Llista de problemes

4.6 Determineu els extrems relatius de $f(x,y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$

4.7 Determineu els extrems relatius de $f(x,y) = x^4 + y^4 - 2(x-y)^2$

4.8 Determineu els extrems relatius de $f(x,y) = 8xy + \frac{1}{x} + \frac{1}{y}$

4.9 Determineu els extrems relatius de $f(x,y) = x^2 + xy + y^3 - y^2 - 3x - 2y + 1$

4.10 Descomposeu 120 com a suma de tres nombres de manera que la suma dels productes dos a dos sigui màxima.

4.11 Trobeu la distància mínima entre la paràbola $y = x^2$ i la recta $x - y - 2 = 0$

Llista de problemes

4.12 Busqueu els extrems absoluts de $f(x,y) = 2xy - x - y + 1$ en $R = [0,1] \times [0,1]$.

4.13 " extrems absoluts de $f(x,y) = x^2 + y^2 - xy + x - y$ en $A = \{x \leq 0, y \geq 0, -x + y \leq 3\}$

4.14 " extrems de $f(x,y) = x - y$ amb la condició $x^2 - y^2 = 2$

4.15 " " de $f(x,y,z) = x + y + z$ amb les condicions $x^2 + y^2 = 1$, $3x + z = 1$

4.16 Determineu les dimensions d'una llauna cilíndrica que contingui un litre amb un mínim de metall

4.17 Troben els punts de la superfície $z^2 - xy = 1$ més propers a l'origen

4.18 Troben el paral·lelepípede rectangular més gran que es pot inscriure en $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

4.6 $f(x,y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$

$$\frac{\partial f}{\partial x} = 2x - 3y + 5 = 0$$

$$\frac{\partial f}{\partial y} = -3x - 2 + 12y = 0$$

$$\begin{cases} 2x - 3y = -5 \\ -3x + 12y = 2 \end{cases}$$

$$\begin{pmatrix} 2 & -3 \\ -3 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} -5 & -3 \\ 2 & 12 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ -3 & 12 \end{vmatrix}} = \frac{-54}{15} = \frac{-18}{5} \; ; \qquad y = \frac{\begin{vmatrix} 2 & -5 \\ -3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ -3 & 12 \end{vmatrix}} = \frac{-11}{15}$$

$$Hf(x,y) = \begin{pmatrix} 2 & -3 \\ -3 & 12 \end{pmatrix} \qquad \begin{cases} a_{11} = 2 > 0 \\ \det = 24 - 9 > 0 \end{cases} \longrightarrow \left(\frac{-18}{5}, \frac{-11}{15}\right)$$

és mín. relatiu estricte

4.7

$$f(x,y) = x^4 + y^4 - 2(x-y)^2$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4(x-y) = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4(x-y) = 0$$

sumant $4x^3 + 4y^3 = 0 \rightarrow x^3 = -y^3 \rightarrow x = -y$

substituint a la 1ª eq.:

$$4x^3 - 4(x+x) = 0$$

$$x^3 - 2x = 0$$

$$x(x^2-2) = 0 \quad \begin{cases} x = 0 \\ x = \pm\sqrt{2} \end{cases}$$

Punts crítics $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

$$Hf(x,y) = \begin{pmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{pmatrix}$$

$$Hf(\sqrt{2}, -\sqrt{2}) = Hf(-\sqrt{2}, \sqrt{2}) = \begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix}$$

$\left.\begin{array}{l} a_{11} > 0 \\ \det = 20^2 - 4^2 > 0 \end{array}\right\}$ $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ són mín. relatius estrictes

4.7 (2)

$$Hf(0,0) = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}$$

$\det = 0 \Rightarrow$ hi ha un vap $= 0$

(No podem aplicar la cond. suficient)

