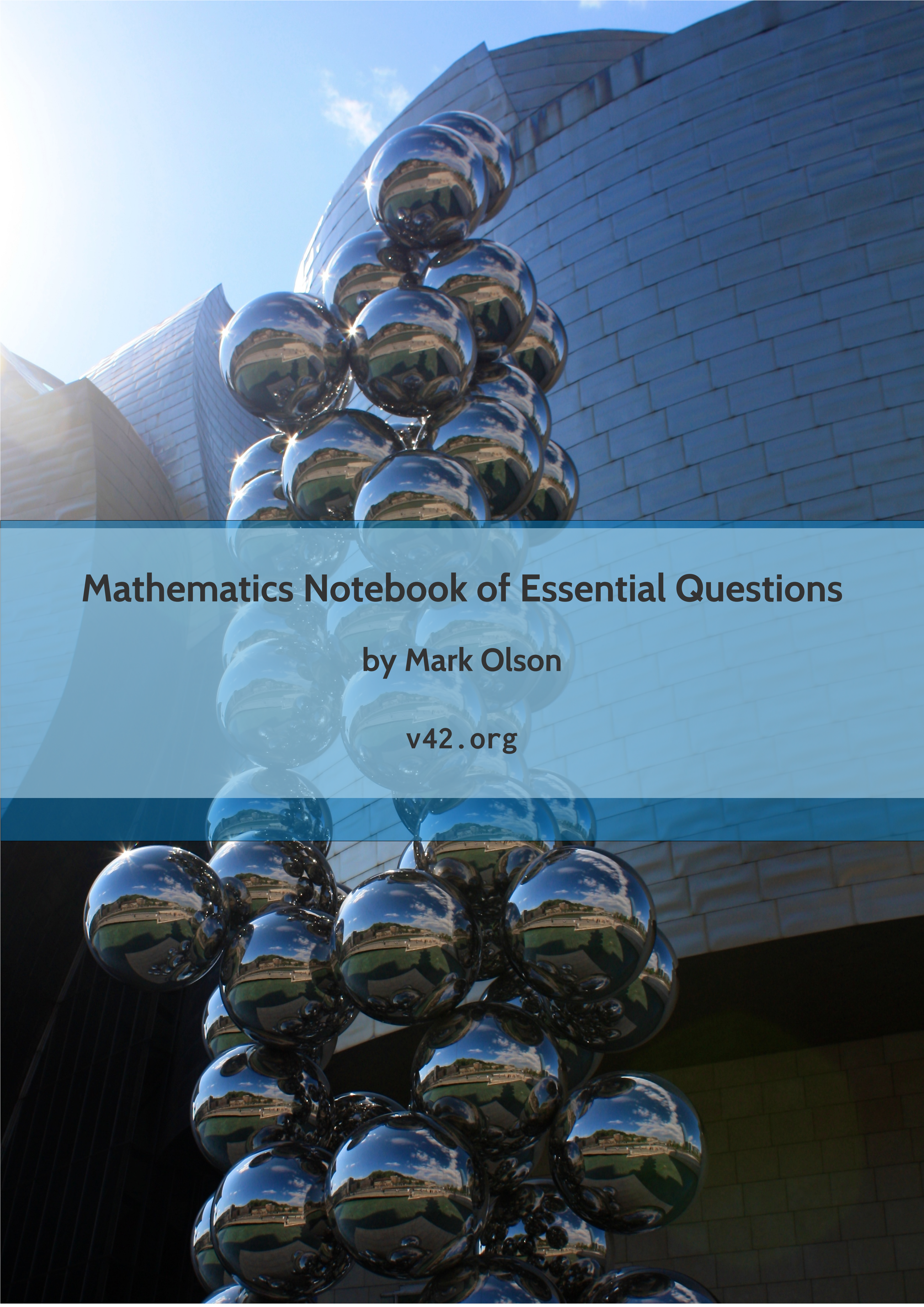

Mathematics Notebook of Essential Questions

by Mark Olson

v42.org

Contents

Reference

1. Operations

1.1 Dyadic Operations

Definition 1.1.1 – Operation of Addition (OOA).

$$\underbrace{\underbrace{a}_{\text{Augend}} + \underbrace{b}_{\text{Addend}}}_{\text{Sum}} \tag{1.1}$$

More generally,

$$\underbrace{\underbrace{a}_{\text{Summand}} + \underbrace{b}_{\text{Summand}}}_{\text{Sum}} \tag{1.2}$$

Definition 1.1.2 – Operation of Multiplication (OOM).

$$\underbrace{\underbrace{a}_{\text{Multiplicand}} \times \underbrace{b}_{\text{Multiplier}}}_{\text{Product}} \tag{1.3}$$

More generally,

$$\underbrace{\underbrace{a}_{\text{Factor}} \times \underbrace{b}_{\text{Factor}}}_{\text{Product}} \tag{1.4}$$

Definition 1.1.3 – Common Denominator (CD).

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \tag{1.5a}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \tag{1.5b}$$

Rule 1.1.1 – Fraction Operation of Addition (FOOA).

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \tag{1.6a}$$

$$\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d} \tag{1.6b}$$

2. Field Properties

2.1 Summary of Field Properties

Name	Addition	Multiplication
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive	$a(b + c) = ab + ac$	$(a + b)c = ac + bc$
Identity	$a + 0 = a = 0 + a$	$a \cdot 1 = a = 1 \cdot a$
Inverse	$a + (-a) = 0 = (-a) + a$	$a \cdot a^{-1} = 1 = a^{-1} \cdot a$

Table 2.1: Summary of the Field Properties

2.2 Properties of Addition

Definition 2.2.1 – Commutative Property of Addition (CPA).

$$ab = ba \tag{2.1}$$

Definition 2.2.2 – Associative Property of Addition (APA).

$$a + b + c = (a + b) + c \tag{2.2a}$$
$$a + b + c = a + (b + c) \tag{2.2b}$$

Definition 2.2.3 – Distributive Property Factoring (DPF).

$$ba + ca = (b + c)a \tag{2.3a}$$
$$ab + ac = a(b + c) \tag{2.3b}$$

Definition 2.2.4 – Additive Identity (AId).

$$a + 0 = a \tag{2.4a}$$
$$a = a + 0 \tag{2.4b}$$

Definition 2.2.5 – Additive Inverse (AI).

$$a + (-a) = 0 \tag{2.5a}$$

2.3 Properties of Multiplication

Definition 2.3.1 – Commutative Property of Multiplication (CPM).

$$a \cdot b = b \cdot a \tag{2.6}$$

Definition 2.3.2 – Associative Property of Multiplication (APM).

$$a \cdot b \cdot c = (a \cdot b) \cdot c \tag{2.7a}$$
$$a \cdot b \cdot c = a \cdot (b \cdot c) \tag{2.7b}$$

Definition 2.3.3 – Distributive Property Expanding (DPE).

$$a(b + c) = ab + ac \tag{2.8a}$$
$$(b + c)a = ba + ca \tag{2.8b}$$

Definition 2.3.4 – Multiplicative Identity (MId).

$$1a = a \tag{2.9a}$$
$$a = 1a \tag{2.9b}$$

R If the coefficient of a univariate monomial is the multiplicative identity 2.9a, 1, then it is not shown in it's canonical form.

$$\begin{aligned} C_k x^k &= C_k x^k \\ &= 1x^k \\ &= x^k \end{aligned}$$

Definition 2.3.5 – Multiplicative Inverse (MI).

$$a \cdot \frac{1}{a} = 1 \tag{2.10a}$$
$$a \cdot a^{-1} = 1 \tag{2.10b}$$

2.4 Properties of Subtraction

Definition 2.4.1 – Definition of Subtraction (DOS).

$$a - b = a + \neg b \tag{2.11a}$$
$$a + \neg b = a - b \tag{2.11b}$$

Definition 2.4.2 – Natural Numbers.

$$\mathbb{N} = \{0, 1, 2, 3 \dots\}$$

R It is not uncommon for zero to be excluded from the natural numbers. In fact, some exclude zero from the natural numbers and then describe the set of natural numbers that include zero the whole numbers.

$$\mathbb{W} = \{0, 1, 2, 3, \dots\}$$

For the purposes of these notes, zero will be included within the set of natural numbers.

Definition 2.4.3 – Operation of Addition (OOA).

$$\underbrace{\underbrace{a}_{\text{Augend}} + \underbrace{b}_{\text{Addend}}}_{\text{Sum}} \tag{2.12}$$

More generally,

$$\underbrace{\underbrace{a}_{\text{Summand}} + \underbrace{b}_{\text{Summand}}}_{\text{Sum}} \tag{2.13}$$

Definition 2.4.4 – Operation of Multiplication (OOM).

$$\underbrace{\underbrace{a}_{\text{Multiplicand}} \times \underbrace{b}_{\text{Multiplier}}}_{\text{Product}} \tag{2.14}$$

More generally,

$$\underbrace{\underbrace{a}_{\text{Factor}} \times \underbrace{b}_{\text{Factor}}}_{\text{Product}} \tag{2.15}$$

Definition 2.4.5 – Integers.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Definition 2.4.6 – Positive Integers.

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$$

Definition 2.4.7 – Greatest Common Divisor. Suppose that m and n are positive integers. The greatest common divisor is the largest divisor (factor) common to both m and n.

Definition 2.4.8 – Relatively Prime. Two integers m and n are relatively prime to each other, $m \perp n$, if they share no common positive integer divisors (factors) except 1.

$$m \perp n \text{ if } \gcd(m, n) = 1.$$

Definition 2.4.9 – Rational Numbers.

$$\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z}, n \neq 0\}$$

Definition 2.4.10 – Proper Fraction. Given $m < n$, then the fraction m/n is called proper.

Definition 2.4.11 – Improper Faction. Given $m > n$, then the fraction m/n is called improper.

Definition 2.4.12 – Common Denominator (CD).

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \tag{2.16a}$$

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b} \tag{2.16b}$$

Rule 2.4.1 – Fraction Operation of Addition (FOOA).

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \tag{2.17a}$$

$$\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d} \tag{2.17b}$$

II

Number Sense

Algebra

IV

Differential Calculus

V

MYP

3. Univariate Polynomial Expressions

3.1 Classification of Univariate Polynomial Expressions

Definition 3.1.1 – Indeterminate.

$$x$$

An indeterminate is a symbol that is treated as a variable, but does not stand for anything else but itself and is used as a placeholder.

- it does **not** designate a constant or a parameter
- it is **not** an unknown that could be solved for
- it is **not** a variable designating a function argument

[**wikipediate:indeterminate**]

Definition 3.1.2 – Coefficient.

$$Cx^k$$

A coefficient, C is a real number multiplicative factor.

