The solution is given ~~explicitly~~ in [2], via explicit construction of an integral ~~transform~~ inverse transform pair. In this section, we provide that explicit mathematical construction, and adapt it into an algorithm that can be implemented in software. The transform pair is displayed in equation (3.5), using notation developed throughout §3.2.

3 Algorithm

Consider an IBVP of the form (1.6). We now outline the algorithm to find its solution (1.7).

The adjoint boundary condition associated with the adjoint problem relates to that of the original problem in a way that is important to making the Fokas transform pairs work as we would like them to. In fact, the first step in implementing the Fokas method is to construct the adjoint of a homogeneous boundary condition.

To this end, the following algorithm to construct adjoint boundary conditions has been devised and implemented in Julia, outlined as follows.

3.1 Constructing the Adjoint of a Homogeneous Boundary Condition

Define the differential operator

$$L := S$$

to be the spatial differential operator (1.5), the interval

$$[a, b] := [0, 1]$$

to be the unit interval in (1.6), and U to be the vector boundary form associated with two $n \times n$ matrices M, N given by

$$M_{jk} := b_{jk}, \quad N_{jk} := \beta_{jk}$$

for $b_{jk}, \beta jk$ in (1.3). Then we have defined a boundary-value problem in the spatial variable from the original IBVP, namely

$$Lx = 0 \quad Ux = 0.$$

We seek to construct a valid adjoint boundary condition $U^+x = 0$.

3.1.1 Checking input

We first check that $U_1, \ldots, U_n$ are linearly independent. Recall from (2.14) that

$$Ux = M\xi(a) + N\xi(b).$$

Thus, it suffices to check whether $\operatorname{rank}(M : N) = n$. U would be considered an invalid input if $\operatorname{rank}(M : N) \neq n$.

3.1.2 Finding a candidate adjoint

Recall from Definition 2.10 that extending $U_1, \ldots, U_n$ to $U_1, \ldots, U_{2n}$ (where $(U_{n+1}, \ldots, U_{2n})$ is a complementary boundary form U_c) is equivalent to embedding $(M : N)$ in a $2n \times 2n$ nonsingular matrix (where the newly added rows constitute $(\tilde{M}, \tilde{N})$ associated with U_c). We construct this $2n \times 2n$ nonsingular matrix from the identity matrix E_{2n} and $(M : N)$ as follows. We append the rows of E_{2n} one by one to $(M : N)$ and discard any row that does not make the rank of the identifying matrix

resulting matrix increase, as shown below.

Algorithm 1: Algorithm to find U_c.

Data: $(M : N)_{n\times 2n}$ with rank n, E_{2n} with rank $2n$

Result: A $2n \times 2n$ matrix with rank $2n$ where the first n rows are $(M : N)$

```
begin
    mat ⟵ (M : N);
    for i in range(nrow(E)) do
        mat1 ⟵ vcat(mat, E[i,]) ; ▷ vcat := vertical concatenation, or joining
          two matrices vertically
        if rank(mat1) == rank(mat)+1 then
            mat ⟵ mat1
        else
            E ⟵ E[-i,]
    return vcat(mat, E)
```

The output from the above algorithm is a $2n \times 2n$ matrix of the form

$$\begin{bmatrix} M & N \\ E' & E'' \end{bmatrix}$$

where the rows of E', E'' are the first n entries and the last n entries of the retained rows of E_{2n}, respectively. We identify this matrix with the matrix

$$H = \begin{bmatrix} M & N \\ \tilde{M} & \tilde{N} \end{bmatrix}$$

in Theorem 2.14 (where $\tilde{M}, \tilde{N}$ are associated with the complementary boundary form $U_c x = \tilde{M}\xi(a) + \tilde{N}\xi(b)$).

Recall that the linear differential operator L is characterized by the functions $p_0, \ldots, p_n$ and the interval $[a, b]$. Let B be as in (2.6) which depends on $p_0, \ldots, p_n$. Construct $\hat{B}$ from B and $[a, b]$ as in Theorem 2.14. Let $J := (\hat{B}H^{-1})^*$. By Theorem 2.14 and Proposition 2.12, the matrix J is of the form

$$J = \begin{bmatrix} M' & N' \\ \tilde{M}' & \tilde{N}' \end{bmatrix}$$

where $\tilde{M}', \tilde{N}'$ are associated with an adjoint U^+ and M', N' with its complement U_c^+. Thus, we can identify $(P^\star : Q^\star)$ with the last n rows of J. That is, identify $P^\star$ with $\tilde{M}'$, the lower-left $n \times n$ submatrix of J, and $Q^\star$ with $\tilde{N}'$, the lower-right $n \times n$ submatrix of J. Define U^+ by

$$U^+ x = P^\star \xi(a) + Q^\star \xi(b),$$

then we have found an adjoint U^+ to U.

