Invited article to appear in the special issue
for the 25th year of Inverse Problems

Lax Pairs: A Novel Type of Separability

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June 2009

Dedicated to V. E. Zakharov on the occasion of his 70th birthday, for his seminal contributions to the field of Integrability.

Abstract

An attempt is made to place into historical context the fundamental concept of Lax pairs. For economy of presentation, emphasis is placed on the effectiveness of Lax pairs for the analysis of integrable nonlinear evolution PDEs. It is argued that Lax pairs provide a deeper type of separability than the classical separation of variables. Indeed, it is shown that: (a) The solution of the Cauchy problem of evolution equations is based on the derivation of a nonlinear Fourier transform pair, and this is achieved by employing the spectral analysis of *one* of the two eigenvalue equations forming a Lax pair; thus, although this methodology still follows the reverent philosophy of the classical separation of variables - transform methods, it can be applied to a class of *nonlinear* PDEs. (b) The solution of initial - boundary value problems of evolution equations is based on the simultaneous spectral analysis of *both* equations forming a Lax pair and hence, in a sense, it employs the synthesis instead of the separation of variables; this methodology does not have a direct classical analogue, however, it can be considered as the nonlinearisation of a method which combines the Green's functions classical integral representations with an analogue of the method of images, but which are now formulated in the spectral (Fourier) instead of the physical space.

In addition to presenting a general methodology for analysing initial and initial - boundary value problems for nonlinear integrable evolution equations in one and two spatial variables, recent progress is reviewed for the derivation and the solution of integrable nonlinear evolution PDEs formulated in higher than two spatial dimensions.

Introduction

$$q(x,t): \qquad iq_t + q_{xx} + 2\lambda|q|^2 q = 0, \quad \lambda = \pm 1,\ x \in \mathbb{R},\ t > 0.$$

The associated Lax pair consists of the following two linear equations satisfied by the 2×2-matrix valued function $M(x,t,k)$:

$$M_x + ik[\sigma_3, M] = QM, \quad \sigma_3 = \mathrm{diag}(1,-1), \tag{1a}$$

$$M_t + 2ik^2[\sigma_3, M] = \tilde{Q}M, \quad k \in \mathbb{C}, \tag{1b}$$

where [,] denotes the usual matrix commutation,

$$Q = \begin{pmatrix} 0 & q \\ \lambda\bar{q} & 0 \end{pmatrix}, \quad \tilde{Q} = 2kQ - iQ_x\sigma_3 - \lambda i|q|^2\sigma_3.$$

IVPs: *Spectral Analysis* of (1a) $\Rightarrow$ Nonlinear Fourier Transform pair via the solution of a Riemann - Hilbert problem.

BVPs: *Simultaneous spectral analysis* of (1a) and (1b).

From Nonlinear to Linear

$u(x,t):$ $$iu_t + u_{xx} = 0$$

The associated Lax pair consists of the following two linear equations satisfied by the scalar function $\mu(x,t,k)$:

$$\mu_x + ik\mu = u \tag{2a}$$

$$\mu_t + ik^2\mu = iu_x + ku, \;\; k \in \mathbb{C} \tag{2b}$$

Compare with the classical separation of variables

$$u(x,t) = X(x;k)T(t;k).$$

$$X^{''} + k^2 X = 0 \tag{3a}$$

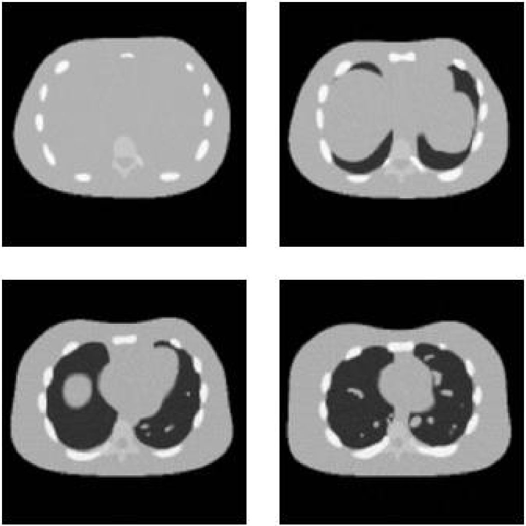
$$T^{'} - ik^2 T = 0 \tag{3b}$$

IVP: Spectral Analysis of (2a) or spectral analysis of (3a) yields the Fourier Transform pair, thus **new way of deriving transforms**.

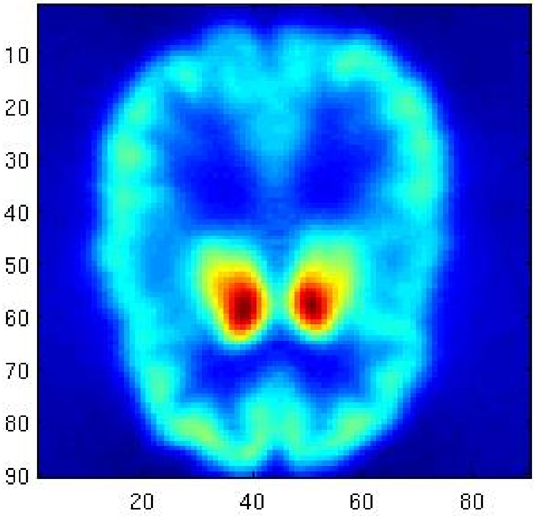
BVPs: **New method for solving BVPs.**

I. PET - SPECT

CT:



PET-SPECT:



F-Gelfand (1992) : Novel derivation of 2D FT via $\bar{\partial}$

$$(\partial_{x_1} + i\partial_{x_2} - k)\,\mu(x_1, x_2, k) = q(x_1, x_2)$$

F-Novikov (1992) : Novel derivation of Radon transform

$$\left[\frac{1}{2}\left(k + \frac{1}{k}\right)\partial_{x_1} + \frac{1}{2\mathrm{i}}\left(k - \frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1, x_2, k) = f(x_1, x_2)$$

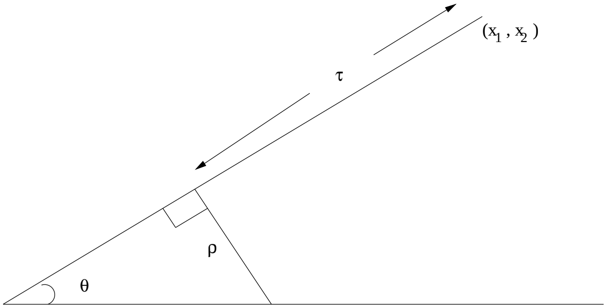
Novikov (2002) : Derivation of attenuated Radon transform

$$\begin{aligned}\left[\frac{1}{2}\left(k + \frac{1}{k}\right)\partial_{x_1} + \frac{1}{2\mathrm{i}}\left(k - \frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1, x_2, k)\\ + f(x_1, x_2)\mu(x_1, x_2, k) = g(x_1, x_2)\end{aligned}$$

F-Iserles-Marinakis (2006) : *Rederivation via Radon and Novel Numerical Implementation*
F-Hutton-Kacperski (2008) : *Modeling of Collimator Response in Analytical SPECT Image Reconstruction*
Barbano-F-Kastis (2009) : *Analytical Reconstructions for PET and SPECT Employing L_1 Denoising*

I.1 RADON TRANSFORM

Reconstruct f from its line integrals.



$$x_1 = \tau\cos\theta - \rho\sin\theta \qquad \tau = x_1\cos\theta + x_2\sin\theta$$
$$\leftrightarrow$$
$$x_2 = \tau\sin\theta + \rho\cos\theta \qquad \rho = -x_1\sin\theta + x_2\cos\theta$$

$$F(\tau,\rho,\theta) = f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta).$$

Direct Radon transform

$$\hat{f}(\rho,\theta) = \int_{-\infty}^{\infty} F(\tau,\rho,\theta)\mathrm{d}\tau.$$

Inverse Radon transform

$$H(\theta,\rho) = \frac{1}{i\pi} \oint_{-\infty}^{\infty} \frac{\hat{f}(\rho',\theta)}{\rho'-\rho}\mathrm{d}\rho',$$

$$f(x_1,x_2) = \frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2})\int_0^{2\pi} e^{i\theta} H(\theta, -x_1\sin\theta + x_2\cos\theta)\mathrm{d}\theta.$$

I.2 MATHEMATICS OF SPECT

Reconstruct g from its weighted line integrals

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

Direct Attenuated Radon transform

$$\hat{g}_f(\rho,\theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty F(s,\rho,\theta)ds} G(\tau,\rho,\theta) d\tau$$

Inverse Attenuated Radon transform

$$P^{\pm} g(\rho) = \pm \frac{g(\rho)}{2} + \frac{1}{2i\pi} \oint_{-\infty}^{\infty} \frac{g(\rho')}{\rho' - \rho} d\rho'$$

$$H(\theta,\tau,\rho)=e^{\int_\tau^\infty F(s,\rho,\theta)ds}\Big\{e^{P^-\hat{f}(\rho,\theta)}P^-e^{-P^-\hat{f}(\rho,\theta)}+$$
$$+e^{-P^+\hat{f}(\rho,\theta)}P^+e^{P^+\hat{f}(\rho,\theta)}\Big\}\,\hat{g}_f(\rho,\theta),$$

$$g(x_1,x_2)=\frac{1}{4\pi}(\partial_{x_1}-i\partial_{x_2})\cdot$$
$$\cdot\int_0^{2\pi}e^{i\theta}H(\theta,x_1\cos\theta+x_2\sin\theta,x_2\cos\theta-x_1\sin\theta)d\theta.$$

Spectral analysis of a SINGLE equation$\to$ Analytic inversion of integrals

$$\left\{\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2i}\left(k-\frac{1}{k}\right)\partial_{x_2}\right\}\mu(x_1,x_2,k)=f(x_1,x_2)$$

$$k\in\mathbb{C},\quad (x_1,x_2)\in\mathbb{R}^2,\quad f(x)\in S(\mathbb{R}^2)$$

(i) Solve for μ in terms of f for ALL $k\in\mathbb{C}$

$$z\doteqdot\frac{1}{2i}\left(k-\frac{1}{k}\right)x_1-\frac{1}{2}\left(k+\frac{1}{k}\right)x_2$$

$$\frac{1}{2i}\left(\frac{1}{|k|^2}-|k|^2\right)\frac{\partial\mu}{\partial\bar{z}}=f(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}^2$$

Impose: $\mu=O\left(\frac{1}{z}\right),\quad z\to\infty.$

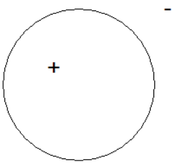
$$\boxed{\mu(x_1,x_2,k)=\frac{1}{2\pi i}\mathrm{sgn}\left(\frac{1}{|k|^2}-|k|^2\right)\iint_{\mathbb{R}^2}\frac{f(x_1',x_2')dx_1'dx_2'}{z'-z},\ \ |k|\neq 1}$$

(ii) Solve for μ in terms of $\hat{f}$

$$\mu^{\pm}=\mp\left(P^{\mp}\hat{f}\right)-\int_{\tau}^{\infty}F(\rho,s,\theta)ds,\quad (\rho,\tau)\in\mathbb{R}^2,\quad \theta\in(0,2\pi)$$

$$\boxed{\mu(x_1,x_2,k)=-\frac{1}{2i\pi^2}\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-k}\left(\oint_{-\infty}^{\infty}\frac{\hat{f}(\rho,\theta)\mathrm{d}\rho}{\rho-(x_2\cos\theta-x_1\sin\theta)}\right)\mathrm{d}\theta}$$

