Llista de problemes (9)

3.13 Expresseu la integral $\int_D f(x,y)\,dx\,dy$ com a integral iterada, en els dos ordres quan

(a) D és el triangle de vèrtexs $A=(0,0)$, $B=(1,1)$, $C=(1,2)$

(b) D està determinat per $y^2-x^2\leq 2$, $x^2+y^2\leq 4$

3.14 Canvieu l'ordre d'integració en $\int_0^1 \left(\int_y^{\sqrt{2-y^2}} f(x,y)\,dx\right)dy$

3.15 Calculeu $\iint_A xy^2\,dx\,dy$ on $A=\{(x,y) \mid y^2\leq 2ax,\ x\leq a\}$, $a>0$

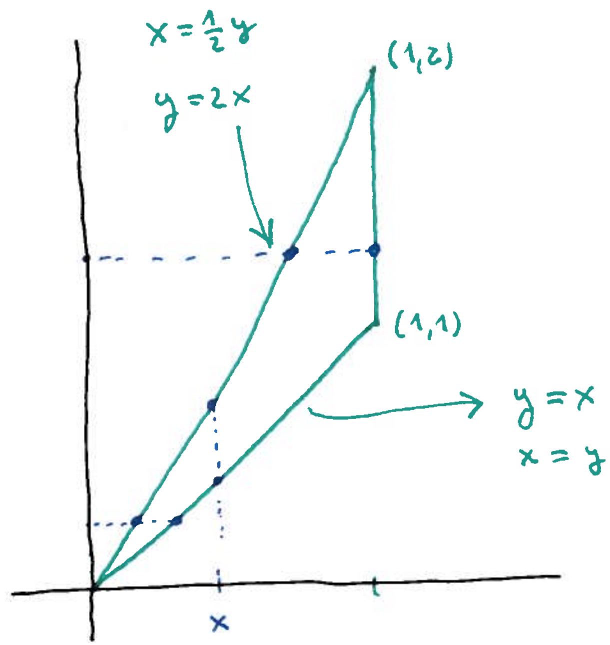
3.16 Calculeu $\iint_A y^2\sqrt{x}\,dx\,dy$ on $A=\{(x,y) \mid x\geq 0,\ y\geq x^2,\ y\leq 10-x^2\}$

3.17 Calculeu $\iint_A e^{x-y}\,dx\,dy$ on A és el triangle de vèrtexs $(0,0)$, $(1,3)$, $(2,2)$.

3.13

$$\int_D f(x,y)\,dx\,dy$$

a) triangle de vèrtexs $A = (0,0)$, $B = (1,1)$, $C = (1,2)$

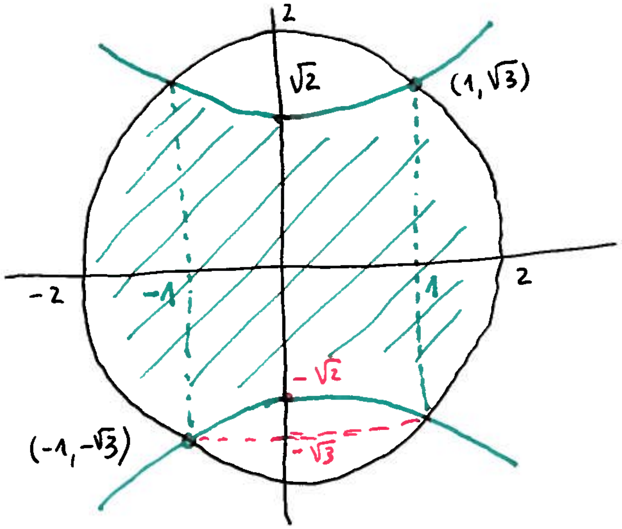


$$\int_0^1 \left(\int_x^{2x} f(x,y)\,dy \right) dx$$

$$\int_0^2 \left(\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx \right) dy = \int_0^1 \left(\int_{\frac{1}{2}y}^{y} f(x,y)\,dx \right) dy + \int_1^2 \left(\int_{\frac{1}{2}y}^{1} f(x,y)\,dx \right) dy$$