3.1.3 Checking the validity of the candidate adjoint

By Theorem 2.19, with

$$Ux = M\xi(a) + N\xi(b), \quad U^+ x = P^\star \xi(a) + Q^\star \xi(b),$$

we can check whether the U^+ found above is indeed a valid adjoint to U by checking

$$MB^{-1}(a)P = NB^{-1}(b)Q$$

where B is as in (2.6).

M is used for two different ~~concepts~~ objects on consecutive pages. Change notation for one of them.

3.2 Constructing the Fokas Transform Pair [2]

Define the matrices

$$b^\star := P^\star, \quad \beta^\star := Q^\star,$$

where $P^\star, Q^\star$ are the matrices associated with the adjoint vector boundary form U^+ constructed above. Let $\alpha := e^{2\pi i/n}$. For complex variable λ, define $M^+(\lambda)$, $M^-(\lambda)$ as the $n \times n$ matrices

$$M^+_{kj}(\lambda) := \sum_{r=0}^{n-1}(-i\alpha^{k-1}\lambda)^r b^\star_{jr} \tag{3.1a}$$

$$M^-_{kj}(\lambda) := \sum_{r=0}^{n-1}(-i\alpha^{k-1}\lambda)^r \beta^\star_{jr}, \tag{3.1b}$$

where $b^\star_{jr}$, $\beta^\star_{jr}$ are indexed from 0 to $n-1$ in r. Define M as the $n \times n$ matrix

$$M_{kj}(\lambda) := M^+_{kj}(\lambda) + M^-_{kj}(\lambda)e^{-\alpha^{k-1}\lambda}. \tag{3.2}$$

Let $\Delta(\lambda) := \det M(\lambda)$. Then by the definition of $M(\lambda)$, Δ is an exponential polynomial [8] in λ. That is,

$$\Delta(\lambda) = \sum_{k=1}^{n} a_k \lambda^k e^{b_k \lambda},$$

where $a_k, b_k \in \mathbb{C}$ for $k \in \{1, \dots, n\}$.

Let X^{lj} be the $(n-1) \times (n-1)$ submatrix of the block matrix $\mathbb{M} := \begin{bmatrix} M & M \\ M & M \end{bmatrix}$, where X^{lj}_{11} is $\mathbb{M}_{l+1,j+1}$.

For $\lambda \in \mathbb{C}$ such that $\Delta(\lambda) \neq 0$, define

$$F^+_\lambda(f) := \frac{1}{2\pi\Delta(\lambda)} \sum_{l=1}^{n}\sum_{j=1}^{n}(-1)^{(n-1)(l+j)} \det X^{lj}(\lambda) M^+_{1j}(\lambda) \int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\, dx \tag{3.3a}$$

$$F^-_\lambda(f) := \frac{-e^{-i\lambda}}{2\pi\Delta(\lambda)} \sum_{l=1}^{n}\sum_{j=1}^{n}(-1)^{(n-1)(l+j)} \det X^{lj}(\lambda) M^-_{1j}(\lambda) \int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\, dx. \tag{3.3b}$$

By the theory of exponential polynomials [8], the infimum of the distances between every two distinct zeroes of $\Delta(\lambda)$ is positive. Let 5ϵ be such minimum, then ϵ is positive. ~~Let ∂S denote boundary of set S.~~ open Let $\operatorname{cl} S$ denote the closure of set S. Let $\mathbb{C}^\pm$ denote the ~~upper and lower~~ open halves of the complex plane, respectively. Let $D(x_0, \epsilon)$ denote the disk $\{x \in \mathbb{C} : |x - x_0| < \epsilon\}$ centered at x_0 with radius ϵ. Let $C(x_0, \epsilon)$ denote the circle $\{x \in \mathbb{C} : |x - x_0| = \epsilon\}$ centered at x_0 with radius ϵ. Define the contours

$$\Gamma^\pm_a := \partial(\{\lambda \in \mathbb{C}^\pm : \operatorname{Re}(a\lambda^n) > 0\} \setminus \bigcup_{\substack{\sigma\in\mathbb{C};\\ \Delta(\sigma)=0}} D(\sigma, 2\epsilon)) \tag{3.4a}$$

$$\Gamma_a := \Gamma^+_a \cup \Gamma^-_a \tag{3.4b}$$

$$\Gamma^+_0 := \bigcup_{\substack{\sigma\in\operatorname{cl}\mathbb{C}^+;\\ \Delta(\sigma)=0}} C(\sigma, \epsilon) \tag{3.4c}$$

