<u>Integrals impròpies</u>: integrals de funcions no acotades i/o integrals en dominis no acotats.

Cas d'una variable

* $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\delta\to 0}\int_\delta^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\delta\to 0} 2\sqrt{x}\Big]_\delta^1 = \lim_{\delta\to 0}\left(2\cdot 1 - 2\sqrt{\delta}\right) = 2$$

* $$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{\delta\to\infty}\int_1^\delta \frac{1}{x^3}\,dx = \lim_{\delta\to\infty} \frac{1}{-2}\,\frac{1}{x^2}\Big]_1^\delta = \lim_{\delta\to\infty}\left(\frac{1}{-2}\,\frac{1}{\delta^2} + \frac{1}{2}\right) = \frac{1}{2}$$

* $$\int_0^\infty \frac{1}{\sqrt{x}+x^{3/2}}\,dx = \lim_{\substack{\delta\to 0\\ \eta\to\infty}}\int_\delta^\eta \frac{1}{\sqrt{x}+x^{3/2}}\,dx \underset{\text{canvi } x=t^2}{=} \lim_{\substack{\delta\to 0\\ \eta\to\infty}}\int_{\sqrt{\delta}}^{\sqrt{\eta}} \frac{1}{t+t^3}\,2t\,dt =$$

$$= \lim_{\substack{\delta\to 0\\ \eta\to\infty}} 2\int_{\sqrt{\delta}}^{\sqrt{\eta}} \frac{1}{1+t^2}\,dt = \lim_{\substack{\delta\to 0\\ \eta\to\infty}} 2\,\mathrm{arctg}\, t\Big]_{\sqrt{\delta}}^{\sqrt{\eta}} = \lim_{\substack{\delta\to 0\\ \eta\to\infty}}\left(2\,\mathrm{arctg}\sqrt{\eta} - 2\,\mathrm{arctg}\sqrt{\delta}\right) = 2\cdot\frac{\pi}{2}$$

Cas de diverses variables

Considerem $\displaystyle\int_D f$.

Si f i/o D no són acotats, prenem un domini depenent d'un o més paràmetres D_δ (o $D_{\delta\eta}$) t.q. D_δ és acotat, f és acotada en D_δ i D és el "límit" de D_δ quan $\delta \to 0$ (o $\delta \to \infty$)

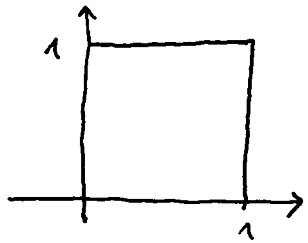
Llavors $\exists \displaystyle\int_{D_\delta} f$ i prenem

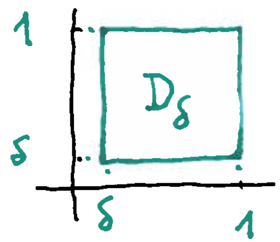
$$\boxed{\int_D f = \lim_{\delta \to 0} \int_{D_\delta} f \qquad \left(\text{si } D_\delta \to D \text{ quan } \delta \to 0\right)}$$

Si el límit no existeix o és ∞ diem que f no és integrable en D.
Si el límit existeix es diu que és la integral impròpia de f en D.

<u>Ex</u>

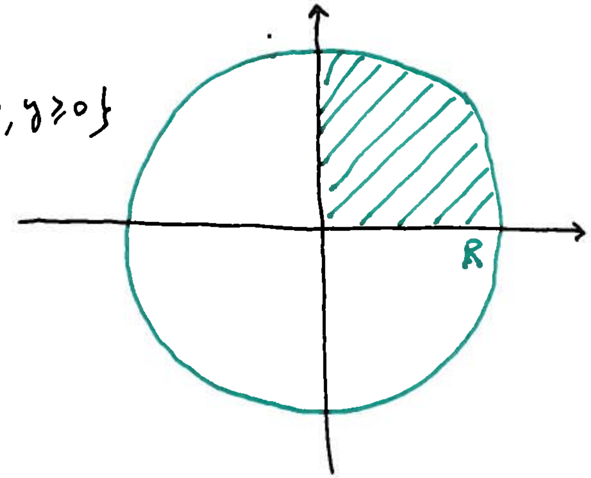
$$\int_D \frac{1}{\sqrt{xy}}\, dx\, dy, \qquad D = [0,1] \times [0,1]$$





