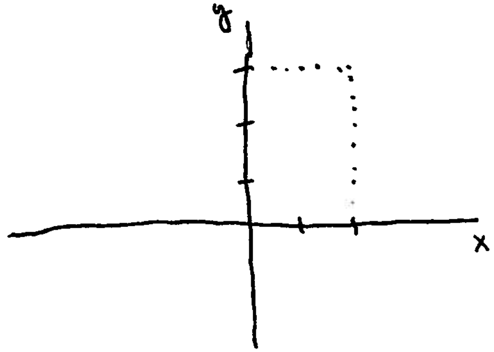
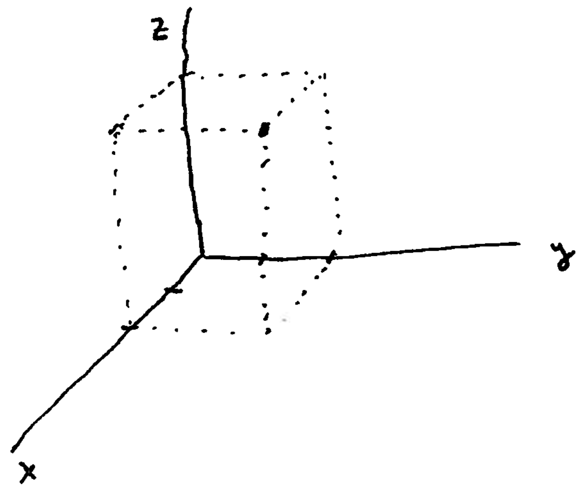
# Els espais $\mathbb{R}^2$ i $\mathbb{R}^3$

$\mathbb{R}^2 = \{(x,y)\ ;\ x \in \mathbb{R}\ ,\ y \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(x,y,z)\ ;\ x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}$




**suma**

$(x,y) + (x',y') = (x+x', y+y')$

$(x,y,z) + (x',y',z') = (x+x', y+y', z+z')$

**producte per escalar**

$\alpha(x,y) = (\alpha x, \alpha y)$

$\alpha(x,y,z) = (\alpha x, \alpha y, \alpha z)$

**element oposat de $(x,y)$**

$(-x,-y)$

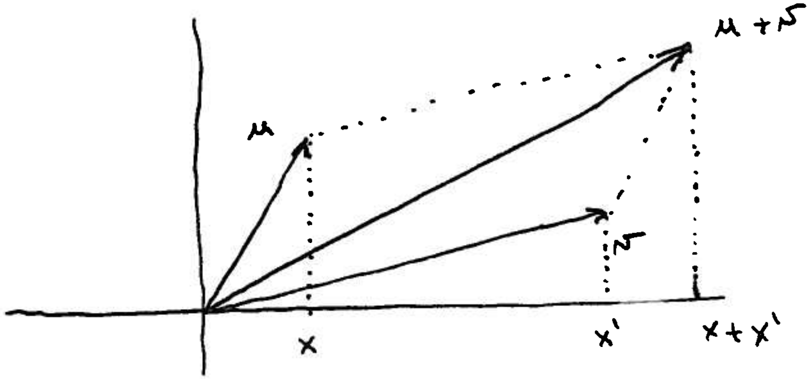
$(-x,-y,-z)$

**resta**

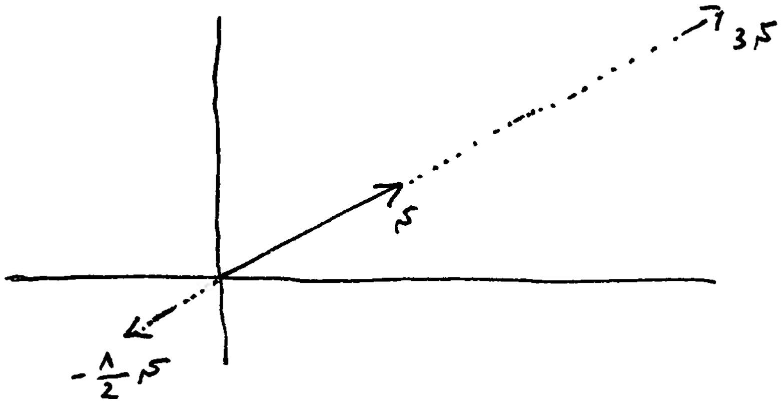
$(x,y) - (x',y') = (x,y) + (-x',-y') = (x-x', y-y')$