$$\Gamma^-_0 := \bigcup_{\substack{\sigma\in\mathbb{C}^-;\\ \Delta(\sigma)=0}} C(\sigma, \epsilon) \tag{3.4d}$$

$$\Gamma_0 := \Gamma^+_0 \cup \Gamma^-_0 \tag{3.4e}$$

$$\Gamma := \Gamma_0 \cup \Gamma_a. \tag{3.4f}$$

The Fokas transform pair is given by

$$F_\lambda : f(x) \mapsto F(\lambda) \quad F_\lambda(f) = \begin{cases} F_\lambda^+(f) & \text{if } \lambda \in \Gamma_0^+ \cup \Gamma_a^+, \\ F_\lambda^-(f) & \text{if } \lambda \in \Gamma_0^- \cup \Gamma_a^-, \end{cases} \tag{3.5a}$$

$$f_x : F(\lambda) \mapsto f(x) \quad f_x(F) = \int_\Gamma e^{i\lambda x} F(\lambda)\, d\lambda,\ x \in [0,1], \tag{3.5b}$$

which allows computing the IBVP solution $q(x,t)$ in (1.7) by

$$\begin{aligned} q(x,t) &= f_x\left(e^{-a\lambda^n t} F_\lambda(f)\right) \\ &= \int_{\Gamma_0^+} e^{i\lambda x} e^{-a\lambda^n t} F_\lambda^+(f)\, d\lambda + \int_{\Gamma_a^+} e^{i\lambda x} e^{-a\lambda^n t} F_\lambda^+(f)\, d\lambda \\ &\quad + \int_{\Gamma_0^-} e^{i\lambda x} e^{-a\lambda^n t} F_\lambda^-(f)\, d\lambda + \int_{\Gamma_a^-} e^{i\lambda x} e^{-a\lambda^n t} F_\lambda^-(f)\, d\lambda. \end{aligned} \tag{3.6}$$

All objects defined above except for the contours, namely $M^+(\lambda)$, $M^-(\lambda)$, $M(\lambda)$, $\Delta(\lambda)$, X^{lj}, $F_\lambda^+(f)$, $F_\lambda^-(f)$, can be constructed directly from their explicit expressions. Thus, to construct the Fokas transform pair F_λ and f_x, it remains to find an explicit characterization of the contour Γ.

3.2.1 Contour-tracing

Given a complex constant a, we find the boundary of

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(a\lambda^n) > 0\}.$$

For complex number z, define $\theta_z := \arg(z)$ and $r_z := |z|$. Let $\lambda \in \mathbb{C}$. Then

$$a\lambda^n = r_a e^{i\theta_a} \cdot r_\lambda^n e^{ni\theta_\lambda} = r_a r_\lambda^n e^{i(\theta_a + n\theta_\lambda)} = r_a r_\lambda^n (\cos(\theta_a + n\theta_\lambda) + i\sin(\theta_a + n\theta_\lambda)).$$

Thus,

$$\mathrm{Re}(a\lambda^n) = r_a r_\lambda^n \cos(\theta_a + n\theta_\lambda),$$

where

$$\mathrm{Re}(a\lambda^n) > 0 \iff \cos(\theta_a + n\theta_\lambda) > 0.$$

But

$$\cos(\theta_a + n\theta_\lambda) > 0 \iff \theta_a + n\theta_\lambda \in \bigcup_{k\in\mathbb{Z}} \left(2\pi k - \frac{\pi}{2},\, 2\pi k + \frac{\pi}{2}\right),$$

and so

$$\theta_\lambda \in \bigcup_{k\in\mathbb{Z}} \left(\frac{2\pi k - \frac{\pi}{2} - \theta_a}{n},\, \frac{2\pi k + \frac{\pi}{2} - \theta_a}{n}\right).$$

For example, when $a = -i$ and $n = 3$,

$$\theta_\lambda \in \bigcup_{k\in\mathbb{Z}} \left(\frac{2\pi k - \frac{\pi}{2} - (-\frac{\pi}{2})}{3},\, \frac{2\pi k + \frac{\pi}{2} - (-\frac{\pi}{2})}{3}\right) = \bigcup_{k\in\mathbb{Z}} \left(\frac{2\pi k}{3},\, \frac{2\pi k + \pi}{3}\right).$$

