (4.9) equals

$$-\frac{1}{2}\rho(0) + \frac{1}{2\pi\mathrm{i}}\,⨍_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}}{k}\left(\frac{\mathrm{i}}{k}\int_0^\infty \mathrm{e}^{\mathrm{i}t/k}q_2(t)\,\mathrm{d}t\right)\mathrm{d}k \\ -\frac{1}{2\pi\mathrm{i}}\,⨍_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}}{k}\left(\int_0^l \mathrm{e}^{\mathrm{i}kx'}\dot{q}_1(x')\,\mathrm{d}x'\right)\mathrm{d}k.$$

Using

$$\rho(0) = -q_1(l), \quad \frac{\mathrm{i}}{k}\int_0^\infty \mathrm{e}^{\mathrm{i}t/k}q_2(t)\,\mathrm{d}t = -q_2(0) + \tilde{q}_2(k),$$

$$\int_0^l \mathrm{e}^{\mathrm{i}kx'}\dot{q}_1(x')\,\mathrm{d}x' = \mathrm{e}^{\mathrm{i}kx}q_1(l) - q_1(0) - \mathrm{i}k\int_0^l \mathrm{e}^{\mathrm{i}kx'}q_1(x')\,\mathrm{d}x',$$

where $\tilde{q}_2(k)$ is analytic and bounded in the lower half complex $k$-plane and of order $k$ as $k \to 0$, it follows that the right-hand side of equation (4.9) equals

$$\frac{1}{2}q_1(l) - \frac{q_2(0)}{2\pi\mathrm{i}}\fint_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\frac{\mathrm{d}k}{k} - \frac{q_1(l)}{2\pi\mathrm{i}}\fint_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(l-x)}\frac{\mathrm{d}k}{k} + \frac{q_1(0)}{2\pi\mathrm{i}}\fint_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\frac{\mathrm{d}k}{k} \\ +\frac{1}{2\pi}\fint_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\left(\int_0^{\infty}\mathrm{e}^{\mathrm{i}kx'}q_1(x')\,\mathrm{d}x'\right)\mathrm{d}k.$$

This yields $q_1(x)$, since $q_2(0) = q_1(0)$ and

$$\fint_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(l-x)}\,\mathrm{d}k/k = \mathrm{i}\pi.$$

We now prove that $q(0,t) = q_2(t)$. The definitions of $q(x,t)$ and of $\rho(k)$ imply

$$q(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}t/k}\left(\frac{1}{k^2}\int_0^{\infty}\mathrm{e}^{\mathrm{i}\tau/k}q_2(\tau)\,\mathrm{d}\tau\right)\mathrm{d}k \\ -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}t/k}\left(\frac{1}{k}\int_0^{l}\mathrm{e}^{\mathrm{i}kx'}\dot{q}_1(x')\,\mathrm{d}x\right)\mathrm{d}k.$$

Using the substitution $l = 1/k$, it follows that the integrand of the second integral is analytic and bounded in the lower half complex $l$-plane and it vanishes at $l = 0$. Thus, the second integral vanishes and the right-hand side of the above equation yields $q_2(t)$.

**Remark 4.1.** The definition of $q(x,t)$ implies

$$q(x,t) = \tfrac{1}{2}q_2(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(kx+(t/k))}}{k^2}\left(\int_0^t \mathrm{e}^{\mathrm{i}t'/k}q_2(t')\,\mathrm{d}t'\right)\mathrm{d}k \\ -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}(kx+(t/k))}}{k}\left(\int_0^x \mathrm{e}^{-\mathrm{i}kx'}q_1(x')\,\mathrm{d}x'\right)\mathrm{d}k. \qquad (4.11)$$

Indeed, using equation (1.20), it follows that $q(x,t)$, as defined by equation (1.19), differs from the right-hand side of equation (4.11) by $I_1 + I_2$, where

$$I_1 = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}t/k}}{k}\left(\int_x^l \mathrm{e}^{\mathrm{i}k(x'-x)}q_1(x')\,\mathrm{d}x'\right)\mathrm{d}k$$

and

$$I_2 = -\tfrac{1}{2}q_2(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}}{k^2}\left(\int_t^{\infty}\mathrm{e}^{\mathrm{i}(t'-t)/k}q_2(t')\,\mathrm{d}t'\right)\mathrm{d}k.$$

Using $l = 1/k$, $I_1$ becomes

$$I_1 = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}tl}}{l}\left(\int_x^{\infty}\mathrm{e}^{\mathrm{i}(x'-x)/l}q_1(x')\,\mathrm{d}x'\right)\mathrm{d}l,$$

which equals zero, since its integrand consists of a function which is bounded and analytic in the lower half complex $l$-plane and which vanishes at $l = 0$. Also, using integration by parts, the integral appearing in the right-hand side of $I_2$ becomes

$$\frac{\mathrm{i}q_2(0)}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}}{k}\,\mathrm{d}k + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}}{k}\left(\int_t^{\infty}\mathrm{e}^{\mathrm{i}(t'-t)/k}q_2(t')\,\mathrm{d}t'\right)\mathrm{d}k.$$

The first term above equals $\frac{1}{2}q_2(0)$ (since $x > 0$) and the second term vanishes. Thus, $I_2 = 0$.

The representation (4.11) shows that $q(x,t)$ depends only on the values of $q_1(x')$, $0 \leqslant x' \leqslant x$ and of $q_2(t')$ for $0 \leqslant t' \leqslant t$. This agrees with the classical result for this IBV problem obtained by a Picard iteration (Garabedian 1964).

## 5. The sine–Gordon in light-cone coordinates

The sine–Gordon equation (1.22) possesses the Lax pair (Ablowitz *et al.* 1973)

$$\mu(x,t,k)_x + \mathrm{i}k[\sigma_3, \mu(x,t,k)] = Q(x,t)\mu, \tag{5.1 a}$$

$$\mu(x,t,k)_t + \frac{\mathrm{i}}{4k}[\sigma_3, \mu(x,t,k)] = -\frac{\mathrm{i}}{4k}\tilde{Q}(x,t)\mu, \tag{5.1 b}$$

where

$$\sigma_3 = \mathrm{diag}(1,-1), \quad Q = \begin{pmatrix} 0 & -\frac{1}{2}q_x \\ \frac{1}{2}q_x & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} -1+\cos q & \sin q \\ \sin q & 1-\cos q \end{pmatrix}, \tag{5.2}$$

$\mu$ is a $2 \times 2$ matrix valued function, and $[,]$ denotes the usual commutator.