Definition 3.1.3 – Univariate Monomial.

$$C_k x^k$$

A univariate monomial is made up of two factors. The first factor of a monomial, C_k, is the coefficient. The second factor of each monomial, x^k, is an indeterminate raised to a non-negative integer power k.

Example 3.1 – id:20141120-202842.

Express $1x^2$ in canonical form.

S

Solution:

$$x^2 \qquad \text{MId(2.9b)}$$

■

Definition 3.1.4 – Univariate Polynomial Expression.

$$\sum_{k=0}^{n} C_k x^n = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0} \tag{3.1}$$

A univariate polynomial in an indeterminate x is an expression made up of one or more summands of the form $C_k x^k$, which are called monomials. The first factor of each monomial, C_k, is a numerical factor called the coefficient where $C_k \in$. The second factor of each monomial, x^k, is an indeterminate raised to a non-negative integer power i.
[wikipedia:polynomial]

Definition 3.1.5 – Degree of the Indeterminate.

$$x^k$$

The exponent of an indeterminate power, k is called the degree of the indeterminate.
[wikipedia:polynomial]

Definition 3.1.6 – Degree of the Univarite Polynomial.

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

The degree of the univariate polynomial is determined by the monomial with the largest degree of the indeterminate.

3.2 Degree -1 Univariate Polynomials

Monomials

3.3 Degree 0 Univariate Polynomials

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

Degree 0 univariate polynomial expressions are made up of univariate monomials, C_0, called constants. The power identity is an indeterminate raised to a power of 0 has a value of 1. Thus, $x^0 = 1$ and results in the monomial $C_0 \cdot 1$. The canonical form of a product does not show the multiplicative identity factor, so what remains of this monomial product is only the coefficient factor C_0 and from now on will be referred to as a constant.

Example 3.2 – id:20141121-093747.

Express $13x^0$ in canonical form.

S

Solution:

13

Monomials

Degree 0 univariate polynomial expressions are usually a monomial in their canonical form if C_0 is a non-zero real number. The exception is if $C_0 = 0$, the additive identity, then the result is the zero polynomial, which can be considered a degree -1 polynomial.

The expression can be manipulated into its monomial canonical form by simplifying the expression. Simplifying the expression can be defined as evaluating the expression by following order of operations, which is the same as evaluating an arithmetic expression.

3.4 Degree 1 Univariate Polynomials

$$C_n x^n + C_{n-1} x^{n-1} + \cdots + C_k x^k + \cdots + C_2 x^2 + C_1 x^1 + \underbrace{C_0}_{C_0 x^0}$$

Degree 1 univariate polynomial expressions can be expressed with at most two different terms and consequently this expression in its canonical form has at most two monomial summands – called a binomial.

Definition 3.4.1 – Univariate Like Terms.

$$C_1 x^k = C_2 x^k$$

Two or more univariate monomials are defined as having like terms if each monomial has the same term, which will be the same indeterminate raised to the same positive integer power.

Sometimes the word term is used to describe monomials (including both the coefficient and the term), which may be confusing when trying to define like terms. For this reason, we will refer to the summands of a polynomial as monomials.

The monomials $5x^1$ and $3x^1$ can be described as having like terms because they share the common term x^1. One could also say that $5x^1$ and $3x^1$ are like terms by definition and consequenlty giving the reader an impression that $5x^1$ and $3x^1$ are terms themselves.

R A degree 1 indeterminate does not display the multiplicative identity in the exponent when its in canonical form.

Example 3.3 – id:20141120-202042.

Express $5x^1$ in canonical form

S

Solution:

$5x$ MId(2.9b)

Example 3.4 – id:20141121-093439.

Express $7x^1 + 5$ in canonical form.

S

Solution:

$7x + 5$ MId(2.9b)

Monomials

Essential Questions 3.1

1. How do we simplify univariate polynomial expressions?

If the constant monomial is 0, the additive identity, then the canonical form of a degree 1 univariate polynomial is a degree 1 monomial.

Simplifying Univariate Monomial Expressions

The definition of a univariate monomial expression is based on the expanded canonical form of some polynomial expression. It might be that the original expression might not be in the expanded canonical form, so a process called simplifying by expanding will be introduced to manipulate the expression such that it can be written in its expanded canonical form.

This process of simplifying by expanding polynomial expressions will be developed to the extent that it will be used to simplify multivariate polynomials. We will start by simplifying univariate monomial expressions.

Example 3.5 – id:20141120-203846.

Simplify $6x + 7x$

S

Solution:

Notice that the indeterminate of each monomial is of degree 1; however, the exponent 1 is not shown. The monomials $6x$ and $7x$ have a like term of x.

$(6+7)x$ DPF(2.3a)

$13x$ OOA(2.12)

Notice that the sum of two monomials that have like terms can be found by adding the coefficients of the monomials. The distributive property in the factoring direction provides some insight to why we can add the coefficients of monomials that have like terms.

S

Less Steps Solution:

$13x$ OOA(2.12)

Example 3.6 – id:20141027-075159.

Simplify $7\,\text{cm} + 8\,\text{cm}$

S

Solution:

$(7+8)\,\text{cm}$ DPF(2.3a)

$15\,\text{cm}$ OOA(2.12)

R Remember, if a monomial does not have a coefficient factor, then it's implied that the coefficient factor is 1, the multiplicative identity, and consequently its not explicitly shown.

Example 3.7 – id:20141121-185558.

Simplify $x+5x$

S

Solution:

It can be useful when simplifying expressions to make the multiplicative identity (MId) factor explicit.

$1x + 5x$	MId(2.9a)
$(1+5)x$	DPF(2.3a)
$6x$	OOA(2.12)

Less Steps Solution:

$1x + 5x$	MId(2.9a)
$6x$	OOA(2.12)

As one becomes more experienced, there is no reason to make the multiplicative identity coefficient explicit.

Less Steps Solution:

$6x$	OOA(2.12)

Notation 3.1 (Operation of Negation (ONeg))**.**

$$-a = \neg a \tag{3.2a}$$

$$\neg a = -a \tag{3.2b}$$

I have used a different symbol, ¬, as the prefix negation operator only to differentiate it from the minus sign infix operator symbol, −, which is also used as the infix operator for the dyadic operation of subtraction. I will refer to this change of symbol as ONeg. This is used only as a teaching tool and should not be confused with the logic negation operator. Another advantage of using this symbol is that it reduces the number of delimiters used in an expression for example, $\neg a$ *versus* $(-a)$.

- *Negative five:* -5
- *Negative five:* $\neg 5$
- *Four minus five:* $4 - 5$
- *Four minus negative five:* $4 - -5$
- *Four minus negative five:* $4 - (-5)$
- *Four minus negative five:* $4 - \neg 5$
- *Negative four minus five:* $-4 - 5$
- *Negative four minus five:* $\neg 4 - 5$

Example 3.8 – id:20141121-190857.

Simplify $8x - 6x$

Solution:

$8x + \neg 6x$	DOS(2.11a)
$(8 + \neg 6)x$	DPF(2.3a)
$2x$	OOA(2.12)

S

Less Steps Solution:

$8x + \neg 6x$	DOS(2.11a)
$2x$	OOA(2.12)

Example 3.9 – id:20141121-193636.

Simplify $3x - 5x$

S

Solution:

$3x + \neg 5x$	DOS(2.11a)
$(3 + \neg 5)x$	DPF(2.3a)
$\neg 2x$	OOA(2.12)
$-2x$	ONeg(3.2b)

S

Less Steps Solution:

$3x + \neg 5x$	DOS(2.11a)
$\neg 2x$	OOA(2.12)
$-2x$	ONeg(3.2b)

S

Less Steps Solution:

$-2x$	OOA(2.12)

Example 3.10 – id:20141106-150622.

Simplify $13x - x$

Solution:

$13x - 1x$	MId(2.9a)
$13x + \neg 1x$	DOS(2.11a)
$(13 + \neg 1)x$	DPF(2.3a)
$12x$	OOA(2.12)

Less Steps Solution:

$13x + \neg x$	DOS(2.11a)
$12x$	OOA(2.12)

It is possible for a univariate monomial to have more than two terms in its non-canonical form. The associative property of addition will be used to help simplify these expressions.

Example 3.11 – id:20141121-184652.

Simplify the expression $3x + 7x + 8x$

$(3x + 7x) + 8x$	APA(2.2a)
$(3 + 7)x + 8x$	DPF(2.3a)
$10x + 8x$	OOA(2.12)
$(10 + 8)x$	DPF(2.3a)
$18x$	OOA(2.12)

Less Steps Solution:

$(3x + 7x) + 8x$	APA(2.2a)
$10x + 8x$	OOA(2.12)
$18x$	OOA(2.12)

You might have noticed that this expression could be simplified in one step by adding the coefficient of the three monomials $3x$, $7x$ and $8x$, which have the like term x.

S

Less Steps Solution:

$18x$	OOA(2.12)

Example 3.12 – id:20141106-152020.

Simplify $4x - 2x - x$

S

Solution:

$4x - 2x - 1x$	MId(2.9a)
$4x + \neg 2x + \neg 1x$	DOS(2.11a)
$(4 + \neg 2)x + \neg 1x$	DPF(2.3a)
$2x + \neg 1x$	OOA(2.12)
$(2 + \neg 1)x$	DPF(2.3a)
$1x$	OOA(2.12)
x	MId(2.9b)

S

Less Steps Solution:

$4x + \neg 2x + \neg x$	DOS(2.11a)
x	OOA(2.12)

Example 3.13 – id:20141108-194431.

Simplify $-3 \cdot 7x - 2x \cdot 4$

S

Solution:

$$
\begin{aligned}
\neg 3 \cdot 7x - 2x \cdot 4 &\qquad \text{ONeg(3.2a)}\\
\neg 3 \cdot 7x + \neg 2x \cdot 4 &\qquad \text{DOS(2.11a)}\\
\neg 3 \cdot 7 \cdot x + \neg 2 \cdot x \cdot 4 &\qquad \text{JTC(\textbf{??})}\\
\neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x &\qquad \text{CPM(2.6)}\\
(\neg 3 \cdot 7) \cdot x + (\neg 2 \cdot 4) \cdot x &\qquad \text{APM(2.7a)}\\
\neg 21 \cdot x + \neg 8 \cdot x &\qquad \text{OOM(2.14)}\\
\neg 21x + \neg 8x &\qquad \text{CTJ(\textbf{??})}\\
(\neg 21 + \neg 8)x &\qquad \text{DPF(2.3a)}\\
\neg 29x &\qquad \text{OOA(2.12)}\\
-29x &\qquad \text{ONeg(3.2b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
-3 \cdot 7x + \neg 2x \cdot 4 &\qquad \text{DOS(2.11a)}\\
\neg 3 \cdot 7 \cdot x + \neg 2 \cdot 4 \cdot x &\qquad \text{CPM(2.6)}\\
\neg 21x + \neg 8x &\qquad \text{OOM(2.14)}\\
-29x &\qquad \text{OOA(2.12)}
\end{aligned}
$$

■

Example 3.14 – id:20141108-194156.