Thus, in $[0, 2\pi)$,

$$\theta_\lambda = \arg(\lambda) \in \bigcup_{k\in\{0,1,2\}} \left(\frac{2\pi k}{3},\, \frac{2\pi k + \pi}{3}\right) = \left(0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right) \cup \left(\frac{4\pi}{3}, \frac{5\pi}{3}\right).$$

figs too small

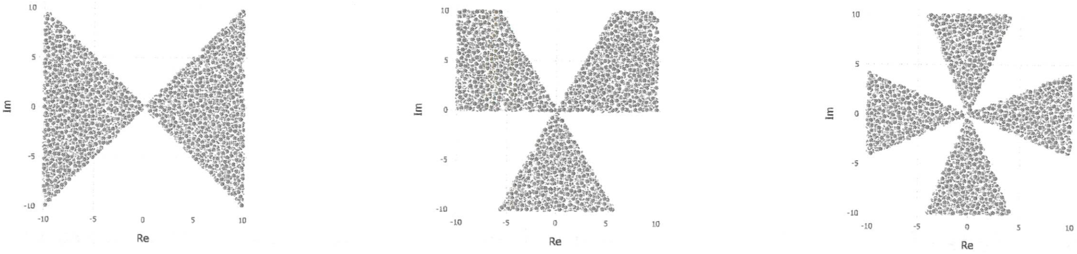


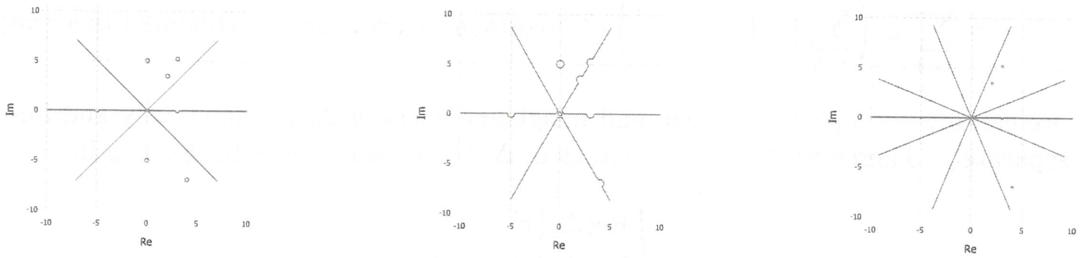
Figure 3: Simulation of sectors in the domain $\{\lambda \in \mathbb{C} : \mathrm{Re}(a\lambda^n) > 0\}$ by sampling 10^4 points for $a = 1, n = 2$, $a = -i, n = 3$, and $a = 1, n = 4$, respectively.

has

Figure 3 shows some simulations of the domain $\{\lambda \in \mathbb{C} : \mathrm{Re}(a\lambda^n) > 0\}$.

We proceed to deform the contours obtained above to avoid zeroes of $\Delta(\lambda)$. By the theory of exponential polynomials [8], $\Delta(\lambda)$ infinitely many zeroes, but any finite region contains only finitely many of these zeroes. In practice, we would choose a sufficiently large real number d to represent infinity in the algorithm's context. Thus, in the disk $D(0, d)$ where we operate, there are only finitely many zeroes of $\Delta(\lambda)$. With this setting, one complication that arises in practice is that each circle around a zero of $\Delta(\lambda)$ not only cannot overlap with the circle around another zero but also cannot overlap with the boundary of any sector. That is, the 5ϵ defined above in fact needs to be the minimum of the pairwise distances between zeroes and the distances between each zero and the boundary of each sector. But since we are working with a finite number of zeroes of $\Delta(\lambda)$, such ϵ exists and is positive.

Suppose first that we have been given finitely many zeroes of $\Delta(\lambda)$ (we will approximate these zeroes in the next section). Each sector in the boundary is characterized by the angles of the two rays that mark its boundaries, the starting ray and the ending ray. The contour Γ is built from the boundaries of these sectors with possible deformations to include the contours encircling the zeroes. Figures 4 and 5 show the contour Γ for some examples.



figs too small

Figure 4: Plots of Γ for $a = 1, n = 2$, $a = -i, n = 3$, and $a = 1, n = 4$, respectively, with zeroes of $\Delta(\lambda)$ at $3 + 3\sqrt{3}i$, $2 + 2\sqrt{3}i$, $0 + 0i$, $0 + 5i$, $0 - 5i$, 3, -5, and $4 - 4\sqrt{3}i$.

Note that each sector will be assigned to Γ_a^+ or Γ_a^- depending on whether it belongs to the upper or lower half plane. If a sector overlaps with the real line (Figure 5), we divide it into the upper and lower half planes, which can then be assigned to Γ_a^+, Γ_a^-, respectively.