For the analysis of equations (5.1), it is useful to observe that the general solution of the equation

$$f(x)_x + \alpha[\sigma_3, f(x)] = G(x),$$

where $\alpha$ is a constant scalar and $G(x)$ is a $2 \times 2$ matrix, is given by

$$f(x) = \mathrm{e}^{-\alpha x\hat{\sigma}_3}A + \int_0^x \mathrm{e}^{-\alpha(x-x')\hat{\sigma}_3}G(x')\,\mathrm{d}x,$$

where $A$ is a constant $2 \times 2$ matrix and

$$\mathrm{e}^{x\hat{\sigma}_3}A \doteqdot \mathrm{e}^{x\sigma_3}A\mathrm{e}^{-x\sigma_3}. \tag{5.3}$$

In order to derive the formula which expresses $q(x,t)$ in terms of appropriate spectral data and the formula which expresses the spectral data in terms of the given initial and boundary data, we first *assume* that there exists a solution $q(x,t)$ which decays to zero as $x \to \infty$ and as $t \to \infty$. After we have obtained these formulae, we will discuss how it can be verified directly that this $q(x,t)$ satisfies equations (1.22) and (1.23) *without* the *a priori* assumption that $q(x,t)$ exists.

Let the $2 \times 2$ matrix value functions $\Phi$ and $\Psi$ be defined by

$$\begin{aligned}\Phi(x,t,k) = I_2 &+ \frac{\mathrm{i}}{4k}\mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_t^\infty \mathrm{e}^{(-\mathrm{i}/4k)(t-t')\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t',k)\,\mathrm{d}t' \\ &+ \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')\hat{\sigma}_3}(Q\Phi)(x',t,k)\,\mathrm{d}x', \quad I_2 = \mathrm{diag}(1,1),\end{aligned} \tag{5.4}$$

$$\begin{aligned}\Psi(x,t,k) = I_2 &- \frac{\mathrm{i}}{4k}\mathrm{e}^{-\mathrm{i}k(x-l)\hat{\sigma}_3}\int_0^t \mathrm{e}^{(-\mathrm{i}/4k)(t-t')\hat{\sigma}_3}(\tilde{Q}\Psi)(l,t',k)\,\mathrm{d}t' \\ &- \int_x^l \mathrm{e}^{-\mathrm{i}k(x-x')\hat{\sigma}_3}(Q\Psi)(x',t,k)\,\mathrm{d}x'.\end{aligned} \tag{5.5}$$

Using the above observation, it follows that $\Phi$ and $\Psi$ satisfy equations (5.1). Furthermore, using the fact that

$$\mathrm{e}^{x\hat{\sigma}_3}A = \begin{pmatrix} A_{11} & A_{12}\mathrm{e}^{2x} \\ A_{21}\mathrm{e}^{-2x} & A_{22}, \end{pmatrix}$$

where $A_{ij}$ denote the entries of the matrix $A$, it follows that

$$\Phi = (\Phi^+, \Phi^-), \quad \Psi = (\Psi^-, \Psi^+). \tag{5.6}$$

In these equations, $\Phi^+$ ($\Phi^-$) and $\Psi^-$ ($\Psi^+$) denote the first (second) column vectors of the matrices $\Phi$ and $\Psi$. In equations (5.6), the superscripts + and − denote analyticity and boundness in the upper and lower half complex $k$-plane, respectively.

The functions $\Phi$ and $\Psi$ are related by the equation

$$\Phi(x,t,k) = \Psi(x,t,k)\mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3-(\mathrm{i}/4k)t\hat{\sigma}_3}\rho(k), \quad k \in \mathbb{R}, \tag{5.7}$$

where the $2 \times 2$ matrix $\rho(k)$ is defined by

$$\rho(k) = I_2 + \frac{\mathrm{i}}{4k}\int_0^\infty \mathrm{e}^{(\mathrm{i}/4k)t\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t,k)\,\mathrm{d}t + \int_0^l \mathrm{e}^{\mathrm{i}kx\hat{\sigma}_3}(Q\Phi)(x,0,k)\,\mathrm{d}x. \tag{5.8}$$

Indeed, any two solutions of equations (5.1) are related by an equation of the form (5.7), where $\rho(k)$ is some $2 \times 2$ matrix depending only on $k$. Evaluating equation (5.7) at $t = 0$, $x = l$, equation (5.8) follows.

Evaluating equation (5.4) at $x = 0$, and at $t = 0$, one finds

$$\Phi(0,t,k) = I_2 + \frac{\mathrm{i}}{4k}\int_t^\infty \mathrm{e}^{-(\mathrm{i}/4k)(t-t')\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t',k)\,\mathrm{d}t', \tag{5.9}$$

$$\begin{aligned} \Phi(x,0,k) = I_2 + \frac{\mathrm{i}}{4k}\mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_0^\infty \mathrm{e}^{(\mathrm{i}/4k)t\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t,k)\,\mathrm{d}t \\ + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')\hat{\sigma}_3}(Q\Phi)(x',0,k)\,\mathrm{d}x'. \end{aligned} \tag{5.10}$$

Given $q(0,t)$, equation (5.9) yields $\Phi(0,t,k)$. Then, having $\Phi(0,t,k)$ and $q(x,0)$, equation (5.10) yields $\Phi(x,0,k)$. Using $\Phi(0,t,k)$ and $\Phi(x,0,k)$, equation (5.8) yields $\rho(k)$.

We note that $\Phi(0,t,k)$ is the unique solution of (5.1 *b*) with $q(x,t)$ replaced by $q(0,t)$, which satisfies

$$\lim_{t\to\infty} \exp\left[\frac{it}{4k}\hat{\sigma}_3\right]\Phi(0,t,k) = I_2.$$

Also, $\Phi(x,0,k)$ is the unique solution of (5.1 *a*) with $q(x,t)$ replaced by $q(x,0)$, which satisfies

$$\lim_{x\to 0} \exp[\mathrm{i}kx\hat{\sigma}_3]\Phi(x,0,k) = \Phi(0,0,k).$$

Furthermore,

$$\rho(k) = \lim_{x\to l} \exp[\mathrm{i}k\hat{\sigma}_3 x]\Phi(x,0,k).$$

The functions $\Phi$ and $\Psi$ exhibit the following asymptotic behaviour:

$$\mu = I_2 + \frac{\mu_1}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \quad \operatorname{Im} k \neq 0, \tag{5.11}$$

$$\mu = \mu_0 + O(k), \quad k \to 0, \tag{5.12}$$

where, if $(\mu_1)_{ij}$ denote the entries of the matrix $\mu_1$, then

$$q_x = -4\mathrm{i}(\mu_1)_{12} = 4\mathrm{i}(\mu_1)_{21}. \tag{5.13}$$

Both equations (5.1) are traceless. Thus, $\det \mu$ is constant. Then equations (5.4) and (5.5) yield

$$\det \Phi = \det \Psi = 1. \tag{5.14}$$

These equations and equation (5.7) imply that $\det \rho(k) = 1$, i.e.

$$\rho_{11}\rho_{22} - \rho_{12}\rho_{21} = 1. \tag{5.15}$$

Equations (5.6), (5.7), (5.11) and (5.12) define a Riemann–Hilbert problem. In order to write this Riemann–Hilbert problem in a canonical form, we note that equation (5.7) can be written as

$$(\Phi^+, \Phi^-) = (\Psi^-, \Psi^+)\begin{pmatrix} \rho_{11} & \rho_{12}\bar{E} \\ \rho_{21}E & \rho_{22} \end{pmatrix}, \quad E(x,t,k) \doteq \mathrm{e}^{2\mathrm{i}kx+(\mathrm{i}t/2k)}, \quad k \in \mathbb{R},$$

where bar denotes complex conjugation. Thus

$$\Phi^+ = \rho_{11}\Psi^- + \rho_{21}E\Psi^+, \tag{5.16 a}$$

$$\Phi^- = \rho_{22}\Psi^+ + \rho_{12}\bar{E}\Psi^-. \tag{5.16 b}$$

Equation (5.16 *b*) yields

$$\Psi^+ = \frac{\Phi^-}{\rho_{22}} - \frac{\rho_{12}\bar{E}\Psi^-}{\rho_{22}}. \tag{5.17 a}$$

