Factorial Methods

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Factorial Methods

- Find the isomorph transformation from original space

  keeps the adjacency relationships among variables

- Results expressed in a ficticious space

- Might produce interpretation problems

- Methods
  - PCA (Principal components analysis)
  - Simple correspondence analysis
  - Multiple correspondence analysis

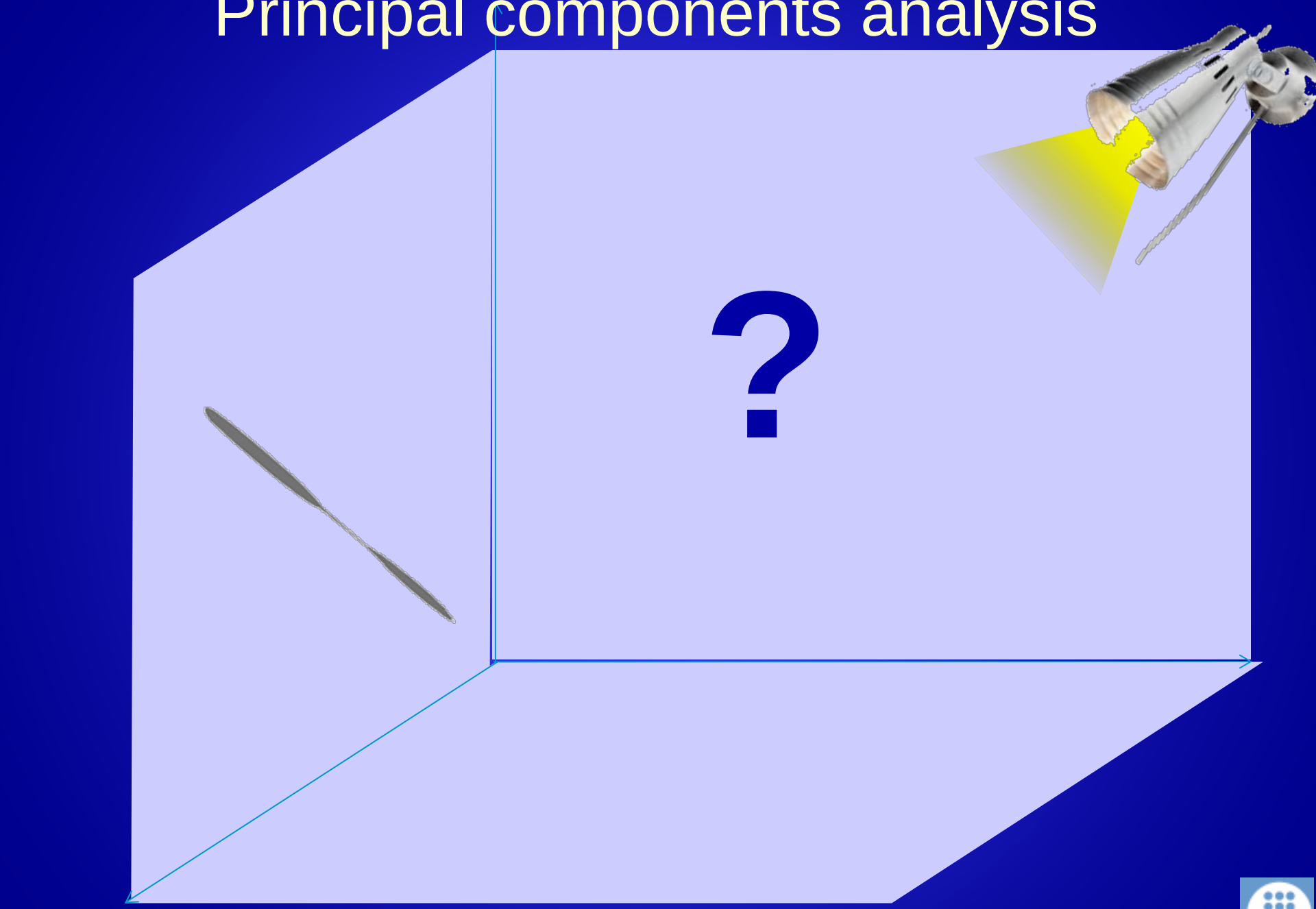
Factorial Methods

- Principal Components Analysis
  - Only numerical variables
  - Find the most informative projection planes

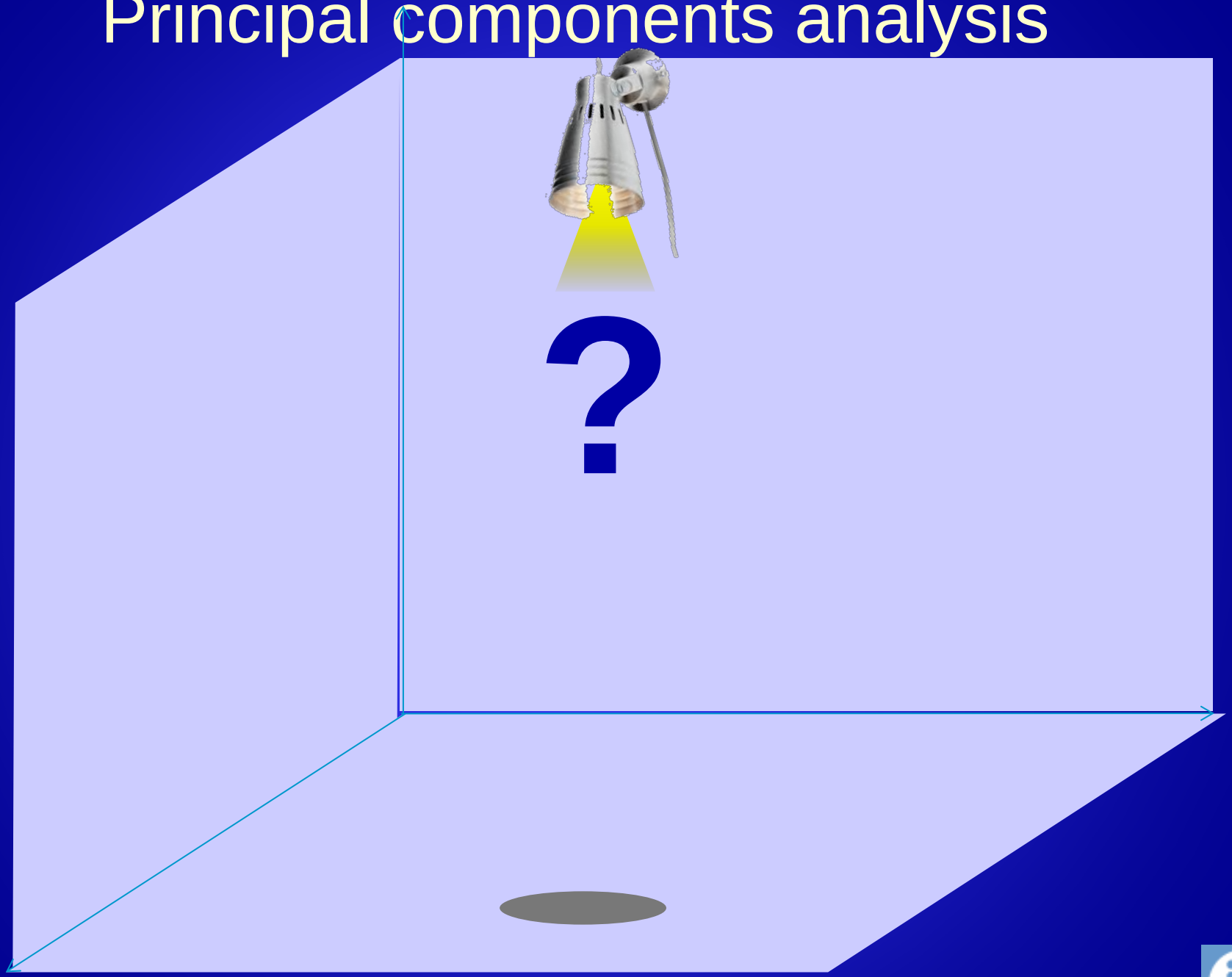
    (factorial planes)

Example “Copas”

Principal components analysis



Principal components analysis



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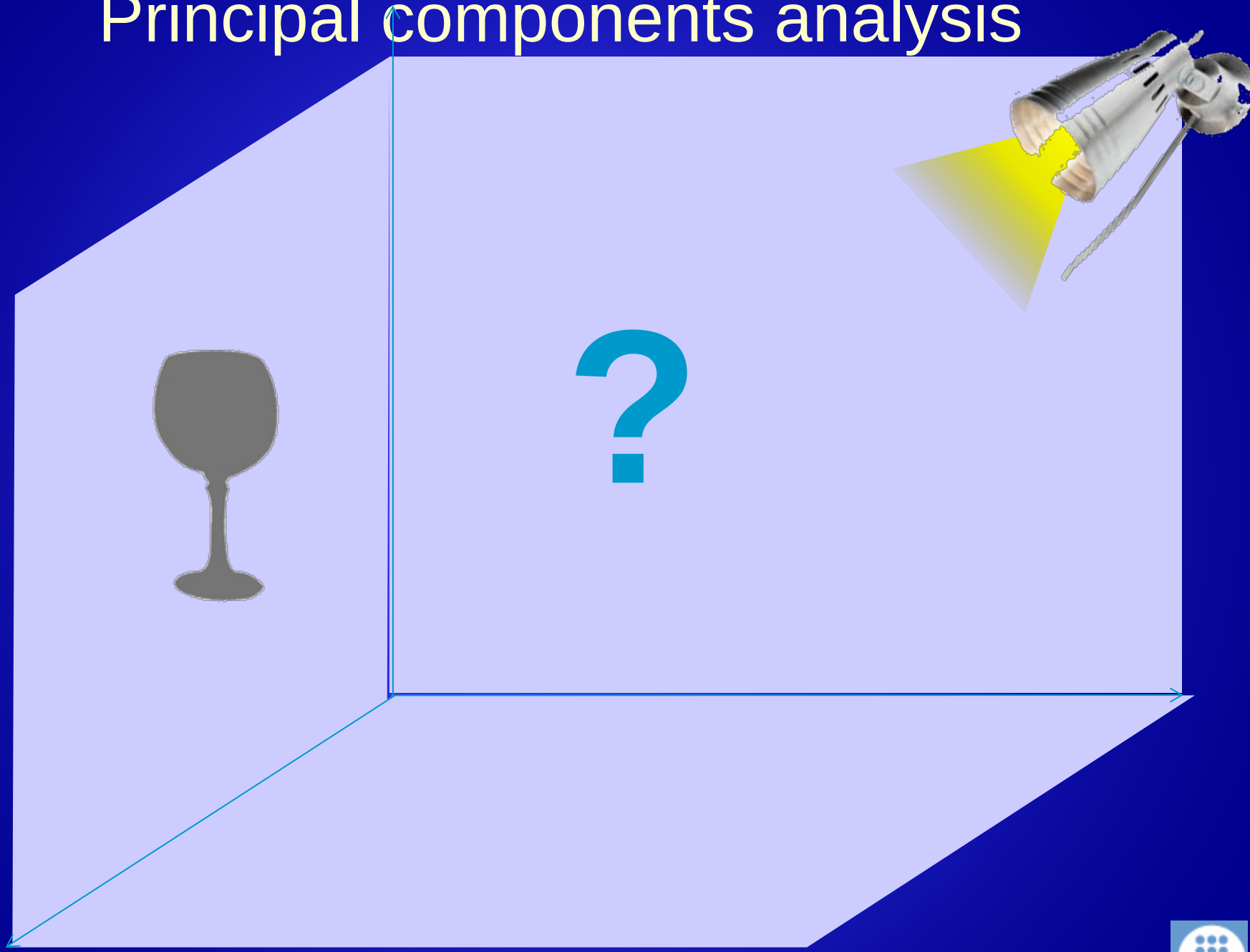

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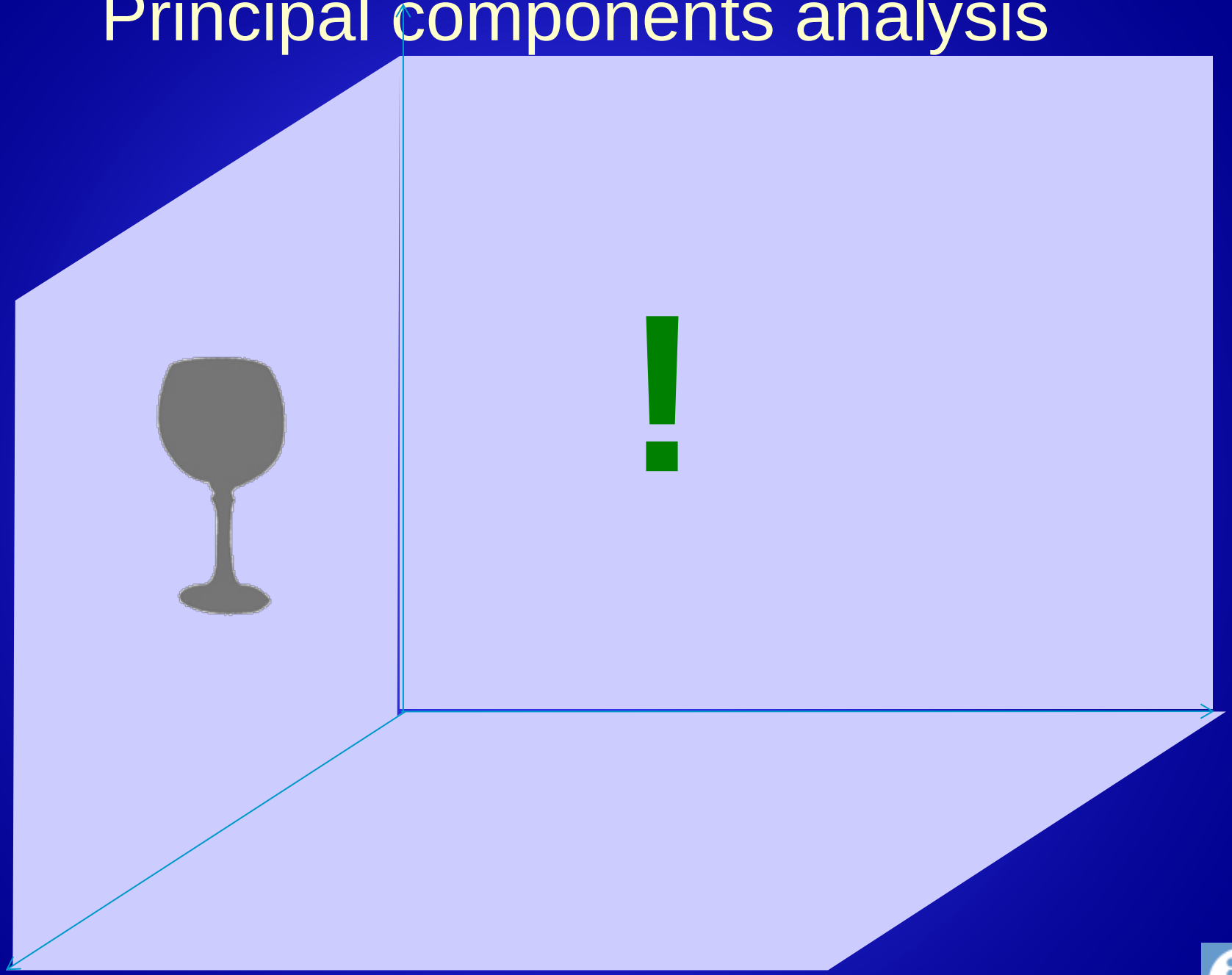

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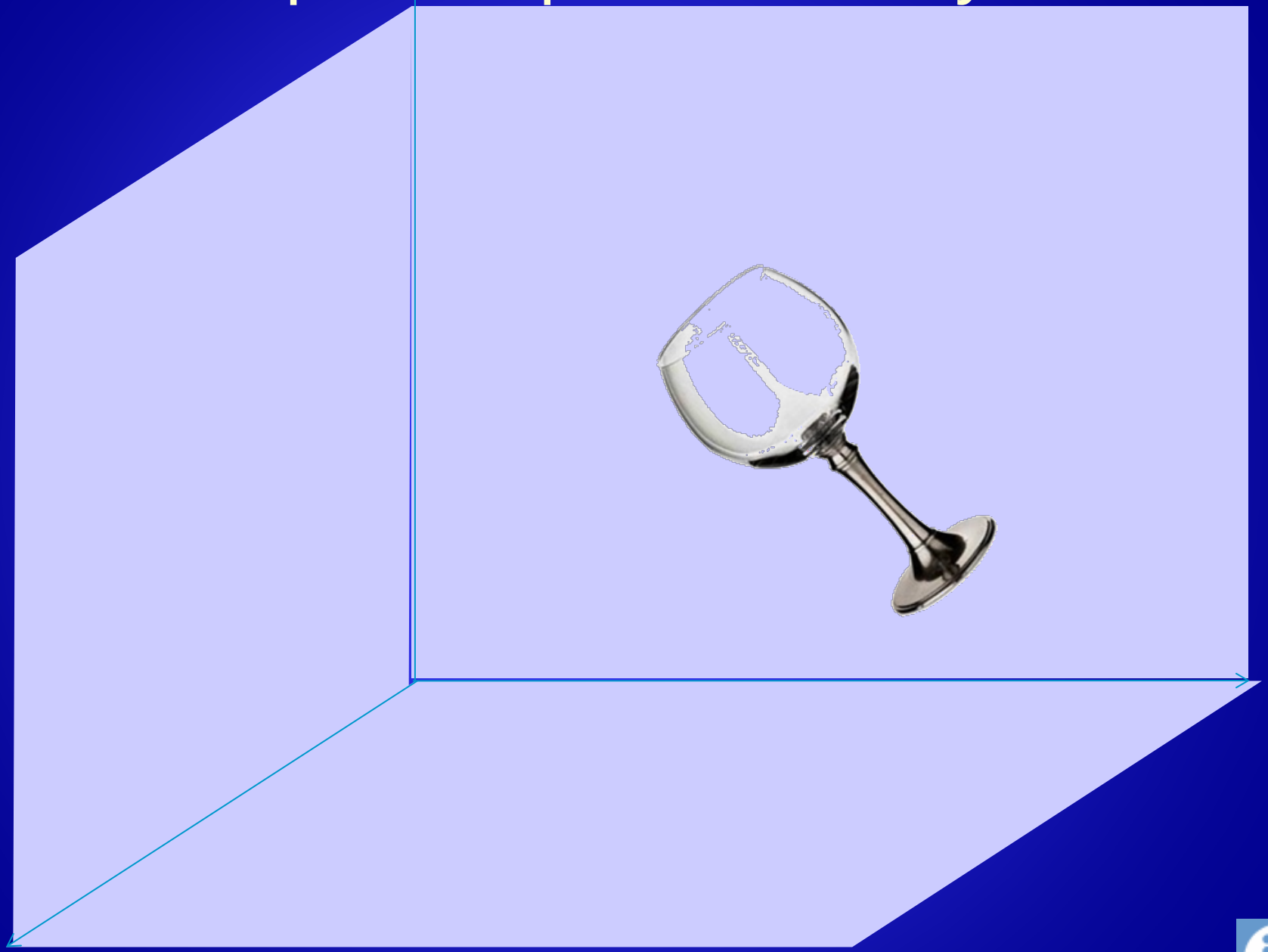

