PROBLEMA 3:

Dada unha *m.a.s.* de tamaño n dunha VA con función de densidade $f(x) = \theta \exp\{-\theta(x-1)\}$ se $x > 1$, calcula o EMV de θ.

SOLUCION.

Primeiro calculamos a función de verosimilitude.

$$\begin{aligned} V(X_1,...,X_n;\theta) &= \prod_{i=1}^{n} f(X_i;\theta) = \prod_{i=1}^{n} \theta \exp\{-\theta(X_i-1)\} \\ &= \theta \exp\{-\theta(X_1-1)\} \times \theta \exp\{-\theta(X_2-1)\} \times ... \times \theta \exp\{-\theta(X_n-1)\} \\ &= \theta^n \exp\left\{-\theta \sum_{i=1}^{n}(X_i-1)\right\}. \end{aligned}$$

Vamos a maximizar esta función.

$$\begin{aligned} \ln V(X_1,...,X_n;\theta) &= \ln\theta^n + \ln\exp\left\{-\theta\sum_{i=1}^{n}(X_i-1)\right\} \\ &= n\ln\theta - \theta\sum_{i=1}^{n}(X_i-1) \Rightarrow \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln V(X_1,...,X_n;\theta)}{\partial\theta} &= \frac{n}{\theta} - \sum_{i=1}^{n}(X_i-1) = 0 \Rightarrow n - \theta\sum_{i=1}^{n}(X_i-1) = 0 \Rightarrow \\ \theta &= \frac{n}{\sum\limits_{i=1}^{n}(X_i-1)} = \frac{n}{\sum\limits_{i=1}^{n}X_i - n} = \frac{1}{\overline{X}-1}. \end{aligned}$$

Comprobamos que é un máximo.

$$\begin{aligned} \frac{\partial^2 \ln V(X_1,...,X_n;\theta)}{\partial\theta^2} &= \frac{-n}{\theta^2} < 0 \Rightarrow \text{ é un máximo} \\ &\Rightarrow \widehat{\theta}_{MV} = \frac{1}{\overline{X}-1}. \end{aligned}$$

PROBLEMA 4:
Consideremos a poboación discreta X definida por:

$$P(X=-1)=\frac{1-2\theta}{3},\ P(X=0)=\frac{\theta+\lambda}{3},\ P(X=1)=\frac{2+\theta-\lambda}{3}$$

onde $0<\theta<\frac{1}{2}$, $0<\lambda<1$.

Obter polo método dos momentos unha estimación de θ e λ a partir dos resultados da seguinte *m.a.s*: $0,-1,1,1,0,-1,1$.

SOLUCION.
(a)

$$E(X)=-1\frac{1-2\theta}{3}+1\frac{2+\theta-\lambda}{3}=\frac{1}{3}(-1+2\theta+2+\theta-\lambda)=\frac{1+3\theta-\lambda}{3}.$$

Como non somos capaces de despexar θ e λ, calculamos $E\left(X^2\right)$.

$$E\left(X^2\right)=\frac{1-2\theta}{3}+\frac{2+\theta-\lambda}{3}=\frac{3-\theta-\lambda}{3}.$$

Temos un sistema de dúas ecuacións con dúas incógnitas (θ e λ).

$$\begin{cases} 3E(X)-1=3\theta-\lambda \\ 3E\left(X^2\right)-3=-\theta-\lambda \end{cases}$$

Restando as dúas ecuación obtemos

$$\begin{aligned}
3E(X)-1-3E\left(X^2\right)+3 &= 4\theta \Rightarrow \theta=\frac{2+3E(X)-3E\left(X^2\right)}{4}\Rightarrow \\
\lambda &= -3E(X)+1+3\theta \Rightarrow \\
\widehat{\theta}_{Mom} &= \frac{2+3\overline{X}-3\frac{\sum_{i=1}^{n}X_i^2}{n}}{4} \text{ e} \\
\widehat{\lambda}_{Mom} &= -3\overline{X}+1+3\widehat{\theta}_{Mom}.
\end{aligned}$$

