Llista de problemes (8)

3.7 Calculeu l'àrea d'un cercle de radi r, usant integrals dobles

3.8 Calculeu l'àrea d'una el·lipse de semieixos a, b

3.9 Calculeu

$$\int_D x^3 y \, dx\, dy\,,$$

on D és la regió acotada per l'eix y i la paràbola $x = -4y^2 + 3$.

3.10 Calculeu

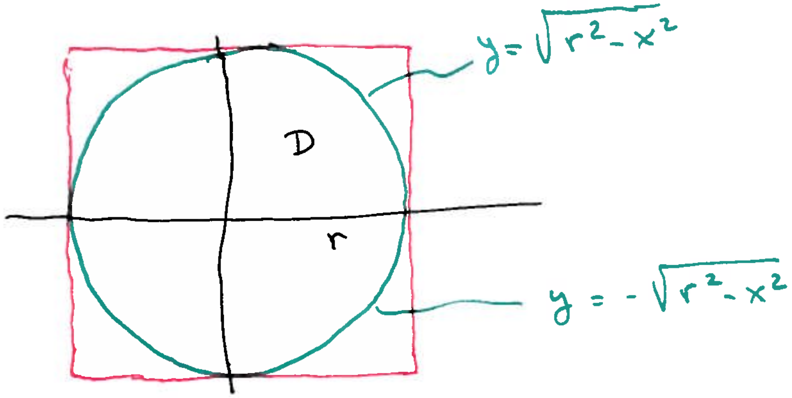
$$\int_D (1+xy)\, dx\, dy\,,$$

on D és la regió determinada per $1 \leq x^2+y^2 \leq 2$, $y \geq 0$.

3.11 Trobeu el volum de l'interior de la superfície $z = x^2+y^2$ entre $z=0$ i $z=9$

3.12 Calculeu el volum d'un con de base de radi r i altura h.

3.7



$$\text{àrea } D = \int_D 1 \, dx\, dy = \int_{-r}^{r} \left(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy \right) dx = \int_{-r}^{r} y \Big]_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dx$$

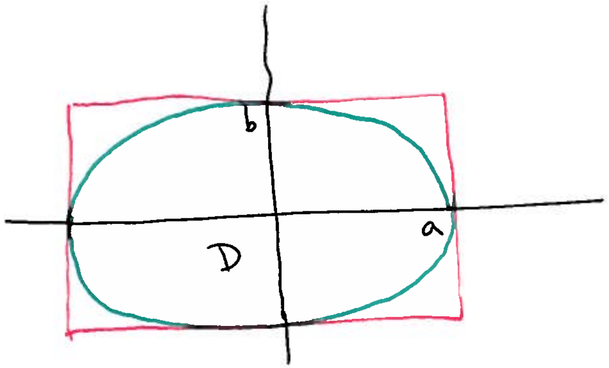
$$= \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx \underset{\uparrow}{=} 2 \int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2 \sin^2 t} \;\; r \cos t \, dt$$

canvi $x = r \sin t$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 t} \, \cos t \, dt = 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt = 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2t}{2} \, dt$$

$$= 2r^2 \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_{-\frac{\pi}{2}}^{\pi/2} = 2r^2 \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \pi r^2$$

3.8



Eq. el·lipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

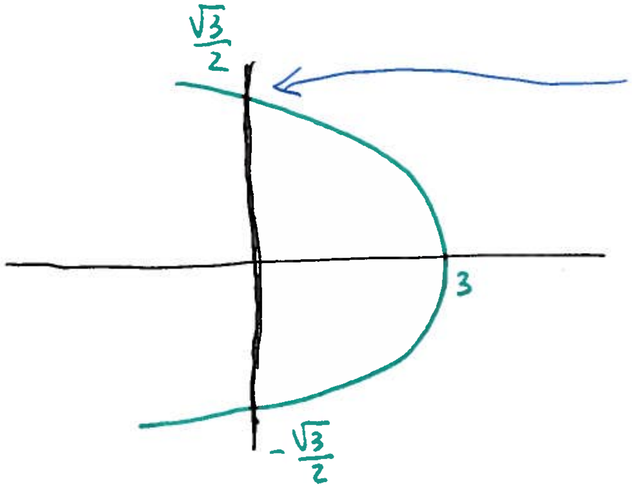
$$\text{Area } D = \int_D 1\,dx\,dy = \int_{-a}^{a}\left[\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} 1\,dy\right]dx = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} 2b\sqrt{1-\frac{a^2\sin^2 t}{a^2}}\; a\cos t\,dt = 2ab\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt$$