Simplify $3 \cdot 5x + 3x \cdot 4$

S

Solution:

$$
\begin{aligned}
3 \cdot 5 \cdot x + 3 \cdot x \cdot 4 &\qquad \text{JTC(\textbf{??})}\\
3 \cdot 5 \cdot x + 3 \cdot 4 \cdot x &\qquad \text{CPM(2.6)}\\
(3 \cdot 5) \cdot x + (3 \cdot 4) \cdot x &\qquad \text{APM(2.7a)}\\
15 \cdot x + 12 \cdot x &\qquad \text{OOM(2.14)}\\
15x + 12x &\qquad \text{CTJ(\textbf{??})}\\
(15 + 12)x &\qquad \text{DPF(2.3a)}\\
27x &\qquad \text{OOA(2.12)}
\end{aligned}
$$

S

Less Steps Solution:

$3 \cdot 5 \cdot x + 3 \cdot 4 \cdot x$	CPM(2.6)
$15x + 12x$	OOM(2.14)
$27x$	OOA(2.12)

■

Example 3.15 – id:20141108-173613.

Simplify $8x \cdot 5$

S

Solution:

$8 \cdot x \cdot 5$	JTC(**??**)
$8 \cdot 5 \cdot x$	CPM(2.6)
$(8 \cdot 5) \cdot x$	APM(2.7a)
$40 \cdot x$	OOM(2.14)
$40x$	CTJ(**??**)

■

Binomials

Example 3.16 – id:20141109-090809.

Simplify by expanding $5(x + 4)$

S

Solution:

$5(1x + 4)$	MId(2.9a)
$5 \cdot 1x + 5 \cdot 5$	DPF(2.3a)
$5 \cdot 1 \cdot x + 5 \cdot 5$	JTC(**??**)
$5 \cdot x + 25$	OOM(2.14)
$5x + 25$	CTJ(**??**)

S

Less Steps Solution:

$5x + 20$	DPE(2.8a)

■

Example 3.17 – id:20141109-091015.

Simplify by expanding $5(3x-9)$

S

Solution:

$$\begin{aligned}
5(3x+\neg 9) && \text{DOS(2.11a)}\\
5\cdot 3x+5\cdot\neg 9 && \text{DPE(2.8a)}\\
5\cdot 3\cdot x+5\cdot\neg 9 && \text{JTC(??)}\\
15\cdot x+\neg 45 && \text{OOM(2.14)}\\
15x+\neg 45 && \text{CTJ(??)}\\
15x-45 && \text{DOS(2.11b)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
5(3x+\neg 9) && \text{DOS(2.11a)}\\
15x+\neg 40 && \text{DPE(2.8a)}\\
15x-40 && \text{DOS(2.11b)}
\end{aligned}$$

Example 3.18 – id:20141109-092448.

Simplify by expanding $-(5x+7)$

S

Solution:

$$\begin{aligned}
\neg 1(5x+7) && \text{MId(2.9a)}\\
\neg 1\cdot 5x+\neg 1\cdot 7 && \text{DPE(2.8a)}\\
\neg 1\cdot 5\cdot x+\neg 1\cdot 7 && \text{JTC(??)}\\
\neg 5\cdot x+\neg 7 && \text{OOM(2.14)}\\
\neg 5x+\neg 7 && \text{CTJ(??)}\\
\neg 5x-7 && \text{DOS(2.11b)}\\
-5x-7 && \text{ONeg(3.2b)}
\end{aligned}$$

S

Less Steps Solution:

$$-5x - 7 \qquad \text{DPE(2.8a)}$$

■

Example 3.19 – id:20141109-092651.

Simplify by expanding $-13(7x - 9)$

S

Solution:

$$
\begin{aligned}
\neg 13(7x + \neg 9) & \qquad \text{DOS(2.11a)} \\
\neg 13 \cdot 7x + \neg 13 \cdot \neg 9 & \qquad \text{DPE(2.8a)} \\
\neg 13 \cdot 7 \cdot x + \neg 13 \cdot \neg 9 & \qquad \text{JTC(\textbf{??})} \\
\neg 91 \cdot x + 117 & \qquad \text{OOM(2.14)} \\
\neg 91x + 117 & \qquad \text{CTJ(\textbf{??})} \\
-91x + 117 & \qquad \text{ONeg(3.2b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
-13(7x + \neg 9) & \qquad \text{DOS(2.11a)} \\
-91x + 117 & \qquad \text{DPE(2.8a)}
\end{aligned}
$$

■

Example 3.20 – id:20141109-092910.

Simplify by expanding $a(x + b)$, where $a, b \in \mathbb{Z}$

S

Solution:

$$
\begin{aligned}
a(1x + b) & \qquad \text{MId(2.9a)} \\
a \cdot 1x + a \cdot b & \qquad \text{DPE(2.8a)} \\
a \cdot 1 \cdot x + a \cdot b & \qquad \text{JTC(\textbf{??})} \\
1 \cdot a \cdot x + a \cdot b & \qquad \text{CPM(2.6)} \\
1ax + ab & \qquad \text{JTC(\textbf{??})} \\
ax + ab & \qquad \text{MId(2.9b)}
\end{aligned}
$$

S

Less Steps Solution:

$$ax + ab \qquad \text{DPE(2.8a)}$$

Example 3.21 – id:20141109-093220.

Simplify by expanding $5(x+2)+4$

S

Solution:

$$\begin{aligned}
5(1x+2)+4 &\qquad \text{MId(2.9a)}\\
5\cdot 1x+5\cdot 2+4 &\qquad \text{DPE(2.8a)}\\
5\cdot 1\cdot x+5\cdot 2+4 &\qquad \text{JTC(??)}\\
5\cdot x+10+4 &\qquad \text{OOM(2.14)}\\
5x+10+4 &\qquad \text{CTJ(??)}\\
5x+14 &\qquad \text{OOA(2.12)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
5x+10+4 &\qquad \text{DPE(2.8a)}\\
5x+14 &\qquad \text{OOA(2.12)}
\end{aligned}$$

Example 3.22 – id:20141109-093419.

Simplify by expanding $7x+5(4x+8)$

S

Solution:

$$\begin{aligned}
7x + 5 \cdot 4x + 5 \cdot 8 &\quad& \text{DPE(2.8a)}\\
7 \cdot x + 5 \cdot 4 \cdot x + 5 \cdot 8 &\quad& \text{JTC(\textbf{??})}\\
7 \cdot x + 20 \cdot x + 40 &\quad& \text{OOM(2.14)}\\
7x + 20x + 40 &\quad& \text{CTJ(\textbf{??})}\\
(7 + 20)x + 40 &\quad& \text{DPF(2.3a)}\\
27x + 40 &\quad& \text{OOA(2.12)}
\end{aligned}$$

■

Example 3.23 – id:20141109-094928.

Simplify by expanding $4(3x + 4) + x + 6$

S

Solution:

$$\begin{aligned}
4(3x + 4) + 1x + 6 &\quad& \text{MId(2.9a)}\\
4 \cdot 3x + 4 \cdot 4 + 1x + 6 &\quad& \text{DPE(2.8a)}\\
4 \cdot 3 \cdot x + 4 \cdot 4 + 1 \cdot x + 6 &\quad& \text{JTC(\textbf{??})}\\
12 \cdot x + 16 + 1 \cdot x + 6 &\quad& \text{OOM(2.14)}\\
12x + 16 + 1x + 6 &\quad& \text{CTJ(\textbf{??})}\\
12 + 1x + 16 + 6 &\quad& \text{CPA(2.1)}\\
(12 + 1)x + 16 + 6 &\quad& \text{DPF(2.3a)}\\
13x + 22 &\quad& \text{OOA(2.12)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
12x + 16 + x + 6 &\quad& \text{DPE(2.8a)}\\
12x + x + 16 + 6 &\quad& \text{CPA(2.1)}\\
13x + 22 &\quad& \text{OOA(2.12)}
\end{aligned}$$

■

Example 3.24 – id:20141109-095151.

Simplify by expanding $5(x - 4) + 3x - 5$

S

Solution:

$$
\begin{aligned}
5(1x-4)+3x-5 &\quad \text{MId(2.9a)}\\
5(1x+\neg 4)+3x+\neg 5 &\quad \text{DOS(2.11a)}\\
5\cdot 1x+5\cdot\neg 4+3x+\neg 5 &\quad \text{DPE(2.8a)}\\
5\cdot 1\cdot x+5\cdot\neg 4+3\cdot x+\neg 5 &\quad \text{JTC(??)}\\
5\cdot x+\neg 20+3\cdot x+\neg 5 &\quad \text{OOM(2.14)}\\
5x+\neg 20+3x+\neg 5 &\quad \text{JTC(??)}\\
5x+3x+\neg 20+\neg 5 &\quad \text{CPA(2.1)}\\
(5+3)x+\neg 20+\neg 5 &\quad \text{DPF(2.3a)}\\
8x+\neg 25 &\quad \text{OOA(2.12)}\\
8x-25 &\quad \text{DOS(2.11b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
5(x+\neg 4)+3x+\neg 5 &\quad \text{DOS(2.11a)}\\
5x+\neg 20+3x+\neg 5 &\quad \text{DPE(2.8a)}\\
5x+3x+\neg 20+\neg 5 &\quad \text{CPA(2.1)}\\
8x+\neg 25 &\quad \text{OOA(2.12)}\\
8x-25 &\quad \text{DOS(2.11b)}
\end{aligned}
$$

Example 3.25 – id:20141109-095536.

Simplify by expanding $8x-5-4(x-3)$

S

Solution:

$$
\begin{aligned}
8x-5-4(1x-3) &\quad \text{MId(2.9a)}\\
8x+\neg 5+\neg 4(1x+\neg 3) &\quad \text{DOS(2.11a)}\\
8x+\neg 5+\neg 4\cdot 1x+\neg 4\cdot\neg 3 &\quad \text{DPE(2.8a)}\\
8\cdot x+\neg 5+\neg 4\cdot 1\cdot x+\neg 4\cdot\neg 3 &\quad \text{JTC(??)}\\
8\cdot x+\neg 5+\neg 4\cdot x+12 &\quad \text{OOM(2.14)}\\
8x+\neg 5+\neg 4x+12 &\quad \text{CTJ(??)}\\
8x+\neg 4x+\neg 5+12 &\quad \text{CPA(2.1)}\\
(8+\neg 4)x+\neg 5+12 &\quad \text{DPF(2.3a)}\\
4x+7 &\quad \text{OOA(2.12)}
\end{aligned}
$$

S

Less Steps Solution:

$$\begin{aligned} 8x + \neg 5 + \neg 4(x + \neg 3) && \text{DOS(2.11a)} \\ 8x + \neg 5 + \neg 4x + 12 && \text{DPE(2.8a)} \\ 8x + \neg 4x + \neg 5 + 12 && \text{CPA(2.1)} \\ 4x + 7 && \text{OOA(2.12)} \end{aligned}$$

■

Example 3.26 – id:20141109-095842.