Se calculamos estes estimadores na mostra obtemos que $\overline{X}=\frac{1}{7}$ e $\frac{\sum_{i=1}^{n}X_i^2}{n}=\frac{5}{7}$ polo que

$$\widehat{\theta}_{Mom}=\frac{1}{14} \text{ e } \widehat{\lambda}_{Mom}=\frac{11}{14}.$$

PROBLEMA 5:
Sexa $\{X_1, ..., X_n\}$ unha $m.a.s.$ dunha $Bi(10, p)$.
(a) Calcula $\widehat{p}_{Mom}$, é decir, o estimador de momentos de p.
(b) Estudiar se $\widehat{p}_{Mom}$ é insesgado.
(c) Calcula o Erro Cuadrático Medio (ECM) de $\widehat{p}_{Mom}$.
(d) Estudiar se $\widehat{p}_{Mom}$ é consistente.

SOLUCION.
(a) Aplicamos o método dos momentos.

$$E(X) = 10p \Rightarrow$$

$$p = \frac{E(X)}{10} \Rightarrow$$

$$\widehat{p}_{Mom} = \frac{\overline{X}}{10}.$$

(b) $\widehat{p}_{Mom}$ é insesgado xa que

$$E(\widehat{p}_{Mom}) = E\left(\frac{\overline{X}}{10}\right) = \frac{E(X)}{10} = \frac{10p}{p} = p.$$

(c)

$$ECM(\widehat{p}_{Mom}) = (E(\widehat{p}_{Mom}) - p)^2 + Var(\widehat{p}_{Mom})$$

Polo apartado (b),

$$(E(\widehat{p}_{Mom}) - p)^2 = (p-p)^2 = 0$$

$$\begin{aligned} Var(\widehat{p}_{Mom}) &= Var\left(\frac{\overline{X}}{10}\right) = \frac{1}{100}Var(\overline{X}) = \frac{1}{100}\frac{Var(X)}{n} \\ &= \frac{1}{100}\frac{10p(1-p)}{n} = \frac{p(1-p)}{10n}. \end{aligned}$$

Logo,

$$ECM(\widehat{p}_{Mom}) = \frac{p(1-p)}{10n}.$$

(d) $\widehat{p}_{Mom}$ é consistente cando é asintóticamente insesgado e $\lim_{n\to\infty} Var(\widehat{p}_{Mom}) = 0$.
No apartado (b) vimos que $\widehat{p}_{Mom}$ é insesgado. Logo é asintóticamente insesgado.
Usando o apartado (c),

$$\lim_{n\to\infty} Var(\widehat{p}_{Mom}) = \lim_{n\to\infty} \frac{p(1-p)}{10n} = 0.$$

PROBLEMA 6:

Sexa $\{X_1, ..., X_n\}$ unha *m.a.s.* dunha *V.A.* X con función de densidade $f(x) = \frac{2x}{\alpha^2}$ se $0 \leq x \leq \alpha$.

(a) Calcula o EMV de α.

(b) Calcula o estimador de momentos de α.

(c) ¿Qué estimador é mellor, o de momentos ou a media mostral?

SOLUCION.

(a) Primeiro calculamos a función de verosimilitude.

$$\begin{aligned} V(X_1, ..., X_n; \alpha) &= \prod_{i=1}^{n} f(X_i; \alpha) = \prod_{i=1}^{n} \frac{2X_i}{\alpha^2} \\ &= \frac{2X_1}{\alpha^2} \times \frac{2X_2}{\alpha^2} \times ... \times \frac{2X_n}{\alpha^2} \\ &= \frac{2^n}{\alpha^{2n}} \prod_{i=1}^{n} X_i. \end{aligned}$$

Vamos a maximizar esta función en α. Como α está no denominador esta función farase máxima cando α sexa o máis pequeno posible. Como $X_i \leq \alpha$ para todo $i = 1, ..., n$ deducimos que o valor máis pequeno de α é $\max_i \{X_i\}$. Logo $\widehat{\alpha}_{MV} = \max_i \{X_i\}$.