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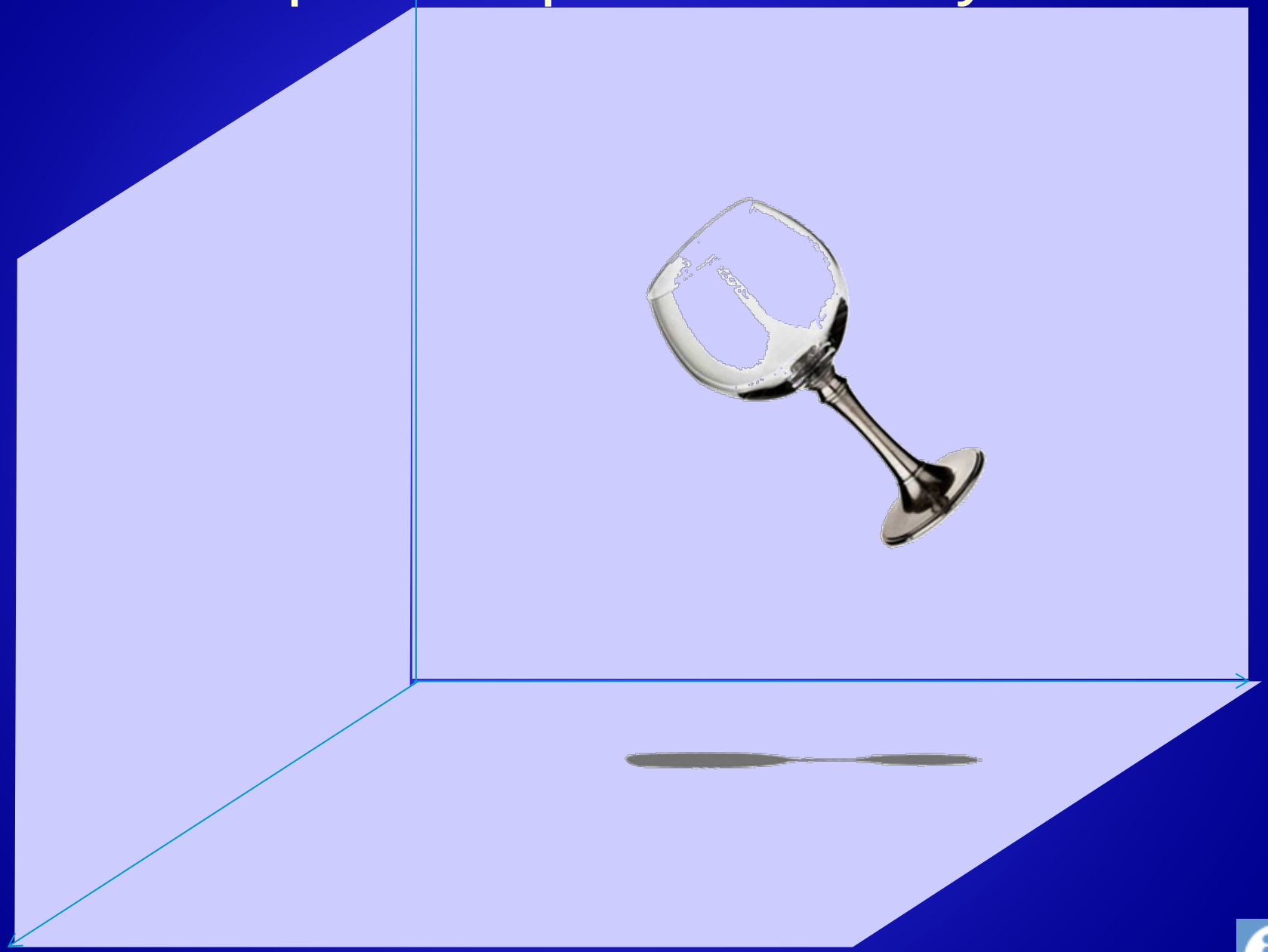

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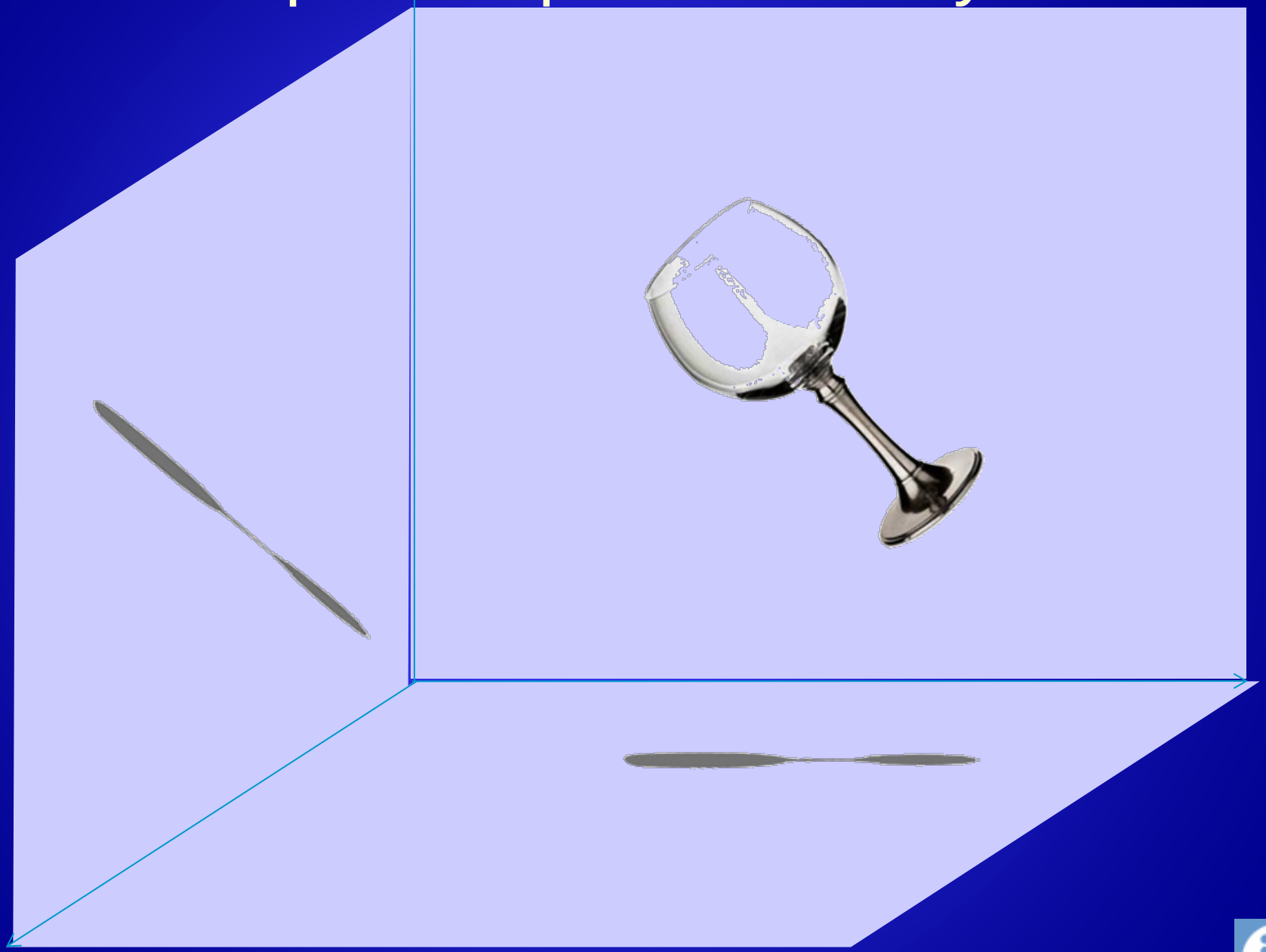

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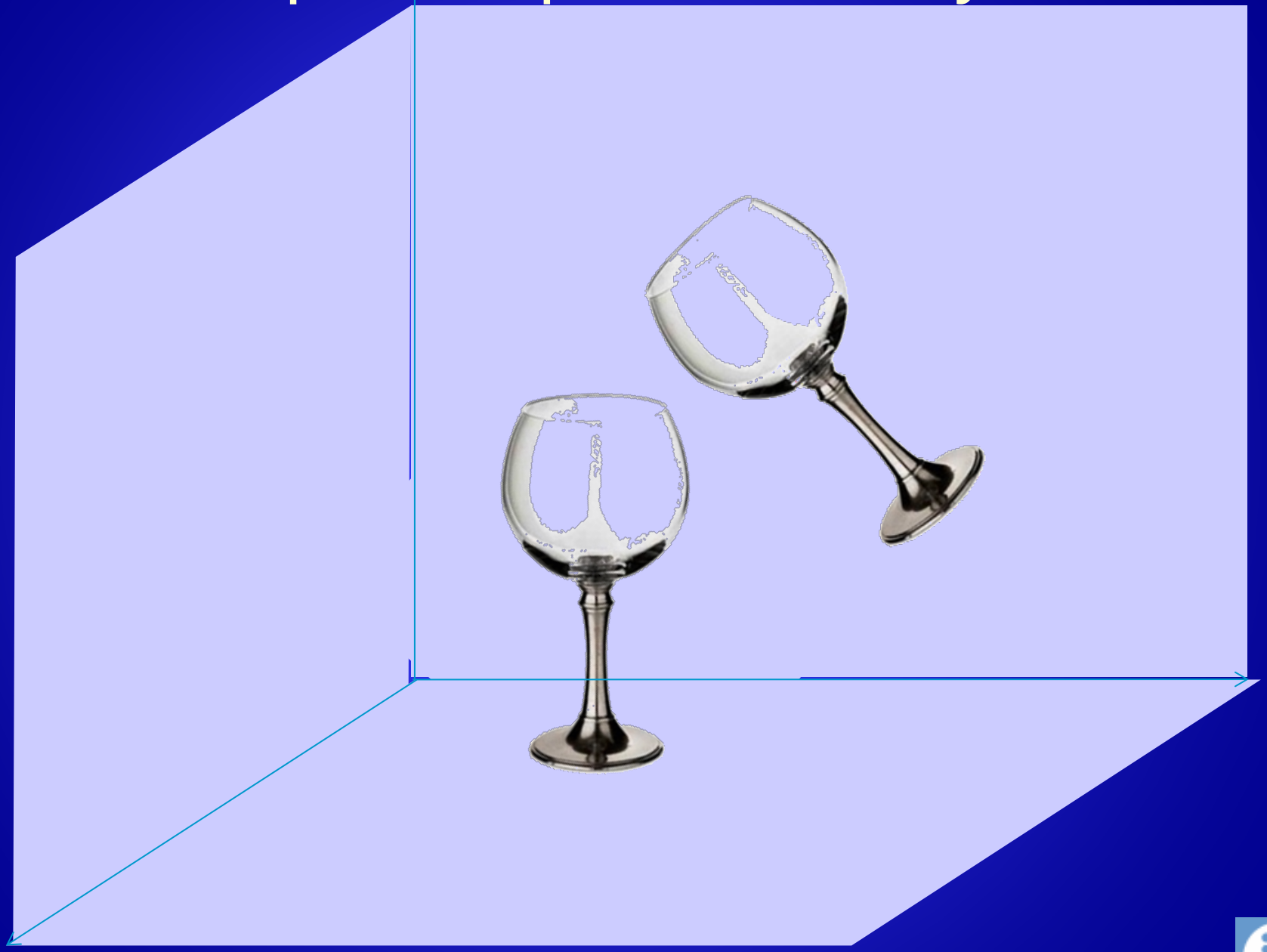

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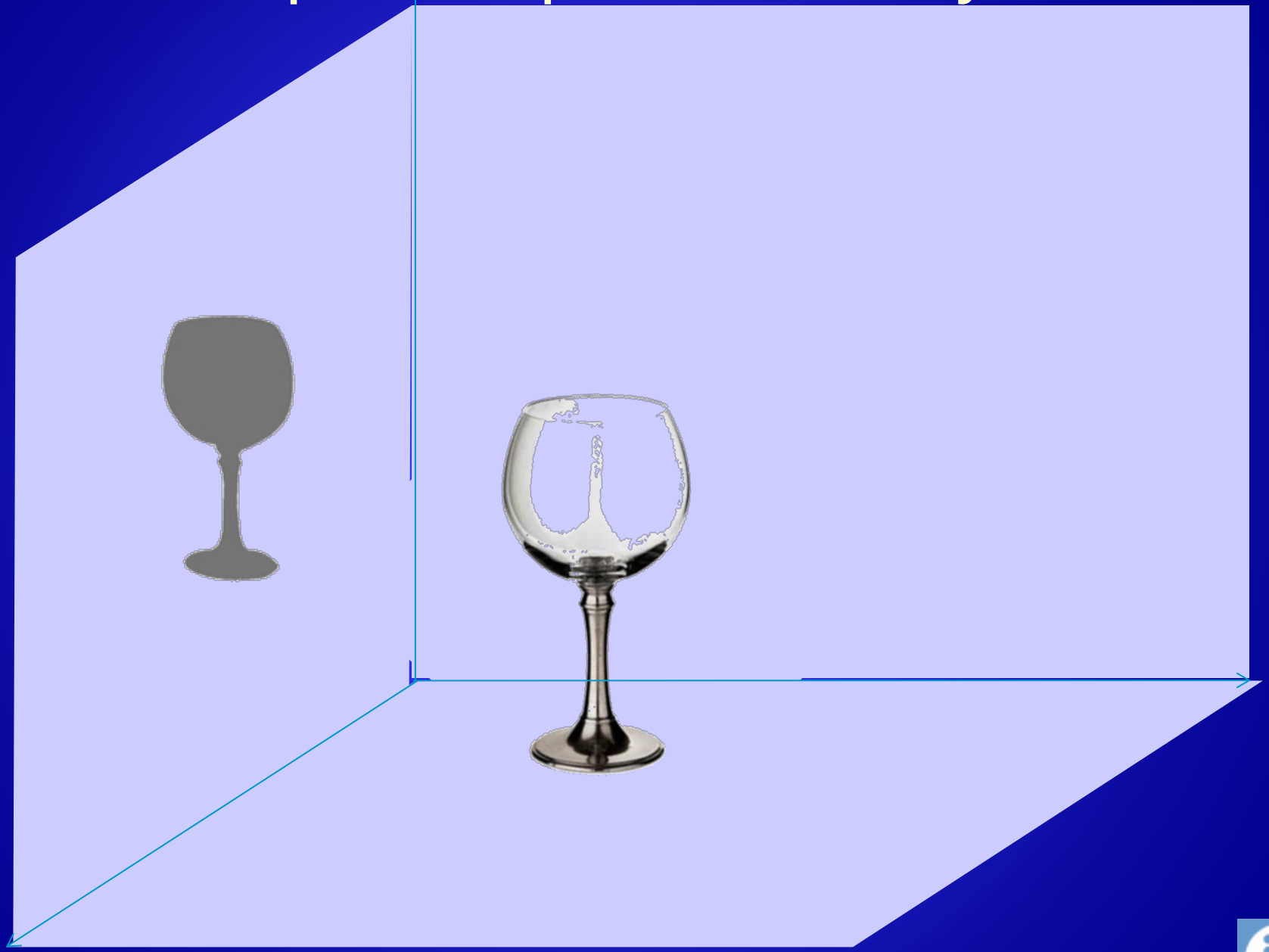

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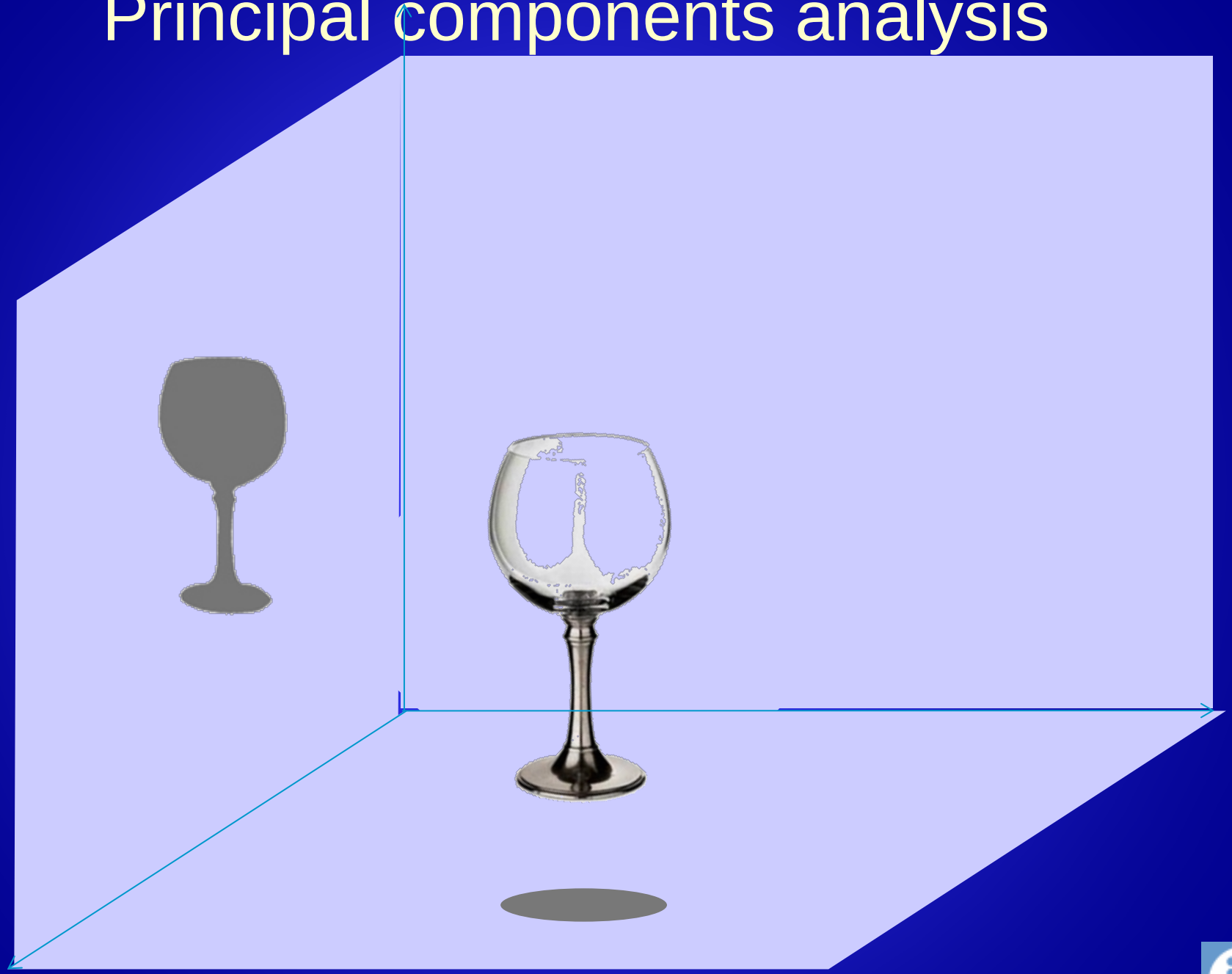

