Reading notes

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This document contains all notes taken while reading materials (e.g., textbooks, literature) in preparation for capstone. Black text are information consolidated from the readings; blue text are notes; red text are questions.

Contents

Todo list

1 Theory of Ordinary Differential Equations (Coddington Levinson)

1.1 Chapter 11: Algebraic properties of linear boundary-value problems on a finite interval

1.1.1 Introduction

Definition 1.1.1. Let L be the **linear differential operator of oder** n **$(n \geq 1)$** defined by

$$Lx = p_0 x^{(n)} + p_1 x^{(n-1)} + \cdots + p_{n-1} x' + p_n x$$

where the p_k are complex-valued functions of class C^{n-k} on a closed bounded interval $[a,b]$ (i.e., derivatives $p_k, p_k', \ldots, p_k^{(n-k)}$ exist on $[a,b]$ and are continuous) and $p_0(t) \neq 0$ on $[a,b]$.

Definition 1.1.2. **Homogeneous boundary conditions** refer to a set of equations/constraints of the type

$$\sum_{k=1}^{n} (M_{jk} x^{(k-1)}(a) + N_{jk} x^{(k-1)}(b)) = 0 \quad (j = 1, \ldots, m) \tag{1.1.1}$$

where M_{jk}, N_{jk} are complex constants.

Definition 1.1.3. A **homogeneous boundary-value problem** concerns finding the solutions of

$$Lx = 0$$

on $[a,b]$ which satisfy some homogeneous boundary conditions defined above.

Definition 1.1.4. For any homogeneous boundary value problem, an **adjoint problem** refers to the problem of finding the solutions of

$$L^+ x := (-1)^n (\bar{p}_0 x)^{(n)} + (-1)^{n-1} (\bar{p}_1 x)^{(n-1)} + \cdots + \bar{p}_n x = 0$$

on $[a,b]$ which satisfy some homogeneous boundary conditions “complementary” to the conditions associated with the solutions of $Lx = 0$.

Theorem 1.1.5. *(Green's formula) For $u, v \in C^n$ on $[a,b]$,*

$$\int_{t_1}^{t_2} (Lu)\bar{v}\, dt - \int_{t_1}^{t_2} u(\bar{L^+ v})\, dt = [uv](t_2) - [uv](t_1) \tag{1.1.2}$$

where $a \leq t_1 < t_2 \leq b$ and $[uv](t)$ is the form in $(u, u', \ldots, u^{(n-1)})$ and $(v, v', \ldots, v^{(n-1)})$ given by

$$[uv](t) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^i u^{(k)}(t) (p_{n-m}\bar{v})^{(j)}(t).$$

Remark 1.1.6. Alternatively, $[uv](t)$ can be written as (checked for $n = 2$)

$$[uv](t) = \sum_{j,k=1}^{n} B_{jk}(t) u^{(k-1)}(t) \bar{v}^{(j-1)}(t) \tag{1.1.3}$$

where B_{jk} are the j,k-entry of the $n \times n$ matrix

$$B(t) = \begin{bmatrix} B_{11} & B_{12} & \cdots & \cdots & p_0(t) \\ \vdots & \vdots & \cdots & -p_0(t) & 0 \\ (-1)^{n-1} p_0(t) & 0 & \cdots & 0 & 0. \end{bmatrix}$$

Since $B(t)$ is square with $\det B(t) = (p_0(t))^n$ where $p_0(t) \neq 0$ on $[a,b]$ (as in the definition of L), $B(t)$ is nonsingular/invertible for $t \in [a,b]$.

Definition 1.1.7. For vectors $f = (f_1, \ldots, f_k)$, $g = (g_1, \ldots, g_k)$, define the product

$$f \cdot g := \sum_{i=1}^{k} f_i \bar{g}_i.$$

Definition 1.1.8. A **semibilinear form** is a complex-valued function $\mathcal{S}$ defined for pairs of vectors $f = (f_1, \ldots, f_k)$, $g = (g_1, \ldots, g_k)$ satisfying

$$\begin{aligned} \mathcal{S}(\alpha f + \beta g, h) &= \alpha \mathcal{S}(f,h) + \beta \mathcal{S}(g,h) \\ \mathcal{S}(f, \alpha g + \beta h) &= \bar{\alpha} \mathcal{S}(f,g) + \bar{\beta} \mathcal{S}(f,h) \end{aligned}$$

for any complex numbers α, β and vectors f, g, h.

