Reading notes

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This document contains all notes taken while reading materials (e.g., textbooks, literature) in preparation for the capstone project. Black text are information consolidated from the readings; blue text are notes (proofs, explanations); red text are questions.

Contents

Todo list

1 Theory of Ordinary Differential Equations (Coddington Levinson)

1.1 Chapter 11: Algebraic properties of linear boundary-value problems on a finite interval

1.1.1 Introduction

Definition 1.1. Let L be the **linear differential operator of oder n ($n \geq 1$)** defined by

$$Lx = p_0 x^{(n)} + p_1 x^{(n-1)} + \cdots + p_{n-1} x' + p_n x$$

where the p_k are complex-valued functions of class C^{n-k} on a closed bounded interval $[a, b]$ (i.e., derivatives $p_k, p_k', \ldots, p_k^{(n-k)}$ exist on $[a, b]$ and are continuous) and $p_0(t) \neq 0$ on $[a, b]$.

Definition 1.2. Homogeneous boundary conditions refer to a set of equations/constraints of the type

$$\sum_{k=1}^{n} (M_{jk} x^{(k-1)}(a) + N_{jk} x^{(k-1)}(b)) = 0 \quad (j = 1, \ldots, m) \tag{1.1.1}$$

where M_{jk}, N_{jk} are complex constants.

Definition 1.3. A **homogeneous boundary-value problem** concerns finding the solutions of

$$Lx = 0$$

on $[a, b]$ which satisfy some homogeneous boundary conditions defined above.

Definition 1.4. For any homogeneous boundary value problem, an **adjoint problem** refers to the problem of finding the solutions of

$$L^+ x := (-1)^n (\bar{p}_0 x)^{(n)} + (-1)^{n-1} (\bar{p}_1 x)^{(n-1)} + \cdots + \bar{p}_n x = 0$$

on $[a, b]$ which satisfy some homogeneous boundary conditions "complementary" to the conditions associated with the solutions of $Lx = 0$.

Theorem 1.5. *(Green's formula) For $u, v \in C^n$ on $[a, b]$, Think about why we are asking for so much smoothness as C^n. Can you get away with any less? Consider looking up functions of bounded variation or absolutely continuous functions. We need the $(n-1)$st derivative of u, v to exist for the form $[uv]$ to be defined. We need the nth derivatives to exist for Lu, $L^+ v$ to be defined. Note that $u, v \in C^{n-1}$ ensures that the $(n-1)$st derivatives exist and are continuous, but not that the nth derivatives exist.*

Do we need the nth derivatives to be continuous though?

By Proposition 1.22, f u, v satisfy the corresponding boundary conditions, the equation below would be zero.

$$\int_{t_1}^{t_2} (Lu) \bar{v} \, dt - \int_{t_1}^{t_2} u (\overline{L^+ v}) \, dt = [uv](t_2) - [uv](t_1) \tag{1.1.2}$$

where $a \leq t_1 < t_2 \leq b$ and $[uv](t)$ is the form in $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$ given by

$$[uv](t) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(t) (p_{n-m} \bar{v})^{(j)}(t) \tag{1.1.3}$$

Remark 1.6. Alternatively, $[uv](t)$ can be written as

$$\begin{aligned}[uv](t) &= \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(t)(p_{n-m}\bar{v})^{(j)}(t) \\ &= \sum_{j,k=1}^{n} B_{jk}(t)u^{(k-1)}(t)\bar{v}^{(j-1)}(t)\end{aligned} \tag{1.1.4}$$

for some B_{jk} to be found later.

Note that

$$\begin{aligned}[uv](t) &= \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(t)(p_{n-m}\bar{v})^{(j)}(t) \\ &= \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j u^{(k)}(t) \left(\sum_{l=0}^{j} \binom{j}{l} p_{n-m}^{(j-l)}(t)\bar{v}^{(l)}(t) \right) \\ &= \sum_{m=1}^{n} \sum_{k=0}^{m-1} (-1)^{m-1-k} u^{(k)}(t) \left(\sum_{l=0}^{m-1-k} \binom{m-1-k}{l} p_{n-m}^{(m-1-k-l)}(t)\bar{v}^{(l)}(t) \right) \\ &= \sum_{m=1}^{n} \sum_{k=1}^{m} (-1)^{m-k} \left(\sum_{l=0}^{m-k} \binom{m-k}{l} p_{n-m}^{(m-k-l)}(t)\bar{v}^{(l)}(t) \right) u^{(k-1)}(t) \quad \text{(shifting } k \text{ to } k+1) \\ &= \sum_{k=1}^{n} \sum_{m=k}^{n} (-1)^{m-k} \left(\sum_{l=0}^{m-k} \binom{m-k}{l} p_{n-m}^{(m-k-l)}(t)\bar{v}^{(l)}(t) \right) u^{(k-1)}(t).\end{aligned}$$

To find B_{jk}, we need to pull out the coefficients of $u^{(k-1)}\bar{v}^{(j-1)}$. We first note that, fixing m and k, when $l = j-1$, the coefficient of $\bar{v}^{(j-1)}$ is

$$\binom{m-k}{j-1} p_{n-m}^{(m-k-j+1)}(t).$$

To find the coefficient of $u^{(k-1)}\bar{v}^{(j-1)}$, we need to fix k and collect the above coefficient across all values of m. Since m goes up to n, $m-k$ goes up to $n-k$. Since $l \le m-k$, $l = j-1$ implies $j-1 \le m-k$. Thus, $m-k$ ranges from $j-1$ to $n-k$.

Let $l' := m-k$, then $m = k + l'$, and the above equation becomes

$$\begin{aligned}[uv](t) &= \sum_{k=1}^{n} \sum_{m=k}^{n} (-1)^{m-k} \left(\sum_{l=0}^{m-k} \binom{m-k}{l} p_{n-m}^{(m-k-l)}(t)\bar{v}^{(l)}(t) \right) u^{(k-1)}(t) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} (-1)^{l'} \left(\sum_{l'=j-1}^{n-k} \binom{l'}{j-1} p_{n-(k+l')}^{(l'-(j-1))} \right) \bar{v}^{(j-1)}(t)u^{(k-1)}(t) \\ &= \sum_{j,k=1}^{n} \left(\sum_{l=j-1}^{n-k} \binom{l}{j-1} p_{n-k-l}^{(l-j+1)}(-1)^l \right) u^{(k-1)}(t)\bar{v}^{(j-1)}(t) \quad \text{(replace } l' \text{ by } l).\end{aligned}$$

Thus,

$$B_{jk} = \sum_{l=j-1}^{n-k} \binom{l}{j-1} p_{n-k-l}^{(l-j+1)} (-1)^l.$$

Define $B(t)$ to be the $n \times n$ matrix whose entry is B_{jk}:

$$B(t) = \begin{bmatrix} B_{11} & B_{12} & \cdots & \cdots & p_0(t) \\ \vdots & \vdots & \cdots & -p_0(t) & 0 \\ (-1)^{n-1} p_0(t) & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{1.1.5}$$

Since $B(t)$ is square with $\det B(t) = (p_0(t))^n$ where $p_0(t) \neq 0$ on $[a, b]$ (as in the definition of L), $B(t)$ is nonsingular/invertible for $t \in [a, b]$.

Definition 1.7. For vectors $f = (f_1, \ldots, f_k)$, $g = (g_1, \ldots, g_k)$, define the product

$$f \cdot g := \sum_{i=1}^{k} f_i \bar{g}_i.$$

Note that $f \cdot g = g^* f$.

Definition 1.8. A **semibilinear form** is a complex-valued function $\mathcal{S}$ defined for pairs of vectors $f = (f_1, \ldots, f_k)$, $g = (g_1, \ldots, g_k)$ satisfying

$$\mathcal{S}(\alpha f + \beta g, h) = \alpha \mathcal{S}(f, h) + \beta \mathcal{S}(g, h)$$
$$\mathcal{S}(f, \alpha g + \beta h) = \bar{\alpha} \mathcal{S}(f, g) + \bar{\beta} \mathcal{S}(f, h)$$

for any complex numbers α, β and vectors f, g, h.

Note that $\mathcal{S}$ is linear in the first argument but not the second one. If $\mathcal{S}$ were bilinear, it would be linear in each argument. Indeed, because it is semilinear in the second argument, it is also called "sesquilinear" (Latin for one and a half is sesquus).

Remark 1.9. If

$$S = \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \cdots & s_{kk} \end{bmatrix},$$

then $Sf \cdot g$ is a semibilinear form

$$\begin{aligned}
\mathcal{S}(f,g) &:= Sf \cdot g \\
&= \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \cdots & s_{kk} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^k s_{1j} f_j \\ \vdots \\ \sum_{j=1}^k s_{kj} f_j \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} s_{ij} f_j \right) \bar{g}_i \\
&= \sum_{i,j=1}^{k} s_{ij} f_i \bar{g}_i .
\end{aligned} \tag{1.1.6}$$

Indeed:

$$\begin{aligned}
\mathcal{S}(\alpha f + \beta g, h) &= \sum_{i,j=1}^{k} s_{ij} (\alpha f_j + \beta g_j) \bar{h}_i \\
&= \alpha \sum_{i,j=1}^{k} s_{ij} f_j \bar{h}_i + \beta \sum_{i,j=1}^{k} g_j \bar{h}_i \\
&= \alpha S f \cdot h + \beta S g \cdot h \\
&= \alpha \mathcal{S}(f,h) + \beta \mathcal{S}(g,h).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{S}(f, \alpha g + \beta h) &= \sum_{i,j=1}^{k} s_{ij} f_j (\alpha g_i \bar{+} \beta h_i) \\
&= \bar{\alpha} \sum_{i,j=1}^{k} s_{ij} f_j \bar{g}_i + \bar{\beta} \sum_{i,j=1}^{k} f_j \bar{h}_i \\
&= \bar{\alpha} S f \cdot g + \bar{\beta} S f \cdot h \\
&= \bar{\alpha} \mathcal{S}(f,g) + \bar{\beta} \mathcal{S}(f,h).
\end{aligned}$$