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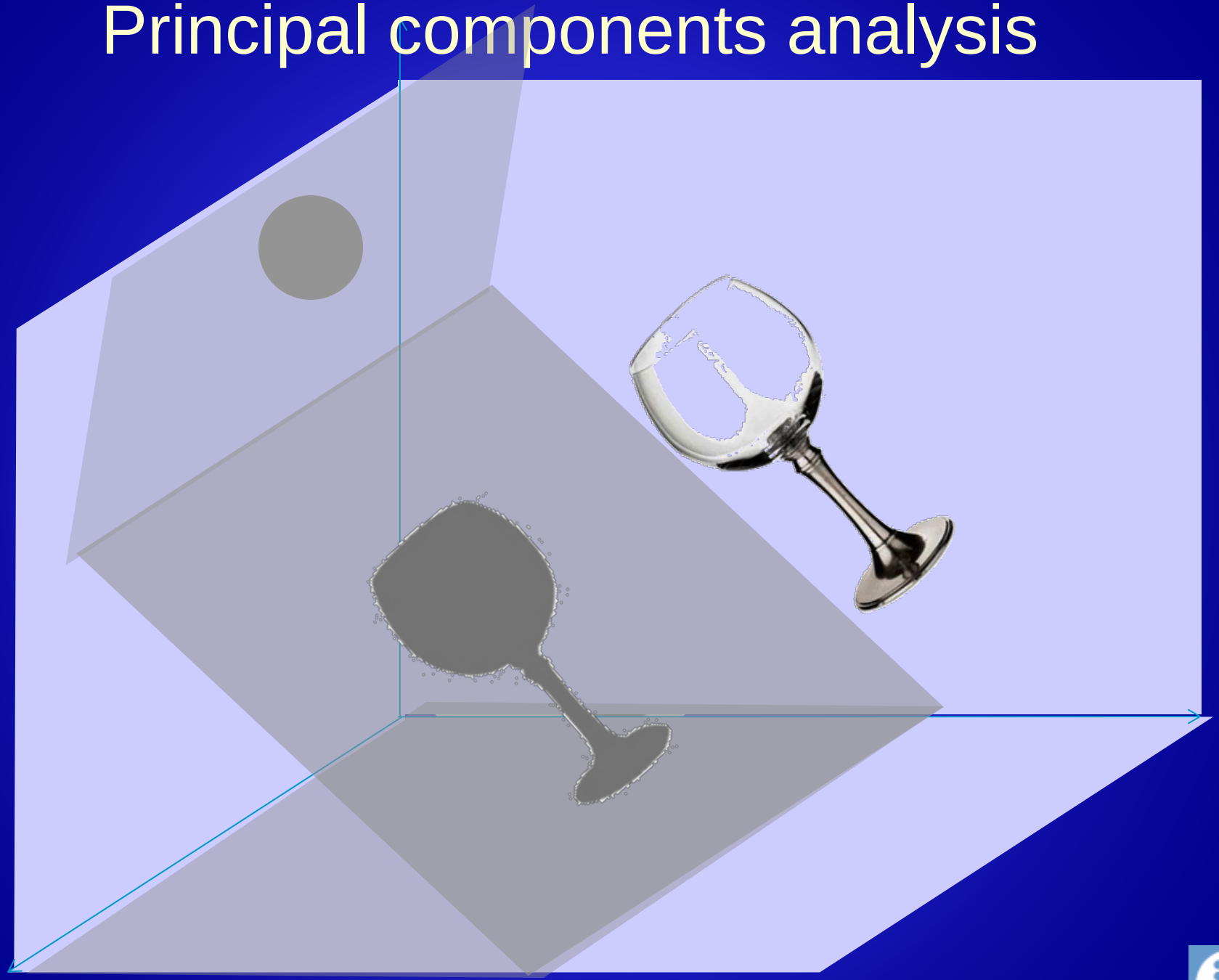

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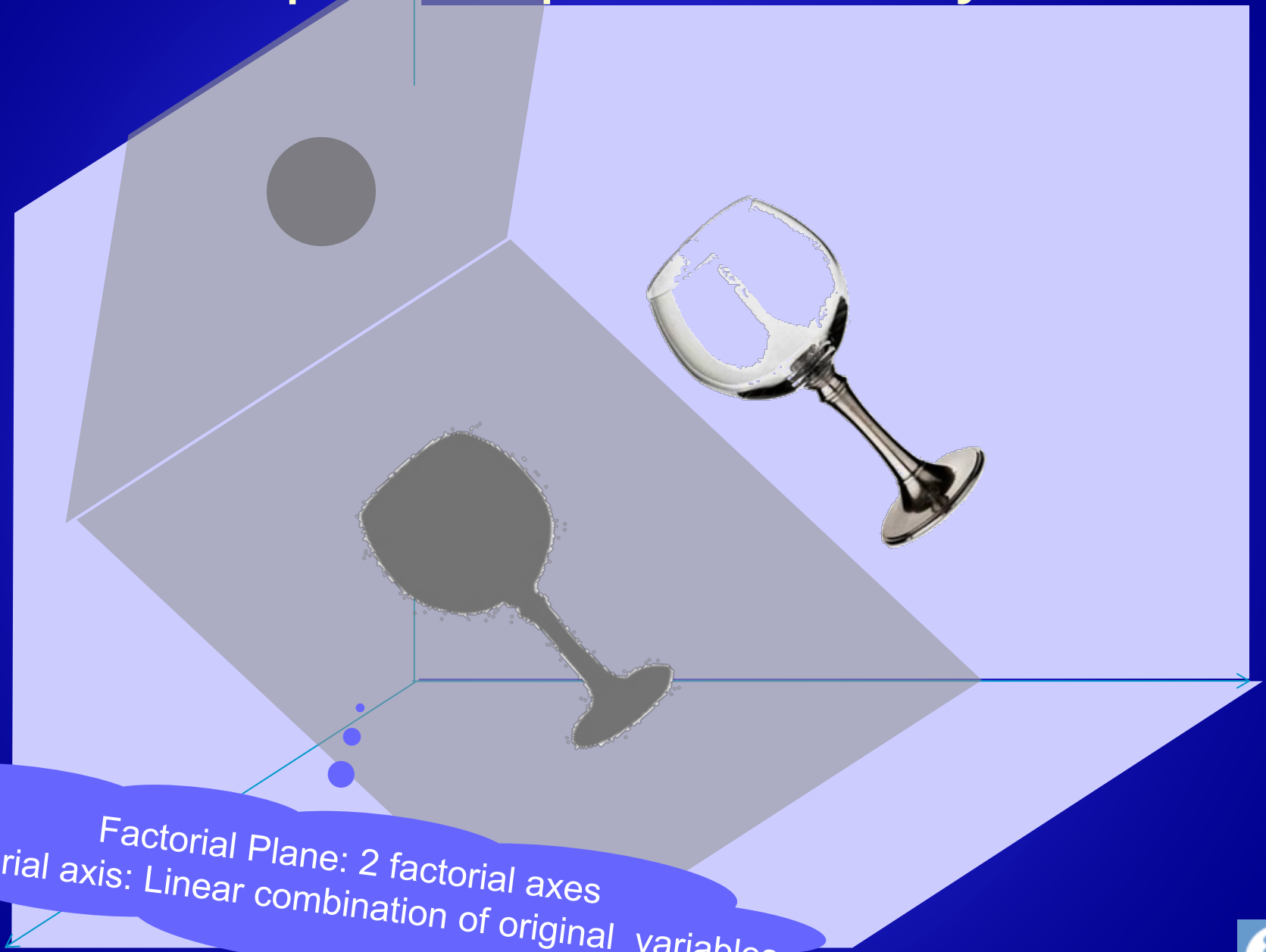
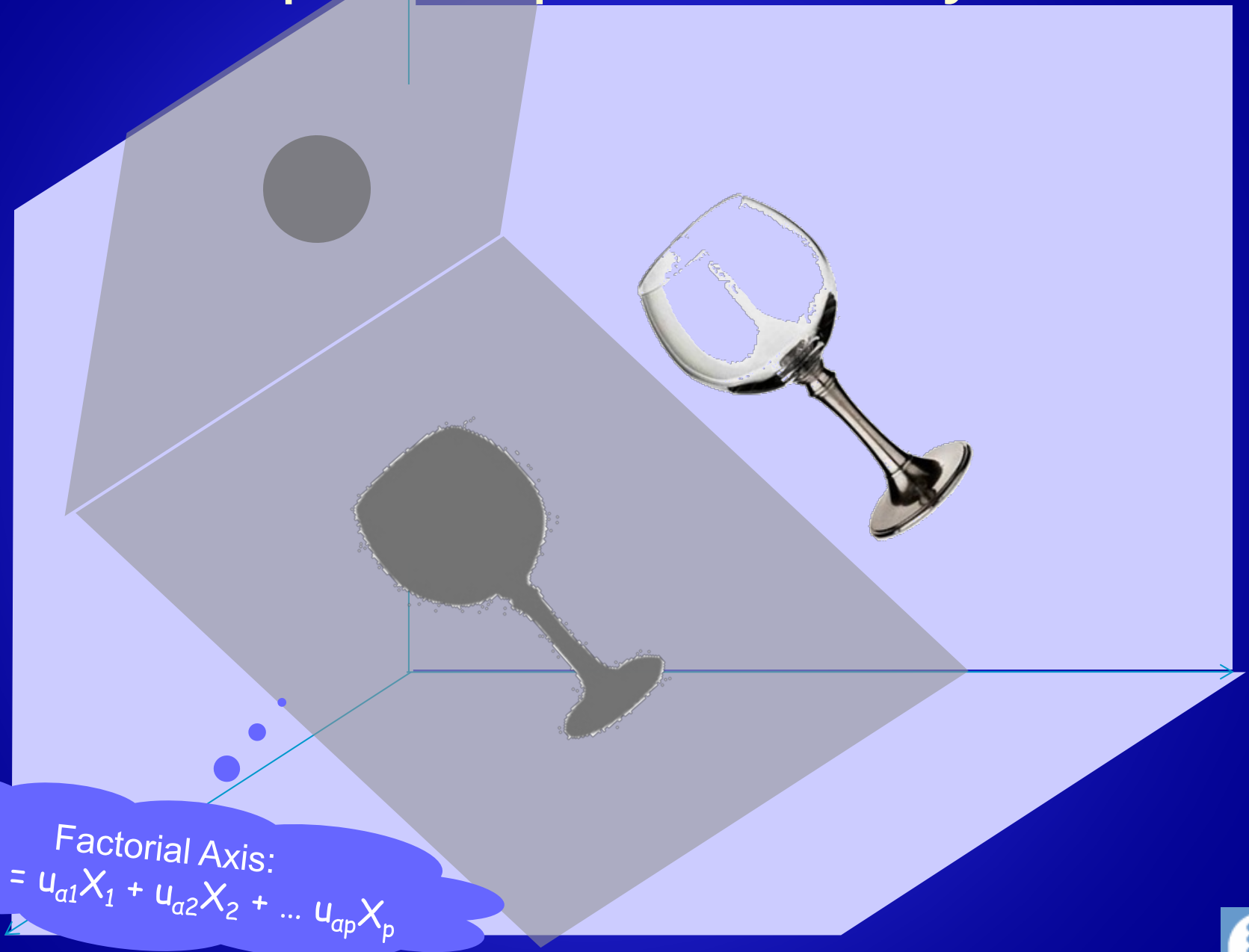


Factorial Plane: 2 factorial axes
Factorial axis: Linear combination of original variables

Principal components analysis



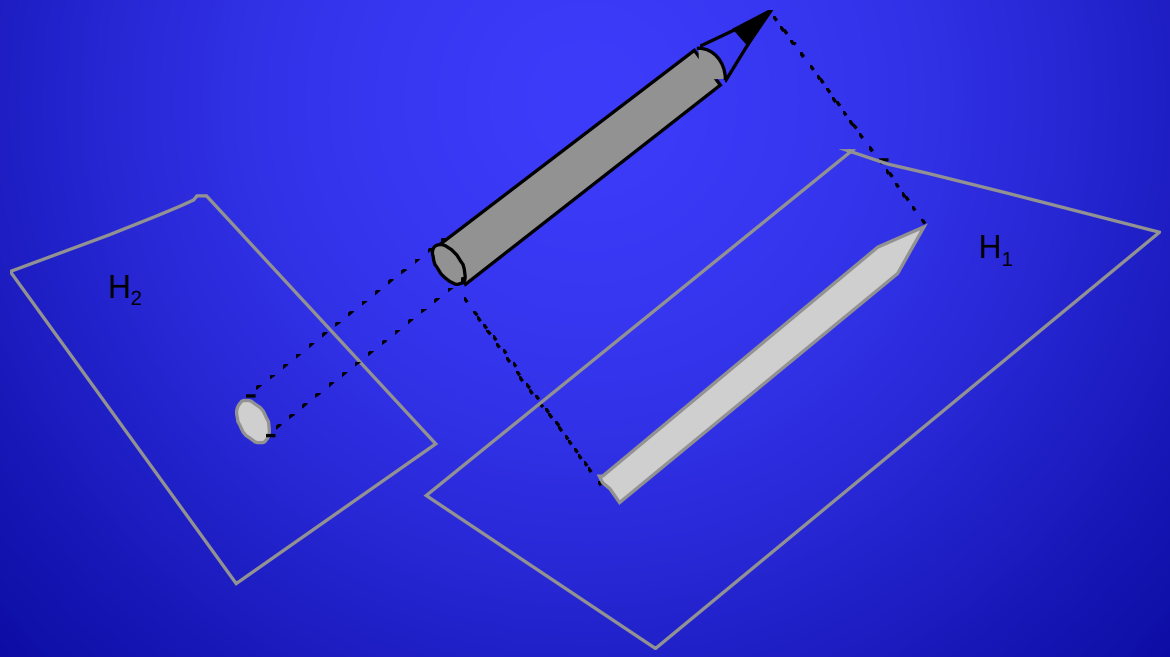
Factorial Axis:

$PC_a = u_{a1}X_1 + u_{a2}X_2 + \ldots u_{ap}X_p$

Principal components analysis

- Purpose:
  - To project the cloud of points upon a subspace (plane) retaining as much original cloud information.

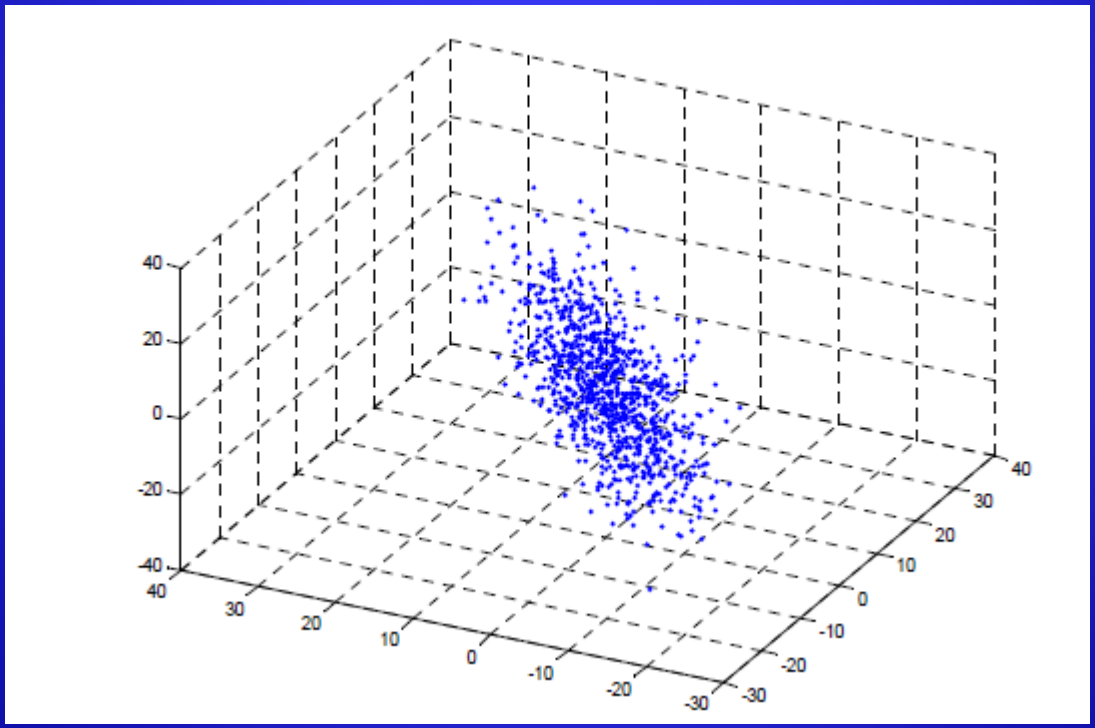
    (see video)



Principal components analysis

- Find the most informative projection planes of data cloud

  (factorial planes)



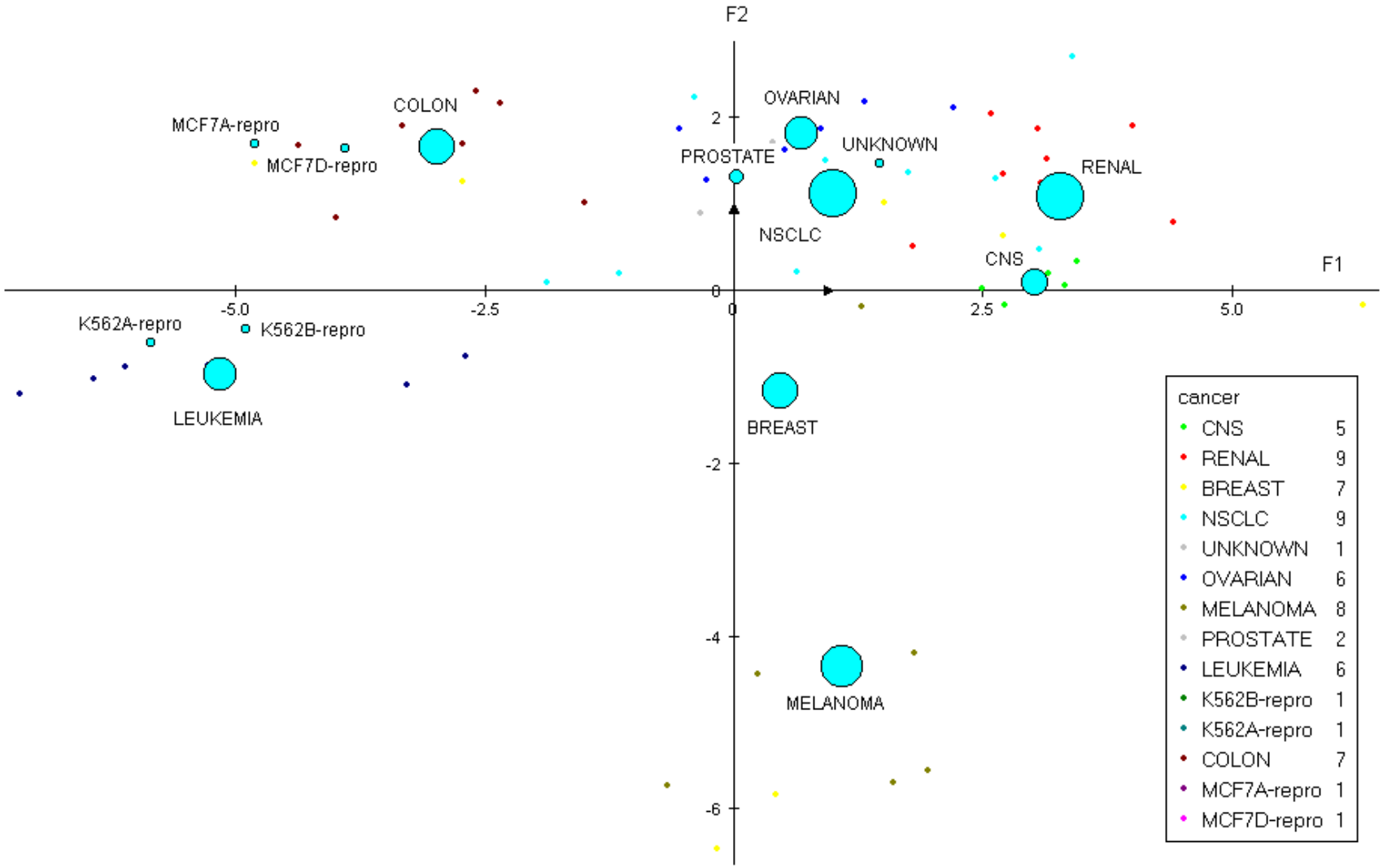
Factorial Methods

- Output: K factors rotating original X variables
- Factors: Linear combinations of original variables

