

# **Mathematics Notebook**

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# Contents

I	MYP	
<b>1</b>	<b>Algebraic Expressions .....</b>	<b>7</b>
1.1	Expressions	7
1.2	Polynomial Expressions	7
1.2.1	Monomials of Like Terms .....	11
1.2.2	Surds .....	11
<b>2</b>	<b>Univariate Polynomial Expressions .....</b>	<b>13</b>
2.1	Classification of Univariate Polynomial Expressions	13
2.2	Degree -1 Univariate Polynomials	15
2.3	Degree 0 Univariate Polynomials	15
2.4	Degree 1 Univariate Polynomials	15
2.5	Degree 2 Univariate Polynomials	30
2.6	Degree 3 Univariate Polynomials	41
2.7	Degree $n$ Univariate Polynomials	41
<b>3</b>	<b>Equations .....</b>	<b>43</b>
3.1	Equality	43
3.2	Solving Linear Equations	43
3.3	Solving Quadratic Equations	48
3.3.1	Completing the Square .....	49

<b>4</b>	<b>Functions .....</b>	<b>55</b>
4.1	Evaluating Functions	55
4.2	Quadratic Functions	56
4.2.1	Completing the Square .....	56
4.3	Trigonometric Functions	57
4.3.1	Graphing Trigonometric Function .....	57
4.4	Rational Functions	64
<b>5</b>	<b>Differentiation .....</b>	<b>67</b>
5.1	Limit of the Difference Quotient	67
5.2	Derivative of a Monomial Functions	68
5.3	Derivative of Polynomial Functions	69
5.4	Derivative of Rational Functions	72
5.5	Derivative of Trigonometric Functions	72
5.6	Derivative of Exponential Functions	73
5.7	Derivative of Logarithmic Functions	73
5.8	Derivative of Composite Functions	73
<b>6</b>	<b>Integration .....</b>	<b>77</b>
6.1	Integration Rules	77
6.2	Volumes of Revolution	78
	<b>Bibliography .....</b>	<b>83</b>
	<b>Bibliography .....</b>	<b>83</b>
	Websites	83
	Articles	83
	<b>Index .....</b>	<b>85</b>

<b>1</b>	<b>Algebraic Expressions .....</b>	<b>7</b>
1.1	Expressions	
1.2	Polynomial Expressions	
<b>2</b>	<b>Univariate Polynomial Expressions ....</b>	<b>13</b>
2.1	Classification of Univariate Polynomial Expressions	
2.2	Degree -1 Univariate Polynomials	
2.3	Degree 0 Univariate Polynomials	
2.4	Degree 1 Univariate Polynomials	
2.5	Degree 2 Univariate Polynomials	
2.6	Degree 3 Univariate Polynomials	
2.7	Degree $n$ Univariate Polynomials	
<b>3</b>	<b>Equations .....</b>	<b>43</b>
3.1	Equality	
3.2	Solving Linear Equations	
3.3	Solving Quadratic Equations	





# 1. Algebraic Expressions

## 1.1 Expressions

### Essential Questions 1.1

1. What is an algebraic expression?

## 1.2 Polynomial Expressions

**Definition 1.2.1 — Operation of Addition (OOA).**

$$\underbrace{a}_{\text{Augend}} + \underbrace{b}_{\text{Addend}} = \underbrace{\text{Sum}}_{(1.1)}$$

	Arithmetic	Polynomial	Algebraic
Constant	Yes	Yes	Yes
Factorial	Yes	Yes	Yes
Variable: parameter/coefficient	Yes	Yes	Yes
Variable: unknown/indeterminate	No	Yes	Yes
Power with $\mathbb{Z}^+$ exponent	No	Yes	Yes
Power with $\mathbb{Z}$ exponent	No	No	Yes
$n$ -th root	No	No	Yes
Power with $\mathbb{Q}$ exponent	No	No	Yes

Table 1.1: Names of different types of expressions

**Definition 1.2.2 — Common Denominator (CD).**

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad (1.2a)$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \quad (1.2b)$$

**Rule 1.2.3 — Fraction Operation of Addition (FOOA).**

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad (1.3a)$$

$$\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d} \quad (1.3b)$$

**Definition 1.2.4 — Operation of Multiplication (OOM).**

$$\underbrace{\begin{array}{c} a \\ \text{Multiplicand} \end{array}}_{\text{Product}} \cdot \underbrace{\begin{array}{c} b \\ \text{Multiplier} \end{array}}_{\text{Product}} \quad (1.4)$$

**Definition 1.2.5 — Operation of Exponentiation (OOE).**

$$\underbrace{\begin{array}{c} b \\ \text{base} \end{array}}_{\text{Power}}^{\text{Exponent}} \quad (1.5)$$

**Definition 1.2.6 — Juxtaposition to Center-Dot (JTC).**

$$ab = a \cdot b \quad (1.6)$$

**Definition 1.2.7 — Center-Dot to Justaposition (CTJ).**

$$a \cdot b = ab \quad (1.7)$$

**Definition 1.2.8 — Commutative Property of Multiplication (CPM).**

$$\textcolor{red}{a} \cdot b = b \cdot \textcolor{red}{a} \quad (1.8)$$

**Definition 1.2.9 — Multiplicative Inverse (MI).**

$$a \cdot \frac{1}{a} = 1 \quad (1.9a)$$

$$a \cdot \textcolor{red}{a}^{-1} = 1 \quad (1.9b)$$

**Definition 1.2.10 — Associative Property of Multiplication (APM).**

$$a \cdot b \cdot c = (a \cdot b) \cdot c \quad (1.10a)$$

$$a \cdot b \cdot c = a \cdot (b \cdot c) \quad (1.10b)$$

## Powers

**Rule 1.2.11 — Power of a Quotient of Powers (PoQPo).**

$$\left( \frac{a^m}{b^n} \right)^k = \frac{a^{m \cdot k}}{b^{n \cdot k}} \quad (1.11a)$$

$$\frac{a^{m \cdot k}}{b^{n \cdot k}} = \left( \frac{a^m}{b^n} \right)^k \quad (1.11b)$$

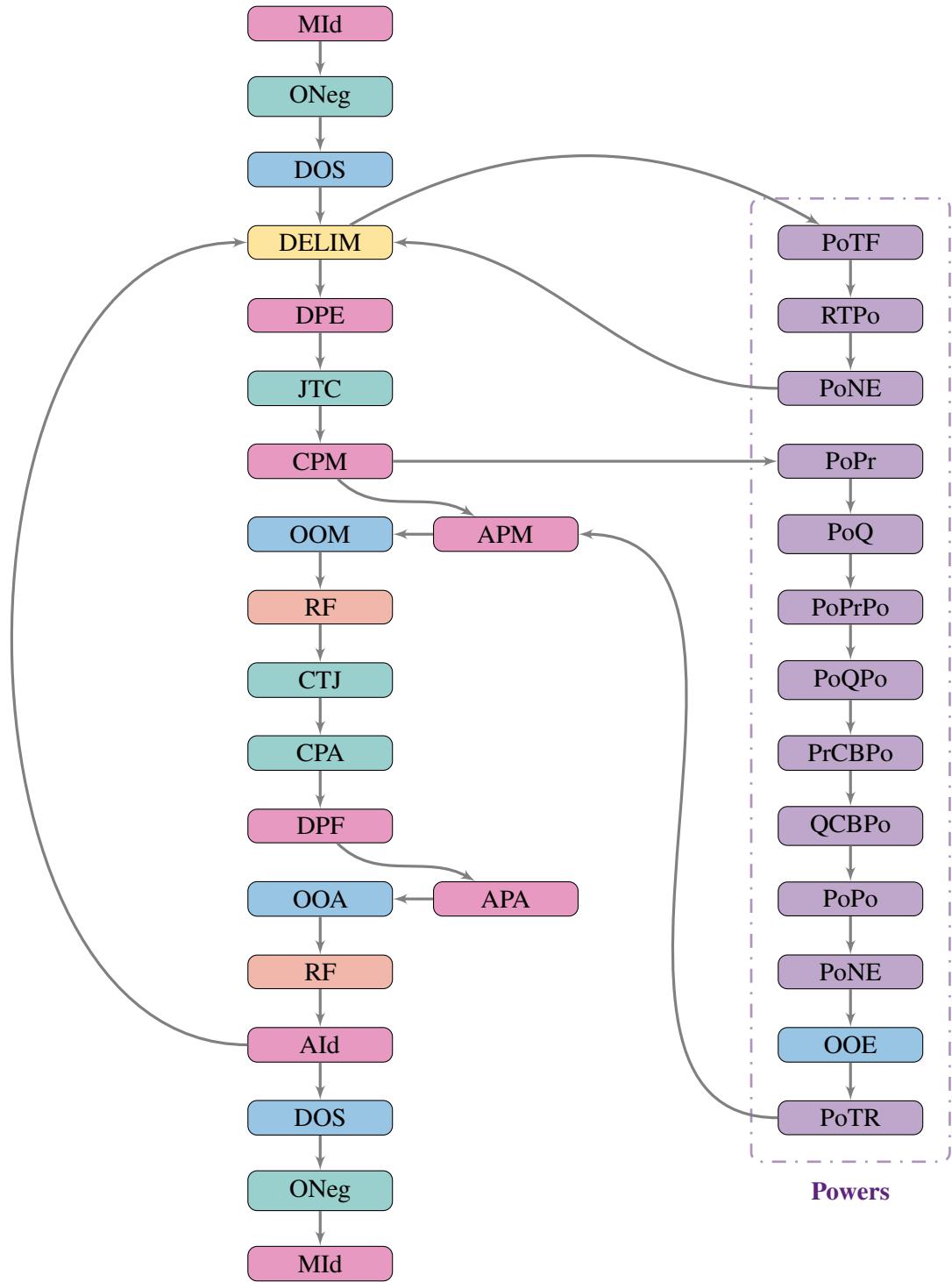


Figure 1.1: Simplifying Expressions Workflow:

■ Property, ■ Operation, ■ Notation, ■ Powers, ■ Delimiters, ■ Process, ■ Not Used

**Rule 1.2.12 — Power of a Product of Powers (PoPrPo).**

$$(a^m \cdot b^n)^k = a^{m \cdot k} \cdot b^{n \cdot k} \quad (1.12a)$$

$$a^{m \cdot k} \cdot b^{n \cdot k} = (a^m \cdot b^n)^k \quad (1.12b)$$

**Definition 1.2.13 — Power To Factor (PoTF).**

$$a^n = a_1 \cdot a_2 \cdot \dots \cdot a_{n-1} \cdot a_n \quad (1.13)$$

**Definition 1.2.14 — Factor To Power (FTPo).**

$$a_1 \cdot a_2 \cdot \dots \cdot a_{n-1} \cdot a_n = a^n \quad (1.14)$$

**Definition 1.2.15 — Power Inverse (Pol).**

$$(b^m)^{\frac{1}{m}} = b \quad (1.15a)$$

**Definition 1.2.16 — Power Inverse (Pold).**

$$1 = b^0 \quad (1.16a)$$

$$b^0 = 1 \quad (1.16b)$$

**Notation 1.1** (Radical To Power (RTPo)).

$$\sqrt[m]{b^n} = b^{\frac{n}{m}} \quad (1.17)$$

**Notation 1.2** (Power To Radical (PoTR)).

$$b^{\frac{n}{m}} = \sqrt[m]{b^n} \quad (1.18)$$

## 1.2.1 Monomials of Like Terms

**Definition 1.2.17 — Additive Inverse (AI).**

$$a + (-a) = 0 \quad (1.19a)$$

## 1.2.2 Surds

**Example 1.1 — id:20141108-085327.**

$$\text{Simplify } 2\sqrt{2} - \frac{(\sqrt{2})^3}{3} - \left( 2(-\sqrt{2}) - \frac{(-\sqrt{2})^3}{3} \right)$$

(S)

**Solution:**

$$2\sqrt{2} - \frac{1(1\sqrt{2})^3}{3} - 1 \left( 2(-1\sqrt{2}) - \frac{1(-1\sqrt{2})^3}{3} \right) \quad \text{MId(2.1a)}$$

$$2\sqrt{2} - \frac{1(1\sqrt{2})^3}{3} - 1 \left( 2(-1\sqrt{2}) - \frac{1(-1\sqrt{2})^3}{3} \right) \quad \text{ONeg(2.6a)}$$

$$2\sqrt{2} + \frac{-1(1\sqrt{2})^3}{3} + -1 \left( 2(-1\sqrt{2}) + \frac{-1(-1\sqrt{2})^3}{3} \right) \quad \text{DOS(2.5a)}$$

$$2 \cdot 2^{1/2} + \frac{-1(1 \cdot 2^{1/2})^3}{3} + -1 \left( 2(-1 \cdot 2^{1/2}) + \frac{-1(-1 \cdot 2^{1/2})^3}{3} \right) \quad \text{RTPo(1.17)}$$

$$2 \cdot 2^{1/2} + \frac{-1(1 \cdot 2^{1/2})^3}{3} + -1 \left( 2 \cdot -1 \cdot 2^{1/2} + \frac{-1(-1 \cdot 2^{1/2})^3}{3} \right) \quad \text{JTC(1.6)}$$

$$2 \cdot 2^{1/2} + \frac{-1 \cdot 1 \cdot 2^{3/2}}{3} + -1 \left( 2 \cdot -1 \cdot 2^{1/2} + \frac{-1 \cdot -1 \cdot 2^{3/2}}{3} \right) \quad \text{PoPrPo(1.12a)}$$

$$2 \cdot 2^{1/2} + \frac{-1 \cdot 1 \cdot 2^{2/2} \cdot 2^{1/2}}{3} + -1 \left( 2 \cdot -1 \cdot 2^{1/2} + \frac{-1 \cdot -1 \cdot 2^{2/2} \cdot 2^{1/2}}{3} \right) \quad \text{PrCBPo(2.10b)}$$

$$2 \cdot 2^{1/2} + \frac{-1 \cdot 1 \cdot 2 \cdot 2^{1/2}}{3} + -1 \left( 2 \cdot -1 \cdot 2^{1/2} + \frac{-1 \cdot -1 \cdot 2 \cdot 2^{1/2}}{3} \right) \quad \text{MId(2.1b)}$$

$$2 \cdot \sqrt{2} + \frac{-1 \cdot 1 \cdot 2 \cdot \sqrt{2}}{3} + -1 \left( 2 \cdot -1 \cdot \sqrt{2} + \frac{-1 \cdot -1 \cdot 2 \cdot \sqrt{2}}{3} \right) \quad \text{PoTR(1.18)}$$

$$2 \cdot \sqrt{2} + \frac{-1 \cdot 1 \cdot 2 \cdot \sqrt{2}}{3} + -1 \cdot 2 \cdot -1 \cdot \sqrt{2} + \frac{-1 \cdot -1 \cdot -1 \cdot 2 \cdot \sqrt{2}}{3} \quad \text{DPE(2.9a)}$$

$$2 \cdot \sqrt{2} + \frac{-2 \cdot \sqrt{2}}{3} + 2 \cdot \sqrt{2} + \frac{-2 \cdot \sqrt{2}}{3} \quad \text{OOM(1.4)}$$

$$2\sqrt{2} + \frac{-2\sqrt{2}}{3} + 2\sqrt{2} + \frac{-2\sqrt{2}}{3} \quad \text{CTJ(1.7)}$$

$$\left( 2 + \frac{-2}{3} + 2 + \frac{-2}{3} \right) \sqrt{2} \quad \text{DPF(2.4a)}$$

$$\frac{8}{3}\sqrt{2} \quad \text{OOA(1.1)}$$

(D)

Dependencies:example 6.5-20141108-083108





## 2. Univariate Polynomial Expressions

### 2.1 Classification of Univariate Polynomial Expressions

**Definition 2.1.1 — Indeterminate.**

$x$

An indeterminate is a symbol that is treated as a variable, but does not stand for anything else but itself and is used as a placeholder.

- it does **not** designate a constant or a parameter
- it is **not** an unknown that could be solved for
- it is **not** a variable designating a function argument

[[wikipedia:indeterminate](#) ]

**Definition 2.1.2 — Coefficient.**

$Cx^k$

A coefficient,  $C$  is a real number multiplicative factor.

**Definition 2.1.3 — Univariate Monomial.**

$C_k x^k$

A univariate monomial is made up of two factors. The first factor of a monomial,  $C_k$ , is the **coefficient**. The second factor of each monomial,  $x^k$ , is an indeterminate raised to a non-negative integer power  $k$ .

**Definition 2.1.4 — Multiplicative Identity (MId).**

$$1a = a \quad (2.1a)$$

$$a = 1a \quad (2.1b)$$

- R** If the coefficient of a univariate monomial is the multiplicative identity 2.1a, 1, then it is not shown in its canonical form.

$$\begin{aligned} C_k x^k &= C_k x^k \\ &= 1x^k \\ &= x^k \end{aligned}$$

**Example 2.1 — id:20141120-202842.**

Express  $1x^2$  in canonical form.