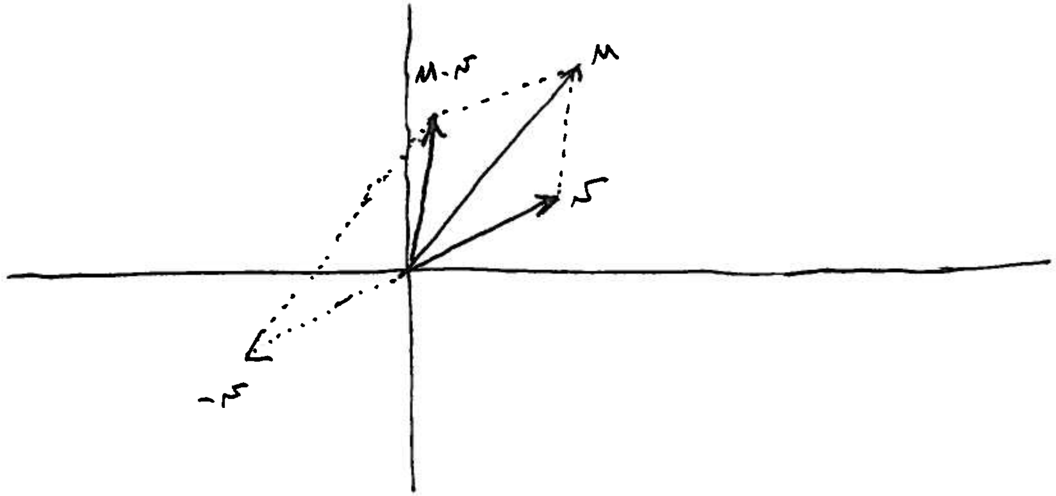
$(x,y,z) - (x',y',z') = (x-x', y-y', z-z')$

# Vectors i escalars




$u = (x, y)$ , $v = (x', y')$



## El vector 0

$0 = (0,0)$ en $\mathbb{R}^2$, $\quad 0 = (0,0,0)$ en $\mathbb{R}^3$

Tenim que $\quad x - x = 0$

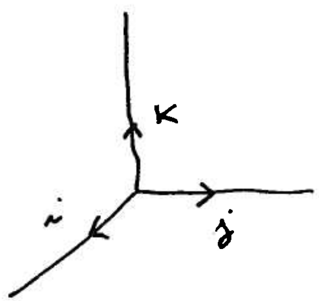
## Base canònica (o estandard)

En $\mathbb{R}^2$ $\quad i = (1,0)\,,\quad j = (0,1)\,.\qquad$ També $\quad e_1 = i\,,\quad e_2 = j$

$$x = (x_1, x_2) = (x_1, 0) + (0, x_2) = x_1(1,0) + x_2(0,1) = x_1 i + x_2 j$$

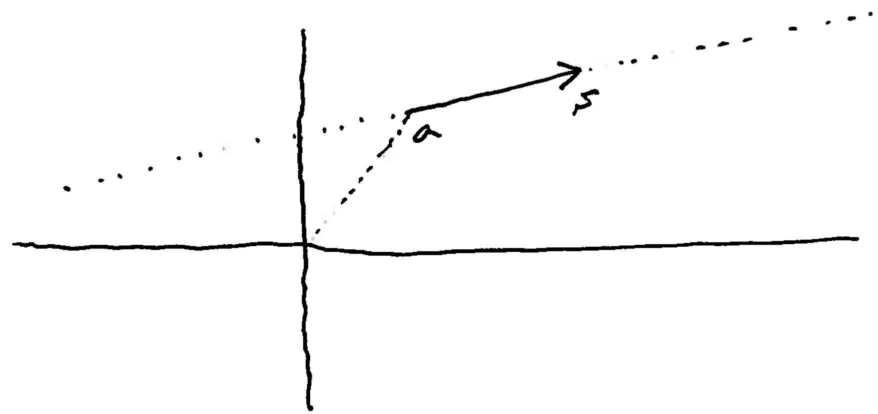
En $\mathbb{R}^3$ $\quad i = (1,0,0)\,,\quad j = (0,1,0)\,,\quad K = (0,0,1)\,.\qquad e_1 = i,\ e_2 = j,\ e_3 = k$

$$x = (x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3) = x_1 i + x_2 j + x_3 k$$



## rectes a $\mathbb{R}^2$

recta que passa per $a$ i té vector director $v$




$$\boxed{\ell(t) = a + tv, \quad t \in \mathbb{R}}$$

Ex $\quad a = (1,1), \quad v = (3,1)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

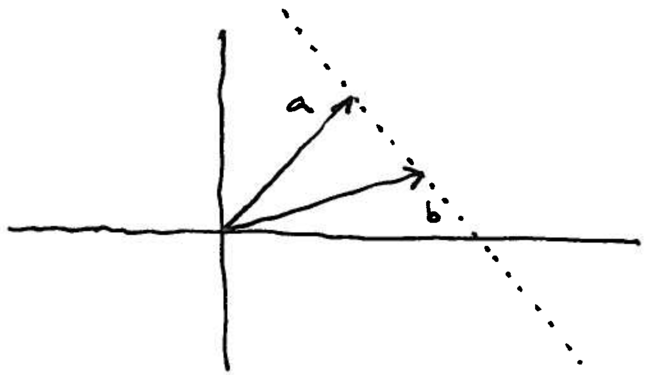
$$\begin{cases} x = 1 + 3t \\ y = 1 + t \end{cases}$$