Simplify by expanding $5(x+3)+3(x+2)$

S

Solution:

$$\begin{aligned} 5 \cdot x + 5 \cdot 3 + 3 \cdot x + 3 \cdot 2 && \text{DPE(2.8a)} \\ 5 \cdot x + 15 + 3 \cdot x + 6 && \text{OOM(2.14)} \\ 5x + 15 + 3x + 6 && \text{CTJ(??)} \\ 5x + 3x + 15 + 6 && \text{CPA(2.1)} \\ (3+5)x + 15 + 6 && \text{DPF(2.3a)} \\ 8x + 21 && \text{OOA(2.12)} \end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned} 5x + 15 + 3x + 6 && \text{DPE(2.8a)} \\ 5x + 3x + 15 + 6 && \text{CPA(2.1)} \\ 8x + 21 && \text{OOA(2.12)} \end{aligned}$$

■

3.5 Degree 2 Univariate Polynomials

Monomials

Example 3.27 – id:20141106-151138.

Simplify $4x^2 + 12x^2$

S

Solution:

$(4+12)x^2$	DPF(2.3a)
$16x^2$	OOA(2.12)

S

Less Steps Solution:

$16x^2$	OOA(2.12)

Example 3.28 – id:20141106-154547.

Simplify $x^2 - x + x^2 + x$

S

Solution:

$1x^2 - 1x + 1x^2 + 1x$	MId(2.9a)
$1x^2 + \neg 1x + 1x^2 + 1x$	DOS(2.11a)
$1x^2 + 1x^2 + \neg 1x + 1x$	CPA(2.1)
$(1+1)x^2 + (\neg 1 + 1)x$	DPF(2.3a)
$2x^2 + 0x$	OOA(2.12)
$2x^2$	MId(2.9b)

S

Less Steps Solution:

$x^2 + \neg x + x^2 + x$	DOS(2.11a)
$x^2 + x^2 + \neg x + x$	CPA(2.1)
$2x^2$	OOA(2.12)

Example 3.29 – id:20141108-194709.

Simplify $-2x4x - x \cdot -x3$

S

Solution:

$$
\begin{aligned}
&-2x4x - 1x \cdot 1x3 && \text{MId(2.9a)}\\
&\neg 2x4x - 1x \cdot 1x3 && \text{ONeg(3.2a)}\\
&\neg 2 \cdot x \cdot 4 \cdot x + \neg 1 \cdot x \cdot 1 \cdot x \cdot 3 && \text{JTC(??)}\\
&\neg 2 \cdot 4 \cdot x \cdot x + \neg 1 \cdot \neg 1 \cdot 3 \cdot x \cdot x && \text{CPM(2.6)}\\
&\neg 2 \cdot 4 \cdot x^2 + \neg 1 \cdot \neg 1 \cdot 3 \cdot x^2 && \text{PrCBPo(3.3a)}\\
&(\neg 2 \cdot 4) \cdot x^2 + (\neg 1 \cdot \neg 1 \cdot 3)x^2 && \text{APM(2.7a)}\\
&\neg 8x^2 + 3x^2 && \text{OOM(2.14)}\\
&(\neg 8 + 3)x^2 && \text{DPF(2.3a)}\\
&\neg 5x^2 && \text{OOA(2.12)}\\
&-5x^2 && \text{ONeg(3.2b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
&-2x4x + \neg x \cdot -x3 && \text{DOS(2.11a)}\\
&\neg 2 \cdot 4 \cdot x \cdot x + 3 \cdot \neg x \cdot \neg x && \text{CPM(2.6)}\\
&\neg 2 \cdot 4 \cdot x^2 + 3 \cdot x^2 && \text{PrCBPo(3.3a)}\\
&\neg 8x^2 + 3x^2 && \text{OOM(2.14)}\\
&-5x^2 && \text{OOA(2.12)}
\end{aligned}
$$

■

Rule 3.5.1 – Product of a Common Base Powers (PrCBPo).

$$b^m \cdot b^n = b^{m+n} \tag{3.3a}$$
$$b^{m+n} = b^m \cdot b^n \tag{3.3b}$$

Rule 3.5.2 – Quotient of a Common Base Powers (QCBPo).

$$\frac{b^m}{b^n} = b^{m-n} \tag{3.4a}$$
$$b^{m-n} = \frac{b^m}{b^n} \tag{3.4b}$$

Rule 3.5.3 – Power of a Power (PoPo).

$$(b^m)^k = b^{m \cdot k} \tag{3.5a}$$
$$b^{m \cdot k} = (b^m)^k \tag{3.5b}$$

Example 3.30 – id:20141108-191616.

Simplify $-5x \cdot 4x$

Solution:

$\neg 5x \cdot 4x$	ONeg(3.2a)
$\neg 5 \cdot x \cdot 4 \cdot x$	JTC(??)
$\neg 5 \cdot 4 \cdot x \cdot x$	CPM(2.6)
$\neg 5 \cdot 4 \cdot x^2$	PrCBPo(3.3a)
$(\neg 5 \cdot 4) \cdot x^2$	APM(2.7a)
$\neg 20 \cdot x^2$	OOM(2.14)
$\neg 20x^2$	CTJ(??)
$-20x^2$	ONeg(3.2b)

Less Steps Solution:

$\neg 5 \cdot 4 \cdot x \cdot x$	CPM(2.6)
$\neg 5 \cdot 4 \cdot x^2$	PrCBPo(3.3a)
$-20x^2$	OOM(2.14)

Binomials

Example 3.31 – id:20141106-152339.

Simplify $3x^2 + 2x + 5x^2 + 4x$

Solution:

$3x^2 + 5x^2 + 2x + 4x$	CPA(2.1)
$(3+5)x^2 + (2+4)x$	DPF(2.3a)
$8x^2 + 6x$	OOA(2.12)

If needed we could continue and express it in the simplified factored form using the distributive property

$(4x + 3)2x$ DPF(2.3a)

Less Steps Solution:

$$3x^2 + 5x^2 + 2x + 4x \quad \text{CPA(2.1)}$$
$$8x^2 + 6x \quad \text{OOA(2.12)}$$

Example 3.32 – id:20141107-121834.

Simplify $\left(\sqrt{9 - x^2}\right)^2$

Solution:

$$\left(\sqrt{9 - 1x^2}\right)^2 \quad \text{MId(2.9a)}$$
$$\left(\sqrt{9 + \neg 1x^2}\right)^2 \quad \text{DOS(2.11a)}$$
$$\left[\left(9 + \neg x^2\right)^{\frac{1}{2}}\right]^2 \quad \text{RTPo(??)}$$
$$9 + \neg 1x^2 \quad \text{PoPo(3.5a)}$$
$$\neg 1x^2 + 9 \quad \text{CPA(2.1)}$$
$$\neg x^2 + 9 \quad \text{MId(2.9a)}$$
$$-x^2 + 9 \quad \text{ONeg(3.2b)}$$

Less Steps Solution:

$$9 - x^2 \quad \text{PoPo(3.5a)}$$

It might be easier to view this using a variable substitution for the radicand, $9 - x^2$. *Let* $k = 9 + \neg 1x^2$.

$\left(\sqrt{k}\right)^2$	MId(2.9a)
$\left(\sqrt{k}\right)^2$	DOS(2.11a)
$\left[(k)^{\frac{1}{2}}\right]^2$	RTPo(??)
k	PoPo(3.5a)
$9 + \neg 1x^2$	CPA(2.1)
$\neg 1x^2 + 9$	CPA(2.1)
$\neg x^2 + 9$	MId(2.9a)
$-x^2 + 9$	ONeg(3.2b)

D

Dependencies:example ??-20141105-144223

Example 3.33 – id:20141209-145211.

Simplify $2x(2x+4) + x^2 \cdot 2 \cdot 1 + 0$

S

Solution:

$2x \cdot 2x + 2x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0$	DPE(2.8a)
$2 \cdot x \cdot 2 \cdot x + 2 \cdot x \cdot 4 + x^2 \cdot 2 \cdot 1 + 0$	JTC(??)
$2 \cdot 2 \cdot x \cdot x + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0$	CPM(2.6)
$2 \cdot 2 \cdot x^2 + 2 \cdot 4 \cdot x + 2 \cdot 1 \cdot x^2 + 0$	PrCBPo(3.3a)
$4 \cdot x^2 + 8 \cdot x + 2 \cdot x^2 + 0$	OOM(2.14)
$4x^2 + 8x + 2x^2 + 0$	CTJ(??)
$4x^2 + 2x^2 + 8x$	APA(2.2a)
$(4+2)x^2 + 8x$	DPF(2.3b)
$6x^2 + 8x$	OOA(2.12)

D

Dependencies:example 5.8-20141209-144203

Trinomials

Example 3.34 – id:20141109-133008.

Simplify by expanding $(x+5)(x-8)$

Solution:

$$
\begin{aligned}
(1x+5)(1x-8) &\qquad \text{MId(2.9a)}\\
(1x+5)(1x+\neg 8) &\qquad \text{DOS(2.11a)}\\
1x(1x+\neg 8)+5(1x+\neg 8) &\qquad \text{DPE(2.8b)}\\
1x\cdot 1x+1x\cdot\neg 8+5\cdot 1x+5\cdot\neg 8 &\qquad \text{DPE(2.8a)}\\
1\cdot x\cdot 1\cdot x+1\cdot x\cdot\neg 8+5\cdot 1\cdot x+5\cdot\neg 8 &\qquad \text{JTC(\textbf{??})}\\
1\cdot 1\cdot x\cdot x+\neg 8\cdot 1\cdot x+1\cdot 5\cdot x+\neg 8\cdot 5 &\qquad \text{CPM(2.6)}\\
1\cdot 1\cdot x^2+\neg 8\cdot 1\cdot x+1\cdot 5\cdot x+\neg 8\cdot 5 &\qquad \text{PrCBPo(3.3a)}\\
1\cdot x^2+\neg 8\cdot x+5\cdot x+\neg 40 &\qquad \text{OOM(2.14)}\\
1x^2+\neg 8x+5x+\neg 40 &\qquad \text{CTJ(\textbf{??})}\\
1x^2+\neg 3x+\neg 40 &\qquad \text{OOA(2.12)}\\
1x^2-3x-40 &\qquad \text{DOS(2.11b)}\\
x^2-3x-40 &\qquad \text{MId(2.9a)}
\end{aligned}
$$

Less Steps Solution:

$$
\begin{aligned}
(x+5)(x+\neg 8) &\qquad \text{DOS(2.11a)}\\
x(x+\neg 8)+5(x+\neg 8) &\qquad \text{DPE(2.8b)}\\
x^2+\neg 8x+5x+\neg 40 &\qquad \text{DPE(2.8a)}\\
x^2-3x-40 &\qquad \text{OOA(2.12)}
\end{aligned}
$$

Example 3.35 – id:20141109-133316.