Substituting equation (5.17 *a*) in (5.16 *a*), and using equation (5.15), equation (5.16 *a*) becomes

$$\Phi^+ = \frac{\Psi^-}{\rho_{22}} + \frac{\rho_{21}E\Phi^-}{\rho_{22}}. \tag{5.17 b}$$

Equations (5.17) can be rewritten in the matrix form

$$\left.\begin{aligned} \left(\Psi^+, \frac{\Phi^+}{\rho_{11}}\right) = \left(\frac{\Phi^-}{\rho_{22}}, \Psi^-\right)G, \quad G \doteq \begin{pmatrix} 1 & \dfrac{\rho_{21}E}{\rho_{11}} \\ -\dfrac{\rho_{12}}{\rho_{22}}\bar{E} & \dfrac{1}{\rho_{11}\rho_{22}} \end{pmatrix}, \\ E = \mathrm{e}^{2\mathrm{i}kx+(2\mathrm{i}t/k)}, \quad k \in \mathbb{R}. \end{aligned}\right\} \tag{5.18}$$

(Equation (5.17 *b*) has been divided by $\rho_{11}$ in order for the jump matrix above to have unit determinant.)

The matrices $Q(x,t)$ and $\tilde{Q}(x,t)$ satisfy certain symmetry conditions. Using these conditions, it can be shown that the functions $\Phi$ and $\Psi$ satisfy the following relations:

$$\left.\begin{aligned} \Phi_2^-(k) = \Phi_1^+(-k), \quad \Phi_1^-(k) = -\Phi_2^+(-k), \\ \Psi_2^+(k) = \Psi_1^-(-k), \quad \Psi_1^+(k) = -\Psi_2^-(-k), \end{aligned}\right\} \tag{5.19}$$

where, for convenience of notation, we have suppressed the $x$ and $t$ dependence. Furthermore, if $q \in \mathbb{R}$, in addition to (5.19), the following relations are also valid:

$$\Phi_2^-(k) = \overline{\Phi_1^+(\bar{k})}, \quad \Phi_1^-(k) = -\overline{\Phi_2(\bar{k})}, \quad \Psi_2^+(k) = \overline{\Psi_1^-(\bar{k})}, \quad \Psi_1^+(k) = -\overline{\Psi_2^-(\bar{k})}. \tag{5.20}$$

Equations (5.19) imply that

$$\rho_{22}(k) = \rho_{11}(-k), \quad \rho_{21}(k) = -\rho_{12}(-k). \tag{5.21}$$

Furthermore, if $q \in \mathbb{R}$, in addition to (5.21), the following relations are valid:

$$\rho_{22}(k) = \overline{\rho_{11}(\bar{k})}, \quad \rho_{21}(k) = \overline{-\rho_{12}(\bar{k})}. \tag{5.22}$$

For completeness, we note that there exists a nonlinear version of equation (1.21), namely

$$\begin{aligned}\mathrm{off}\bigg\{ -\frac{\mathrm{i}}{4k}\mathrm{e}^{-\mathrm{i}kl\hat{\sigma}_3}\int_0^\infty \mathrm{e}^{(\mathrm{i}/4k)t\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t,k)\,\mathrm{d}t + \int_0^l \mathrm{e}^{-\mathrm{i}k(x-l)\hat{\sigma}_3}(Q\Phi)(x,0,k)\,\mathrm{d}x \\ +\frac{\mathrm{i}}{4k}\int_0^\infty \mathrm{e}^{(\mathrm{i}/4k)t\hat{\sigma}_3}(\tilde{Q}\Phi)(l,t,k)\,\mathrm{d}t\bigg\} = 0, \quad \text{in } (\mathrm{I}\cup\mathrm{II}, \mathrm{III}\cup\mathrm{IV}),\end{aligned} \tag{5.23}$$

where off$\{A\}$ denotes the off-diagonal part of the matrix $A$. Indeed, using arguments similar to the ones used in the linear problem, it follows that

$$\begin{aligned}\mathrm{off}\bigg\{ -\frac{\mathrm{i}}{4k}\mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_0^\infty \mathrm{e}^{(\mathrm{i}/4k)t\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t,k)\,\mathrm{d}t + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')\hat{\sigma}_3}(Q\Phi)(x',0,k)\,\mathrm{d}x' \\ +\frac{\mathrm{i}}{4k}\int_0^\infty \mathrm{e}^{(\mathrm{i}/4k)t\hat{\sigma}_3}(\tilde{Q}\Phi)(x,t,k)\,\mathrm{d}t\bigg\} = 0, \quad \text{in } (\mathrm{I}\cup\mathrm{II}, \mathrm{III}\cup\mathrm{IV}).\end{aligned} \tag{5.24}$$

The evaluation of this equation at $x = 0$ yields an identity, while its evaluation at $x = l$ yields equation (5.23).

*Proof of theorem 1.3.*

(*i*) *Definition of* $\rho_1$ *and* $\rho_2$. We define $\Phi^{(1)}(t,k) = (\Phi_1(t,k), \Phi_2(t,k))^{\mathrm{T}}$, $\Psi^{(1)}(x,k) = (\Psi_1(x,k), \Psi_2(x,k))^{\mathrm{T}}$, $\rho_1(k)$ and $\rho_2(k)$ by equations (1.26)–(1.28)†. We first show that these definitions make sense. Let

$$\tilde{Q}_2(t) = \begin{pmatrix} -1+\cos q_2(t) & \sin q_2(t) \\ \sin q_2(t) & 1-\cos q_2(t) \end{pmatrix}, \quad Q_1(x) = \begin{pmatrix} 0 & -\frac{1}{2}\dot{q}_1(x) \\ \frac{1}{2}\dot{q}_1(x) & 0 \end{pmatrix}. \tag{5.25}$$

The vector $\Phi^{(1)}(t,k)$ satisfies

$$\Phi^{(1)}(t,k) = (1,0)^{\mathrm{T}} - \frac{\mathrm{i}}{4k}\int_t^\infty \mathrm{diag}(1, \mathrm{e}^{(\mathrm{i}/2k)(t-t')})\tilde{Q}_2(t')\,\Phi^{(1)}(t',k)\,\mathrm{d}t', \tag{5.26}$$

which is a Volterra integral equation for $\Phi^{(1)}(t,k)$, thus, if $q_2(t) \in S(\mathbb{R}^+)$, it has a unique solution. The vector $\Psi^{(1)}(x,k)$ satisfies

$$\begin{aligned}\Psi^{(1)}(x,k) = (1,0)^{\mathrm{T}} - \frac{\mathrm{i}}{4k}\mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_0^\infty \mathrm{diag}(1, \mathrm{e}^{-(\mathrm{i}/2k)t})\tilde{Q}_2(t)\,\Phi^{(1)}(t,k)\,\mathrm{d}t \\ + \int_0^x \mathrm{diag}(1, \mathrm{e}^{-2\mathrm{i}k(x-x')})Q_1(x')\,\Psi^{(1)}(x',k)\,\mathrm{d}x',\end{aligned} \tag{5.27}$$

which is also a Volterra integral equation for $\Psi^{(1)}(x,k)$, thus, if $q_1(x) \in C^1[0,l]$, it