For each sector, we initialize its boundary contour by a path consisting of three points, namely a point on the ending ray with distance d from the origin, the origin, and the ~~a~~ point on the starting ray with ~~norm~~ modulus d (this would be the contour if no zero of $\Delta(\lambda)$ were on the sector boundary). For each zero, if it is on the boundary of some sector, we deform the sector's boundary contour to include a small n-gon of radius ϵ around the zero. If the zero is exterior to all sectors, we add an n-gon contour around it to $\Gamma_0^\pm$, depending on whether the zero is in the upper or lower half plane. If the zero is interior to any sector, we ignore it, since its contribution to the contour integral will be cancelled out. Expand on this

this seems like the reverse order.

After we have looped through all zeroes in this process, Γ_a would contain all sector boundary

this n is not the same as the ~~order~~ order of the problem. change notation.

these figs too small

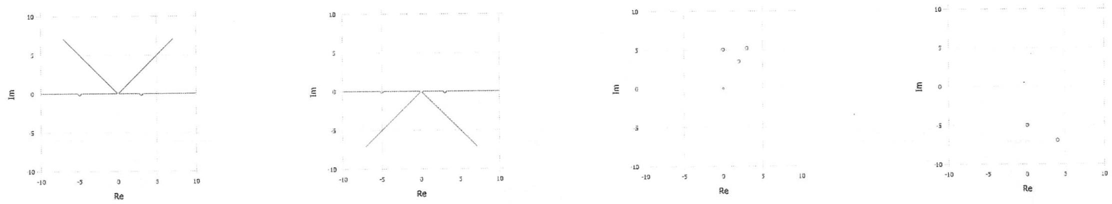


Figure 5: Plots of Γ_a^+, Γ_a^-, Γ_0^+, and Γ_0^- for $a = 1$ and $n = 2$, respectively, with the same zeroes as in Figure 4.

contours now appropriately deformed, and Γ_0 would contain all the n-gon contours encircling isolated zeroes. Although not reflected in the figures, care is taken to ensure that the points in Γ are listed in the order in which the contour will be integrated over.

Approximating the Roots of an Exponential Polynomial

As promised, we now describe a method to approximate the roots of $\Delta(\lambda)$. We observe that since $\Delta(\lambda)$ is an exponential polynomial of the form

$$\Delta(\lambda) = \sum_{k=1}^{n} a_k \lambda^k e^{b_k \lambda},$$

writing $\lambda = x + iy$ where $x, y \in \mathbb{R}$ gives

$$\begin{aligned}
\Delta(x,y) &= \sum_{k=1}^{n} a_k (x+iy)^k e^{b_k(x+iy)} \\
&= \sum_{k=1}^{n} a_k (x+iy)^k e^{b_k x}(\cos(b_k y) + i\sin(b_k y)) \quad \text{(Euler's formula)} \\
&= \sum_{k=1}^{n} a_k \left(\sum_{l=0}^{k} \binom{k}{l} x^{k-l} (iy)^l \right) e^{b_k x}(\cos(b_k y) + i\sin(b_k y)) \quad \text{(Binomial theorem).}
\end{aligned}$$

Thus, we can always separate the real and imaginary parts of $\Delta(\lambda)$ analytically and find their roots separately. Therefore, to find the zeroes of $\Delta(\lambda)$, it suffices to solve for x, y in

$$\begin{cases} \mathrm{Re}(\Delta)(x) &= 0, \\ \mathrm{Im}(\Delta)(y) &= 0. \end{cases}$$

In practice, we find it convenient to visualize the zeroes of $\Delta(\lambda)$ using level curves of the above equations. In the examples below (Figure 6), the red lines correspond to the zeroes of $\mathrm{Re}(\Delta)(x)$, and the blue lines correspond to those of $\mathrm{Im}(\Delta)(y)$. Thus, we can approximately locate the zeroes of $\Delta(\lambda)$ at the intersections of the red and blue lines.

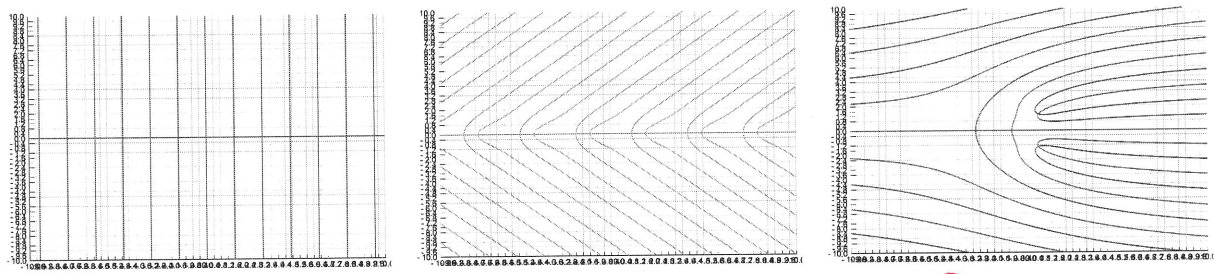


Figure 6: Level curves for $\Delta(\lambda) = \cos(\lambda)$, $\cos(\lambda)e^{\lambda}$, and $(\lambda^3 + \lambda + 2) * e^{\lambda}$, respectively.