$$t = \frac{x-1}{3} = y - 1 \longrightarrow \boxed{x - 3y + 2 = 0}$$

$$\downarrow$$

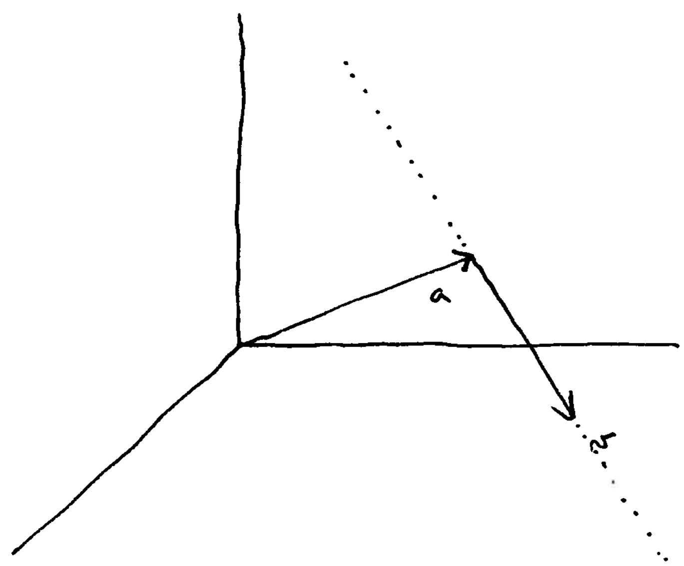
$$y = \frac{1}{3}x + \frac{2}{3}$$

## recta que passa per dos punts




$$\boxed{\ell(t) = a + t(b - a)}$$

## rectes a $\mathbb{R}^3$




$$\boxed{\ell(t) = a + t v}$$

<u>Ex:</u> $a = (1, 2, 1)$, $v = (-1, 2, -3)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$\begin{cases} x = 1 - t \\ y = 2 + 2t \\ z = 1 - 3t \end{cases}$$

$$t = \frac{x-1}{-1} = \frac{y-2}{2} = \frac{z-1}{-3}$$

$$\begin{cases} 2x + y - 4 = 0 \\ -3y - 2z + 8 = 0 \end{cases}$$

## recta que passa per dos punts a, b

$$\boxed{\ell(t) = a + t(b-a)}$$

siguen $a = (x_1, y_1, z_1)$, $b = (x_2, y_2, z_2)$

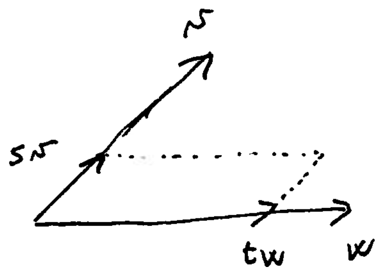
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

$$\begin{cases} x = x_1 + t\,(x_2 - x_1) \\ y = y_1 + t\,(y_2 - y_1) \\ z = z_1 + t\,(z_2 - z_1) \end{cases}$$

$$t = \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

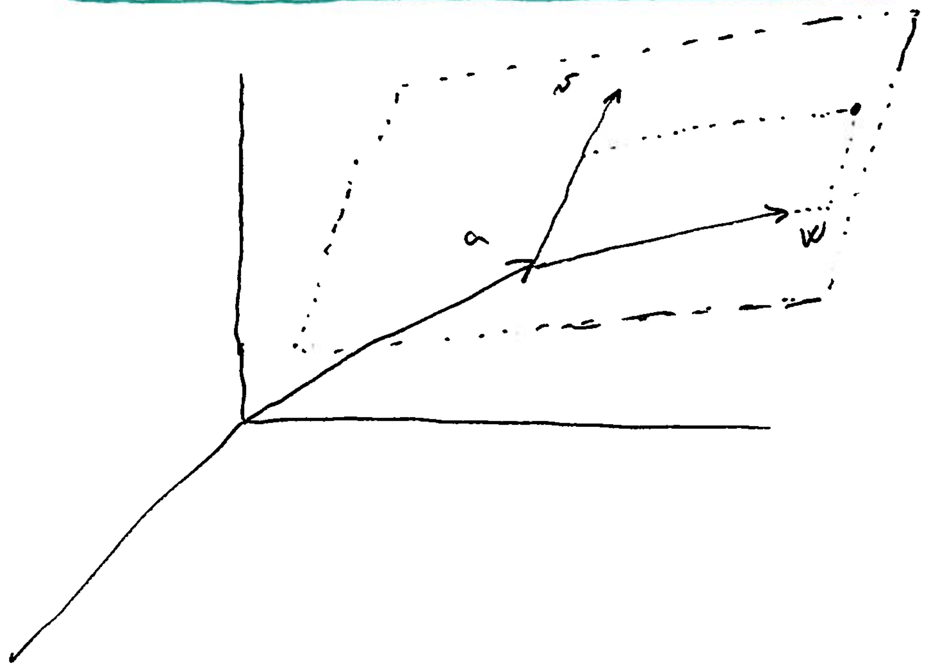
# Plans a $\mathbb{R}^3$

Necessitem dos vectors no paral·lels (linealment independents)