Simplify by expanding $(x+a)(x+b)$, where $a, b \in \mathbb{Z}$

Solution:

$$\begin{aligned}
&(1x + a)(1x + b) && \text{MId(2.9a)}\\
&1x(1x + b) + a(1x + b) && \text{DPE(2.8b)}\\
&1x \cdot 1x + 1x \cdot b + a \cdot 1x + a \cdot b && \text{DPE(2.8a)}\\
&1 \cdot x \cdot 1 \cdot x + 1 \cdot x \cdot b + a \cdot 1 \cdot x + a \cdot b && \text{JTC(\textbf{??})}\\
&1 \cdot 1 \cdot x \cdot x + 1 \cdot b \cdot x + 1 \cdot a \cdot x + a \cdot b && \text{CPM(2.6)}\\
&1 \cdot 1 \cdot x^2 + 1 \cdot b \cdot x + 1 \cdot a \cdot x + a \cdot b && \text{PrCBPo(3.3a)}\\
&1 \cdot x^2 + 1 \cdot b \cdot x + 1 \cdot a \cdot x + a \cdot b && \text{OOM(2.14)}\\
&1x^2 + 1bx + 1ax + ab && \text{CTJ(\textbf{??})}\\
&1x^2 + (1b + 1a)x + ab && \text{DPF(2.3a)}\\
&x^2 + (b + a)x + ab && \text{MId(2.9b)}
\end{aligned}$$

S

Less Steps Solution:

$$\begin{aligned}
&x(x + b) + a(x + b) && \text{DPE(2.8b)}\\
&x^2 + (b + a)x + ab && \text{DPE(2.8a)}
\end{aligned}$$

■

Example 3.36 – id:20141109-140659.

Simplify by expanding $(2x + 3)(5x + 13)$

S

Solution:

$$\begin{aligned}
&2x(5x + 13) + 3(5x + 13) && \text{DPE(2.8b)}\\
&2x \cdot 5x + 2x \cdot 13 + 3 \cdot 5x + 3 \cdot 13 && \text{DPE(2.8a)}\\
&2 \cdot x \cdot 5 \cdot x + 2 \cdot x \cdot 13 + 3 \cdot 5 \cdot x + 3 \cdot 13 && \text{JTC(\textbf{??})}\\
&2 \cdot 5 \cdot x \cdot x + 2 \cdot 13 \cdot x + 5 \cdot 3 \cdot x + 3 \cdot 13 && \text{CPM(2.6)}\\
&2 \cdot 5 \cdot x^2 + 2 \cdot 13 \cdot x + 5 \cdot 3 \cdot x + 3 \cdot 13 && \text{PrCBPo(3.3a)}\\
&10 \cdot x^2 + 26 \cdot x + 16 \cdot x + 39 && \text{OOM(2.14)}\\
&10x^2 + 26x + 15x + 39 && \text{CTJ(\textbf{??})}\\
&10x^2 + 41x + 39 && \text{OOA(2.12)}
\end{aligned}$$

S

Less Steps Solution:

$2x(5x+13)+3(5x+13)$	DPE(2.8b)
$10x^2+26x+15x+39$	DPE(2.8a)
$10x^2+41x+39$	OOA(2.12)

Example 3.37 – id:20141109-141019.

Simplify by expanding $(-3x-5)(7x+8)$

Solution:

$(\neg 3x-5)(7x+8)$	ONeg(3.2a)
$(\neg 3x+\neg 5)(7x+8)$	DOS(2.11a)
$\neg 3x(7x+8)+\neg 5(7x+8)$	DPE(2.8b)
$\neg 3x\cdot 7x+\neg 3x\cdot 8+\neg 5\cdot 7x+\neg 5\cdot 8$	DPE(2.8a)
$\neg 3\cdot x\cdot 7\cdot x+\neg 3\cdot x\cdot 8+\neg 5\cdot 7\cdot x+\neg 5\cdot 8$	JTC(**??**)
$\neg 3\cdot 7\cdot x\cdot x+\neg 3\cdot 8\cdot x+\neg 5\cdot 7\cdot x+\neg 5\cdot 8$	CPM(2.6)
$\neg 3\cdot 7\cdot x^2+\neg 3\cdot 8\cdot x+\neg 5\cdot 7\cdot x+\neg 5\cdot 8$	PrCBPo(3.3a)
$\neg 21\cdot x^2+\neg 24\cdot x+\neg 35\cdot x+\neg 40$	OOM(2.14)
$\neg 21x^2+\neg 24x+\neg 35x+\neg 40$	CTJ(**??**)
$\neg 21x^2+\neg 59x+\neg 40$	OOA(2.12)
$\neg 21x^2-59x-40$	DOS(2.11b)
$-21x^2-59x-40$	ONeg(3.2b)

Less Steps Solution:

$(-3x+\neg 5)(7x+8)$	DOS(2.11a)
$\neg 3x(7x+8)+\neg 5(7x+8)$	DPE(2.8b)
$\neg 21x^2+\neg 24x+\neg 35x+\neg 40$	CTJ(2.8a)
$-21x^2-59x-40$	OOA(2.12)

Example 3.38 – id:20141109-141347.

Simplify by expanding $(ax+b)(cx+d)$, where $a,b,c,d\in\mathbb{Z}$

S

Solution:

$$
\begin{aligned}
ax(cx+d)+b(cx+d) &\quad \text{DPE(2.8b)}\\
ax\cdot cx+ax\cdot d+b\cdot cx+b\cdot d &\quad \text{DPE(2.8a)}\\
a\cdot x\cdot c\cdot x+a\cdot x\cdot d+b\cdot c\cdot x+b\cdot d &\quad \text{JTC(??)}\\
a\cdot c\cdot x\cdot x+a\cdot d\cdot x+b\cdot c\cdot x+b\cdot d &\quad \text{CPM(2.6)}\\
a\cdot c\cdot x^2+a\cdot d\cdot x+b\cdot c\cdot x+b\cdot d &\quad \text{PrCBPo(3.3a)}\\
acx^2+adx+bcx+bd &\quad \text{CTJ(??)}\\
acx^2+(ad+bc)x+bd &\quad \text{DPF(2.3a)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
&ax(cx+d)+b(cx+d) &\quad \text{DPE(2.8b)}\\
&acx^2+(ad+bc)x+bd &\quad \text{DPE(2.8a)}
\end{aligned}
$$

■

Example 3.39 – id:20141105-161225.

Simplify $\left(2-\frac{x}{2}\right)^2$ by expanding.

S

Solution:

$$
\begin{aligned}
\left(2 - \frac{1x}{2}\right)^2 &\quad \text{MId(2.9a)}\\
\left(2 + \neg\frac{1x}{2}\right)^2 &\quad \text{DOS(2.11a)}\\
\left(2 + \neg\frac{1x}{2}\right)\left(2 + \neg\frac{1x}{2}\right) &\quad \text{PoTF(??)}\\
2\left(2 + \neg\frac{1x}{2}\right) + \neg\frac{1x}{2}\left(2 + \neg\frac{1x}{2}\right) &\quad \text{DPE(2.8b)}\\
2 \cdot 2 + 2 \cdot \neg\frac{1x}{2} + \neg\frac{1x}{2} \cdot 2 + \neg\frac{1x}{2} \cdot \neg\frac{1x}{2} &\quad \text{DPE(2.8a)}\\
2 \cdot 2 + 2 \cdot \neg\frac{1}{2} \cdot x + \neg\frac{1}{2} \cdot x \cdot 2 + \neg\frac{1}{2} \cdot x \cdot \neg\frac{1}{2} \cdot x &\quad \text{JTC(??)}\\
2 \cdot 2 + 2 \cdot \neg\frac{1}{2} \cdot x + \neg\frac{1}{2} \cdot 2 \cdot x + \neg\frac{1}{2} \cdot \neg\frac{1}{2} \cdot x \cdot x &\quad \text{CPM(2.6)}\\
2 \cdot 2 + 2 \cdot \neg\frac{1}{2} \cdot x + \neg\frac{1}{2} \cdot 2 \cdot x + \neg\frac{1}{2} \cdot \neg\frac{1}{2} \cdot x^2 &\quad \text{PrCBPo(3.3a)}\\
4 + \neg 1 \cdot x + \neg 1 \cdot x + \frac{1}{4} \cdot x^2 &\quad \text{OOM(2.14)}\\
4 + \neg 1x + \neg 1x + \frac{1}{4}x^2 &\quad \text{CTJ(??)}\\
\frac{1}{4}x^2 + \neg 1x + \neg 1x + 4 &\quad \text{CPA(2.1)}\\
\frac{1}{4}x^2 + \neg 2x + 4 &\quad \text{OOA(2.12)}\\
\frac{1}{4}x^2 - 2x + 4 &\quad \text{DOS(2.11b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
7x + 20x + 40 &\quad \text{DPE(2.8a)}\\
27x + 40 &\quad \text{OOA(2.12)}
\end{aligned}
$$

■

3.6 Degree 3 Univariate Polynomials

Monomials

Binomials

Trinomials

Polynomials

3.7 Degree n Univariate Polynomials

Monomials

Binomials

Trinomials

Polynomials

4. Equations

4.1 Equality

Property 4.1.1 – Reflexive Property of Eqaulity (RPE).

$$a = a \tag{4.1a}$$

Property 4.1.2 – Substitution Property of Equality (SPE).

Given $a = b$, then

$$E(a) = E(b) \tag{4.2}$$

$E(x)$ represents any expression.

Property 4.1.3 – Symmetric Property of Equality (SyPE).

$$a = b \quad \text{then} \quad b = a \tag{4.3a}$$

Property 4.1.4 – Transitive Property of Equality (TPE).

$$\text{if} \quad a = b \quad \text{and} \quad b = c \quad \text{then} \quad a = c \tag{4.4a}$$

4.2 Solving Linear Equations

Example 4.1 – id:20141206-102142.