Note that $\mathcal{S}$ is linear in the first argument but not the second one. If $\mathcal{S}$ were bilinear, it would be linear in each argument.

Remark 1.1.9. If

$$S = \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \cdots & s_{kk} \end{bmatrix},$$

then $Sf \cdot g$ is a semibilinear form

$$\begin{aligned} \mathcal{S}(f,g) &:= Sf \cdot g \\ &= \begin{bmatrix} s_{11} & \cdots & s_{1k} \\ \vdots & & \vdots \\ s_{k1} & \cdots & s_{kk} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{k} s_{1j} f_j \\ \vdots \\ \sum_{j=1}^{k} s_{kj} f_j \end{bmatrix} \cdot \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix} \\ &= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} s_{ij} f_j \right) \bar{g}_i \\ &= \sum_{i,j=1}^{k} s_{ij} f_i \bar{g}_i. \end{aligned} \tag{1.1.4}$$

Indeed:

$$\begin{aligned} \mathcal{S}(\alpha f + \beta g, h) &= \sum_{i,j=1}^{k} s_{ij} (\alpha f_j + \beta g_j) \bar{h}_i \\ &= \alpha \sum_{i,j=1}^{k} s_{ij} f_j \bar{h}_i + \beta \sum_{i,j=1}^{k} g_j \bar{h}_i \\ &= \alpha S f \cdot h + \beta S g \cdot h \\ &= \alpha \mathcal{S}(f,h) + \beta \mathcal{S}(g,h). \end{aligned}$$

Similarly,

$$\mathcal{S}(f, \alpha g + \beta h) = \sum_{i,j=1}^{k} s_{ij} f_j (\alpha g_i \bar{+} \beta h_i)$$

$$
\begin{aligned}
&= \bar{\alpha} \sum_{i,j=1}^{k} s_{ij} f_j \bar{g}_i + \bar{\beta} \sum_{i,j=1}^{k} f_j \bar{h}_i \\
&= \bar{\alpha} S f \cdot g + \bar{\beta} S f \cdot h \\
&= \bar{\alpha} \mathcal{S}(f, g) + \bar{\beta} \mathcal{S}(f, h).
\end{aligned}
$$

Remark 1.1.10. Under a similar matrix framework, we see that $[uv](t)$ is a semibilinear form with matrix $B(t)$:
Let $\vec{u} = (u, u', \ldots, u^{(n-1)})$ and $\vec{v} = (v, v', \ldots, v^{(n-1)})$. Then we have

$$
\begin{aligned}
[uv](t) &= \sum_{j,k=1}^{n} B_{jk}(t) u^{(k-1)}(t) \bar{v}^{(j-1)}(t) \quad \text{(by (1.1.3))} \\
&= \sum_{i,j=1}^{n} (B_{ij} u^{(j-1)} \bar{v}^{(i-1)})(t) \\
&= (B\vec{u} \cdot \vec{v})(t) \\
&= \mathcal{S}(\vec{u}, \vec{v})(t).
\end{aligned}
$$

With this notation, we can rewrite the right hand side of Green's formula as

$$
\begin{aligned}
[uv](t_2) - [uv](t_1) &= \sum_{j,k=1}^{n} B_{jk}(t_2) u^{(k-1)}(t_2) \bar{v}^{(j-1)}(t_2) - \sum_{j,k=1}^{n} B_{jk}(t_2) u^{(k-1)}(t_1) \bar{v}^{(j-1)}(t_1) \\
&= B(t_2)\vec{u}(t_2) \cdot \vec{v}(t_2) - B(t_1)\vec{u}(t_1) \cdot \vec{v}(t_1) \\
&= \begin{bmatrix} B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots \\ B_{n1}(t_2) & \ldots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} - \\
&\quad \begin{bmatrix} B_{11}(t_1) & \cdots & B_{1n}(t_1) \\ \vdots & & \vdots \\ B_{n1}(t_1) & \ldots & B_{nn}(t_1) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \end{bmatrix} \\
&= \begin{bmatrix} -B_{11}(t_1) & \cdots & -B_{1n}(t_1) \\ \vdots & & \vdots \\ -B_{n1}(t_1) & \ldots & -B_{nn}(t_1) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \end{bmatrix} + \\
&\quad \begin{bmatrix} B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots \\ B_{n1}(t_2) & \ldots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix} \\
&= \begin{bmatrix} -B_{11}(t_1) & \cdots & -B_{1n}(t_1) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -B_{n1}(t_1) & \cdots & -B_{nn}(t_1) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_{11}(t_2) & \cdots & B_{1n}(t_2) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & B_{n1}(t_2) & \cdots & B_{nn}(t_2) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \\ u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \\ \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} -B(t_1) & 0_n \\ 0_n & B(t_2) \end{bmatrix} \begin{bmatrix} u(t_1) \\ \vdots \\ u^{(n-1)}(t_1) \\ u(t_2) \\ \vdots \\ u^{(n-1)}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \bar{v}(t_1) \\ \vdots \\ \bar{v}^{(n-1)}(t_1) \\ \bar{v}(t_2) \\ \vdots \\ \bar{v}^{(n-1)}(t_2) \end{bmatrix}$$