**S**

**Solution:**

$$x^2 \quad \text{MId}(2.1b)$$

■

**Definition 2.1.5 — Univariate Polynomial Expression.**

$$\sum_{k=0}^n C_k x^n = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0} \quad (2.2)$$

A univariate polynomial in an indeterminate  $x$  is an expression made up of one or more summands of the form  $C_k x^k$ , which are called monomials. The first factor of each monomial,  $C_k$ , is a numerical factor called the **coefficient** where  $C_k \in \mathbb{R}$ . The second factor of each monomial,  $x^k$ , is an indeterminate raised to a non-negative integer power  $i$ .

[[wikipedia:polynomial](#)]

**Definition 2.1.6 — Degree of the Indeterminate.**

$$x^k$$

The exponent of an indeterminate power,  $k$  is called the degree of the indeterminate.

[[wikipedia:polynomial](#)]

**Definition 2.1.7 — Degree of the Univariate Polynomial.**

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

The degree of the univariate polynomial is determined by the monomial with the largest degree of the indeterminate.

## 2.2 Degree -1 Univariate Polynomials

### Monomials

## 2.3 Degree 0 Univariate Polynomials

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

Degree 0 univariate polynomial expressions are made up of univariate monomials,  $C_0$ , called **constants**. The power identity is an indeterminate raised to a power of 0 has a value of 1. Thus,  $x^0 = 1$  and results in the monomial  $C_0 \cdot 1$ . The canonical form of a product does not show the multiplicative identity factor, so what remains of this monomial product is only the coefficient factor  $C_0$  and from now on will be referred to as a **constant**.

### Example 2.2 — id:20141121-093747.

Express  $13x^0$  in canonical form.

(S)

**Solution:**

13

■

### Monomials

Degree 0 univariate polynomial expressions are usually a monomial in their canonical form if  $C_0$  is a non-zero real number. The exception is if  $C_0 = 0$ , the additive identity, then the result is the zero polynomial, which can be considered a degree -1 polynomial.

The expression can be manipulated into its monomial canonical form by simplifying the expression. Simplifying the expression can be defined as evaluating the expression by following order of operations, which is the same as evaluating an arithmetic expression.

## 2.4 Degree 1 Univariate Polynomials

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

Degree 1 univariate polynomial expressions can be expressed with at most two different terms and consequently this expression in its canonical form has at most two monomial summands – called a binomial.

### Definition 2.4.1 — Univariate Like Terms.

$$C_1 x^k = C_2 x^k$$

Two or more univariate monomials are defined as having like terms if each monomial has the

same term, which will be the same indeterminate raised to the same positive integer power.

Sometimes the word **term** is used to describe monomials (including both the coefficient and the term), which may be confusing when trying to define like terms. For this reason, we will refer to the summands of a polynomial as monomials.

The monomials  $5x^1$  and  $3x^1$  can be described as having like terms because they share the common term  $x^1$ . One could also say that  $5x^1$  and  $3x^1$  are like terms by definition and consequently giving the reader an impression that  $5x^1$  and  $3x^1$  are terms themselves.



A degree 1 indeterminate does not display the multiplicative identity in the exponent when its in canonical form.

### Example 2.3 — id:20141120-202042.

Express  $5x^1$  in canonical form




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**Solution:**

$5x$

MId(2.1b)

■

### Example 2.4 — id:20141121-093439.

Express  $7x^1 + 5$  in canonical form.




---

**Solution:**

$7x + 5$

MId(2.1b)

■

## Monomials

### Essential Questions 2.1

- How do we simplify univariate polynomial expressions?

### Definition 2.4.2 — Additive Identity (Ald).

$$a + 0 = a \tag{2.3a}$$

$$a = a + 0 \tag{2.3b}$$

If the constant monomial is 0, the additive identity, then the canonical form of a degree 1

univariate polynomial is a degree 1 monomial.

### Simplifying Univariate Monomial Expressions

The definition of a univariate monomial expression is based on the expanded canonical form of some polynomial expression. It might be that the original expression might not be in the expanded canonical form, so a process called **simplifying by expanding** will be introduced to manipulate the expression such that it can be written in its expanded canonical form.

This process of simplifying by expanding polynomial expressions will be developed to the extent that it will be used to simplify multivariate polynomials. We will start by simplifying univariate monomial expressions.

#### Definition 2.4.3 — Distributive Property Factoring (DPF).

$$ba + ca = (b+c)a \quad (2.4a)$$

$$ab + ac = a(b+c) \quad (2.4b)$$

#### Example 2.5 — id:20141120-203846.

Simplify  $6x + 7x$

(S)

#### Solution:

Notice that the indeterminate of each monomial is of degree 1; however, the exponent 1 is not shown. The monomials  $6x$  and  $7x$  have a like term of  $x$ .

$$\begin{array}{ll} (6+7)x & \text{DPF(2.4a)} \\ 13x & \text{OOA(1.1)} \end{array}$$

Notice that the sum of two monomials that have like terms can be found by adding the coefficients of the monomials. The distributive property in the factoring direction provides some insight to why we can add the coefficients of monomials that have like terms.

(S)

#### Less Steps Solution:

$$13x \quad \text{OOA(1.1)}$$

■

#### Example 2.6 — id:20141027-075159.

Simplify  $7\text{cm} + 8\text{cm}$

(S)

**Solution:**

$$(7 + 8) \text{ cm}$$

$$15 \text{ cm}$$

DPF(2.4a)

OOA(1.1)



Remember, if a monomial does not have a coefficient factor, then it's implied that the coefficient factor is 1, the multiplicative identity, and consequently its not explicitly shown.

**Example 2.7 — id:20141121-185558.**

Simplify  $x + 5x$

**Solution:**

It can be useful when simplifying expressions to make the multiplicative identity (MId) factor explicit.

$$1x + 5x$$

MId(2.1a)

$$(1 + 5)x$$

DPF(2.4a)

$$6x$$

OOA(1.1)

**Less Steps Solution:**

$$1x + 5x$$

MId(2.1a)

$$6x$$

OOA(1.1)

As one becomes more experienced, there is no reason to make the multiplicative identity coefficient explicit.

**Less Steps Solution:**

$$6x$$

OOA(1.1)

**Definition 2.4.4 — Definition of Subtraction (DOS).**

$$a - b = a + \neg b \tag{2.5a}$$

$$a + \neg b = a - b \tag{2.5b}$$

**Notation 2.1** (Operation of Negation (ONeg)).

$$-a = \neg a \quad (2.6a)$$

$$\neg a = -a \quad (2.6b)$$

I have used a different symbol,  $\neg$ , as the prefix negation operator only to differentiate it from the minus sign infix operator symbol,  $-$ , which is also used as the infix operator for the dyadic operation of subtraction. I will refer to this change of symbol as ONeg. This is used only as a teaching tool and should not be confused with the logic negation operator. Another advantage of using this symbol is that it reduces the number of delimiters used in an expression for example,  $\neg a$  versus  $(-a)$ .

- Negative five:  $-5$
- Negative five:  $\neg 5$
- Four minus five:  $4 - 5$
- Four minus negative five:  $4 - \neg 5$
- Four minus negative five:  $4 - (-5)$
- Four minus negative five:  $4 - \neg 5$
- Negative four minus five:  $-4 - 5$
- Negative four minus five:  $\neg 4 - 5$

#### Example 2.8 — id:20141121-190857.

Simplify  $8x - 6x$

(S)

**Solution:**

$$\begin{array}{ll} 8x + \neg 6x & \text{DOS(2.5a)} \\ (8 + \neg 6)x & \text{DPF(2.4a)} \\ 2x & \text{OOA(1.1)} \end{array}$$

(S)

**Less Steps Solution:**

$$\begin{array}{ll} 8x + \neg 6x & \text{DOS(2.5a)} \\ 2x & \text{OOA(1.1)} \end{array}$$

■

#### Example 2.9 — id:20141121-193636.

Simplify  $3x - 5x$

(S)

**Solution:**

$$\begin{array}{ll}
 3x + -5x & \text{DOS(2.5a)} \\
 (3 + -5)x & \text{DPF(2.4a)} \\
 -2x & \text{OOA(1.1)} \\
 -2x & \text{ONeg(2.6b)}
 \end{array}$$

**S****Less Steps Solution:**

$$\begin{array}{ll}
 3x + -5x & \text{DOS(2.5a)} \\
 -2x & \text{OOA(1.1)} \\
 -2x & \text{ONeg(2.6b)}
 \end{array}$$

**S****Less Steps Solution:**

$$\begin{array}{ll}
 -2x & \text{OOA(1.1)}
 \end{array}$$

■

**Example 2.10 — id:20141106-150622.**Simplify  $13x - x$ **S****Solution:**

$$\begin{array}{ll}
 13x - 1x & \text{MId(2.1a)} \\
 13x + -1x & \text{DOS(2.5a)} \\
 (13 + -1)x & \text{DPF(2.4a)} \\
 12x & \text{OOA(1.1)}
 \end{array}$$

**S****Less Steps Solution:**

$$\begin{array}{ll}
 13x + -x & \text{DOS(2.5a)} \\
 12x & \text{OOA(1.1)}
 \end{array}$$

■

It is possible for a univariate monomial to have more than two terms in its non-canonical form. The associative property of addition will be used to help simplify these expressions.

**Definition 2.4.5 — Associative Property of Addition (APA).**

$$a + b + c = (a + b) + c \quad (2.7a)$$

$$a + b + c = a + (b + c) \quad (2.7b)$$

**Example 2.11 — id:20141121-184652.**

Simplify the expression  $3x + 7x + 8x$

**S**

$$\begin{array}{ll} (3x + 7x) + 8x & \text{APA(2.7a)} \\ (3 + 7)x + 8x & \text{DPF(2.4a)} \\ 10x + 8x & \text{OOA(1.1)} \\ (10 + 8)x & \text{DPF(2.4a)} \\ 18x & \text{OOA(1.1)} \end{array}$$

**S**

**Less Steps Solution:**

$$\begin{array}{ll} (3x + 7x) + 8x & \text{APA(2.7a)} \\ 10x + 8x & \text{OOA(1.1)} \\ 18x & \text{OOA(1.1)} \end{array}$$

You might have noticed that this expression could be simplified in one step by adding the coefficient of the three monomials  $3x$ ,  $7x$  and  $8x$ , which have the like term  $x$ .

**S**

**Less Steps Solution:**

$$18x \quad \text{OOA(1.1)}$$

■

**Example 2.12 — id:20141106-152020.**

Simplify  $4x - 2x - x$

**S**

**Solution:**

$4x - 2x - 1x$	MId(2.1a)
$4x + \neg 2x + \neg 1x$	DOS(2.5a)
$(4 + \neg 2)x + \neg 1x$	DPF(2.4a)
$2x + \neg 1x$	OOA(1.1)
$(2 + \neg 1)x$	DPF(2.4a)
$1x$	OOA(1.1)
$x$	MId(2.1b)

(S) \_\_\_\_\_

**Less Steps Solution:**

$4x + \neg 2x + \neg x$	DOS(2.5a)
$x$	OOA(1.1)

■

**Example 2.13 — id:20141108-194431.**Simplify  $-3 \cdot 7x - 2x \cdot 4$ 

(S) \_\_\_\_\_

**Solution:**

$\neg 3 \cdot 7x - 2x \cdot 4$	ONeg(2.6a)
$\neg 3 \cdot 7x + \neg 2x \cdot 4$	DOS(2.5a)
$\neg 3 \cdot 7 \cdot x + \neg 2 \cdot x \cdot 4$	JTC(1.6)
$\neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x$	CPM(1.8)
$(\neg 3 \cdot 7) \cdot x + (\neg 2 \cdot 4) \cdot x$	APM(1.10a)
$\neg 21 \cdot x + \neg 8 \cdot x$	OOM(1.4)
$\neg 21x + \neg 8x$	CTJ(1.7)
$(\neg 21 + \neg 8)x$	DPF(2.4a)
$\neg 29x$	OOA(1.1)
$-29x$	ONeg(2.6b)

(S) \_\_\_\_\_

**Less Steps Solution:**

$$\begin{array}{ll} -3 \cdot 7x + -2x \cdot 4 & \text{DOS(2.5a)} \\ -3 \cdot 7 \cdot x + -2 \cdot 4 \cdot x & \text{CPM(1.8)} \\ -21x + -8x & \text{OOM(1.4)} \\ -29x & \text{OOA(1.1)} \end{array}$$

■

**Example 2.14 — id:20141108-194156.**Simplify  $3 \cdot 5x + 3x \cdot 4$ 

(S) \_\_\_\_\_

**Solution:**

$$\begin{array}{ll} 3 \cdot 5 \cdot x + 3 \cdot x \cdot 4 & \text{JTC(1.6)} \\ 3 \cdot 5 \cdot x + 3 \cdot 4 \cdot x & \text{CPM(1.8)} \\ (3 \cdot 5) \cdot x + (3 \cdot 4) \cdot x & \text{APM(1.10a)} \\ 15 \cdot x + 12 \cdot x & \text{OOM(1.4)} \\ 15x + 12x & \text{CTJ(1.7)} \\ (15 + 12)x & \text{DPF(2.4a)} \\ 27x & \text{OOA(1.1)} \end{array}$$

(S) \_\_\_\_\_

**Less Steps Solution:**

$$\begin{array}{ll} 3 \cdot 5 \cdot x + 3 \cdot 4 \cdot x & \text{CPM(1.8)} \\ 15x + 12x & \text{OOM(1.4)} \\ 27x & \text{OOA(1.1)} \end{array}$$

■

**Example 2.15 — id:20141108-173613.**Simplify  $8x \cdot 5$ 

(S) \_\_\_\_\_

**Solution:**

$8 \cdot x \cdot 5$	JTC(1.6)
$8 \cdot 5 \cdot x$	CPM(1.8)
$(8 \cdot 5) \cdot x$	APM(1.10a)
$40 \cdot x$	OOM(1.4)
$40x$	CTJ(1.7)

■

**Binomials****Definition 2.4.6 — Commutative Property of Addition (CPA).**

$$\textcolor{red}{ab} = \textcolor{blue}{ba} \quad (2.8)$$

**Definition 2.4.7 — Distributive Property Expanding (DPE).**

$$\textcolor{red}{a}(b+c) = \textcolor{blue}{ab} + \textcolor{blue}{ac} \quad (2.9a)$$

$$(b+c)\textcolor{red}{a} = \textcolor{blue}{ba} + \textcolor{blue}{ca} \quad (2.9b)$$

**Example 2.16 — id:20141109-090809.**Simplify by expanding  $5(x+4)$ 

S

**Solution:**

$5(1x+4)$	MId(2.1a)
$5 \cdot 1x + 5 \cdot 5$	DPF(2.4a)
$5 \cdot 1 \cdot x + 5 \cdot 5$	JTC(1.6)
$5 \cdot x + 25$	OOM(1.4)
$5x + 25$	CTJ(1.7)

S

**Less Steps Solution:**

$$5x + 20 \quad \text{DPE}(2.9a)$$

■

**Example 2.17 — id:20141109-091015.**Simplify by expanding  $5(3x-9)$

**S****Solution:**

$5(3x + -9)$	DOS(2.5a)
$5 \cdot 3x + 5 \cdot -9$	DPE(2.9a)
$5 \cdot 3 \cdot x + 5 \cdot -9$	JTC(1.6)
$15 \cdot x + -45$	OOM(1.4)
$15x + -45$	CTJ(1.7)
$15x - 45$	DOS(2.5b)

**S****Less Steps Solution:**

$5(3x + -9)$	DOS(2.5a)
$15x + -40$	DPE(2.9a)
$15x - 40$	DOS(2.5b)

■

**Example 2.18 — id:20141109-092448.**Simplify by expanding  $-(5x + 7)$ **S****Solution:**

$-1(5x + 7)$	MId(2.1a)
$-1 \cdot 5x + -1 \cdot 7$	DPE(2.9a)
$-1 \cdot 5 \cdot x + -1 \cdot 7$	JTC(1.6)
$-5 \cdot x + -7$	OOM(1.4)
$-5x + -7$	CTJ(1.7)
$-5x - 7$	DOS(2.5b)
$-5x - 7$	ONeg(2.6b)

**S****Less Steps Solution:**

$-5x - 7$	DPE(2.9a)
-----------	-----------

■

**Example 2.19 — id:20141109-092651.**

Simplify by expanding  $-13(7x - 9)$

**S**

---

**Solution:**

$$\begin{array}{ll}
 -13(7x + -9) & \text{DOS(2.5a)} \\
 -13 \cdot 7x + -13 \cdot -9 & \text{DPE(2.9a)} \\
 -13 \cdot 7 \cdot x + -13 \cdot -9 & \text{JTC(1.6)} \\
 -91 \cdot x + 117 & \text{OOM(1.4)} \\
 -91x + 117 & \text{CTJ(1.7)} \\
 -91x + 117 & \text{ONeg(2.6b)}
 \end{array}$$