Solve the equation $x + a = b$ for x

S

Solution:

$$
\begin{aligned}
[x + a] + \neg a &= [b] + \neg a && \text{SPE(4.2) + AI(2.5a)} \\
x + (a + \neg a) &= b + \neg a && \text{APA(2.2b)} \\
x + 0 &= b + \neg a && \text{OOA(2.12)} \\
x &= b + \neg a && \text{AId(2.4b)} \\
x &= b - a && \text{DOS(2.11b)}
\end{aligned}
$$

Example 4.2 – id:20141111-222931.

Solve the equations $x + 8 = 0$

S

Solution:

$$
\begin{aligned}
[x + 8] + \neg 8 &= [0] + \neg 8 && \text{SPE(4.2) + AI(2.5a)} \\
x + (8 + \neg 8) &= 0 + \neg 8 && \text{APA(2.2b)} \\
x + 0 &= \neg 8 && \text{OOA(2.12)} \\
x &= \neg 8 && \text{AId(2.4b)} \\
x &= -8 && \text{ONeg(3.2b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
[x + 8] + \neg 8 &= [0] + \neg 8 && \text{SPE(4.2) + AI(2.5a)} \\
x &= -8 && \text{OOA(2.12)}
\end{aligned}
$$

D

Dependencies:
example **??**-20141111-190212

Example 4.3 – id:20141206-101632.

Solve the equation $x + 4 = 7$

S

Solution:

$$
\begin{aligned}
[x+4] + \neg 4 &= [7] + \neg 4 && \text{SPE(4.2) + AI(2.5a)}\\
x + (4 + \neg 4) &= 7 + \neg 4 && \text{APA(2.2b)}\\
x + 0 &= 3 && \text{OOA(2.12)}\\
x &= 3 && \text{AId(2.4b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
[x] + 4 + \neg 4 &= [7] + \neg 4 && \text{SPE(4.2) + AI(2.5a)}\\
x &= 3 && \text{OOA(2.12)}
\end{aligned}
$$

■

Example 4.4 – id:20141206-101107.

Solve the equation $x - 8 = 15$ for x

S

Solution:

$$
\begin{aligned}
x + \neg 8 &= 15 && \text{DOS(2.11a)}\\
[x + \neg 8] + 8 &= [15] + 8 && \text{SPE(4.2) + AI(2.5a)}\\
x + (\neg 8 + 8) &= 15 + 8 && \text{APA(2.2b)}\\
x + 0 &= 23 && \text{OOA(2.12)}\\
x &= 23 && \text{AId(2.4b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
[x + \neg 8] + 8 &= [15] + 8 && \text{SPE(4.2) + AI(2.5a)}\\
x &= 23 && \text{OOA(2.12)}
\end{aligned}
$$

■

Example 4.5 – id:20141206-102404.

Solve the equation $5x = 9$ for x.

Solution:

$$
\begin{aligned}
\frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(4.2) + MI(2.10a)}\\
\frac{1}{5}\cdot[5\cdot x] &= \frac{1}{5}\cdot 9 && \text{JTC(??)}\\
\left(\frac{1}{5}\cdot 5\right)\cdot x &= \frac{1}{5}\cdot 9 && \text{APM(2.7b)}\\
1\cdot x &= \frac{9}{5} && \text{OOM(2.14)}\\
x &= \frac{9}{5} && \text{MId(2.9b)}
\end{aligned}
$$

Less Steps Solution:

$$
\begin{aligned}
\frac{1}{5}[5x] &= \frac{1}{5}[9] && \text{SPE(4.2) + MI(2.10a)}\\
x &= \frac{9}{5} && \text{OOM(2.14)}
\end{aligned}
$$

■

Example 4.6 – id:20141206-104404.

Solve the equation $ax = b$ for x.

Solution:

$$
\begin{aligned}
\frac{1}{a}[ax] &= \frac{1}{a}[b] && \text{SPE(4.2) + MI(2.10a)}\\
\frac{1}{a}\cdot(a\cdot x) &= \frac{1}{a}\cdot b && \text{JTC(??)}\\
\left(\frac{1}{a}\cdot a\right)\cdot x &= \frac{1}{a}\cdot b && \text{APM(2.7b)}\\
1\cdot x &= \frac{b}{a} && \text{OOM(2.14)}\\
x &= \frac{b}{a} && \text{MId(2.9b)}
\end{aligned}
$$

■

Example 4.7 – id:20141206-102723.

Solve the equation $-2x = 7$ for x

S

Solution:

$$
\begin{aligned}
\neg 2x &= 7 && \text{ONeg(3.2a)}\\
\neg\frac{1}{2}[\neg 2x] &= \neg\frac{1}{2}[7] && \text{SPE(4.2) + MI(2.10b)}\\
\neg\frac{1}{2}\cdot(\neg 2\cdot x) &= \neg\frac{1}{2}\cdot 7 && \text{JTC(??)}\\
\left(\neg\frac{1}{2}\cdot\neg 2\right)\cdot x &= \neg\frac{1}{2}\cdot 7 && \text{APM(2.7b)}\\
1\cdot x &= \neg\frac{7}{2} && \text{OOM(2.14)}\\
1\cdot x &= -\frac{7}{2} && \text{ONeg(3.2b)}\\
x &= -\frac{7}{2} && \text{MId(2.9b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
\neg\frac{1}{2}[\neg 2x] &= \neg\frac{1}{2}[7] && \text{SPE(4.2) + MI(2.10b)}\\
x &= -\frac{7}{2} && \text{OOM(2.14)}
\end{aligned}
$$

Example 4.8 – id:20141111-215726.

Solve the equation $2x + 5 = 0$ for x

S

Solution:

$$
\begin{aligned}
[2x+5] + \neg 5 &= [0] + \neg 5 && \text{SPE(4.2) + AI(2.5a)}\\
2x + (5 + \neg 5) &= 0 + \neg 5 && \text{APA(2.2a)}\\
2x + 0 &= \neg 5 && \text{OOA(2.12)}\\
2x &= \neg 5 && \text{AId(2.4a)}\\
\frac{1}{2}[2x] &= \frac{1}{2}[\neg 5] && \text{SPE(4.2) + MI(2.10a)}\\
\frac{1}{2}\cdot 2\cdot x &= \frac{1}{2}\cdot \neg 5 && \text{JTC(??)}\\
\left(\frac{1}{2}\cdot 2\right)\cdot x &= \frac{1}{2}\cdot \neg 5 && \text{APM(2.7a)}\\
1\cdot x &= \frac{\neg 5}{2} && \text{OOM(2.14)}\\
1x &= -\frac{5}{2} && \text{ONeg(3.2b)}\\
x &= -\frac{5}{2} && \text{MId(2.9b)}
\end{aligned}
$$

S

Less Steps Solution:

$$
\begin{aligned}
[2x+5] + \neg 5 &= [0] + \neg 5 && \text{SPE(4.2) + AI(2.5a)} && (4.5)\\
2x &= \neg 5 && \text{OOA(2.12)} && (4.6)\\
\frac{1}{2}[2x] &= \frac{1}{2}[\neg 5] && \text{SPE(4.2) + MI(2.10a)} && (4.7)\\
x &= -\frac{5}{2} && \text{OOM(2.14)} && (4.8)
\end{aligned}
$$

D

Dependencies:
example **??**-20141111-192213

4.3 Solving Quadratic Equations

Example 4.9 – id:20141107-131748.

Solve the equation $2 - x^2 = 0$ for x

S

Solution:

$$
\begin{aligned}
2 - 1x^2 &= 0 && \text{MId(2.9a)}\\
2 + \neg 1x^2 &= 0 && \text{DOS(2.11a)}\\
\left[2 + \neg 1x^2\right] + 1x^2 &= [0] + 1x^2 && \text{SPE(4.2)} + \text{AI(2.5a)}\\
2 + \left(\neg 1x^2 + 1x^2\right) &= 0 + 1x^2 && \text{APA(2.2a)}\\
2 + 0 &= 0 + 1x^2 && \text{OOA(2.12)}\\
2 &= 1x^2 && \text{AId(2.4a)}\\
2 &= x^2 && \text{MId(2.9b)}\\
\pm [2]^{\frac{1}{2}} &= \left[x^2\right]^{\frac{1}{2}} &&\\
\pm 2^{\frac{1}{2}} &= x && \text{PoPo(3.5a)}\\
\pm\sqrt{2} &= x && \text{PoTR(\textbf{??})}\\
x &= \pm\sqrt{2} && \text{SyPE(4.3a)}
\end{aligned}
$$

■

4.3.1 Completing the Square

Completing the square is an algebraic algorithm used to find the solutions of quadratic equations of the form, $ax^2 + bx + c = 0$. Essentially, we want to manipulate this equation such that $x =$ some values. To gain some understanding of how this algorithm works, we will consider each step individually. Let's begin with a quadratic equation in the general form: $ax^2 + bx + c = 0$.

Since we are trying to manipulate the equation $ax^2+bx+c = 0$ such that $x =$ some value, we first want the coefficient factor a to be equal to 1. This is done by multiplying both expressions by the multiplicative inverse of a (step 1) followed by simplifying both expressions.

$$
\begin{aligned}
\frac{1}{a}\left[\boldsymbol{ax^2 + bx + c}\right] &= \frac{1}{a}[0] && \text{SPE(4.2)} + \text{MI(2.10a)}\\
\frac{1}{a}\cdot ax^2 + \frac{1}{a}\cdot bx + \frac{1}{a}\cdot c &= \frac{1}{a}\,[0] && \text{DPE(2.8a)}\\
\frac{1}{a}\cdot a\cdot x^2 + \frac{1}{a}\cdot b\cdot x + \frac{1}{a}\cdot c &= \frac{1}{a}\,[0] && \text{JTC(\textbf{??})}\\
x^2 + \frac{b}{a}\cdot x + \frac{c}{a} &= 0 && \text{OOM(2.14)}\\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{CTJ(\textbf{??})}
\end{aligned}
$$

We now have three terms in the left hand expression where the first two terms have at least one variable factor, x. The problem is that the third term is a constant and we want $x =$ some value. This text step is to get rid of the $\frac{c}{a}$ term, which can be done by using the additive inverse followed by simplifying both expressions.

$$\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] + \neg\frac{c}{a} = [0] + \neg\frac{c}{a} \qquad \text{SPE(4.2) + AI(2.5a)}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} + \neg\frac{c}{a} = 0 + \neg\frac{c}{a} \qquad \text{APA(2.2a)}$$

$$x^2 + \frac{b}{a}x = \neg\frac{c}{a} \qquad \text{OOA(2.12)}$$

The next step is called completing the square. The idea is to add a *NeW* constant, k, to the left-hand expression, $x^2 + \frac{b}{a}x + k$, such that the quadratic expression can then be factored as two identical factors, $(x + m)(x + m) = (x + m)^2$, where $k = m \cdot m$. Notice that since we are adding a constant term, k, to the left-hand expression, then we must also add this constant, k, to the right-hand expression, $x^2 + \frac{b}{a}x + k = \neg\frac{c}{a} + k$. To determine the values of both m and k we should refer to the the organization of the two factors that make up the product of a quadratic expression, $x^2 + \frac{b}{a}x + k = (x + m)^2$.