(b) Aplicamos o método dos momentos.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\alpha} \frac{2x^2}{\alpha^2} dx = \frac{2}{\alpha^2} \left[\frac{x^3}{3}\right]_0^{\alpha} = \frac{2}{3}\alpha \\ &\Rightarrow \widehat{\alpha}_{Mom} = \frac{3\overline{X}}{2}. \end{aligned}$$

(c) Será mellor o estimador que teña menor ECM. Facemos cálculos:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^{\alpha} \frac{2x^3}{\alpha^2} dx = \frac{2}{\alpha^2} \left[\frac{x^4}{4}\right]_0^{\alpha} = \frac{\alpha^2}{2}$$

polo que

$$\begin{aligned} E(\overline{X}) &= E(X) = \frac{2}{3}\alpha. \\ Var(\overline{X}) &= \frac{Var(X)}{n} = \frac{E(X^2) - (E(X))^2}{n} = \frac{\frac{\alpha^2}{2} - \frac{4\alpha^2}{9}}{n} = \frac{\alpha^2}{18n} \\ E(\widehat{\alpha}_{Mom}) &= \frac{3}{2} E(\overline{X}) = \frac{3}{2} E(X) = \frac{3}{2}\frac{2}{3}\alpha = \alpha. \\ Var(\widehat{\alpha}_{Mom}) &= \frac{9}{4} Var(\overline{X}) = \frac{9}{4}\frac{\alpha^2}{18n} = \frac{\alpha^2}{8n}. \end{aligned}$$

Logo,

$$\begin{aligned} ECM(\widehat{\alpha}_{Mom}) &= (E(\widehat{\alpha}_{Mom}) - \alpha)^2 + Var(\widehat{\alpha}_{Mom}) = \frac{\alpha^2}{8n} \text{ e} \\ ECM(\overline{X}) &= \left(E(\overline{X}) - \alpha\right)^2 + Var(\overline{X}) = \frac{\alpha^2}{9} + \frac{\alpha^2}{18n} = \frac{(2n+1)\alpha^2}{18n}. \end{aligned}$$

Entón,

$$\begin{aligned} ECM(\widehat{\alpha}_{Mom}) &< ECM(\overline{X}) \Leftrightarrow \frac{\alpha^2}{8n} < \frac{(2n+1)\alpha^2}{18n} \Leftrightarrow \frac{1}{8} < \frac{(2n+1)}{18} \\ &\Leftrightarrow 18 < 16n + 8 \Leftrightarrow \frac{10}{16} < n. \end{aligned}$$

Como $n \geq 1$, a anterior condición cúmplese sempre. Polo tanto é mellor $\widehat{\alpha}_{Mom}$ por ter menor ECM.

PROBLEMA 7:

Sexa X unha VA ca seguinte función de densidade $f(x) = \frac{\theta}{x^{\theta+1}}$ se $x \geq 1$. Dada unha *m.a.s.* de tamaño n,

(a) Calcula $\widehat{\theta}_{MV}$, é decir, o estimador de máxima verosimilitude de θ.

(b) Calcula $\widehat{\theta}_{Mom}$, é decir, o estimador de momentos de θ.

SOLUCION.

(a) Calculamos a función de versosimilitude.

$$\begin{aligned} V(X_1, ..., X_n; \theta) &= \prod_{i=1}^{n} f(X_i; \theta) = \prod_{i=1}^{n} \frac{\theta}{X_i^{\theta+1}} \\ &= \frac{\theta}{X_1^{\theta+1}} \times \frac{\theta}{X_2^{\theta+1}} \times ... \times \frac{\theta}{X_n^{\theta+1}} = \frac{\theta^n}{\prod_{i=1}^{n} X_i^{\theta+1}}. \end{aligned}$$

Maximizamos dita función.