**S**

---

**Less Steps Solution:**

$$\begin{array}{ll}
 -13(7x + -9) & \text{DOS(2.5a)} \\
 -91x + 117 & \text{DPE(2.9a)}
 \end{array}$$

■

**Example 2.20 — id:20141109-092910.**

Simplify by expanding  $a(x + b)$ , where  $a, b \in \mathbb{Z}$

**S**

---

**Solution:**

$$\begin{array}{ll}
 a(1x + b) & \text{MId(2.1a)} \\
 a \cdot 1x + a \cdot b & \text{DPE(2.9a)} \\
 a \cdot 1 \cdot x + a \cdot b & \text{JTC(1.6)} \\
 1 \cdot a \cdot x + a \cdot b & \text{CPM(1.8)} \\
 1ax + ab & \text{JTC(1.6)} \\
 ax + ab & \text{MId(2.1b)}
 \end{array}$$

**S**

---

**Less Steps Solution:**

$$ax + ab \quad \text{DPE(2.9a)}$$

**Example 2.21 — id:20141109-093220.**

Simplify by expanding  $5(x + 2) + 4$

**S****Solution:**

$5(1x + 2) + 4$	MId(2.1a)
$5 \cdot 1x + 5 \cdot 2 + 4$	DPE(2.9a)
$5 \cdot 1 \cdot x + 5 \cdot 2 + 4$	JTC(1.6)
$5 \cdot x + 10 + 4$	OOM(1.4)
$5x + 10 + 4$	CTJ(1.7)
$5x + 14$	OOA(1.1)

**S****Less Steps Solution:**

$5x + 10 + 4$	DPE(2.9a)
$5x + 14$	OOA(1.1)

**Example 2.22 — id:20141109-093419.**

Simplify by expanding  $7x + 5(4x + 8)$

**S****Solution:**

$7x + 5 \cdot 4x + 5 \cdot 8$	DPE(2.9a)
$7 \cdot x + 5 \cdot 4 \cdot x + 5 \cdot 8$	JTC(1.6)
$7 \cdot x + 20 \cdot x + 40$	OOM(1.4)
$7x + 20x + 40$	CTJ(1.7)
$(7 + 20)x + 40$	DPF(2.4a)
$27x + 40$	OOA(1.1)

**Example 2.23 — id:20141109-094928.**

Simplify by expanding  $4(3x + 4) + x + 6$

(S)

**Solution:**

$4(3x + 4) + 1x + 6$	MId(2.1a)
$4 \cdot 3x + 4 \cdot 4 + 1x + 6$	DPE(2.9a)
$4 \cdot 3 \cdot x + 4 \cdot 4 + 1 \cdot x + 6$	JTC(1.6)
$12 \cdot x + 16 + 1 \cdot x + 6$	OOM(1.4)
$12x + 16 + 1x + 6$	CTJ(1.7)
$12 + 1x + 16 + 6$	CPA(2.8)
$(12 + 1)x + 16 + 6$	DPF(2.4a)
$13x + 22$	OOA(1.1)

(S)

**Less Steps Solution:**

$12x + 16 + x + 6$	DPE(2.9a)
$12x + x + 16 + 6$	CPA(2.8)
$13x + 22$	OOA(1.1)

■

**Example 2.24 — id:20141109-095151.**

Simplify by expanding  $5(x - 4) + 3x - 5$

(S)

**Solution:**

$5(1x - 4) + 3x - 5$	MId(2.1a)
$5(1x + \neg 4) + 3x + \neg 5$	DOS(2.5a)
$5 \cdot 1x + 5 \cdot \neg 4 + 3x + \neg 5$	DPE(2.9a)
$5 \cdot 1 \cdot x + 5 \cdot \neg 4 + 3 \cdot x + \neg 5$	JTC(1.6)
$5 \cdot x + \neg 20 + 3 \cdot x + \neg 5$	OOM(1.4)
$5x + \neg 20 + 3x + \neg 5$	JTC(1.6)
$5x + 3x + \neg 20 + \neg 5$	CPA(2.8)
$(5 + 3)x + \neg 20 + \neg 5$	DPF(2.4a)
$8x + \neg 25$	OOA(1.1)
$8x - 25$	DOS(2.5b)

(S)

**Less Steps Solution:**

$5(x + \neg 4) + 3x + \neg 5$	DOS(2.5a)
$5x + \neg 20 + 3x + \neg 5$	DPE(2.9a)
$5x + 3x + \neg 20 + \neg 5$	CPA(2.8)
$8x + \neg 25$	OOA(1.1)
$8x - 25$	DOS(2.5b)

■

**Example 2.25 — id:20141109-095536.**Simplify by expanding  $8x - 5 - 4(x - 3)$ 

(S)

**Solution:**

$8x - 5 - 4(1x - 3)$	MId(2.1a)
$8x + \neg 5 + \neg 4(1x + \neg 3)$	DOS(2.5a)
$8x + \neg 5 + \neg 4 \cdot 1x + \neg 4 \cdot \neg 3$	DPE(2.9a)
$8 \cdot x + \neg 5 + \neg 4 \cdot 1 \cdot x + \neg 4 \cdot \neg 3$	JTC(1.6)
$8 \cdot x + \neg 5 + \neg 4 \cdot x + 12$	OOM(1.4)
$8x + \neg 5 + \neg 4x + 12$	CTJ(1.7)
$8x + \neg 4x + \neg 5 + 12$	CPA(2.8)
$(8 + \neg 4)x + \neg 5 + 12$	DPF(2.4a)
$4x + 7$	OOA(1.1)

**S****Less Steps Solution:**

$$\begin{array}{ll} 8x + -5 + -4(x + -3) & \text{DOS(2.5a)} \\ 8x + -5 + -4x + 12 & \text{DPE(2.9a)} \\ 8x + -4x + -5 + 12 & \text{CPA(2.8)} \\ 4x + 7 & \text{OOA(1.1)} \end{array}$$

■

**Example 2.26 — id:20141109-095842.**Simplify by expanding  $5(x + 3) + 3(x + 2)$ **S****Solution:**

$$\begin{array}{ll} 5 \cdot x + 5 \cdot 3 + 3 \cdot x + 3 \cdot 2 & \text{DPE(2.9a)} \\ 5 \cdot x + 15 + 3 \cdot x + 6 & \text{OOM(1.4)} \\ 5x + 15 + 3x + 6 & \text{CTJ(1.7)} \\ 5x + 3x + 15 + 6 & \text{CPA(2.8)} \\ (3 + 5)x + 15 + 6 & \text{DPF(2.4a)} \\ 8x + 21 & \text{OOA(1.1)} \end{array}$$

**S****Less Steps Solution:**

$$\begin{array}{ll} 5x + 15 + 3x + 6 & \text{DPE(2.9a)} \\ 5x + 3x + 15 + 6 & \text{CPA(2.8)} \\ 8x + 21 & \text{OOA(1.1)} \end{array}$$

■

**2.5 Degree 2 Univariate Polynomials****Monomials****Example 2.27 — id:20141106-151138.**Simplify  $4x^2 + 12x^2$ **S**

**Solution:**

$$\begin{array}{ll} (4 + 12)x^2 & \text{DPF(2.4a)} \\ 16x^2 & \text{OOA(1.1)} \end{array}$$

**S****Less Steps Solution:**

$$16x^2 \quad \text{OOA(1.1)}$$

■

**Example 2.28 — id:20141106-154547.**Simplify  $x^2 - x + x^2 + x$ **S****Solution:**

$$\begin{array}{ll} 1x^2 - 1x + 1x^2 + 1x & \text{MId(2.1a)} \\ 1x^2 + \neg 1x + 1x^2 + 1x & \text{DOS(2.5a)} \\ 1x^2 + 1x^2 + \neg 1x + 1x & \text{CPA(2.8)} \\ (1 + 1)x^2 + (\neg 1 + 1)x & \text{DPF(2.4a)} \\ 2x^2 + 0x & \text{OOA(1.1)} \\ 2x^2 & \text{MId(2.1b)} \end{array}$$

**S****Less Steps Solution:**

$$\begin{array}{ll} x^2 + \neg x + x^2 + x & \text{DOS(2.5a)} \\ x^2 + x^2 + \neg x + x & \text{CPA(2.8)} \\ 2x^2 & \text{OOA(1.1)} \end{array}$$

■

**Example 2.29 — id:20141108-194709.**Simplify  $-2x^4x - x \cdot -x^3$ **S**

**Solution:**

$-2x^4x - 1x \cdot 1x^3$	MId(2.1a)
$\neg 2x^4x - 1x \cdot 1x^3$	ONeg(2.6a)
$\neg 2 \cdot x \cdot 4 \cdot x + \neg 1 \cdot x \cdot 1 \cdot x \cdot 3$	JTC(1.6)
$\neg 2 \cdot 4 \cdot x \cdot x + \neg 1 \cdot \neg 1 \cdot 3 \cdot x \cdot x$	CPM(1.8)
$\neg 2 \cdot 4 \cdot x^2 + \neg 1 \cdot \neg 1 \cdot 3 \cdot x^2$	PrCBPo(2.10a)
$(\neg 2 \cdot 4) \cdot x^2 + (\neg 1 \cdot \neg 1 \cdot 3)x^2$	APM(1.10a)
$\neg 8x^2 + 3x^2$	OOM(1.4)
$(\neg 8 + 3)x^2$	DPF(2.4a)
$\neg 5x^2$	OOA(1.1)
$-5x^2$	ONeg(2.6b)

**S**

**Less Steps Solution:**

$-2x^4x + \neg x \cdot \neg x^3$	DOS(2.5a)
$\neg 2 \cdot 4 \cdot x \cdot x + 3 \cdot \neg x \cdot \neg x$	CPM(1.8)
$\neg 2 \cdot 4 \cdot x^2 + 3 \cdot x^2$	PrCBPo(2.10a)
$\neg 8x^2 + 3x^2$	OOM(1.4)
$-5x^2$	OOA(1.1)

#### Rule 2.5.1 — Product of a Common Base Powers (PrCBPo).

$$b^m \cdot b^n = b^{m+n} \quad (2.10a)$$

$$b^{m+n} = b^m \cdot b^n \quad (2.10b)$$

#### Rule 2.5.2 — Quotient of a Common Base Powers (QCBPo).

$$\frac{b^m}{b^n} = b^{m-n} \quad (2.11a)$$

$$b^{m-n} = \frac{b^m}{b^n} \quad (2.11b)$$

#### Rule 2.5.3 — Power of a Power (PoPo).

$$(b^m)^k = b^{m \cdot k} \quad (2.12a)$$

$$b^{m \cdot k} = (b^m)^k \quad (2.12b)$$

**Example 2.30 — id:20141108-191616.**

Simplify  $-5x \cdot 4x$

**S**

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**Solution:**

$-5x \cdot 4x$	ONeg(2.6a)
$-5 \cdot x \cdot 4 \cdot x$	JTC(1.6)
$-5 \cdot 4 \cdot x \cdot x$	CPM(1.8)
$-5 \cdot 4 \cdot x^2$	PrCBPo(2.10a)
$(-5 \cdot 4) \cdot x^2$	APM(1.10a)
$-20 \cdot x^2$	OOM(1.4)
$-20x^2$	CTJ(1.7)
$-20x^2$	ONeg(2.6b)

**S**

---

**Less Steps Solution:**

$-5 \cdot 4 \cdot x \cdot x$	CPM(1.8)
$-5 \cdot 4 \cdot x^2$	PrCBPo(2.10a)
$-20x^2$	OOM(1.4)

■

**Binomials****Example 2.31 — id:20141106-152339.**

Simplify  $3x^2 + 2x + 5x^2 + 4x$

**S**

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**Solution:**

$3x^2 + 5x^2 + 2x + 4x$	CPA(2.8)
$(3 + 5)x^2 + (2 + 4)x$	DPF(2.4a)
$8x^2 + 6x$	OOA(1.1)

If needed we could continue and express it in the simplified factored form using the distributive property

$$(4x + 3)2x \quad \text{DPF(2.4a)}$$

(S)

**Less Steps Solution:**

$$\begin{array}{rcl} 3x^2 + 5x^2 + 2x + 4x & & \text{CPA(2.8)} \\ 8x^2 + 6x & & \text{OOA(1.1)} \end{array}$$

■

**Example 2.32 — id:20141107-121834.**

$$\text{Simplify } (\sqrt{9 - x^2})^2$$

(S)

**Solution:**

$$\begin{array}{rcl} (\sqrt{9 - x^2})^2 & & \text{MId(2.1a)} \\ (\sqrt{9 + \neg 1x^2})^2 & & \text{DOS(2.5a)} \\ [(9 + \neg x^2)^{\frac{1}{2}}]^2 & & \text{RTPo(1.17)} \\ 9 + \neg 1x^2 & & \text{PoPo(2.12a)} \\ \neg 1x^2 + 9 & & \text{CPA(2.8)} \\ \neg x^2 + 9 & & \text{MId(2.1a)} \\ -x^2 + 9 & & \text{ONeg(2.6b)} \end{array}$$

(S)

**Less Steps Solution:**

$$9 - x^2 \quad \text{PoPo(2.12a)}$$

*It might be easier to view this using a variable substitution for the radicand,  $9 - x^2$ . Let  $k = 9 + \neg 1x^2$ .*

$(\sqrt{k})^2$	MId(2.1a)
$(\sqrt{k})^2$	DOS(2.5a)
$[(k)^{\frac{1}{2}}]^2$	RTPo(1.17)
$k$	PoPo(2.12a)
$9 + -1x^2$	CPA(2.8)
$-1x^2 + 9$	CPA(2.8)
$-x^2 + 9$	MId(2.1a)
$-x^2 + 9$	ONeg(2.6b)

**D**

Dependencies:example 6.2-20141105-144223

**Example 2.33 — id:20141209-145211.**Simplify  $2x(2x + 4) + x^2 \cdot 2 \cdot 1 + 0$ **S****Solution:**

$2x \cdot 2x + 2x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0$	DPE(2.9a)
$2 \cdot x \cdot 2 \cdot x + 2 \cdot x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0$	JTC(1.6)
$2 \cdot 2 \cdot x \cdot x + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0$	CPM(1.8)
$2 \cdot 2 \cdot x^2 + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0$	PrCBPo(2.10a)
$4 \cdot x^2 + 8 \cdot x + 2 \cdot x^2 + 0$	OOM(1.4)
$4x^2 + 8x + 2x^2 + 0$	CTJ(1.7)
$4x^2 + 2x^2 + 8x$	APA(2.7a)
$(4 + 2)x^2 + 8x$	DPF(2.4b)
$6x^2 + 8x$	OOA(1.1)

**D**

Dependencies:example 5.6-20141209-144203

**Trinomials**

**Example 2.34 — id:20141109-133008.**

Simplify by expanding  $(x + 5)(x - 8)$

**S**

---

**Solution:**

$$\begin{aligned}
 & (1x + 5)(1x - 8) && \text{MId(2.1a)} \\
 & (1x + 5)(1x + \neg 8) && \text{DOS(2.5a)} \\
 & 1x(1x + \neg 8) + 5(1x + \neg 8) && \text{DPE(2.9b)} \\
 & 1x \cdot 1x + 1x \cdot \neg 8 + 5 \cdot 1x + 5 \cdot \neg 8 && \text{DPE(2.9a)} \\
 & 1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot \neg 8 + 5 \cdot 1 \cdot x + 5 \cdot \neg 8 && \text{JTC(1.6)} \\
 & 1 \cdot 1 \cdot x \cdot x + \neg 8 \cdot 1 \cdot x + 1 \cdot 5 \cdot x + \neg 8 \cdot 5 && \text{CPM(1.8)} \\
 & 1 \cdot 1 \cdot x^2 + \neg 8 \cdot 1 \cdot x + 1 \cdot 5 \cdot x + \neg 8 \cdot 5 && \text{PrCBPo(2.10a)} \\
 & 1 \cdot x^2 + \neg 8 \cdot x + 5 \cdot x + \neg 40 && \text{OOM(1.4)} \\
 & 1x^2 + \neg 8x + 5x + \neg 40 && \text{CTJ(1.7)} \\
 & 1x^2 + \neg 3x + \neg 40 && \text{OOA(1.1)} \\
 & 1x^2 - 3x - 40 && \text{DOS(2.5b)} \\
 & x^2 - 3x - 40 && \text{MId(2.1a)}
 \end{aligned}$$

**S**

---

**Less Steps Solution:**

$$\begin{aligned}
 & (x + 5)(x + \neg 8) && \text{DOS(2.5a)} \\
 & x(x + \neg 8) + 5(x + \neg 8) && \text{DPE(2.9b)} \\
 & x^2 + \neg 8x + 5x + \neg 40 && \text{DPE(2.9a)} \\
 & x^2 - 3x - 40 && \text{OOA(1.1)}
 \end{aligned}$$