† These definitions are motivated by equations (5.9) and (5.10), where $\Phi_{11}(0,t,k) = \Phi_1(t,k)$, $\Phi_{21}(0,t,k) = \Phi_2(t,k)$, $\Phi_{11}(x,0,k) = \Psi_1(x,k)$, $\Phi_{21}(x,0,k) = \Psi_2(x,k)$, $\rho_1 = \rho_{12}$ and $\rho_2 = \rho_{22}$.

has a unique solution. The definition of $\rho_2(k)$ (equations (1.28)) implies that $\rho_2(k)$ is analytic in the lower half complex $k$-plane and, also (since $q_1(x)$ and $q_2(t)$ are real),

$$\overline{\rho_2(\bar{k})} = \rho_2(-k). \tag{5.28}$$

(*ii*) *Definitions of* $M_1$, $M_2$. We define $\rho_{11}$, $\rho_{12}$, $\rho_{21}$, $\rho_{22}$ by

$$\rho_{12} = \rho_1, \quad \rho_{22} = \rho_2, \quad \rho_{11}(k) = \overline{\rho_{22}(\bar{k})}, \quad \rho_{21}(k) = -\overline{\rho_{12}(\bar{k})}. \tag{5.29}$$

We define the vectors $\Psi^+ = (\Psi_1^+, \Psi_2^+)^{\mathrm{T}}$, $\Psi^- = (\Psi_1^-, \Psi_2^-)^{\mathrm{T}}$, $\Phi^+ = (\Phi_1^+, \Phi_2^+)^{\mathrm{T}}$, $\Phi^- = (\Phi_1^-, \Phi_2^-)^{\mathrm{T}}$ as the solution of the matrix Riemann–Hilbert (RH) problem satisfying the jump condition (5.18) together with the boundary condition

$$(\Phi^+, \Phi^-) = I_2 + O\left(\frac{1}{k}\right), \quad (\Psi^-, \Psi^+) = I_2 + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad \operatorname{Im} k \neq 0. \tag{5.30}$$

The investigation of this type of RH problem can be found in Fokas & Gel'fand (1994) and Fokas & Its (1992, 1994, 1996). Here we will only give a brief description of the main steps needed for the relevant analysis. There exist three main steps. (i) Obtaining a system of linear integral equations for the case that $\rho_{11}$ and $\rho_{22}$ have no zeros in $\mathbb{C}^+$ and in $\mathbb{C}^-$, respectively. In this case, taking the plus projection on the first column of equation (5.18), we find

$$\Psi^+(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2\mathrm{i}\pi} \int_{-\infty}^{\infty} \overline{E(k')} \frac{\rho_{12}(k')}{\rho_{22}(k')} \Psi^-(k') \frac{\mathrm{d}k'}{k' - (k + \mathrm{i}0)},$$

where, for simplicty of notation, we have suppressed the $x$ and $t$ dependence. The symmetry relations (5.29) imply

$$\Psi_2^- = -\overline{\Psi_1^+} \quad \text{and} \quad \Psi_1^- = \overline{\Psi_2^+}.$$

Renaming $\Psi_1^+$ and $\Psi_2^+$ as $M_1$ and $M_2$, the above equation becomes equation (1.29). (ii) Establishing the solvability of equation (1.29). It turns out that equation (1.29) can be solved uniquely without having to assume that $\rho_{12}$ and $\rho_{21}$ are small. This is a consequence of the symmetry relations (5.29). Indeed, using these relations, it is possible to prove that the homogeneous version of the RH problem (5.18) and (5.30) has only the trivial solution (Fokas & Its 1992, 1994, 1996). (iii) Investigating the case that $\rho_{11}$ and $\rho_{22}$ have zeros in $\mathbb{C}^+$ and $\mathbb{C}^-$. In this case, the solution of the RH problem (5.18) and (5.30) can be reduced to solving a regular RH problem (i.e. a RH problem of the type (i) above) and a system of algebraic equations. Indeed, assume that $\rho_{11}(\lambda_j) = 0$, $j = 1, \ldots, N$, $\operatorname{Im} \lambda_j > 0$. Define $k_j$ by

$$k_1 = \lambda_1, \quad k_2 = -\bar{\lambda}_1, \ldots, \quad k_{2N-1} = \lambda_N, \quad k_{2N} = -\bar{\lambda}_N, \quad \rho_{11}(\lambda_j) = 0, \quad \operatorname{Im} \lambda_j > 0. \tag{5.31}$$

Solve the RH problem

$$Z^+(x,t,k) = Z^-(x,t,k)\tilde{G}(x,t,k), \quad k \in \mathbb{R}, \tag{5.32 a}$$

$$Z = I_2 + O\left(\frac{1}{k}\right), \quad k \to \infty, \tag{5.32 b}$$

where $Z$ is a $2 \times 2$ matrix and $\tilde{G}$ is defined by equation (5.18), but with $\rho_{11}$ and $\rho_{22}$ replaced by

$$\tilde{\rho}_{11} = \rho_{11} \prod_{j=1}^{2N} \left(\frac{k - \bar{k}_j}{k - k_j}\right), \quad \tilde{\rho}_{22} = \rho_{22} \prod_{j=1}^{2N} \left(\frac{k - k_j}{k - \bar{k}_j}\right). \tag{5.33}$$

Let $B_1(x,t),\ldots,B_{2N}(x,t)$, be the $2\times 2$ matrices which are determined recursively by solving the following algebraic equations:

$$(k_jI_2+B_j)Z^+_{j-1}(x,t,k_j)\begin{pmatrix}-d_j\\1\end{pmatrix}=0,\quad \left(\bar{k}_jI_2+B_j\right)Z^-_{j-1}(x,t,\bar{k}_j)\begin{pmatrix}1\\\bar{d}_j\end{pmatrix}=0, \tag{5.34}$$

where $Z^+_j$ and $Z^-_j$ satisfy

$$Z_j(x,t,k)=(kI_2+B_j)Z_{j-1}(x,t,k),\quad j=1,\ldots,2N-1,\quad Z_0=Z, \tag{5.35}$$

and $d_j$ are determined by

$$d_j(x,t)=\frac{\rho_{21}(k_j)}{\dot{\rho}_{11}(k_j)}\frac{\prod_{l=1,l\neq j}^{N}(k_j-k_l)}{\prod_{l=1}^{N}(k_j-\bar{k}_l)}\exp\left(2\mathrm{i}k_jx+\frac{\mathrm{i}t}{2k_j}\right),\quad j=1,\ldots,2N. \tag{5.36}$$