You should explain clearly that a human still has to read these plots and ~~go~~ record all the ~~zeros, so that the~~ within $D(0,d)$ so that the list can be used as an input for the function that produces Γ. Maybe at the beginning of this section, or may be here.

Note that when constructing Γ, we need to deform the contours to avoid zeroes of $\Delta(\lambda)$ with n-gons of radius ϵ. Thus, there is relatively high tolerance for our approximation of the locations of the zeroes.

3.2.2 Approximating Integral Using Chebyshev Polynomials

With the above contour construction, theoretically we can already compute the Fokas transform pair F_λ and f_x and solve for $q(x,t)$. Yet there is one more complication in practice. As shown in (3.6), the definition of the solution $q(x,t)$ involves integration of F_λ^+, F_λ^- over the Γ contours, but as shown in (3.3a) ~~&~~ (3.3b), F_λ^+ and F_λ^- themselves involve an integral of the initial data $f(x)$. Double integrals are computationally expensive, and so to improve algorithm efficiency, we replace the inner integral involving $f(x)$ with an approximation of it that has explicit formula, developed as follows.

The inner integral we wish to compute is

$$\int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\, dx.$$

Note that this is the Fourier transform of f defined on $[0,1]$ evaluated at $\alpha^{l-1}\lambda$. The idea is to approximate $f(x)$ using Chebyshev polynomials and compute the integral using explicit formula of the Fourier Transform of Chebyshev polynomials [9].

The Chebyshev polynomials are given by

$$T_n(x) = \begin{cases} \cos(n \cdot \arccos(x)) & |x| \leq 1 \\ \frac{1}{2}\left((x-\sqrt{x^2-1})^n + (x+\sqrt{x^2-1})^n\right) & |x| > 1. \end{cases}$$

Given a function f on the interval $[a,b]$, we can approximate f using Chebyshev polynomial expansion. The Chebyshev approximation is best on $[-1,1]$. Thus, we scale $[a,b]$ to $[-1,1]$ via $g: [a,b] \to [-1,1]$ given by $g(x) = 2(x-a)/(b-a) - 1$, obtain the Chebyshev coefficients $\{b_n\}_{n=0}^N$ there, and write the approximation of f on $[a,b]$ as

$$f \approx \sum_{n=0}^N b_n \cdot T_n(g(x)),$$

which is a finite sum of Chebyshev polynomials.

Now, suppose we are given the IBVP's initial data $f(x)$. Define $g(x): [0,1] \to [-1,1]$ by $g(x) = 2x - 1$. With the change of variable $t = g(x) = 2x - 1$, we have $x = g^{-1}(t) = \frac{t+1}{2}$. Writing $c = \frac{\alpha^{l-1}\lambda}{2}$, we have

$$\begin{aligned}\int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\, dx &= \int_0^1 e^{-i2c\frac{t+1}{2}} f(\frac{t+1}{2})\, d(\frac{t+1}{2}) \\ &= \int_{-1}^1 e^{-ict}e^{-ic} f(\frac{t+1}{2})\frac{1}{2}\, dt \\ &= \frac{1}{2e^{ic}}\int_{-1}^1 e^{-ict} f(\frac{t+1}{2})\, dt \\ &= \frac{1}{2e^{ic}}\int_{-1}^1 e^{-ict} q(t)\, dt,\end{aligned}$$

\left, \right

where $q(t) := f \circ g^{-1}(t)$. Thus, it suffices to find

$$\int_{-1}^1 e^{-ict} q(t)\, dt.$$

Suppose that on $[-1,1]$,

$$q(t) \approx \sum_{n=0}^{N} b_n T_n(t).$$

(Note that since $t = g(x)$, the Chebyshev coefficients of q are exactly the coefficients of f obtained on $[-1,1]$.) Then since $t \in [-1,1]$,

$$q(t) \approx \sum_{n=0}^{N} b_n \cos(n \cdot \cos^{-1}(t)),$$

and so

$$\begin{aligned}
\int_{-1}^{1} e^{-ict} q(t)\, dt &= \int_{-1}^{1} e^{-ict} \sum_{n=0}^{N} b_n \cos(n \cdot \cos^{-1}(t))\, dt \\
&= \sum_{n=0}^{N} b_n \int_{-1}^{1} e^{-ict} \cos(n \cdot \cos^{-1}(t))\, dt.
\end{aligned}$$