■

**Example 2.35 — id:20141109-133316.**

Simplify by expanding  $(x + a)(x + b)$ , where  $a, b \in \mathbb{Z}$

**S**

---

**Solution:**

$(1x + a)(1x + b)$	MId(2.1a)
$1x(1x + b) + a(1x + b)$	DPE(2.9b)
$1x \cdot 1x + 1x \cdot b + a \cdot 1x + a \cdot b$	DPE(2.9a)
$1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot b + a \cdot 1 \cdot x + a \cdot b$	JTC(1.6)
$1 \cdot 1 \cdot x \cdot x + 1 \cdot b \cdot x + 1 \cdot a \cdot x + a \cdot b$	CPM(1.8)
$1 \cdot 1 \cdot x^2 + 1 \cdot b \cdot x + 1 \cdot a \cdot x + a \cdot b$	PrCBPo(2.10a)
$1 \cdot x^2 + 1 \cdot b \cdot x + 1 \cdot a \cdot x + a \cdot b$	OOM(1.4)
$1x^2 + 1bx + 1ax + ab$	CTJ(1.7)
$1x^2 + (1b + 1a)x + ab$	DPF(2.4a)
$x^2 + (b + a)x + ab$	MId(2.1b)

**S****Less Steps Solution:**

$$\begin{aligned} x(x + b) + a(x + b) & \quad \text{DPE(2.9b)} \\ x^2 + (b + a)x + ab & \quad \text{DPE(2.9a)} \end{aligned}$$

■

**Example 2.36 — id:20141109-140659.**Simplify by expanding  $(2x + 3)(5x + 13)$ **S****Solution:**

$2x(5x + 13) + 3(5x + 13)$	DPE(2.9b)
$2x \cdot 5x + 2x \cdot 13 + 3 \cdot 5x + 3 \cdot 13$	DPE(2.9a)
$2 \cdot x \cdot 5 \cdot x + 2 \cdot x \cdot 13 + 3 \cdot 5 \cdot x + 3 \cdot 13$	JTC(1.6)
$2 \cdot 5 \cdot x \cdot x + 2 \cdot 13 \cdot x + 5 \cdot 3 \cdot x + 3 \cdot 13$	CPM(1.8)
$2 \cdot 5 \cdot x^2 + 2 \cdot 13 \cdot x + 5 \cdot 3 \cdot x + 3 \cdot 13$	PrCBPo(2.10a)
$10 \cdot x^2 + 26 \cdot x + 16 \cdot x + 39$	OOM(1.4)
$10x^2 + 26x + 15x + 39$	CTJ(1.7)
$10x^2 + 41x + 39$	OOA(1.1)

**S**

**Less Steps Solution:**

$$2x(5x + 13) + 3(5x + 13) \quad \text{DPE(2.9b)}$$

$$10x^2 + 26x + 15x + 39 \quad \text{DPE(2.9a)}$$

$$10x^2 + 41x + 39 \quad \text{OOA(1.1)}$$

■

**Example 2.37 — id:20141109-141019.**

Simplify by expanding  $(-3x - 5)(7x + 8)$

(S)

**Solution:**

$$(-3x - 5)(7x + 8) \quad \text{ONeg(2.6a)}$$

$$(-3x + -5)(7x + 8) \quad \text{DOS(2.5a)}$$

$$-3x(7x + 8) + -5(7x + 8) \quad \text{DPE(2.9b)}$$

$$-3x \cdot 7x + -3x \cdot 8 + -5 \cdot 7x + -5 \cdot 8 \quad \text{DPE(2.9a)}$$

$$-3 \cdot x \cdot 7 \cdot x + -3 \cdot x \cdot 8 + -5 \cdot 7 \cdot x + -5 \cdot 8 \quad \text{JTC(1.6)}$$

$$-3 \cdot 7 \cdot x \cdot x + -3 \cdot 8 \cdot x + -5 \cdot 7 \cdot x + -5 \cdot 8 \quad \text{CPM(1.8)}$$

$$-3 \cdot 7 \cdot x^2 + -3 \cdot 8 \cdot x + -5 \cdot 7 \cdot x + -5 \cdot 8 \quad \text{PrCBPo(2.10a)}$$

$$-21 \cdot x^2 + -24 \cdot x + -35 \cdot x + -40 \quad \text{OOM(1.4)}$$

$$-21x^2 + -24x + -35x + -40 \quad \text{CTJ(1.7)}$$

$$-21x^2 + -59x + -40 \quad \text{OOA(1.1)}$$

$$-21x^2 - 59x - 40 \quad \text{DOS(2.5b)}$$

$$-21x^2 - 59x - 40 \quad \text{ONeg(2.6b)}$$

(S)

**Less Steps Solution:**

$$(-3x + -5)(7x + 8) \quad \text{DOS(2.5a)}$$

$$-3x(7x + 8) + -5(7x + 8) \quad \text{DPE(2.9b)}$$

$$-21x^2 + -24x + -35x + -40 \quad \text{CTJ(2.9a)}$$

$$-21x^2 - 59x - 40 \quad \text{OOA(1.1)}$$

■

**Example 2.38 — id:20141109-141347.**

Simplify by expanding  $(ax + b)(cx + d)$ , where  $a, b, c, d \in \mathbb{Z}$

**S****Solution:**

$$ax(cx+d) + b(cx+d) \quad \text{DPE(2.9b)}$$

$$ax \cdot cx + ax \cdot d + b \cdot cx + b \cdot d \quad \text{DPE(2.9a)}$$

$$a \cdot x \cdot c \cdot x + a \cdot x \cdot d + b \cdot c \cdot x + b \cdot d \quad \text{JTC(1.6)}$$

$$a \cdot c \cdot x \cdot x + a \cdot d \cdot x + b \cdot c \cdot x + b \cdot d \quad \text{CPM(1.8)}$$

$$a \cdot c \cdot x^2 + a \cdot d \cdot x + b \cdot c \cdot x + b \cdot d \quad \text{PrCBPo(2.10a)}$$

$$acx^2 + adx + bcx + bd \quad \text{CTJ(1.7)}$$

$$acx^2 + (ad + bc)x + bd \quad \text{DPF(2.4a)}$$

**S****Less Steps Solution:**

$$ax(cx+d) + b(cx+d) \quad \text{DPE(2.9b)}$$

$$acx^2 + (ad + bc)x + bd \quad \text{DPE(2.9a)}$$

■

**Example 2.39 — id:20141105-161225.**

Simplify  $\left(2 - \frac{x}{2}\right)^2$  by expanding.

**S**

**Solution:**

$$\begin{aligned}
 & \left(2 - \frac{1x}{2}\right)^2 && \text{MId(2.1a)} \\
 & \left(2 + -\frac{1x}{2}\right)^2 && \text{DOS(2.5a)} \\
 & \left(2 + -\frac{1x}{2}\right) \left(2 + -\frac{1x}{2}\right) && \text{PoTF(1.13)} \\
 & 2 \left(2 + -\frac{1x}{2}\right) + -\frac{1x}{2} \left(2 + -\frac{1x}{2}\right) && \text{DPE(2.9b)} \\
 & 2 \cdot 2 + 2 \cdot -\frac{1x}{2} + -\frac{1x}{2} \cdot 2 + -\frac{1x}{2} \cdot -\frac{1x}{2} && \text{DPE(2.9a)} \\
 & 2 \cdot 2 + 2 \cdot -\frac{1}{2} \cdot x + -\frac{1}{2} \cdot x \cdot 2 + -\frac{1}{2} \cdot x \cdot -\frac{1}{2} \cdot x && \text{JTC(1.6)} \\
 & 2 \cdot 2 + 2 \cdot -\frac{1}{2} \cdot x + -\frac{1}{2} \cdot 2 \cdot x + -\frac{1}{2} \cdot -\frac{1}{2} \cdot x \cdot x && \text{CPM(1.8)} \\
 & 2 \cdot 2 + 2 \cdot -\frac{1}{2} \cdot x + -\frac{1}{2} \cdot 2 \cdot x + -\frac{1}{2} \cdot -\frac{1}{2} \cdot x^2 && \text{PrCBPo(2.10a)} \\
 & 4 + -1 \cdot x + -1 \cdot x + \frac{1}{4} \cdot x^2 && \text{OOM(1.4)} \\
 & 4 + -1x + -1x + \frac{1}{4}x^2 && \text{CTJ(1.7)} \\
 & \frac{1}{4}x^2 + -1x + -1x + 4 && \text{CPA(2.8)} \\
 & \frac{1}{4}x^2 + -2x + 4 && \text{OOA(1.1)} \\
 & \frac{1}{4}x^2 - 2x + 4 && \text{DOS(2.5b)}
 \end{aligned}$$

(S)

**Less Steps Solution:**

$$\begin{aligned}
 & 7x + 20x + 40 && \text{DPE(2.9a)} \\
 & 27x + 40 && \text{OOA(1.1)}
 \end{aligned}$$

■

**2.6 Degree 3 Univariate Polynomials****Monomials****Binomials****Trinomials****Polynomials****2.7 Degree  $n$  Univariate Polynomials****Monomials****Binomials****Trinomials****Polynomials**





## 3. Equations

### 3.1 Equality

**Property 3.1.1 — Reflexive Property of Equality (RPE).**

$$a = a \quad (3.1a)$$

**Property 3.1.2 — Substitution Property of Equality (SPE).**

Given  $a = b$ , then

$$E(a) = E(b) \quad (3.2)$$

$E(x)$  represents any expression.

**Property 3.1.3 — Symmetric Property of Equality (SyPE).**

$$a = b \quad \text{then} \quad b = a \quad (3.3a)$$

**Property 3.1.4 — Transitive Property of Equality (TPE).**

$$\text{if } a = b \quad \text{and} \quad b = c \quad \text{then} \quad a = c \quad (3.4a)$$

### 3.2 Solving Linear Equations

**Example 3.1 — id:20141206-102142.**

Solve the equation  $x + a = b$  for  $x$

**S**

---

**Solution:**

$$\begin{array}{ll} [x+a] + \neg a = [b] + \neg a & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x + (a + \neg a) = b + \neg a & \text{APA(2.7b)} \\ x + 0 = b + \neg a & \text{OOA(1.1)} \\ x = b + \neg a & \text{AId(2.3b)} \\ x = b - a & \text{DOS(2.5b)} \end{array}$$

■

**Example 3.2 — id:20141111-222931.**

Solve the equations  $x + 8 = 0$

**S**

---

**Solution:**

$$\begin{array}{ll} [x+8] + \neg 8 = [0] + \neg 8 & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x + (8 + \neg 8) = 0 + \neg 8 & \text{APA(2.7b)} \\ x + 0 = \neg 8 & \text{OOA(1.1)} \\ x = \neg 8 & \text{AId(2.3b)} \\ x = -8 & \text{ONeg(2.6b)} \end{array}$$

**S**

---

**Less Steps Solution:**

$$\begin{array}{ll} [x+8] + \neg 8 = [0] + \neg 8 & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x = -8 & \text{OOA(1.1)} \end{array}$$

**D**

---

**Dependencies:**

example 4.3-20141111-190212

■

**Example 3.3 — id:20141206-101632.**

Solve the equation  $x + 4 = 7$

(S) \_\_\_\_\_

**Solution:**

$$\begin{array}{ll} [x+4] + \neg 4 = [7] + \neg 4 & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x + (4 + \neg 4) = 7 + \neg 4 & \text{APA(2.7b)} \\ x + 0 = 3 & \text{OOA(1.1)} \\ x = 3 & \text{AId(2.3b)} \end{array}$$

(S) \_\_\_\_\_

**Less Steps Solution:**

$$\begin{array}{ll} [x] + 4 + \neg 4 = [7] + \neg 4 & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x = 3 & \text{OOA(1.1)} \end{array}$$

■

### Example 3.4 — id:20141206-101107.

Solve the equation  $x - 8 = 15$  for  $x$

(S) \_\_\_\_\_

**Solution:**

$$\begin{array}{ll} x + \neg 8 = 15 & \text{DOS(2.5a)} \\ [x + \neg 8] + 8 = [15] + 8 & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x + (-8 + 8) = 15 + 8 & \text{APA(2.7b)} \\ x + 0 = 23 & \text{OOA(1.1)} \\ x = 23 & \text{AId(2.3b)} \end{array}$$

(S) \_\_\_\_\_

**Less Steps Solution:**

$$\begin{array}{ll} [x + \neg 8] + 8 = [15] + 8 & \text{SPE(3.2)} + \text{AI(1.19a)} \\ x = 23 & \text{OOA(1.1)} \end{array}$$

■

**Example 3.5 — id:20141206-102404.**

Solve the equation  $5x = 9$  for  $x$ .

**S**

---

**Solution:**

$$\begin{aligned} \frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(3.2) + MI(1.9a)} \\ \frac{1}{5} \cdot [5 \cdot x] &= \frac{1}{5} \cdot 9 && \text{JTC(1.6)} \\ \left(\frac{1}{5} \cdot 5\right) \cdot x &= \frac{1}{5} \cdot 9 && \text{APM(1.10b)} \\ 1 \cdot x &= \frac{9}{5} && \text{OOM(1.4)} \\ x &= \frac{9}{5} && \text{MId(2.1b)} \end{aligned}$$

**S**

---

**Less Steps Solution:**

$$\begin{aligned} \frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(3.2) + MI(1.9a)} \\ x &= \frac{9}{5} && \text{OOM(1.4)} \end{aligned}$$

**Example 3.6 — id:20141206-104404.**

Solve the equation  $ax = b$  for  $x$ .

**S**

---

**Solution:**

$$\begin{aligned} \frac{1}{a}[ax] &= \frac{1}{a}[b] && \text{SPE(3.2) + MI(1.9a)} \\ \frac{1}{a} \cdot (a \cdot x) &= \frac{1}{a} \cdot b && \text{JTC(1.6)} \\ \left(\frac{1}{a} \cdot a\right) \cdot x &= \frac{1}{a} \cdot b && \text{APM(1.10b)} \\ 1 \cdot x &= \frac{b}{a} && \text{OOM(1.4)} \\ x &= \frac{b}{a} && \text{MId(2.1b)} \end{aligned}$$

**Example 3.7 — id:20141206-102723.**

Solve the equation  $-2x = 7$  for  $x$

(S)

**Solution:**

$$-2x = 7 \quad \text{ONeg(2.6a)}$$

$$-\frac{1}{2}[-2x] = -\frac{1}{2}[7] \quad \text{SPE(3.2) + MI(1.9b)}$$

$$-\frac{1}{2} \cdot (-2 \cdot x) = -\frac{1}{2} \cdot 7 \quad \text{JTC(1.6)}$$

$$\left(-\frac{1}{2} \cdot -2\right) \cdot x = -\frac{1}{2} \cdot 7 \quad \text{APM(1.10b)}$$

$$1 \cdot x = -\frac{7}{2} \quad \text{OOM(1.4)}$$

$$1 \cdot x = -\frac{7}{2} \quad \text{ONeg(2.6b)}$$

$$x = -\frac{7}{2} \quad \text{MId(2.1b)}$$

(S)

**Less Steps Solution:**

$$\begin{aligned} -\frac{1}{2}[-2x] &= -\frac{1}{2}[7] && \text{SPE(3.2) + MI(1.9b)} \\ x &= -\frac{7}{2} && \text{OOM(1.4)} \end{aligned}$$

■

**Example 3.8 — id:20141111-215726.**

Solve the equation  $2x + 5 = 0$  for  $x$

(S)

**Solution:**

$$\begin{aligned}
 [2x + 5] + \neg 5 &= [0] + \neg 5 && \text{SPE(3.2) + AI(1.19a)} \\
 2x + (5 + \neg 5) &= 0 + \neg 5 && \text{APA(2.7a)} \\
 2x + 0 &= \neg 5 && \text{OOA(1.1)} \\
 2x &= \neg 5 && \text{AId(2.3a)} \\
 \frac{1}{2}[2x] &= \frac{1}{2}[\neg 5] && \text{SPE(3.2) + MI(1.9a)} \\
 \frac{1}{2} \cdot 2 \cdot x &= \frac{1}{2} \cdot \neg 5 && \text{JTC(1.6)} \\
 \left(\frac{1}{2} \cdot 2\right) \cdot x &= \frac{1}{2} \cdot \neg 5 && \text{APM(1.10a)} \\
 1 \cdot x &= \frac{\neg 5}{2} && \text{OOM(1.4)} \\
 1x &= -\frac{5}{2} && \text{ONeg(2.6b)} \\
 x &= -\frac{5}{2} && \text{MId(2.1b)}
 \end{aligned}$$

(S)

**Less Steps Solution:**

$$\begin{aligned}
 [2x + 5] + \neg 5 &= [0] + \neg 5 && \text{SPE(3.2) + AI(1.19a)} \quad (3.5) \\
 2x &= \neg 5 && \text{OOA(1.1)} \quad (3.6) \\
 \frac{1}{2}[2x] &= \frac{1}{2}[\neg 5] && \text{SPE(3.2) + MI(1.9a)} \quad (3.7) \\
 x &= -\frac{5}{2} && \text{OOM(1.4)} \quad (3.8)
 \end{aligned}$$

(D)

**Dependencies:**

example 4.4-20141111-192213

■

**3.3 Solving Quadratic Equations****Example 3.9 — id:20141107-131748.**Solve the equation  $2 - x^2 = 0$  for  $x$ 

(S)

**Solution:**

$2 - 1x^2 = 0$	MId(2.1a)
$2 + \neg 1x^2 = 0$	DOS(2.5a)
$[2 + \neg 1x^2] + 1x^2 = [0] + 1x^2$	SPE(3.2) + AI(1.19a)
$2 + (-1x^2 + 1x^2) = 0 + 1x^2$	APA(2.7a)
$2 + 0 = 0 + 1x^2$	OOA(1.1)
$2 = 1x^2$	AId(2.3a)
$2 = x^2$	MId(2.1b)
$\pm [2]^{\frac{1}{2}} = [x^2]^{\frac{1}{2}}$	
$\pm 2^{\frac{1}{2}} = x$	PoPo(2.12a)
$\pm \sqrt{2} = x$	PoTR(1.18)
$x = \pm \sqrt{2}$	SyPE(3.3a)

■

**3.3.1 Completing the Square**

Completing the square is an algebraic algorithm used to find the solutions of quadratic equations of the form,  $ax^2 + bx + c = 0$ . Essentially, we want to manipulate this equation such that  $x = \text{some value}$ . To gain some understanding of how this algorithm works, we will consider each step individually. Let's begin with a quadratic equation in the general form:  $ax^2 + bx + c = 0$ .