Then the solution of the RH problem (5.18) and (5.30) is given by

$$\left(\Psi^+,\frac{\Phi^+}{\rho_{11}}\right)=PZ^+D,\quad \left(\frac{\Phi^-}{\rho_{22}},\Psi^-\right)=PZ^-D, \tag{5.37}$$

where

$$\begin{aligned}P(x,t,k)&\doteqdot(kI_2+B_{2N})(kI_2+B_{2N-1})\cdots(kI+B_1),\\ D(k)&=\operatorname{diag}\left(\frac{1}{\prod_{j=1}^{2N}(k-\bar{k}_j)},\frac{1}{\prod_{j=1}^{2N}(k-k_j)}\right).\end{aligned} \tag{5.38}$$

We now discuss briefly how this result can be obtained. The symmetry relations (5.28) and (5.29) imply that if $\lambda$ is a zero of $\rho_{11}$ then $-\bar{\lambda}$ is also a zero of $\rho_{11}$ and $\bar{\lambda}$, $-\lambda$ are zeros of $\rho_{22}$. The definitions (5.33) and (5.38) imply that if $Z$ solves the RH problem (5.32), then the left-hand sides of equations (5.37) satisfy equations (5.18) and (5.30). The final step is to show that the requirement that the left and the right-hand side of equations (5.37) have the same residues implies equations (5.34)–(5.36). To achieve this, we first note that $G$ can be factorized as

$$G=\begin{pmatrix}1&0\\-\dfrac{\rho_{12}}{\rho_{22}}\bar{E}&1\end{pmatrix}\begin{pmatrix}1&\dfrac{\rho_{21}}{\rho_{11}}E\\0&1\end{pmatrix}.$$

Thus, equation (5.18) becomes

$$\left(\Psi^+,\frac{\Phi^+}{\rho_{11}}\right)\begin{pmatrix}1&-\dfrac{\rho_{21}}{\rho_{11}}E\\0&1\end{pmatrix}=\left(\frac{\Phi^-}{\rho_{22}},\Psi^-\right)\begin{pmatrix}1&0\\-\dfrac{\rho_{12}}{\rho_{22}}\bar{E}&1\end{pmatrix}. \tag{5.39}$$

Assume that $\lambda$, $\operatorname{Im}\lambda>0$, is a simple zero of $\rho_{11}$ and $\hat{\lambda}$, $\operatorname{Im}\hat{\lambda}<0$, is a simple zero of $\rho_{22}$. The function $\Psi$ is entire in $\mathbb{C}/\{0\}$ and $\rho_{21}$ admits analytic continuation into the finite upper half plane. This implies that the left-hand side of equation (5.39) has no singularity at $k=\lambda$, thus

$$\operatorname{res}_\lambda\left(\frac{\Phi^+}{\rho_{11}}\right)=cE(\lambda)\,\Psi^+|_{k=\lambda},\quad c(\lambda)\doteqdot\frac{\rho_{21}(\lambda)}{\dot{\rho}_{11}(\lambda)}.$$

This equation can be written in the form

$$\lim_{k\to\lambda}\left(\Psi^+(\lambda), \frac{\Phi^+(k)(k-\lambda)}{\rho_{11}(k)}\right)\begin{pmatrix} -c(\lambda)E(\lambda) \\ 1 \end{pmatrix} = 0. \tag{5.40}$$

Similarly,

$$\lim_{k\to\hat\lambda}\left(\frac{\Phi^-(k)(k-\hat\lambda)}{\rho_{22}(k)}, \Psi^-(\hat\lambda)\right)\begin{pmatrix} 1 \\ -\hat c(\hat\lambda)\overline{E(\hat\lambda)} \end{pmatrix} = 0, \quad \hat c(\hat\lambda) \doteqdot \frac{\rho_{12}(\hat\lambda)}{\dot\rho_{22}(\hat\lambda)}. \tag{5.41}$$

Letting $\lambda = k_j$, $\hat\lambda = \bar k_j$, $\hat c = -\bar c$, multiplying equations (5.37) from the right by $D^{-1}$ and using equations (5.41), equations (5.34) follow.

In the above discussion, we have assumed that the zeros of $\rho_{11}$ are simple, finite and do not occur in the real axis. The case of infinite zeros can be considered as the limit $N \to \infty$.

Having defined $\rho_1$, $\rho_2$, $M_1$, $M_2$, $q_x$ is defined by equation (1.25). Then the equation $\sin q = -q_{xt}$ implies $q(x,t)$.

(*iii*) *q solves* (*1.21*), (*1.22*). We first show that if $M$ and $Q$ are defined by

$$M = \left(\Psi^-, \frac{\Phi^-}{\rho_{22}}\right), \quad Q = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\sigma_3, Mg]\,\mathrm{d}k, \tag{5.42}$$

where the $2\times 2$ matrix $g$ has components

$$g_{11} = -\frac{\rho_{12}\rho_{21}}{\rho_{11}\rho_{22}}, \quad g_{12} = -\frac{\rho_{12}}{\rho_{22}}\bar E, \quad g_{21} = \frac{\rho_{21}}{\rho_{11}}E, \quad g_{22} = 0,$$

then $M$ and $Q$ satisfy the $x$-part of the Lax pair, i.e. equation (5.1 *a*). Indeed, equation (5.18) can be written as

$$\left(\frac{\Phi^+}{\rho_{11}} \quad \Psi^+\right) - M = Mg.$$

The minus projection of this equation implies

$$M = I_2 + \frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{\infty} Mg\frac{\mathrm{d}k'}{k'-k}, \quad k \in \mathbb{C}^-. \tag{5.43}$$

The expression of $Q$ is a consequence of the definition of $q_x$. Using the fact that

$$g_x = -\mathrm{i}k[\sigma_3, g], \tag{5.44}$$

equation (5.43) implies that the quantity

$$(\partial_x + \mathrm{i}k\hat\sigma_3 - Q)M$$

satisfies the homogeneous verison of equation (5.43). Thus, since equation (5.43) has a unique solution, the above quantity equals zero.

We now show that if $M$ solves equation (5.43), and if $\tilde Q(x,t)$ is defined by

$$\tilde Q(x,t) = -(\hat\sigma_3 M_0)M_0^{-1}, \quad M_0(x,t) = M(x,t,0), \tag{5.45}$$

then $M$ and $\tilde Q$ solve the $t$-part of the Lax pair, i.e. equation (5.1 *b*). Indeed, using the fact that

$$g_t = -\frac{\mathrm{i}}{k}[\sigma_3, g], \tag{5.46}$$

equation (5.43) implies that the quantity

$$\left(\partial_t + \frac{\mathrm{i}}{4k}\hat{\sigma}_3 + \frac{\mathrm{i}}{4k}\tilde{Q}\right) M$$

solves the homogeneous version of equation (5.43), provided that $\tilde{Q}$ satisfies

$$\tilde{Q} + \frac{\tilde{Q}}{2\mathrm{i}\pi}\int_{-\infty}^{\infty} Mg\frac{\mathrm{d}k}{k} + \frac{1}{2\mathrm{i}\pi}\int_{-\infty}^{\infty}[\sigma_3, Mg]\frac{\mathrm{d}k}{k} = 0.$$

Evaluating equation (5.43) at $k = 0$, and then applying $\hat{\sigma}_3$ on the resulting equation, we find that $\tilde{Q}$ is given by equation (5.45).