3.13 (2)

b) $D = \{ (x,y) \mid \quad y^2 - x^2 \leq 2, \quad x^2 + y^2 \leq 4 \}$



$y^2 - x^2 \leq 2 \quad \rightarrow \quad y^2 \leq 2 + x^2$

$\rightarrow \; |y| \leq \sqrt{2 + x^2}$

$\rightarrow \; -\sqrt{2 + x^2} \leq y \leq \sqrt{2 + x^2}$

$\begin{cases} y^2 - x^2 = 2 \\ x^2 + y^2 = 4 \end{cases} \rightarrow \; 2y^2 = 6 \; \rightarrow \; y^2 = 3 \; \rightarrow \; y = \pm\sqrt{3}$

$\downarrow$

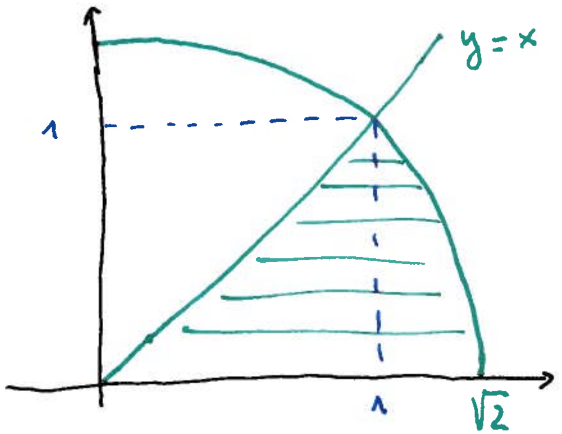
$x = \pm\sqrt{4 - y^2} = \pm 1$

$$I = \int_{-2}^{2} \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy \right) dx = \int_{-2}^{-1} \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\, dy \right) dx + \int_{-1}^{1} \left(\int_{-\sqrt{2+x^2}}^{\sqrt{2+x^2}} f(x,y)\, dy \right) dx + \int_{1}^{2} \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\, dy \right) dx$$

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \right) dy = \int_{-\sqrt{3}}^{-\sqrt{2}} \underbrace{\left(\int_{-\sqrt{4-y^2}}^{-\sqrt{y^2-2}} f\, dx + \int_{\sqrt{y^2-2}}^{\sqrt{4-y^2}} f\, dx \right)}\, dy + \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f\, dx \right) dy + \int_{\sqrt{2}}^{\sqrt{3}} \Big(\qquad \Big) dy$$