Since we are trying to manipulate the equation  $ax^2 + bx + c = 0$  such that  $x = \text{some value}$ , we first want the coefficient factor  $a$  to be equal to 1. This is done by multiplying both expressions by the multiplicative inverse of  $a$  (step 1) followed by simplifying both expressions.

$$\begin{aligned} \frac{1}{a}[ax^2 + bx + c] &= \frac{1}{a}[0] && \text{SPE(3.2) + MI(1.9a)} \\ \frac{1}{a} \cdot ax^2 + \frac{1}{a} \cdot bx + \frac{1}{a} \cdot c &= \frac{1}{a}[0] && \text{DPE(2.9a)} \\ \frac{1}{a} \cdot a \cdot x^2 + \frac{1}{a} \cdot b \cdot x + \frac{1}{a} \cdot c &= \frac{1}{a}[0] && \text{JTC(1.6)} \\ x^2 + \frac{b}{a} \cdot x + \frac{c}{a} &= 0 && \text{OOM(1.4)} \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{CTJ(1.7)} \end{aligned}$$

We now have three terms in the left hand expression where the first two terms have at least one variable factor,  $x$ . The problem is that the third term is a constant and we want  $x = \text{some value}$ . This text step is to get rid of the  $\frac{c}{a}$  term, which can be done by using the additive inverse followed by simplifying both expressions.

$$\begin{aligned} \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right] + -\frac{c}{a} &= [0] + -\frac{c}{a} && \text{SPE(3.2) + AI(1.19a)} \\ x^2 + \frac{b}{a}x + \frac{c}{a} + -\frac{c}{a} &= 0 + -\frac{c}{a} && \text{APA(2.7a)} \\ x^2 + \frac{b}{a}x &= -\frac{c}{a} && \text{OOA(1.1)} \end{aligned}$$

The next step is called completing the square. The idea is to add a *NeW* constant,  $k$ , to the left-hand expression,  $x^2 + \frac{b}{a}x + k$ , such that the quadratic expression can then be factored as two identical factors,  $(x + m)(x + m) = (x + m)^2$ , where  $k = m \cdot m$ . Notice that since we are adding a constant term,  $k$ , to the left-hand expression, then we must also add this constant,  $k$ , to the right-hand expression,  $x^2 + \frac{b}{a}x + k = -\frac{c}{a} + k$ . To determine the values of both  $m$  and  $k$  we should refer to the organization of the two factors that make up the product of a quadratic expression,  $x^2 + \frac{b}{a}x + k = (x + m)^2$ .

Since both factors of this new quadratic expression are the same, both terms that make up the middle term, must also be the same. We know that  $mx + mx = \frac{b}{a}x$ , so we should be able to determine the value of  $m$  from this equation. If we can determine the value of  $m$ , then we can determine the value of  $k$ .

$$\begin{aligned} \frac{b}{a}x &= mx + mx \\ &= 2mx && \text{OOA(1.1)} \end{aligned}$$

Solving for  $m$ ,

$$\begin{aligned} 2mx &= \frac{b}{a}x && \\ 2 \cdot m \cdot x &= \frac{b}{a} \cdot x && \text{JTC(1.6)} \\ \frac{1}{2} [2 \cdot m \cdot x] &= \frac{1}{2} \left[ \frac{b}{a} \cdot x \right] && \text{SPE(3.2) + MI(1.9a)} \\ m \cdot x &= \frac{b}{2a} \cdot x && \text{OOM(1.4)} \\ [m \cdot x] \frac{1}{x} &= \left[ \frac{b}{2a} \cdot x \right] \frac{1}{x} && \text{SPE(3.2) + MI(1.9a)} \\ m &= \frac{b}{2a} && \text{OOM(1.4)} \end{aligned}$$

$\left[ x^2 + \frac{b}{a}x \right] + \left( \frac{b}{2a} \right)^2 = \left[ -\frac{c}{a} \right] + \left( \frac{b}{2a} \right)^2$	SPE(3.2)
$x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 = -\frac{c}{a} + \left( \frac{b}{2a} \right)^2$	APA(2.7b)
$\left( x + \frac{b}{2a} \right) \left( x + \frac{b}{2a} \right) = -\frac{c}{a} + \left( \frac{b}{2a} \right)^2$	DPF(2.4b)
$\left( x + \frac{b}{2a} \right)^2 = -\frac{c}{a} + \left( \frac{b}{2a} \right)^2$	PoTF(1.13)
$\left( x + \frac{b}{2a} \right)^2 = -\frac{c}{a} + \frac{(b)^2}{(2a)^2}$	PoQPo(1.11a)
$\left( x + \frac{b}{2a} \right)^2 = -\frac{c}{a} + \frac{b^2}{2^2 a^2}$	PoPrPo
$\left( x + \frac{b}{2a} \right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$	OOE
$\left( x + \frac{b}{2a} \right)^2 = -\frac{c}{a} \cdot \frac{\textcolor{red}{4a}}{\textcolor{red}{4a}} + \frac{b^2}{4a^2}$	MId
$\left( x + \frac{b}{2a} \right)^2 = -\frac{c \cdot 4 \cdot a}{a \cdot 4 \cdot a} + \frac{b^2}{4a^2}$	JTC
$\left( x + \frac{b}{2a} \right)^2 = -\frac{4 \cdot a \cdot c}{4 \cdot a \cdot a} + \frac{b^2}{4a^2}$	CPM
$\left( x + \frac{b}{2a} \right)^2 = -\frac{4 \cdot a \cdot c}{4 \cdot a^2} + \frac{b^2}{4a^2}$	PrCBPo
$\left( x + \frac{b}{2a} \right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$	CTJ
$\left( x + \frac{b}{2a} \right)^2 = \frac{-4ac + b^2}{4a^2}$	CD
$\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}$	CPA
$\left[ \left( x + \frac{b}{2a} \right)^2 \right]^{\frac{1}{2}} = \pm \left[ \frac{b^2 - 4ac}{4a^2} \right]^{\frac{1}{2}}$	PoI

$$x + \frac{b}{2a} = \pm \left[ \frac{b^2 + \neg 4ac}{4a^2} \right]^{\frac{1}{2}} \quad \text{PoPrPo}$$

$$x + \frac{b}{2a} = \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{[4a^2]^{\frac{1}{2}}} \quad \text{PoQPo}$$

$$x + \frac{b}{2a} = \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{4^{\frac{1}{2}}a} \quad \text{PoPrPo}$$

$$x + \frac{b}{2a} = \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{2a} \quad \text{OOE}$$

$$\left[ x + \frac{b}{2a} \right] + \neg \frac{b}{2a} = \left[ \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{2a} \right] + \neg \frac{b}{2a} \quad \text{AI}$$

$$x + \frac{b}{2a} + \neg \frac{b}{2a} = \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{2a} + \neg \frac{b}{2a} \quad \text{APA}$$

$$x + \frac{b}{2a} + \neg \frac{b}{2a} = \neg \frac{b}{2a} \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{2a} \quad \text{CPA}$$

$$x = \neg \frac{b}{2a} \pm \frac{[b^2 + \neg 4ac]^{\frac{1}{2}}}{2a} \quad \text{OOA}$$

$$x = \frac{\neg b \pm [b^2 + \neg 4ac]^{\frac{1}{2}}}{2a} \quad \text{CD}$$

$$x = \frac{\neg b \pm [b^2 - 4ac]^{\frac{1}{2}}}{2a} \quad \text{DOS}$$

$$x = \frac{\neg b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{ETR}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{ONeg}$$

<b>4</b>	<b>Functions</b>	55
4.1	Evaluating Functions	
4.2	Quadratic Functions	
4.3	Trigonometric Functions	
4.4	Rational Functions	
<b>5</b>	<b>Differentiation</b>	67
5.1	Limit of the Difference Quotient	
5.2	Derivative of a Monomial Functions	
5.3	Derivative of Polynomial Functions	
5.4	Derivative of Rational Functions	
5.5	Derivative of Trigonometric Functions	
5.6	Derivative of Exponential Functions	
5.7	Derivative of Logarithmic Functions	
5.8	Derivative of Composite Functions	
<b>6</b>	<b>Integration</b>	77
6.1	Integration Rules	
6.2	Volumes of Revolution	
	<b>Bibliography</b>	83
	<b>Bibliography</b>	83
	Websites	
	Articles	
	<b>Index</b>	85





## 4. Functions

### 4.1 Evaluating Functions

**Example 4.1 — id:20141106-083703.**

Given  $f(x) = \frac{1}{12}x^3 - x^2 + 4x$ , evaluate  $f(3) - f(2)$ .

(S) \_\_\_\_\_

**Solution:**

$$\begin{aligned}
 f(3) - f(2) &= \left( \frac{[3]^3}{12} - [3]^2 + 4[3] \right) - \left( \frac{[2]^3}{12} - [2]^2 + 4[2] \right) \\
 &= \left( \frac{(3)^3}{12} - (3)^2 + 4(3) \right) - 1 \left( \frac{(2)^3}{12} - (2)^2 + 4(2) \right) && \text{MId(2.1a)} \\
 &= \left( \frac{(3)^3}{12} + \neg(3)^2 + 4(3) \right) + \neg 1 \left( \frac{(2)^3}{12} + \neg(2)^2 + 4(2) \right) && \text{DOS(2.5a)} \\
 &= \left( \frac{27}{12} + \neg 9 + 4(3) \right) + \neg 1 \left( \frac{8}{12} + \neg 4 + 4(2) \right) && \text{OOE(1.5)} \\
 &= \left( \frac{27}{12} + \neg 9 + 12 \right) + \neg 1 \left( \frac{8}{12} + \neg 4 + 8 \right) && \text{OOM(1.4)} \\
 &= \left( \frac{27}{12} + 3 \right) + \neg 1 \left( \frac{8}{12} + 4 \right) && \text{OOA(1.1)} \\
 &= \left( \frac{27+36}{12} \right) + \neg 1 \left( \frac{8+48}{12} \right) && \text{FOOA(1.3a)} \\
 &= \left( \frac{63}{12} \right) + \neg 1 \left( \frac{56}{12} \right) && \text{OOA(1.1)} \\
 &= \left( \frac{63}{12} \right) + \neg \frac{56}{12} && \text{OOM(1.4)} \\
 &= \frac{7}{12} && \text{OOA(1.1)}
 \end{aligned}$$

(D)

**Dependencies:**

example 6.2-20141105-144223

■

## 4.2 Quadratic Functions

### 4.2.1 Completing the Square

**Definition 4.2.1 — Completing The Square.** Completing the square is the process used to convert a quadratic polynomial function  $f(x) = ax^2 + bx + c$  to the form

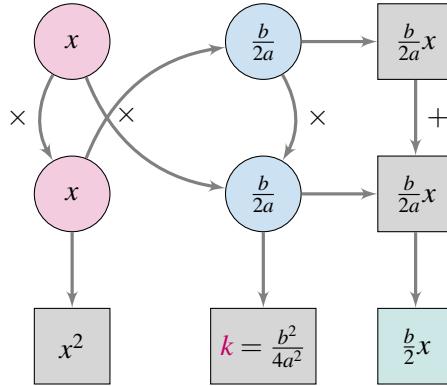
$$f(x) = a \left( x + \frac{b}{2a} \right)^2 + \left( c - \frac{b^2}{4a} \right)$$

We can simplify this form by defining  $B = \frac{b}{2a}$  and  $C = c - \frac{b^2}{4a}$ , which gives us

$$f(x) = a(x + B) + C$$

[mathworld:completethesquare ]

$$\begin{aligned}
 f(x) &= ax^2 + bx + c \\
 &= a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right] && \text{DPF(2.4b)} \\
 &= a \left[ x^2 + \frac{b}{a}x + \cancel{k} + \cancel{-k} + \frac{c}{a} \right] && \text{AId(2.3b)} \\
 &= a \left[ \left( x^2 + \frac{b}{a}x + \cancel{k} \right) + \left( \cancel{-k} + \frac{c}{a} \right) \right] && \text{APA(2.7b)}
 \end{aligned}$$

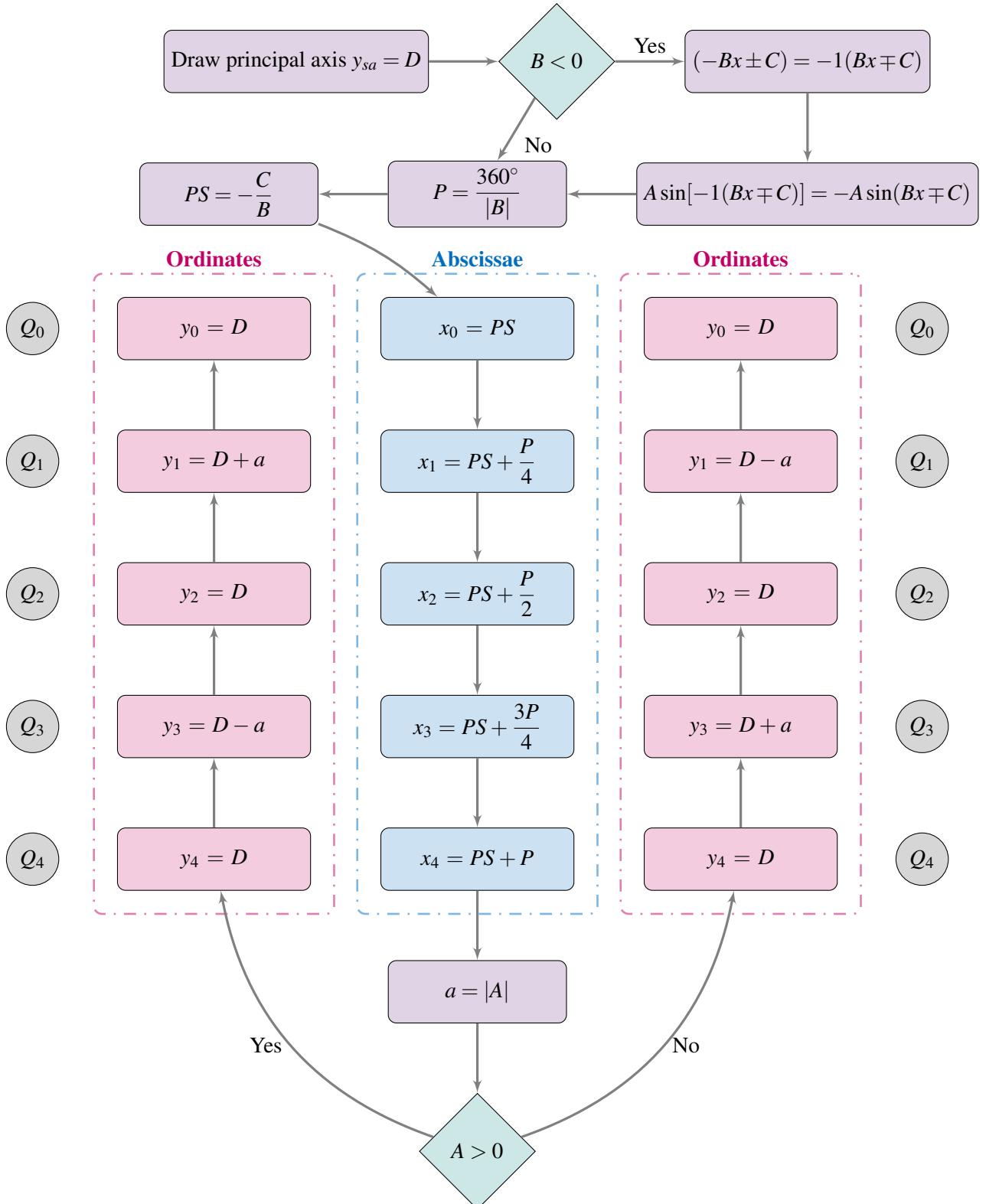
Figure 4.1: Factoring Organizer used to find the value of  $k$ 

$$\begin{aligned}
 f(x) &= a \left[ \left( x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) + \left( \frac{-b^2}{4a^2} + \frac{c}{a} \right) \right] \\
 &= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{-b^2}{4a^2} + \frac{c}{a} \right] && \text{DPF(2.4b)} \\
 &= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{-b^2}{4a^2} + \frac{4ac}{4a^2} \right] \\
 &= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right] && \text{OOA(1.1)} \\
 &= a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} && \text{DPE(2.9a)} \\
 &= a \left( x + \frac{b}{2a} \right)^2 + \left( c - \frac{b^2}{4a} \right)
 \end{aligned}$$