Several uses:

- As an associative data mining method:
  analyze relationships among variables
  Project variables and modalities and find associations

Microarray data: 64 cancers 6830 gen cromotografy



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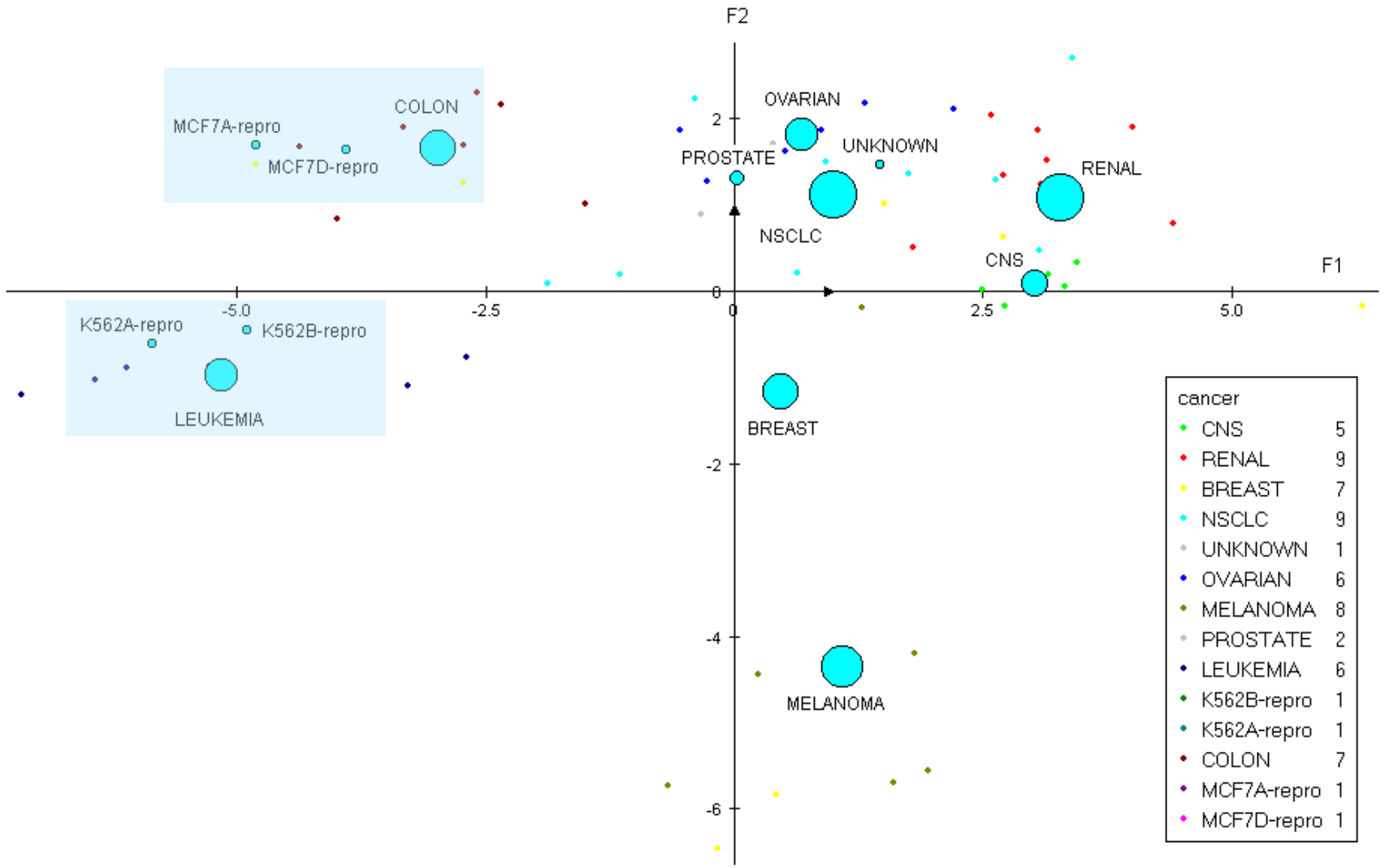
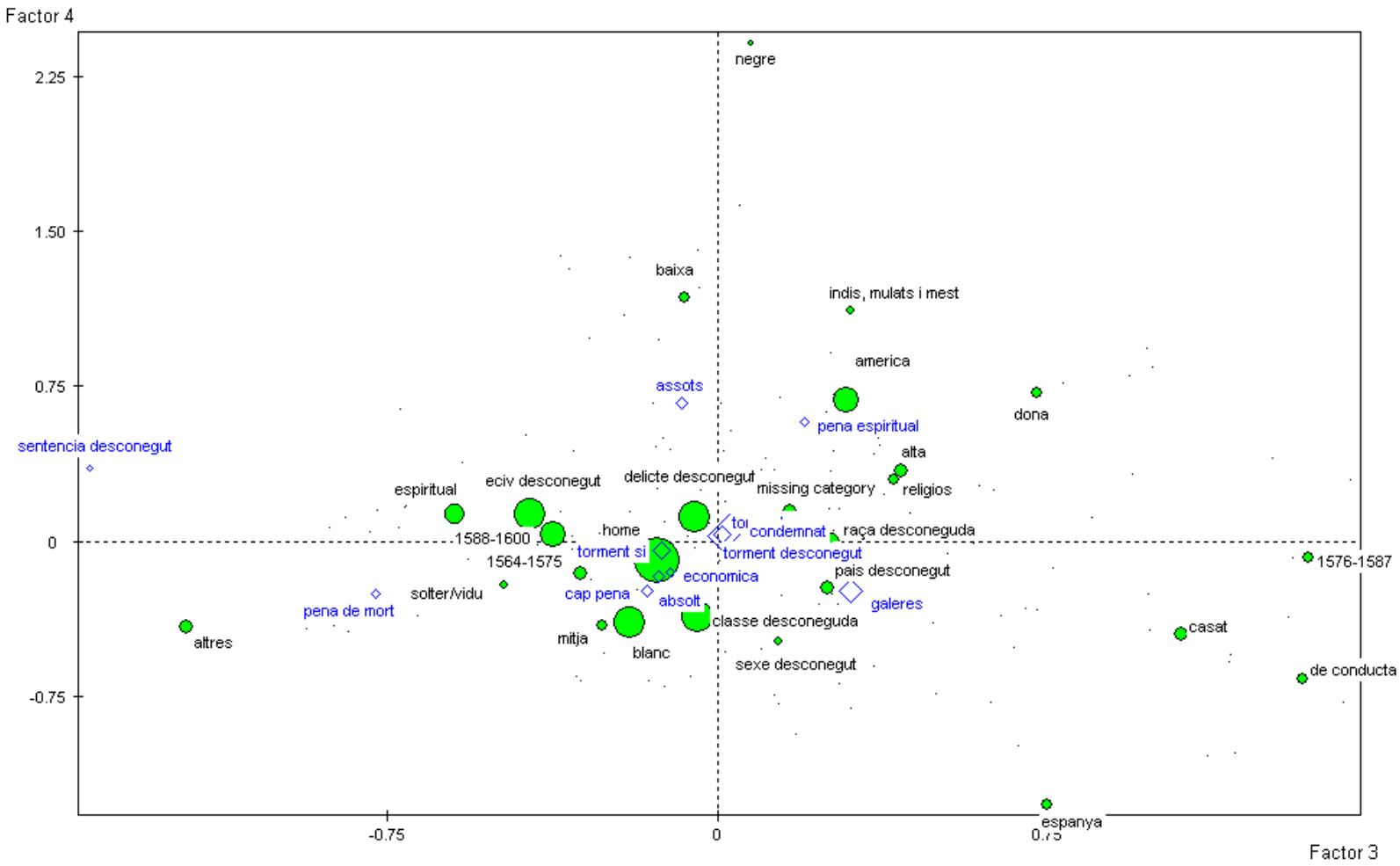


Spanish inquisition 1567-1600

sentences & crimes



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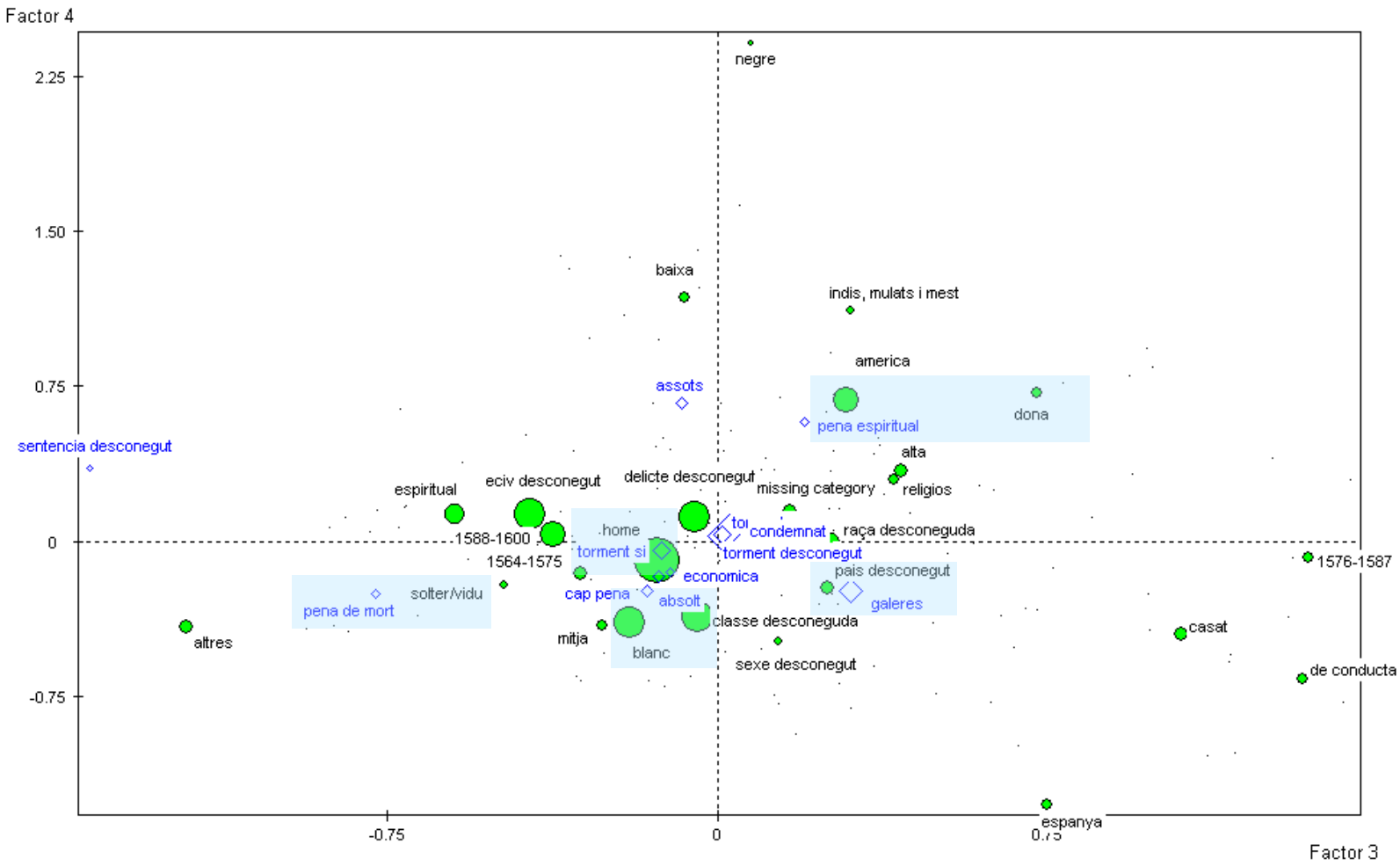

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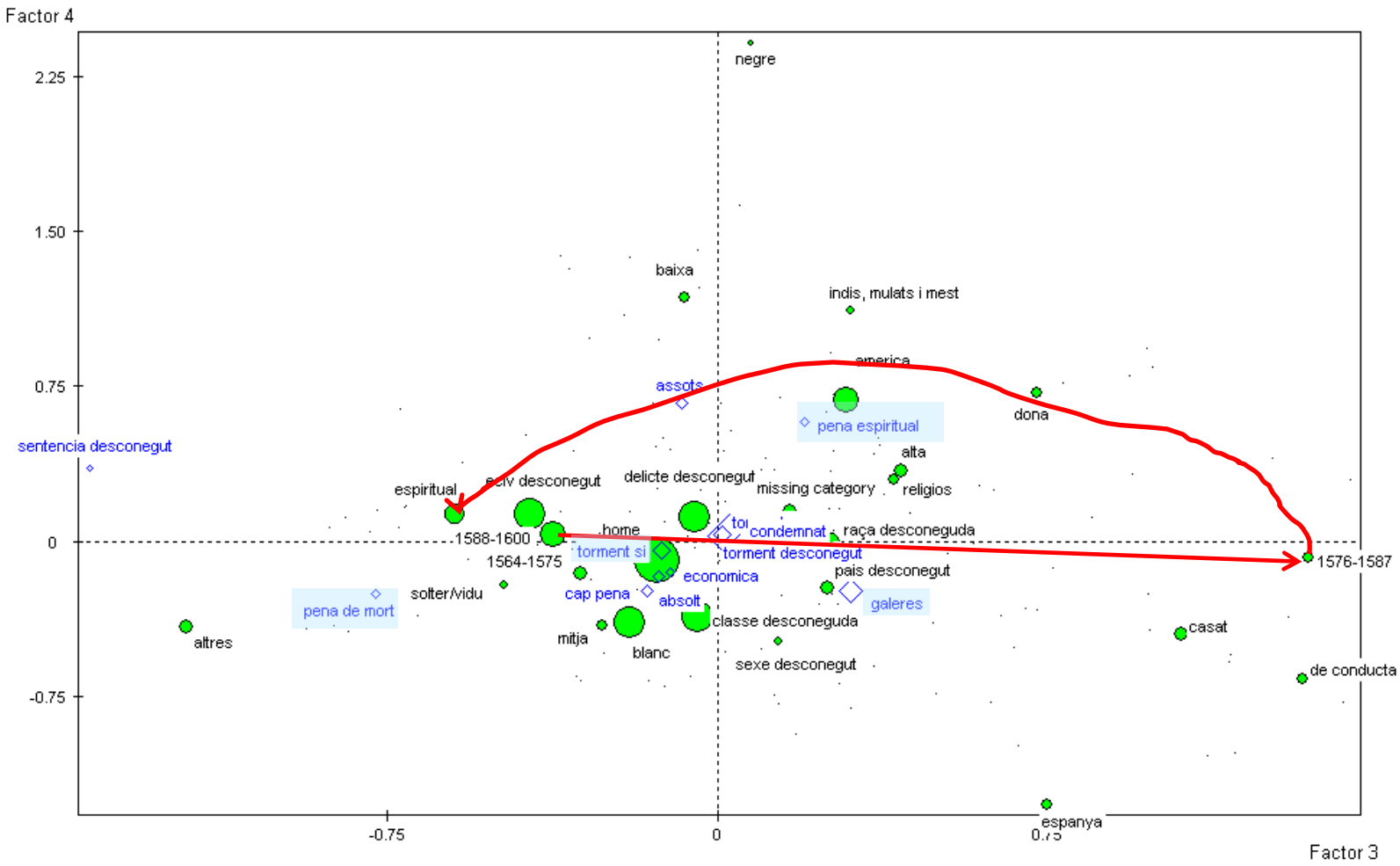
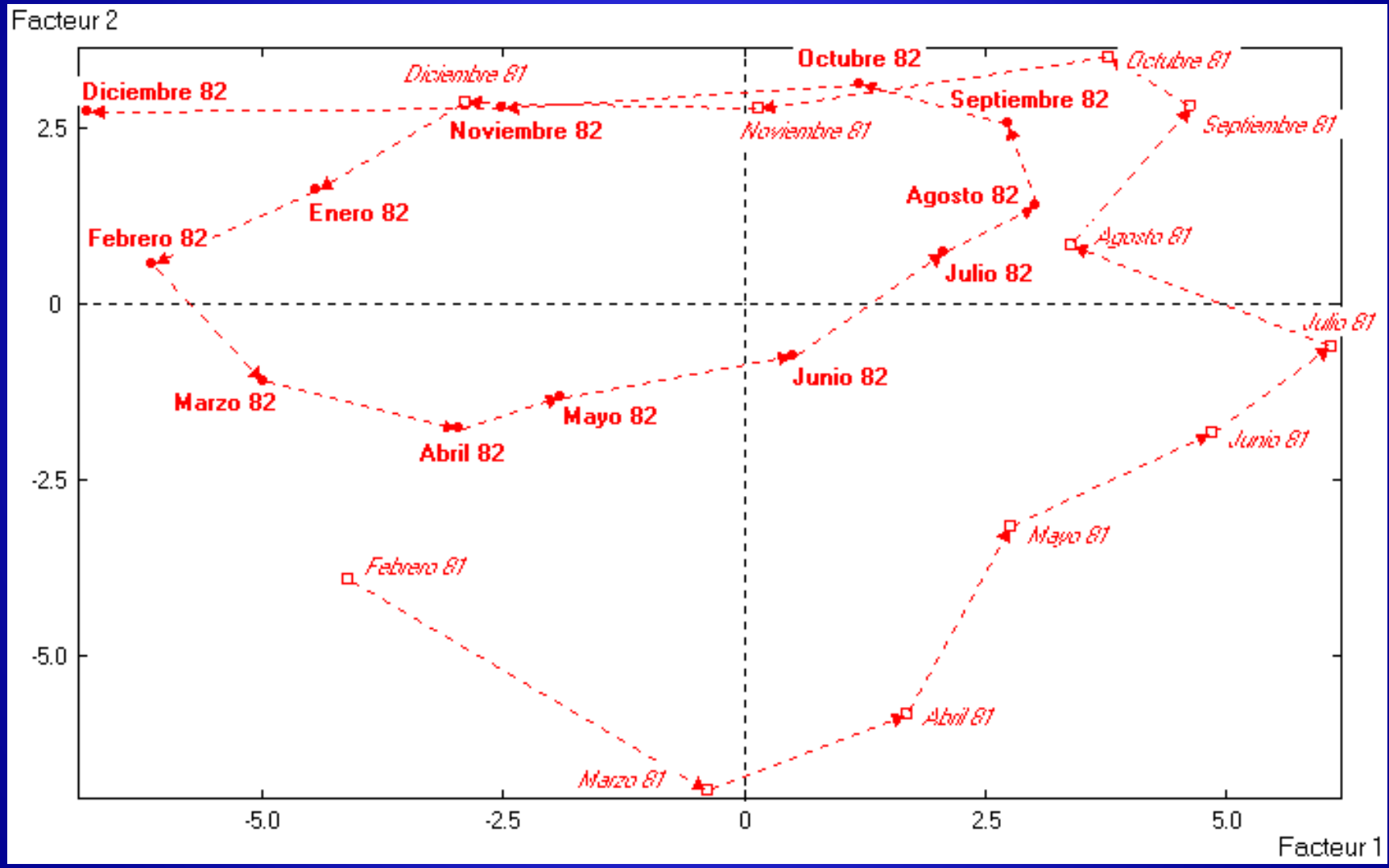