canvi $x = a\sin t$

$$= 2ab\,\frac{\pi}{2} = ab\pi$$

3.9 D regió acotada per l'eix y i la paràbola $x = -4y^2 + 3$



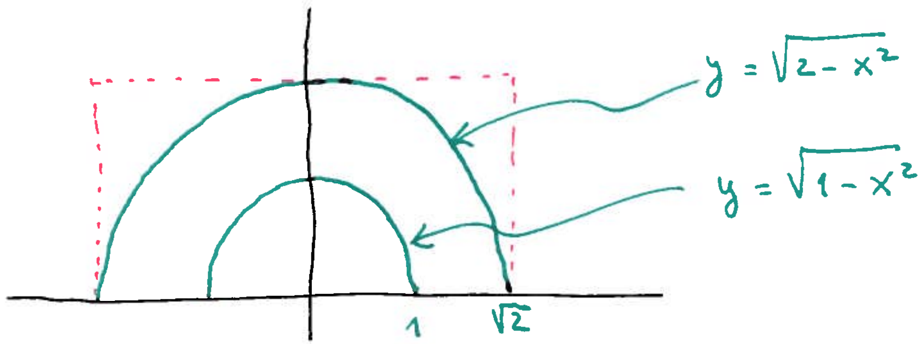
intersecció $\begin{cases} x = -4y^2 + 3 \\ x = 0 \end{cases} \rightarrow \begin{array}{l} -4y^2 + 3 = 0 \\ 4y^2 = 3 \rightarrow y = \pm \frac{\sqrt{3}}{2} \end{array}$

$$\int_D x^3 y \, dx \, dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\int_0^{-4y^2+3} x^3 y \, dx \right) dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left. \frac{x^4 y}{4} \right]_0^{-4y^2+3} dy$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{(-4y^2+3)^4}{4} y \, dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{4} \frac{1}{-8} (-4y^2+3)^4 (-8y) \, dy = \frac{-1}{32} \left. \frac{(-4y^2+3)^5}{5} \right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} = 0$$

3.10 $\quad D = \{ (x,y) \mid 1 \leqslant x^2 + y^2 \leqslant 2, \quad y \geqslant 0 \}$

$$I = \int_D (1 + xy)\, dx\, dy$$



$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{\phi_1(x)}^{\phi_2(x)} (1+xy)\, dy \right) dx$$

$$= \int_{-\sqrt{2}}^{-1} \left(\int_0^{\sqrt{2-x^2}} (1+xy)\, dy \right) dx + \int_{-1}^{1} \left(\int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} (1+xy)\, dy \right) dx + \int_{1}^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} (1+xy)\, dy \right) dx$$

$$= \int_{-\sqrt{2}}^{-1} \left[y + x \frac{y^2}{2} \right]_0^{\sqrt{2-x^2}} dx + \int_{-1}^{1} \left[y + x \frac{y^2}{2} \right]_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} dx + \int_{1}^{\sqrt{2}} \left[y + x \frac{y^2}{2} \right]_0^{\sqrt{2-x^2}} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\sqrt{2-x^2} + x \frac{(2-x^2)}{2} \right] dx - \int_{-1}^{1} \left[\sqrt{1-x^2} + x \frac{1-x^2}{2} \right] dx$$