## 4.3 Trigonometric Functions

### 4.3.1 Graphing Trigonometric Function

## Graphing the Sine Function



**Example 4.2 — id:20150327-115953.**

Sketch the function  $g(x) = \sin(x) + 2$

**S**

**Solution:**

$$f(x) = A \sin(Bx + C) + D$$

$$g(x) = \sin(x) + 2$$

$$g(x) = 1 \sin(1x) + 2$$

MId(2.1a)

$$g(x) = 1 \sin(1x + 0) + 2$$

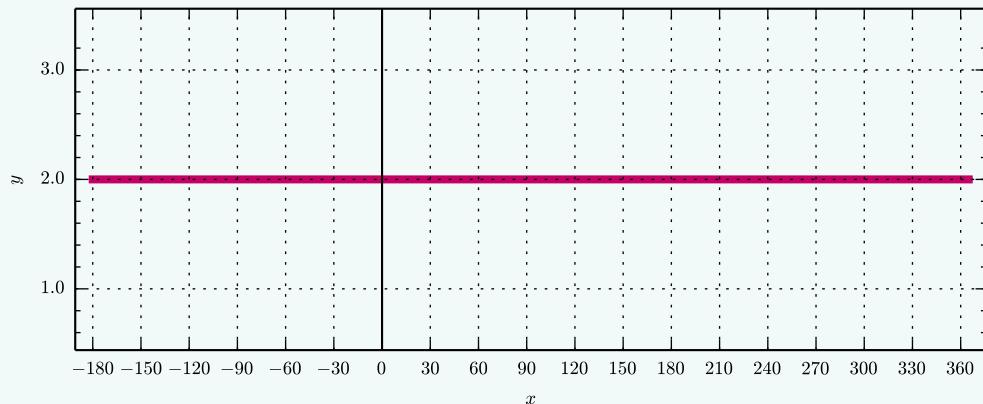
AId(2.3a)

Thus, A=1, B=1, C=0 and D=2

**Step 1:** Find and draw the sinusoidal-axis.

$$y_{sa} = D$$

$$y_{sa} = 2$$



**Step 2:** Check if the value of  $B < 0$

Since  $B > 0$ , there is nothing to do here.

**Step 3:** Calculate the period  $P$

$$P = \frac{360^\circ}{|B|}$$

$$P = \frac{360^\circ}{|[1]|}$$

SPE(3.2)

$$P = 360^\circ$$

**Step 4:** Calculate the phase shift  $PS$

$$\begin{aligned}
 Bx_{ps} + C &= 0 \\
 1x_{px} + 0 &= 0 && \text{SPE(3.2)} \\
 1x_{px} &= 0 && \text{AId(2.3b)} \\
 x_{ps} &= 0 && \text{MId(2.1b)}
 \end{aligned}$$

**Step 5:** State the abscissae of the some points of interest.

$$x_0 = PS$$

$$x_0 = 0^\circ$$

$$x_1 = PS + \frac{P}{4}$$

$$x_1 = 0^\circ + \frac{[360^\circ]}{4} && \text{SPE(3.2)}$$

$$x_1 = 0^\circ + 90^\circ$$

$$x_1 = 90^\circ$$

$$x_2 = PS + \frac{P}{2}$$

$$x_2 = 0^\circ + \frac{[360^\circ]}{2} && \text{SPE(3.2)}$$

$$x_2 = 0^\circ + 180^\circ$$

$$x_2 = 180^\circ$$

$$x_3 = PS + \frac{3P}{4}$$

$$x_3 = 0^\circ + \frac{3[360^\circ]}{4} && \text{SPE(3.2)}$$

$$x_3 = 0^\circ + 270^\circ$$

$$x_3 = 270^\circ$$

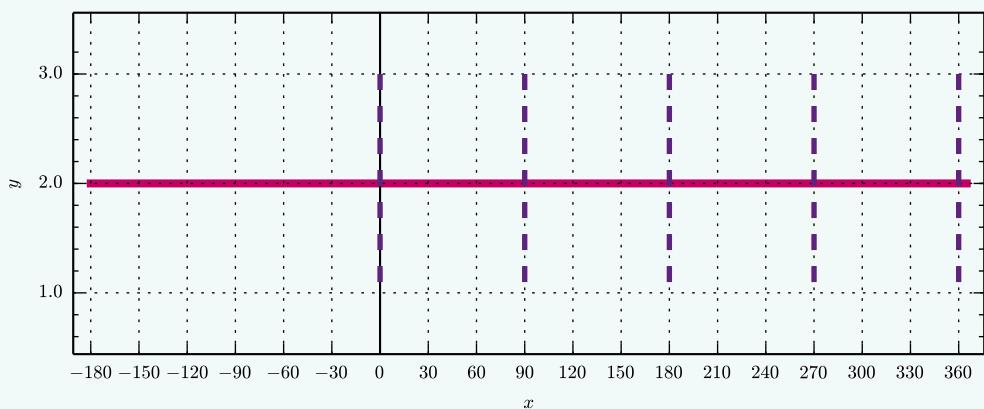
$$x_4 = PS + P$$

$$x_4 = 0^\circ + [360^\circ] && \text{SPE(3.2)}$$

$$x_4 = 0^\circ + 360^\circ$$

$$x_4 = 360^\circ$$

**Step 6:** Sketch the vertical equations of the lines to indicate the  $x$  values of the points that we will plot.



**Step 7:** Find the amplitude of the function.

$$a = |A|$$

$$a = ||[1]||$$

$$a = 1$$

**Step 8:** Determine if the functions  $y$  values are reflected about the  $x$ -axis.

Since the value of  $A = 1$  therefore  $A > 0$  and consequently there will be no reflection of the  $y$ -values about the  $x$ -axis.

**Step 9:** Find the corresponding ordinates to the abscissae found in step 5.

$$y_4 = D$$

$$y_4 = [2]$$

$$y_4 = 2$$

$$y_3 = D - a$$

$$y_3 = [2] - [1]$$

$$y_3 = 1$$

$$y_2 = D$$

$$y_2 = [2]$$

$$y_2 = 2$$

$$y_1 = D + a$$

$$y_1 = [2] + [1]$$

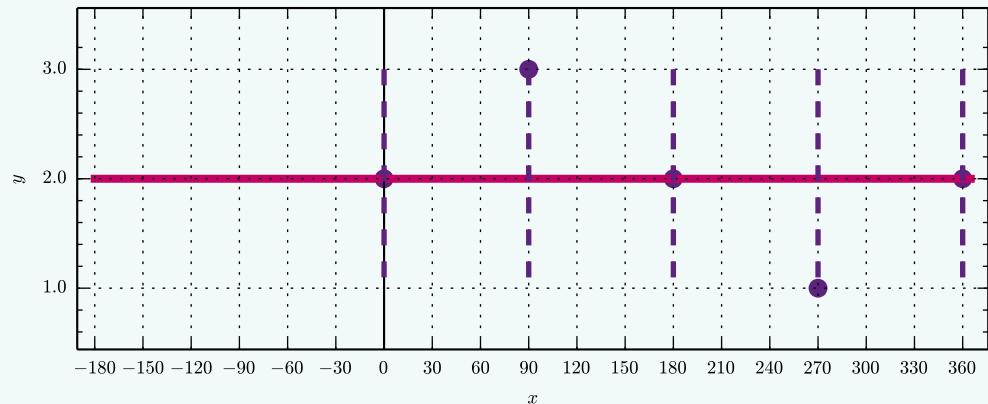
$$y_1 = 3$$

$$y_0 = D$$

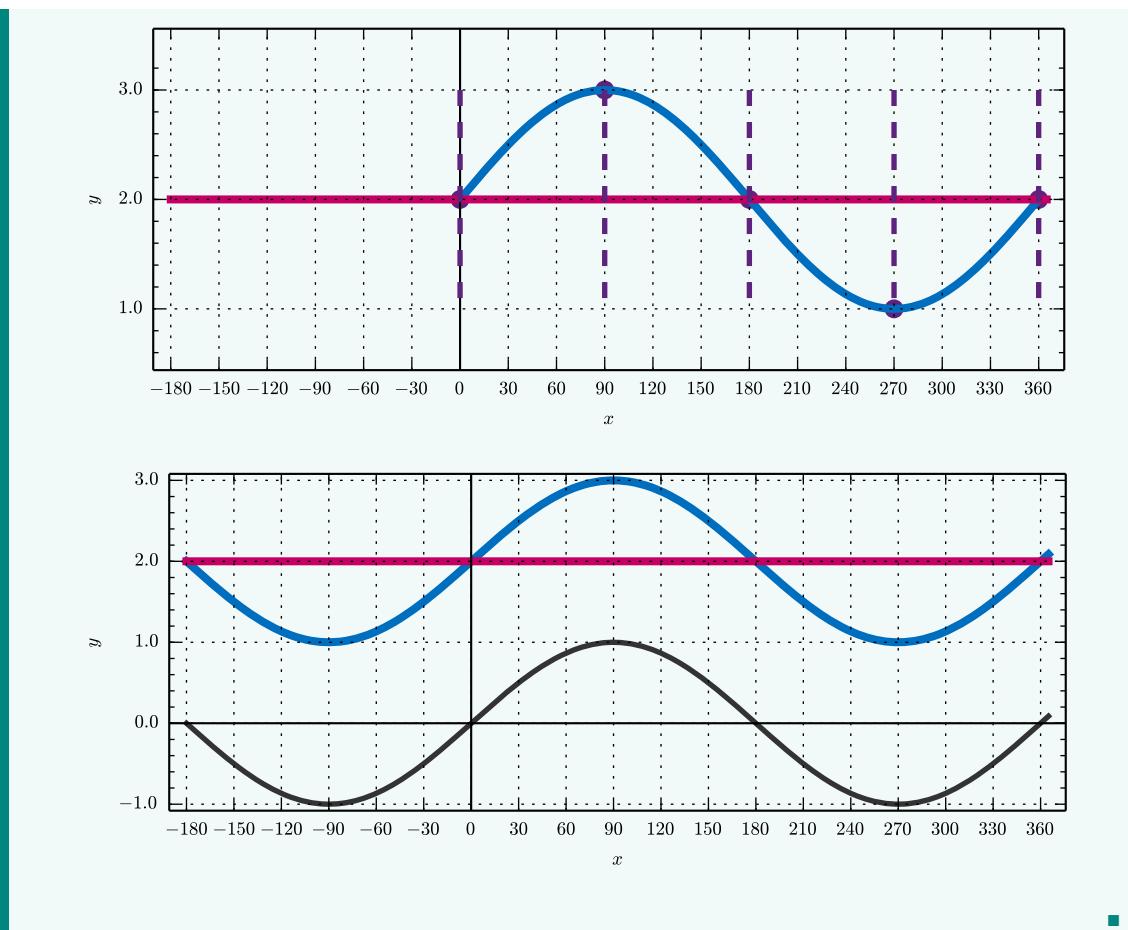
$$y_0 = [2]$$

$$y_0 = 2$$

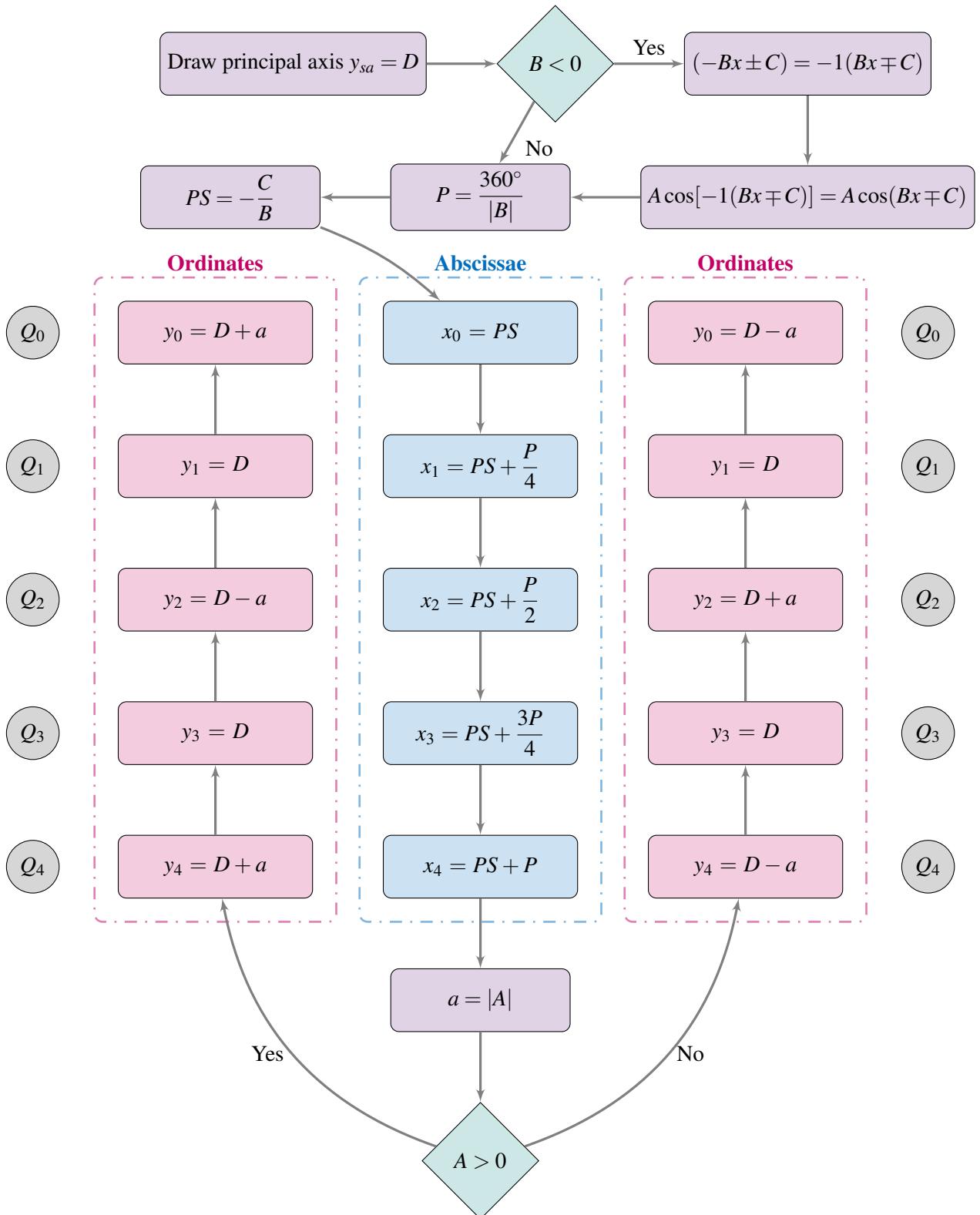
**Step 10:** Plot the points of interest.



**Step 10:** Sketch the function.



## Graphing the Cosine Function



## 4.4 Rational Functions

**Definition 4.4.1 — Asymptote.** An asymptote is a line or curve that approaches a given curve arbitrarily close.[mathworld:asymptote ]

A vertical asymptote is a vertical line  $x_{va} = c$ , that is used to visualize the values of  $x$  for which the function is not defined.