Monitoring of the inner temperatures of Lascaux cave (France)



Factorial Methods

- Output: K factors rotating original X variables
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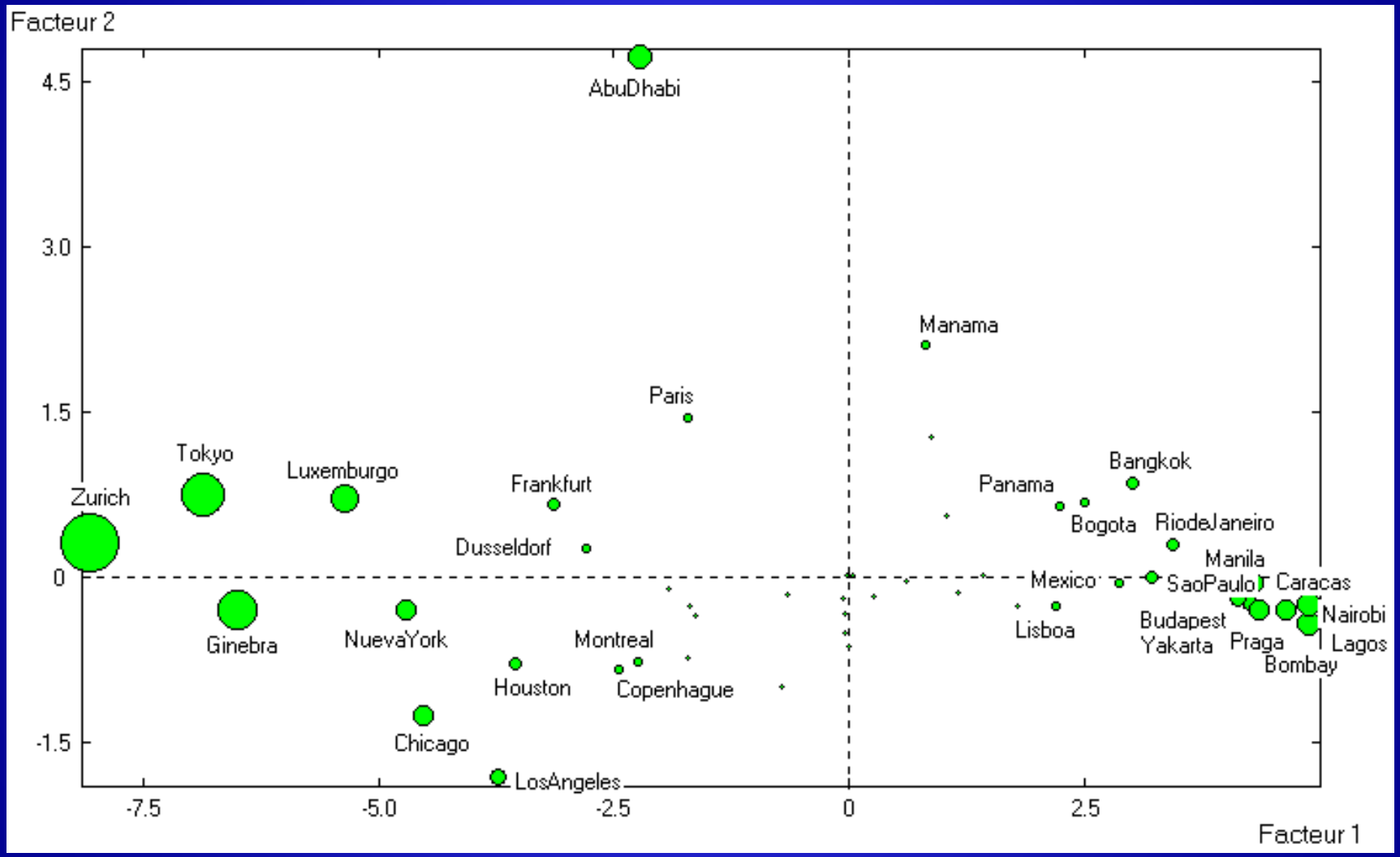
Several uses:

- As an associative data mining method to analyze relationships among variables
  Project variables and modalities and find associations

- As a preprocessing method for elicitation of latent variables
  Project active and illustrative variables/individuals on first/second factorial plane and interpret factors (find latent variables)

Visualisation of international cities according their salaries. USB 1994.



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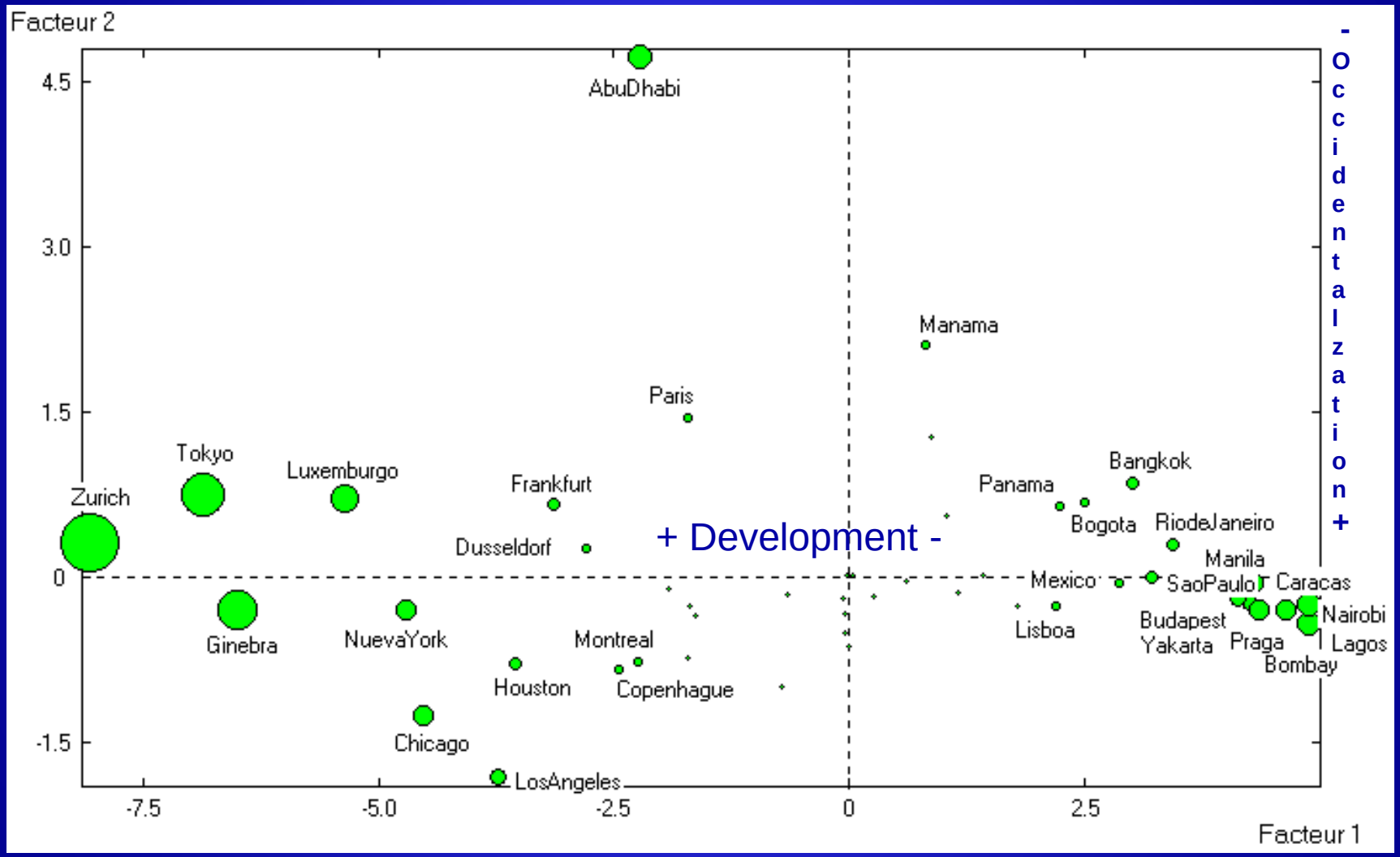
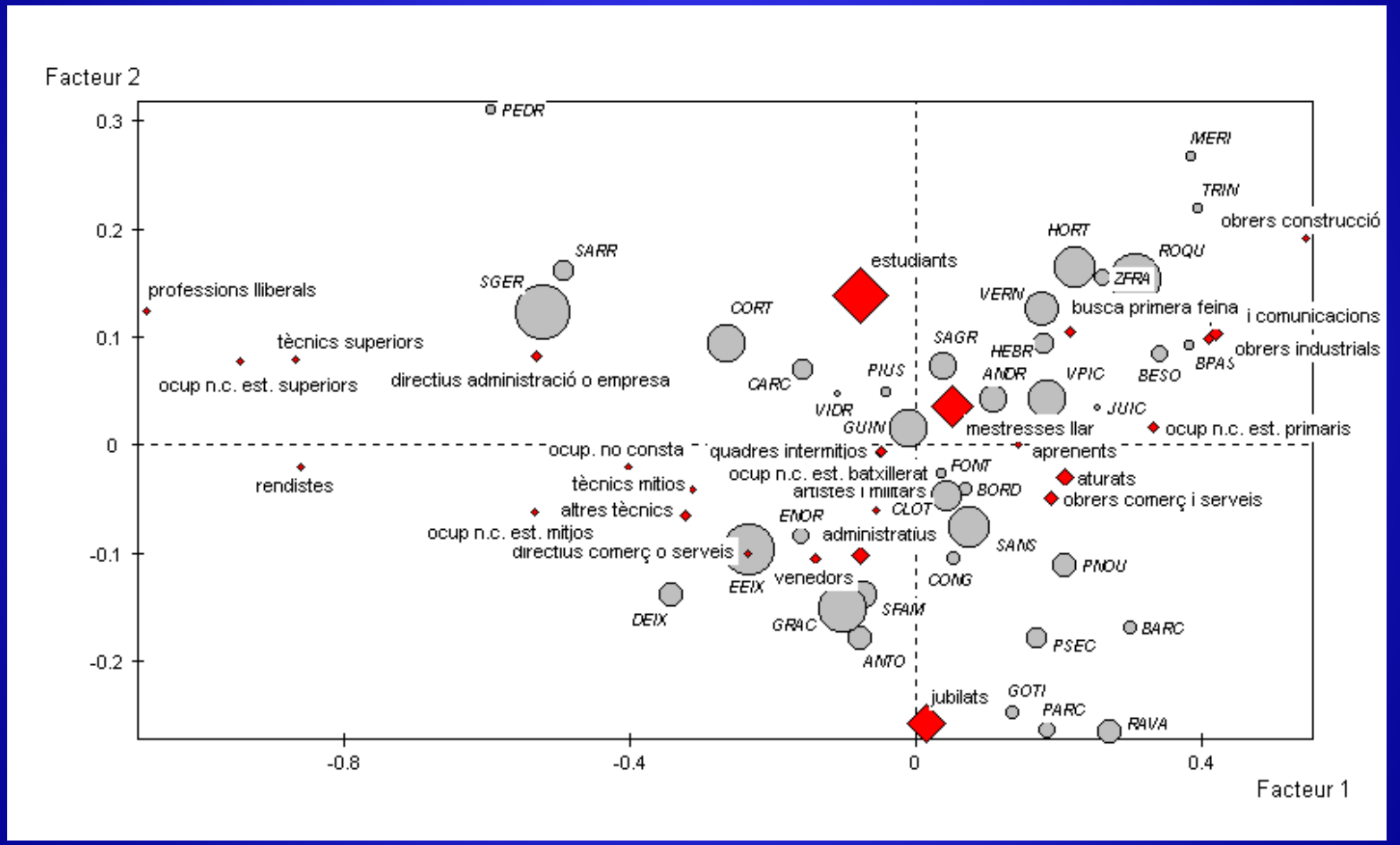


Visualization of the table *BCN* Quarters x *Profession of inhabitants*



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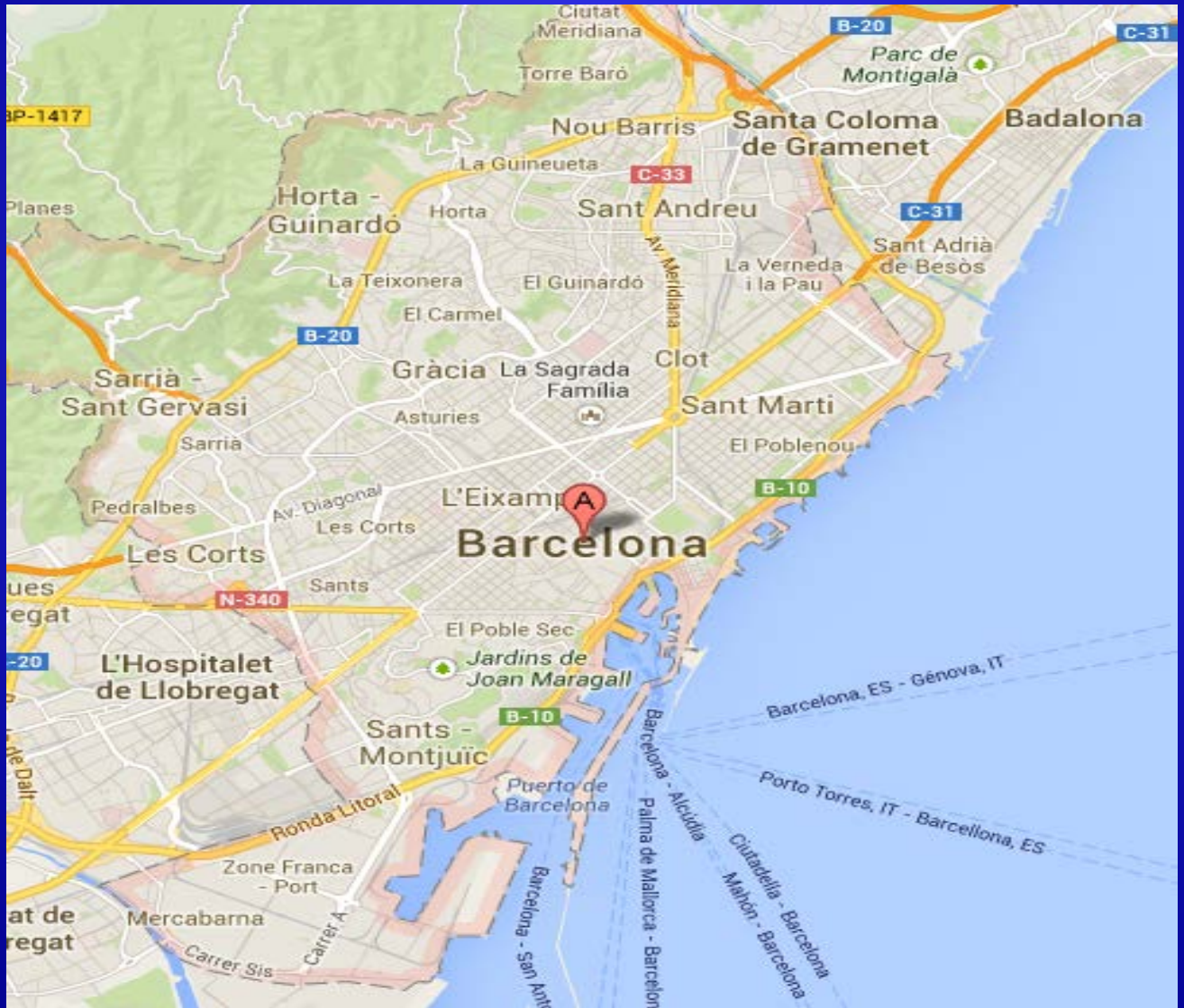

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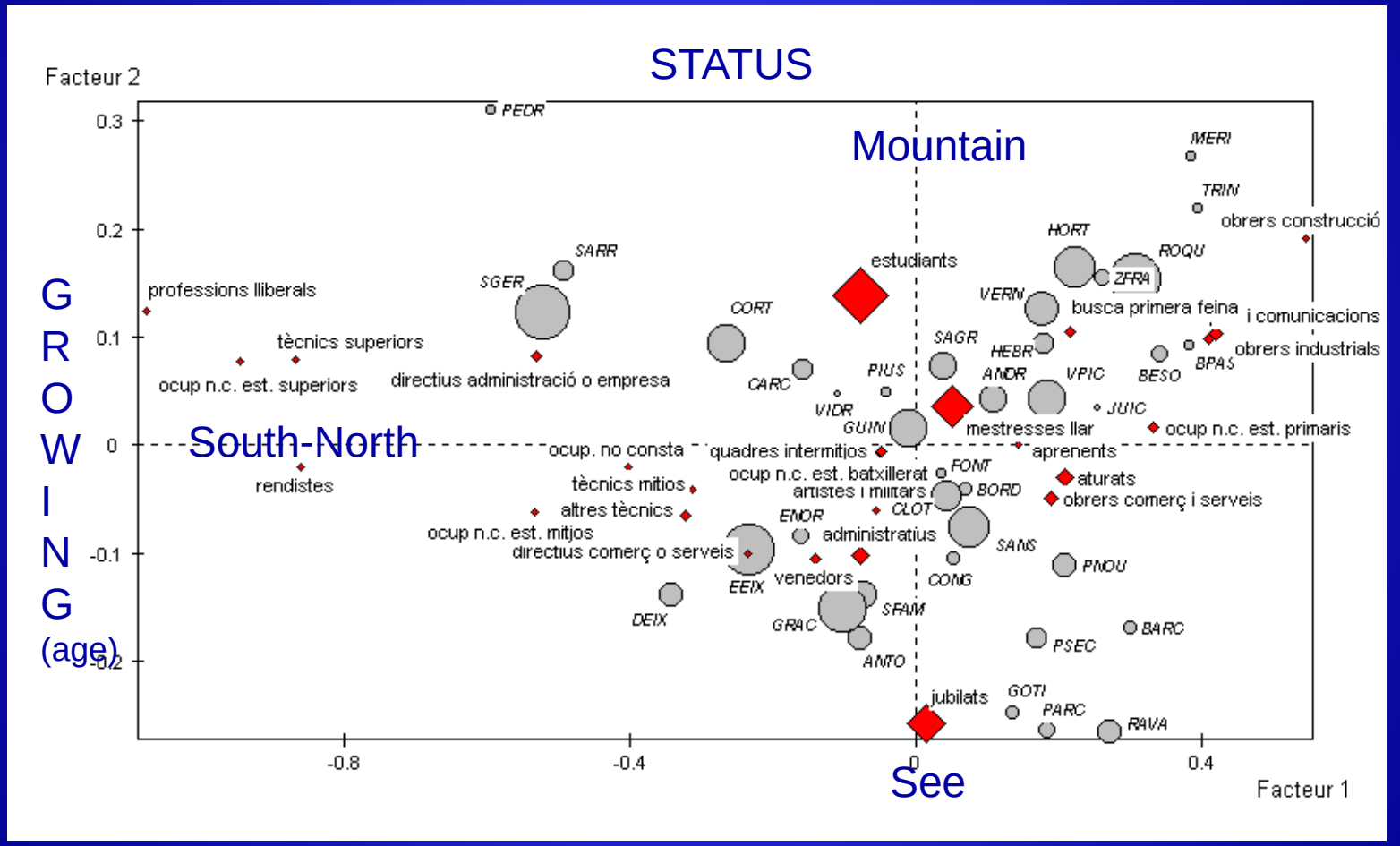