$$I = \lim_{\delta \to 0} \int_{D_\delta} \frac{1}{\sqrt{xy}}\, dx\, dy = \lim_{\delta \to 0} \int_\delta^1 \int_\delta^1 \frac{1}{\sqrt{xy}}\, dy\, dx = \lim_{\delta \to 0} \int_\delta^1 \frac{1}{\sqrt{x}}\, dx \int_\delta^1 \frac{1}{\sqrt{y}}\, dy = \lim_{\delta \to 0} \left[2\sqrt{x}\right]_\delta^1 \left[2\sqrt{y}\right]_\delta^1 = 4$$

Ex $\int_D xy\, e^{-(x^2+y^2)}\, dx\, dy$, $\quad D = \{(x,y) \mid x \geq 0,\ y \geq 0\}$



$$= \lim_{R\to\infty} \int_{D_R} xy\, e^{-(x^2+y^2)}\, dx\, dy$$

$D_R = \{(x,y) \in D \mid x^2+y^2 \leq R^2\}$, "$D_R \to D$"

$$= \lim_{R\to\infty} \int_0^{\pi/2}\int_0^R r\cos\theta\, r\sin\theta\, e^{-r^2}\cdot r\, dr\, d\theta = \lim_{R\to\infty} \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^R r^3 e^{-r^2}\, dr$$

$$= \lim_{R\to\infty} \left.\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \int_0^R r^3 e^{-r^2}\, dr = \frac{1}{2}\lim_{R\to\infty} \frac{-1}{2}\left[r^2 e^{-r^2} + e^{-r^2}\right]_0^R = -\frac{1}{4}\lim_{R\to\infty}\left[R^2 e^{-R^2} + e^{-R^2} - 1\right] = \frac{1}{4}$$

Primitiva

$$\int r^3 e^{-r^2}\, dr = \frac{1}{-2}\int \underbrace{r^2}_{v}\underbrace{(-2r)\, e^{-r^2}}_{u'}\, dr = -\frac{1}{2}\left[r^2 e^{-r^2} - \int 2r e^{-r^2}\, dr\right] = -\frac{1}{2}\left[r^2 e^{-r^2} + e^{-r^2}\right]$$

Funcions definides per integrals o integrals depenents de paràmetres

1) Comencem considerant integrals $\int_a^b f(x,y)\,dx$, depenent de y. $a,b \in \mathbb{R}$

Ex. La funció

$$F(y) = \int_0^1 x^{y-1} \sin x \, dx$$

Resultats bàsics

a) si $f(x,y)$ és contínua llavors $F(y) = \int_a^b f(x,y)\,dx$ és contínua

b) si $f(x,y)$ i $\frac{\partial f}{\partial y}(x,y)$ són contínues llavors $F(y)$ és derivable i

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

2) Pel teorema fonamental del càlcul, si f és contínua, $F(y) = \int_a^y f(x)\,dx$ és derivable i

$$F'(y) = f(y)$$

Anàlogament $G(y) = \int_y^b f(x)\,dx$ és derivable i

$$G'(y) = -f(y)$$

3) Cas en que l'integrant i els límits d'integració depenen d'un paràmetre:

$$F(y) = \int_{a(y)}^{b(y)} f(x,y)\,dx$$

Si f, $\frac{\partial f}{\partial y}$ són contínues i $a(y), b(y)$ són derivables llavors F és derivable i

$$\boxed{F'(y) = \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y}(x,y)\,dx + f(b(y),y)\,b'(y) - f(a(y),y)\,a'(y)}$$