With the change of variable $t = \cos\theta$, — or be consistent earlier

arccos

$$\begin{aligned}
\int_{-1}^{1} e^{-ict} \cos(n \cdot \cos^{-1}(t))\, dt &= \int_{-1}^{1} e^{-ic\cos\theta} \cos(n \cdot \cos^{-1}(\cos\theta))\, d\cos\theta \\
&= \int_{\pi}^{0} e^{-ic\cos\theta} \cos(n\theta)(-\sin\theta)\, d\theta \\
&= \int_{0}^{\pi} e^{-ic\cos\theta} \cos(n\theta)\sin\theta\, d\theta.
\end{aligned}$$

Define

$$\tilde{T}_n(c) := \int_0^{\pi} e^{-ic\cos\theta} \cos(n\theta)\sin\theta\, d\theta.$$

Then

$$\begin{aligned}
\tilde{T}_n(0) = \int_0^{\pi} \cos(n\theta)\sin\theta\, d\theta &= \begin{cases} 2 & \text{if } n = 0 \\ \left[\frac{1}{2}\sin^2\theta + \frac{0}{2}\cos^2\theta\right]_0^{\pi} & \text{if } n = 1 \\ \left[\frac{n}{(n-1)(n+1)}\sin\theta\sin(n\theta) + \frac{1}{(n-1)(n+1)}\cos\theta\cos(n\theta)\right]_0^{\pi} & \text{if } n \geq 2 \end{cases} \\
&= \begin{cases} 2 & \text{if } n = 0 \\ \frac{1}{2}\sin^2(\pi) - \frac{1}{2}\sin^2(0) & \text{if } n = 1 \\ \frac{n}{n^2-1}\sin(\pi)\sin(n\pi) + \frac{1}{n^2-1}\cos(\pi)\cos(n\pi) & \\ -\frac{n}{n^2-1}\sin(0)\sin(0) - \frac{1}{n^2-1}\cos(0)\cos(0) & \text{if } n \geq 2 \end{cases} \\
&= \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ \frac{(-1)^{n+1}-1}{n^2-1} & \text{if } n \geq 2 \end{cases}.
\end{aligned}$$

indent, so it is clear this is part of the same ~~[illegible]~~ expression on two lines

For $c \in \mathbb{C} \setminus \{0\}$, using integration by parts,

$$\begin{aligned}
\tilde{T}_n(c) &= \int_0^{\pi} e^{-ic\cos\theta} \cos(n\theta)\sin\theta\, d\theta \\
&= \left[\cos(n\theta)\frac{1}{ic}e^{-ic\cos\theta}\right]_0^{\pi} - \int_0^{\pi} \frac{1}{ic}e^{-ic\cos\theta} n(-\sin(n\theta))\, d\theta
\end{aligned}$$

$$= \left[\cos(n\pi)\frac{1}{ic}e^{-ic\cos(\pi)} - \cos(0)\frac{1}{ic}e^{-ic\cos 0}\right] + \frac{n}{ic}\int_0^\pi e^{-ic\cos\theta}\sin(n\theta)\,d\theta$$
$$= \frac{1}{ic}\left(n\int_0^\pi e^{-ic\cos\theta}\sin(n\theta)\,d\theta + (-1)^n e^{ic} - e^{-ic}\right).$$

Let $z \in \mathbb{C}\setminus\{0\}$, define

$$K_n(z) := \int_0^\pi e^{z\cos\theta}\sin(n\theta)\,d\theta.$$

Then

$$K_0(z) = \int_0^\pi e^{z\cos\theta}\sin 0\,d\theta = 0,$$

$$\begin{aligned}
K_1(z) &= \int_0^\pi e^{z\cos\theta}\sin\theta\,d\theta \\
&= \left[-\frac{1}{z}e^{z\cos\theta}\right]_0^\pi \\
&= -\frac{1}{z}e^{z\cos(\pi)} + \frac{1}{z}e^{z\cos 0} \\
&= \frac{e^z - e^{-z}}{z}
\end{aligned}$$

To compute $K_n(z)$ for $n \geq 2$, we first note that

$$\begin{aligned}
\sin((n+1)\theta) - \sin((n-1)\theta) &= \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{2i} - \frac{e^{i(n-1)\theta} - e^{-i(n-1)\theta}}{2i} \\
&= \frac{e^{in\theta}e^{i\theta} - e^{in\theta}e^{-\theta} - e^{-in\theta}e^{-i\theta} + e^{-in\theta}e^{i\theta}}{2i} \\
&= \frac{(e^{i\theta} - e^{-i\theta})(e^{in\theta} + e^{-in\theta})}{2i} \\
&= 2\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\left(\frac{e^{in\theta} + e^{-in\theta}}{2}\right) \\
&= 2\sin\theta\cos(n\theta).
\end{aligned}$$