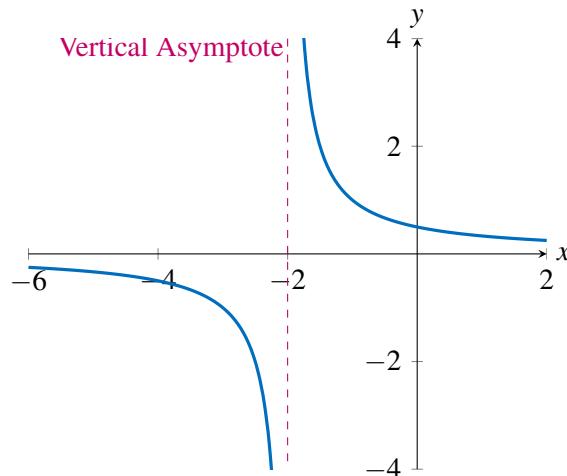


Figure 4.2: Vertical Asymptote

**Example 4.3 — id:20141111-190212.**

Find the vertical asymptote of the function  $R(x) = \frac{7}{x+8}$

(S)

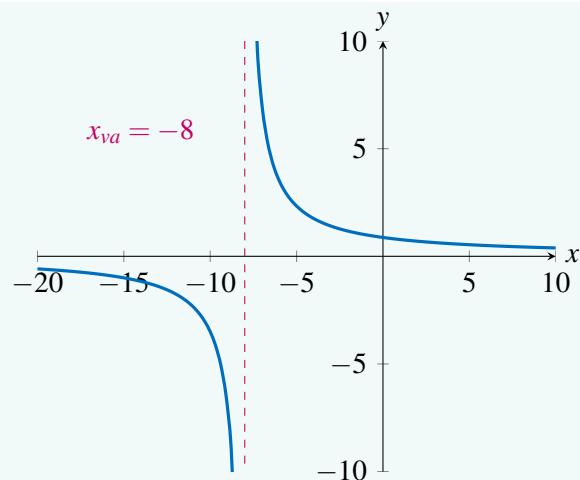
**Solution:**

We are interested in the values of  $x$  for which the denominator of  $R(x)$  has a value of zero.

$$x_{va} + 8 = 0$$

$$x_{va} = -8$$

solving


**Example 4.4 — id:20141111-192213.**

Find the vertical asymptote of the function  $R(x) = \frac{x+4}{2x+5}$

(S)

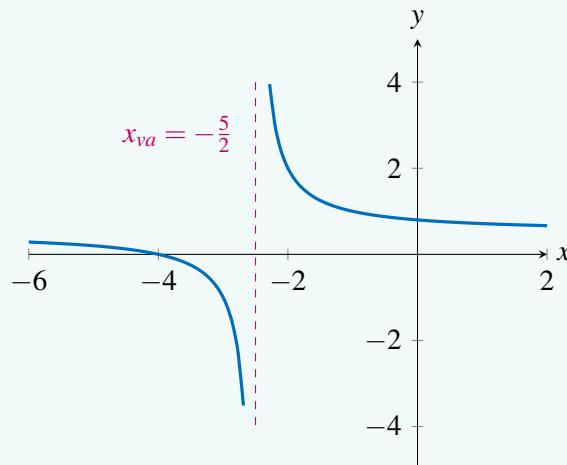
**Solution:**

We are interested in the values of  $x$  for which the denominator of  $R(x)$  has a value of zero.

$$2x_{va} + 5 = 0$$

$$x_{va} = -\frac{5}{2}$$

solving 3.8



## 5. Differentiation

### 5.1 Limit of the Difference Quotient

**Definition 5.1.1 — Derivative.** The derivative of a function  $f(x)$  with respect to the variable  $x$  is defined as

$$f'(x) \equiv \lim_{\Delta x \rightarrow 0} \underbrace{\frac{f(x + \Delta x) - f(x)}{\Delta x}}_{\text{Difference Quotient}} \quad (5.1)$$

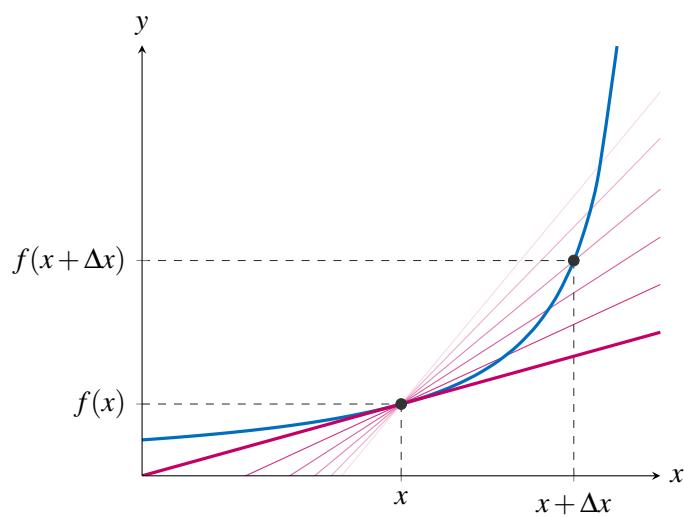


Figure 5.1: [mooculus:textbook ]

**Example 5.1 — id:20141219-212546.**

Differentiate the function  $f(x) = 5$

**S**

**Solution:**

$$f(x) = 5x^0 \quad \text{PoID(1.16a)}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5[x + \Delta x]^0 - 5[x]^0}{\Delta x} \quad \text{SPE(3.2)&DBFP(5.1)}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5(1) - 5(1)}{\Delta x} \quad \text{PoID(1.16b)}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} 0 \quad \text{OOM(1.4)}$$

$$f'(x) = 0$$

■

## 5.2 Derivative of a Monomial Functions

### Rule 5.2.1 — Derivative of a Constant (DC).

$$[c]' = 0 \quad (5.2)$$

$$\frac{d}{dx} [c] = 0 \quad (5.3)$$

### Rule 5.2.2 — Derivative of a Constant Multiple (DCM).

$$[cf(x)]' = c[f(x)]' \quad (5.4)$$

$$\frac{d}{dx} [cf(x)] = c \frac{d}{dx} [f(x)] \quad (5.5)$$

### Rule 5.2.3 — Derivative of a Power (DPo).

$$[x^n]' = nx^{n-1} \quad (5.6)$$

$$\frac{d}{dx} [x^n] = nx^{n-1} \quad (5.7)$$

**Example 5.2 — id:20141124-153017.**

Differentiate  $f(x) = -3$

**S**

**Solution:**

$$f'(x) = [-3]'$$

SPE(3.2)

$$f'(x) = 0$$

DC(5.2)

**D****Dependencies:**

example 5.4-20141124-152503

■

**Example 5.3 — id:20141124-141850.**Differentiate  $f(x) = x^2$ **S****Solution:**

$$f'(x) = [x^2]'$$

SPE(3.2)

$$f'(x) = 2x^{2-1}$$

DPo(5.6)

$$f'(x) = 2x^1$$

OOA(1.1)

$$f'(x) = 2x$$

MId(2,1b)

**S****Less Steps Solution:**

$$f'(x) = 2x$$

DPo(5.6)

**D****Dependencies:**

example 5.4-20141124-152503

■

**5.3 Derivative of Polynomial Functions****Rule 5.3.1 — Derivative of a Sum (DS).**

$$[f(x) + g(x)]' = f'(x) + g'(x) \quad (5.8)$$

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)] \quad (5.9)$$

**Example 5.4 — id:20141124-152503.**

Differentiate  $f(x) = x^2 - 3$

(S)

**Solution:**

$$\begin{array}{ll} f(x) = x^2 + -3 & \text{DOS(2.5a)} \\ f'(x) = [x^2 + -3]' & \text{SPE(3.2)} \\ f'(x) = [x^2]' + [-3]' & \text{DS(5.8)} \\ f'(x) = [x^2]' + 0 & \text{DC(5.2)} \\ f'(x) = [x^2]' & \text{AId(2.3a)} \\ f'(x) = 2x & \text{DPo(5.6) goto 5.3} \end{array}$$

(S)

**Less Steps Solution:**

$$f(x) = 2x^2 \quad \text{DPo(5.6)&DC(5.6)}$$

(D)

**Dependencies:**

example 5.12-20141124-205219

**Example 5.5 — id:20141128-151834.**

Differentiate  $f(x) = 3x^2 - 6x + 4$

(S)

**Solution:**

$$\begin{aligned}
 f(x) &= 3x^2 + -6x + 4 && \text{DOS(2.5a)} \\
 f'(x) &= [3x^2 + -6x + 4]' && \text{SPE(3.2)} \\
 f'(x) &= [3x^2]' + [6x]' + [4]' && \text{DS(5.8)} \\
 f'(x) &= [3x^2]' + [6x]' + 0 && \text{DC(5.2)} \\
 f'(x) &= [3x^2]' + [6x]' && \text{AId(2.3a)} \\
 f'(x) &= 3[x^2]' + 6[x]' && \text{DCM(5.4)} \\
 f'(x) &= 3(2x) + 6(1) && \text{DPo(5.6)} \\
 f'(x) &= 6x + 6 && \text{OOM(1.4)}
 \end{aligned}$$

(S)

$$f'(x) = 6x + 6 \quad \text{DS(5.8)}$$

■

**Rule 5.3.2 — Derivative of a Product (DPr).**

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \quad (5.10)$$

$$\frac{d}{dx} [f(x)g(x)] = \frac{d}{dx} [f(x)]g(x) + f(x)\frac{d}{dx} [g(x)] \quad (5.11)$$

**Example 5.6 — id:20141209-144203.**Differentiate  $f(x) = x^2(2x + 4)$ 

(S)

**Solution:**

$$\begin{aligned}
 f'(x) &= [x^2(2x + 4)]' && \text{SPE(3.2)} \\
 f'(x) &= [x^2]'(2x + 4) + x^2[2x + 4]' && \text{DPr(5.10)} \\
 f'(x) &= [x^2]'(2x + 4) + x^2[2x] + [4]' && \text{DS(5.8)} \\
 f'(x) &= [x^2]'(2x + 4) + x^2 \cdot 2[x] + [4]' && \text{DCM(5.4)} \\
 f'(x) &= 2x(2x + 4) + x^2 \cdot 2 \cdot 1 + [4]' && \text{DPo(5.6)} \\
 f'(x) &= 2x(2x + 4) + x^2 \cdot 2 \cdot 1 + 0 && \text{DC(5.2)} \\
 f'(x) &= 6x^2 + 8x && \text{simplify goto 2.33}
 \end{aligned}$$

■

**Example 5.7 — id:20141209-142321.**

Differentiate  $f(x) = x^2 \cos(x)$

(S)

**Solution:**

$$\begin{aligned} f'(x) &= [x^2 \cos(x)]' && \text{SPE(3.2)} \\ f'(x) &= [x^2]' \cos(x) + x^2 [\cos(x)]' && \text{DPr(5.10)} \\ f'(x) &= 2x \cos(x) + x^2 [\cos(x)]' && \text{DPo(5.6)} \\ f'(x) &= 2x \cos(x) + x^2 (-1 \sin(x)) && \text{DCos(5.16)} \\ f'(x) &= 2x \cos(x) - x^2 \sin(x) && \text{OOM(1.4)} \end{aligned}$$

■

**Example 5.8 — id:20141209-151354.**

Differentiate  $f(x) = \sin(x) \sin(x)$

(S)

**Solution:**

$$\begin{aligned} f'(x) &= [\sin(x) \sin(x)] && \text{SPE(3.2)} \\ f'(x) &= [\sin(x)]' \sin(x) + \sin(x) [\sin(x)]' && \text{DPr(5.10)} \\ f'(x) &= \cos(x) \sin(x) + \sin(x) \cos(x) && \text{DSin(5.14)} \\ f'(x) &= \cos(x) \sin(x) + \cos(x) \sin(x) && \text{CPM(1.8)} \\ f'(x) &= 2 \cos(x) \sin(x) && \text{OOA(1.1)} \end{aligned}$$

■

## 5.4 Derivative of Rational Functions

**Rule 5.4.1 — Derivative of a Quotient (DQ).**

$$\left[ \frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad (5.12)$$

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\frac{d}{dx} [f(x)] g(x) - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2} \quad (5.13)$$

## 5.5 Derivative of Trigonometric Functions

**Rule 5.5.1 — Derivative of Sine (DSin).**

$$[\sin(x)]' = \cos(x) \quad (5.14)$$

$$\frac{d}{dx} [\sin(x)] = \cos(x) \quad (5.15)$$

**Rule 5.5.2 — Derivative of Cosine (DCos).**

$$[\cos(x)]' = -\sin(x) \quad (5.16)$$

$$\frac{d}{dx} [\cos(x)] = -\sin(x) \quad (5.17)$$

## 5.6 Derivative of Exponential Functions

## 5.7 Derivative of Logarithmic Functions

**Rule 5.7.1 — Derivative of a Natural Logarithm (DNL).**

$$[\ln x]' = \frac{1}{x} \quad (5.18)$$

$$\frac{d}{dx} [\ln x] = \frac{1}{x} \quad (5.19)$$

## 5.8 Derivative of Composite Functions

**Rule 5.8.1 — Derivative of a Composite Function (DComp).**

$$[f(g(x))]' = [g(x)]' [f(g(x))]' \quad (5.20)$$

$$\frac{d}{dx} [f(g(x))] = \frac{d}{dx} [g(x)] \frac{d}{dx} [f(g(x))] \quad (5.21)$$

**Example 5.9 — id:20141124-203850.**

Differentiate  $y = \ln(3x)$

(S)

**Solution:**

After identifying that  $y = \ln(3x)$  is a composite function, we let  $u = 3x$  and thus we get a new function  $y = \ln(u)$ .

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{DComp}(5.21)$$

We need to find the factors  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

$$y = \ln(u) \quad u = 3x$$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = 3$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{DComp(5.21)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} && \text{DNL(5.18)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot 3 && \text{DPo(5.6)} \\ \frac{dy}{dx} &= \frac{1}{3x} \cdot 3 \\ \frac{dy}{dx} &= \frac{3}{3x} && \text{OOM(1.4)} \\ \frac{dy}{dx} &= \frac{1}{x}\end{aligned}$$

(D)

**Dependencies:**

example 5.12-20141124-205219

■

**Example 5.10 — id:20141128-160248.**Differentiate  $y = \ln(3x^2 - 6x + 4)$ 

(S)

**Solution:** After identifying that  $y = \ln(3x^2 - 6x + 4)$  is a composite function, we let  $u = 3x^2 - 6x + 4$  and thus we get a new function  $y = \ln(u)$ .

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{DComp(5.21)}$$

We need to find the factors  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

$$\begin{aligned}y &= \ln(u) \quad u = 3x^2 - 6x + 4 \\ \frac{dy}{du} &= \frac{1}{u} \quad \frac{du}{dx} = 6x - 6 \quad \text{goto 5.5}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{DComp(5.21)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} && \text{DNL(5.18)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot (6x - 6) && \text{DPo(5.6)} \\ \frac{dy}{dx} &= \frac{1}{3x^2 - 6x + 4} \cdot (6x - 6) \\ \frac{dy}{dx} &= \frac{6x - 6}{3x^2 - 6x + 4} && \text{OOM(1.4)}\end{aligned}$$

■

**Example 5.11 — id:20141128-155506.**Differentiate  $y = \ln(\cos x)$ 

(S)

**Solution:** After identifying that  $y = \ln(\cos x)$  is a composite function, we let  $u = \cos x$  and thus we get a new function  $y = \ln(u)$ .