$$= \underbrace{\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-x^2}\,dx}_{I_1} + \underbrace{\int_{-\sqrt{2}}^{\sqrt{2}} x\,\frac{(2-x^2)}{2}\,dx}_{I_2} - \underbrace{\int_{-1}^{1} \sqrt{1-x^2}\,dx}_{I_3} - \underbrace{\int_{-1}^{1} x\,\frac{1-x^2}{2}\,dx}_{I_4}$$

$$I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2-2\sin^2 t}\,\sqrt{2}\cos t\,dt = 2\int_{-\frac{\pi}{2}}^{\pi/2} \cos^2 t\,dt = 4\int_0^{\pi/2} \cos^2 t\,dt = 4\int_0^{\pi/2} \frac{1+\cos 2t}{2}\,dt$$

canvi $x = \sqrt{2}\sin t$

$$= 4\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\pi/2} = \pi$$

$$I_2 = \frac{1}{2}\,\frac{1}{-2}\int_{-\sqrt{2}}^{\sqrt{2}} -2x\,(2-x^2)\,dx = -\frac{1}{4}\,\frac{(2-x^2)^2}{2}\Bigg]_{-\sqrt{2}}^{\sqrt{2}} = 0$$

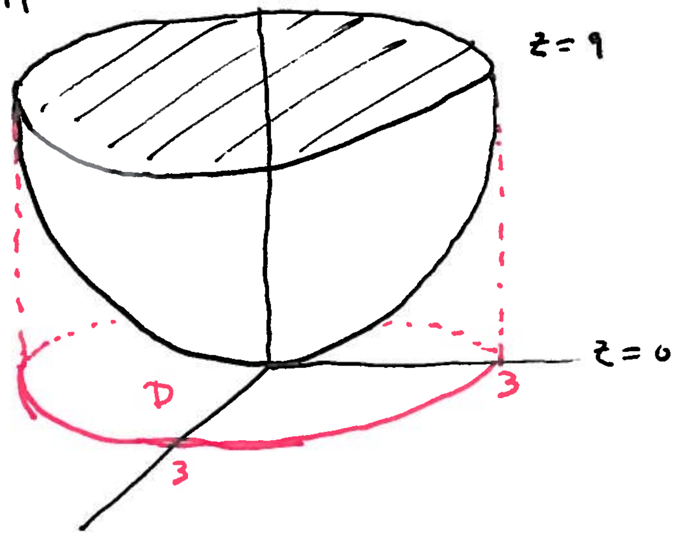
$$I_3 = \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 t}\,\cos t\,dt = \int_{-\pi/2}^{\pi/2} \cos^2 t\,dt = 2\int_0^{\pi/2} \cos^2 t\,dt = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

canvi $x = \sin t$

$$I_4 = 0$$

$$\Longrightarrow \quad I = I_1 + I_2 - I_3 - I_4 = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

3.11



$$z = x^2 + y^2$$

$x^2 + y^2 = 9 \rightarrow$ circumferència de radi 3

$$I = \int_D (x^2+y^2)\,dx\,dy = \int_{-3}^{3} \left(\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2+y^2)\,dy \right) dx = \int_{-3}^{3} \left[x^2 y + \frac{y^3}{3} \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} \left[x^2\sqrt{9-x^2} + \frac{(9-x^2)^{3/2}}{3} + x^2\sqrt{9-x^2} + \frac{(9-x^2)^{3/2}}{3} \right] dx$$

$$= 2\int_{-3}^{3} \left(x^2 + \frac{9-x^2}{3} \right) \sqrt{9-x^2}\,dx = 2\int_{-3}^{3} \left(3 + \frac{2}{3}x^2 \right) \sqrt{9-x^2}\,dx =$$

canvi $x = 3\sin t$

$dx = 3\cos t\,dt$

$$= 2\int_{-\frac{\pi}{2}}^{\pi/2} \left(3 + \frac{2}{3}\,9\sin^2 t \right) \sqrt{9 - 9\sin^2 t}\; 3\cos t\,dt = 18 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3 + 6\sin^2 t)\cos^2 t\,dt$$