$$\begin{aligned} \ln V(X_1, ..., X_n; \theta) &= \ln \theta^n - \ln \left(\prod_{i=1}^{n} X_i^{\theta+1} \right) = \ln \theta^n - \sum_{i=1}^{n} \ln X_i^{\theta+1} \\ &= n \ln \theta - (\theta + 1) \sum_{i=1}^{n} \ln X_i \Rightarrow \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial \theta} \ln V(X_1, ..., X_n; \theta) &= \frac{n}{\theta} - \sum_{i=1}^{n} \ln X_i = \frac{n - \theta \left(\sum_{i=1}^{n} \ln X_i \right)}{\theta} = 0 \\ &\Rightarrow \theta = \frac{n}{\sum_{i=1}^{n} \ln X_i}. \end{aligned}$$

Comprobamos que é un máximo

$$\frac{\partial^2 \ln V(X_1, ..., X_n; \theta)}{\partial \theta^2} = -\frac{n}{2\theta^2} < 0 \Rightarrow \text{ é un máximo.}$$

Logo

$$\widehat{\theta}_{MV} = \frac{n}{\sum_{i=1}^{n} \ln X_i}.$$

(b) Aplicamos o método dos momentos.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx = \int_{1}^{\infty} x \frac{\theta}{x^{\theta+1}} dx = \theta \int_{1}^{\infty} x \frac{1}{x^{\theta+1}} dx \\ &= \theta \int_{1}^{\infty} \frac{1}{x^{\theta}} dx = \theta \int_{1}^{\infty} x^{-\theta} dx = \theta \left[\frac{x^{-\theta+1}}{-\theta+1} \right]_{1}^{\infty} \\ &= \theta \left[0 - \frac{1}{-\theta+1} \right] = -\frac{\theta}{-\theta+1} = \frac{\theta}{\theta-1}. \end{aligned}$$

Despexamos θ en función de $E(X)$.

$$(\theta - 1) E(X) = \theta \Rightarrow \theta (E(X) - 1) = E(X) \Rightarrow \theta = \frac{E(X)}{E(X) - 1}$$

logo $\boxed{\theta_{mom} = \frac{\bar{X}}{\bar{X}-1}}$

PROBLEMA 8:

Sexa $\{X_1, ..., X_n\}$ unha *m.a.s.* dunha poboación $\Gamma\left(\frac{1}{\theta}, 2\right)$ onde $\theta > 0$. Sabemos que se $X \sim \Gamma\left(\frac{1}{\theta}, 2\right)$ entón $f(x) = \frac{1}{\theta^2} x \exp\left\{-\frac{x}{\theta}\right\}$ se $x > 0$. Ademáis $E(X) = 2\theta$ e $Var(X) = 2\theta^2$.

(a) Calcula o EMV de θ.

(b) Calcula $ECM\left(\widehat{\theta}_{MV}\right)$.

(c) Demostra que $\frac{\overline{X}^2}{2}$ é un estimador asintóticamente insesgado de $Var(X) = 2\theta^2$.

SOLUCION.

(a) Primeiro calculamos a función de verosimilitude.

$$\begin{aligned}
V(X_1, ..., X_n; \theta) &= \prod_{i=1}^{n} f(X_i; \theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} X_i \exp\left\{-\frac{X_i}{\theta}\right\} \\
&= \frac{1}{\theta^2} X_1 \exp\left\{-\frac{X_1}{\theta}\right\} \times \frac{1}{\theta^2} X_2 \exp\left\{-\frac{X_2}{\theta}\right\} \times ... \times \frac{1}{\theta^2} X_n \exp\left\{-\frac{X_n}{\theta}\right\} \\
&= \frac{1}{\theta^{2n}} \left(\prod_{i=1}^{n} X_i\right) \exp\left\{-\frac{\sum\limits_{i=1}^{n} X_i}{\theta}\right\}.
\end{aligned}$$

Vamos a maximizar esta función.