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{DComp(5.21)}$$

We need to find the factors  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

$$y = \ln(u) \quad u = \cos x$$

$$\frac{dy}{du} = \frac{1}{u} \quad \frac{du}{dx} = -\sin x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{DComp(5.21)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot \frac{du}{dx} && \text{DNL(5.18)} \\ \frac{dy}{dx} &= \frac{1}{u} \cdot -\sin x && \text{DPo(5.6)} \\ \frac{dy}{dx} &= \frac{1}{\cos x} \cdot -\sin x \\ \frac{dy}{dx} &= \frac{-\sin x}{\cos x} && \text{OOM(1.4)} \\ \frac{dy}{dx} &= -\tan x\end{aligned}$$

■

**Example 5.12 — id:20141124-205219.**

Differentiate  $y = (x^2 - 1) \ln(3x)$

(S)

**Solution:**

$$y' = [x^2 - 3]' \cdot \ln x + (x^2 - 3) \cdot [\ln 3x]' \quad \text{DPr(5.10)}$$

$$y' = 2x \cdot \ln x + (x^2 - 3) \cdot [\ln 3x]' \quad \text{differentiate goto 5.4}$$

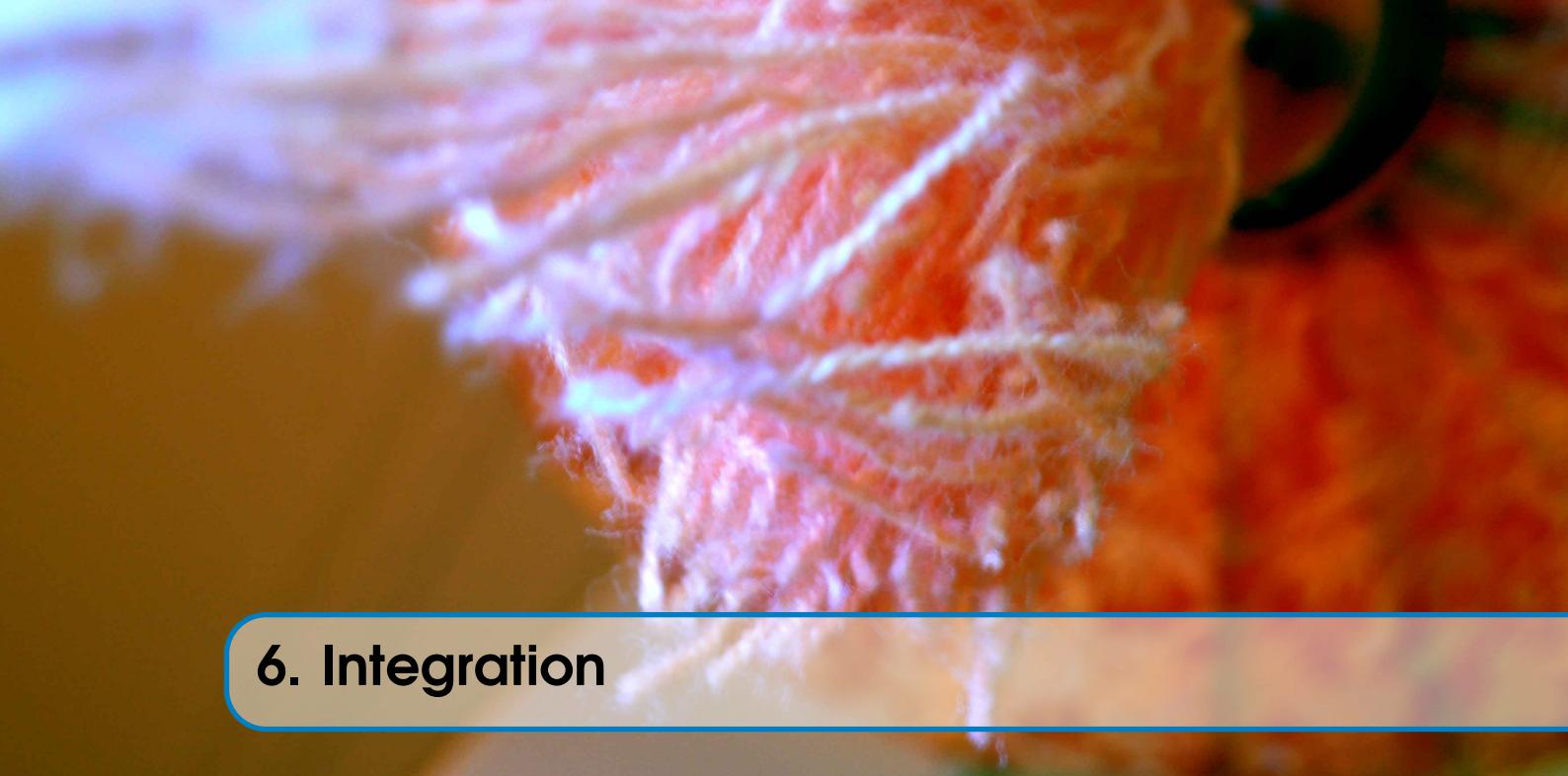
$$y' = 2x \cdot \ln x + (x^2 - 3) \cdot \frac{1}{x} \quad \text{differentiate goto 5.9}$$

$$y' = 2x \cdot \ln x + \frac{x^2 - 3}{x} \quad \text{OOM(1.4)}$$

$$y' = 2x^2 \ln x + \frac{x^2 - 3}{x} \quad \text{JTC(1.6)}$$

$$y' = \frac{2x^2 \ln x + (x^2 - 3)}{x} \quad \text{OOA(1.1)}$$

■



## 6. Integration

### 6.1 Integration Rules

**Rule 6.1.1 — Integral of a Constant (IC).**

$$\int c \, dx = cx + k \quad (6.1)$$

**Rule 6.1.2 — Integral of a Constant Multiple (ICM).**

$$\int cx \, dx = c \int x \, dx \quad (6.2)$$

**Rule 6.1.3 — Integral of a Power (IPo).**

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad \text{for } x \neq -1 \quad (6.3)$$

**Rule 6.1.4 — Integral of a Sum (IS).**

$$\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx \quad (6.4)$$

**Example 6.1 — id:20141105-172103.**

Integrate  $\int \frac{1}{4}x^2 - 2x + 4 \, dx$

(S) \_\_\_\_\_

**Solution:**

$$\begin{aligned}
 \int \frac{1}{4}x^2 - 2x + 4 \, dx &= \int \frac{1}{4}x^2 + -2x + 4 \, dx && \text{DOS(2.5a)} \\
 &= \int \frac{1}{4}x^2 \, dx + \int -2x \, dx + \int 4 \, dx && \text{IS(6.4)} \\
 &= \frac{1}{4} \int x^2 \, dx + -2 \int x \, dx + \int 4 \, dx && \text{ICM(6.2)} \\
 &= \frac{1}{4} \int x^2 \, dx + -2 \int x \, dx + 4(x) + c_1 && \text{IC(6.1)} \\
 &= \frac{1}{4} \left( \frac{x^3}{3} \right) + c_2 + -2 \left( \frac{x^2}{2} \right) + c_3 + 4(x) + c_1 && \text{IPo(6.3)} \\
 &= \frac{x^3}{12} + c_2 - x^2 + c_3 + 4x + c_1 && \text{OOM(1.4)} \\
 &= \frac{x^3}{12} - x^2 + 4x + \underbrace{c_1 + c_2 + c_3}_{\text{Just a constant}} && \text{CPA(2.8)} \\
 &= \frac{x^3}{12} - x^2 + 4x + C && \text{OOA(1.1)}
 \end{aligned}$$

(D)

**Dependencies:**

example 6.2-20141105-144223

■

**Theorem 6.1.5 — Fundamental Theorem of Calculus (FTThmC).** If  $f$  is continuous on the closed interval  $[a, b]$  and  $F$  is the indefinite integral of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) \, dx = F(b) - F(a) \quad (6.5a)$$

$$= [F(x)]_a^b \quad (6.5b)$$

## 6.2 Volumes of Revolution

**Definition 6.2.1 — Volume of Revolution: Method of Disks.** Let  $f$  be a nonnegative and continuous function on the closed interval  $[a, b]$ , then the solid of revolution obtained by rotating the curve  $f(x)$  about the  $x$ -axis from  $x = a$  to  $x = b$  has volume given by

$$V = \pi \int_a^b [f(x)]^2 \, dx \quad (6.6)$$

**Example 6.2 — id:20141105-144223.**

Find the volume of the solid formed by rotating the region enclosed by  $y = 2 - \frac{x}{2}$  and  $y = 0$  through  $2\pi$  about the  $x$ -axis on the closed interval  $[2, 3]$ .

**S****Solution:**

$$\begin{aligned}
 V &= \pi \int_a^b [f(x)]^2 dx && \text{VRD(6.6)} \\
 &= \pi \int_2^3 \left[2 - \frac{x}{2}\right]^2 dx && \text{SPE(3.2)} \\
 &= \pi \int_2^3 \frac{1}{4}x^2 + -2x + 4 dx && \text{simplify: goto 2.39} \\
 &= \pi \left[ \frac{x^3}{12} - x^2 + 4x \right]_2^3 && \text{integrate goto 6.1} \\
 &= \pi \left[ \frac{[3]^3}{12} - [3]^2 + 4[3] - \left( \frac{[2]^3}{12} - [2]^2 + 4[2] \right) \right] && \text{FThmC(6.5a)} \\
 &= \pi \left( \frac{7}{12} \right) && \text{evaluate: example 4.1} \\
 &= \frac{7}{12}\pi && \text{CPM(1.8)}
 \end{aligned}$$

■

**Example 6.3 — id:20141106-114907.**

Find the volume of the solid formed by rotating the region enclosed by  $y = \sqrt{9 - x^2}$  and  $y = 0$  through  $2\pi$  about the  $x$ -axis on the closed interval  $[1, 3]$ .

**S****Solution:**

$$\begin{aligned}
 V &= \pi \int_a^b [f(x)]^2 dx && \text{VRD(6.6)} \\
 &= \pi \int_1^3 \left[\sqrt{9-x^2}\right]^2 dx && \text{SPE(3.2)} \\
 &= \pi \int_1^3 9 - x^2 dx && \text{simplify: goto 2.32} \\
 &= \pi \left[ 9x - \frac{x^3}{3} \right]_1^3 && \text{integrate} \\
 &= \pi \left[ 9[3] - \frac{[3]^3}{3} - \left( 9[1] - \frac{[1]^3}{3} \right) \right] && \text{FThmC(6.5a)} \\
 &= \pi \left( \frac{28}{3} \right) && \text{evaluate} \\
 &= \frac{28}{3}\pi && \text{CPM(1.8)}
 \end{aligned}$$

■

**Example 6.4 — id:20141106-122528.**

Find the volume of the solid formed by rotating the region enclosed by  $y = \sqrt{\sin x}$  and  $y = 0$  through  $2\pi$  about the  $x$ -axis on the closed interval  $[\frac{\pi}{2}, \pi]$ .

(S)

**Solution:**

$$\begin{aligned}
 V &= \pi \int_a^b [f(x)]^2 dx && \text{VRD(6.6)} \\
 &= \pi \int_{\pi/2}^{\pi} [\sqrt{\sin \theta}]^2 d\theta && \text{SPE(3.2)} \\
 &= \pi \int_{\pi/2}^{\pi} \sin \theta d\theta && \text{PoPo(2.12a)} \\
 &= \pi [-1 \cos \theta]_{\pi/2}^{\pi} && \text{integrate} \\
 &= \pi \left[ -1 \cos [\pi] - \left( -1 \cos \left[ \frac{\pi}{2} \right] \right) \right] && \text{FThmC(6.5a)} \\
 &= \pi(1) && \text{evaluate} \\
 &= \pi && \text{MId(2.1b)}
 \end{aligned}$$

■

**Example 6.5 — id:20141108-083108.**

Find the volume of the solid formed by rotating the region enclosed by  $y = \sqrt{2 - x^2}$  and  $y = 0$  through  $2\pi$  about the  $x$ -axis.

(S)

Since no closed interval is stated, we will need to find where the curves  $y = \sqrt{2 - x^2}$  and  $y = 0$  intersect to determine the closed interval. Solving this system of equations (see example 3.9), we find that  $x = \pm\sqrt{2}$ .

$$\begin{aligned}
 V &= \pi \int_a^b [f(x)]^2 dx && \text{VRD(6.6)} \\
 &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [\sqrt{2-x^2}]^2 dx \\
 &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} 2-x^2 dx && \text{PoPo(2.12a)} \\
 &= \pi \left[ 2x - \frac{x^3}{3} \right]_{-\sqrt{2}}^{\sqrt{2}} && \text{integrate} \\
 &= \pi \left[ 2[\sqrt{2}] - \frac{[\sqrt{2}]^3}{3} - \left( 2[-\sqrt{2}] - \frac{[-\sqrt{2}]^3}{3} \right) \right] && \text{FThmC(6.5a)} \\
 &= \pi \left( \frac{8}{2} \sqrt{2} \right) && \text{simplify goto 2.32} \\
 &= \frac{8}{2} \pi \sqrt{2} && \text{CPM(1.8)}
 \end{aligned}$$

■

**Example 6.6 — id:20141125-084541.**

Find the volume of the solid formed by rotating the region enclosed by  $y = \sqrt{2-x^2}$  and  $y = 0$  through  $2\pi$  about the  $x$ -axis.

(S)

Since no closed interval is stated, we will need to find where the curves  $y = 4 - x^2$  and  $y = 0$  intersect to determine the closed interval. Solving this system of equations (see the similar example 3.9), we find that  $x = \pm 2$ .

Since the parabola  $y = 4 - x^2$  is symmetric about the  $y$ -axis, we can find the volume on the closed interval  $[0,2]$  and multiply this by a factor of two to include the volume on the closed interval  $[-2,0]$ .

(S)

**Solution:**

$$\begin{aligned} V &= 2 \cdot \pi \int_a^b [f(x)]^2 dx && \text{VRD(6.6)} \\ &= 2 \cdot \pi \int_0^2 [4 - x^2]^2 dx \\ &= 2 \cdot \pi \int_0^2 x^4 - 8x^2 + 16 dx \\ &= 2 \cdot \pi \left[ \frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2 \\ &= 2 \cdot \pi \left( \frac{[2]^5}{5} - \frac{8[2]^3}{3} + 16[2] \right) && \text{FThmC(6.5a)} \\ &= \frac{512\pi}{15} \end{aligned}$$

■



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# Index

- Additive Identity, 16
- Additive Inverse, 11
- Algebraic Expressions, 7
- Asymptote, 57
- Classification of Univariate Polynomial Expressions, 13
- Coefficient, 13
- Common Denominator, 8
- Definition of Subtraction, 18
- Degree  $n$  Univariate Polynomials, 41
- Degree -1 Univariate Polynomials, 15
- Degree 0 Univariate Polynomials, 15
- Degree 1 Univariate Polynomials, 15
- Degree 2 Univariate Polynomials, 30
- Degree 3 Univariate Polynomials, 41
- Degree of the Indeterminate, 14
- Degree of the Univariate Polynomial, 14
- Derivative, 61
- Derivatives
  - Derivative of a Composite Function, 67
  - Derivative of a Constant, 62
  - Derivative of a Constant Multiple, 62
  - Derivative of a Natural Logarithm, 67
  - Derivative of a Power, 62
  - Derivative of a Product, 65
  - Derivative of a Quotient, 66
  - Derivative of a Sum, 63
  - Derivative of Cosine, 67
  - Derivative of Sine, 67
- Equality, 43
- Example
  - 20141027-075159, 17
  - 20141105-144223, 72
  - 20141105-161225, 39
  - 20141105-172103, 71
  - 20141106-083703, 55
  - 20141106-114907, 73
  - 20141106-122528, 74
  - 20141106-150622, 20
  - 20141106-151138, 30
  - 20141106-152020, 21
  - 20141106-152339, 33
  - 20141106-154547, 31
  - 20141107-121834, 34
  - 20141107-131748, 48
  - 20141108-083108, 74
  - 20141108-085327, 11
  - 20141108-173613, 23
  - 20141108-191616, 33
  - 20141108-194156, 23
  - 20141108-194431, 22
  - 20141108-194709, 31
  - 20141109-090809, 24
  - 20141109-091015, 24
  - 20141109-092448, 25
  - 20141109-092651, 26
  - 20141109-092910, 26

- 20141109-093220, 27  
 20141109-093419, 27  
 20141109-094928, 28  
 20141109-095151, 28  
 20141109-095536, 29  
 20141109-095842, 30  
 20141109-133008, 36  
 20141109-133316, 36  
 20141109-140659, 37  
 20141109-141019, 38  
 20141109-141347, 38  
 20141111-190212, 58  
 20141111-192213, 59  
 20141111-215726, 47  
 20141111-222931, 44  
 20141120-202042, 16  
 20141120-202842, 14  
 20141120-203846, 17  
 20141121-093439, 16  
 20141121-093747, 15  
 20141121-184652, 21  
 20141121-185558, 18  
 20141121-190857, 19  
 20141121-193636, 19  
 20141124-141850, 63  
 20141124-152503, 64  
 20141124-153017, 62  
 20141124-203850, 67  
 20141124-205219, 70  
 20141125-084541, 75  
 20141128-151834, 64  
 20141128-155506, 69  
 20141128-160248, 68  
 20141206-101107, 45  
 20141206-101632, 44  
 20141206-102142, 44  
 20141206-102404, 46  
 20141206-102723, 47  
 20141206-104404, 46  
 20141209-142321, 66  
 20141209-144203, 65  
 20141209-145211, 35  
 20141209-151354, 66  
 20141219-212546, 62
- Fraction Operation of Addition, 8
- Indeterminate, 13
- Integrals
- Integral of a Constant, 71
- Integral of a Constant Multiple, 71  
 Integral of a Power, 71  
 Integral of a Sum, 71
- Like Terms, 15
- Method of Disks, 72
- Monomials of Like Terms, 11
- Multiplicative Identity, 14
- Multiplicative Inverse, 9
- Notation
- Center-Dot to Juxtaposition, 9
  - Juxtaposition to Center-Dot, 9
- Operation
- Operation of Addition, 7
  - Operation of Exponentiation, 9
  - Operation of Multiplication, 9
  - Operation of Negation, 19
- Polynomial Expressions, 7
- Power Identity, 11
- Power Inverse, 11
- Powers, 9
- Factor to Power, 11
  - Power of a Power, 32
  - Power of a Product of Powers, 11
  - Power of a Quotient of Powers, 9
  - Power to Factor, 11
  - Power to Radical, 11
  - Product of Common Base Powers, 32
  - Quotient of Common Base Powers, 32
  - Radical to Power, 11
- Property
- Associative Property of Addition, 21
  - Associative Property of Multiplication, 9
  - Commutative Property of Addition, 24
  - Commutative Property of Multiplication, 9
  - Distributive Property Factoring, 17, 24
- Reflexive Property of Equality, 43
- Substitution Property of Equality, 43
- Surds, 11
- Symmetric Property of Equality, 43
- Theorems
- Fundamental Theorem of Calculus, 72
- Transitive Property of Equality, 43

Univariate Monomial, 13

Univariate Polynomial Expression, 14