Remark 1.10. Under a similar matrix framework, we see that $[uv](t)$ is a semibilinear form with matrix $B(t)$:

Let $\vec{u} = (u, u', \ldots, u^{(n-1)})$ and $\vec{v} = (v, v', \ldots, v^{(n-1)})$. Then we have

$$\begin{aligned}
[uv](t) &= \sum_{j,k=1}^{n} B_{jk}(t) u^{(k-1)}(t)\bar{v}^{(j-1)}(t) \quad \text{(by (1.1.4))} \\
&= \sum_{i,j=1}^{n} (B_{ij} u^{(j-1)} \overline{v}^{(i-1)})(t) \qquad (1.1.7) \\
&= (B\vec{u} \cdot \vec{v})(t) \\
&= \mathcal{S}(\vec{u}, \vec{v})(t).
\end{aligned}$$

With this notation, we can rewrite the right hand side of Green's formula as a semibilinear form below:

$$
\begin{aligned}
[uv](t_2) - [uv](t_1) &= \sum_{j,k=1}^{n} B_{jk}(t_2)u^{(k-1)}(t_2)\bar{v}^{(j-1)}(t_2) - \sum_{j,k=1}^{n} B_{jk}(t_2)u^{(k-1)}(t_1)\bar{v}^{(j-1)}(t_1) \\
&= B(t_2)\vec{u}(t_2)\cdot\vec{v}(t_2) - B(t_1)\vec{u}(t_1)\cdot\vec{v}(t_1) \\
&= \begin{bmatrix} B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots \\ B_{n1}(t_2) & \dots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} - \\
&\quad \begin{bmatrix} B_{11}(t_1) & \cdots & B_{1n}(t_1) \\ \vdots & & \vdots \\ B_{n1}(t_1) & \dots & B_{nn}(t_1) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \end{bmatrix} \\
&= \begin{bmatrix} -B_{11}(t_1) & \cdots & -B_{1n}(t_1) \\ \vdots & & \vdots \\ -B_{n1}(t_1) & \dots & -B_{nn}(t_1) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \end{bmatrix} + \\
&\quad \begin{bmatrix} B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots \\ B_{n1}(t_2) & \dots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&= \begin{bmatrix} -B_{11}(t_1) & \cdots & -B_{1n}(t_1) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -B_{n1}(t_1) & \cdots & -B_{nn}(t_1) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & B_{n1}(t_2) & \cdots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \\ u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \\ \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&= \begin{bmatrix} -B(t_1) & 0_n \\ 0_n & B(t_2) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \\ u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \\ \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&=: \hat{B} \begin{bmatrix} \vec{u}(t_1) \\ \vec{u}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \vec{v}(t_1) \\ \vec{v}(t_2) \end{bmatrix}.
\end{aligned}
\tag{1.1.8}
$$

Since $\det \hat{B} = (-1)^n \det B(t_1) \det B(t_2)$ (note that $\det(\lambda A) = \lambda^n \det(A)$ for $n \times n$ matrix A. In this case, $\det(-B(t_1)) = (-1)^n \det(B(t_1))$ since $B(t_1)$ is $n \times n$), $\hat{B}$ is nonsingular for $t_1, t_2 \in [a, b]$ (since $B(t)$ is nonsingular for $t \in [a, b]$, as shown before).

1.1.2 Boundary form formula

Definition 1.11. Given any set of $2mn$ complex constants M_{ij}, N_{ij} $(i = 1, \ldots, m; j = 1, \ldots, n)$, define m **boundary operators (boundary forms)** $U_1, \ldots, U_m$ for functions x on $[a, b]$, for which $x^{(j)}$ $(j = 1, \ldots, n-1)$ exists at a and b, by

$$U_i x = \sum_{j=1}^{n} (M_{ij} x^{(j-1)}(a) + N_{ij} x^{(j-1)}(b)) \quad (i = 1, \ldots, m) \tag{1.1.9}$$

U_i are **linearly independent** if the only set of complex constants $c_1, \ldots, c_m$ for which

$$\sum_{i=1}^{m} c_i U_i x = 0$$

for all $x \in C^{n-1}$ on $[a, b]$ is $c_1 = c_2 = \cdots = c_m = 0$.

Remark 1.12. Note that for $\alpha, \beta \in \mathbb{C}$ and $x_1, x_2 \in C^{n-1}$ on $[a, b]$,

$$\begin{aligned} U_i(\alpha x_1 + \beta x_2) &= \sum_{j=1}^{n} (M_{ij}(\alpha x_1 + \beta x_2)^{(j-1)}(a) + N_{ij}(\alpha x_1 + \beta x_2)^{(j-1)}(b)) \\ &= \alpha \sum_{j=1}^{n} (M_{ij} x_1^{(j-1)}(a) + N_{ij} x_1^{(j-1)}(b)) + \\ &\quad \beta \sum_{j=1}^{n} (M_{ij} x_2^{(j-1)}(a) + N_{ij} x_2^{(j-1)}(b)) \quad \text{(by linearity of derivatives)} \\ &= \alpha U_i x_1 + \beta U_i x_2. \end{aligned}$$

So U_i are linear operators.

Remark 1.13. To describe (1.1.9) with matrices, define

$$\xi := \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix}; \quad U := \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}; \quad M := \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix}; \quad N := \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix}.$$

Then (1.1.9) can be written as

$$Ux = M\xi(a) + N\xi(b).$$

Indeed:

$$\begin{aligned} M\xi(a) + N\xi(b) &= \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} x(a) \\ x'(a) \\ \vdots \\ x^{(n-1)}(a) \end{bmatrix} + \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} x(b) \\ x'(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{n} M_{1j} x^{(j-1)}(a) \\ \vdots \\ \sum_{j=1}^{n} M_{mj} x^{(j-1)}(a) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n} N_{1j} x^{(j-1)}(b) \\ \vdots \\ \sum_{j=1}^{n} N_{mj} x^{(j-1)}(b) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \sum_{j=1}^{n}(M_{1j}x^{(j-1)}(a) + N_{1j}x^{(j-1)}(b)) \\ \vdots \\ \sum_{j=1}^{n}(M_{mj}x^{(j-1)}(a) + N_{mj}x^{(j-1)}(b)) \end{bmatrix}$$

$$= \begin{bmatrix} U_1 x \\ \vdots \\ U_m x \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix} x = Ux.$$

Define the $m \times 2n$ matrix

$$(M : N) := \begin{bmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{bmatrix}.$$

Then $U_1, \ldots, U_m$ are linearly independent if and only if $\operatorname{rank}(M : N) = m$, or equivalently, $\operatorname{rank}(U) = m$. Recall that the rank of a matrix is the largest number of linearly independent rows or columns in it. For a matrix $A_{m\times n}$, $\operatorname{rank}(A) \le \min\{m, n\}$ and $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Ux can be written as

$$Ux = \begin{bmatrix} \sum_{j=1}^{n}(M_{1j}x^{(j-1)}(a) + N_{1j}x^{(j-1)}(b)) \\ \vdots \\ \sum_{j=1}^{n}(M_{mj}x^{(j-1)}(a) + N_{mj}x^{(j-1)}(b)) \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} x(a) \\ \vdots \\ x^{(n-1)}(a) \\ x(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix}$$

$$= (M : N) \begin{bmatrix} \xi(a) \\ \xi(b) \end{bmatrix}.$$

Definition 1.14. If $U = (U_1, \ldots, U_m)$ is any boundary form with $\operatorname{rank}(U) = m$ and $U_c = (U_{m+1}, \ldots, U_{2n})$ any form with $\operatorname{rank}(U_c) = 2n - m$ such that $(U_1, \ldots, U_{2n})$ has rank $2n$, then U and U_c are **complementary boundary forms**. "Adjoining" $U_{m+1}, \ldots, U_{2n}$ to $U_1, \ldots, U_m$ is equivalent to imbedding the matrix $(M : N)$ in a $2n \times 2n$ nonsingular matrix (recall that for square matrices, nonsingular $\iff$ full rank).

We wish to describe the right hand side of Green's formula (1.1.2) as a linear combination of a boundary form U and a complementary form U_c. To do so, we consider the following results about the semibilinear form (1.1.6).

Definition 1.15. For a matrix $A = (a_{ij})$, its **adjoint** is defined as the conjugate transpose $A^* = (\bar{a}_{ij})$.

Proposition 1.16. *In the context of the semibilinear form* (1.1.6)*, we have*

$$Sf \cdot g = f \cdot S^* g. \tag{1.1.10}$$

Proof.

$$\begin{aligned}
Sf \cdot g &= \sum_{i,j=1}^{k} s_{ij} f_j \bar{g}_i \quad \text{(by (1.1.6))}; \\
f \cdot S^* g &= \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} \bar{s}_{11} & \cdots & \bar{s}_{k1} \\ \vdots & & \vdots \\ \bar{s}_{1k} & \cdots & \bar{s}_{kk} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\
&= \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} \sum_{j=1}^{k} \bar{s}_{j1} g_j \\ \vdots \\ \sum_{j=1}^{k} \bar{s}_{jk} g_j \end{bmatrix} \\
&= \sum_{i=1}^{k} f_i \cdot \overline{\left(\sum_{j=1}^{k} \bar{s}_{ji} g_j \right)} \\
&= \sum_{i=1}^{k} f_i \cdot \left(\sum_{j=1}^{k} s_{ji} \bar{g}_j \right) \\
&= \sum_{i,j=1}^{k} s_{ji} f_i \bar{g}_j = Sf \cdot g.
\end{aligned}$$

□

Proposition 1.17. *Let $\mathcal{S}$ be the semibilinear form associated with a nonsingular matrix S. Suppose $\bar{f} := Ff$ where F is a nonsingular matrix. Then there exists a unique nonsingular matrix G such that if $\bar{g} = Gg$, then $\mathcal{S}(f, g) = \bar{f} \cdot \bar{g}$ for all f, g.*