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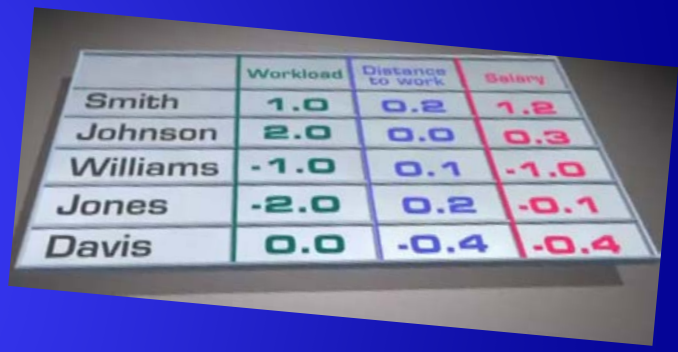
- As a preprocessing method for multidimensionality reduction

Factorial Methods

Data	Factorial Method
Continuous variables	PCA - Principal Component Analysis
Count variables	CA - (Simple) Correspondence Analysis
Categorical variables	MCA - Multiple Correspondence Analysis

Factorial Methods

- Principal Components Analysis
  - Only numerical variables
  - Find the most informative projection planes (factorial planes, maximize projected inertia)



	Workload	Distance to work	Salary
Smith	1.0	0.2	1.2
Johnson	2.0	0.0	0.3
Williams	-1.0	0.1	-1.0
Jones	-2.0	0.2	-0.1
Davis	0.0	-0.4	-0.4

Given <X,M,D>

- A data matrix X (nxp) centered
- A matrix of individuals weights D (nxn)
- Assume euclidean metrics to compare individuals (M= $\mathbb{I}_p$)

Matrix $M^{1/2}\, X'DXM^{1/2}$

- Product of data with the two metrics
- Simetric,
- Semidefinite
- Catches relationships and opositions of data

Factorial Methods

- Principal Components Analysis

  $M^{1/2}X'DXM^{1/2}$ catches well the data structures

  $Rang(M^{1/2}X'DXM^{1/2}) = r$, $r = rang(X)$ r positive vaps and p-r null vaps

  $Trace(M^{1/2}X'DXM^{1/2}) = \sum_{\alpha=1}^{r} \lambda_\alpha$ (λ_α, the r non null vaps)

  $M = \mathbb{I}_p$: $M^{1/2}X'DXM^{1/2} = X'DX$

  X centered and D diagonal : $X'DX = Cov(X)$

  Build variances and covariances matrix: $X'DX$

  Diagonalize $X'DX$ (i.e. solving the equation) $X'DXu = \lambda u$

  provides eigen values λ_α and

  eigenvectors $u_\alpha = (u_{\alpha 1} \ldots u_{\alpha p})$

Factorial Methods

- Principal Components Analysis

Diagonalize X'DX (i.e. solving the equation) $X'DXu = \lambda u$ (1)

det(X'DX-λ)=0 (find roots of characteristic polynomial)

provides eigen values λ_α (α =1:r, r=rang(X))

substituting in (1) provides eigenvectors $u_\alpha = (u_{\alpha 1} \ldots u_{\alpha p})$

$u^{-1}X'DXu = \lambda$ *is a diagonal matrix*

(X'DX becomes diagonal when pre/post multiplied by u)

$u^{-1}=u'$ *in orthonormal basis:* $u'X'DXu = \lambda$

X'DX decompose in a product by a diagonal matrix $X'DX = u\lambda u'$

$X'DX = u\lambda u' = u\lambda^{1/2}\lambda^{1/2} u' = u\lambda^{1/2}\mathbb{I}\lambda^{1/2} u' = u\lambda^{1/2} u'u\ \lambda^{1/2} u' = A^{1/2} A^{1/2}$

X'DX decompose in a product of something by itself (A square root)

Trace(X'DX)=Trace(λ*)* (property of diagonalization)

Factorial Methods

- Given <X,M,D>

Diagonalize covariances matrix (with centered data X'DX)

Get r eigen values λ_α *and sort decreasingly*

$$\{\lambda_\alpha\}_{\alpha=1:r} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_r$$

Corresponding eigenvectors $u_\alpha = (u_{\alpha 1} \ldots u_{\alpha p})$

$$|u_\alpha| = 1$$

$$u_\alpha u_{\alpha'} = 0$$

$\{u_\alpha\}_{\alpha=1:r}$ *orthonormal base for individuals*

The subspace generated by $\{u_\alpha\}_{\alpha=1:r}$ *is the same as the subspace generated by the rows of X*

Factorial Methods

- Given <X,M,D>

In general *Diagonalize* $M^{½}X'DXM^{½}$)

Get r eigen values λ_α *and sort decreasingly (vaps are conserved!!!!)*

$$\{\lambda_\alpha\}_{\alpha=1:r} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \geq \lambda_r$$

Corresponding eigenvectors $u^*_\alpha = (u^*_{\alpha 1} \ldots u^*_{\alpha p})$

by algebraic properties, u^*_α *can be found from* u^*

$$u^*_\alpha = M^{-½} u_\alpha$$

$\{u^*_\alpha\}_{\alpha=1:r}$ *orthonormal base for individuals*

$$|u^*_\alpha|_M = 1 : \quad u^{*\prime}_\alpha M u^*_\alpha = u'_\alpha M^{-½} M M^{-½} u_\alpha = 1$$

$$u^*_\alpha M u^*_{\alpha'} = 0: \quad u^*_\alpha M u^*_{\alpha'} = u'_\alpha M^{-½} M M^{-½} u_{\alpha'} = 0$$

Subspace generated by $\{u^*_\alpha\}_{\alpha=1:r}$ = *Subspace generated by X rows*

Factorial Methods

- Given <X,M,D>

Diagonalize Covariance matrix X'DX

Get r eigen values λ_α *and sort decreasingly*

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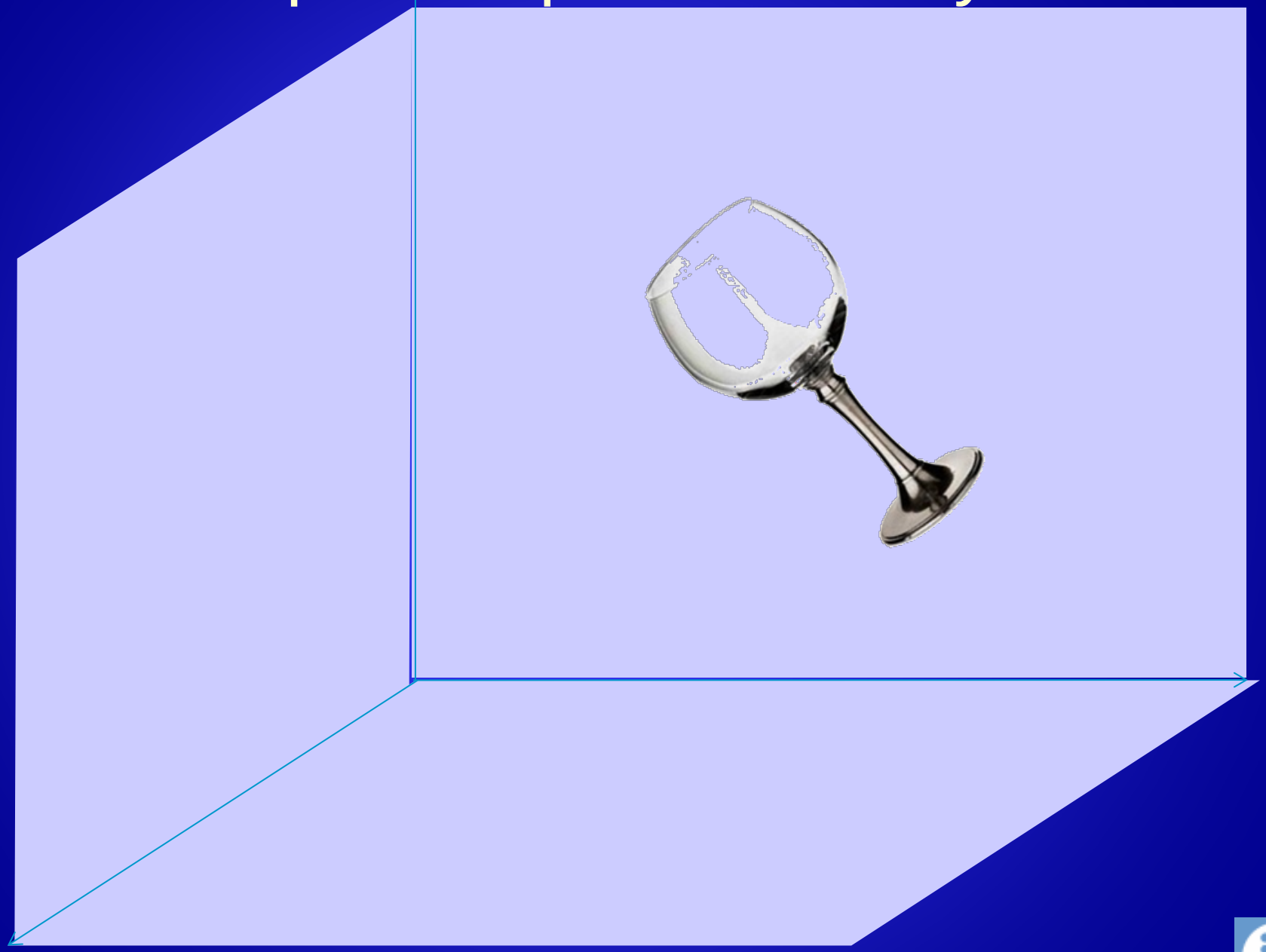
for $M = \mathbb{I}_p$: $u^*_\alpha = u_\alpha$; *for* $M \neq \mathbb{I}_p$: $u^*_\alpha = M^{-1/2} u_\alpha$

$\{u^*_\alpha\}_{\alpha=1:r}$ *orthonormal base for individuals*

u^*_α *are the principal factors of X : good rotation directions*

$U^* = ([u^*_1]\ [u^*_2] \ldots\ldots\ [u^*_r])$ *is the basis for the projection space*

Principal components analysis

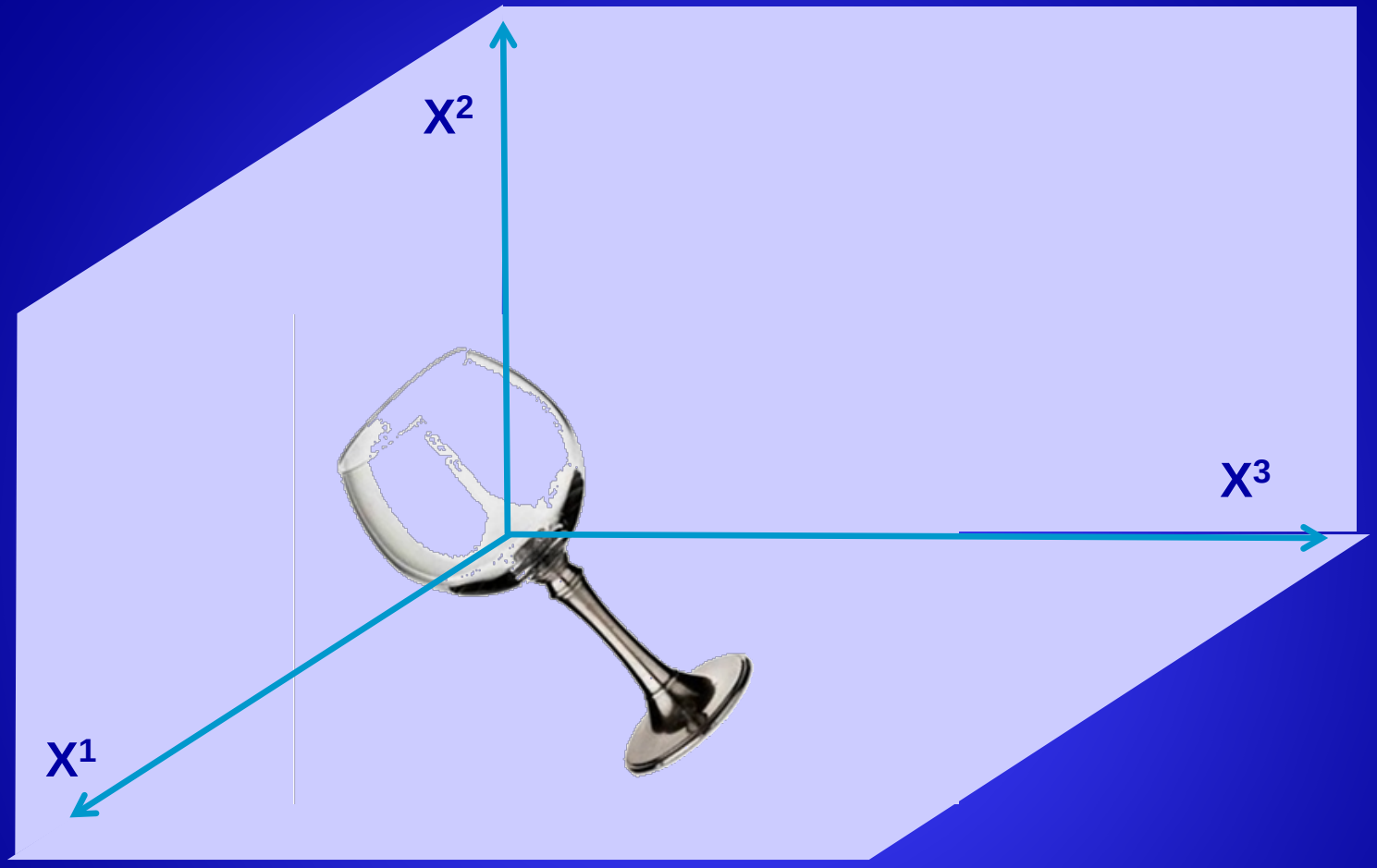


Principal components analysis

Centering X

(0,0,0)

Principal components analysis



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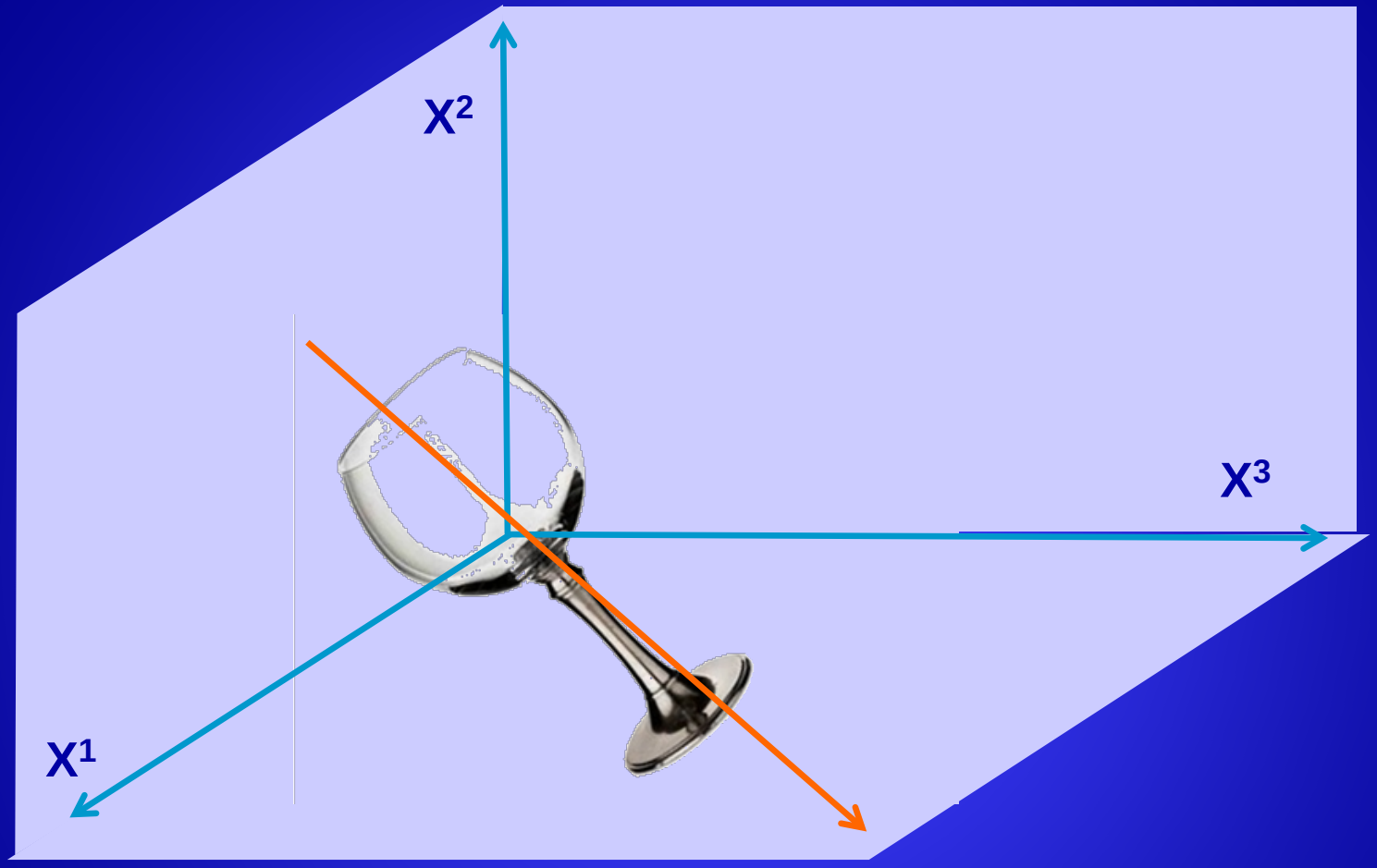