$s \in \mathbb{R}$
$t \in \mathbb{R}$

pla que passa per $a$ i té vectors directors $v, w$ (no paral·lels)




$$\boxed{p(t,s) = a + tv + sw}$$

$$x = a_1 + v_1 t + w_1 s$$

$$y = a_2 + v_2 t + w_2 s$$

$$z = a_3 + v_3 t + w_3 s$$

## Producte escalar

És un producte que donats dos vectors ens dóna un escalar (nombre real)

En $\mathbb{R}^2$ : si $a = (a_1, a_2)$ , $b = (b_1, b_2)$

$$a \cdot b = \langle a, b \rangle = a_1 b_1 + a_2 b_2$$

En $\mathbb{R}^3$ : si $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$

$$a \cdot b = \langle a, b \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

**Propietats** $\forall a, b, c \in \mathbb{R}^n$ , $\forall \alpha \in \mathbb{R}$ $n=2$ o $n=3$

(i) $a \cdot a \geqslant 0$

(ii) $a \cdot a = 0 \iff a = 0$

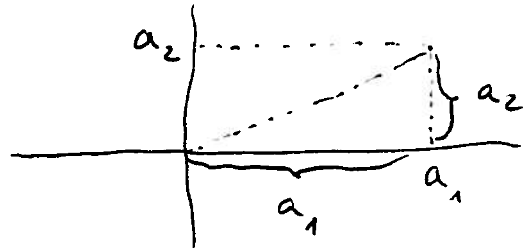
(iii) $\alpha a \cdot b = \alpha (a \cdot b) = a \cdot \alpha b$

(iv) $a \cdot (b + c) = a \cdot b + a \cdot c$ , $(a + b) \cdot c = a \cdot c + b \cdot c$

(v) $a \cdot b = b \cdot a$

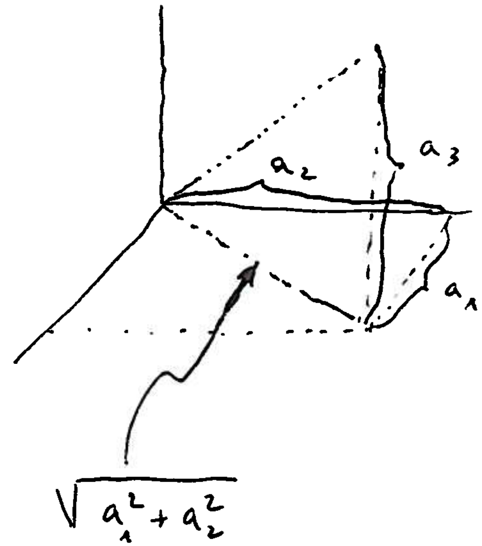
# Norma

Ens dóna la longitud d' un vector




En $\mathbb{R}^2$ : si $a = (a_1, a_2)$

$$\|a\| = \sqrt{a_1^2 + a_2^2} = \sqrt{a \cdot a}$$




En $\mathbb{R}^3$ : si $a = (a_1, a_2, a_3)$

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a \cdot a}$$

## Propietats

$\forall a, b \in \mathbb{R}^n$, $\forall \alpha \in \mathbb{R}$

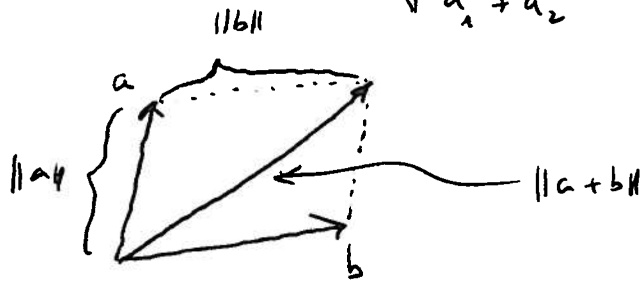
(i) $\|a\| \geq 0$

(ii) $\|a\| = 0 \iff a = 0$

(iii) $\|\alpha a\| = |\alpha| \, \|a\|$

(iv) $\|a+b\| \leq \|a\| + \|b\|$

(desigualtat triangular)