Proof. Let $G := (SF^{-1})^*$, then

$$\begin{aligned}
\mathcal{S}(f, g) &= Sf \cdot g \\
&= S(F^{-1}F)f \cdot g \\
&= SF^{-1}(Ff) \cdot g \\
&= SF^{-1}\bar{f} \cdot g \\
&= \bar{f} \cdot (SF^{-1})^* g \quad \text{(by (1.1.10))} \\
&= \bar{f} \cdot G * g \\
&= \bar{f} \cdot \bar{g}.
\end{aligned}$$

To see that G is nonsingular, note that $\det G = \det((\overline{SF^{-1}})^T) = \det(\overline{SF^{-1}}) = \overline{\det(SF^{-1})} = \overline{\det(S)\det(F)^{-1}} \neq 0$ since S, F are nonsingular. □

Proposition 1.18. *Suppose $\mathcal{S}$ is associated with the unit matrix E, i.e., $\mathcal{S}(f,g) = f \cdot g$. Let F be a nonsingular matrix such that the first j $(1 \leq j < k)$ components of $\bar{f} = Ff$ are the same as those of f. Then the unique nonsingular matrix G such that $\bar{g} = Gg$ and $\bar{f} \cdot \bar{g} = f \cdot g$ (as in Proposition 1.17) is such that the last $k-j$ components of $\bar{g}$ are linear combinations of the last $k-j$ components of g with nonsingular coefficient matrix.*

Proof. We note that for the condition on F to hold, F must have the form

$$\begin{bmatrix} E_j & 0_+ \\ F_+ & F_{k-j} \end{bmatrix}_{k\times k}$$

where E_j is the $j \times j$ identity matrix, 0_+ is the $j \times (k-j)$ zero matrix, F_+ is a $(k-j) \times j$ matrix, and F_{k-j} a $(k-j) \times (k-j)$ matrix. Let G be the unique nonsingular matrix in Proposition 1.17. Write G as

$$\begin{bmatrix} G_j & G_- \\ G_= & G_{k-j} \end{bmatrix}_{k\times k}$$

where $G_j, G_-, G_=, G_{k-j}$ are $j \times j, j \times (k-j), (k-j) \times j, (k-j) \times (k-j)$ matrices, respectively. By the definition of G,

$$f \cdot g = Ff \cdot Gg = \bar{f} \cdot Gg = G^* \bar{f} \cdot g = G^* Ff \cdot g,$$

(where the third equality follows from a reverse application of (1.1.10) with $\bar{f}$ as f, G^* as S) which implies

$$G^* F = E_k.$$

Since

$$\begin{aligned} G^* F &= \begin{bmatrix} G_j^* & G_=^* \\ G_-^* & G_{k-j}^* \end{bmatrix} \begin{bmatrix} E_j & 0_+ \\ F_+ & F_{k-j} \end{bmatrix} \\ &= \begin{bmatrix} G_j^* + G_=^* F_+ & G_=^* F_{k-j} \\ G_-^* + G_{k-j}^* F_+ & G_{k-j}^* F_{k-j} \end{bmatrix} \\ &= \begin{bmatrix} E_j & 0_{j\times(k-j)} \\ 0_{(k-j)\times j} & E_{k-j} \end{bmatrix}. \end{aligned}$$

Thus, $G_=^* F_{k-j} = 0_+$, the $j \times (k-j)$ zero matrix. But $\det F = \det(E_j) \cdot \det(F_{k-j}) \neq 0$, so $\det F_{k-j} \neq 0$ and we must have $G_=^* = 0_+$, i.e., $G_= = 0_{(k-j)\times j}$. Thus, G is upper-triangular, and so $\det G = \det G_j \cdot \det G_{k-j} \neq 0$, which implies $\det G_{k-j} \neq 0$ and G_{k-j} is nonsingular. Hence,

$$\bar{g} = Gg = \begin{bmatrix} G_j & G_- \\ 0_{(k-j)\times j} & G_{k-j} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}$$

where G_{k-j} is the nonsingular coefficient matrix such that

$$\begin{bmatrix} \bar{g}_{j-1} \\ \vdots \\ \bar{g}_k \end{bmatrix} = G_{k-j} \begin{bmatrix} g_{j-1} \\ \vdots \\ g_k \end{bmatrix}.$$

$\square$

Theorem 1.19. *(Boundary-form formula) Given any boundary form U of rank m (Definition 1.11), and any complementary form U_c (Definition 1.14), there exist unique boundary forms U_c^+, U^+ of rank m and $2n-m$, respectively, such that*

$$[xy](b) - [xy](a) = Ux \cdot U_c^+ y + U_c x \cdot U^+ y. \tag{1.1.11}$$

If $\tilde{U}_c$ is any other complementary form to U, and $\tilde{U}_c^+, \tilde{U}^+$ the corresponding forms of rank m and $2n-m$, then

$$\tilde{U}^+ y = C^* U^+ y \tag{1.1.12}$$

for some nonsingular matrix C.

Remark 1.20. This mean that, given a boundary form, its adjoint boundary forms are related to each other by linear transformation.

Yes. It is a kind of uniqueness result. It says that the adjoint boundary forms are unique *up to linear transformation*. This can be understood further as saying that you don't need to find the "right" adjoint boundary form; any adjoint boundary form is good enough because they are all, from the point of view of their action, the same object. In practice, we will usually aim to have our boundary forms in (some sense of) reduced row-echelon form for convenience and comparability. Theorem 1.19 says that such a form is only canonical because it is something we have decided upon, not because it is mathematically better.

Proof. Recall from (1.1.8) that the left hand side of (1.1.11) can be considered as a semibilinear form $\mathcal{S}(f,g) = \hat{B}f \cdot g$ for vectors

$$f = \begin{bmatrix} x(a) \\ \vdots \\ x^{(n-1)}(a) \\ x(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix}, \; g = \begin{bmatrix} y(a) \\ \vdots \\ y^{(n-1)}(a) \\ y(b) \\ \vdots \\ y^{(n-1)}(b) \end{bmatrix}$$

with the nonsingular matrix

$$\hat{B} = \begin{bmatrix} -B(a) & 0_n \\ 0_n & B(b) \end{bmatrix}.$$

Recall from Remark 1.13 that

$$Ux = M\xi(a) + N\xi(b) = (M : N)\begin{bmatrix} \xi(a) \\ \xi(b) \end{bmatrix}$$

for M, N, ξ are as defined there. With the definition of f, we have $f = \begin{bmatrix} \xi(a) \\ \xi(b) \end{bmatrix}$ and thus

$$Ux = (M : N)f.$$

By Definition 1.14, $U_c x = (\tilde{M} : \tilde{N})f$ for two appropriate matrices $\tilde{M}, \tilde{N}$ for which

$$H = \begin{bmatrix} M & N \\ \tilde{M} & \tilde{N} \end{bmatrix}_{2n \times 2n}$$

has rank $2n$. Thus,

$$\begin{bmatrix} Ux \\ U_c x \end{bmatrix} = \begin{bmatrix} (M:N)f \\ (\tilde{M}:\tilde{N})f \end{bmatrix} = \begin{bmatrix} M & N \\ \tilde{M} & \tilde{N} \end{bmatrix} f = Hf.$$

By Proposition 1.17, there exists a unique $2n \times 2n$ nonsingular matrix J such that $\mathcal{S}(f,g) = Hf \cdot Jg$. Let U^+, U_c^+ be such that

$$Jg = \begin{bmatrix} U_c^+ y \\ U^+ y \end{bmatrix},$$

then (1.1.11) holds since

$$[xy](b) - [xy](a) = \mathcal{S}(f,g) = Hf \cdot Jg = \begin{bmatrix} Ux \\ U_c x \end{bmatrix} \cdot \begin{bmatrix} U_c^+ y \\ U^+ y \end{bmatrix} = Ux \cdot U_c^+ y + U_c x \cdot U^+ y.$$

The second statement in the theorem follows from Proposition 1.18 with Hf and Jg corresponding to f and g.

Proposition 1.18 poses condition on F and invokes the existence of G; what are the objects corresponding to F and G here?

Not sure. It seems to me that unicity of J is only once H has been chosen, so a different H (ie different complementary boundary form) will give you a different J, hence different adjoint boundary form and different complementary adjoint boundary form. So the task is to understand how the chain of changes new complementary boundary form $\to$ new H $\to$ new J $\to$ new adjoint boundary forms provides this linearity result. I guess Proposition 1.18 must help with this in some way. □

1.1.3 Homogeneous Boundary-value Problems and Adjoint Problems

Definition 1.21. For any boundary form U of rank m there is associated the **homogeneous boundary condition**

$$Ux = 0 \tag{1.1.13}$$

for functions $x \in C^{n-1}$ on $[a,b]$. If U^+ is any boundary form of rank $2n-m$ determined as in Theorem 1.19, then the homogeneous boundary condition

$$U^+ x = 0 \tag{1.1.14}$$

is an **adjoint boundary condition** to 1.1.13.

Proposition 1.22. *By Green's formula* (1.1.2) *and the boundary-form formula* (1.1.11)*, for* $(u,v) := \int_a^b u\bar{v}\,dt$*,*

$$(Lu, v) = (u, L^+ v)$$

for all $u \in C^n$ *on* $[a,b]$ *satisfying* (1.1.13) *and all* $v \in C^n$ *on* $[a,b]$ *satisfying* (1.1.14).

Proof.

$$\begin{aligned} (Lu,v) - (u, L^+v) &= \int_a^b Lu\bar{v}\,dt - \int_a^b u(\overline{L^+v})\,dt \\ &= [uv](a) - [uv](b) \quad \text{(by Green's formula (1.1.2))} \\ &= Uu \cdot U_c^+ v + U_c u \cdot U^+ v \quad \text{(by boundary-form formula (1.1.11))} \\ &= 0 \cdot U_c^+ v + U_c u \cdot 0 \quad \text{(by (1.1.13) and (1.1.14))} \\ &= 0. \end{aligned} \tag{1.1.15}$$

$\square$

Remark 1.23. Let D, D^+ be the set of functions $u \in C^n$ satisfying (1.1.13) and (1.1.14), respectively. Then Theorem 1.19 shows that D^+ is uniquely determined by U, although U^+ is not. Note that U^+ is not uniquely determined by U because adjoint boundary forms are only unique up to linear transformation. Yet D^+ is uniquely determined by U because given U, its different adjoint boundary forms U^+ related to each other by linear transformations constitute the same condition on D^+: Since $\tilde{U}^+ = C^*U^+$ for some linear map C, $\tilde{U}^+x = 0 \iff U^+x = 0$.