Estudiem directament la funció

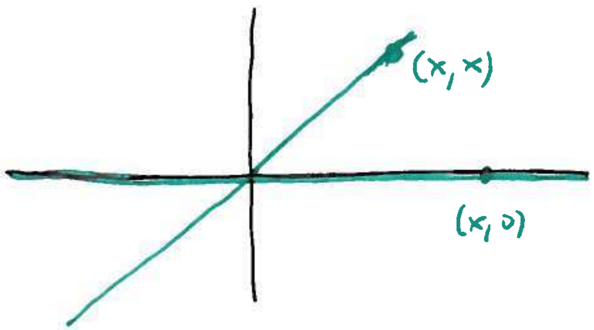
$f(x,0) = x^4 - 2x^2 \equiv \varphi(x)$

$\varphi'(x) = 4x^3 - 4x \longrightarrow \varphi'(0) = 0$

$\varphi''(x) = 12x^2 - 4 \longrightarrow \varphi''(0) = -4$

$\Rightarrow$ f restringida a la recta $(x,0)$ té un màx. en $(0,0)$

$f(x,x) = 2x^4 \Rightarrow$ f restringida a la recta (x,x) té un mín en $(0,0)$



$(0,0)$ és un punt sella

4.8 $\quad f(x,y) = 8xy + \frac{1}{x} + \frac{1}{y}$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 8y - \dfrac{1}{x^2} = 0 & \longrightarrow \quad y = \dfrac{1}{8x^2} \\[2ex] \dfrac{\partial f}{\partial y} = 8x - \dfrac{1}{y^2} = 0 & \longrightarrow \quad 8x = \dfrac{1}{y^2} \longrightarrow 8x = 64x^4 \longrightarrow 1 = 8x^3 \longrightarrow x = \dfrac{1}{2} \end{cases}$$

$$y = \frac{1}{8\frac{1}{4}} = \frac{1}{2}$$

$$Hf(x,y) = \begin{pmatrix} \frac{2}{x^3} & 8 \\ 8 & \frac{2}{y^3} \end{pmatrix} \qquad Hf\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{pmatrix} 16 & 8 \\ 8 & 16 \end{pmatrix}$$

$$\left.\begin{array}{l} a_{11} = 16 > 0 \\ \det = 16^2 - 8^2 > 0 \end{array}\right\} \longrightarrow \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \text{ és mín. rel. estricte}$$

4.9 $f(x,y) = x^2 + xy + y^3 - y^2 - 3x - 2y + 1$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + y - 3 = 0 \\[2ex] \dfrac{\partial f}{\partial y} = x + 3y^2 - 2y - 2 = 0 \rightarrow x = 2 + 2y - 3y^2 \end{cases}$$

$4 + 4y - 6y^2 + y - 3 = 0 \rightarrow 6y^2 - 5y - 1 = 0$

$$y = \frac{5 \pm \sqrt{25+24}}{12} = \frac{5 \pm 7}{12} = \begin{cases} 1 \longrightarrow x = 1 \\ -\frac{1}{6} \longrightarrow x = \frac{19}{12} \end{cases}$$

$$Hf(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 6y-2 \end{pmatrix}$$

$$Hf(1,1) = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \qquad \left.\begin{matrix} a_{11} = 2 > 0 \\ \det = 8 - 1 > 0 \end{matrix}\right\} \Rightarrow (1,1) \text{ mín. rel. estricte}$$

$$Hf\left(\frac{19}{12}, \frac{-1}{6}\right) = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \qquad \left.\begin{matrix} a_{11} = 2 > 0 \\ \det = -6 - 1 < 0 \end{matrix}\right\} \Rightarrow \left(\frac{19}{12}, \frac{-1}{6}\right) \text{ punt sella}$$

4.10 $\quad x+y+z=120 \quad , \quad xy+xz+yz \quad$ màx

$$f(x,y) = xy + x(120-x-y) + y(120-x-y)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \cancel{y} + 120 - 2x - \cancel{y} - y = 0 \\ \dfrac{\partial f}{\partial y} = \cancel{x} - \cancel{x} + 120 - x - 2y = 0 \end{cases}$$