Dem Considerem $F(y) = \int_{a(y)}^{b(y)} f(x,y)\,dx$ com a composició

$$I \subset \mathbb{R} \xrightarrow{\phi} \mathcal{U} \subset \mathbb{R}^3 \xrightarrow{H} \mathbb{R}$$

$$y \longmapsto (b(y), a(y), y)$$

$$(b, a, t) \longmapsto \int_a^b f(x,t)\,dx$$

$$F(y) = H \circ \phi(y) \longrightarrow F'(y) = DH(\phi(y))\, D\phi(y)$$

$$DH(b,a,t) = \left(f(b,t), \; -f(a,t), \; \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx \right)$$

$$D\phi(y) = \begin{pmatrix} b'(y) \\ a'(y) \\ 1 \end{pmatrix}$$

$$F'(y) = f(b(y),y)\,b'(y) - f(a(y),y)\,a'(y) + \int_{a(y)}^{b(y)} \frac{\partial f}{\partial t}(x,y)\,dx$$

Ex

$$F(a)=\int_0^{\pi/2}\cos ax\,dx=\frac{\sin ax}{a}\Big]_0^{\pi/2}=\frac{\sin\frac{a\pi}{2}}{a}$$

$$F'(a)=\int_0^{\pi/2}-x\sin ax\,dx=\frac{\cos\frac{a\pi}{2}\cdot\frac{\pi}{2}a-\sin\frac{a\pi}{2}}{a^2}\Rightarrow\underbrace{\int_0^{\pi/2}x\sin ax\,dx}_{\parallel\ G(a)}=-\frac{\pi}{2a}\cos\frac{a\pi}{2}+\frac{1}{a^2}\sin\frac{a\pi}{2}$$

$$G'(a)=\int_0^{\pi/2}x^2\cos ax\,dx=\frac{d}{da}\left[-\frac{\pi}{2a}\cos\frac{a\pi}{2}+\frac{1}{a^2}\sin\frac{a\pi}{2}\right]$$

Ex Calculeu les integrals $\int_0^a\frac{dx}{(x^2+a^2)^2}$ i $\int_0^a\frac{dx}{(x^2+a^2)^3}$ a partir de $\int_0^a\frac{dx}{x^2+a^2}$, $a>0$.

$$F(a)=\int_0^a\frac{dx}{x^2+a^2}\underset{x=ay}{=}\int_0^1\frac{a\,dy}{a^2y^2+a^2}=\frac{1}{a}\int_0^1\frac{dy}{1+y^2}=\frac{1}{a}\operatorname{arctg}y\Big]_0^1=\frac{1}{a}(\operatorname{arctg}1-\operatorname{arctg}0)=\frac{\pi}{4a}$$

$$F'(a)=\frac{1}{a^2+a^2}+\int_0^a\frac{-1\cdot 2a}{(x^2+a^2)^2}dx=\frac{-\pi}{4}\frac{1}{a^2}\to -2a\int_0^a\frac{1}{(x^2+a^2)^2}dx=-\frac{\pi}{4a^2}-\frac{1}{2a^2}\to\underbrace{\int_0^a\frac{dx}{(x^2+a^2)^2}}_{=G(a)}=\frac{2+\pi}{8a^3}$$

$$G'(a)=\frac{1}{(a^2+a^2)^2}+\int_0^a\frac{-2\cdot 2a}{(x^2+a^2)^3}dx=\frac{2+\pi}{8}\frac{-3}{a^4}\to -4a\int_0^a\frac{dx}{(x^2+a^2)^3}=-\frac{1}{4a^4}-\frac{3}{8}\frac{2+\pi}{a^4}\to\int_0^a\frac{dx}{(x^2+a^2)^3}=\frac{8+3\pi}{32a^5}$$

Funcions definides per integrals impròpies

<u>Ex</u> La funció Γ d'Euler : $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx$, $y > 0$.