Just like how U is associated with two $m \times n$ matrices M, N (Remark 1.13), U^+ is associated with two $n \times (2n-m)$ matrices P, Q such that $(P^* : Q^*)$ has rank $2n - m$ and

$$U^+x = P^*\xi(a) + Q^*\xi(b) \tag{1.1.16}$$

Note that imbedding M, N, P^*, Q^* in the same matrix gives

$$\begin{bmatrix} (M : N)_{m\times 2n} \\ (P^* : Q^*)_{(2n-m)\times 2n} \end{bmatrix}_{2n\times 2n} = \begin{bmatrix} M & N \\ P^* & Q^* \end{bmatrix}$$

is an $2n \times 2n$ matrix of full rank.

We want to characterize the adjoint condition (1.1.14) in terms of the matrices M, N, P, Q.

Theorem 1.24. *The boundary condition $U^+x = 0$ is adjoint to $Ux = 0$ if and only if*

$$MB^{-1}(a)P = NB^{-1}(b)Q \tag{1.1.17}$$

where $B(t)$ is the $n \times n$ matrix associated with the form $[xy](t)$ ((1.1.5)).

Proof. Let $\eta := (y, y', \dots, y^{(n-1)})$, then $[xy](t) = B(t)\xi(t) \cdot \eta(t)$ by (1.1.7).

Suppose $U^+x = 0$ is adjoint to $Ux = 0$. By definition of adjoint boundary condition 1.1.14, U^+ is determined as in Theorem 1.19. But by Theorem 1.19, in determining U^+, there exist boundary forms U_c, U_c^+ of rank $2n-m$ and m, respectively, such that 1.1.11 holds.

Put

$$\begin{aligned} U_c x &= M_c\xi(a) + N_c\xi(b) \quad && \text{rank}(M_c : N_c) = 2n - m \\ U_c^+ y &= P_c^*\eta(a) + Q_c^*\eta(b) \quad && \text{rank}(P_c^* : Q_c^*) = m. \end{aligned}$$

Then by the boundary-form formula (1.1.11),

$$\begin{aligned} B(b)\xi(b)\cdot\eta(b) - B(a)\xi(a)\cdot\eta(a) = {} & (M\xi(a) + N\xi(b)) \cdot (P_c^*\eta(a) + Q_c^*\eta(b)) + \\ & (M_c\xi(a) + N_c\xi(b)) \cdot (P^*\eta(a) + Q^*\eta(b)). \end{aligned}$$

By 1.1.10,

$$M\xi(a) \cdot P_c^*\eta(a) = P_c M\xi(a) \cdot \eta(a).$$

Thus,

$$\begin{aligned} B(b)\xi(b)\cdot\eta(b) - B(a)\xi(a)\cdot\eta(a) = {} & (P_cM + PM_c)\xi(a)\cdot\eta(a) + (Q_cM + QM_c)\xi(a)\cdot\eta(b) \\ & (P_cN + PN_c)\xi(b)\cdot\eta(a) + (Q_cN + QN_c)\xi(b)\cdot\eta(b). \end{aligned}$$

Thus, we have

$$P_cM + PM_c = -B(a) \qquad\qquad P_cN + PN_c = 0_n$$
$$Q_cM + QM_c = 0_n \qquad\qquad Q_cN + QN_c = B(b).$$

Since $\det B(t) \neq 0$ on $t \in [a,b]$, $B^{-1}(a)$, $B^{-1}(b)$ exist, and thus

$$\begin{bmatrix} -B^{-1}(a)P_c & -B^{-1}(a)P \\ B^{-1}(b)Q_c & B^{-1}(b)Q \end{bmatrix} \begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix} = \begin{bmatrix} E_n & 0_n \\ 0_n & E_n \end{bmatrix}.$$

Recall that $\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix}$ has full rank, which means that it is nonsingular (Definition 1.14). Thus, the two matrices on the left are inverses of each other. So we also have

$$\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix} \begin{bmatrix} -B^{-1}(a)P_c & -B^{-1}(a)P \\ B^{-1}(b)Q_c & B^{-1}(b)Q \end{bmatrix} = \begin{bmatrix} E_m & 0_+ \\ 0_- & E_{2n-m} \end{bmatrix}.$$

Therefore,

$$-MB^{-1}(a)P + NB^{-1}(b)Q = 0_+,$$

which is (1.1.17).

Conversely, let U_1^+ is a boundary form of rank $2n-m$ such that

$$U_1^+y = P_1^*\eta(a) + Q_1^*\eta(b)$$

for appropriate P_1^*, Q_1^* with $\operatorname{rank}(P_1^* : Q_1^*) = 2n-m$. Suppose

$$MB^{-1}(a)P_1 = NB^{-1}(b)Q_1 \tag{1.1.18}$$

holds.

Recall that dim(solution space) + rank(matrix) = # of unknown variables. Let u be a $2n \times 1$ vector, then there exist exactly $2n-m$ linearly independent $2n \times 1$ vector solutions of the linear system $(M : N)_{m\times 2n}u = 0$. By (1.1.18),

$$MB^{-1}(a)P_1 - NB^{-1}(b)Q = 0,$$

and thus

$$(M : N)_{m\times 2n} \begin{bmatrix} B^{-1}(a)P_1 \\ -B^{-1}(b)Q_1 \end{bmatrix}_{2n\times(2n-m)} = 0_{m\times(2n-m)}.$$

So the $2n-m$ columns of the matrix

$$H_1 := \begin{bmatrix} B^{-1}(a)P_1 \\ -B^{-1}(b)Q_1 \end{bmatrix}$$

are solutions of this system. Since $\operatorname{rank}(P_1^* : Q_1^*) = 2n-m$,

$$\operatorname{rank} \begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} = 2n-m.$$

Since $B(a)$, $B(b)$ are nonsingular, $\operatorname{rank}(H_1) = 2n-m$.

If $U^+x = P^*\xi(a) + Q^*\xi(b) = 0$ is a boundary condition adjoint to $Ux = 0$, then the matrix

$$\begin{bmatrix} -B^{-1}(a)P_c & -B^{-1}(a)P \\ B^{-1}(b)Q_c & B^{-1}(b)Q \end{bmatrix}_{2n\times 2n}$$

is nonsingular (because it has inverse $\begin{bmatrix} M & N \\ M_c & N_c \end{bmatrix}$), i.e., it has full rank. Thus, if

$$H = \begin{bmatrix} -B^{-1}(a)P \\ B^{-1}(b)Q \end{bmatrix}_{n\times(2n-m)},$$

then $\operatorname{rank}(H) = 2n - m$. Therefore, by (1.1.17), the $2n - m$ columns of H also form $2n = m$ linearly independent solutions of $(M : N)u = 0$, as in the case of H_1. Hence, there exists a nonsingular $(2n - m) \times (2n - m)$ matrix A such that $H_1 = HA$ (change of basis in the solution space). Thus we have

$$\begin{bmatrix} B^{-1}(a)P_1 \\ -B^{-1}(b)Q_1 \end{bmatrix} = H_1 = HA = \begin{bmatrix} B^{-1}(a)PA \\ -B^{-1}(b)QA \end{bmatrix},$$

or $P_1 = PA$, $Q_1 = QA$. Thus,

$$U_1^+y = P_1^*\eta(a) + Q_1^*\eta(b) = A^*P^*\eta(a) + A^*Q^*\eta(b) = A^*U^+y.$$

This implies that $U_1^+y = 0$ is an adjoint boundary condition to $Ux = 0$.

Is this by Theorem 1.19? But it says adjoint boundary forms are related by multiplication by nonsingular matrices, not that the multiplication of an adjoint boundary form by a nonsingular matrix is still an adjoint boundary form.

□

Definition 1.25. If U is a boundary form of rank m, the problem of finding solutions of

$$\pi_m : \ Lx = 0 \quad Ux = 0$$

on $[a, b]$ is a **homogeneous boundary-value problem of rank** m. The problem

$$\pi_{2n-m}^+ : \ L^+x = 0 \quad U^+x = 0$$

on $[a, b]$ is the **adjoint boundary-value problem to** π_m.

Note the "the" in the above statement. In the language of Remark 1.23, since D^+ is uniquely determined by U, the problem is unique in terms of its solutions, even though U^+ is not unique.

In fact, π_m and π_{2n-m}^+ are adjoint problems to each other. The zero function on $[a, b]$ is a solution to both π_m and π_{2n-m}^+, known as the **trivial solution**.