$$\begin{cases} 2x + y = 120 \\ x + 2y = 120 \end{cases}$$

$$-2x - 4y = -240$$

$$-3y = -120 \rightarrow \boxed{y = 40}$$

$$x = 120 - 2y = 120 - 80 = 40$$

$$Hf(x,y) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

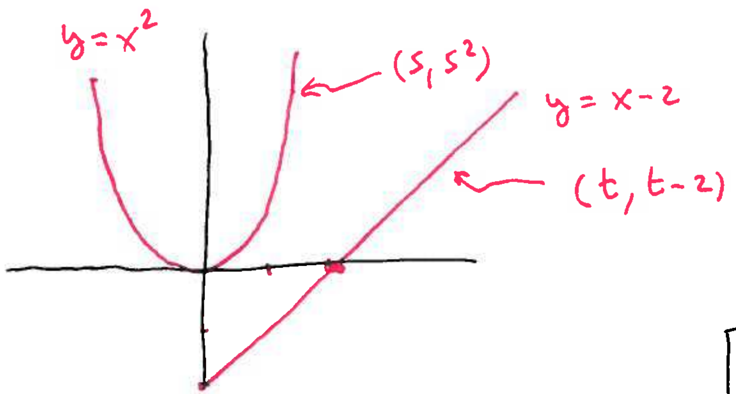
$$\left. \begin{array}{l} a_{11} = -2 < 0 \\ \det = 4 - 1 > 0 \end{array} \right\} \rightarrow (40,40) \text{ és màx. rel. estricte}$$

Els nombres han de ser $x = 40$

$y = 40$

$z = 40$

4.11 $\qquad y = x^2 \quad , \qquad x - y - 2 = 0$



$$\text{distància}\left((s, s^2), (t, t-2) \right) =$$

$$= \sqrt{(s-t)^2 + (s^2 - t + 2)^2}$$

$$\boxed{f(s,t) = (s-t)^2 + (s^2 - t + 2)^2}$$

$$(1) \begin{cases} \dfrac{\partial f}{\partial s} = 2(s-t) + 2(s^2 - t + 2)\,2s = 0 \\[2ex] \dfrac{\partial f}{\partial t} = -2(s-t) - 2(s^2 - t + 2) = 0 \end{cases}$$

$$\begin{array}{r} 2(s-t) + 2(s^2 - t + 2)\,2s = 0 \\ -2(s-t)\,2s - 2(s^2 - t + 2)\,2s = 0 \\ \hline 2(s-t)(1-2s) = 0 \end{array} \quad \begin{array}{l} \rightarrow s = \frac{1}{2} \\ \searrow s = t \end{array}$$

Si $s = \frac{1}{2}$, de (1) $\quad \cancel{2}\left(\frac{1}{2} - t\right) + \cancel{2}\left(\frac{1}{4} - t + 2\right) = 0 \rightarrow \frac{11}{4} - 2t = 0 \rightarrow t = \frac{11}{8}$

Si $t = s$, de (2) $\quad -2 \cdot 0 - 2(s^2 - s + 2) = 0 \rightarrow s = \dfrac{1 \pm \sqrt{1-8}}{2} \notin \mathbb{R}$

$$Hf(s,t) = \begin{pmatrix} 2 + 12s^2 - 4t + 8 & -2 - 4s \\ -2 - 4s & 2 + 2 \end{pmatrix} \rightarrow Hf\left(\frac{1}{2}, \frac{11}{8}\right) = \begin{pmatrix} 15/2 & -4 \\ -4 & 4 \end{pmatrix}$$

$a_{11} > 0$

$\det = 14 > 0 \quad \rightarrow$ mínim

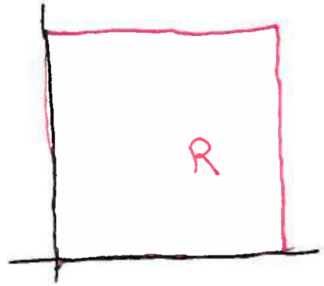
4.12 Extrems absoluts de $f(x,y) = 2xy - x - y + 1$ en $R = [0,1] \times [0,1]$