3.14

$$\int_0^1 \left(\int_y^{\sqrt{2-y^2}} f(x,y)\, dx \right) dy$$



$x = \sqrt{2-y^2} \rightarrow x^2 = 2 - y^2$

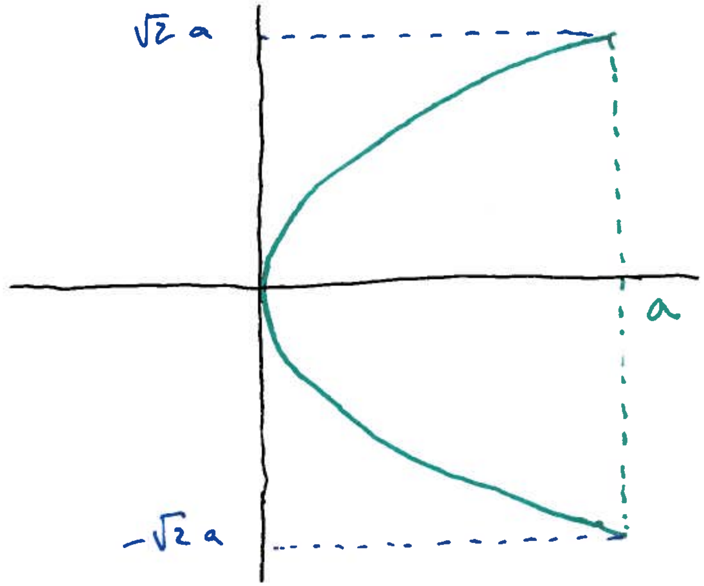
$\rightarrow x^2 + y^2 = 2$

intersecció $\left.\begin{array}{l} x^2+y^2=2 \\ x = y \end{array}\right\}$ $2x^2 = 2 \rightarrow x = \pm 1$

$y = \pm 1$

$$\int_0^{\sqrt{2}} \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\, dy \right) dx = \int_0^1 \left(\int_0^x f(x,y)\, dy \right) dx + \int_1^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} f(x,y)\, dy \right) dx$$

3.15 $\quad \iint_A x y^2 \, dx \, dy \qquad A = \{(x,y) \mid y^2 \le 2ax, \quad x \le a\}$



intersecció $\begin{cases} y^2 = 2ax \\ x = a \end{cases}$

$\to y^2 = 2a^2 \to y = \pm\sqrt{2}\,a$

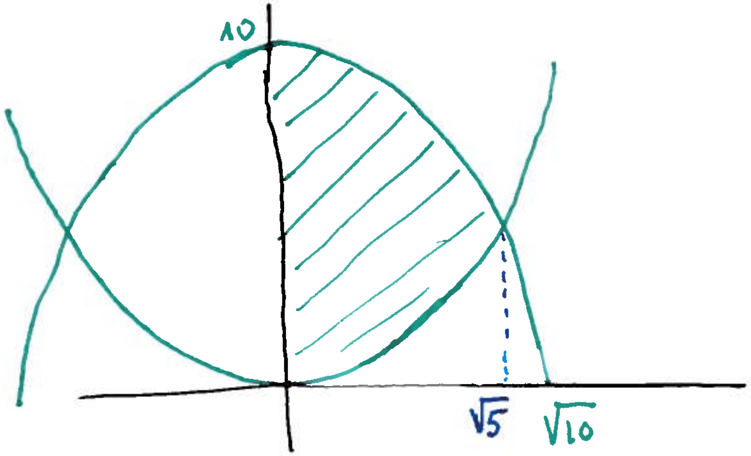
$$I = \int_{-\sqrt{2}a}^{\sqrt{2}a} \left(\int_{y^2/2a}^{a} x y^2 \, dx \right) dy = \int_{-\sqrt{2}a}^{\sqrt{2}a} \left[\frac{1}{2} x^2 y^2 \right]_{y^2/2a}^{a} dy = \int_{-\sqrt{2}a}^{\sqrt{2}a} \left(\frac{1}{2} a^2 y^2 - \frac{1}{2} \frac{y^4}{4a^2} y^2 \right) dy$$

$$= 2 \int_0^{\sqrt{2}a} \left(\frac{1}{2} a^2 y^2 - \frac{1}{8a^2} y^6 \right) dy = 2 \left[\frac{1}{2} a^2 \frac{y^3}{3} - \frac{1}{8a^2} \frac{y^7}{7} \right]_0^{\sqrt{2}a} = 2 \left(\frac{1}{6} 2\sqrt{2}\, a^5 - \frac{1}{8 \cdot 7} 8\sqrt{2}\, a^5 \right)$$

$$= \frac{8\sqrt{2}}{21} a^5$$

$$I = \int_0^a \left(\int_{-\sqrt{2ax}}^{\sqrt{2ax}} x y^2 \, dy \right) dx$$

3.16 $\iint_A y^2\sqrt{x}\,dx\,dy$ $\qquad A=\{(x,y) \mid x\geq 0,\ y\geq x^2,\ y\leq 10-x^2\}$