Theorem 1.26. *If $m = n$, the boundary condition $Ux = 0$ is adjoint to itself if and only if*

$$MB^{-1}(a)M^* = NB^{-1}(b)N^*.$$

Proof. Replace P, Q with M, N in Theorem 1.24. □

Theorem 1.27. *If $Ux = 0$ is self-adjoint and $L^+ = L$, the boundary-value problem π_m is self-adjoint, i.e., if $u, v \in C^n$ on $[a, b]$ and satisfy $Ux = 0$, then*

$$(Lu, v) = (u, Lv).$$

Proof. The equation follows as a special case of Proposition 1.22. □

Definition 1.28. Let $\varphi_1, \dots, \varphi_n$ be a fundamental set (basis of the solution space to $Lx = 0$). Let Φ denote the nonsingular matrix

$$\Phi := \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \vdots & \varphi_n^{(n-1)} \end{bmatrix}.$$

Then Φ is a **fundamental matrix associated with** $Lx = 0$. Similarly, if $\psi_1, \dots, \psi_n$ is a fundamental set fo $L^+x = 0$, then the corresponding fundamental matrix is

$$\Psi := \begin{bmatrix} \psi_1 & \cdots & \psi_n \\ \psi_1' & \cdots & \psi_n' \\ \vdots & & \vdots \\ \psi_1^{(n-1)} & \vdots & \psi_n^{(n-1)} \end{bmatrix}.$$

The meanings of U, U^+ can be extended from vectors (Remark 1.13) to matrices as follows:

$$U\Phi := M\Phi(a) + N\Phi(b)$$
$$U^+\Psi := P^*\Psi(a) + Q^*\Psi(b).$$

Remark 1.29. We note that

$$\begin{aligned}
U\Phi &= M\Phi(a) + N\Phi(b) \\
&= \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} \varphi_1(a) & \cdots & \varphi_n(a) \\ \varphi_1'(a) & \cdots & \varphi_n'(a) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(a) & \vdots & \varphi_n^{(n-1)}(a) \end{bmatrix} + \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} \varphi_1(b) & \cdots & \varphi_n(b) \\ \varphi_1'(b) & \cdots & \varphi_n'(b) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(b) & \vdots & \varphi_n^{(n-1)}(b) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n M_{1j}\varphi_1^{(j-1)}(a) & \cdots & \sum_{j=1}^n M_{1j}\varphi_n^{(j-1)}(a) \\ \vdots & & \vdots \\ \sum_{j=1}^n M_{mj}\varphi_1^{(j-1)}(a) & \cdots & \sum_{j=1}^n M_{mj}\varphi_n^{(j-1)}(a) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n N_{1j}\varphi_1^{(j-1)}(b) & \cdots & \sum_{j=1}^n N_{1j}\varphi_n^{(j-1)}(b) \\ \vdots & & \vdots \\ \sum_{j=1}^n N_{mj}\varphi_1^{(j-1)}(b) & \cdots & \sum_{j=1}^n N_{mj}\varphi_n^{(j-1)}(b) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n (M_{1j}\varphi_1^{(j-1)}(a) + N_{1j}\varphi_1^{(j-1)}(b)) & \cdots & (\sum_{j=1}^n M_{1j}\varphi_n^{(j-1)}(a) + N_{1j}\varphi_n^{(j-1)}(b)) \\ \vdots & & \vdots \\ \sum_{j=1}^n (M_{mj}\varphi_1^{(j-1)}(a) + N_{mj}\varphi_1^{(j-1)}(b)) & \cdots & (\sum_{j=1}^n M_{mj}\varphi_n^{(j-1)}(a) + N_{mj}\varphi_n^{(j-1)}(b)) \end{bmatrix} \\
&= \begin{bmatrix} U_1\varphi_1 & \cdots & U_1\varphi_n \\ \vdots & & \vdots \\ U_m\varphi_1 & \cdots & U_m\varphi_n \end{bmatrix}.
\end{aligned}$$

Theorem 1.30. *The problem π_m has exactly k $(0 \le k \le n)$ linearly independent solutions if and only if $U\Phi$ has rank $n-k$, where Φ is any fundamental matrix associated with $Lx = 0$.*

Proof. A function φ satisfies $Lx = 0$ if and only if the corresponding vector $\vec{\varphi} = (\varphi, \varphi', \dots, \varphi^{(n-1)})$ is of the form $\vec{\varphi} = \Phi\vec{c}$, where $\vec{c} = (c_1, \dots, c_n)$ is a constant vector.

Indeed: Suppose φ is a solution to $Lx = 0$. Then by definition of fundamental set $\varphi_1, \dots, \varphi_n$, $\varphi = c_1\varphi_1 + \cdots + c_n\varphi_n$ for some $c_1, \dots, c_n \in \mathbb{C}$. By linearity of derivatives, $\varphi^{(j)} = c_1\varphi_1^{(j)} + \cdots + c_n\varphi_n^{(j)}$. Thus,

$$\vec{\varphi} = \begin{bmatrix} \varphi \\ \varphi' \\ \vdots \\ \varphi^{(n-1)} \end{bmatrix} = \begin{bmatrix} c_1\varphi_1 + \cdots + v_n\varphi_n \\ c_1\varphi_1' + \cdots + v_n\varphi_n' \\ \vdots \\ c_1\varphi_1^{(n-1)} + \cdots + v_n\varphi_n^{(n-1)} \end{bmatrix}$$
$$= \begin{bmatrix} \varphi_1 & \cdots & \varphi_n \\ \varphi_1' & \cdots & \varphi_n' \\ \vdots & & \vdots \\ \varphi_1^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \Phi\vec{c}.$$

Thus, $U\varphi = 0$

This is the definition of Ux in Remark 1.13?

if and only if

$$U(\Phi c) = (U\Phi)c = 0.$$

Since dim(solution space) + rank(matrix) = # of unknown variables, the number of linearly independent vectors $\vec{c}$ satisfying $(U\Phi)c = 0$ is $n - \operatorname{rank}(U\Phi)$. Thus, the number of solutions φ to $Lx = 0$ is $n - \operatorname{rank}(U\Phi)$.

If Φ_1 is any other fundamental matrix associated with $Lx = 0$, then $\Phi_1 = \Phi C$, where C is a nonsingular constant matrix. Therefore

By change of basis?

$$\operatorname{rank}(U\Phi_1) = \operatorname{rank}(U\Phi).$$

□

Theorem 1.31. *If π_m has exactly k linearly independent solutions, then π^+_{2n-m} has exactly $k + m - n$ linearly independent solutions.*

Proof. Let $\varphi_1, \dots, \varphi_k$ be k linearly independent solutions of π_m. Suppose U_c where

$$U_c x = M_c\xi(a) + N_c\xi(b)$$

is a boundary form of rank $2n - m$ complementary to U. We show that the vectors $U_c\varphi_i$ $(i = 1, \dots, k)$ are linearly independent. Suppose not, then for some constants $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ not all zero,

$$\sum_{i=1}^{k} \alpha_i U_c \varphi_i = 0,$$

which implies

$$U_c\left(\sum_{i=1}^{k}\alpha_i\varphi_i\right)=0.$$

But since each φ_i is a solution to π_m, they each satisfy $Ux=0$. Thus,

$$U\left(\sum_{i=1}^{k}\alpha_i\varphi_i\right)=0.$$

Let $\bar{\varphi}=\sum_{i=1}^{k}\alpha_i\varphi_i$. Let $\bar{\xi}=(\bar{\varphi},\bar{\varphi}',\ldots,\bar{\varphi}^{(n-1)})$. Then by Remark 1.13, the above equations imply

$$\begin{aligned} M\bar{\xi}(a)+N\bar{\xi}(b)&=U\bar{\xi}=0\\ M_c\bar{\xi}(a)+N_c\bar{\xi}(b)&=U_c\bar{\xi}=0. \end{aligned}$$

Or

$$\begin{bmatrix} M & N\\ M_c & N_c\end{bmatrix}\begin{bmatrix}\bar{\xi}(a)\\ \bar{\xi}(b)\end{bmatrix}=0_{2n\times 1}.$$

But rank $\begin{bmatrix} M & N\\ M_c & N_c\end{bmatrix}=2n$, which implies it is nonsingular. Thus $\bar{\xi}(a)=\bar{\xi}(b)=0_{n\times 1}$. But since $\varphi_1,\ldots,\varphi_k$ are solutions to $Lx=0$, we have

$$L\bar{\varphi}=L\left(\sum_{i=1}^{k}\alpha_i\varphi_i\right)=\sum_{i=1}^{k}\alpha_i L\varphi_i=0.$$

We show that this implies $\bar{\varphi}=0$. Indeed: If not, then L maps a nonzero function to 0, which means if two distinct functions x_1, x_2 are such that $x_1-x_2=\bar{\varphi}$, then $Lx_1-Lx_2=L(x_1-x_2)=0$, i.e., the pre-image of 0 under L is not unique.

This is how I interpreted "uniqueness" in the next line. But why is this a problem / where is the contradiction?

Thus by uniqueness, $\bar{\varphi}(t)=0$ for $t\in[a,b]$. This contradicts the definition of $\bar{\varphi}$ as a nontrivial linear combination of $\varphi_1,\ldots,\varphi$ (i.e., not all α_i are 0). Hence

$$\alpha_1=\alpha_2=\cdots=\alpha_k=0$$

and $U_c\varphi_i$ are linearly independent.

Let $\psi_1,\ldots,\psi_n$ be n linearly independent solutions of $L^+x=0$. Suppose Ψ is the corresponding fundamental matrix. Since φ_i, ψ_j are solutions to π_m and $L^+x=0$,

This is not the same as requiring ψ_j to be solutions to π_{2n-m}, is it? Since there is an extra $U^+\psi_j=0$ to fulfill?

respectively, by Proposition 1.22,

Proposition 1.22 requires that $U\varphi_i=0$ and $U^+\psi_j=0$; are these conditions fulfilled?

$$(L\varphi_i,\psi_j)=(\varphi_i,L^+\psi_j).$$

By Green's formula (1.1.2),

$$0 = (L\varphi_i, \psi_j) - (\varphi_i, L^+\psi_j) = [\varphi_i\psi_j](b) - [\varphi_i\psi_j](a)$$

for $i = 1, \ldots, k$, $j = 1, \ldots, n$. By the boundary-form formula (1.1.11),

$$[\varphi_i\psi_j](b) - [\varphi_i\psi_j](a) = U_{\varphi_i} \cdot U_c^+\psi_j + U_{c\varphi_i} \cdot U^+\psi_j.$$

Since φ_i are solutions to π_m, we have $U\varphi_i = 0$ for $i = 1, \ldots, k$. Thus,

$$U_{c\varphi_i} \cdot U^+\psi_j = 0.$$

Does this mean we don't know $U^+\psi_j = 0$? If so, how could we use Proposition 1.22 above?