(a) extrems en $(0,1) \times (0,1)$

$$\frac{\partial f}{\partial x} = 2y - 1 = 0 \longrightarrow y = \frac{1}{2}$$

$$\frac{\partial f}{\partial y} = 2x - 1 = 0 \longrightarrow x = \frac{1}{2}$$

$$f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \tfrac{1}{2}$$



(b) extrems quan $x = 0$. $g(y) = f(0,y) = 1 - y$

$x = 1$ $\quad g(y) = f(1,y) = y$

$y = 0$ $\quad g(x) = f(x,0) = -x + 1$

$y = 1$ $\quad g(x) = f(x,1) = x$

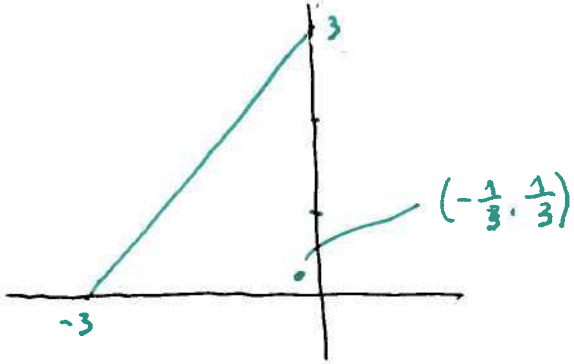
(c) $f(0,0) = 1$, $\quad f(1,0) = 0$

$f(0,1) = 0$, $\quad f(1,1) = 1$

Màxim en $(0,0)$, $(1,1)$

Mínim en $(0,1)$, $(1,0)$

4.13 Extrems absoluts de $f(x,y) = x^2+y^2-xy+x-y$ en $A = \{(x,y)\in\mathbb{R}^2 \mid x\leq 0,\ y\geq 0, -x+y\leq 3\}$



$$\boxed{\begin{aligned} f(0,0) &= 0 \\ f(-3,0) &= 6 \\ f(0,3) &= 6 \end{aligned}}$$

(a) extrems a l'interior de $A = A \setminus \partial A$

$$\frac{\partial f}{\partial x} = 2x - y + 1 = 0 \qquad\qquad \frac{\partial f}{\partial y} = 2y - x - 1 = 0$$

$$\begin{array}{r} 2x - y + 1 = 0 \\ 4y - 2x - 2 = 0 \\ \hline 3y - 1 = 0 \end{array} \longrightarrow \quad y = \frac{1}{3}, \quad x = 2y - 1 = -\frac{1}{3}$$

$$\boxed{f\left(-\frac{1}{3}, \frac{1}{3}\right) = -\frac{1}{3}}$$

(b) extrems quan

$x = 0$. $\quad g(y) = f(0,y) = y^2 - y;\quad g'(y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$

$$\boxed{f\left(0, \frac{1}{2}\right) = -\frac{1}{4}}$$

$y = 0$, $\quad g(x) = f(x,0) = x^2 + x;\quad g'(x) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

$$\boxed{f\left(-\frac{1}{2}, 0\right) = -\frac{1}{4}}$$

$y = x+3$, $\quad g(x) = f(x, x+3) = x^2 + (x+3)^2 - x^2 - 3x + x - x - 3$

$= x^2 + 3x + 6;\quad g'(x) = 2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

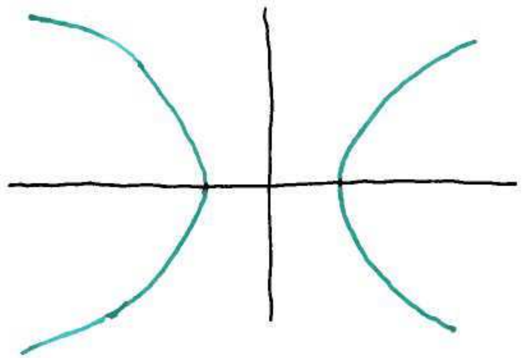
$y = \frac{3}{2}$

$$\boxed{f\left(-\frac{3}{2}, \frac{3}{2}\right) = \frac{15}{4}}$$

Màxim en $(-3,0)$, $(0,3)$

Mínim en $\left(-\frac{1}{3}, \frac{1}{3}\right)$

4.14 Extrems de $f(x,y) = x - y$ amb la condició $x^2 - y^2 = 2$



$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$$

$1 = \lambda 2x \quad \rightarrow \quad x = \dfrac{1}{2\lambda}$

$-1 = -\lambda 2y \quad \rightarrow \quad y = \dfrac{1}{2\lambda}$

$x^2 - y^2 = 2 \quad \rightarrow \quad \left(\dfrac{1}{2\lambda}\right)^2 - \left(\dfrac{1}{2\lambda}\right)^2 = 2$

no té solució

$\Rightarrow$ No existeixen extrems relatius

Un altre mètode:

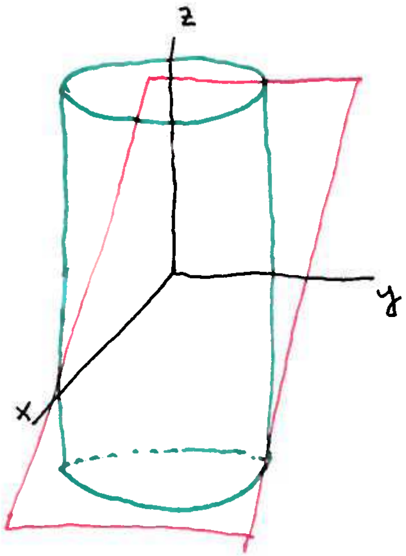
$x^2 = 2 + y^2 \quad \rightarrow \quad x = \sqrt{2+y^2}$ (branca de la dreta)

$g(y) = f(\sqrt{2+y^2}, y) = \sqrt{2+y^2} - y \quad ; \quad g'(y) = \dfrac{2y}{2\sqrt{2+y^2}} - 1 = 0$

$\Rightarrow \dfrac{y}{\sqrt{2+y^2}} = 1 \quad \Rightarrow \quad \dfrac{y^2}{2+y^2} = 1$ (no té solució)

4.15 Extrems de $f(x,y,z) = x+y+z$ amb les condicions $x^2+y^2=1$

$3x+z=1$



$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$1 = \lambda_1 2x + \lambda_2 \cdot 3$ $\qquad -2 = \lambda_1 2x \rightarrow x = \frac{-1}{\lambda_1}$

$1 = \lambda_1 2y + \lambda_2 \cdot 0$ $\qquad \rightarrow y = \frac{1}{2\lambda_1}$

$1 = \lambda_1 \cdot 0 + \lambda_2 \cdot 1 \rightarrow \lambda_2 = 1$

$x^2 + y^2 = 1$

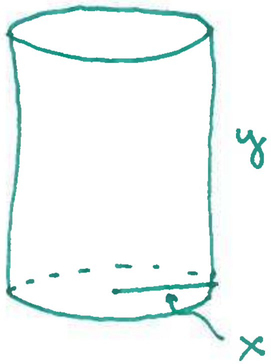
$3x + z = 1$

$$\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{2\lambda_1}\right)^2 = 1 \rightarrow \frac{1}{\lambda_1^2}\left(1+\frac{1}{4}\right) = 1 \rightarrow \lambda_1^2 = \frac{5}{4} \rightarrow \lambda_1 = \frac{\pm\sqrt{5}}{2}$$

Candidats a extrems $P_1 = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1+\frac{6}{\sqrt{5}}\right)$ $\qquad f(P_1) = 1 + \frac{5}{\sqrt{5}}$

$P_2 = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 1-\frac{6}{\sqrt{5}}\right)$ $\qquad f(P_2) = 1 - \frac{5}{\sqrt{5}}$

4.16 Determinar les dimensions d'una llauna cilíndrica que contingui un litre amb un mínim de metall.



$$\text{Àrea} = 2\pi x^2 + 2\pi x y = f(x,y)$$

$$\text{Volum} = \pi x^2 y = g(x,y)$$

$$\nabla f = \lambda \nabla g$$

$$4\pi x + 2\pi y = \lambda\, 2\pi x y \longrightarrow 2x + y = \lambda x y \qquad 2x+y = 2y \rightarrow y = 2x$$