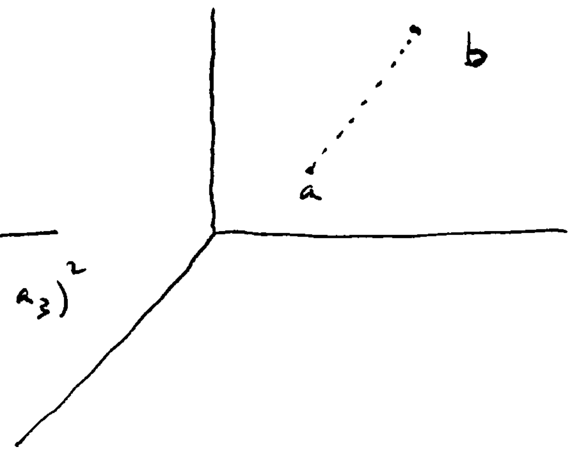


## Distància

$d(a,b) = \| b - a \|$

$= \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2}$

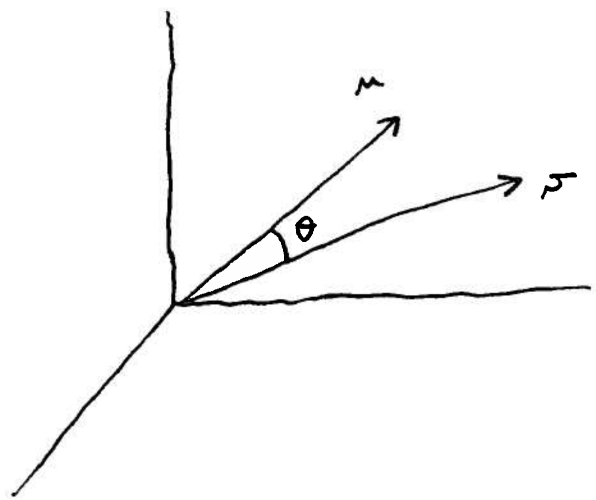
en $\mathbb{R}^3$




## Angle entre dos vectors

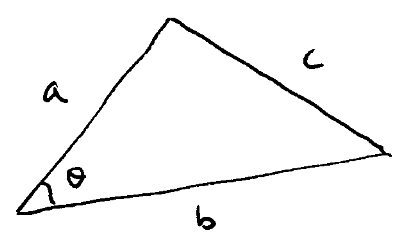
Si $u, v \in \mathbb{R}^n$ $(n = 2, 3)$ $u, v \neq 0$ l'angle $\theta \in [0, \pi]$ que formen verifica

$$u \cdot v = \|u\| \, \|v\| \cos\theta \longrightarrow \theta = \arccos\left(\frac{u \cdot v}{\|u\| \, \|v\|}\right)$$



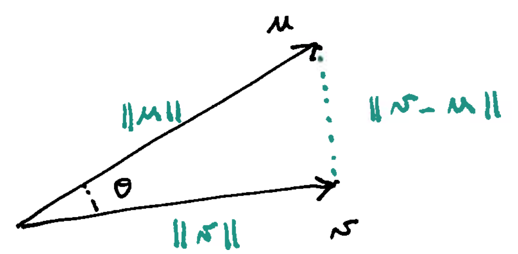
La fórmula $u \cdot v = \|u\| \, \|v\| \cos\theta$ es dedueix amb un càlcul a partir del

teorema del cosinus :




$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

En el cas que ens ocupa




$$\longrightarrow \quad \|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\| \cos\theta$$

D'altre banda

$$\|v-u\|^2 = (v-u)\cdot(v-u) = v\cdot v - v\cdot u - u\cdot v + u\cdot u = \|v\|^2 + \|u\|^2 - 2u\cdot v$$

Notem que la fórmula també és certa si $u$ o $v$ són zero

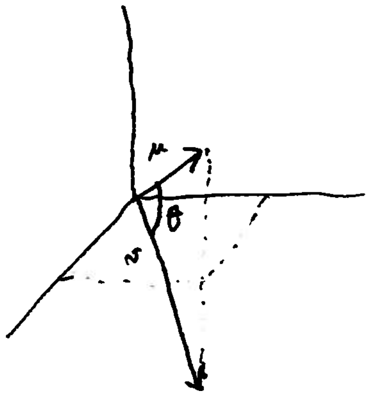
# Desigualtat de Cauchy - Schwarz

$\forall u, v \in \mathbb{R}^n \qquad |u \cdot v| \leq \|u\| \, \|v\|$

És conseqüència de $\quad u \cdot v = \|u\| \, \|v\| \cos \theta$

i de $\quad |\cos \theta| \leq 1$

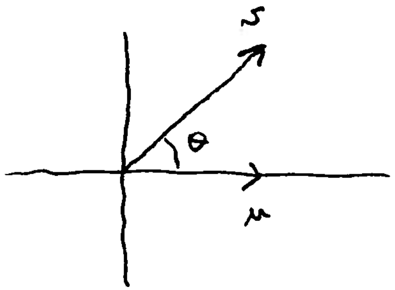
$\boxed{\text{Ex}}$ Càlcul de l'angle entre $u = (1,1,1)$ i $v = (1,1,-1)$




$$u \cdot v = 1\cdot 1 + 1\cdot 1 + 1\cdot(-1) = 1$$
$$\|u\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$
$$\|v\| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$$

$$\cos\theta = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}, \qquad \theta = \arccos\frac{1}{3} = 1.2309\ldots$$

angle entre $u=(1,0)$ i $v=(1,1)$



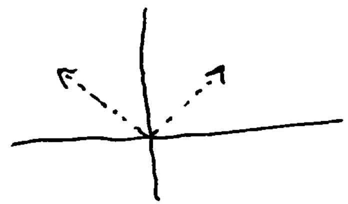

$$u \cdot v = 1\cdot 1 + 0\cdot 1 = 1$$
$$\|u\| = \sqrt{1^2+0^2} = 1$$
$$\|v\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

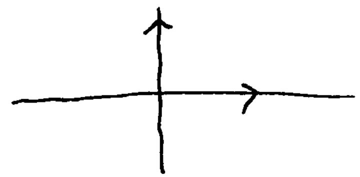
Dos vectors són <u>perpendiculars</u> o <u>ortogonals</u> si l'angle que formen és $\pi/2$