intersecció $\begin{cases} y=x^2 \\ y=10-x^2\end{cases}$

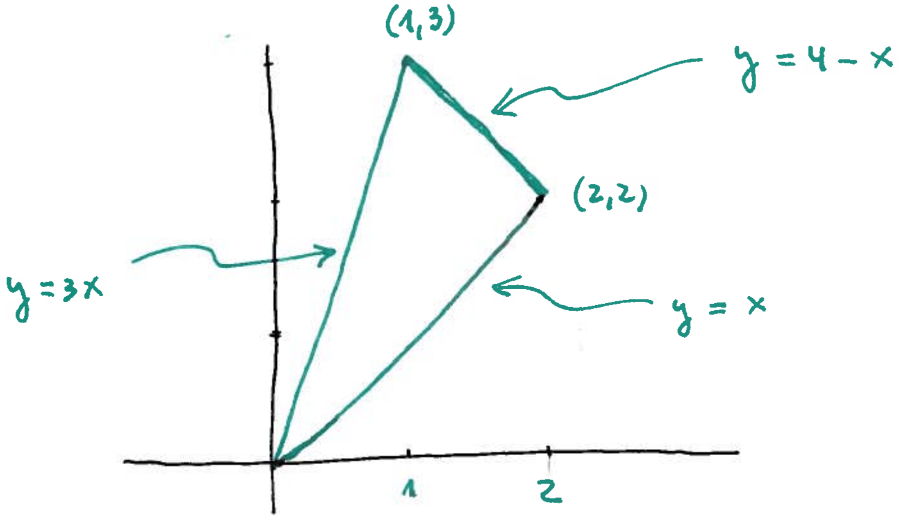
$$x^2=10-x^2 \longrightarrow 2x^2=10 \rightarrow x=\pm\sqrt{5}$$

$$I=\int_0^{\sqrt{5}}\left(\int_{x^2}^{10-x^2} y^2\sqrt{x}\,dy\right)dx=\int_0^{\sqrt{5}}\left[\frac{y^3}{3}\sqrt{x}\right]_{x^2}^{10-x^2}dx=\frac{1}{3}\int_0^{\sqrt{5}}\left((10-x^2)^3\sqrt{x}-x^6\sqrt{x}\right)dx$$

$$=\frac{1}{3}\int_0^{\sqrt{5}}\left[(1000-3\cdot 100x^2+3\cdot 10x^4-x^6)\sqrt{x}-x^6\sqrt{x}\right]dx$$

$$=\frac{1}{3}\left[1000\,\frac{x^{3/2}}{\frac{3}{2}}-300\,\frac{x^{7/2}}{\frac{7}{2}}+30\,\frac{x^{11/2}}{\frac{11}{2}}-2\,\frac{x^{15/2}}{\frac{15}{2}}\right]_0^{\sqrt{5}}$$

3.17 $\displaystyle\iint_A e^{x-y}\,dx\,dy \qquad A \qquad (0,0),\ (1,3),\ (2,2)$



$$I = \int_0^1 \left(\int_x^{3x} e^{x-y}\,dy \right) dx + \int_1^2 \left(\int_x^{4-x} e^{x-y}\,dy \right) dx = \int_0^1 \left[-e^{x-y} \right]_x^{3x} dx + \int_1^2 \left[-e^{x-y} \right]_x^{4-x} dx$$

$$= -\int_0^1 \left(e^{-2x} - 1 \right) dx - \int_1^2 \left(e^{2x-4} - 1 \right) dx = -\left[\frac{e^{-2x}}{-2} - x \right]_0^1 - \left[\frac{e^{2x-4}}{2} - x \right]_1^2$$

$$= -\left(\frac{e^{-2}}{-2} - 1 - \frac{1}{-2} \right) - \left(\frac{1}{2} - 2 - \frac{e^{-2}}{2} + 1 \right) = 1 + e^{-2}.$$