<u>Resultats bàsics</u> : Siguin I interval acotat o no i $f : \underset{(x,\, y)}{I \times U} \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
acotada o no.

a) si $f(x,y)$ és contínua i existeix $g(x)$ t.q.

* $g(x) \geq 0$
* $|f(x,y)| \leq g(x)$, $\forall y, \forall x$
* existeix la integral (impròpia) $\int_I g(x)\, dx$

llavors $F(y) = \int_I f(x,y)\, dx$ és contínua

b) si $f(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ són contínues i $\exists\, h(x)$ t.q.

* $h(x) \geq 0$
* $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq h(x)$, $\forall y, \forall x$
* existeix la integral (impròpia) $\int_I h(x)\, dx$

$\Longrightarrow$ F és derivable i

$$F'(y) = \int_I \frac{\partial f}{\partial y}(x,y)\, dx$$

Propietats de la funció Γ , $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx$

* Γ és derivable (i també C^∞) , $\Gamma'(y) = \int_0^\infty \frac{\partial}{\partial y}\left[x^{y-1} e^{-x}\right] dx = \int_0^\infty x^{y-1} \log x \, e^{-x}\, dx$

* $\Gamma(1) = \int_0^\infty x^0 e^{-x}\, dx = -e^{-x}\Big]_0^\infty = -(0-1) = 1$

* $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx \underset{x=t^2}{=} \int_0^\infty t^{2y-2} e^{-t^2} 2t\, dt = 2\int_0^\infty t^{2y-1} e^{-t^2}\, dt$

* $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty t^0 e^{-t^2}\, dt = 2\int_0^\infty e^{-t^2}\, dt$

* $\left(\int_0^\infty e^{-x^2} dx\right)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_D \overbrace{e^{-x^2} e^{-y^2}}^{e^{-(x^2+y^2)}} dx\, dy$, $D = \{(x,y) \mid x \geq 0,\ y \geq 0\}$

canvi a polars $= \int_B e^{-r^2} r\, dr\, d\theta = \int_0^{\pi/2}\int_0^\infty \frac{1}{-2}(-2r) e^{-r^2} dr\, d\theta = -\frac{1}{2}\int_0^{\pi/2} d\theta \left[e^{-r^2}\right]_0^\infty = -\frac{1}{2}\,\frac{\pi}{2}(-1) = \frac{\pi}{4}$

$\Rightarrow \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

* $\Gamma(y) = \int_0^\infty \underbrace{x^{y-1}}_{u} \underbrace{e^{-x}}_{v'} \, dx = \left[x^{y-1} \frac{e^{-x}}{-1} \right]_0^\infty - \int_0^\infty (y-1) x^{y-2} \frac{e^{-x}}{-1} \, dx$

$= (y-1) \int_0^\infty x^{y-2} e^{-x} \, dx = (y-1)\, \Gamma(y-1)$, si $y > 1$

* si $n \in \mathbb{N}$ $\quad \Gamma(n) = (n-1)\, \Gamma(n-1) = (n-1)(n-2)\, \Gamma(n-2) = (n-1)(n-2) \cdots 1 \cdot \Gamma(1)$

$= (n-1)!$

<u>Ex</u> Algunes integrals es poden reduir a la funció Γ

$$\int_0^\infty \sqrt{x}\, e^{-x^3} \, dx \underset{\uparrow}{=} \int_0^\infty t^{1/6} e^{-t} \frac{1}{3} t^{-2/3} \, dt = \frac{1}{3} \int_0^\infty t^{-3/6} e^{-t} \, dt = \frac{1}{3} \Gamma\left(\frac{1}{2}\right)$$

canvi $x^3 = t$

$x = t^{1/3}$, $dx = \frac{1}{3} t^{-2/3} \, dt$

$$\int_0^1 (\log x)^4 \, dx \underset{\uparrow}{=} \int_\infty^0 (\log e^{-t})^4 (-e^{-t}) \, dt = \int_0^\infty (-t)^4 e^{-t} \, dt = \Gamma(5) = 4!$$

canvi $x = e^{-t}$

Propietats de la funció B , $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, $x,y>0$.