Per tant $u, v$ són perpendiculars $\iff$ $u \cdot v = 0$

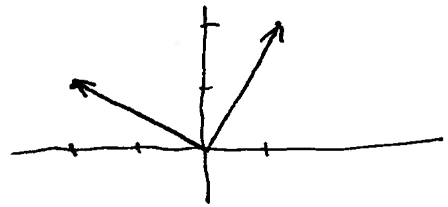
<u>Ex</u> $u = (1,1)$, $v = (-1,1)$



$u\,(1,0)$, $v\,(0,1)$



$u = (1,2)$, $v\,(-2,1)$



$u = (\alpha, \beta)$, $v = (-\beta, \alpha)$

$u = (1,1,1)$, $v = (1,1,-2)$

# Producte vectorial

És un producte a $\mathbb{R}^3$ que associa a dos vectors un altre vector

Si $a = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 K$, $\quad b = (b_1, b_2, b_3) = b_1 i + b_2 j + b_3 K$

$$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} K$$

- formalment

$$\boxed{a \times b = \begin{vmatrix} i & j & K \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

Ex $\quad a = (1, 1, 0) \qquad b = (0, 1, 1)$

$$a \times b = \begin{vmatrix} i & j & K \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} K = (1, -1, 1)$$

**Propietats** $\forall a, b, c \in \mathbb{R}^3, \quad \forall \alpha, \beta \in \mathbb{R}$

(1) $a \times b = -b \times a$

(2) $\begin{cases} (\alpha a + \beta b) \times c = \alpha\, a \times c + \beta\, b \times c \\ a \times (\alpha b + \beta c) = \alpha\, a \times b + \beta\, a \times c \end{cases}$

**Conseqüències**

$a \times a = 0$ ja que $a \times a = -a \times a$

$i \times i = j \times j = k \times k = 0$

$$i \times j = (1,0,0) \times (0,1,0) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} k = k$$

Anàlogament

$j \times k = i, \quad k \times i = j$ (regla mnemotècnica $i \to j \to k$)

Ara buscarem una interpretació geomètrica de $a \times b$

**Triple producte**

Donats $a, b, c \in \mathbb{R}^3$ definim el triple producte de $a, b, c$ com $\boxed{a \cdot (b \times c)}$

Si $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$

$$a \cdot (b \times c) = (a_1, a_2, a_3) \cdot \left( \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right)$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

**Perpendicularitat:** $b \cdot (b \times c) = 0$, $c \cdot (b \times c) = 0$

$\forall \beta, \gamma \in \mathbb{R}$ $(\beta b + \gamma c) \cdot (b \times c) = \beta b \cdot (b \times c) + \gamma c \cdot (b \times c) = 0$

$\Rightarrow$ $b \times c$ és perpendicular al pla generat pels vectors $b, c$.

Notem que $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) = -b \cdot (a \times c) = \ldots$

## Càlcul de la longitud de $b \times c$

$$\|b \times c\|^2 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}^2$$

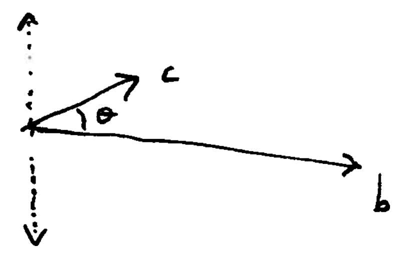
$$= (b_2 c_3 - b_3 c_2)^2 + (b_1 c_3 - b_3 c_1)^2 + (b_1 c_2 - b_2 c_1)^2$$

$$= (b_1^2 + b_2^2 + b_3^2)(c_1^2 + c_2^2 + c_3^2) - (b_1 c_1 + b_2 c_2 + b_3 c_3)^2$$

$$= \|b\|^2 \|c\|^2 - (b \cdot c)^2 = \|b\|^2 \|c\|^2 - \|b\|^2 \|c\|^2 \cos^2\theta = \|b\|^2 \|c\|^2 (1 - \cos^2\theta)$$
$$= \|b\|^2 \|c\|^2 \sin^2\theta .$$

$$\Longrightarrow \quad \|b \times c\| = \|b\| \, \|c\| \, |\sin\theta|$$

La perpendicularitat i la longitud determinen dos possibles vectors per a $b \times c$