3.11 (2)

$$= 18\int_{-\frac{\pi}{2}}^{\pi/2} (3+6\sin^2 t)\cos^2 t\, dt = 18\cdot 6\int_0^{\pi/2} \cos^2 t\, dt + 18\cdot 12\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt$$

Primitive de $\sin^2 t\cos^2 t$:

$$\int \sin^2 t\cos^2 t\, dt = \int \underbrace{\sin^2 t\cos t}_{u'}\ \underbrace{\cos t}_{v}\ dt = \frac{\sin^3 t}{3}\cos t - \int \frac{\sin^3 t}{3}(-\sin t)\, dt$$

$$\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt = \frac{1}{3}\int_0^{\pi/2} \sin^4 t\, dt$$

$$\sin^2 t = \frac{1-\cos 2t}{2}, \quad \sin^4 t = \left(\frac{1-\cos 2t}{2}\right)^2 = \frac{1-2\cos 2t+\cos^2 2t}{4} =$$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

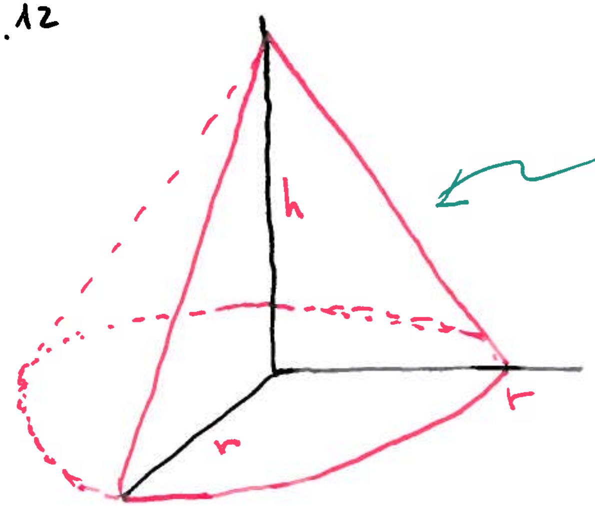
$$= \frac{1}{4}\left(1-2\cos 2t + \frac{1+\cos 4t}{2}\right) = \frac{1}{8}(3-4\cos 2t+\cos 4t)$$

$$\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt = \frac{1}{3}\left[\frac{3}{8}t - 4\,\frac{\sin 2t}{2} + \frac{\sin 4t}{4}\right]_0^{\pi/2} = \frac{1}{8}\,\frac{\pi}{2}$$

$$\Rightarrow\ I = 18\cdot 6\,\frac{\pi}{4} + 18\cdot 12\,\frac{1}{8}\,\frac{\pi}{2} = \left(9\cdot 3 + 9\cdot 3\,\frac{1}{2}\right)\pi = \frac{81}{2}\pi$$

$$\text{volum} = \text{volum cilindre} - I = \pi\cdot 3^2\cdot 9 - \frac{81}{2}\pi = \frac{81}{2}\pi$$

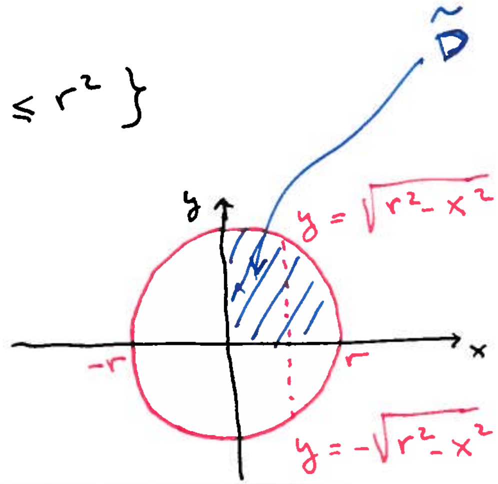
3.12



$$z = h + \frac{-h}{r}\, y$$

Eq. del con $\qquad z = h - \frac{h}{r}\sqrt{x^2+y^2}$

$$D = \{ (x,y) \mid x^2+y^2 \le r^2 \}$$



$$V = \iint_D \left(h - \frac{h}{r}\sqrt{x^2+y^2} \right) dx\, dy =$$

$$= 4\int_{\tilde{D}} \left(h - \frac{h}{r}\sqrt{x^2+y^2} \right) dx\, dy = 4h \int_{\tilde{D}} dx\, dy - \frac{4h}{r} \int_{\tilde{D}} \sqrt{x^2+y^2}\, dx\, dy .$$