* B és diferenciable (i també C^∞) en $\{(x,y) \mid x>0, y>0\}$

* $B(x,y)=B(y,x)$

canvi $t=1-s$

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_1^0 (1-s)^{x-1} s^{y-1}(-1)\,ds = \int_0^1 s^{y-1}(1-s)^{x-1}\,ds = B(y,x)$$

* $B(x,y)=\frac{y-1}{x}B(x+1,y-1)$, $(x>0,\ y>1)$

$$B(x,y)=\int_0^1 \underbrace{t^{x-1}}_{u'}\underbrace{(1-t)^{y-1}}_{v}\,dt = \left[\frac{t^x}{x}(1-t)^{y-1}\right]_0^1 - \int_0^1 \frac{t^x}{x}(y-1)(1-t)^{y-2}(-1)\,dt$$

$$= \frac{y-1}{x}\int_0^1 t^x (1-t)^{y-2}\,dt = \frac{y-1}{x}B(x+1,y-1)$$

* $B(x,1)=\int_0^1 t^{x-1}(1-t)^0\,dt = \left.\frac{t^x}{x}\right]_0^1 = \frac{1}{x}$

* Si $\begin{cases} y\in\mathbb{N} \\ x>0 \end{cases}$, $B(x,y)=\frac{y-1}{x}B(x+1,y-1)=\frac{(y-1)(y-2)}{x\ (x+1)}B(x+2,y-2)=\ \cdots$

$$=\frac{(y-1)(y-2)(y-3)\cdots 1}{x(x+1)(x+2)\cdots(x+y-2)}B(x+y-1,1)=\frac{(y-1)!}{x(x+1)\cdots(x+y-1)}$$

Relació entre Γ i B

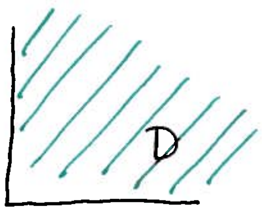
$$\boxed{B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \qquad x>0,\ y>0.}$$

recordem que $\boxed{\Gamma(z)} = \displaystyle\int_0^\infty t^{z-1} e^{-t}\, dt \boxed{= 2\int_0^\infty s^{2z-1} e^{-s^2}\, ds}$

A més $\boxed{B(x,y)} = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\, dt \underset{\text{canvi } t=\sin^2\theta}{=} \int_0^{\pi/2} \sin^{2x-2}\theta\,(1-\sin^2\theta)^{y-1}\, 2\sin\theta\cos\theta\, d\theta$

$$\boxed{= 2\int_0^{\pi/2} \sin^{2x-1}\theta\, \cos^{2y-1}\theta\, d\theta}$$

$$\Gamma(x)\Gamma(y) = 2\int_0^\infty s^{2x-1} e^{-s^2}\, ds \cdot 2\int_0^\infty t^{2y-1} e^{-t^2}\, dt = 4\iint_D s^{2x-1} t^{2y-1} e^{-s^2} e^{-t^2}\, ds\, dt$$



(canvi a polars)

$t = r\cos\theta$

$s = r\sin\theta$

$$= 4\int_0^{\pi/2}\int_0^\infty r^{2x-1}(\sin\theta)^{2x-1}\, r^{2y-1}(\cos\theta)^{2y-1}\, e^{-r^2}\, r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\, d\theta\; 2\int_0^\infty r^{2x+2y-1} e^{-r^2}\, dr = B(x,y)\,\Gamma(x+y)$$

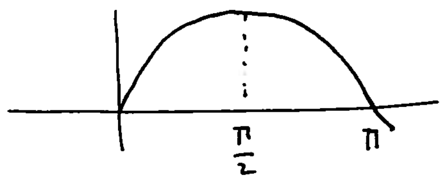
Aplicacions al càlcul d'algunes integrals