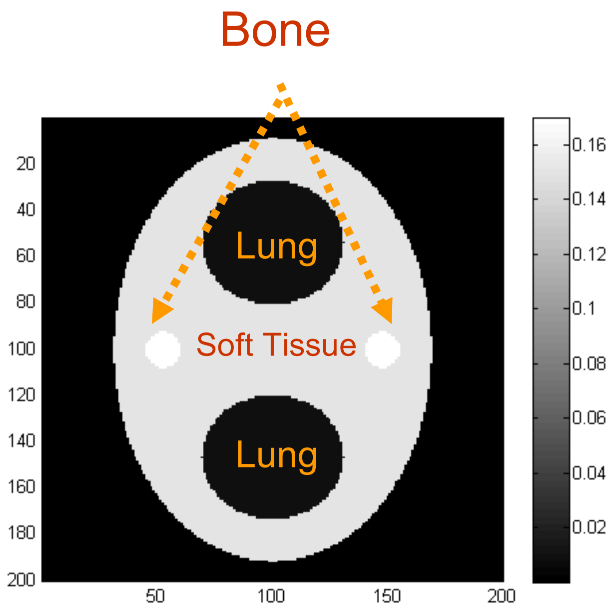
complex k-plane :



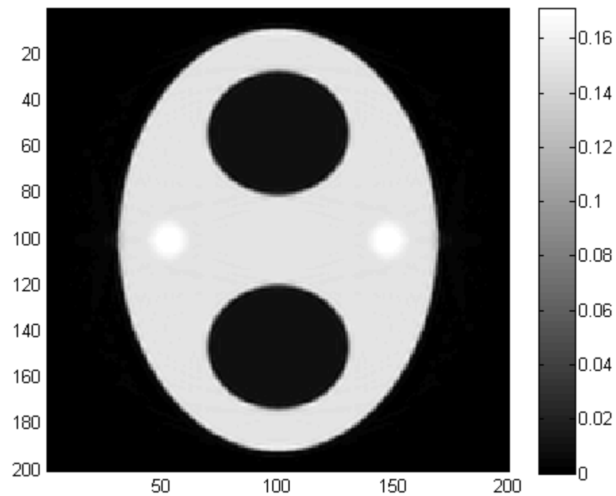
E. Miqueles and A.R. De Pierro (2009) : *X-Ray Fluorescence Tomography*

Lung Phantom *(360 projections)*

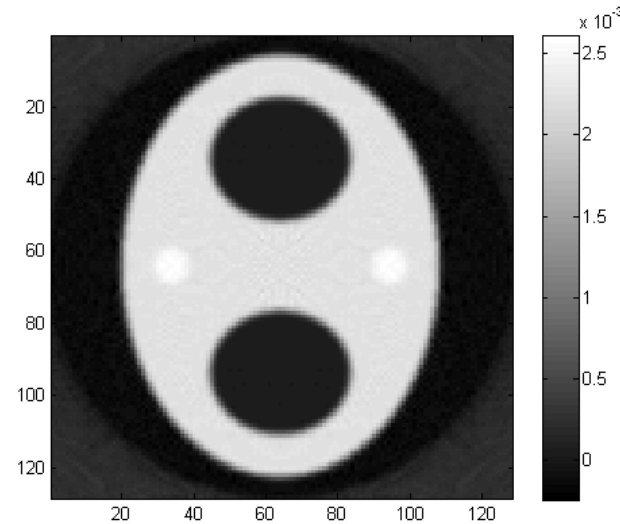
Simulated Data



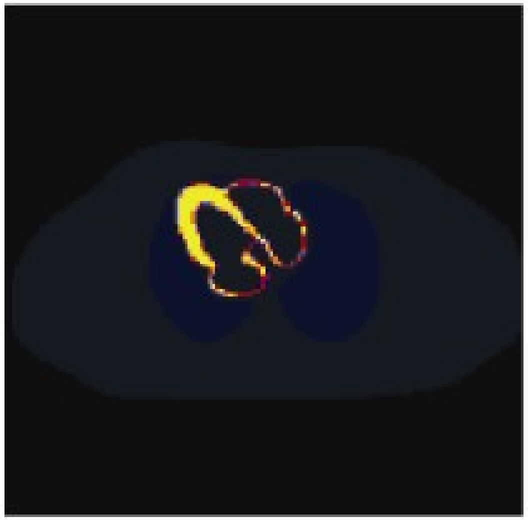
Digital Phantom



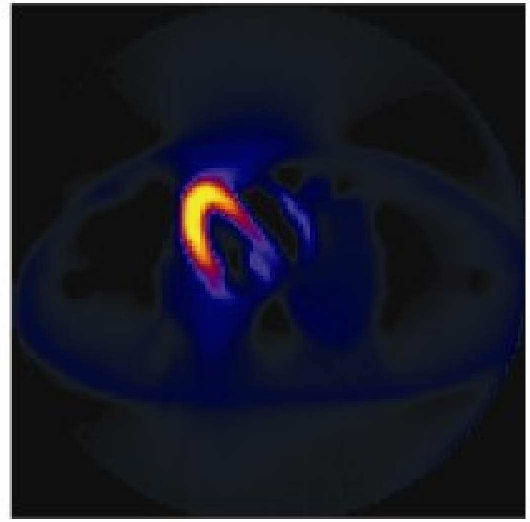
Splines Expansion



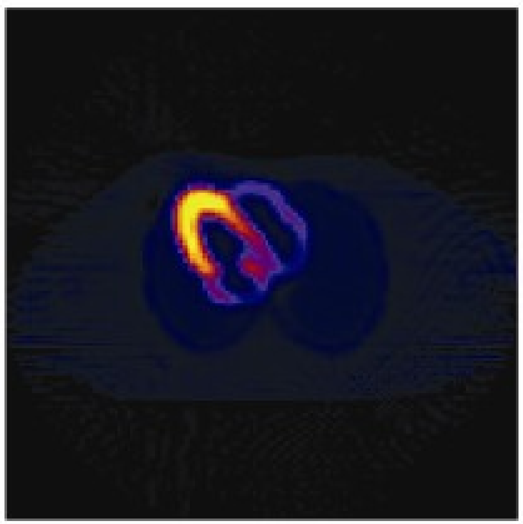
Filtered-Back-Projection



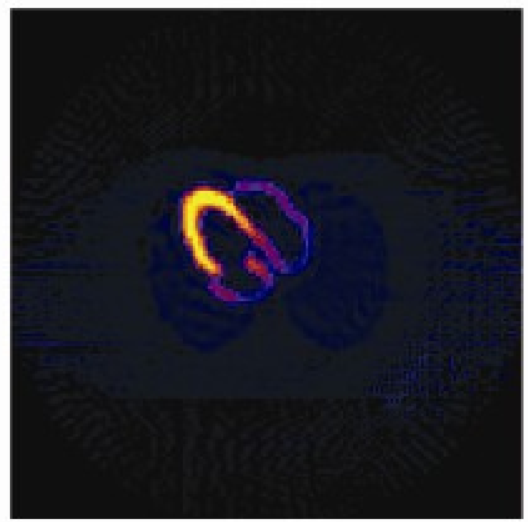
(a)



(b)



(c)



(d)

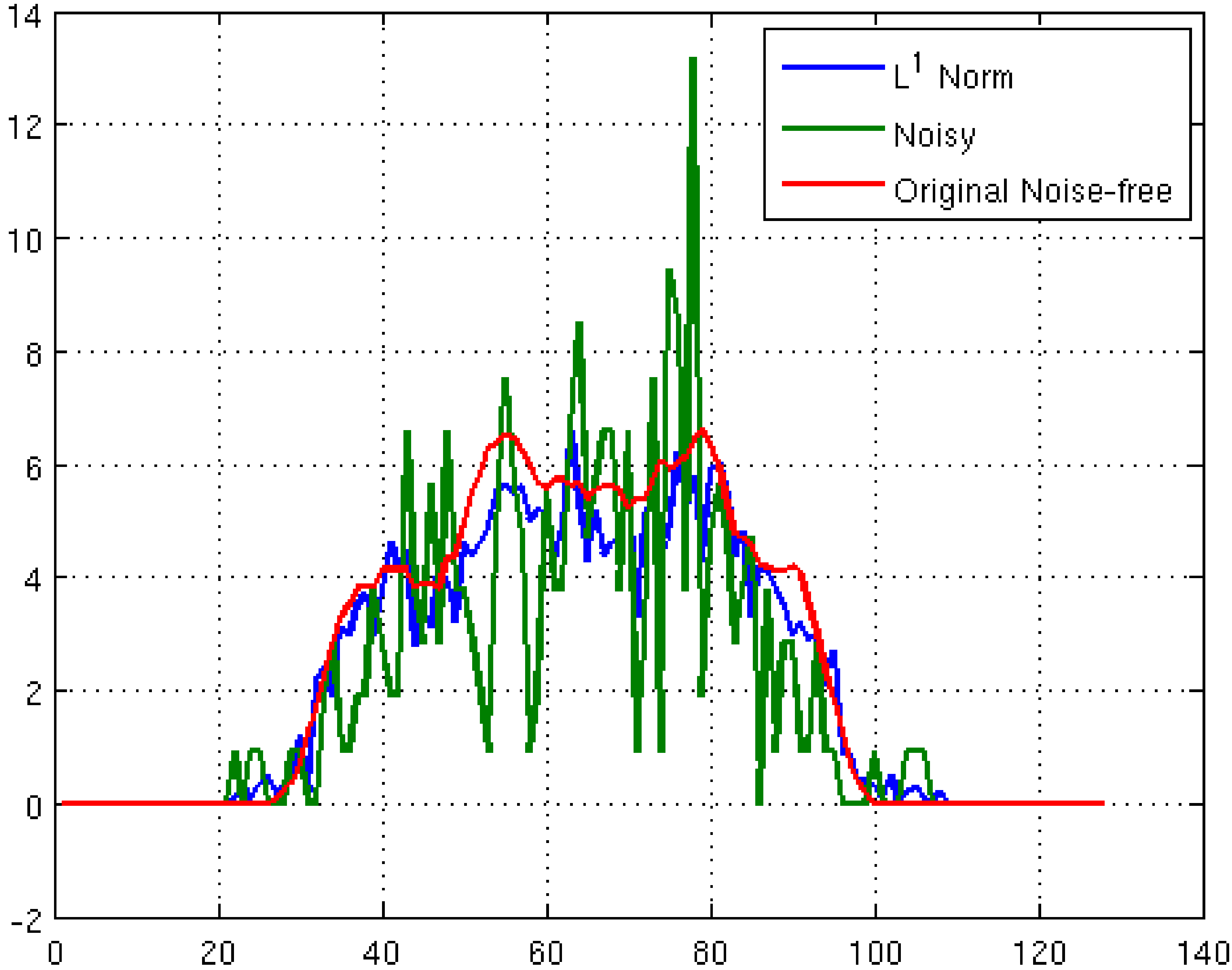
In the following Slides, we illustrate by means of concrete examples, how the L1-based de-noising techniques produce an good approximation of the original noise-less emission count.
The approximation is constructed by solving, for some choice of a non-smooth norm $|| \cdot ||_{N}$:

(1) $$f^* = \text{argmin} \left(|| f ||_N \;\middle|\; M(f) = y \right).$$

It is important to note here that the quality of the approximation in (1) is good even when (like in the case of real data) the recording is corrupted by a significant level of "Noise" (= mixture of Poisson *and other* Stochastic Perturbations)

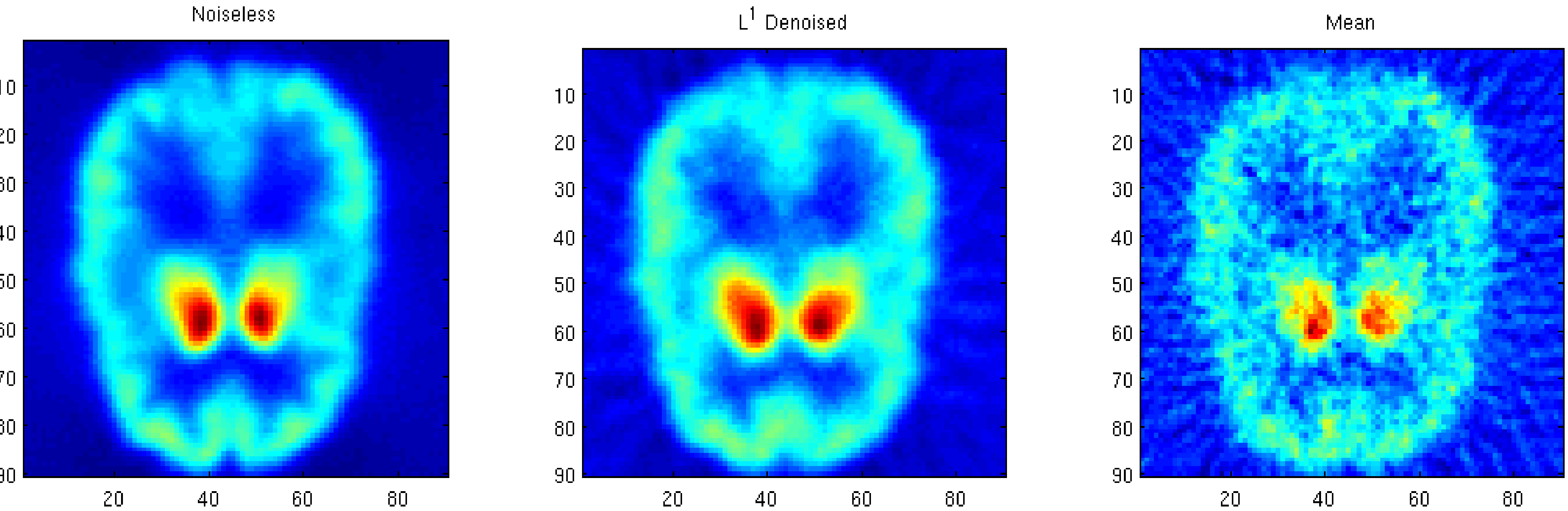
I – Simulated Data

For clarity purposes, we begin by displaying the 1D cross section of a 2D projection”slice”, where the original, simulated Brain Data has been corrupted by high levels of Poisson Noise.