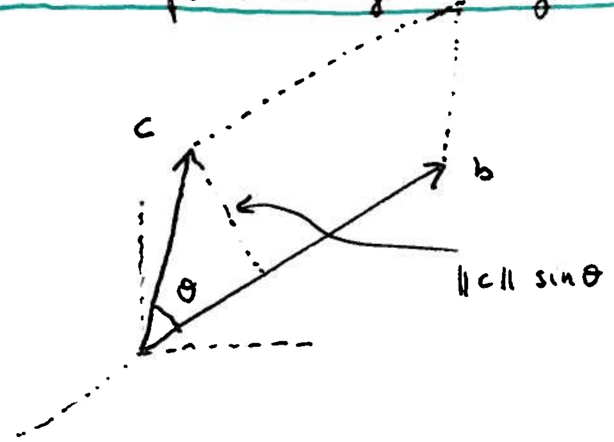



<u>regla de la ma dreta</u>

<u>regla del tornavís</u>

# Aplicacions geomètriques

- àrea del paral·lelogram generat pels vectors $b, c$




$$\text{àrea} = \text{base} \cdot \text{altura}$$

$$= \|b\| \, \|c\| \underbrace{\sin\theta}_{\geq 0} = \|b \times c\|$$

- determinació d'un vector perpendicular a dos donats

el producte vectorial

ex: $b = (1, 2, 0)$, $c = (1, -1, 1)$

$$b \times c = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 2i - j - 3k = (2, -1, -3)$$

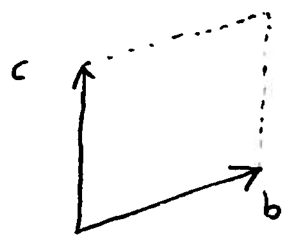
comprovació

$$b \cdot (b \times c) = (1, 2, 0) \cdot (2, -1, -3) = 0$$

$$c \cdot (b \times c) = (1, -1, 1) \cdot (2, -1, -3) = 0$$

- interpretació del determinant $2 \times 2$

Donats $b = (b_1, b_2)$, $c = (c_1, c_2) \in \mathbb{R}^2$, podem imaginar-los a $\mathbb{R}^3$ com




$b' = (b_1, b_2, 0)$, $c' = (c_1, c_2, 0)$

L'àrea del paral·lelogram que generen és la longitud (norma) del vector $b' \times c'$

$$b' \times c' = \begin{vmatrix} i & j & k \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} = \begin{vmatrix} b_2 & 0 \\ c_2 & 0 \end{vmatrix} i - \begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix} j + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} k$$

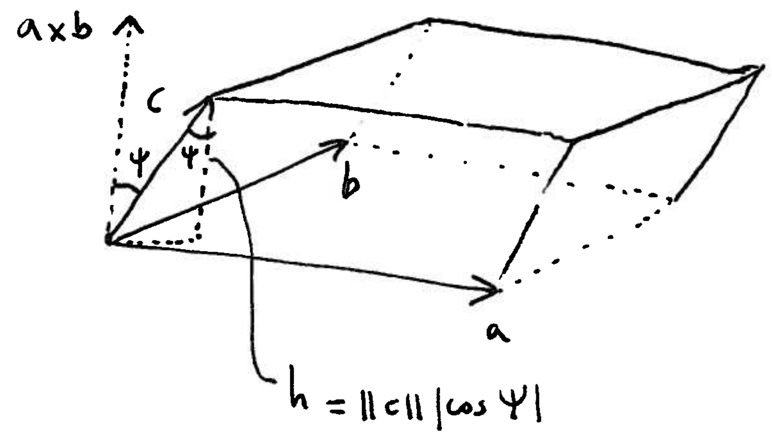
$$= \left( 0, 0, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right)$$

Per tant, el valor absolut de $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ és l'àrea del paral·lelogram generat pels vectors $b, c$.

- <u>interpretació del determinant 3x3</u>

Donats $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$ linealment independents

considerem el paral·lelepíped

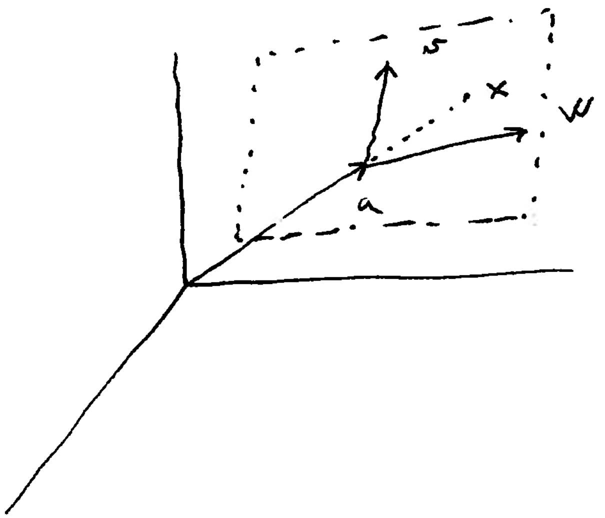



volum = àrea base · altura

àrea base = $\|a \times b\|$

$$\text{volum} = \|a \times b\| \, \|c\| \, |\cos \psi| = |c \cdot (a \times b)| = |a \cdot (b \times c)| = \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|$$