* $$\int_0^{\pi/2} \sin^m\theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

$$= \begin{cases} (\text{si } m \text{ és parell}) = \dfrac{1}{2} \dfrac{\frac{m-1}{2} \frac{m-3}{2} \cdots \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\frac{m}{2} \frac{m-2}{2} \cdots \frac{2}{2} \Gamma(1)} = \dfrac{(m-1)(m-3)\cdots 1}{m\,(m-2)\cdots 2} \dfrac{\sqrt{\pi}\sqrt{\pi}}{2} = \dfrac{(m-1)!!}{m!!} \dfrac{\pi}{2} \\[2ex] (\text{si } m \text{ és senar}) = \dfrac{1}{2} \dfrac{\frac{m-1}{2} \frac{m-3}{2} \cdots \frac{2}{2} \Gamma(1)\, \Gamma\left(\frac{1}{2}\right)}{\frac{m}{2} \frac{m-2}{2} \cdots \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)} = \dfrac{(m-1)!!}{m!!} \end{cases}$$

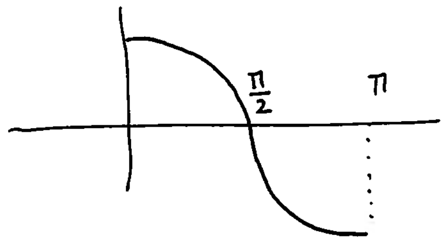
* $$\int_0^{\pi/2} \cos^m\theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{m+1}{2}\right) = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right)$$

* $$\int_0^{\pi} \sin^m\theta \, d\theta = 2 \int_0^{\pi/2} \sin^m\theta \, d\theta$$



* $$\int_0^{\pi} \cos^m\theta \, d\theta = 2 \int_0^{\pi/2} \cos^m\theta \, d\theta \qquad \text{si } m \text{ parell}$$

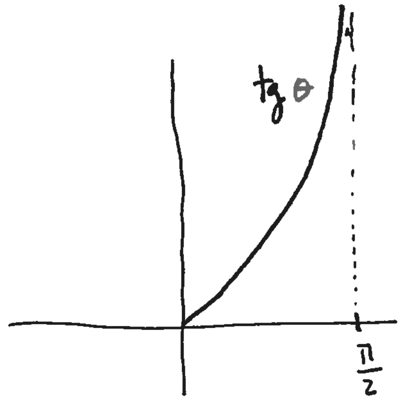
* $$\int_0^{\pi} \cos^m\theta \, d\theta = 0 \qquad \text{si } m \text{ senar}$$



$$* \quad \int_0^{\infty} \frac{dx}{(1+x^2)^m} = \int_0^{\pi/2} \frac{1}{(1+\operatorname{tg}^2\theta)^m} \frac{1}{\cos^2\theta}\, d\theta$$

canvi $\quad x = \operatorname{tg}\theta$

$$dx = \frac{1}{\cos^2\theta}\, d\theta$$



$$= \int_0^{\pi/2} \frac{1}{\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)^m} \frac{1}{\cos^2\theta}\, d\theta = \int_0^{\pi/2} \frac{1}{\left(\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right)^m} \frac{1}{\cos^2\theta}\, d\theta = \int_0^{\pi/2} \cos^{2n-2}\theta\, d\theta$$

$$= \frac{1}{2} B\left(\frac{2m-1}{2}, \frac{1}{2}\right)$$

$$* \quad \int_0^1 \frac{x^m}{\sqrt{1-x^2}}\, dx = \int_0^1 x^m (1-x^2)^{-1/2}\, dx = \int_0^1 t^{m/2} (1-t)^{-1/2} \frac{1}{2} t^{-1/2}\, dt$$

canvi $\quad x = \sqrt{t}$

$$dx = \frac{1}{2} t^{-1/2}\, dt$$

$$= \frac{1}{2} \int_0^1 t^{\frac{m-1}{2}} (1-t)^{-1/2}\, dt = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{1}{2}\right)$$