$$=: \hat{B} \begin{bmatrix} \vec{u}(t_1) \\ \vec{u}(t_2) \end{bmatrix} \cdot \begin{bmatrix} \vec{v}(t_1) \\ \vec{v}(t_2) \end{bmatrix}.$$

Since $\det \hat{B} = (-1)^n \det B(t_1) \det B(t_2)$, $\hat{B}$ is nonsingular for $t_1, t_2 \in [a, b]$ (since $B(t)$ is nonsingular for $t \in [a, b]$, as shown before).

Why the $(-1)^n$?

1.1.2 Boundary form formula

Definition 1.1.11. Given any set of $2mn$ complex constants M_{ij}, N_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$, define m **boundary operators (boundary forms)** $U_1, \ldots, U_m$ for functions x on $[a, b]$, for which $x^{(j)}$ $(j = 1, \ldots, n-1)$ exists at a and b, by

$$U_i x = \sum_{j=1}^{n} (M_{ij} x^{(j-1)}(a) + N_{ij} x^{(j-1)}(b)) \quad (i = 1, \ldots, m) \tag{1.1.5}$$

U_i are **linearly independent** if the only set of complex constants $c_1, \ldots, c_m$ for which

$$\sum_{i=1}^{m} c_i U_i x = 0$$

for all $x \in C^{n-1}$ on $[a, b]$ is $c_1 = c_2 = \cdots = c_m = 0$.

Remark 1.1.12. Note that for $\alpha, \beta \in \mathbb{C}$ and $x_1, x_2 \in C^{n-1}$ on $[a, b]$,

$$\begin{aligned} U_i(\alpha x_1 + \beta x_2) &= \sum_{j=1}^{n} (M_{ij}(\alpha x_1 + \beta x_2)^{(j-1)}(a) + N_{ij}(\alpha x_1 + \beta x_2)^{(j-1)}(b)) \\ &= \alpha \sum_{j=1}^{n} (M_{ij} x_1^{(j-1)}(a) + N_{ij} x_1^{(j-1)}(b)) + \beta \sum_{j=1}^{n} (M_{ij} x_2^{(j-1)}(a) + N_{ij} x_2^{(j-1)}(b)) \quad \text{(by linearity of derivatives)} \\ &= \alpha U_i x_1 + \beta U_i x_2. \end{aligned}$$

So U_i are linear operators.

Remark 1.1.13. To describe (1.1.5) with matrices, define

$$\xi := \begin{bmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{bmatrix}; \quad U := \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}; \quad M := \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix}; \quad N := \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix}.$$

Then (1.1.5) can be written as

$$Ux = M\xi(a) + N\xi(b).$$

Indeed:

$$M\xi(a) + N\xi(b) = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix} \begin{bmatrix} x(a) \\ x'(a) \\ \vdots \\ x^{(n-1)}(a) \end{bmatrix} + \begin{bmatrix} N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots \\ N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} x(b) \\ x'(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^n M_{1j}x^{(j-1)}(a) \\ \vdots \\ \sum_{j=1}^n M_{mj}x^{(j-1)}(a) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n N_{1j}x^{(j-1)}(b) \\ \vdots \\ \sum_{j=1}^n N_{mj}x^{(j-1)}(b) \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{j=1}^n (M_{1j}x^{(j-1)}(a) + N_{1j}x^{(j-1)}(b)) \\ \vdots \\ \sum_{j=1}^n (M_{mj}x^{(j-1)}(a) + N_{mj}x^{(j-1)}(b)) \end{bmatrix}$$
$$= \begin{bmatrix} U_1 x \\ \vdots \\ U_m x \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix} x = Ux.$$