The asymptotic analysis of equation (5.43) in the neighbourhood of $k = 0$ implies that

$$M_0 = \begin{pmatrix} \cos\frac{1}{2}q & -\sin\frac{1}{2}q \\ \sin\frac{1}{2}q & \cos\frac{1}{2}q \end{pmatrix}. \tag{5.47}$$

Then equation (5.45) yields that $\tilde{Q}$ is given by the expression appearing in (5.2).

The proof that $q(x,t)$ satisfies the given initial and boundary conditions uses arguments similar to the ones used for the linearized version of the sine–Gordon equation. ■

**Remark 5.1.** Equation (5.4) implies that the large $k$ behaviour of $\Phi$ is given by

$$\Phi = I_2 + \frac{1}{k}\begin{pmatrix} a(x,t) & -\dfrac{q_x(x,t)}{4\mathrm{i}} + b(x,t)\mathrm{e}^{-2\mathrm{i}kx} \\ -\dfrac{q_x(x,t)}{4\mathrm{i}} + b(x,t)\mathrm{e}^{2\mathrm{i}kx} & -a(x,t) \end{pmatrix} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \tag{5.48}$$

where

$$a(x,t) = \frac{1}{4}\mathrm{i}\int_t^{\infty}(-1 + \cos q_2(t'))\,\mathrm{d}t' + \frac{1}{8\mathrm{i}}\int_0^x q_x^2(x',t)\,\mathrm{d}x'$$
$$b(x,t) = \frac{1}{4}\mathrm{i}\int_t^{\infty}\sin q_2(t')\,\mathrm{d}t' - \tfrac{1}{4}\mathrm{i}q_x(0,t).$$

This result is consistent with the large $k$ behaviour of $\Phi$ obtained from equation (5.43). Also, equation (5.4) implies that the small $k$ behaviour of $\Phi$ is given by the right-hand side of equation (5.47), which is consistent with the small $k$ behaviour of $\Phi$ obtained from equation (5.43).

**Remark 5.2.** We briefly discuss a particular example of constant data,

$$q_1(x) = q_1, \quad q_2(t) = q_1 \quad \text{if } t \in [0,T], \quad q_2(t) = 0 \quad \text{if } t \in (T,\infty), \tag{5.49}$$

where $q_1$ is a constant. In this case, the spectral data can be computed explicitly

$$\rho_{11}(k) = \tfrac{1}{2}(1 - \cos q_1\mathrm{e}^{-\mathrm{i}T/(2k)}) + \tfrac{1}{2}(1 + \cos q_1) \tag{5.50}$$

$$\rho_{21}(k) = -\tfrac{1}{2}\sin q_1\mathrm{e}^{-\mathrm{i}T/(2k)} + \tfrac{1}{2}\sin q_1. \tag{5.51}$$

The expression for $\rho_{11}$ shows that solitons exist (i.e. $\rho_{11}$ has zeros in the upper half complex $k$-plane) iff $\cos q_1 < 0$. It is interesting that if solitons exist, then there exist infinitely many of them, since the roots of $\rho_{11}(k) = 0$ are given by

$$k_j = \tfrac{1}{2}T\frac{(2j-1)\pi - \mathrm{i}\ln\sigma}{[(2j-1)\pi]^2 + (\ln\sigma)^2}, \quad \sigma = \frac{1 + \cos q_1}{1 - \cos q_1}, \quad j \in N. \tag{5.52}$$

## 6. The NLS equation

We will use the example of equation (1.1) to show how the deformation of the Lax pair of a linear equation can be used to obtain the Lax pair of some nonlinear equation. The left-hand side of the Lax pair (1.9) is given by

$$\phi_x + \mathrm{i}k\phi, \quad \phi_t + \mathrm{i}k^2\phi.$$

The complex conjugation of these expressions yields

$$\bar{\phi}_x - \mathrm{i}k\bar{\phi}, \quad \bar{\phi}_t - \mathrm{i}k^2\bar{\phi}.$$

Rewriting the above in a matrix form, one finds

$$\mu_x + \mathrm{i}k\sigma_3\mu, \quad \mu_t + \mathrm{i}k^2\sigma_3\mu, \quad \sigma_3 = \mathrm{diag}(1,-1),$$

where $\mu(x,t,k)$ is a $2\times 2$ matrix valued function. This suggests the introduction of the dressing operators (Fokas & Zakharov 1992)

$$D_x\mu = \mu_x + \mathrm{i}k\mu\sigma_3, \quad D_t\mu = \mu_t + \alpha \mathrm{i}k^2\mu\sigma_3, \quad \alpha \text{ const.} \tag{6.1}$$

Starting from these operators and using $\mu = I + O(1/k)$, $k \to \infty$, it is straightforward to obtain the usual Lax pair of the NLS equation (Zakharov & Shabat 1972), namely

$$\mu_x + \mathrm{i}k[\sigma_3, \mu] = Q\mu, \quad Q \doteqdot \begin{pmatrix} 0 & q \\ \lambda\bar{q} & 0 \end{pmatrix}, \quad \lambda = \pm 1, \tag{6.2 a}$$

$$\mu_t + 2\mathrm{i}k^2[\sigma_3, \mu] = \tilde{Q}\mu, \quad \tilde{Q} \doteqdot 2kQ - \mathrm{i}Q_x\sigma_3 - \lambda\mathrm{i}|q|^2\sigma_3. \tag{6.2 b}$$

Similarly, the matrix form of the left-hand side of the Lax pair (1.10) yields

$$\mu_t + \mathrm{i}k^2 I_2\mu, \quad \mu_x + \begin{pmatrix} 0 & k^2 \\ -1 & 0 \end{pmatrix}\mu, \quad I_2 = \mathrm{diag}(1,1),$$

where $\mu(x,t,k)$ is a $2\times 2$ matrix valued function. The complex conjugation of these expressions yields

$$\bar{\mu}_t - \mathrm{i}k^2 I_2\bar{\mu}, \quad \bar{\mu}_x + \begin{pmatrix} 0 & k^2 \\ -1 & 0 \end{pmatrix}\bar{\mu}.$$

Rewriting the above in a matrix form suggests the introduction of the dressing operators

$$D_t\mu = \mu_t + \mu\begin{pmatrix} -\mathrm{i}k^2 I_2 & 0 \\ 0 & \mathrm{i}k^2 I_2 \end{pmatrix}$$

and

$$D_x\mu = \mu_x + \mu\begin{pmatrix} C(k) & 0 \\ 0 & C(k) \end{pmatrix}, \quad C(k) = \begin{pmatrix} 0 & \alpha k^2 \\ \beta & 0 \end{pmatrix}, \quad \alpha, \beta \text{ const.},$$

where $\mu(x,t,k)$ is a $4\times 4$ matrix valued function. Using these operators together with

$$\mu = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} + O\left(\frac{1}{k^2}\right),$$

it is possible to obtain a new Lax pair for the nonlinear Schrödinger equation.