L^1 Norm
Noisy
Original Noise-free
14
12
10
8
6
4
2
0
-2
0
20
40
60
80
100
120
140

L1-Based Reconstruction of synthetically generated data of a human brain shows the improvement over simple smoothing techniques. The picture below compares noise-free, L1-de-noised and smoothed average with direct FBP reconstruction.

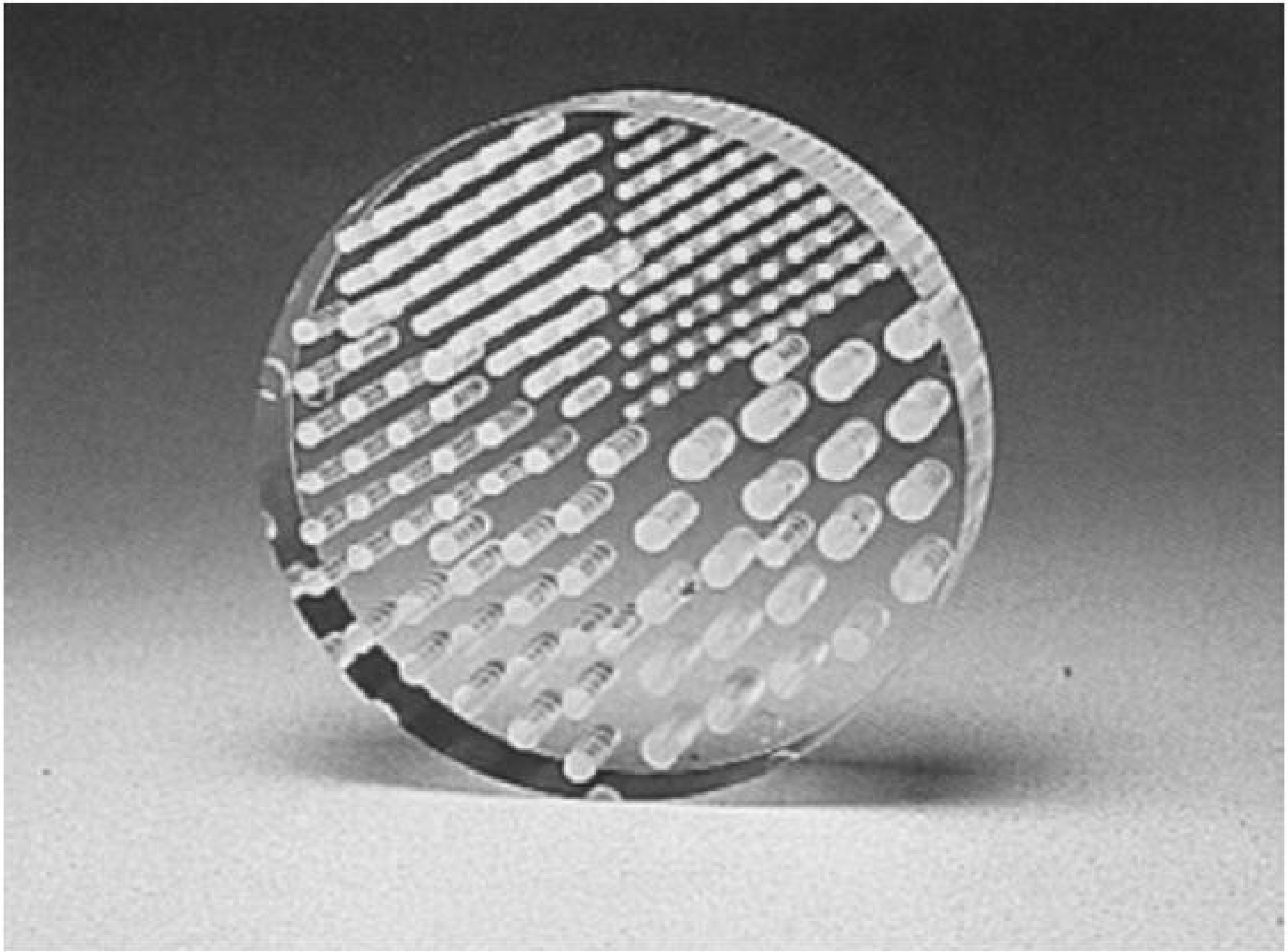


II – Real Data

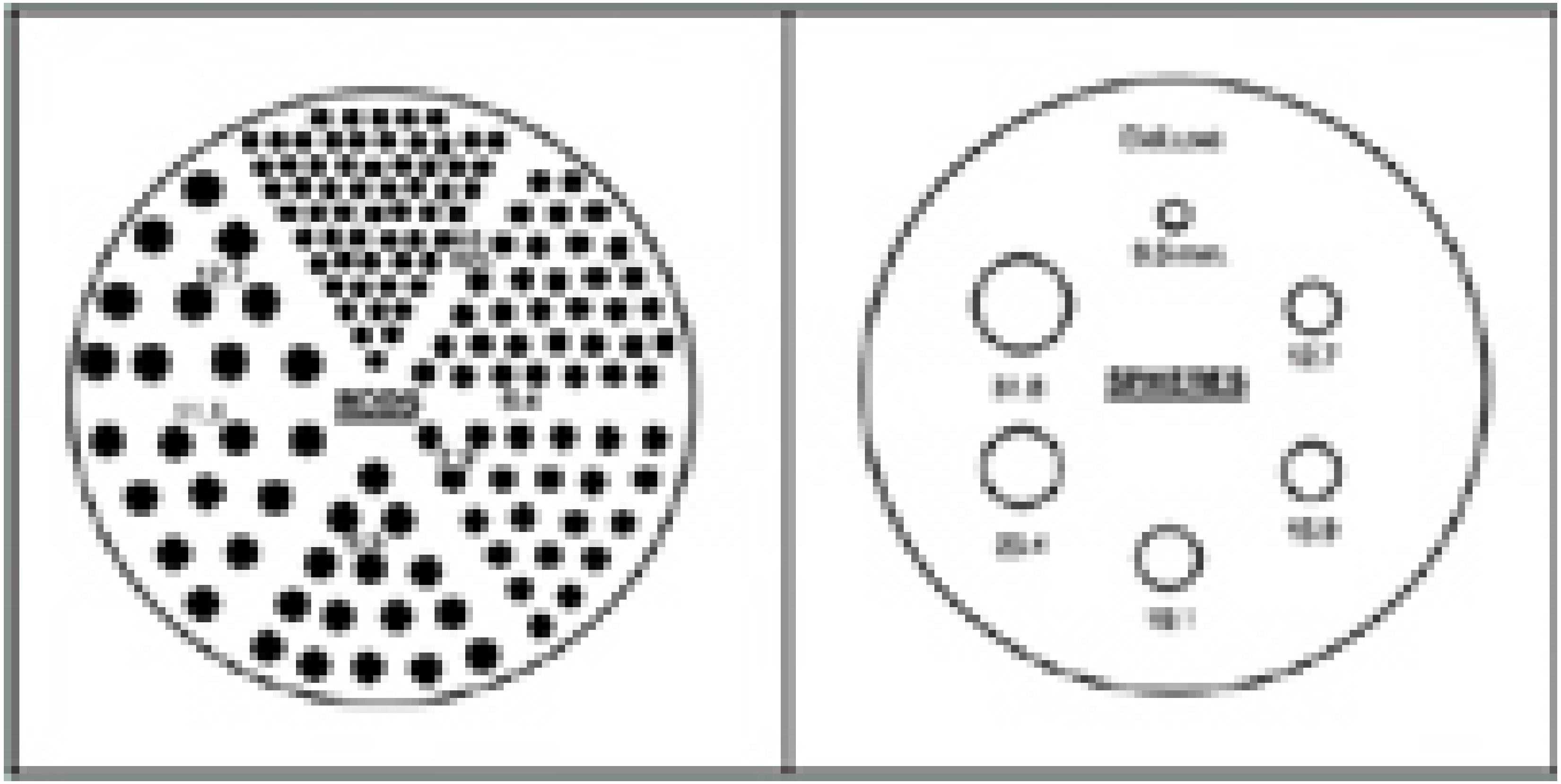
By implementing L1-based de-noising techniques in the experimental setting, we noticed that the impact on real data image reconstructions is far more significant than in the case of Computer generated Projections

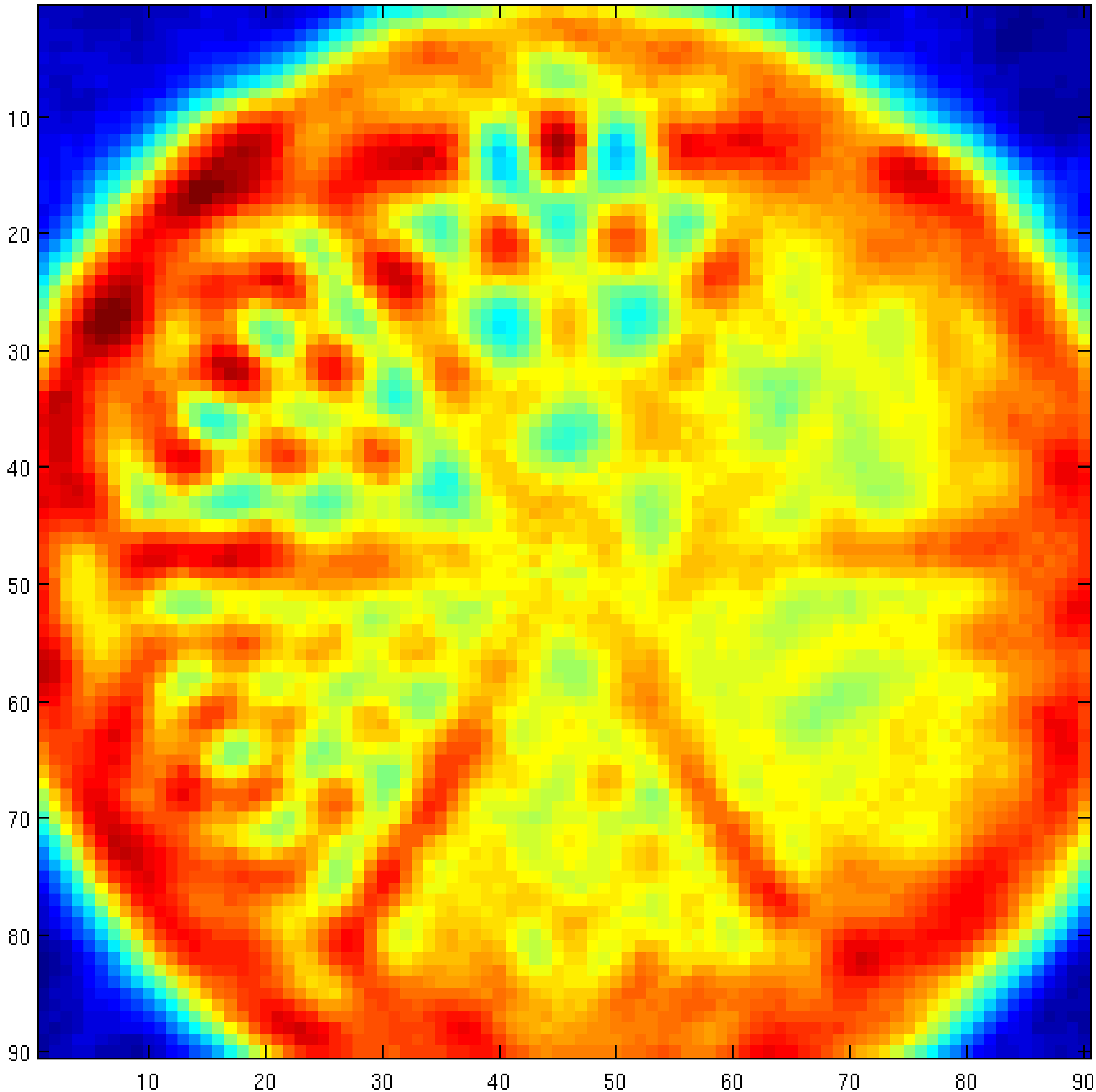
L1-Based Reconstructions or the real projections of the so-called “Jaszsek Phantom” Provide a measure of the real potential of Analytical Formulas, once the Clutter-related obstructions to their application are removed