By definition of $f \cdot g$1.7, $f \cdot g = g^* f$ for any column vectors f, g of the same dimension, so

$$(U^+\psi_j)^* U_{c\varphi_i} = 0 \quad (i = 1, \ldots, k).$$

We have shown before that $U_c\varphi_i$ are linearly independent. So the system $(U^+\psi_j)^* v = 0$ has (at least) the k linearly independent $(2n - m \times 1)$ vectors $U_c\varphi_1, \ldots, U_c\varphi_k$ as solutions. Therefore,

$$\operatorname{rank}(U^+\Psi) = \operatorname{rank}(U^+\Psi)^* \leq (2n - m) - k.$$

Suppose $\operatorname{rank}(U^+\psi) = r < (2n - m) - k$. Then by similar reasoning it can be shown that, if Φ is any fundamental matrix associated with $Lx = 0$, $\operatorname{rank}(U\phi) \leq m - (n - r) < n - k$. By Theorem 1.30, this contradicts with the assumption that π_m has exactly k linearly independent solutions. Thus, we must have

$$\operatorname{rank}(U^+\Psi) = 2n - m - k.$$

By Theorem 1.30, there exist exactly $k + m - n$ linearly independent solutions of π^+_{2n-m}. □

Corollary 1.32. *π_n and π_n^+ have the same number of independent solutions.*

Proof. Apply Theorem 1.31 on $m = n$. □

Coddington & Levinson go to a lot of trouble here to count the dimension of the solution space of problems in general. In practice, we will be working in problems in which the rank of $(M : N)$ is equal to the order of L, so this can be greatly simplified. If you like, one later direction of the capstone would be to understand how this kind of stuff works in some more general setting, but my feeling is that is not the most need. Maybe don't spend too much time on this for now.

1.1.4 Nonhomogeneous Boundary-value Problems and Green's Function

Definition 1.33. A **nonhomogeneous boundary-value problem** associated with π_m is a problem of the form

$$Lx = f \quad Ux = \gamma \tag{1.1.19}$$

on $t \in [a, b]$, where f is a complex-valued continuous function on $[a, b]$ and γ is a complex constant vector such that either f is not the zero function or $\gamma \neq 0$.

Remark 1.34. If φ and $\bar{\varphi}$ are two solutions of 1.1.19, their difference $\varphi - \bar{\varphi}$ is a solution of π_m. Hence, if π_m has k linearly independent solutions $\varphi_1, \ldots, \varphi_k$, then $\varphi = \bar{\varphi} + \sum_{i=1}^k c_i \varphi_i$ for some constants $c_i \in \mathbb{C}$ (since $\varphi_1, \ldots, \varphi_k$ are a basis for the solution space of π_m).

Proposition 1.35. *Let A be a matrix and b a vector. $Ax = b$ has a solution if and only if $b \cdot u = u^* b = 0$ for every solution u of $A^* x = 0$.*

Theorem 1.36. *The nonhomogeneous problem 1.1.19 has a solution if and only if*

$$(f, \psi) = \gamma \cdot U_c^+ \psi \tag{1.1.20}$$

holds for every solution ψ of the adjoint homogeneous problem π_{2n-m}^+.

Remark 1.37. Since (f, ψ) is an inner product, $\gamma = 0$ implies f is orthogonal to all solutions ψ of π_{2n-m}^+.

Proof. Let φ be a solution of 1.1.19. Let ψ be a solution of the adjoint homogeneous problem π_{2n-m}^+. Then by Green's formula (1.1.2) and the boundary-form formula (1.1.11),

$$(L\varphi, \psi) - (\varphi, L^+\psi) = U\varphi \cdot U_c^+ \psi + U_c \varphi \cdot U^+ \psi.$$

Since $L\varphi = f$, $U\varphi = \gamma$, $L^+\psi = 0$, and $U^+\psi = 0$, the above implies that

$$(f, \psi) - (\varphi, 0) = \gamma \cdot U_c^+ \psi + 0,$$

or

$$(f, \psi) = \gamma \cdot U_c^+ \psi.$$

Now suppose 1.1.20 holds for all solutions ψ of π_{2n-m}^+. Let $\varphi_1, \ldots, \varphi_n$ be a fundamental set for $Lx = 0$. Let $\bar{\varphi}$ be a solution of $Lx = f$. Then every solution φ of $Lx = f$ is of the form

$$\varphi = \bar{\varphi} + \sum_{i=1}^n c_i \varphi_i$$

for some constants $c_i \in \mathbb{C}$. Applying U to both sides, we have that 1.1.19 has a solution if and only if there exist c_i such that

$$U\varphi = U\bar{\varphi} + \sum_{i=1}^n c_i U \varphi_i$$

is equal to γ. Equivalently,

$$(U\Phi)c = \gamma - U\bar{\varphi} \tag{1.1.21}$$

where Φ is the fundamental matrix corresponding to $\varphi_1, \ldots, \varphi_n$, and c a constant vector. Note that by Remark 1.29,

$$\begin{aligned}(U\Phi)c &= \begin{bmatrix} U_1\varphi_1 & \cdots & U_1\varphi_n \\ \vdots & & \vdots \\ U_m\varphi_1 & \cdots & U_m\varphi_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^n c_i U_1 \varphi_i \\ \vdots \\ \sum_{i=1}^n c_i U_m \varphi_i \end{bmatrix}\end{aligned}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} c_i U_1 \varphi_i \\ \vdots \\ c_i U_m \varphi_i \end{bmatrix}$$
$$= \sum_{i=1}^{n} c_i U \varphi_i$$

By Proposition 1.35, the above has a solution c if and only if $\gamma - U\bar{\varphi}$ is orthogonal to every solution u of

$$(U\Phi)^* u = 0, \tag{1.1.22}$$

that is,

$$(\gamma - U\bar{\varphi}) \cdot u = 0 \tag{1.1.23}$$

Suppose π^+_{2n-m} have exactly k^+ linearly independent solutions $\psi_1, \ldots, \psi_{k^+}$. By the same argument in Theorem 1.31, the k^+ vectors $U_c^+\psi_1, \ldots, U_c^+\psi_{k^+}$ are linearly independent m-dimensional vectors which are solutions to 1.1.22. Let k be the number of linearly independent solutions of π_m. Then by Theorem 1.30, $\text{rank}(U\Phi) = n - k$. Thus, the number of linearly independent solutions of 1.1.22 is $m - \text{rank}(U\Phi^*) = m - \text{rank}(U\Phi) = m - (n - k)$. But by Theorem 1.31, $k^+ = m - n + k$. Thus, since $U_c^+\varphi_i$ are basis for the solution space of 1.1.22, 1.1.23 holds for every u satisfying 1.1.22 if and only if

$$(\gamma - U\bar{\varphi}) \cdot U_c^+\psi_i = 0 \quad (i = 1, \ldots, k^+) \tag{1.1.24}$$

By Green's formula (1.1.2) and the boundary-form formula (1.1.11),

$$(L\bar{\varphi}, \psi_i) - (\bar{\varphi}, L^+\psi_i) = U_{\bar{\varphi}} \cdot U_c^+\psi_i + U_c\bar{\varphi} \cdot U^+\psi_i. \tag{1.1.25}$$

But ψ_i is a solution to π^+_{2n-m}, so $L^+\psi_i = 0$ and $U^+\psi_i = 0$. Thus, the above becomes

$$(f, \psi_i) = (L\bar{\varphi}, \psi_i) = (L\bar{\varphi}, \psi_i) - 0 = U_{\bar{\varphi}} \cdot U_c^+\psi_i + 0 = U_{\bar{\varphi}} \cdot U_c^+\psi_i. \tag{1.1.26}$$

But by the hypothesis, 1.1.20 holds. Thus,

$$U_{\bar{\varphi}} \cdot U_c^+\psi_i \overset{1.1.26}{=} (f, \psi_i) \overset{1.1.20}{=} \gamma \cdot U_c^+\psi_i.$$

Hence,

$$(\gamma - U\bar{\varphi}) \cdot U_c^+\psi_i = 0.$$

So 1.1.24 is satisfied. Reversing the direction of the argument from 1.1.24, we have that 1.1.23 holds. This implies that 1.1.21 has a solution c, which then implies that 1.1.19 has a solution. □

Corollary 1.38. *1.1.19 has a unique solution if $m = n$ and the only solution of π_n is the trivial one.*

Proof. Suppose $m = n$ and π_n only has the trivial solution (note that π_n is the homogeneous problem). Let φ, $\bar{\varphi}$ be two solutions of 1.1.19. Then $\varphi - \bar{\varphi}$ is a solution of π_n. By the hypothesis, $\varphi - \bar{\varphi} = 0$. Thus, 1.1.19 has a unique solution φ. □

I suggest we come back to Green's functions, and inhomogeneous BVP if and only if we decide to go in that direction. For now, the below is great progress.

Definition 1.39. The **Dirac's delta function** can be loosely thought of as a function on $\mathbb{R}$ which is zero everywhere except at the origin, where it is infinite. It is given by

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0. \end{cases}$$

It is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

Definition 1.40. [1] Given a linear differential operator $L = L(x)$, a **Green's function** $G = G(x, s)$ at the point $s \in \Omega \in \mathbb{R}^n$ corresponding to L is any solution of

$$LG(x, s) = \delta(x - s) \tag{1.1.27}$$

where δ denotes the Dirac's delta function.

Remark 1.41. A Green's function is an **integral kernel** that can be used to solve differential equations, such as ordinary differential equations with initial or boundary value conditions. Recall the Fourier kernel $e^{i\lambda x}$ in the Fourier transform

$$\mathcal{F}[f] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx.$$

To see the motivation for defining Green's function, we note that multiplying both sides of 1.1.27 by a function $f(s)$ and integrating with respect to s gives

$$\int LG(x, s) f(s)\, ds = \int \delta(x - s) f(s)\, ds.$$

The right-hand side reduces to $f(x)$ due to properties of δ, and because L is a linear operator acting only on x and not on s, the left-hand side can be rewritten as

$$L\left(\int G(x, s) f(s)\, ds\right).$$

This reduction is particularly useful when solving for $u = u(x)$ in differential equations of the form

$$Lu(x) = f(x),$$

since we have

$$Lu(x) = L\left(\int G(x, s) f(s)\, ds\right),$$

which implies

$$u(x) = \int G(x, s) f(s)\, ds. \tag{1.1.28}$$

[1] `http://mathworld.wolfram.com/GreensFunction.html`

Suppose $m = n$. By Theorem 1.31, π_n and π_n^+ have the same number k of linearly independent solutions. If $k = 0$, then π_n has only the trivial solution. In this case, it is possible to solve the nonhomogeneous problem 1.1.19 explicitly in terms of the Green's function:

Proposition 1.42. *The nonhomogeneous problem*

$$Lx = f \quad Ux = 0$$

has a unique solution given by $x(t) = \int_a^b f(s)G(s,t)\,ds$ where $G(s,t)$ is a Green's function satisfying some properties.