$$\begin{aligned}
\ln V(X_1, ..., X_n; \theta) &= \ln 1 - \ln \theta^{2n} + \ln\left(\prod_{i=1}^{n} X_i\right) + \ln \exp\left\{-\frac{\sum\limits_{i=1}^{n} X_i}{\theta}\right\} \\
&= -2n \ln \theta + \ln\left(\prod_{i=1}^{n} X_i\right) - \frac{\sum\limits_{i=1}^{n} X_i}{\theta} \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln V(X_1, ..., X_n; \theta)}{\partial \theta} &= \frac{-2n}{\theta} + \frac{\sum\limits_{i=1}^{n} X_i}{\theta^2} = 0 \Rightarrow \frac{1}{\theta^2}\left(-2n\theta + \sum_{i=1}^{n} X_i\right) = 0 \Rightarrow \\
\theta &= \frac{\sum\limits_{i=1}^{n} X_i}{2n} = \frac{\overline{X}}{2}.
\end{aligned}$$

Comprobamos que é un máximo.

$$\begin{aligned}
\frac{\partial^2 \ln V(X_1, ..., X_n; \theta)}{\partial \theta^2} &= \frac{2n}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^{n} X_i = \frac{1}{\theta^2}\left(2n - \frac{2}{\theta} \sum_{i=1}^{n} X_i\right) < 0 \Leftrightarrow \\
2n - \frac{2}{\theta} \sum_{i=1}^{n} X_i &< 0 \Leftrightarrow 2n - \frac{2}{\frac{\overline{X}}{2}} n\overline{X} < 0 \Leftrightarrow 2n - 4n < 0 \Leftrightarrow -2n < 0 \\
&\Rightarrow \text{ é un máximo} \Rightarrow \widehat{\theta}_{MV} = \frac{\overline{X}}{2}.
\end{aligned}$$

(b) Sabemos que $ECM\left(\widehat{\theta}_{MV}\right) = \left(E\left(\widehat{\theta}_{MV}\right) - \theta\right)^2 + Var\left(\widehat{\theta}_{MV}\right)$. Facemos cálculos.

$$
\begin{aligned}
E\left(\widehat{\theta}_{MV}\right) &= \frac{E\left(\overline{X}\right)}{2} = \frac{E(X)}{2} = \frac{2\theta}{2} = \theta \text{ e} \\
Var\left(\widehat{\theta}_{MV}\right) &= \frac{Var\left(\overline{X}\right)}{4} = \frac{Var(X)}{4n} = \frac{2\theta^2}{4n} = \frac{\theta^2}{2n} \\
&\Rightarrow \quad ECM\left(\widehat{\theta}_{MV}\right) = \frac{\theta^2}{2n}.
\end{aligned}
$$

(c) Temos que probar que $\lim_{n\to\infty} E\left(\frac{\overline{X}^2}{2}\right) = Var(X) = 2\theta^2$.

$$
\begin{aligned}
E\left(\frac{\overline{X}^2}{2}\right) &= \frac{1}{2}E\left(\overline{X}^2\right) = \frac{1}{2}\left(Var\left(\overline{X}\right) + \left(E\left(\overline{X}\right)\right)^2\right) = \frac{1}{2}\left(\frac{Var(X)}{n} + (E(X))^2\right) \\
&= \frac{1}{2}\left(\frac{2\theta^2}{n} + 4\theta^2\right) = \frac{\theta^2}{n} + 2\theta^2.
\end{aligned}
$$

Logo,

$$
\lim_{n\to\infty} E\left(\frac{\overline{X}^2}{2}\right) = 2\theta^2 = Var(X).
$$

PROBLEMA 9:

Sexa $\{X_1, ..., X_n\}$ unha *m.a.s.* dunha poboación xeométrica de parámetro $\frac{1}{1+\theta}$. Sabemos que se $X \sim X\left(\frac{1}{1+\theta}\right)$ entón $f(x) = \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^x$ se $x = 0, 1, 2, ...$ Ademáis $E(X) = \theta$ e $Var(X) = \theta(1+\theta)$.

(a) Calcula o EMV de θ.

(b) Calcula o estimador de momentos de θ.

(c) Comproba que o EMV é eficiente.

SOLUCION.