- Pla que passa per $a$ i té vectors directors $v$, $w$




$n = v \times w$ és perpendicular al pla

$x - a$ ha de ser perpendicular a $n$

$$\Longrightarrow \quad n \cdot (x - a) = 0$$

que pren la forma

$$\boxed{n_1 (x_1 - a_1) + n_2 (x_2 - a_2) + n_3 (x_3 - a_3) = 0}$$

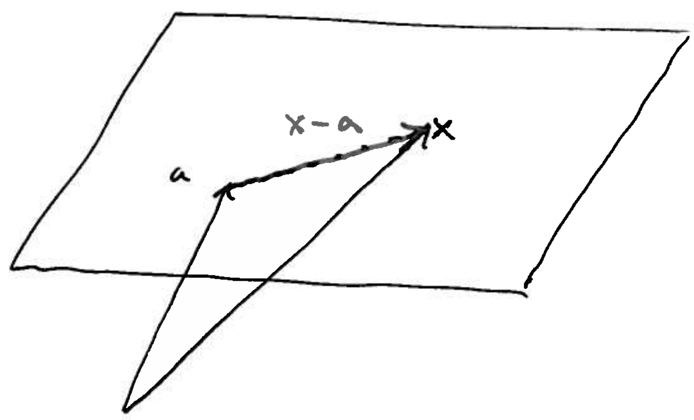
o

$$n_1 x_1 + n_2 x_2 + n_3 x_3 - (n_1 a_1 + n_2 a_2 + n_3 a_3) = 0$$

Usant la notació $(x, y, z)$ l'eq. del pla és de la forma

$$\boxed{Ax + By + Cz + D = 0}$$

una altra possibilitat




Si $x$ pertany al pla

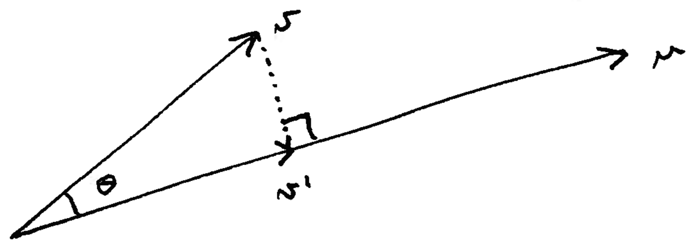
$x - a$ ha de ser combinació lineal de $u$ i $v$,

és a dir, $x-a, u, v$ han de ser l.d.

és a dir

$$\begin{vmatrix} x_1 - a_1 & x_2 - a_2 & x_3 - a_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

## Projecció d'un vector $v$ sobre $u$




Primer notem que (si $u \neq 0$) $\frac{u}{\|u\|}$ té norma 1: $\left\| \frac{u}{\|u\|} \right\| = \frac{1}{\|u\|} \|u\| = 1$

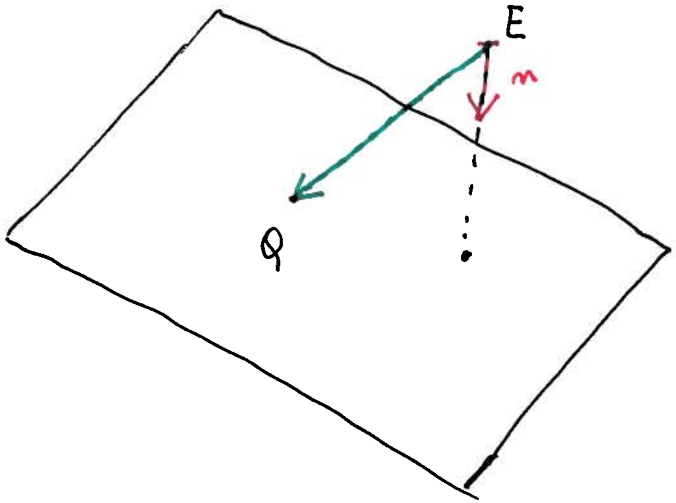
És verifica $u \cdot v = \|u\| \, \|v\| \cos\theta \longrightarrow \cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$

La projecció $v' = \|v\| \cos\theta \, \frac{u}{\|u\|}$

$$v' = \|v\| \cos\theta \, \frac{u}{\|u\|} = \|v\| \frac{u \cdot v}{\|u\| \, \|v\|} \frac{u}{\|u\|} = \left( v \cdot \frac{u}{\|u\|} \right) \frac{u}{\|u\|}$$

- Distància d'un punt a un pla

punt $E = (x_1, y_1, z_1)$ , pla $Ax + By + Cz + D = 0$




Sigui $Q = (x_0, y_0, z_0)$ un punt qualsevol del pla

$m$ vector normal al pla

distància = norma de la projecció de $Q - E$ sobre $m$

Si $\|m\| = 1$, distància $= \|((Q-E)\cdot m)\, m\| = |(Q-E)\cdot m|$

$Q = (x_0, y_0, z_0)$ verifica $Ax_0 + By_0 + Cz_0 + D = 0 \longrightarrow -Ax_0 - By_0 - Cz_0 = D$

$$m = \frac{1}{\sqrt{A^2+B^2+C^2}}(A, B, C)$$

$$\text{distància} = \left| (x_1 - x_0, y_1 - y_0, z_1 - z_0)\frac{1}{\sqrt{A^2+B^2+C^2}}(A,B,C) \right| = \left| \frac{A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)}{\sqrt{A^2+B^2+C^2}} \right|$$

$$\boxed{\text{distància} = \left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2+B^2+C^2}} \right|}$$