Below, we display two of the horizontal sections of the Phantom we are interested in reconstructing



The sections provide a means to accurately estimate the precision of the reconstructions.





10
20
30
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70
80
90
10
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50
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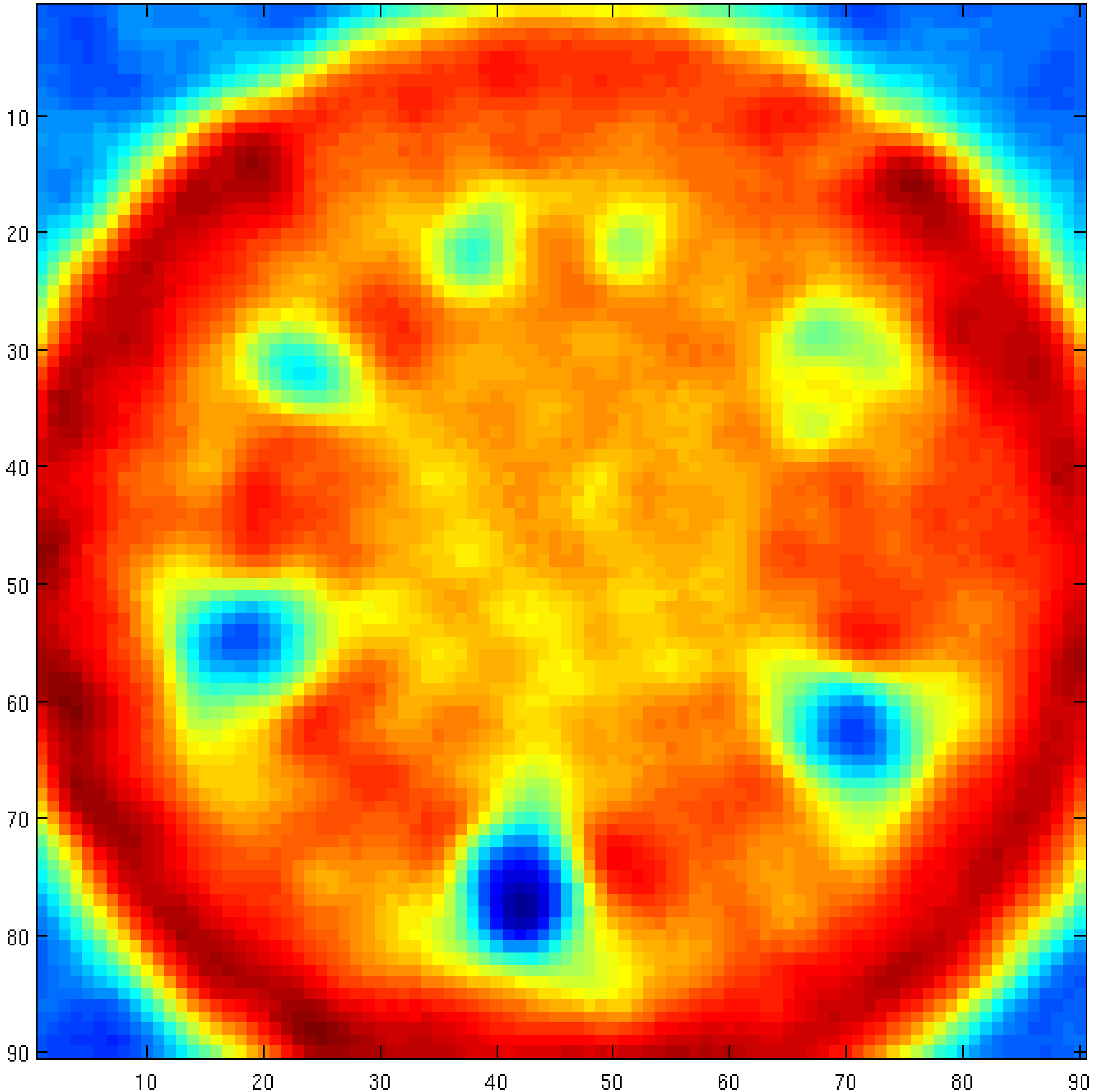
70
80
90


10
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30
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60
70
80
90
10
20
30
40
50
60
70
80
90

In the following four pictures we compare the RAW reconstruction of a real PET image, acquired with the usual techniques, as well as the filtered versions, in both cases, L1-denoised and Filtered.

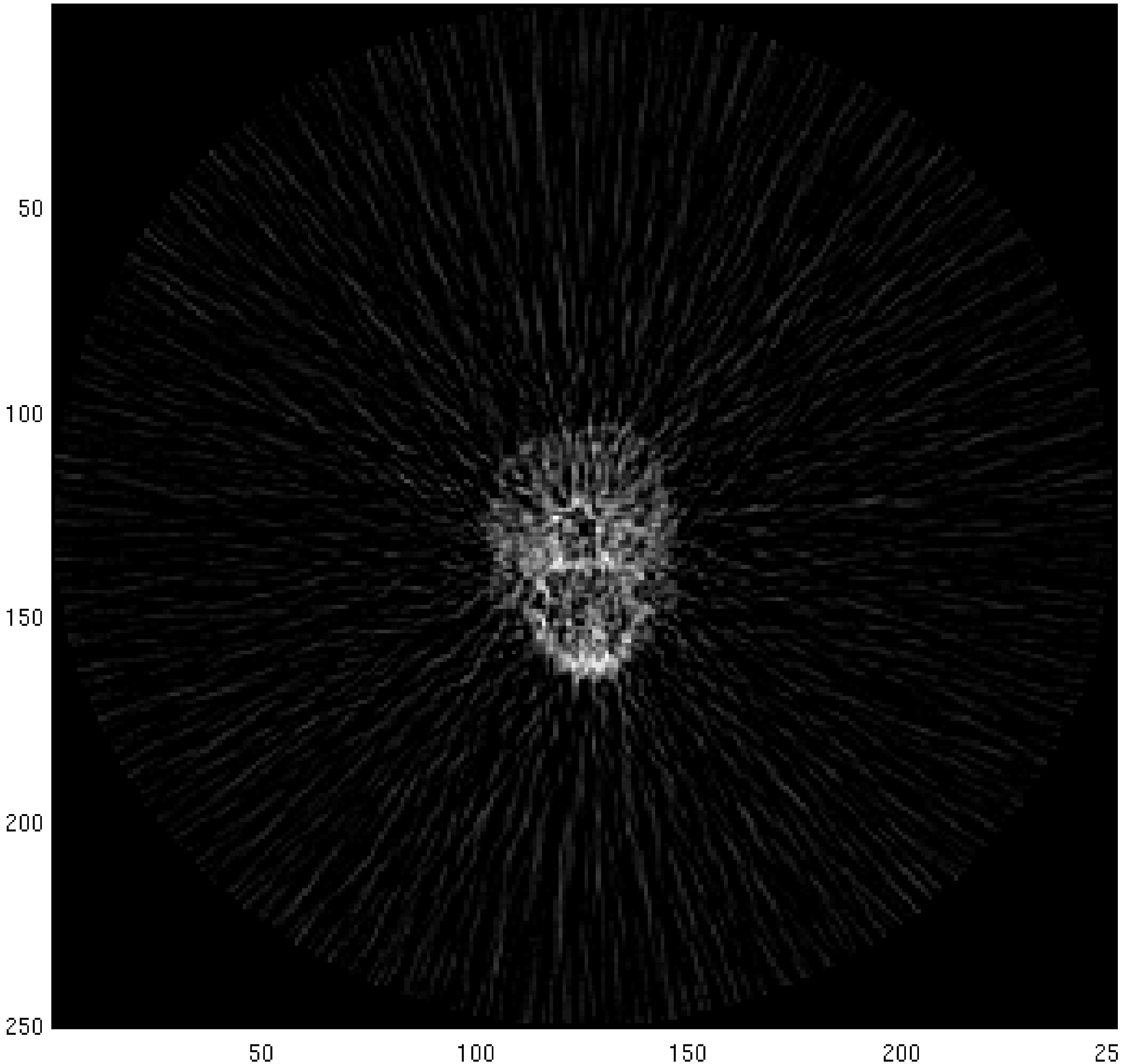
Figures 1 and 2:

RAW reconstruction first with then without L1 De-noising

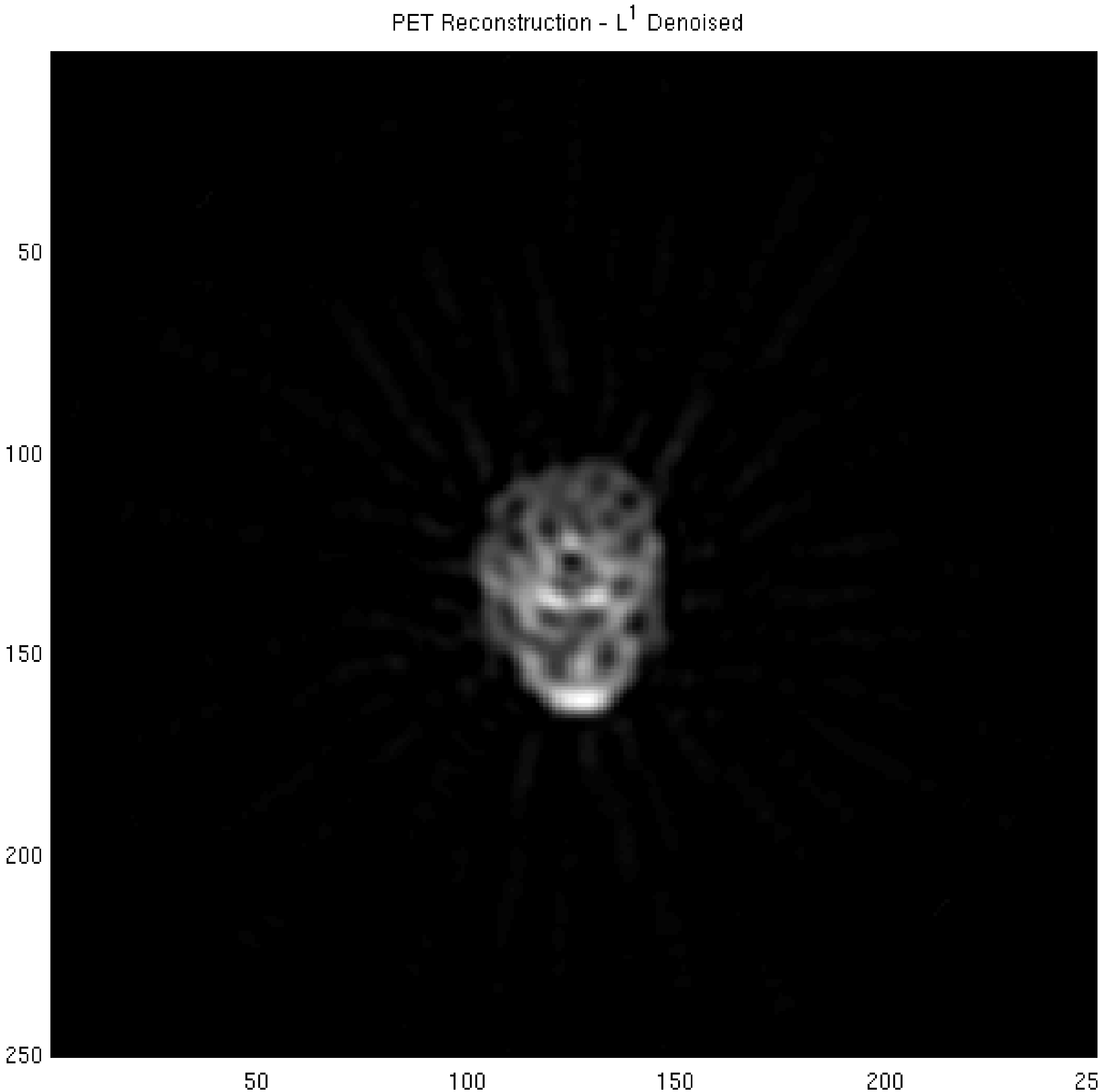
Figures 3 and 4:

FILTERED reconstruction first with then without L1 De-noising

1

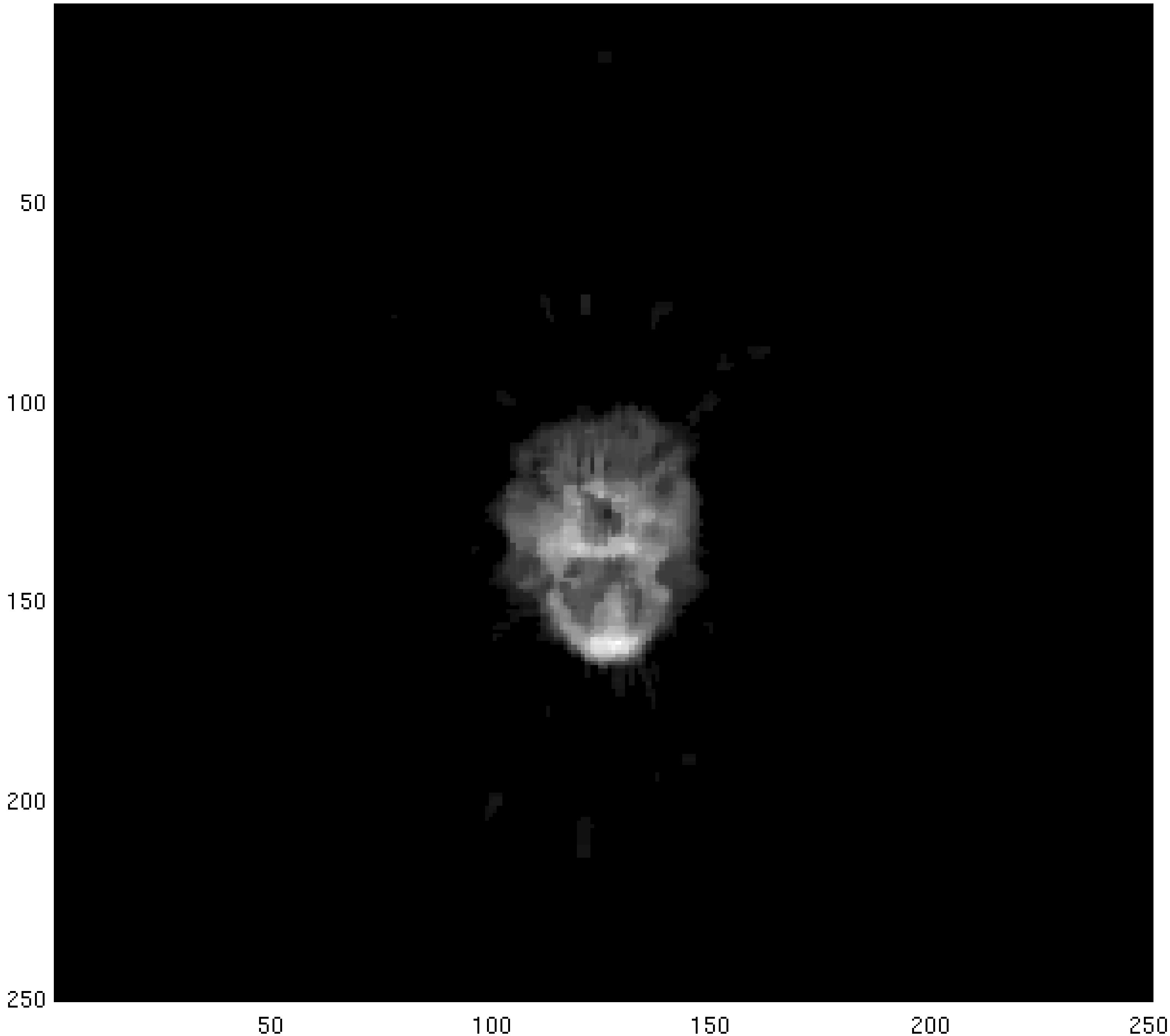


2

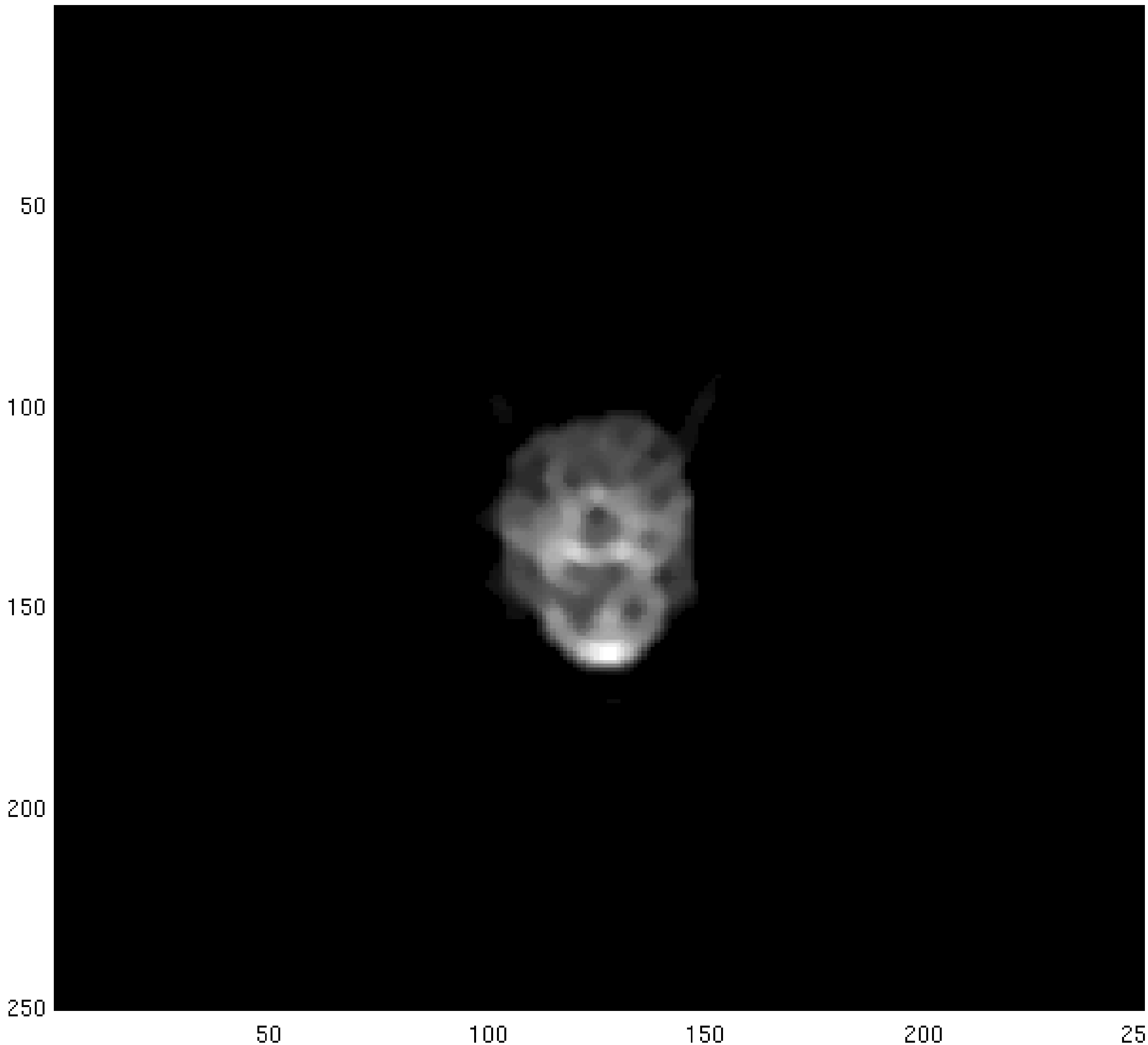


PET Reconstruction - L^1 Denoised
50
100
150
200
250
50
100
150
200
250

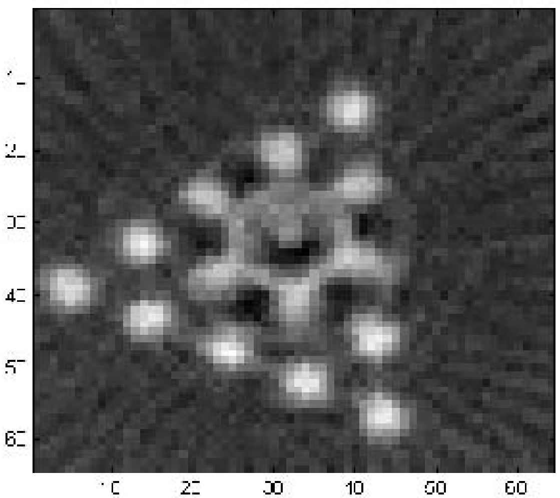
3



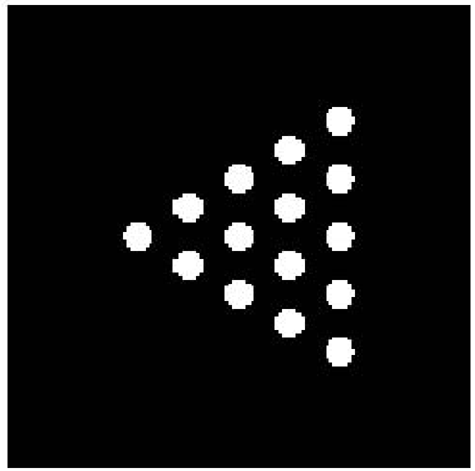
4



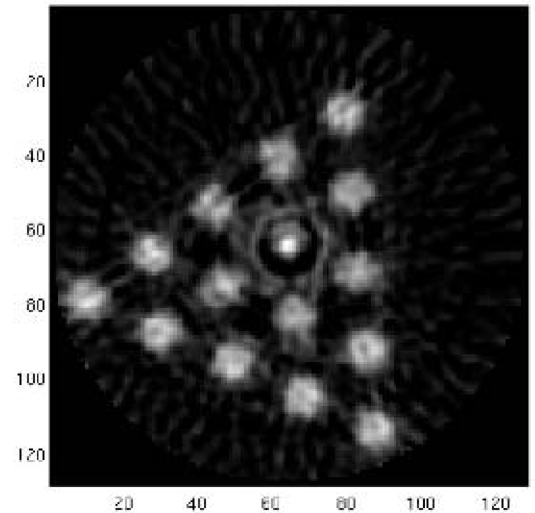
FBP vs. Cubic Spline *(60 projections)*
Real Data



Filtered-Back-Projection



Actual Image



Spline Reconstruction
L1 Denoising

A Unified Approach to Boundary Value Problems

Athanassios S. Fokas

CBMS-NSF Regional Conference Series 78

This book presents a new approach to analyzing initial-boundary value problems for integrable partial differential equations (PDEs) in two dimensions, a method that the author first introduced in 1997 and which is based on ideas of the inverse scattering transform. This method is unique in also yielding novel integral representations for the explicit solution of linear boundary value problems, which include such classical problems as the heat equation on a finite interval and the Helmholtz equation in the interior of an equilateral triangle.