So

$$\sin((n+1)\theta) = \sin((n-1)\theta) + 2\sin\theta\cos(n\theta).$$

Thus,

$$\begin{aligned}
K_{n+1}(z) &= \int_0^\pi e^{z\cos\theta}\sin((n+1)\theta)\,d\theta \\
&= \int_0^\pi e^{z\cos\theta}(\sin((n-1)\theta) + 2\sin\theta\cos(n\theta))\,d\theta \\
&= \int_0^\pi e^{z\cos\theta}\sin((n-1)\theta)\,d\theta + 2\int_0^{\frac{\pi}{2}} e^{z\cos\theta}\sin\theta\cos(n\theta)\,d\theta \\
&= K_{n-1}(z) + 2\int_0^\pi \cos(n\theta)\sin\theta e^{z\cos\theta}\,d\theta.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\int_0^\pi \cos(n\theta)\sin\theta e^{z\cos\theta}\,d\theta &= \left[-\frac{1}{z}e^{z\cos\theta}\cos(n\theta)\right]_0^\pi - \int_0^\pi \frac{1}{z}e^{z\cos\theta}n\sin(n\theta)\,d\theta \\
&= -\frac{1}{z}e^{z\cos(\pi)}\cos(n\pi) + \frac{1}{z}e^{z\cos 0}\cos 0 - \frac{n}{z}\int_0^\pi e^{z\cos\theta}\sin(n\theta)\,d\theta
\end{aligned}$$

$$= \frac{e^z}{z} + (-1)^{n+1}\frac{e^{-z}}{z} - \frac{n}{z}K_n(z).$$

Thus, $K_n(z)$ satisfies the recurrence relation

$$\begin{aligned}
K_0(z) &= 0 \\
K_1(z) &= \frac{e^z - e^{-z}}{z} \\
K_n(z) &= K_{n-2}(z) + 2\left(\frac{e^z}{z} + (-1)^n\frac{e^{-z}}{z} - \frac{n-1}{z}K_{n-1}(z)\right), \quad n \geq 2.
\end{aligned}$$

Hence, for $c \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
\tilde{T}_n(c) &= \frac{1}{ic}\left(n\int_0^\pi e^{-ic\cos\theta}\sin(n\theta)\,d\theta + (-1)^n e^{ic} - e^{-ic}\right) \\
&= \frac{1}{ic}\left(nK_n(-ic) + (-1)^n e^{ic} - e^{-ic}\right).
\end{aligned}$$

Further solving the recurrence relation [9] gives

$$\begin{aligned}
\tilde{T}_n(c) = \int_0^\pi e^{-ic\cos\theta}\cos(n\theta)\sin\theta\,d\theta &= \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ \frac{(-1)^{n+1}-1}{n^2-1} & \text{if } n \geq 2, \end{cases} && \text{if } c = 0, \\
&= \sum_{m=1}^{n+1} \alpha(m,n)\left[\frac{e^{i\lambda}}{(i\lambda)^m} + (-1)^{m+n}\frac{e^{-i\lambda}}{(i\lambda)^m}\right] && \text{if } c \neq 0,
\end{aligned}$$

where

$$\alpha(m,n) = \begin{cases} (-1)^n & \text{if } m = 1, \\ (-1)^{n+1}n^2 & \text{if } m = 2, \\ (-1)^{n+m-1}2^{m-2}n\sum_{k=1}^{n-m+2}\binom{m+k-3}{k-1}\prod_{j=k}^{m+k-3}(n-j) & \text{else.} \end{cases}$$

Thus, in summary,

$$\begin{aligned}
\int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\,dx &= \frac{1}{2e^{ic}}\int_{-1}^1 e^{-ict}q(t)\,dt \\
&= \frac{1}{2e^{ic}}\sum_{n=0}^N b_n \int_{-1}^1 e^{-ict}\cos(n\cdot \overset{\text{arccos}}{\cancel{\cos^{-1}}}(t))\,dt \\
&= \frac{1}{2e^{ic}}\sum_{n=0}^N b_n\hat{T}_n(t)
\end{aligned}$$

4 Implementation

The above algorithm is implemented in Julia 0.6.4. Packages used are “SymPy”, “Distributions”, “ApproxFun”, “Roots”, “QuadGK”, “Plots”, “Gadfly”, and “PyPlot” (TBD: citations).

4.1 Examples

TBD: Problem1, problem2 with formulae verification? [5].

5 Discussion

TBD: Significance (symbolic features, visualizations). Current limits (e.g., speed limit due to compilation time). Need for custom integrator tailored to the IBVP.

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