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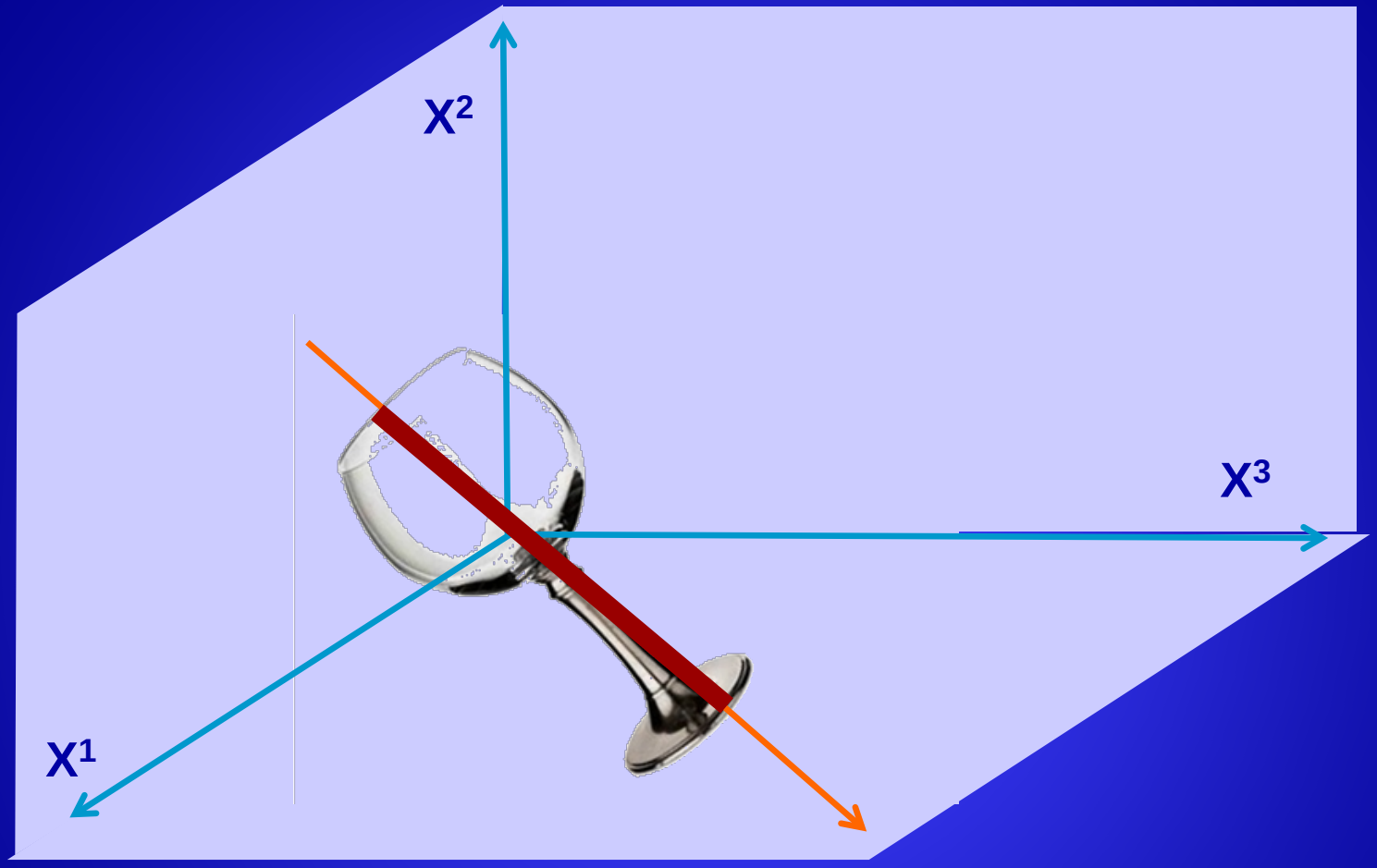

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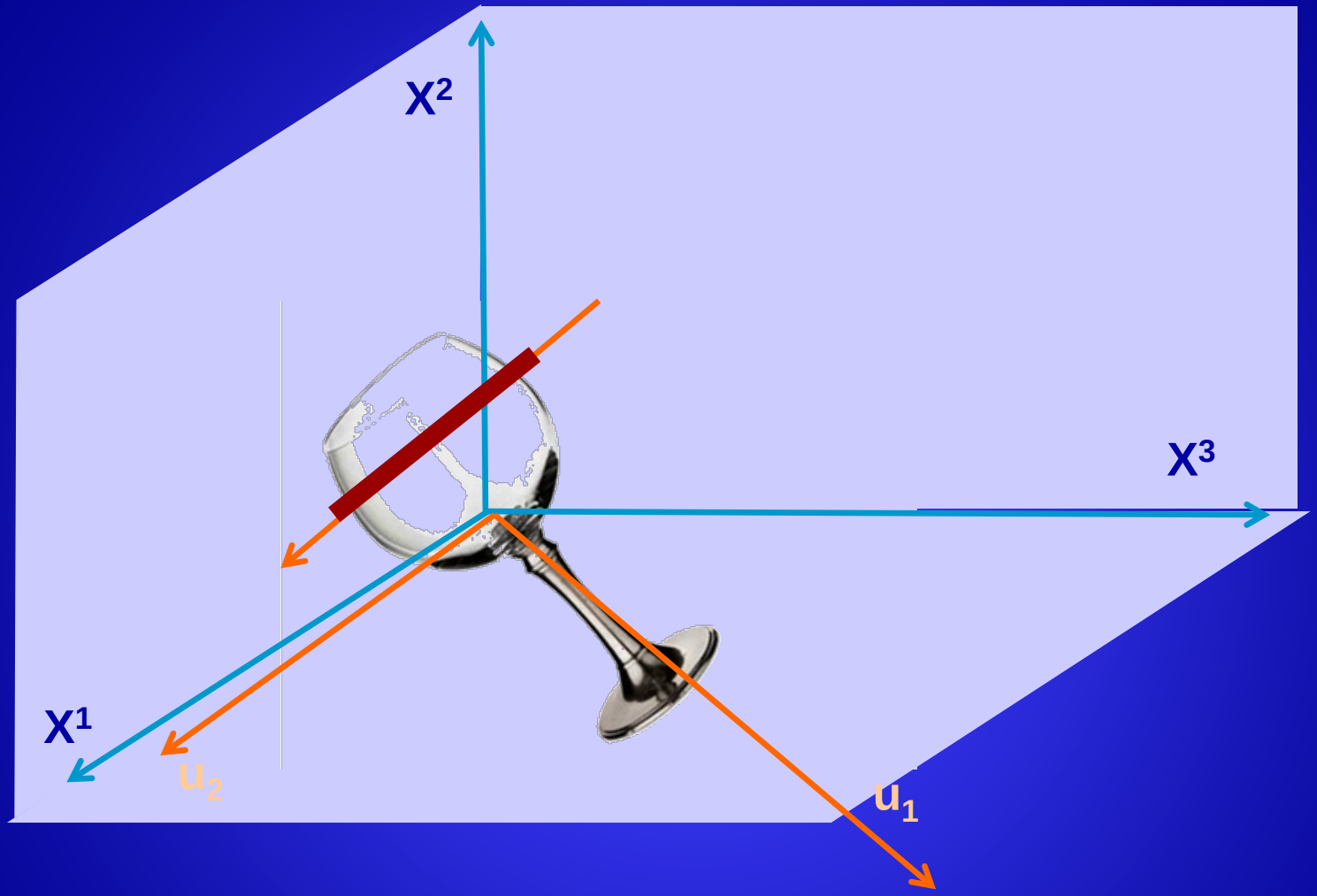

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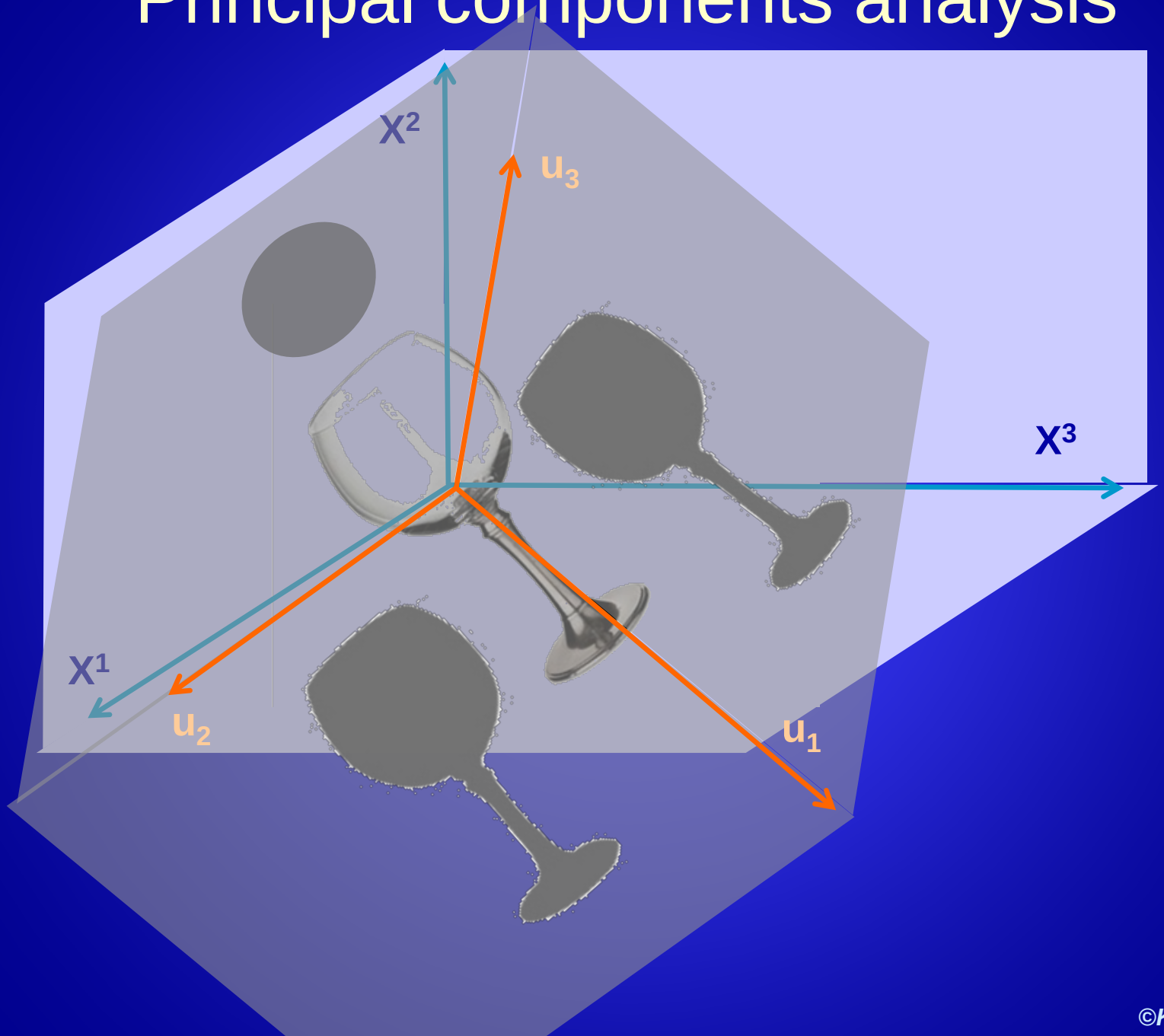

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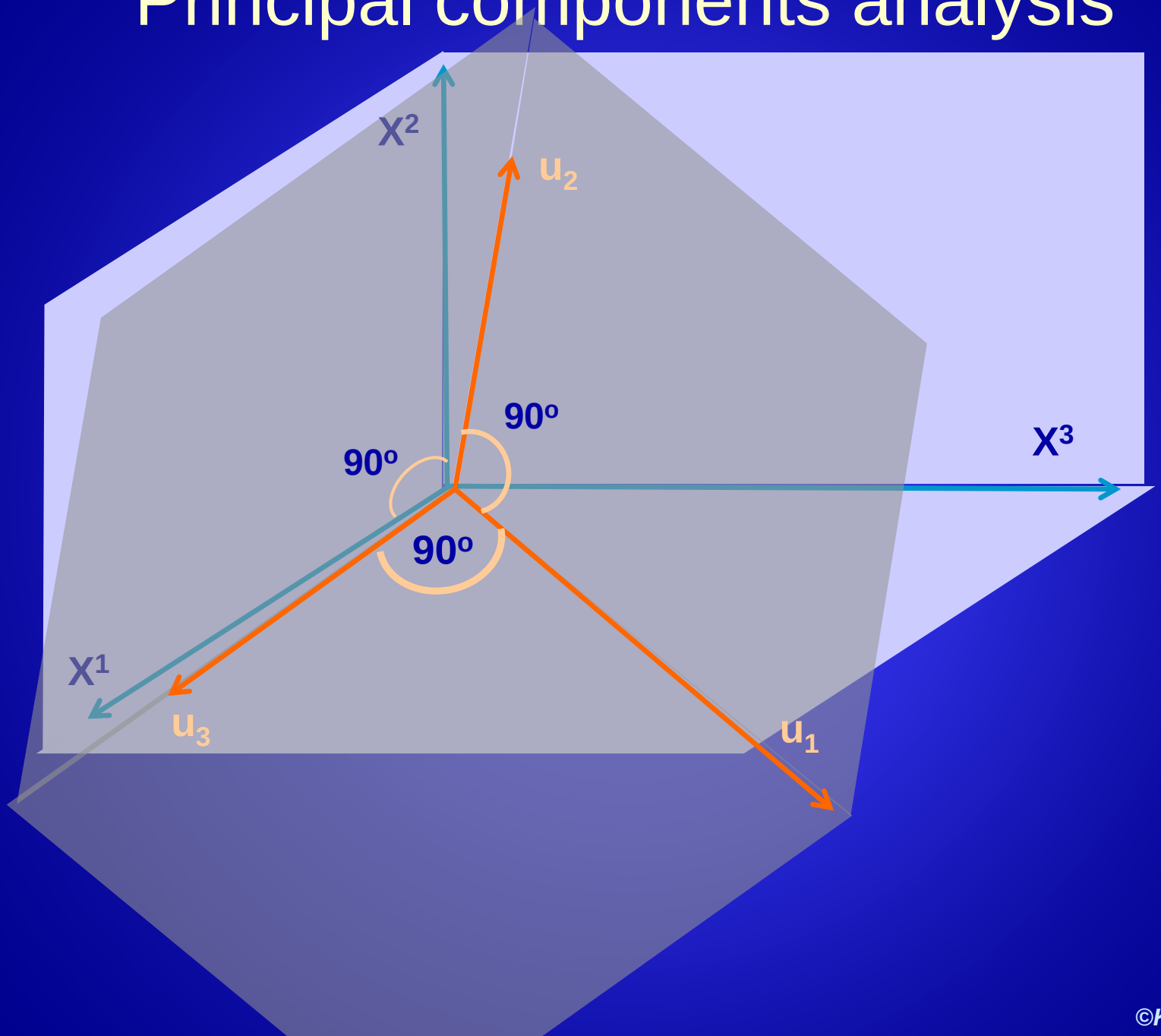

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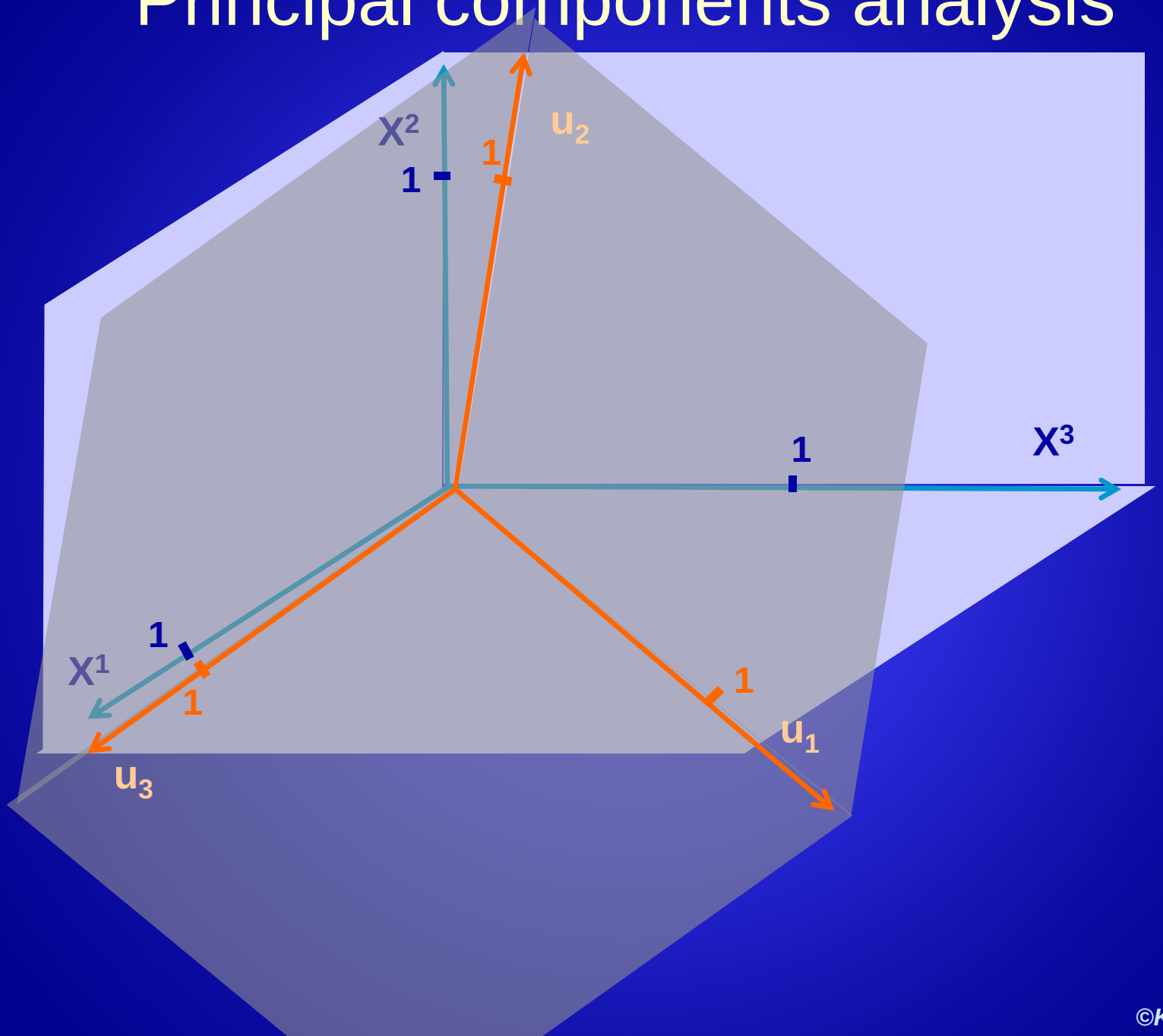

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u^*_α *are the principal factors of X : good rotation directions*