(a) Primeiro calculamos a función de verosimilitude.

$$\begin{aligned}
V(X_1, ..., X_n; \theta) &= \prod_{i=1}^{n} f(X_i; \theta) = \prod_{i=1}^{n} \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{X_i} \\
&= \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{X_1} \times \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{X_2} \times ... \times \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^{X_n} \\
&= \frac{1}{(1+\theta)^n}\left(\frac{\theta}{1+\theta}\right)^{\sum\limits_{i=1}^{n} X_i}.
\end{aligned}$$

Vamos a maximizar esta función.

$$\begin{aligned}
\ln V(X_1, ..., X_n; \theta) &= \ln 1 - \ln(1+\theta)^n + \ln\left(\frac{\theta}{1+\theta}\right)^{\sum\limits_{i=1}^{n} X_i} \\
&= -n\ln(1+\theta) + \sum_{i=1}^{n} X_i \ln\left(\frac{\theta}{1+\theta}\right) \\
&= -n\ln(1+\theta) + \sum_{i=1}^{n} X_i (\ln\theta - \ln(1+\theta)) \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln V(X_1, ..., X_n; \theta)}{\partial\theta} &= \frac{-n}{1+\theta} + \left(\sum_{i=1}^{n} X_i\right)\left(\frac{1}{\theta} - \frac{1}{1+\theta}\right) \\
&= \frac{-n}{1+\theta} + \left(\sum_{i=1}^{n} X_i\right)\left(\frac{1}{\theta(1+\theta)}\right) \\
&= \frac{1}{\theta(1+\theta)}\left(-n\theta + \sum_{i=1}^{n} X_i\right) = 0 \\
&\Rightarrow \quad \theta = \frac{\sum\limits_{i=1}^{n} X_i}{n} = \overline{X}.
\end{aligned}$$

Comprobamos que é un máximo.

$$\begin{aligned}
\frac{\partial^2 \ln V(X_1, ..., X_n; \theta)}{\partial\theta^2} &= \frac{n}{(1+\theta)^2} - \left(\sum_{i=1}^{n} X_i\right)\left(\frac{1+2\theta}{\theta^2(1+\theta)^2}\right) \\
&= \frac{1}{\theta^2(1+\theta)^2}\left(n\theta^2 - \left(\sum_{i=1}^{n} X_i\right)(1+2\theta)\right) < 0 \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
n\theta^2 - \left(\sum_{i=1}^{n} X_i\right)(1+2\theta) &< 0 \Leftrightarrow n\overline{X}^2 - n\overline{X}\left(1+2\overline{X}\right) < 0 \\
&\Leftrightarrow \quad -\overline{X} - \overline{X}^2 < 0 \Rightarrow \text{ é un máximo} \\
&\Rightarrow \quad \widehat{\theta}_{MV} = \overline{X}.
\end{aligned}$$

(b) Calculamos os momentos. Como $E(X) = \theta$ deducimos que $\widehat{\theta}_{Mom} = \overline{X}$.

(c) Sabemos que $E\left(\overline{X}\right) = E(X)$ polo que $\overline{X}$ é insesgado de θ. Ademáis

$$Var\left(\overline{X}\right) = \frac{Var(X)}{n} = \frac{\theta(1+\theta)}{n}.$$

Calculamos a cota de FCR.

$$\begin{aligned}
f(X;\theta) &= \frac{1}{1+\theta}\left(\frac{\theta}{1+\theta}\right)^X \Rightarrow \\
\ln f(X;\theta) &= \ln 1 - \ln(1+\theta) + \ln\left(\frac{\theta}{1+\theta}\right)^X \\
&= -\ln(1+\theta) + X\ln\left(\frac{\theta}{1+\theta}\right) \\
&= -\ln(1+\theta) + X\left[\ln\theta - \ln(1+\theta)\right] \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\ln f(X;\theta) &= -\frac{1}{1+\theta} + X\left(\frac{1}{\theta} - \frac{1}{1+\theta}\right) \\
&= -\frac{1}{1+\theta} + X\left(\frac{1}{\theta(1+\theta)}\right) = \frac{X-\theta}{\theta(1+\theta)} \Rightarrow \\
E\left[\left(\frac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2\right] &= E\left[\frac{(X-\theta)^2}{\theta^2(1+\theta)^2}\right] = \frac{1}{\theta^2(1+\theta)^2}E\left[(X-E(X))^2\right] \\
&= \frac{Var(X)}{\theta^2(1+\theta)^2} = \frac{\theta(1+\theta)}{\theta^2(1+\theta)^2} = \frac{1}{\theta(1+\theta)}.
\end{aligned}$$