The author's thorough Introduction allows the interested reader to quickly assimilate the essential results of the book, avoiding many computational details. Several new developments are addressed in the book, including:

- A new transform method for linear evolution equations on the half-line and on the finite interval
- Analytical inversion of certain integrals such as the attenuated radon transform and the Dirichlet-to-Neumann map for a moving boundary
- Analytical and numerical methods for elliptic PDEs in a convex polygon
- Integrable nonlinear PDEs

An epilogue provides a list of problems on which the author's new approach has been used, offers open problems, and gives a glimpse into how the method might be applied to problems in three dimensions.

Audience — This book is appropriate for courses in boundary value problems at the advanced undergraduate and first-year graduate levels. Applied mathematicians, engineers, theoretical physicists, mathematical biologists, and other scholars who use PDEs will also find the book valuable.

About the Author — *Athanassios S. Fokas* has the chair of Nonlinear Mathematical Sciences in the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge, UK. In 2000 he was awarded the Naylor Prize for his work on which this book is based. He is co-author or co-editor of nine additional books and author or co-author of over 200 papers.

Available August 2008 · Approx. x + 336 pages · Softcover · ISBN 978-0-898716-51-1
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Epilogue; Bibliography; Index

6/08

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II. Evolution PDEs (F, Proc. R. Soc. 1997)

Let $q(x,t)$ satisfy the linear evolution PDE

$$q_t + w(-i\partial_x)q = 0,$$

where $w(k)$ is a polynomial of degree n such that $\mathrm{Re}w(k) \geq 0$ for k real.

1.

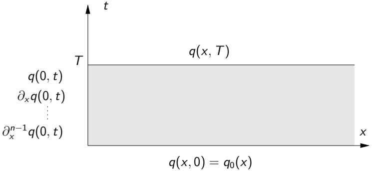
$$\left(e^{-ikx+w(k)t}q(x,t)\right)_t - \left(e^{-ikx+w(k)t}\sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t)\right)_x = 0, \quad k \in \mathbb{C}$$

2.

$$\int_{\partial D} e^{-ikx+w(k)t}\left[q(x,t)dx + \sum_{j=0}^{n-1} c_j(k)\partial_x^j q(x,t)dt\right] = 0$$

Half-Line

$$0 < x < \infty, \quad 0 < t < T$$



$$\hat{q}(k,T) = \int_0^\infty e^{-ikx} q(x,T)dx, \quad \mathrm{Im}k \le 0$$

$$\hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)dx, \quad \mathrm{Im}k \le 0$$

$$\tilde{g}_j(k) = \int_0^T e^{ks}\partial_x^j q(0,s)ds, \quad k \in \mathbb{C}, \quad j = 0, 1, \cdots, n-1$$

Global Relation

$$\hat{q}_0(k) - \sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(w(k)) = e^{w(k)T}\hat{q}(k,T), \quad \mathrm{Im}k \leq 0$$

Integral Representation

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\hat{q}_0(k)dk - \frac{1}{2\pi}\int_{\partial D+} e^{ikx-w(k)t}\tilde{g}(k)dk$$

$$D^+ = \{k \in \mathbb{C}, \quad \mathrm{Im}k > 0, \quad \mathrm{Re}w(k) < 0\}$$

$$\tilde{g}(k) = \sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(w(k))$$

Example 1: The Heat Equation

$$q_t = q_{xx}, \quad w(k) = k^2.$$

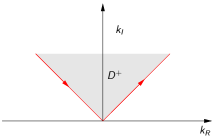
Global Relation

$$\hat{q}_0(k) - ik\tilde{g}_0(k^2) - \tilde{g}_1(k^2) = e^{k^2 T}\hat{q}(k, T), \quad \mathrm{Im} k \le 0$$

The Dirichlet Problem

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2 t}\left[\hat{q}_0(-k) + 2ikG_0(k^2)\right] dk,$$

$$G_0(k) = \int_0^T e^{ks} q(0,s) ds$$



Example 2: The First Stokes Equation

$$q_t + q_{xxx} + q_x = 0, \quad w(k) = ik - ik^3.$$

Global Relation

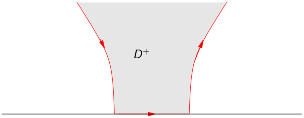
$$\hat{q}_0(k) - (k^2-1)\tilde{g}_0(w(k)) + ik\tilde{g}_1(w(k)) + \tilde{g}_2(w(k)) = e^{(ik-ik^3)T}\hat{q}(k,T), \quad \mathrm{Im}k \le 0$$

The Dirichlet Problem

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(ik-ik^3)t}\hat{q}_0(k) - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-(ik-ik^3)t}\tilde{g}(k)dk,$$

$$\tilde{g}(k) = \frac{1}{\nu_1 - \nu_2}\left[(\nu_1 - k)\hat{q}_0(\nu_2) + (k-\nu_2)\hat{q}_0(\nu_1)\right] + (3k^2-1)G_0(w(k)), \;\; k \in D^+$$

$$G_0(k) = \int_0^T e^{ks}q(0,s)ds$$



$$\nu_1, \nu_2: \quad w(k) = w(\nu k)$$

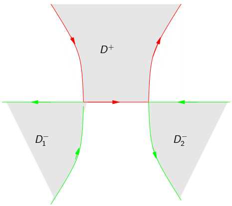
$$\nu_j^2 + k\nu_j + k^2 - 1 = 0, \quad j = 1, 2.$$

Remark: For the Heat equation, the classical sine transform yields

$$q(x,t) = \frac{2}{\pi} \int_0^\infty \sin kx \left[\hat{q}_s(k) - kG_0(k^2, t)\right] dk.$$

For the first Stokes equation, there does not exist such a formula.

Elimination of the Unknown Boundary Values



$$\tilde{g}_2 + ik\tilde{g}_1 = (k^2 - 1)\tilde{g}_0 - \hat{q}_0(k), \quad \mathrm{Im}\, k \leq 0$$

$$\omega(k) = \omega(\nu(k)), \quad \nu - \nu^3 = k - k^3, \quad \nu^2 + \nu k + k^2 = 1$$

$$\tilde{g}_1 = -i(\nu_1 + \nu_2)\tilde{g}_0 + i\frac{\hat{q}_0(\nu_1) - \hat{q}_0(\nu_2)}{\nu_1 - \nu_2},$$

$$\tilde{g}_2 = -(1 + \nu_1\nu_2)\tilde{g}_0 + \frac{\nu_2\hat{q}_0(\nu_1) - \nu_1\hat{q}_0(\nu_2)}{\nu_1 - \nu_2}$$

Theorem (F - Sung)

Let $q(x,t)$ satisfy

$$\begin{aligned} q_t + q_x + q_{xxx} &= 0, \quad 0 < x < \infty, \quad 0 < t < T, \\ q(x,0) &= q_0(x) \in H^1(R^+) \\ q(0,t) &= g_0(t) \in H^1(0,T), \quad g_0(0) = q_0(0). \end{aligned}$$

Then $q(x,t)$ is well defined.

Numerical Implementation (Flyer-F, Proc. R. Soc. 2008)

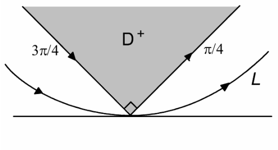
Integral representations can be deformed to give exponential convergence.

Example (The Heat Equation)

$$q_0(x) = xe^{-a^2x}, \quad 0 < x < \infty, \quad q(0,t) = \sin bt, \quad t > 0, \quad a > 0, \quad b > 0.$$

$$\hat{q}_0(k) = \frac{1}{(ik+a^2)^2}, \quad \tilde{g}_0(k,t) = \frac{1}{2i}\left[\frac{e^{(k+ib)t}-1}{k+ib} - \frac{e^{(k-ib)t}-1}{k-ib}\right].$$

$$q(x,t) = \frac{1}{2\pi}\int_{\mathcal{L}}\left\{e^{ikx-k^2t}\left[\frac{1}{(ik+a^2)^2} - \frac{1}{(-ik+a^2)^2}\right]\right. \\ \left. -ke^{ikx}\left[\frac{e^{ibt}-e^{-k^2t}}{k^2+ib} - \frac{e^{-ibt}-e^{-k^2t}}{k^2-ib}\right]\right\}dk.$$



Let $\mathcal{L}$ asymptote to $\arg k \to \alpha$ and $\arg k \to \pi - \alpha$ as $|k| \to \infty$. Then

$$k(\theta) = i\sin(\alpha - i\theta)$$

maps $\mathcal{L}$ to the real line.

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(\theta)x - k^2(\theta)t}\left[\frac{1}{(ik(\theta)+a^2)^2} - \frac{1}{(-ik(\theta)+a^2)^2}\right.$$
$$\left. - k(\theta)e^{ik(\theta)x}\left[\frac{e^{ibt} - e^{-k(\theta)^2 t}}{k(\theta)^2 + ib} - \frac{e^{-ibt} - e^{-k(\theta)^2 t}}{k(\theta)^2 - ib}\right]\cos(\alpha - i\theta)d\theta.\right.$$

Using *Mathematica* and its NIntegrate command, only 4 lines of code, we get

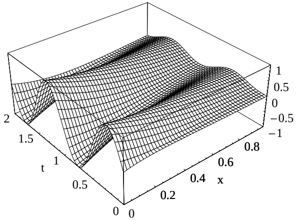


Figure: The solution of the heat equation displayed on $x \in [0, 1]$ and $t \in [0, 2]$.

III. Elliptic PDEs (F, Proc. R. Soc. 2001)

$$q_{z\bar{z}} - \beta^2 q = 0, \quad z \in \Omega, \quad \beta > 0,$$

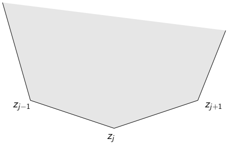


Figure: Part of the Polygon Ω.

Global Relations

$$\sum_{j=1}^{n} \hat{q}_j(k) = 0, \quad \sum_{j=1}^{n} \tilde{q}_j(k) = 0, \quad k \in \mathbb{C}$$

Integral Representation

$$q(z, \bar{z}) = \frac{1}{4i\pi} \sum_{j=1}^{n} \int_{l_j} e^{i\beta(kz - \frac{\bar{z}}{k})} \hat{q}_j(k) \frac{dk}{k}, \quad z \in \Omega,$$

where,

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz - \frac{\bar{z}}{k}\right)} \left[iq_n + i\beta \left(\frac{1}{k}\frac{d\bar{z}}{ds} + k\frac{dz}{ds} \right) q \right] ds, \quad k \in \mathbb{C},$$

$$j = 1, \cdots, n, \quad z_{n+1} = z_1,$$

$$\tilde{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{i\beta\left(k\bar{z} - \frac{z}{k}\right)} \left[iq_n + i\beta \left(\frac{1}{k}\frac{dz}{ds} + k\frac{d\bar{z}}{ds} \right) q \right] ds, \quad k \in \mathbb{C},$$

$$j = 1, \cdots, n, z_{n+1} = z_1,$$

$$l_j = \{k \in \mathbb{C} : \arg(k) = -\arg(z_{j+1} - z_j)\}, j = 1, \cdots, n, z_{n+1} = z_1.$$

Numerical: solve global relation numerically to find unknown boundary values, then use IR to find solution.