**Proposition 6.1.** *The nonlinear Schrödinger equation*

$$\mathrm{i}q_t + q_{xx} - 2\lambda|q|^2 q = 0, \quad \lambda = \pm 1, \tag{6.3}$$

*possesses the Lax pair*

$$\mu_t = \left[\begin{pmatrix} -\mathrm{i}k^2 I_2 & 0 \\ 0 & \mathrm{i}k^2 I_2 \end{pmatrix}, \mu\right] + \begin{pmatrix} 0 & A \\ \lambda\bar{A} & 0 \end{pmatrix}\mu, \quad I_2 = \mathrm{diag}(1,1), \tag{6.4 a}$$

$$\mu_x = \left[\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \mu\right] - \tfrac{1}{2}\mathrm{i}\begin{pmatrix} B & C \\ -\lambda\bar{C} & B \end{pmatrix}\mu, \quad J \doteq \begin{pmatrix} 0 & -k^2 \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{6.4 b}$$

*where $\mu(x,t,k)$ is a $4\times 4$ matrix-valued function, $[\,,]$ denotes the usual commutation, the $2\times 2$ matrices $A(x,t), B(x,t), C(x,t)$ are defined by*

$$A = \begin{pmatrix} a & b \\ q & -a \end{pmatrix}, \quad B = \begin{pmatrix} f & g \\ 0 & -f \end{pmatrix}, \quad C = \begin{pmatrix} q & -2a \\ 0 & -q \end{pmatrix}, \tag{6.5}$$

*the scalar function $f(x,t)$ satisfies the compatible system*

$$f_t = \lambda(\bar{q}q_x - q\bar{q}_x), \quad f_x = -\mathrm{i}\lambda|q|^2, \tag{6.6}$$

*and the scalar functions $a(x,t)$, $b(x,t)$, $g(x,t)$ are explicitly given in terms of $q(x,t)$ and $f(x,t)$ by*

$$a = q_x - \mathrm{i}fq, \quad g = \mathrm{i}\lambda|q|^2 + \mathrm{i}f^2, \quad b = \mathrm{i}q_t - \lambda|q|^2 q + qf^2 + 2\mathrm{i}fq_x. \tag{6.7}$$

We now discuss briefly the IBV problem of the nonlinear Schrödinger equation in the quarter plane. This IBV problem, as well as similar problems for the Korteweg–de Vries and for the sine–Gordon equation in laboratory coordinates, has been investigated in Fokas & Its (1992, 1994, 1996). In these papers, it was realized that, in addition to the spectral analysis of the $x$-part of the Lax pair, one also needs to perform a spectral analysis of the $t$-part. This novel idea led to the determination of the explicit time dependence of the spectral data. It turns out that the *simultaneous* analysis of the two parts of the Lax pair simplifies substantially the derivation of the relevant results. Furthermore, it can be used to obtain an inverse problem for the 'missing' spectral data. Indeed, in analogy with equations (3.2)–(3.4), one defines the $2\times 2$ matrix-valued functions $\Phi(x,t,k)$, $M(x,t,k)$ and $\Psi(x,t,k)$ by

$$\Phi(x,t,k) = I_2 + \mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}\int_0^t \mathrm{e}^{-2\mathrm{i}k^2(t-t')\hat{\sigma}_3}(\tilde{Q}\Phi)(0,t',k)\,\mathrm{d}t' + \int_0^x \mathrm{e}^{-\mathrm{i}k(x-x')\hat{\sigma}_3}(Q\Phi)(x',t,k)\,\mathrm{d}x', \tag{6.8}$$

$$\Psi(x,t,k) = I_2 - \int_x^\infty \mathrm{e}^{-\mathrm{i}k(x-x')\hat{\sigma}_3}(Q\Psi)(x',t,k)\,\mathrm{d}x' \tag{6.9}$$

and $M(x,t,k)$ satisfies an equation similar to $\Phi(x,t,k)$, but with the integral $\int_0^t$ replaced by $-\int_t^\infty$. The column vectors of these functions are analytic and bounded in the domains indicated by their superscripts

$$\Phi = (\Phi^{(1)}, \Phi^{(4)}), \quad M = (M^{(2)}, M^{(3)}), \quad \Psi = (\Psi^{(34)}, \Psi^{(12)}). \tag{6.10}$$

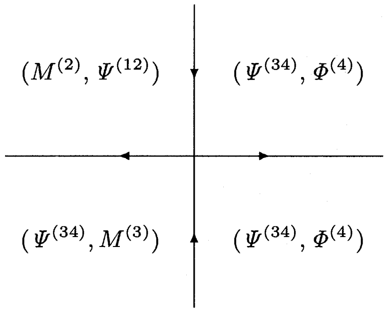


Figure 4. A Riemann–Hilbert problem for the nonlinear Schrödinger in the quarter plane.

Using these functions, it is straightforward to obtain a RH problem for the vectors indicated in figure 4.

The jump matrices for this Riemann–Hilbert problem involve the $\exp[-\mathrm{i}kx\hat{\sigma}_3 - 2\mathrm{i}k^2t\hat{\sigma}_3]$ and a certain matrix-valued function $\rho(k)$. In order to determine $\rho(k)$, one must analyse the analogue of equation (3.12). This equation is given by

$$\mathrm{off}\left\{-\int_0^\infty \mathrm{e}^{-\mathrm{i}kx\hat{\sigma}_3}(Q\,\Psi)(x,0,-k)\,\mathrm{d}x+\int_0^\infty \mathrm{e}^{2\mathrm{i}k^2t\hat{\sigma}_3}(\tilde{Q}\,\Psi)(0,t,-k)\,\mathrm{d}t\right\} = 0 \quad \text{in (II, III)}. \tag{6.11}$$

This equation, in contrast to the linear case, involves the unknown $\Psi(0,t,-k)$; also, $\rho(k)$ involves $\Psi(0,t,k)$. This makes it necessary to formulate an additional inverse problem for the determination of $\Psi(0,t,k)$. This problem is based on equation (6.11) and on the equation

$$\Psi^{-1}(0,t,-k)\,\Psi(0,t,k) = I_2 + 4k\int_0^t \mathrm{e}^{-2\mathrm{i}k^2(t-t')\hat{\sigma}_3}\,\Psi^{-1}(0,t',-k)Q(0,t)\,\Psi(0,t',k)\,\mathrm{d}t'.$$

Details of the rigorous investigation of this problem, as well as the investigation of the NLS for $x \in [0,l]$, will be given elsewhere.

## 7. Discussion

We first discuss linear equations. One first *assumes* that $q(x,y)$ exists and uses the underlying Lax pair to construct an integral representation of $q(x,y)$ in terms of appropriate spectral data $\rho(k)$. This involves the following steps.