Logo

$$\text{cota } FCR = \frac{1}{nE\left[\left(\frac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2\right]} = \frac{\theta(1+\theta)}{n}.$$

Como $\overline{X}$ é insesgado e $Var\left(\overline{X}\right)$ coincide ca cota de FCR deducimos que $\overline{X}$ é eficiente.

PROBLEMA 11:

Sexa $\{X_1, ..., X_n\}$ con $n > 1$ unha *m.a.s.* dunha variable exponencial de parámetro $\frac{1}{\theta}$. Recórdese que se $X \sim exp\left\{\frac{1}{\theta}\right\}$ entón, $f(x) = \frac{1}{\theta}\exp\left\{\frac{-x}{\theta}\right\}$ se $x > 0$, $E(X) = \theta$ e $Var(X) = \theta^2$. Defínense os estimadores $\widehat{\theta_1} = \overline{X}$ e $\widehat{\theta_2} = X_1$. Pídese:

(a) Estudiar a eficiencia de $\widehat{\theta_1}$.

(b) Cal dos dous estimadores é mellor.

(c) Sexa $\widehat{\theta}(w) = w\widehat{\theta_1} + (1-w)\widehat{\theta_2}$ con $0 \leq w \leq 1$. Comprobar para que valores de w, $\widehat{\theta}(w)$ é insesgado.

(d) Calcular o valor de w para o que $\widehat{\theta}(w)$ ten menor varianza.

SOLUCION:

(a) Sabemos que $E\left(\overline{X}\right) = E(X)$. Como $E(X) = \theta$, deducimos que $\overline{X}$ é *insesgado*.

Tamén sabemos que $Var\left(\overline{X}\right) = \dfrac{Var(X)}{n} = \dfrac{\theta^2}{n}$.

Calculamos a cota de FCR.

$$\begin{aligned}
f(X;\theta) &= \frac{1}{\theta}\exp\left\{\frac{-X}{\theta}\right\} \Rightarrow \ln f(X;\theta) \\
&= \ln 1 - \ln\theta + \ln\exp\left\{\frac{-X}{\theta}\right\} = -\ln\theta - \frac{X}{\theta} \Rightarrow \\
\frac{\partial}{\partial\theta}\ln f(X;\theta) &= -\frac{1}{\theta} + \frac{X}{\theta^2} = \frac{X-\theta}{\theta^2} \Rightarrow \\
E\left[\left(\frac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2\right] &= E\left[\frac{(X-\theta)^2}{\theta^4}\right] = \frac{1}{\theta^4}E\left[(X-\theta)^2\right] = \frac{1}{\theta^4}E\left[(X-E(X))^2\right] \\
&= \frac{Var(X)}{\theta^4} = \frac{\theta^2}{\theta^4} = \frac{1}{\theta^2}.
\end{aligned}$$

Logo

$$\text{cota } FCR = \frac{1}{nE\left[\left(\frac{\partial}{\partial\theta}\ln f(X;\theta)\right)^2\right]} = \frac{\theta^2}{n}.$$

Como $\overline{X}$ é insesgado e $Var\left(\overline{X}\right)$ coincide ca cota de FCR deducimos que $\overline{X}$ é *eficiente.*

(b) E mellor o que ten menor ECM. Facemos os cálculos,

$$\begin{aligned}
sesgo(X_1) &= sesgo(X) = E(X) - \theta = 0 \text{ e} \\
Var(X_1) &= Var(X) = \theta^2.
\end{aligned}$$

$$\begin{aligned}
ECM\left(\overline{X}\right) &= \left(sesgo\left(\overline{X}\right)\right)^2 + Var\left(\overline{X}\right) = 0 + \frac{\theta^2}{n} = \frac{\theta^2}{n} \\
ECM(X_1) &= \left(sesgo(X_1)\right)^2 + Var(X_1) = 0 + \theta^2 = \theta^2.
\end{aligned}$$

Como $n > 1$ é mellor $\widehat{\theta_1} = \overline{X}$.