Numerical Implementation (Smitheman-Spence-F, IMA J. Num. Anal. 2008)

Ω a convex polygon with n sides.

- n unknown functions (Neumann boundary value on each side)
- expand each as a series up to N terms - now nN unknowns
- Global relation valid *for all complex k*
- Evaluate global relation at nN collocation points
- Solve linear system for unknown coefficients

Two questions:

- How to choose basis?
- How to choose collocation points?

Polynomial basis - exponential convergence, dense matrices.

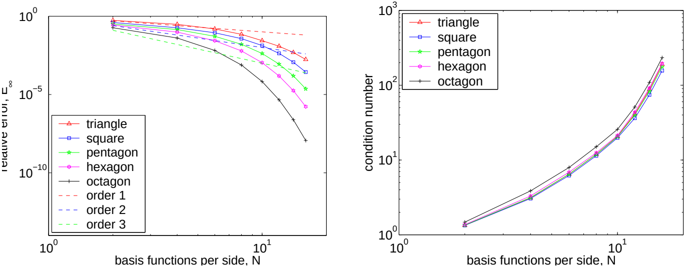
Fourier basis- algebraic convergence (since solutions are non-periodic), less dense matrices.

Dirichlet to Neumann map numerically for interior of convex polygons - results.

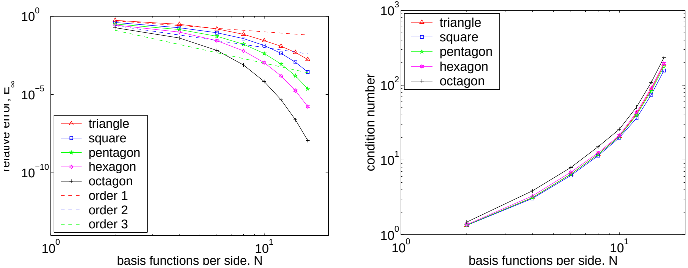
Modified Helmholtz, $\lambda = 100$, regular polygons.

$$E_\infty = \frac{\|N_e - N_c\|_\infty}{\|N_e\|_\infty}$$

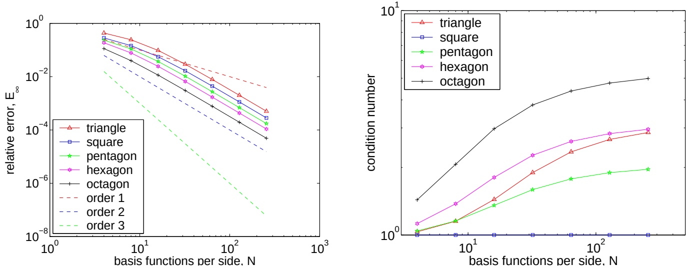
Chebychev basis:



Chebychev basis:



Fourier basis:



IV. Integrable Nonlinear PDEs on the Half Line

	Linear	Nonlinear
$q_0(x)$:	$\hat{q}_0(k)$	$\{a(k), b(k)\}$
$\{q(0,t), q_x(0,t)\}$:	$\tilde{g}(k) = i\tilde{g}_1(k^2) + k\tilde{g}_0(k^2)$	$\{A(k), B(k)\}$
$q(x,t)$:	Explicit Integral Representation	RH problem
Global Relation:	$\tilde{g}_1(k^2)$ in terms of $\hat{q}_0(k)$ and given boundary conditions	(i) $\{A(k), B(k)\}$ in terms of $\{a(k), b(k)\}$ and given boundary conditions (ii) Solve explicitly the global relation in terms of the GLM representation

Integral Representation

Spectral Functions

$q_0(x) \to \{a(k), b(k)\}$

$$a(k) = \phi_2(0, k), \quad b(k) = \phi_1(0, k), \quad \mathrm{Im}\, k \geq 0.$$

The vector $(\phi_1(x, k), \phi_2(x, k))$ is the solution of the system of linear Volterra integral equations

$$\phi_1(x, k) = \int_{\infty}^{x} e^{-2ik(x-x')} q_0(x')\phi_2(x', k)\, dx', \quad \mathrm{Im}\, k \geq 0$$

$$\phi_2(x, k) = 1 + \int_{\infty}^{x} \bar{q}_0(x')\phi_1(x', k)\, dx', \qquad \mathrm{Im}\, k \geq 0.$$

$\{g_0(t), g_1(t)\} \to \{A(k), B(k)\}$

$$A(k) = \overline{\Phi_2(T, \bar{k})}, \qquad B(k) = -e^{4ik^2 T}\Phi_1(T, k), \qquad k \in \mathbb{C}.$$

The vector $(\Phi_1(t, k), \Phi_2(t, k))$ is the solution of the system of linear Volterra integral equations

$$\Phi_1(t, k) = \int_0^t e^{-4ik^2(t-t')}\left[-i|g_0(t')|^2\Phi_1(t', k) + (2kg_0(t') + ig_1(t'))\Phi_2(t', k)\right] dt'$$

$$\Phi_2(t, k) = 1 + \int_0^t \left[(2k\overline{g_0(t')} - i\,\overline{g_1(t')})\Phi_1(t', k) + i|g_0(t')|^2\Phi_2(t', k)\right] dt'.$$

Theorem 1: Define $M(x,t,k)$ in terms of $\{a,b,A,B\}$ as the solution of the following RH problem:

(i) M is analytic in $k \in \mathbb{C} \setminus L$ where L denotes the union of the real and the imaginary axes.

(ii) $M = \operatorname{diag}(1,1) + O(1/k)$ as $k \to \infty$.

(iii) $M^- = M^+ J,\ k \in L$ where

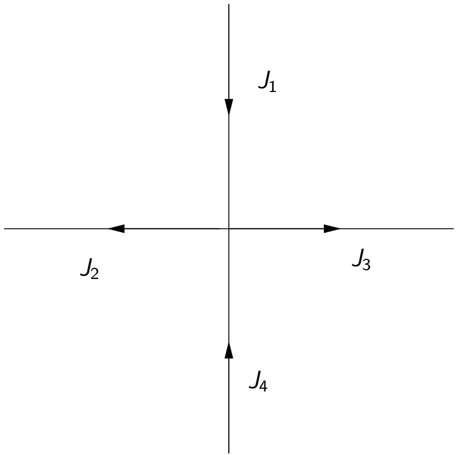
$$J_1 = \begin{bmatrix} 1 & 0 \\ \Gamma(k) e^{2ikx+4ik^2t} & 1 \end{bmatrix}, \qquad J_2 = J_3 J_4^{-1} J_1, \qquad J_3 = \begin{bmatrix} 1 & \overline{\Gamma(k)} e^{-2ikx-4ik^2t} \\ 0 & 1 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 1 & -\gamma(k) e^{-2ikx-4ik^2t} \\ \overline{\gamma(k)} e^{i2kx+4ik^2t} & 1-|\gamma|^2 \end{bmatrix},$$

and

$$\gamma(k) = \frac{b(k)}{\overline{\overline{a(k)}}}, \quad \Gamma(k) = \frac{1}{a(k)} \left[\frac{\overline{A(\bar{k})}}{\overline{\overline{B(\bar{k})}}} a(k) - b(k) \right]^{-1}.$$

Then M exists and is unique.


J_1
J_2
J_3
J_4

Theorem 2: Define $q(x,t)$ in terms of $\{\gamma(k), \Gamma(k), M(x,t,k)\}$ by

$$q(x,t) = -\frac{1}{\pi}\Big\{\int_{\partial I_2} \overline{\Gamma(\bar{k})} e^{-2ikx-4ik^2t} M_{11}^{+}\, dk + \int_{-\infty}^{\infty} \gamma(k) e^{-2ikx-4ik^2t} M_{11}^{+} dk + \int_{0}^{\infty} |\gamma(k)|^2 M_{12}^{+} dk\Big\},$$

where ∂I_2 denotes the boundary of the third quadrant of the complex k-plane. Then $q(x,t)$ solves the defocusing NLS and also satisfies $q(x,0) = q_0(x)$.

Assume that there exist smooth functions $\{g_0(t), g_1(t)\}$ such that $g_0(0) = q_0(0), \quad g_1(0) = \dot{q}_0(0)$, and such that the following global relation is valid,

$$a(k)B(k) - b(k)A(k) = e^{4ik^2T} C_{T}(k), \quad \mathrm{Im}\, k \geq 0,$$

where $C_T(k)$ is a function analytic for $\mathrm{Im} k > 0$ and of $O(1/k)$ as $k \to \infty$. Then

$$q(0,t) = g_0(t), \quad q_x(0,t) = g_1(t).$$

The D-N map

Linearizable Boundary Conditions

Solve global relation by *algebra*.

Example (KdV)
$q_t - q_{xxx} + qq_x = 0$

$$q(0,t) = \chi, \quad q_{xx}(0,t) = \chi + 3\chi^2, \quad \frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))},$$

$$\nu^2 + k\nu + k^2 + \frac{1}{4} = 0, \quad f(k) = \frac{\nu + k}{\nu - k}\left(1 - \frac{4\nu k}{\chi}\right).$$

General case (NLS)

The spectral functions are defined by

$$A(k) = e^{2ik^2 T}\bar{\Phi}_2(T, \bar{k}), \quad B(k) = -e^{2ik^2 T}\Phi_1(T, \bar{k}),$$

where $\Phi(t,k) = (\Phi_1, \Phi_2)$ satisfies :

$$\Phi_t + 2ik^2\sigma_3\Phi = \left(2kG_0(t) + G_1(t)\right)\Phi, \quad t > 0, k \in \mathbb{C}, \Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with

$$G_0(t) = \begin{pmatrix} 0 & g_0(t) \\ \bar{g}_0(t) & 0 \end{pmatrix}, G_1(t) = \begin{pmatrix} -|g_0(t)|^2 & g_1(t) \\ -\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}.$$

For Im$k \geq 0$

$$\int_0^T e^{2ik^2\tau}\left\{i|g_0(\tau)|^2\Phi_1(\tau,k)-[2kg_0(\tau)+ig_1(\tau)]\Phi_2(\tau,k)\right\}d\tau = e^{4ik^2T}c^+(k)$$

Boutet de Monvel, F and Shepelsky (2005)

$$\Phi(t,k)=\begin{pmatrix} 0 \\ e^{if_2t} \end{pmatrix}+\int_{-t}^{t}\begin{pmatrix} L_1(t,s)-\frac{i}{2}g_0(t)M_2(t,s)+kM_1(t,s) \\ L_2(t,s)+\frac{i\rho}{2}\bar{g}_0(t)M_1(t,s)+kM_2(t,s) \end{pmatrix}e^{if_2s}ds,$$

$$g_1(t)=g_0(t)M_2(t,t)-\frac{e^{-\frac{i\pi}{4}}}{\sqrt{\pi}}\int_0^t\frac{\partial M_1}{\partial\tau}(t,2\tau-t)\frac{d\tau}{\sqrt{t-\tau}}.$$

Linear Limit

$M_2(t,t)\sim 0$, $M_1(t,2\tau-t)\sim g_0(\tau)$

Remarks

1.

- C. Zheng, Exact Nonreflecting Boundary Conditions for One-dimensional Cubic Nonlinear Schrödinger Equations, Journal of Computational Physics **215**, 552 - 565 (2006).

2.

- J.L. Bona, and A.S. Fokas, Initial-boundary-value problems for linear and integrable nonlinear dispersive partial differential equations, Nonlinearity **21**, 10, T195–T203 (2008).
- G.M. Dujardin, Asymptotics of Linear Initial Boundary Value Problems with Periodic Boundary Data on the Half-Line and Finite Intervals, Proc. R. Soc. Lond. A (to appear).