Definition 1.43. The problem

$$\pi : \quad Lx = \ell x \quad Ux = 0$$

is an **eigenvalue problem**. If ℓ is such that π has a nontrivial solution, then ℓ is an **eigenvalue** of π, and the nontrivial solutions of π are the **eigenfunctions** of π.

Remark 1.44. Recall from linear algebra that if T is a linear transformation from a vector space V over a field $\mathbb{F}$ into itself and $v \in V$, $v \neq \vec{0}$, then v is an eigenvector of T if $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$, and λ the eigenvalue associated with v.

Here, if L is a linear differential operator, and u is not the zero function, then u is an eigenfunction of π if $Lu = \ell u$ for some ℓ, and ℓ the eigenvalue associated with u.

We introduce the following results.

Proposition 1.45. *Let π denote the problem*

$$Lx - \ell x = 0 \quad Ux = 0.$$

If π has no (nontrivial) solution for at least one value of ℓ, then there exists unique $G = G(t, \tau, \ell)$ defined for (t, τ) on the square $a \leq t, \tau \leq b$ and for all complex ℓ except the eigenvalues of π and having the following properties:

(i) $\frac{\partial^k G}{\partial t^k}$ *$(k = 0, 1, \ldots, n-2)$ exist and are continuous in (t, τ, ℓ) for (t, τ) on the square $a \leq t$, $\tau \leq b$ and ℓ not at an eigenvalue of π. Moreover,* $\frac{\partial^k G}{\partial t^k}$ *for $k = n-1, n$ are continuous in (t, τ, ℓ) for (t, τ) on the triangles $a \leq t \leq \tau \leq b$ and $a \leq \tau \leq t \leq b$ and ℓ not an eigenvalue of π.*

(ii)

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}(r+0, r, \ell) - \frac{\partial^{n-1} G}{\partial t^{n-1}}(r-0, r, \ell) = \frac{1}{p_0(\tau)}.$$

(iii) As a function of t, G satisfies $Lx = \ell x$ if $t \neq \tau$.

(iv) As a function of t, G satisfies the boundary conditions $Ux = 0$ for $a \leq \tau \leq b$.

If for one value of ℓ the homogeneous problem π has no solution, then it would have solutions only for a set of ℓ which are the zeroes of an entire function.

Here, $\ell = 0$ and it is assumed that π_n has only the trivial solution. Denote $G(t, \tau, 0)$ as $G(t, \tau)$. By (1.1.28), the unique solution of 1.1.19 with $\gamma = 0$ is given by

$$\mathcal{G}f(t) := \int_a^b G(t, \tau) f(\tau)\, d\tau.$$

If π_n has only the trivial solution, then by Theorem 1.31, π_n^+ has only the trivial solution (with $k = 1$ and $m = n$). By Proposition 1.45, the Green's function G^+ for π_n^+ exists and is unique.

Theorem 1.46. *If π_n has only the trivial solution, Green's function G^+ for π_n^+ is given by*

$$G^+(t, \tau) = \bar{G}(\tau, t). \tag{1.1.29}$$

Proof. Let $a < \tau_1 < \tau_2 < b$. Consider the functions G_1 and G_2^+ given by $G_1(t) = G(t, \tau_1)$, $G_2^+(t) = G^+(t, \tau_2)$. Then applying Green's formula 1.1.2 to the intervals $[a, \tau_1 - 0]$, $[\tau_1 + 0, \tau_2 - 0]$, $[\tau_2 + 0, b]$, we have

$$[G_1G_2^+](\tau_1-0)-[G_1G_2^+](a)+[G_1G_2^+](\tau_2-0)-[G_1G_2^+](\tau_1+0)+[G_1G_2^+](b)-[G_1G_2^+](\tau_2+0) = 0 \tag{1.1.30}$$

Why is the equation 0?

By the boundary-form formula 1.1.11,

$$[G_1G_2^+](b) - [G_1G_2^+](a) = 0. \tag{1.1.31}$$

Doesn't this follow from 1.1.30?

From the form of $[xy](t)$ 1.1.3 it follows that the only terms of interest in 1.1.30 are those involving the $(n-1)$st derivatives, and these are Why?

$$p_0(t)[(-1)^{n-1}x(t)\bar{y}^{(n-1)}(t) + x^{(n-1)}(t)\bar{y}(t)]. \tag{1.1.32}$$

Now by Proposition 1.45(ii), G satisfies

$$\frac{\partial^{n-1}G}{\partial t^{n-1}}(r+0, r, \ell) - \frac{\partial^{n-1}G}{\partial t^{n-1}}(r-0, r, \ell) = \frac{1}{p_0(\tau)}. \tag{1.1.33}$$

Similarly for G^+,

$$\frac{\partial^{n-1}G^+}{\partial t^{n-1}}(r+0, r, \ell) - \frac{\partial^{n-1}G^+}{\partial t^{n-1}}(r-0, r, \ell) = \frac{1}{(-1)^n\bar{p}_0(\tau)}. \tag{1.1.34}$$

Thus, $\bar{G}^+(\tau_1, \tau_2) - G(\tau_2, \tau_1) = 0$. □ How?

Remark 1.47. To consider $G(t, \tau, \ell)$, the differential operation $(L - l)$ is considered instead of L. Let $L_1 = L - \ell$. Consider the problem

$$L_1 x = 0 \quad Ux = 0 \tag{1.1.35}$$

The adjoint problem is given in $L_1^+ = L^+ - \bar{\ell}$ and U^+. Applying Theorem 1.46 to 1.1.35,

We assume π_n only has the trivial solution?

we have

$$G^+(t, \tau, \ell) = \bar{G}(\tau, t, \bar{\ell}).$$

For the self-adjoint problem where $L^+ = L$ and U equivalent to U^+, it follows that

$$G(t, \tau, \ell) = \bar{G}(\tau, t, \ell).$$

1.2 Chapter 12: Non-self-adjoint boundary-value problems

1.2.1 Introduction

Let L be an nth-order ordinary differential operator which is formally self-adjoint. Consider a self-adjoint boundary-value problem

$$\pi : Lx = \ell x \quad Ux = 0$$

on $[a, b]$ (where $L = L^+$ and $U = U^+$). Then there exists a complete orthonormal set of eigenfunctions $\{\chi_k\}$ and a Green's function $G(t, \tau, \ell)$ for the equation $Lx = \ell x$ where ℓ is not one of the eigenvalues of π.

Theorem 1.48. *(Eigenfunction expansion theorem) Let $f \in C^n$ on $[a, b]$ and satisfy the boundary condition $Uf = 0$. Then on $[a, b]$,*

$$f = \sum_{k=0}^{\infty} (f, \chi_k)\chi_k \tag{1.2.1}$$

where $(f, \chi_k) = \int_a^b f(t)\bar{\chi}_k(t)\, dt$ and the series converges uniformly on $[a, b]$.

2 Finding ways to explicitly construct (a non-unique) U^+ given U

We focus on the case where $m = n$. That is, M and N are $n \times n$ square matrices.

2.1 A simple example of the hard way

For L with lower orders, the adjoint boundary condition can be (feasibly) found using integration by parts.

Consider $Lx = x^{(1)} - \epsilon x^{(2)}$ on $[0, 1]$ with boundary conditions $x(0) = x(1) = 0$. Then

$$Lx = (-\epsilon)x^{(2)} + 1x^{(1)} + 0x^{(0)}$$

with $p_0 = -\epsilon$, $p_1 = 1$, and $p_2 = 0$. And

$$Ux = 0$$

where

$$\begin{aligned}
Ux &= \begin{bmatrix} U_1 x \\ U_2 x \end{bmatrix} \\
&= \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \\
&= \begin{bmatrix} (1 \cdot x^{(0)}(0) + 0 \cdot x^{(0)}(1)) + (0 \cdot x^{(1)}(0) + 0 \cdot x^{(1)}(1)) \\ (0 \cdot x^{(0)}(0) + 1 \cdot x^{(0)}(1)) + (0 \cdot x^{(1)}(0) + 0 \cdot x^{(1)}(1)) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(1) \\ x'(1) \end{bmatrix} = M\xi(0) + N\xi(1) = \vec{0}.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
(y, Lx) &= \int_0^1 y(\overline{Lx})\, dt \\
&= \int_0^1 y(\frac{dx}{dt} - \epsilon \frac{d^2x}{dt^2})\, dt \\
&= \int_0^1 y\frac{dx}{dt}\, dt - \epsilon \int_0^1 y \frac{d^2x}{dt^2}\, dt \\
&= \left([yx]_0^1 - \int_0^1 x\frac{dy}{dt}\, dt \right) - \epsilon \left(\left[y\frac{dx}{dt} \right]_0^1 - \int_0^1 \frac{dx}{dt}\frac{dy}{dt}\, dt \right) \\
&= \left([yx]_0^1 - \int_0^1 x\frac{dy}{dt}\, dt \right) - \epsilon \left(\left[y\frac{dx}{dt} \right]_0^1 - \left(\left[x\frac{dy}{dt} \right]_0^1 - \int_0^1 x\frac{d^2y}{dt^2}\, dt \right) \right) \\
&= \int_0^1 x \left(-\frac{dy}{dt} - \epsilon \frac{d^2y}{dt^2} \right) dt + \left[yx - \epsilon y \frac{dx}{dt} + \epsilon x \frac{dy}{dt} \right]_0^1 \\
&= \int_0^1 x \left(-\frac{dy}{dt} - \epsilon \frac{d^2y}{dt^2} \right) dt + \left[-\epsilon y \frac{dx}{dt} \right]_0^1.
\end{aligned}$$