(c)

$$E\left(\widehat{\theta}(w)\right) = wE\left(\widehat{\theta_1}\right) + (1-w)E\left(\widehat{\theta_2}\right) = w\theta + (1-w)\theta = \theta.$$

Logo $\widehat{\theta}(w)$ é insesgado para todo valor de w.

(d) Como todos os $\widehat{\theta}(w)$ son insesgados e $\widehat{\theta_1}$ é eficiente deducimos que $Var\left(\widehat{\theta}(w)\right) \geq Var\left(\widehat{\theta_1}\right)$ para todo valor de w.

Como $\widehat{\theta_1} = \widehat{\theta}(1)$ a resposta é para $w = 1$.

PROBLEMA 12:

Sexa $\{X_1, ..., X_n\}$ unha *m.a.s.* dunha variable con función de densidade $f(x) = \frac{1}{\theta}\exp\left\{-\frac{x}{\theta}\right\}$ se $x > 0$. Verifícase que $E(X) = \theta$ e $Var(X) = \theta^2$.

(a) Calcula o EMV de θ.

(b) Comproba se o EMV é insesgado

(c) Comproba se o EMV é consistente.

SOLUCION.

(a) Calculamos a función de versosimilitude.

$$\begin{aligned} V(X_1, ..., X_n; \theta) &= \prod_{i=1}^{n} f(X_i; \theta) = \prod_{i=1}^{n} \frac{1}{\theta}\exp\left\{-\frac{X_i}{\theta}\right\} \\ &= \frac{1}{\theta}\exp\left\{-\frac{X_1}{\theta}\right\} \times \frac{1}{\theta}\exp\left\{-\frac{X_2}{\theta}\right\} \times ... \times \frac{1}{\theta}\exp\left\{-\frac{X_n}{\theta}\right\} \\ &= \frac{1}{\theta^n}\exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n} X_i\right\}. \end{aligned}$$

Calculamos o ln da función de verosimilitude

$$\begin{aligned} \ln V(X_1, ..., X_n; \theta) &= \ln 1 - \ln \theta^n + \ln \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n} X_i\right\} \\ &= -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} X_i \end{aligned}$$

Maximizamos o ln da función de verosimilitude.

$$\frac{\partial}{\partial\theta}\ln V(X_1, ..., X_n; \theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} X_i = \frac{1}{\theta^2}\left(-n\theta + \sum_{i=1}^{n} X_i\right) = 0$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} X_i}{n} = \overline{X}.$$

Comprobamos que é un máximo en $\theta = \overline{X}$

$$\begin{aligned} \frac{\partial^2 \ln V(X_1, ..., X_n; \theta)}{\partial\theta^2} &= \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} X_i = \frac{n}{\overline{X}^2} - \frac{2}{\overline{X}^3}n\overline{X} = -\frac{n}{\overline{X}^2} < 0 \\ &\Rightarrow \text{ é un máximo} \Rightarrow \widehat{\theta}_{MV} = \overline{X}. \end{aligned}$$

(b) De teoría sabemos que $E\left(\overline{X}\right) = E(X)$.

Do enunciado sabemos que $E(X) = \theta$. Logo $E\left(\overline{X}\right) = \theta$ polo que $\overline{X}$ é insesgado.

(c) De teoría sabemos que $Var\left(\overline{X}\right) = \dfrac{Var(X)}{n}$. Do enunciado sabemos que $Var(X) = \theta^2$.

$$\lim_{n\to\infty} Var\left(\overline{X}\right) = \lim_{n\to\infty}\frac{\theta^2}{n} = 0.$$

No apartado (b) vimos que era insesgado.

Logo $\overline{X}$ é consistente.