Define the $m \times 2n$ matrix

$$(M : N) := \begin{bmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{bmatrix}.$$

Then $U_1, \dots, U_m$ are linearly independent if and only if $\operatorname{rank}(M : N) = m$, or equivalently, $\operatorname{rank}(U) = m$. Recall that the rank of a matrix is the largest number of linearly independent rows or columns in it. For a matrix $A_{m\times n}$, $\operatorname{rank}(A) \le \min\{m, n\}$ and $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

Ux can be written as

$$Ux = \begin{bmatrix} \sum_{j=1}^n (M_{1j}x^{(j-1)}(a) + N_{1j}x^{(j-1)}(b)) \\ \vdots \\ \sum_{j=1}^n (M_{mj}x^{(j-1)}(a) + N_{mj}x^{(j-1)}(b)) \end{bmatrix}$$
$$= \begin{bmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{bmatrix} \begin{bmatrix} x(a) \\ \vdots \\ x^{(n-a)}(a) \\ x(b) \\ \vdots \\ x^{(n-1)}(b) \end{bmatrix}$$
$$= (M : N) \begin{bmatrix} \xi(a) \\ \xi(b) \end{bmatrix}.$$

Definition 1.1.14. If $U = (U_1, \dots, U_m)$ is any boundary form with $\operatorname{rank}(U) = m$ and $U_c = (U_{m+1}, \dots, U_{2n})$ any form with $\operatorname{rank}(U_c) = 2n - m$ such that $(U_1, \dots, U_{2n})$ has rank $2n$, then U and U_c are **complementary boundary forms**. "Adjoining" $U_{m+1}, \dots, U_{2n}$ to $U_1, \dots, U_m$ is equivalent to imbedding the matrix $(M : N)$ in a $2n \times 2n$ nonsingular matrix (recall that for square matrices, nonsingular $\iff$ full rank).

We wish to describe the right hand side of Green's formula (1.1.2) as a linear combination of a boundary form U and a complementary form U_c. To do so, we consider the following results about the semibilinear form (1.1.4).

Definition 1.1.15. For a matrix $A = (a_{ij})$, its **adjoint** is defined as the conjugate transpose $A^* = (\bar{a}_{ij})$.

Proposition 1.1.16. *In the context of the semibilinear form* (1.1.4), *we have*

$$Sf \cdot g = f \cdot S^* g. \tag{1.1.6}$$

Proof.

$$
\begin{aligned}
Sf \cdot g &= \sum_{i,j=1}^{k} s_{ij} f_j \bar{g}_i \quad \text{(by (1.1.4))};\\
f \cdot S^* g &= \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} \bar{s}_{11} & \cdots & \bar{s}_{k1} \\ \vdots & & \vdots \\ \bar{s}_{1k} & \cdots & \bar{s}_{kk} \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}\\
&= \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix} \cdot \begin{bmatrix} \sum_{j=1}^{k} \bar{s}_{j1} g_j \\ \vdots \\ \sum_{j=1}^{k} \bar{s}_{jk} g_j \end{bmatrix}\\
&= \sum_{i=1}^{k} f_i \cdot \overline{\left(\sum_{j=1}^{k} \bar{s}_{ji} g_j \right)}\\
&= \sum_{i=1}^{k} f_i \cdot \left(\sum_{j=1}^{k} s_{ji} \bar{g}_j \right)\\
&= \sum_{i,j=1}^{k} s_{ji} f_i \bar{g}_j = Sf \cdot g.
\end{aligned}
$$

$\square$

Proposition 1.1.17. *Let $\mathcal{S}$ be the semibilinear form associated with a nonsingular matrix S. Suppose $\bar{f} := Ff$ where F is a nonsingular matrix. Then there exists a unique nonsingular matrix G such that if $\bar{g} = Gg$, then $\mathcal{S}(f, g) = \bar{f} \cdot \bar{g}$ for all f, g.*

Proof. Let $G := (SF^{-1})^*$, then

$$
\begin{aligned}
\mathcal{S}(f,g) &= Sf \cdot g\\
&= S(F^{-1}F)f \cdot g\\
&= SF^{-1}(Ff) \cdot g\\
&= SF^{-1}\bar{f} \cdot g\\
&= \bar{f} \cdot (SF^{-1})^* g \quad \text{(by (1.1.6))}\\
&= \bar{f} \cdot G * g\\
&= \bar{f} \cdot \bar{g}.
\end{aligned}
$$