Asymptotics

- Large t asymptotics for decaying boundary conditions at $x=0$ (F, Its)
  Using the Deift-Zhou method

$$q = \frac{1}{\sqrt{t}}\alpha(\xi)\exp\left[\frac{ix^2}{4t} + 2i\alpha^2(\xi)\ln t + i\Phi(\xi)\right] + O(1),\ \ t \to \infty,$$

$$\xi = -\frac{x}{4t} \text{ and } \frac{x}{t} = O(1),$$

  where

$$\alpha^2(k) = \frac{1}{4\pi}\ln\left(1 - |\gamma(k) - \bar{\Gamma}(k)|^2\right),$$

$$\Phi(k) = -6\alpha^2(k)\ln 2 + \frac{\pi}{4} + \arg(\gamma(\bar{k}) - \bar{\Gamma}(\bar{k}))$$
$$+ \arg\boldsymbol{\Gamma}(2i\alpha^2(k)) - 4\int_{-\infty}^{k}\ln(\ell - l)\mathrm{d}(\alpha^2(\ell)),$$

  $\boldsymbol{\Gamma}$: gamma function
- Large t Asymptotics for periodic in t boundary conditions at $x = 0$ (Boutet-de Monvel, Kotlyarov)
- Small dispersion limit (Kamvissis)

Conclusions

1747, d' Alembert and Euler: Separation of Variables
1807, Fourier: Transforms
1814, Cauchy: Analyticity
1828, Green: Green's Representations

Fourier (spectral) Space

Transforms

Physical Space

Green's Integral Representations
Method of Images

New Method

Integral Representations in the Fourier Space
Invariance of Global Relation

Remarks: Green could have discovered Lax pairs

$$u_{xx} + u_{yy} + u = 0$$

$$(\tilde{u}u_x - \tilde{u}_x u)_x - (u\tilde{u}_y - u_y\tilde{u})_y = 0,$$

$$\tilde{u} = e^{k_1 x + k_2 y},\ k_1^2 + k_2^2 = 1, \quad k_1 = \cos\alpha,\ k_2 = \sin\alpha,\ e^{i\alpha} = k,\ e = e^{\frac{1}{2}(k - \frac{1}{k})y + \frac{i}{2}(k + \frac{1}{k})x}.$$

$$\left\{e\left[u_x - \frac{i}{2}\left(k + \frac{1}{k}\right)u\right]\right\}_x - \left\{e\left[-u_y + \frac{1}{2}\left(k - \frac{1}{k}\right)u\right]\right\}_y = 0.$$

Global relation:

$$\int_{\partial D} \left\{e\left[-u_y + \frac{1}{2}\left(k - \frac{1}{k}\right)u\right]\right\} dx + \left\{e\left[u_x - \frac{i}{2}\left(k + \frac{1}{k}\right)u\right]\right\} dy = 0.$$

Integral representations:

$$e\left[u_x - \frac{i}{2}\left(k + \frac{1}{k}\right)u\right] = \tilde{\mu}_y$$

$$e\left[-u_y + \frac{1}{2}\left(k - \frac{1}{k}\right)u\right] = \tilde{\mu}_x$$

V. Nonlinear PDEs in 4+2 and 3+1

(F, PRL, 2006 and J. Phys A, 2008)

Nonlinear FT in 4D

$\underline{f \rightarrow \hat{f}}$

$$\hat{f}(k,\lambda) = \frac{2}{\pi^3}\int_{\mathbb{R}^4}(\bar{\lambda}-\bar{k})\bar{E}(k,\lambda,x,y)f(x,y)\mu(x,y,\lambda)dxdy,$$

$$x = x_1 + ix_2, y = y_1 + iy_2, k = k_1 + ik_2, \lambda = \lambda_1 + i\lambda_2,$$

$$E(k,\lambda,x,y) = e^{2i[(\lambda_2-k_2)x_1+(k_1-\lambda_1)x_2+2(\lambda_1\lambda_2-k_1k_2)y_1+(k_1^2-k_2^2+\lambda_2^2-\lambda_1^2)y_2]}.$$

μ:

$$\mu_{\bar{y}} - \mu_{\bar{x}\bar{x}} - 2k\mu_{\bar{x}} + f\mu = 0,$$

with

$$\mu \rightarrow 1 \quad \text{as} \quad |x|^2 + |y|^2 \rightarrow \infty.$$

$\underline{\hat{f} \to f}$

$$f(x,y) = \frac{2}{\pi}\partial_{\bar{x}} \int_{\mathbb{R}^4} E(k,\lambda,x,y)\hat{f}(k,\lambda)\mu(x,y,\lambda)dkd\lambda.$$

μ:

$$\mu(x,y,k) = 1 + \frac{1}{\pi}\int_{\mathbb{R}^4} E(k',\lambda,x,y)\hat{f}(k',\lambda)\mu(x,y,\lambda)\frac{dk'd\lambda}{k-k'}, \quad k \in \mathbb{C}.$$

KP in 4+2

$$f \to \hat{f},$$
$$\hat{q}(k,\lambda,t) = \hat{f}(k,\lambda)e^{(\lambda^3-k^3)\bar{t}-(\bar{\lambda}^3-\bar{k}^3)t}, \quad t = t_1 + it_2,$$
$$\hat{q} \to q.$$

Then, q solves the following equation

$$q_{\bar{t}} = q_{\bar{x}\bar{x}\bar{x}} - \frac{3}{2}qq_{\bar{x}} + \frac{3}{4}\partial_{\bar{x}}^{-1}q_{\bar{y}\bar{y}}$$

with

$$q(x_1,x_2,y_1,y_2,0,0) = f(x_1,x_2,y_1,y_2),$$

where

$$\partial_{\bar{x}}^{-1}f = \frac{1}{\pi}\int_{\mathbb{R}^2} f(x_1',x_2')\frac{dx_1'dx_2'}{x-x'}.$$

Explicit Reductions to 3+1

Potential KdV

$$q_{\bar{t}} = \frac{1}{4} q_{\bar{x}\bar{x}\bar{x}} - \frac{3}{4} q_{\bar{x}}^2, \quad t = t_1 + it_2, \quad x = x_1 + ix_2. \tag{1}$$

To preserve reality:

$$q_t = \frac{1}{4} q_{xxx} - \frac{3}{4} q_x^2. \tag{2}$$

$(q_{\bar{t}})_t = (q_t)_{\bar{t}}$:

$$\left(q_{\bar{x}\bar{x}\bar{x}} - 3q_{\bar{x}}^2\right)_t = \left(q_{xxx} - 3q_x^2\right)_{\bar{t}}.$$

Use $\partial_{\bar{x}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$:

$$(\Delta q)(\Delta q_{x_2}) = 0, \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2.$$

(1)$\pm$ (2):

$$q_{t_1} = \frac{1}{16}\left(\partial_{x_1}^3 - 3\partial_{x_1}\partial_{x_2}^2\right) q - \frac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right), \tag{3}$$

$$q_{t_2} = \frac{1}{16}\left(-\partial_{x_2}^3 + 3\partial_{x_2}\partial_{x_1}^2\right) q - \frac{3}{4} q_{x_1} q_{x_2}. \tag{4}$$

$$\boxed{q_{t_1} = \frac{1}{4} q_{x_1x_1x_1} - \frac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right), \quad \Delta q = 0.} \tag{5}$$

$$\boxed{q_{t_2} = -\frac{1}{4} q_{x_2x_2x_2} - \frac{3}{4} q_{x_1} q_{x_2}, \quad \Delta q = 0.} \tag{6}$$

Claim : $\Delta q_{t_1} = \Delta q_{t_2} = 0$

Let $q(x_1, x_2, 0) = q_0(x_1, x_2)$ be a harmonic function. Then $q(x_1, x_2, t_1)$ and $q(x_1, x_2, t_2)$ satisfy the integrable systems (5) and (6) respectively. If q_0 is real, then q remains real.

Potential KP

$$q_{\bar t} = \frac{1}{4} q_{\bar x\bar x\bar x} - \frac{3}{4} q_{\bar x}^2 + \frac{3}{4} \overline{L} q, \quad L = \partial_x^{-1} \partial_y^2 \,,$$

where

$$y = y_1 + i y_2, \quad \partial_{\bar x}^{-1} f = \frac{1}{\pi} \int_{\mathbb{R}^2} f(x_1', x_2') \frac{dx_1' dx_2'}{x - x'}.$$

$$(\Delta q)(\Delta q_{x_2}) + q_{\bar x} L q_{\bar x} - q_x \overline{L} q_x + \frac{1}{2} \left(\overline{L} q_x^2 - L q_{\bar x}^2 \right) = 0.$$

$$\Delta q = q_{\bar x y} = 0, \quad \Rightarrow \quad \Delta q = 0, \quad q_{y_2} = -\partial_{x_1}^{-1} q_{x_2 y_1}$$

$$\boxed{q_{t_1} = \tfrac{1}{4} q_{x_1 x_1 x_1} - \tfrac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right) + \tfrac{3}{4}\partial_{x_1}^{-1} q_{y_1 y_1}, \quad \Delta q = 0.}$$

$$\boxed{q_{t_2} = -\tfrac{1}{4} q_{x_2 x_2 x_2} - \tfrac{3}{4} q_{x_1} q_{x_2} - \tfrac{3}{4}\partial_{x_2}^{-1} q_{y_1 y_1}, \quad \Delta q = 0.}$$

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$$

Example: $-\infty < x_1 < \infty, \quad x_2 > 0.$

$q(x_1, 0, y_1, 0) = Q_0(x_1, y_1).$

$q_{x_2}(x_1, 0, y_1, t) = Hq_{x_1}(x_1, 0, y_1, t)$

$q(x_1, 0, y_1, t) = Q(x_1, y_1, t)$

$$Q_{t_1} = \frac{1}{4} Q_{x_1 x_1 x_1} - \frac{3}{8} \left[Q_{x_1}^2 - (HQ_{x_1})^2 \right] + \frac{3}{4} \partial_{x_1}^{-1} Q_{y_1 y_1},$$
$$Q(x_1, y_1, 0) = Q_0(x_1, y_1).$$

Laplace's equation with $q(x_1, 0, y_1, t) = Q(x_1, y_1, t)$, yields $q(x_1, x_2, y, t)$

DS in 4+2

$$(-1)^j \partial_{\bar{t}} q_j + \frac{1}{4}(\partial_{\bar{\xi}}^2 + \partial_{\bar{\eta}}^2) q_{\bar{j}} - q_j(\partial_{\bar{\xi}}^{-1}\partial_{\bar{\eta}} + \partial_{\bar{\eta}}^{-1}\partial_{\bar{\xi}})(q_1 q_2)_{\bar{\eta}} = 0, \quad j = 1, 2,$$

$$\partial_{\bar{\xi}}^{-1} q = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{q(\xi_1', \xi_2')}{\xi - \xi'} d\xi_1 d\xi_2.$$

Let

$$E_1(\xi,\eta,t,k,\lambda) = e^{4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2) + 4i(k_1k_2 + \lambda_1\lambda_2)t_1 - 2i(k_1^2 - k_2^2 + \lambda_1^2 - \lambda_2^2)t_2},$$

$$E_2(\xi,\eta,t,k,\lambda) = e^{4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2) - 4i(k_1k_2 + \lambda_1\lambda_2)t_1 + 2i(k_1^2 - k_2^2 + \lambda_1^2 - \lambda_2^2)t_2}.$$

<u>1-soliton Solution</u>

$$q_1 = 2\pi\left(\frac{\gamma_1 E_1(\alpha_1,\beta_1)}{\pi^2 - D_1 E_1(\alpha_1,\beta_1)E_2(\alpha_2,\beta_2)}\right)$$

$$q_2 = -2\pi\left(\frac{\gamma_2 E_2(\alpha_2,\beta_2)}{\pi^2 - D_1 E_1(\alpha_1,\beta_1)E_2(\alpha_2,\beta_2)}\right)$$

where $\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma_1,\gamma_2$ are arbitrary complex constants and

$$D_1 = \frac{\gamma_1\gamma_2}{(\alpha_1-\beta_2)(\alpha_2-\beta_1)}.$$