$$\int_{\tilde{D}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^r \left(\int_0^{\sqrt{r^2-x^2}} \sqrt{x^2+y^2}\, dy \right) dx \underset{\text{taules}}{=} \int_0^r \left[\frac{1}{2} y \sqrt{x^2+y^2} + \frac{x^2}{2} \log\left(y + \sqrt{x^2+y^2} \right) \right]_0^{\sqrt{r^2-x^2}} dx$$

$$= \int_0^r \left[\frac{1}{2}\sqrt{r^2-x^2}\sqrt{r^2} + \frac{x^2}{2} \log\left(\sqrt{r^2-x^2} + \sqrt{r^2} \right) - \frac{x^2}{2} \log \sqrt{x^2} \right] dx$$

3.12 (2)

$$= \int_0^r \frac{r}{2}\sqrt{r^2-x^2}\,dx + \int_0^r \overbrace{\frac{x^2}{2}}^{u'} \overbrace{\log\frac{\sqrt{r^2-x^2}+r}{x}}^{v}\,dx \equiv I_1 + I_2$$

$$I_1 = \frac{r}{2}\int_0^{\pi/2} \sqrt{r^2-r^2\sin t}\; r\cos t\,dt = \frac{r^3}{2}\int_0^{\pi/2} \cos^2 t\,dt = \frac{r^3}{2}\,\frac{\pi}{4}$$

canvi $x = r\sin t$

$$I_2 = \left[\frac{x^3}{6}\log\frac{\sqrt{r^2-x^2}+r}{x}\right]_0^r - \int_0^r \frac{x^3}{6}\,\frac{1}{\frac{\sqrt{r^2-x^2}+r}{x}}\,\underbrace{\frac{\frac{1}{2\sqrt{r^2-x^2}}(-2x)\,x - \left(\sqrt{r^2-x^2}+r\right)}{x^2}}_{\underbrace{\frac{\frac{-x^2}{\sqrt{r^2-x^2}} - \frac{r^2-x^2+r\sqrt{r^2-x^2}}{\sqrt{r^2-x^2}}}{x^2}}_{\frac{-\left(r^2+r\sqrt{r^2-x^2}\right)}{x^2\sqrt{r^2-x^2}}}}\,dx$$

integració per parts

$$= 0 + \int_0^r \frac{x^3}{6}\,\frac{x}{\sqrt{r^2-x^2}+r}\,\frac{r\left(r+\sqrt{r^2-x^2}\right)}{x^2\sqrt{r^2-x^2}}\,dx$$

$$= \frac{1}{6}r\int_0^r \frac{x^2}{\sqrt{r^2-x^2}}\,dx = \frac{1}{6}r\int_0^{\pi/2} \frac{r^2\sin^2 t}{\sqrt{r^2-r^2\sin^2 t}}\,r\cos t\,dt = \frac{r^3}{6}\int_0^{\pi/2} \sin^2 t\,dt = \frac{r^3}{6}\,\frac{\pi}{4}$$

canvi $x = r\sin t$

3.12 (3)

$$V = 4h \int_{\tilde{D}} dx\,dy - \frac{4h}{r} \int_{\tilde{D}} \sqrt{x^2+y^2}\,dx\,dy = 4h\,\frac{\pi r^2}{4} - \frac{4h}{r}\left(\frac{r^3}{2}\,\frac{\pi}{4} + \frac{r^3}{6}\,\frac{\pi}{4}\right)$$

$$= h\pi r^2\left(1 - \frac{1}{2} - \frac{1}{6}\right) = \frac{1}{3} h\pi r^2$$