To see that G is nonsingular, note that $\det G = \det((\overline{SF^{-1}})^T) = \det(\overline{SF^{-1}}) = \overline{\det(SF^{-1})} = \overline{\det(S)\det(F)^{-1}} \neq 0$ since S, F are nonsingular. $\square$

Proposition 1.1.18. *Suppose $\mathcal{S}$ is associated with the unit matrix E, i.e., $\mathcal{S}(f, g) = f \cdot g$. Let F be a nonsingular matrix such that the first j $(1 \leq j < k)$ components of $\bar{f} = Ff$ are the same as those of f. Then the unique nonsingular matrix G such that $\bar{g} = Gg$ and $\bar{f} \cdot \bar{g} = f \cdot g$ (as in Proposition 1.1.17) is such that the last $k - j$ components of $\bar{g}$ are linear combinations of the last $k - j$ components of g with nonsingular coefficient matrix.*

Proof. We note that for the condition on F to hold, F must have the form

$$\begin{bmatrix} E_j & 0_+ \\ F_+ & F_{k-j} \end{bmatrix}_{k\times k}$$

where E_j is the $j\times j$ identity matrix, 0_+ is the $j\times(k-j)$ zero matrix, F_+ is a $(k-j)\times j$ matrix, and F_{k-j} a $(k-j)\times(k-j)$ matrix. Let G be the unique nonsingular matrix in Proposition 1.1.17. Write G as

$$\begin{bmatrix} G_j & G_- \\ G_= & G_{k-j} \end{bmatrix}_{k\times k}$$

where $G_j, G_-, G_=, G_{k-j}$ are $j\times j, j\times(k-j), (k-j)\times j, (k-j)\times(k-j)$ matrices, respectively. By the definition of G,

$$f\cdot g = Ff\cdot Gg = \bar{f}\cdot Gg = G^*\bar{f}\cdot g = G^*Ff\cdot g,$$

(where the third equality follows from a reverse application of (1.1.6) with $\bar{f}$ as f, G^* as S) which implies

$$G^*F = E_k.$$

Since

$$\begin{aligned} G^*F &= \begin{bmatrix} G_j^* & G_=^* \\ G_-^* & G_{k-j}^* \end{bmatrix}\begin{bmatrix} E_j & 0_+ \\ F_+ & F_{k-j} \end{bmatrix} \\ &= \begin{bmatrix} G_j^* + G_=^*F_+ & G_=^*F_{k-j} \\ G_-^* + G_{k-j}^*F_+ & G_{k-j}^*F_{k-j} \end{bmatrix} \\ &= \begin{bmatrix} E_j & 0_{j\times(k-j)} \\ 0_{(k-j)\times j} & E_{k-j} \end{bmatrix}. \end{aligned}$$

Thus, $G_=^*F_{k-j} = 0_+$, the $j\times(k-j)$ zero matrix. But $\det F = \det(E_j)\cdot\det(F_{k-j}) \neq 0$, so $\det F_{k-j}\neq 0$ and we must have $G_=^* = 0_+$, i.e., $G_= = 0_{(k-j)\times j}$. Thus, G is upper-triangular, and so $\det G = \det G_j \cdot \det G_{k-j} \neq 0$, which implies $\det G_{k-j}\neq 0$ and G_{k-j} is nonsingular. Hence,

$$\bar{g} = Gg = \begin{bmatrix} G_j & G_- \\ 0_{(k-j)\times j} & G_{k-j} \end{bmatrix}\begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}$$

where G_{k-j} is the nonsingular coefficient matrix such that

$$\begin{bmatrix} \bar{g}_{j-1} \\ \vdots \\ \bar{g}_k \end{bmatrix} = G_{k-j}\begin{bmatrix} g_{j-1} \\ \vdots \\ g_k \end{bmatrix}.$$

□

Theorem 1.1.19. *(Boundary-form formula) Given any boundary form U of rank m (Definition 1.1.11), and any complementary form U_c (Definition 1.1.14), there exist unique boundary forms U_c^+, U^+ of rank m and $2n-m$, respectively, such that*

$$[xy](b) - [xy](a) = Ux\cdot U_c^+y + U_cx\cdot U^+y. \tag{1.1.7}$$

If $\tilde{U}_c$ is any other complementary form to U, and $\tilde{U}_c^+, \tilde{U}^+$ the corresponding forms of rank m and $2n-m$, then

$$\tilde{U}^+y = C^*U^+y$$

for some nonsingular matrix C.

Proof. □