Since both factors of this new quadratic expression are the same, both terms that make up the middle term, must also be the same. We know that $mx + mx = \frac{b}{a}x$, so we should be able to determine the value of m from this equation. If we can determine the value of m, then we can determine the value of k.

$$\begin{aligned} \frac{b}{a}x &= mx + mx \\ &= 2mx \qquad \text{OOA(2.12)} \end{aligned}$$

Solving for m,

$$2mx = \frac{b}{a}x$$

$$2 \cdot m \cdot x = \frac{b}{a} \cdot x \qquad \text{JTC(\textbf{??})}$$

$$\frac{1}{2}\,[2 \cdot m \cdot x] = \frac{1}{2}\left[\frac{b}{a} \cdot x\right] \qquad \text{SPE(4.2) + MI(2.10a)}$$

$$m \cdot x = \frac{b}{2a} \cdot x \qquad \text{OOM(2.14)}$$

$$[m \cdot x]\,\frac{1}{x} = \left[\frac{b}{2a} \cdot x\right]\frac{1}{x} \qquad \text{SPE(4.2) + MI(2.10a)}$$

$$m = \frac{b}{2a} \qquad \text{OOM(2.14)}$$

$$\left[x^2 + \frac{b}{a}x\right] + \left(\frac{b}{2a}\right)^2 = \left[\neg\frac{c}{a}\right] + \left(\frac{b}{2a}\right)^2 \qquad \text{SPE(4.2)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{APA(2.2b)}$$

$$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{DPF(2.3b)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{PoTF(\textbf{??})}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \frac{(b)^2}{(2a)^2} \qquad \text{PoQPo(\textbf{??})}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \frac{b^2}{2^2a^2} \qquad \text{PoPrPo}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{OOE}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} \qquad \text{MId}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{c \cdot 4 \cdot a}{a \cdot 4 \cdot a} + \frac{b^2}{4a^2} \qquad \text{JTC}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{4 \cdot a \cdot c}{4 \cdot a \cdot a} + \frac{b^2}{4a^2} \qquad \text{CPM}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{4 \cdot a \cdot c}{4 \cdot a^2} + \frac{b^2}{4a^2} \qquad \text{PrCBPo}$$

$$\left(x + \frac{b}{2a}\right)^2 = \neg\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \qquad \text{CTJ}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{\neg 4ac + b^2}{4a^2} \qquad \text{CD}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 + \neg 4ac}{4a^2} \qquad \text{CPA}$$

$$\left[\left(x + \frac{b}{2a}\right)^2\right]^{\frac{1}{2}} = \pm\left[\frac{b^2 + \neg 4ac}{4a^2}\right]^{\frac{1}{2}} \qquad \text{PoI}$$

$$x + \frac{b}{2a} = \pm\left[\frac{b^2 + \neg 4ac}{4a^2}\right]^{\frac{1}{2}} \qquad \text{PoPrPo}$$

$$x + \frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{[4a^2]^{\frac{1}{2}}} \qquad \text{PoQPo}$$

$$x + \frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{4^{\frac{1}{2}}a} \qquad \text{PoPrPo}$$

$$x + \frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{OOE}$$

$$\left[x + \frac{b}{2a}\right] + \neg\frac{b}{2a} = \left[\pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a}\right] + \neg\frac{b}{2a} \qquad \text{AI}$$

$$x + \frac{b}{2a} + \neg\frac{b}{2a} = \pm\frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} + \neg\frac{b}{2a} \qquad \text{APA}$$

$$x + \frac{b}{2a} + \neg\frac{b}{2a} = \neg\frac{b}{2a} \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{CPA}$$

$$x = \neg\frac{b}{2a} \pm \frac{\left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{OOA}$$

$$x = \frac{\neg b \pm \left[b^2 + \neg 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{CD}$$

$$x = \frac{\neg b \pm \left[b^2 - 4ac\right]^{\frac{1}{2}}}{2a} \qquad \text{DOS}$$

$$x = \frac{\neg b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{ETR}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{ONeg}$$

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5. Differentiation

5.1 Limit of the Difference Quotient

Definition 5.1.1 – Derivative. The derivative of a function $f(x)$ with respect to the variable x is defined as

$$f'(x) \equiv \lim_{\Delta x \to 0} \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\text{Difference Quotient}} \tag{5.1}$$

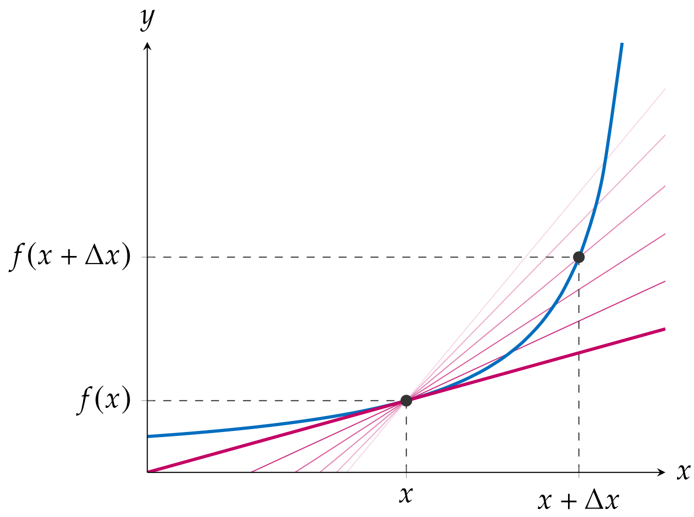

Figure 5.1: [**mooculus:textbook**]

Example 5.1 – id:20141219-212546.

Differentiate the function $f(x) = 5$

S

Solution:

$$
\begin{aligned}
f(x) &= 5x^0 && \text{PoID(??)} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{5[x + \Delta x]^0 - 5[x]^0}{\Delta x} && \text{SPE(4.2)\&DBFP(5.1)} \\
f'(x) &= \lim_{\Delta x \to 0} \frac{5(1) - 5(1)}{\Delta x} && \text{PoID(??)} \\
f'(x) &= \lim_{\Delta x \to 0} 0 && \text{OOM(2.14)} \\
f'(x) &= 0
\end{aligned}
$$

5.2 Derivative of a Monomial Functions

Rule 5.2.1 – Derivative of a Constant (DC).

$$[c]' = 0 \tag{5.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0 \tag{5.3}$$

Rule 5.2.2 – Derivative of a Constant Multiple (DCM).

$$[c\,f(x)]' = c\,[f(x)]' \tag{5.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[c\,f(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \tag{5.5}$$

Rule 5.2.3 – Derivative of a Power (DPo).

$$[x^n]' = nx^{n-1} \tag{5.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1} \tag{5.7}$$

Example 5.2 – id:20141124-153017.

Differentiate $f(x) = -3$

S

Solution:

$$
\begin{aligned}
f'(x) &= [-3]' && \text{SPE(4.2)} \\
f'(x) &= 0 && \text{DC(5.2)}
\end{aligned}
$$

D

Dependencies:
example 5.4-20141124-152503

Example 5.3 – id:20141124-141850.

Differentiate $f(x) = x^2$

S

Solution:

$$
\begin{aligned}
f'(x) &= [x^2]' && \text{SPE(4.2)}\\
f'(x) &= 2x^{2-1} && \text{DPo(5.6)}\\
f'(x) &= 2x^1 && \text{OOA(2.12)}\\
f'(x) &= 2x && \text{MId(2.9b)}
\end{aligned}
$$

S

Less Steps Solution:

$$f'(x) = 2x \qquad \text{DPo(5.6)}$$

D

Dependencies:
example 5.4-20141124-152503

5.3 Derivative of Polynomial Functions

Rule 5.3.1 – Derivative of a Sum (DS).

$$[f(x) + g(x)]' = f'(x) + g'(x) \tag{5.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + \frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{5.9}$$

Example 5.4 – id:20141124-152503.

Differentiate $f(x) = x^2 - 3$

S

Solution:

$$
\begin{aligned}
f(x) &= x^2 + \neg 3 && \text{DOS(2.11a)}\\
f'(x) &= [x^2 + \neg 3]' && \text{SPE(4.2)}\\
f'(x) &= [x^2]' + [\neg 3]' && \text{DS(5.8)}\\
f'(x) &= [x^2]' + 0 && \text{DC(5.2)}\\
f'(x) &= [x^2]' && \text{AId(2.4a)}\\
f'(x) &= 2x && \text{DPo(5.6) goto 5.3}
\end{aligned}
$$

S

Less Steps Solution:

$$f(x) = 2x^2 \qquad \text{DPo(5.6)\&DC(5.6)}$$

D

Dependencies:
example 5.15-20141124-205219

Example 5.5 – id:20141128-151834.

Differentiate $f(x) = 3x^2 - 6x + 4$

S

Solution:

$$
\begin{aligned}
f(x) &= 3x^2 + \neg 6x + 4 && \text{DOS(2.11a)}\\
f'(x) &= [3x^2 + \neg 6x + 4]' && \text{SPE(4.2)}\\
f'(x) &= [3x^2]' + [6x]' + [4]' && \text{DS(5.8)}\\
f'(x) &= [3x^2]' + [6x]' + 0 && \text{DC(5.2)}\\
f'(x) &= [3x^2]' + [6x]' && \text{AId(2.4a)}\\
f'(x) &= 3[x^2]' + 6[x]' && \text{DCM(5.4)}\\
f'(x) &= 3\,(2x) + 6\,(1) && \text{DPo(5.6)}\\
f'(x) &= 6x + 6 && \text{OOM(2.14)}
\end{aligned}
$$

S

$$f'(x) = 6x + 6 \qquad \text{DS(5.8)}$$

5.3.1 Finding the vertex of a quadratic function using differentiation.

We can find the vertex of a quadratic function, $f(x)$ using differentiation by:

1. Differentiate the function: Find $f'(x)$.
2. Set the derivative equal to zero: $f'(x) = 0$.
3. Find the abscissa of the vertex by solving the equation $f'(x) = 0$ for x to find the critical x value: $x = k$.
4. Find the ordinate of the vertex by substituting the value of critical value $x = k$ into the function $f(x)$: Evaluate $f(k)$

Example 5.6 – id:20151008-110208.
Find the vertex of the quadratic function, $f(x) = ax^2 + bx + c$, using differentiation.