(1) Define a function $\mu(x,y,k)$, which solves both parts of the Lax pair, and which is sectionally holomorphic in the complex $k$-plane. This function has different representations in different sectors of the complex $k$-plane.

(2) Establish the relationship between these different representations of $\mu(x,y,k)$. This relationship is uniquely defined in terms of certain spectral data $\rho(k)$, which are explicitly given in terms of the boundary values of $q(x,y)$. Use the fact that $\mu(x,y,k)$ possesses two different representations to establish all possible relations between the boundary values. For a given boundary value problem, use the above relations to express $\rho(k)$ in terms of the given boundary data.

(3) The above relationship between the different representations of $\mu(x,y,k)$, together with the large $k$ behaviour of $\mu(x,y,k)$, define a Riemann–Hilbert problem for $\mu(x,y,k)$ in terms of $\rho(k)$. Solve this Riemann–Hilbert to find $\mu(x,y,k)$ and then $q(x,y)$ in terms of $\rho(k)$.

The integral representation constructed in this way defines $q(x,y)$ *explicitly* in terms of the spectral data $\rho(k)$, which are *explicitly* defined in terms of the given initial and boundary data. One then proves directly that $q(x,y)$ solves the given IBV problem, *without* the *a priori* assumption that $q(x,y)$ exists, by using appropriate contour integrations to show that $q(x,y)$ satisfies the given initial and boundary data. The functional space for the given data is chosen in such a way that $\rho(k)$, $q(x,y)$ and its derivatives are well defined.

It should be noted that if the underlying symbol is $n$-valued then there exist $n$-Lax pairs. Suppose that a linear equation possesses the $n$-Lax pairs

$$\mu_x + \mathrm{i}k\mu = M_l(\partial_x, \partial_y, k)q \tag{7.1 a}$$

$$\mu_y + w_l(k)\mu = \tilde{M}_l(\partial_x, \partial_y, k)q, \quad l = 1, 2, \ldots, n, \tag{7.1 b}$$

where $M_l$ and $\tilde{M}_l$ are linear operators of $\partial_x$ and $\partial_y$ with $k$-dependent coefficients. The spectral analysis of these equations will yield a representation of $q(x,y)$, which involves both the continuous and the discrete spectrum. The continuous spectrum gives rise to

$$\sum_{1}^{n} \int_{L_l} \mathrm{e}^{-\mathrm{i}kx - w_l(k)y} \rho_l(k)\,\mathrm{d}k, \tag{7.2}$$

where, in general, $L_l$ consists of the union of the real axis and of the curves defined by $\mathrm{Re}\, w_l(k) = 0$†. The discrete spectrum will give rise to a discrete version of equation (7.2). Thus,

$$q(x,t) = \sum_{1}^{N} \int_{L_l} \mathrm{e}^{-\mathrm{i}kx - w_l(k)y} \rho_l(k)\,\mathrm{d}k + \sum_{1}^{N} \sum_{j=1}^{N_l} \mathrm{e}^{-\mathrm{i}k_j^{(l)}x - w_l(k_j^{(l)})y} C_j^{(l)}. \tag{7.3}$$

We now discuss nonlinear integrable equations of class (i). One first *assumes* that $q(x,y)$ exists and one uses the underlying Lax pair to construct $q(x,y)$ in terms of appropriate spectral data. This involves the following steps.

(1) Define a matrix-valued function $\mu(x,y,k)$ which solves both parts of the Lax pair and which is sectionally holomorphic in the complex $k$-plane. This function has different representation in different sectors of the complex $k$-plane.

(2) Establish the relationship between these different representations of $\mu(x,y,k)$. This relationship is expressed in terms of a matrix $S(k)$ which involves the spectral data. These spectral data are uniquely defined in terms of the boundary values of $q(x,y)$. Use the fact that $\mu(x,y,k)$ possess two different representations to establish all possible relations between the boundary values. Use these relations to express the spectral data in terms of the given initial and boundary data. This relationship is *not* explicit, but involves the solution of certain linear integral equations.

(3) The above relation between the different representations of $\mu(x,y,k)$, together with the large $k$ behaviour of $\mu(x,y,k)$, define a matrix Riemann–Hilbert problem for $\mu(x,y,k)$ in terms of $S(k)$. The solution of this Riemann–Hilbert problem yields $\mu(x,y,k)$ in terms of $S(k)$. The matrix $\mu(x,y,k)$ *cannot* be given explicitly in terms of $S(k)$, but involves the solution of a linear integral equation. The function $q(x,y)$ can be expressed in terms of $\mu(x,y,k)$ and of $S(k)$.

The above construction can be used to justify $q(x,y)$ *without* the *a priori* assumption that $q(x,y)$ exists. Starting from the given initial and boundary data, define $S(k)$

† Depending on the particular IBV problem, $\rho_l(k)$ may be zero on part of $L_l$. For example, if $x \in [0,\infty)$, then $\rho_l(k) \neq 0$ only for $\mathrm{Im}\, k \leqslant 0$.

through the solution of the linear integral equations mentioned in (2) above. These equations are usually of the Volterra type. Having obtained $S(k)$, define $\mu(x,y,k)$ as the solution of the linear integral equation mentioned in (3) above. One can usually prove solvability of this integral equation by using the fact that the associated matrix Riemann–Hilbert problem possesses certain symmetries. Define $q(x,y)$ in terms of $\mu(x,y,k)$ and $S(k)$ and prove directly that this $q(x,y)$ solves the given nonlinear equation and satisfies the given initial and boundary conditions.

For IBV of class (ii), some of the spectral data cannot be expressed directly in terms of the given data but are characterized through the solution of a certain inverse problem. In this case, one must also establish the solvability of the inverse problem defining these missing spectral data. This is actually the most difficult step since this inverse problem appears to be nonlinear.

**Remark 7.1.** Regarding the analysis of the sine–Gordon equation, it is interesting to note that if the initial and boundary data are constant, $q_1(x) = q_1$, $q_2(t) = q_1 H(T-t)$ ($H$ is the Heaviside function), $q_1$ real constant, then the spectral data can be computed explicitly. The analysis of these explicit formulae shows that if $\cos q_1 > 0$ there exist no solitons, while if $\cos q_1 < 0$ there exist infinitely many solitons. This example suggests that if solitons exist, then, in the generic case, there exist infinitely many of them.

An analogue of the d'Alembert formula for the solution of the sine–Gordon equation is given in Krichever (1981).

**Remark 7.2.** We note that the relationship between the Lax pairs of linear and nonlinear equations elucidated in this paper suggests a promising method for investigating new nonlinear integrable equations. The Lax pair of every physical linear equation in two dimensions should be constructed and proper deformations of these Lax pairs should be investigated. We conjecture that some of the nonlinear integrable equations constructed in this way will be of physical significance.

The solution of the heat equation, of the linearized Korteweg–de Vries equation and of the Laplace equation in the quarter plane, as well as the solution of the linearized NLS equation in the finite $x$-domain, are discussed in Fokas (1996). Certain IBV problems for the stimulated Raman scattering and for the Ernst equation are discussed in Fokas & Menyuk (1996) and Fokas *et al.* (1996).

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