$$2\pi x = \lambda\, \pi x^2 \longrightarrow 2 = \lambda x$$

$$\pi x^2 y = 1 \qquad 2\pi x^3 = 1 \rightarrow x = \frac{1}{\sqrt[3]{2\pi}}$$

$$y = \frac{2}{\sqrt[3]{2\pi}}$$

4.17 Trobeu els punts de la superfície $z^2 - xy = 1$ més propers a l'origen

$$f(x,y) = x^2 + y^2 + z^2$$

$$g(x,y) = z^2 - xy \qquad \nabla f = \lambda \nabla g$$

$$\left.\begin{array}{l} 2x = \lambda(-y) \\ 2y = \lambda(-x) \end{array}\right\} \rightarrow \begin{array}{l} 2x^2 = -\lambda y x \\ 2y^2 = -\lambda x y \end{array} \rightarrow 2x^2 - 2y^2 = 0 \rightarrow x^2 = y^2 \Rightarrow \begin{cases} x = y \\ x = -y \end{cases}$$

(3) $2z = \lambda\, 2z$

$z^2 - xy = 1$

$x = y \Rightarrow 2x = -\lambda x \rightarrow (2+\lambda)x = 0 \rightarrow \left.\begin{array}{l} x = 0 \\ \lambda = -2 \end{array}\right\} \Rightarrow$

$\Rightarrow \begin{cases} y = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1 \underset{(3)}{\Rightarrow} \lambda = 1 \\ \text{Si } \lambda = -2,\ (3) \Rightarrow 2z = -4z \Rightarrow z = 0 \end{cases}$

Si $z = 0$, $xy = -1 \Rightarrow x^2 = -1$ (no hi ha solució)

$x = -y \Rightarrow 2x = \lambda x \rightarrow (2-\lambda)x = 0 \rightarrow \left.\begin{array}{l} x = 0 \\ \lambda = 2 \end{array}\right\} \Rightarrow$

$\Rightarrow \begin{cases} x = 0 \Rightarrow y = 0,\ z = \pm 1 \\ \text{Si } \lambda = 2,\ (3) \Rightarrow 2z = 4z \Rightarrow z = 0 \Rightarrow xy = -1 \end{cases}$

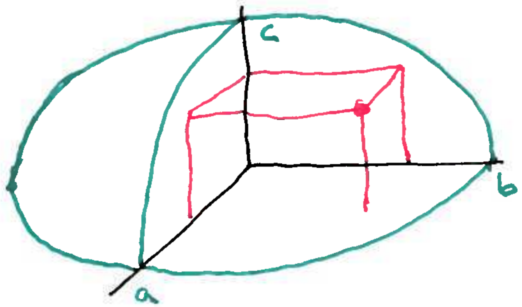
$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1,\ y = \mp 1$

Hi ha 4 candidats $(0,0,\pm 1)$, $(1,-1,0)$, $(-1,1,0)$

Els que tenen distància mínima a l'origen són $(0,0,\pm 1)$

4.18 Paral·lelepípede rectangular més gran que es pot inscriure en

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



$\text{Volum} = 2x\,2y\,2z = f(x,y,z)$; $\tilde{f}(x,y,z) = xyz$

$g(x,y,z) = \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}$ $\qquad x>0, y>0, z>0$

$$\nabla \tilde{f} = \lambda \nabla g$$

$$yz = \lambda \frac{2x}{a^2} \qquad xyz = \lambda\frac{2x^2}{a^2}$$

$$xz = \lambda\frac{2y}{b^2} \qquad xyz = \lambda\frac{2y^2}{b^2}$$

$$xy = \lambda\frac{2z}{c^2} \qquad xyz = \lambda\frac{2z^2}{c^2}$$

$$\lambda\frac{2x^2}{a^2} = \lambda\frac{2y^2}{b^2} \rightarrow \frac{x}{a}=\frac{y}{b}$$

$$\lambda\frac{2y^2}{b^2} = \lambda\frac{2z^2}{c^2} \rightarrow \frac{y}{b}=\frac{z}{c}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow 3\frac{x^2}{a^2}=1 \longrightarrow x=\frac{a}{\sqrt3} \rightarrow y=\frac{b}{\sqrt3} \rightarrow z=\frac{c}{\sqrt3}$$

(únic candidat)

$$\text{Volum}_{\text{màx}} = 8\,\frac{abc}{3\sqrt3}$$