S

Solution:
1. Find the derivative of $f(x)$

$$\begin{aligned}
[f(x)]' &= [ax^2 + bx + c]' && \text{SPE(4.2)}\\
f'(x) &= [ax^2]' + [bx]' + [c]' && \text{DS(5.8)}\\
f'(x) &= a[x^2]' + b[x]' + [c]' && \text{DCM(5.4)}\\
f'(x) &= a \cdot x + b \cdot 1 + c && \text{DPo(5.6)}\\
f'(x) &= ax + b + [c]' && \text{Todo simplify}\\
f'(x) &= ax + b + 0 && \text{DC(5.2)}\\
f'(x) &= ax + b && \text{AId(2.4b)}
\end{aligned}$$

2. Set the derivative equal to zero and solve for x.

$$\begin{aligned}
f'(x) &= 0\\
ax + b &= 0\\
x &= -\frac{b}{a} && \text{Todo Solve}
\end{aligned}$$

The abscissa of the vertex is $x = -\frac{b}{2a}$.
3. Find the ordinate of the vertex by substituting the argument $x = -\frac{b}{2a}$ into $f(x)$

Example 5.7 – id:20150923-152515.
Find the vertex of the parabola $y = x^2 - 2x - 6$ using differentiation.

S

1. Differentiate the function.

$$
\begin{aligned}
f(x) &= x^2 - 2x - 6 && \\
f(x) &= x^2 + \neg 2x + \neg 6 && \text{DOS(2.11a)} \\
[f(x)]' &= [x^2 + \neg 2x + \neg 6]' && \text{SPE(4.2)} \\
f'(x) &= [x^2]' + [\neg 2x]' + [\neg 6]' && \text{DS(5.8)} \\
f'(x) &= [x^2]' + \neg 2[x]' + [\neg 6]' && \text{DCM(5.4)} \\
f'(x) &= 2x + \neg 2 + [\neg 6]' && \text{DPo(5.6)} \\
f'(x) &= 2x + \neg 2 + 0 && \text{DC(5.2)} \\
f'(x) &= 2x + \neg 2 && \text{AId(2.4b)} \\
f'(x) &= 2x - 2 && \text{DOS(2.11b)}
\end{aligned}
$$

2 and 3. Set the derivative equal to zero and solve for x

$$
\begin{aligned}
2x - 2 &= 0 \\
x &= 1
\end{aligned}
$$

4. Find the value of $f(1)$

$$
\begin{aligned}
f(x) &= x^2 - 2x - 6 && \\
f(1) &= [1]^2 - 2[1] - 6 && \text{SPE(4.2)} \\
f(1) &= -7 && \text{Evaluate}
\end{aligned}
$$

The vertex of this parabola is the point $(1, -7)$ ■

Rule 5.3.2 – Derivative of a Product (DPr).

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \tag{5.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)]g(x) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}[g(x)] \tag{5.11}$$

Example 5.8 – id:20141209-144203.

Differentiate $f(x) = x^2(2x + 4)$

S

Solution:

$$
\begin{aligned}
f'(x) &= [x^2(2x+4)]' && \text{SPE(4.2)}\\
f'(x) &= [x^2]'(2x+4) + x^2[2x+4]' && \text{DPr(5.10)}\\
f'(x) &= [x^2]'(2x+4) + x^2[2x] + [4]' && \text{DS(5.8)}\\
f'(x) &= [x^2]'(2x+4) + x^2 \cdot 2[x] + [4]' && \text{DCM(5.4)}\\
f'(x) &= 2x(2x+4) + x^2 \cdot 2 \cdot 1 + [4]' && \text{DPo(5.6)}\\
f'(x) &= 2x(2x+4) + x^2 \cdot 2 \cdot 1 + 0 && \text{DC(5.2)}\\
f'(x) &= 6x^2 + 8x && \text{simplify goto 3.33}
\end{aligned}
$$

■

Example 5.9 – id:20141209-142321.

Differentiate $f(x) = x^2\cos(x)$

S

Solution:

$$
\begin{aligned}
f'(x) &= [x^2\cos(x)]' && \text{SPE(4.2)}\\
f'(x) &= [x^2]'\cos(x) + x^2[\cos(x)]' && \text{DPr(5.10)}\\
f'(x) &= 2x\cos(x) + x^2[\cos(x)]' && \text{DPo(5.6)}\\
f'(x) &= 2x\cos(x) + x^2(-1\sin(x)) && \text{DCos(5.14)}\\
f'(x) &= 2x\cos(x) - x^2\sin x && \text{OOM(2.14)}
\end{aligned}
$$

■

5.4 Derivative of Trigonometric Functions

Rule 5.4.1 – Derivative of Sine (DSin).

$$[\sin(x)]' = \cos(x) \tag{5.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(x)] = \cos(x) \tag{5.13}$$

Rule 5.4.2 – Derivative of Cosine (DCos).

$$[\cos(x)]' = -\sin(x) \tag{5.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)] = -\sin(x) \tag{5.15}$$

Example 5.10 – id:20150910-115935.

Differentiate $f(x) = \sin(x)\cos(x)$

S

Solution:

$$\begin{aligned}
f'(x) &= [\sin(x)\cos(x)]' && \text{SPE(4.2)}\\
f'(x) &= [\sin(x)]'\cos(x) + \sin(x)[\cos(x)]' && \text{DPr(5.10)}\\
f'(x) &= \cos(x)\cos(x) + \sin(x)[\cos(x)]' && \text{DSin(5.12)}\\
f'(x) &= \cos(x)\cos(x) + \sin(x)(-\sin(x)) && \text{DCos(5.14)}\\
f'(x) &= \cos^2(x) - \sin^2(x) && \text{simplify goto ??}
\end{aligned}$$

Example 5.11 – id:20141209-151354.

Differentiate $f(x) = \sin(x)\sin(x)$

S

Solution:

$$\begin{aligned}
f'(x) &= [\sin(x)\sin(x)]' && \text{SPE(4.2)}\\
f'(x) &= [\sin(x)]'\sin(x) + \sin(x)[\sin(x)] && \text{DPr(5.10)}\\
f'(x) &= \cos(x)\sin(x) + \sin(x)\cos(x) && \text{DSin(5.12)}\\
f'(x) &= \cos(x)\sin(x) + \cos(x)\sin(x) && \text{CPM(2.6)}\\
f'(x) &= 2\cos(x)\sin(x) && \text{OOA(2.12)}
\end{aligned}$$

5.5 Derivative of Rational Functions

Rule 5.5.1 – Derivative of a Quotient (DQ).

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \tag{5.16}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\,g(x) - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \tag{5.17}$$

5.6 Derivative of Exponential Functions

5.7 Derivative of Logarithmic Functions

Rule 5.7.1 – Derivative of a Natural Logarithm (DNL).

$$[\ln x]' = \frac{1}{x} \tag{5.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x} \tag{5.19}$$

5.8 Derivative of Composite Functions

Rule 5.8.1 – Derivative of a Composite Function (DComp).

$$[f(g(x))]' = [g(x)]'[f(g(x))]' \tag{5.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = \frac{\mathrm{d}}{\mathrm{d}x}[g(x)]\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] \tag{5.21}$$

Example 5.12 – id:20141124-203850.

Differentiate $y = \ln(3x)$

S

Solution:

After identifying that $y = \ln(3x)$ is a composite function, we let $u = 3x$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{DComp(5.21)}$$

We need to find the factors $\frac{\mathrm{d}y}{\mathrm{d}u}$ and $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$y = \ln(u) \qquad u = 3x$$

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(5.21)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} \qquad \text{DNL(5.18)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot 3 \qquad \text{DPo(5.6)}$$
$$\frac{dy}{dx} = \frac{1}{3x} \cdot 3$$
$$\frac{dy}{dx} = \frac{3}{3x} \qquad \text{OOM(2.14)}$$
$$\frac{dy}{dx} = \frac{1}{x}$$

D

Dependencies:
example 5.15-20141124-205219

Example 5.13 – id:20141128-160248.

Differentiate $y = \ln(3x^2 - 6x + 4)$

S

Solution: After identifying that $y = \ln(3x^2 - 6x + 4)$ is a composite function, we let $u = 3x^2 - 6x + 4$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(5.21)}$$

We need to find the factors $\frac{dy}{du}$ and $\frac{du}{dx}$.

$$y = \ln(u) \qquad u = 3x^2 - 6x + 4$$
$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 6x - 6 \qquad \text{goto } 5.5$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(5.21)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} \qquad \text{DNL(5.18)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot (6x - 6) \qquad \text{DPo(5.6)}$$
$$\frac{dy}{dx} = \frac{1}{3x^2 - 6x + 4} \cdot (6x - 6)$$
$$\frac{dy}{dx} = \frac{6x - 6}{3x^2 - 6x + 4} \qquad \text{OOM(2.14)}$$

Example 5.14 – id:20141128-155506.

Differentiate $y = \ln(\cos x)$

S

Solution: After identifying that $y = \ln(\cos x)$ is a composite function, we let $u = \cos x$ and thus we get a new function $y = \ln(u)$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(5.21)}$$

We need to find the factors $\frac{dy}{du}$ and $\frac{du}{dx}$.

$$y = \ln(u) \qquad u = \cos x$$
$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{DComp(5.21)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} \qquad \text{DNL(5.18)}$$
$$\frac{dy}{dx} = \frac{1}{u} \cdot -\sin x \qquad \text{DPo(5.6)}$$
$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x$$
$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} \qquad \text{OOM(2.14)}$$
$$\frac{dy}{dx} = -\tan x$$

Example 5.15 – id:20141124-205219.

Differentiate $y = \left(x^2 - 1\right)\ln(3x)$

S

Solution:

$$
\begin{aligned}
y' &= \left[x^2 - 3\right]' \cdot \ln x + \left(x^2 - 3\right) \cdot \left[\ln 3x\right]' && \text{DPr(5.10)} \\
y' &= 2x \cdot \ln x + \left(x^2 - 3\right) \cdot \left[\ln 3x\right]' && \text{differentiate goto 5.4} \\
y' &= 2x \cdot \ln x + \left(x^2 - 3\right) \cdot \frac{1}{x} && \text{differentiate goto 5.12} \\
y' &= 2x \cdot \ln x + \frac{x^2 - 3}{x} && \text{OOM(2.14)} \\
y' &= 2x^2 \ln x + \frac{x^2 - 3}{x} && \text{JTC(??)} \\
y' &= \frac{2x^2 \ln x + \left(x^2 - 3\right)}{x} && \text{OOA(2.12)}
\end{aligned}
$$

Bibliography