For the above to equal

$$(L^+ y, x),$$

we ought to define

$$L^+y := -y' - \epsilon y''$$

with $y(0) = y(1) = 0$ for the boundary terms to vanish. Thus, the adjoint boundary condition is

$$U^+y = 0$$

with

$$U^+y = P^*\eta(0) + Q^*\eta(1)$$

where

$$P^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad Q^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

So we have

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix $B(t)$ in this case would be the 2×2 matrix associated with $[xy](t)$. Recall that

$$Lx = -\epsilon x'' + x' = p_0 x^{(2)} + p_1 x^{(1)}.$$

So $p_0 = -\epsilon$, $p_1 = 1$ are constant functions. By Theorem 1.5, $[xy](t)$ is given by

$$\begin{aligned}
[xy](t) &= \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^j x^{(k)}(t)(p_{n-m}\bar{y})^{(j)}(t) \\
&= \sum_{m=1}^{2} \sum_{j+k=m-1} (-1)^j x^{(k)}(t)(p_{2-m}\bar{y})^{(j)}(t) \\
&= (-1)^0 x(t)(p_1\bar{y})(t) + (-1)^0 x^{(1)}(t)(p_0\bar{y})(t) + (-1)^1 x(t)(p_0\bar{y})^{(1)}(t) \\
&= x(t)p_1(t)\bar{y}(t) + x'(t)p_0(t)\bar{y}(t) - x(t)(p_0'(t)\bar{y}(t) + p_0(t)\bar{y}'(t)) \\
&= x(t)\bar{y}(t) + x'(t)(-\epsilon)\bar{y}(t) - x(t)(-\epsilon)\bar{y}'(t) \\
&= x(t)\bar{y}(t) - \epsilon x'(t)\bar{y}(t) + \epsilon x(t)\bar{y}'(t) \\
&= B_{11}(t)x(t)\bar{y}(t) + B_{12}(t)x'(t)\bar{y}(t) + B_{21}(t)x(t)\bar{y}'(t) + B_{22}(t)x'(t)\bar{y}'(t).
\end{aligned}$$

Thus,

$$B(t) = \begin{bmatrix} 1 & -\epsilon \\ \epsilon & 0 \end{bmatrix}.$$

Since p_0, p_1 are constant functions, $B(0) = B(1)$. We verify that

$$\begin{aligned}
MB^{-1}(0)P &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{1+\epsilon^2} \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= NB^{-1}(1)Q.
\end{aligned}$$

So $U^+x = 0$ is indeed adjoint to $Ux = 0$ by Theorem 1.24.

2.2 Construction from existence theorem

Given

$$Lx = 0 \quad Ux = 0$$

on $[a, b]$ where $\mathrm{rank}(U) = m = n$, we use Theorem 1.19 to construct U^+ as follows.

2.2.1 Working with U_i in a vector space

Recall from Linear algebra that the dual space of a vector space V is the vector space of all linear functionals on V, i.e., $\mathcal{L}(V, \mathcal{F})$. Consider the vector space $C^{n-1}_{[a,b]}$. The dual space of $C^{n-1}_{[a,b]}$ is very large, but the subspace spanned by the boundary forms (recall that boundary forms map functions in $C^{n-1}_{[a,b]}$ to the field $\mathbb{C}$) is finite dimensional and has a basis of the $2n$ boundary forms $x \mapsto x^{(j)}(a)$ and $x \mapsto x^{(j)}(b)$ for $j = 0, 1, \cdots, n-1$ and $x \in C^{n-1}_{[a,b]}$ (it is clear that all boundary forms are linear combinations of them).

Let $\mathcal{V}$ denote the subspace with dimension $2n$ spanned by the boundary forms. Then $U = (U_1, \ldots, U_n)$ is a list of n linearly independent vectors in $\mathcal{V}$.

Algorithm 1: Checking linear independence.

Data: A list of boundary operators $U_1, \ldots, U_n$

Result: Whether they are linearly independent

The criterion for linear independence of the U_i is $\sum_{i=1}^{n} c_i U_i x = 0$ implies $c_i = 0$ for all $x \in C^{n-1}_{[a,b]}$. How should I check this (since $C^{n-1}_{[a,b]}$ is infinite)? In general, it seems that I have to deal with criteria like this in the following algorithms working with U_i and their spans.

Find a basis for $\mathcal{V}$ by building a maximally linearly independent set adding one vector at a time until the set spans V, or equivalently, contains $2n$ linearly independent vectors.

Algorithm 2: Finding a basis.

Data: An arbitrary nonzero $V_1 \in \mathcal{V}$

Result: A list $V_1, \ldots, V_{2n}$ of linearly independent vectors in $\mathcal{V}$

```
begin
    l ⟵ [V_1];
    while length(l) < 2n do
        Pick any V ∈ 𝒱 − span(l);
        l ⟵ l + V
    return l
```

By a result from linear algebra, using this basis, $U_1, \ldots, U_n$ can be extended to $U_1, \ldots, U_{2n}$, a basis of $\mathcal{V}$. Thus, we have found $U_c = (U_{n+1}, \ldots, U_{2n})$.

Algorithm 3: Extending to a basis.

```
Data: U_1, ..., U_n and basis V_1, ..., V_{2n}
Result: A basis U_1, ..., U_{2n} of 𝒱
begin
    l ⟵ [U_1, ..., U_n, V_1, ..., V_{2n}];
    i ⟵ 1;
    while length(l) > 2n do
        if i = 1 then
            if V_i ∈ span(U_1, ..., U_n) then
                l ⟵ [U_1, ..., U_n, V_{i+1}, ..., V_{2n}]
        else
            if V_i ∈ span(U_1, ..., U_n, ..., V_{i-1}) then
                l ⟵ [U_1, ..., U_n, V_{i+1}, ..., V_{2n}]
    return l
```

Let $M, N, \tilde{M}, \tilde{N}$ be such that

$$Ux = M\xi(a) + N\xi(b); \quad U_c x = \tilde{M}\xi(a) + \tilde{N}\xi(b).$$

Then we have found

$$H = \begin{bmatrix} M & N \\ \tilde{M} & \tilde{N} \end{bmatrix}.$$

Let $\hat{B}$ be as in 1.1.4 which depends on L. By Proposition 1.17, $J = (\hat{B}H^{-1})^*$ is a nonsingular $2n \times 2n$ matrix such that $\mathcal{S}(f, g) = \hat{B}f \cdot g = Hf \cdot Jg$. Take U^+y to be the last n rows of Jg. Then we have found U^+.

2.2.2 Working with matrices M, N

For $U = (U_1, \ldots, U_n)$, write

$$Ux = M\xi(a) + N\xi(b).$$

Then $U_1, \ldots, U_n$ are linearly independent if and only if $\text{rank}(M : N) = n$. Thus, Algorithm 1 can be replaced by the following.

Algorithm 4: Checking linear independence by checking rank$(M : N)$.

```
Data: A list of boundary operators U_1, ..., U_n
Result: Whether they are linearly independent
begin
    M, N ⟵ n × n matrices associated with U = (U_1, ..., U_n) defined in Remark 1.13;
    mat ⟵ (M : N)
    return rank(mat) == n
```

Extending $U_1, \ldots, U_n$ to $U_1, \ldots, U_{2n}$ is equivalent to imbedding $(M : N)$ in a $2n \times 2n$ nonsingular matrix. Mirroring the approach in the previous section, we may start with any $2n \times 2n$ nonsingular matrix A (for convenience we pick $A = E_{2n}$, the identity matrix) which corresponds to a list of linearly independent boundary forms $V_1, \dots, V_{2n}$ (which is a basis for $\mathcal{V}$), and then reduce some "combination" of $(M : N)$ and A to a $2n \times 2n$ nonsingular matrix containing $(M : N)$. That is, writing

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where all the blocks are $n \times n$ matrices, we seek to turn the $3n \times 2n$ matrix

$$(M : N) : A := \begin{bmatrix} M & N \\ A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

into the $2n \times 2n$ matrix with rank $2n$

$$\begin{bmatrix} M & N \\ A'_{11} & A'_{12} \end{bmatrix}.$$

Since $\text{rank}(A) = 2n$, $(M : N) : A$ has at least $2n$ linearly independent rows, namely its row n to row $3n$ which come from A. But row rank is equal to column rank and $(M : N) : A$ only has $2n$ columns. So $\text{rank}((M : N) : A) = 2n$. This means that we can remove n rows from $(M : N) : A$ without changing its rank, thus obtaining the desired $2n \times 2n$ matrix.

To find which rows to remove in $(M : N) : A$, we could customize the row echelon form algorithm so that the row operations do not overwrite the first n rows.

Alternatively, we can append the rows of A one by one to $(M : N)$ and discard any row that do not make the rank of the resulting matrix increase.

Algorithm 5: Removing rows from $(M : N) : A$ without changing rank: Append and check rank.

Data: $(M : N)_{n\times 2n}$ with rank n, $A_{2n\times 2n}$ with rank $2n$
Result: A $2n \times 2n$ matrix with rank $2n$ where the first n rows are $(M : N)$

```
begin
    mat ⟵ (M : N);
    for i in range(nrow(A)) do
        mat1 ⟵ vcat(mat, A[i,]) ;          ▷ vcat := vertical concatenation
        if rank(mat1) == rank(mat)+1 then
            mat ⟵ mat1
        else
            A ⟵ A[-i,]
    return vcat(mat, A)
```

We then identify it with

$$H = \begin{bmatrix} M & N \\ \tilde{M} & \tilde{N} \end{bmatrix}$$

and define

$$U_c x = \tilde{M}\xi(a) + \tilde{N}\xi(b).$$

Following the steps in the previous section gives U^+.