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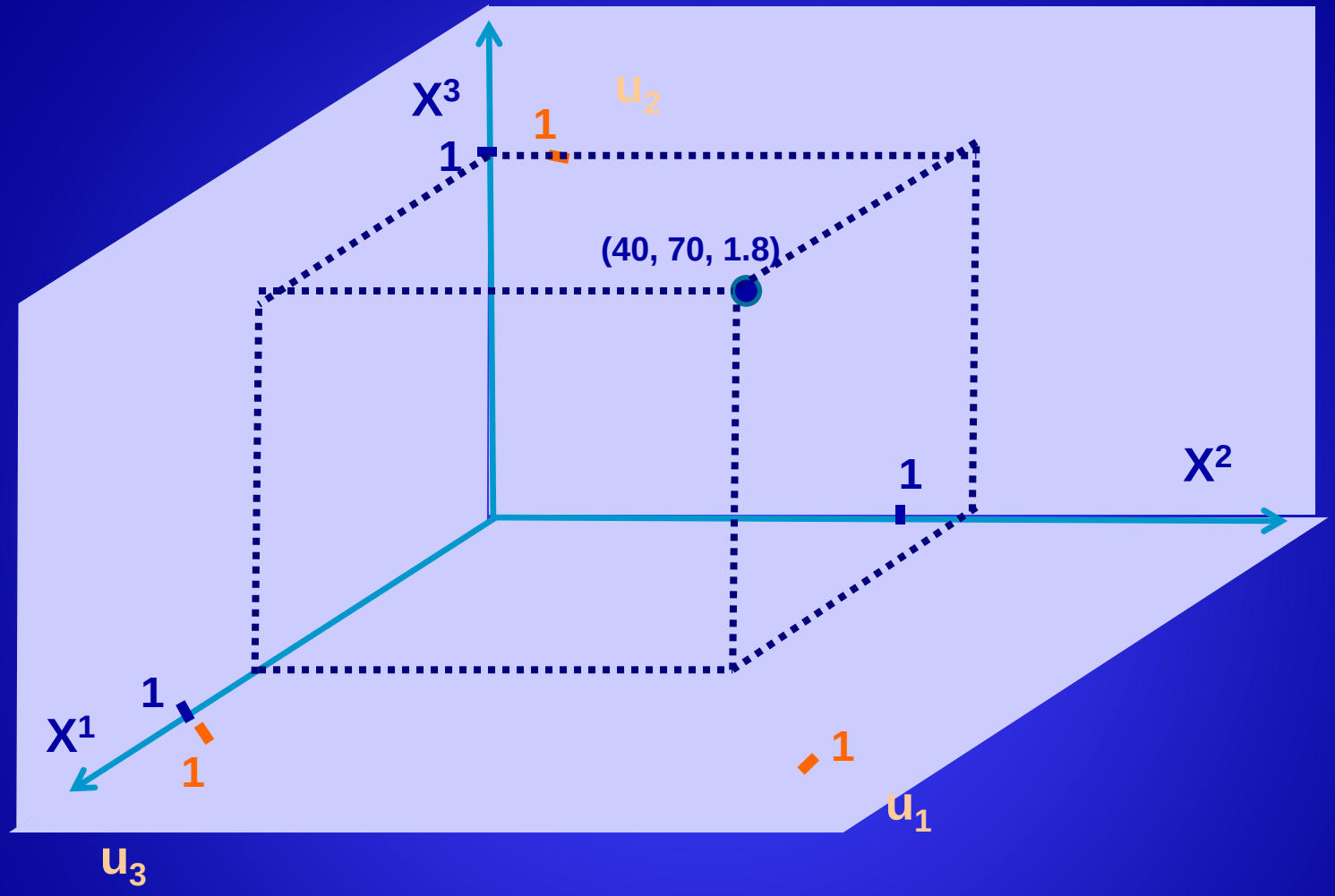
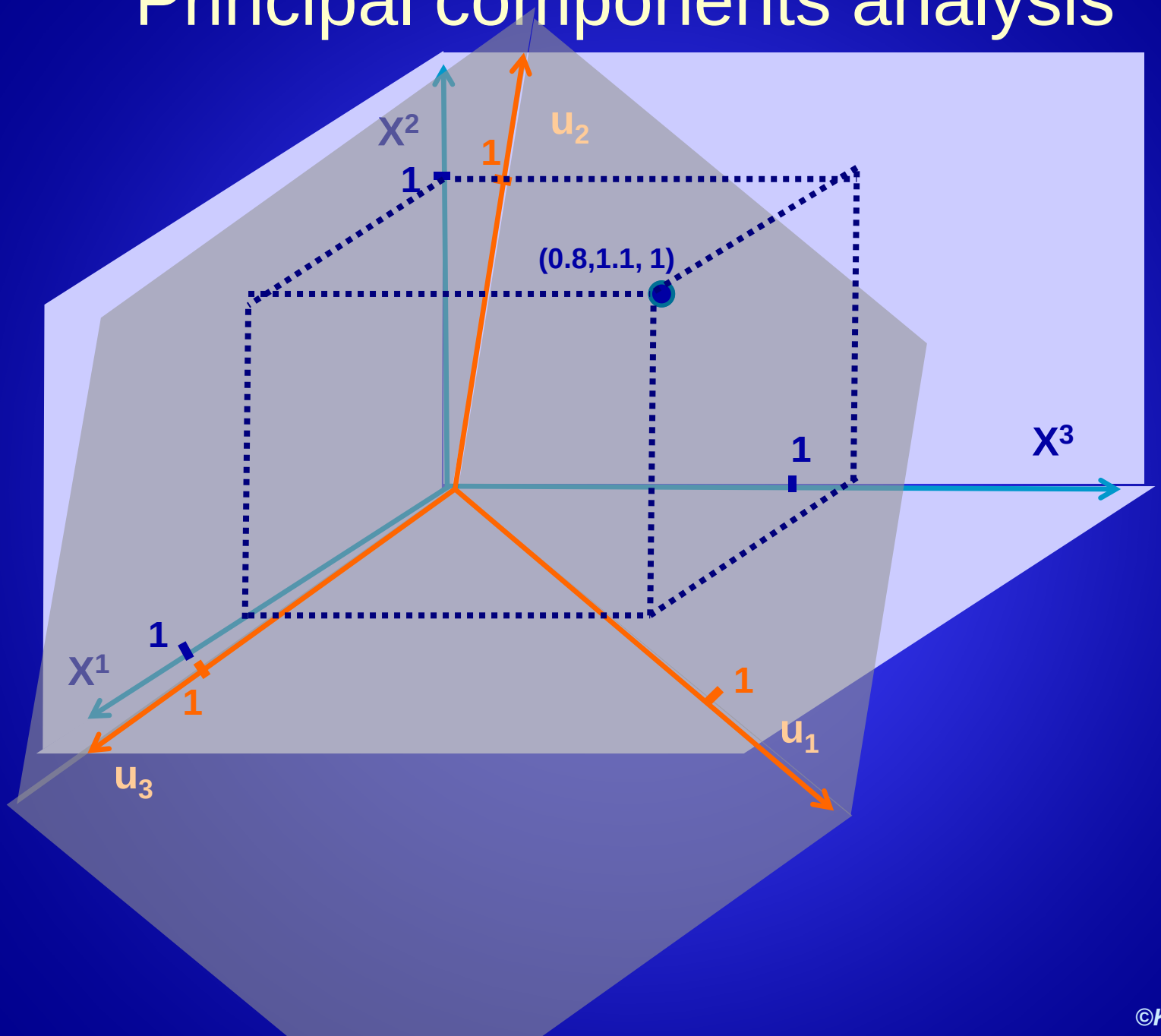
How is i expressed in rotated space?

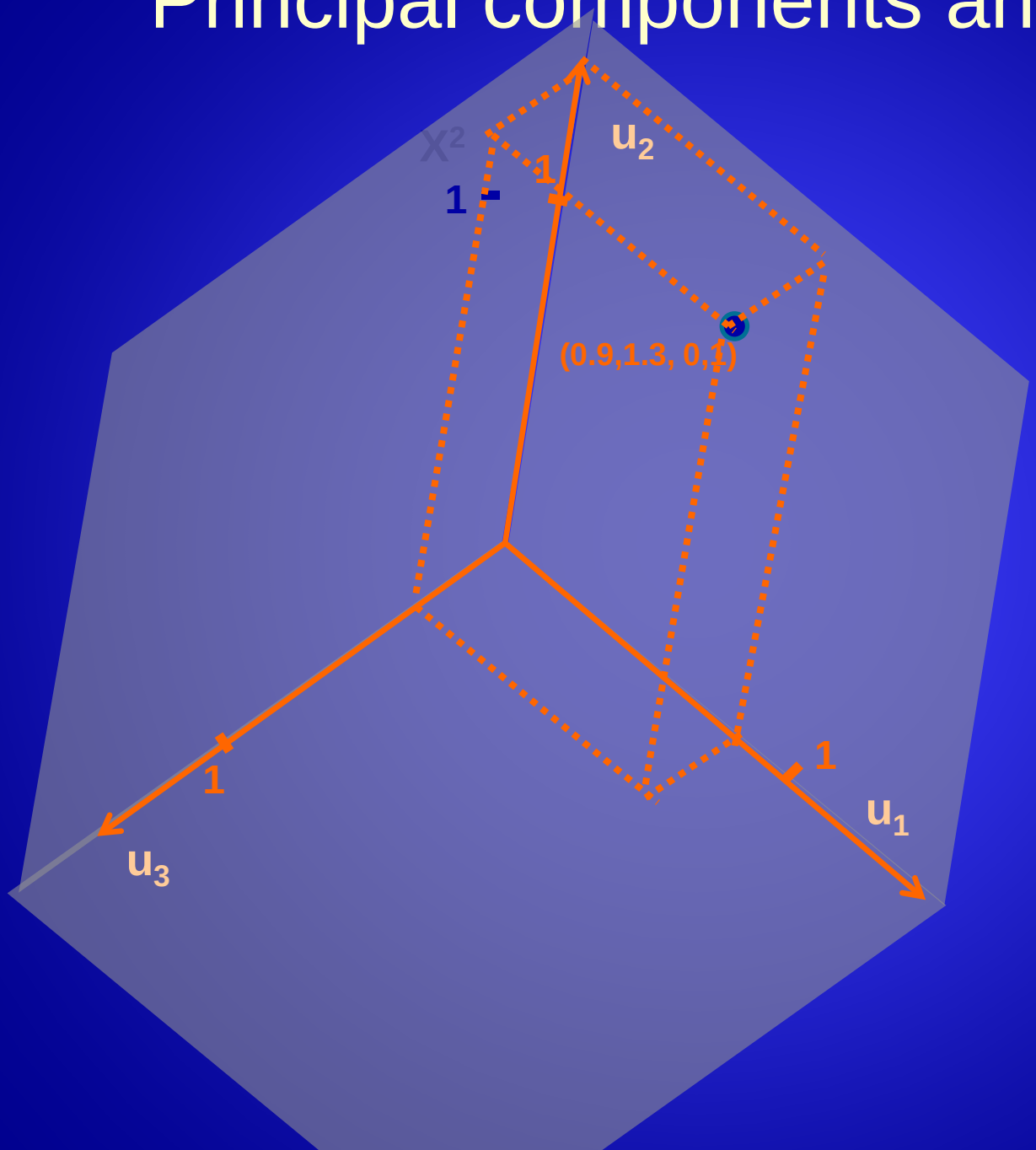
Principal components analysis



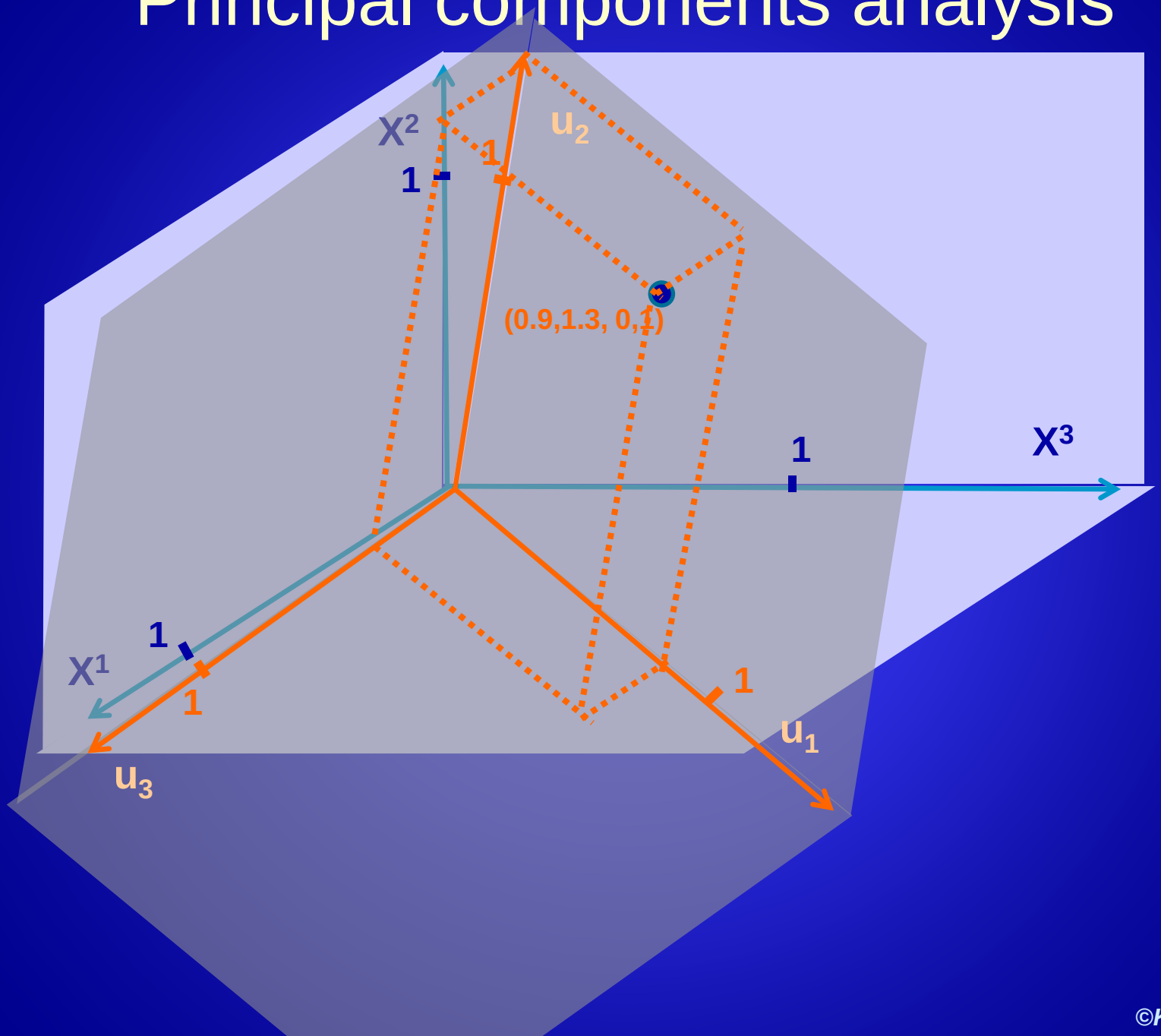
Principal components analysis



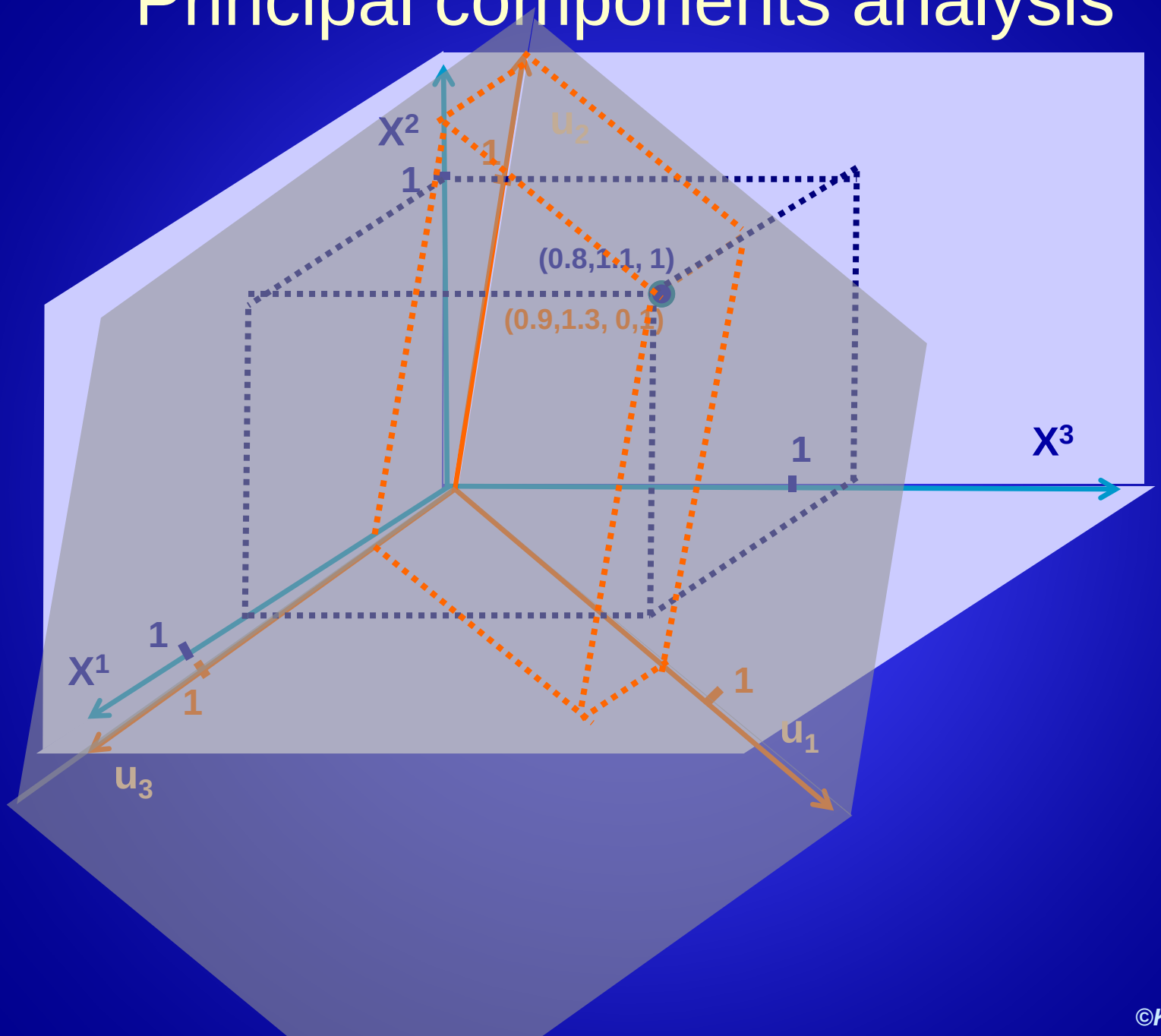
Principal components analysis



Principal components analysis



Principal components analysis



Factorial Methods

- Given <X,M,D>

Can we find coordinates in rotated space from original ones?

The projection matrix $P = U^*_k U^{*\prime}_k M$

Projection of a single individual: $Pr(i) = U^*_k U^{*\prime}_k M x_i$

Projection of all individuals: $Pr(X) = U^*_k U^{*\prime}_k M X'$

Get a matrix with projections in ROWS: $Pr(X)' = XMU^*_k U^{*\prime}_k$

Projections expressed in original vectorial space

The best possible projection over k dimensions

Factorial Methods

- Given <X,M,D>

Matrix $XMU^*_k U^{*\prime}_k$ *provides the best possible k-projection of X*

Silver-Smidth norm: $||X||^2_{MD} = \sum_{\alpha=1}^{r} \lambda_\alpha$

Measures variability, information contained in X

Property: $||XMU^*_k U^{*\prime}_k||^2_{MD} = ||X||^2_{MD}$

Any other k-projection of X

- Provides smallest values of Silver-Smidth norm
- Has less variability
- Keeps smallest information from X

Factorial Methods

- Given <X,M,D>

Diagonalize covariances matrix (with centered data)

eigenvectors $u_\alpha = (u_{\alpha 1} \ldots u_{\alpha p})$ *(direction of factor α, α =1:p)*

$u_{p\,\alpha}$: contribution of variable p to the factor α

$(u_1 \ldots\ldots\ldots u_k)$ *ortonormal*

eigen values λ_k *(quantity of information converved by factor k)*

(Projected inertia)

$\{\lambda_\alpha\}_{\alpha=1:r} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \geq \lambda_r$

$\Sigma_{\forall \alpha} \lambda_\alpha$ = Total inertia of X (information in data)

Factorial Methods

- Given <X,M,D>

eigenvectors $u_\alpha = (u_{1\alpha} \ldots u_{p\alpha})$ *(direction of factor k)*

$u_{p\alpha}$: contribution of variable p to the factor α

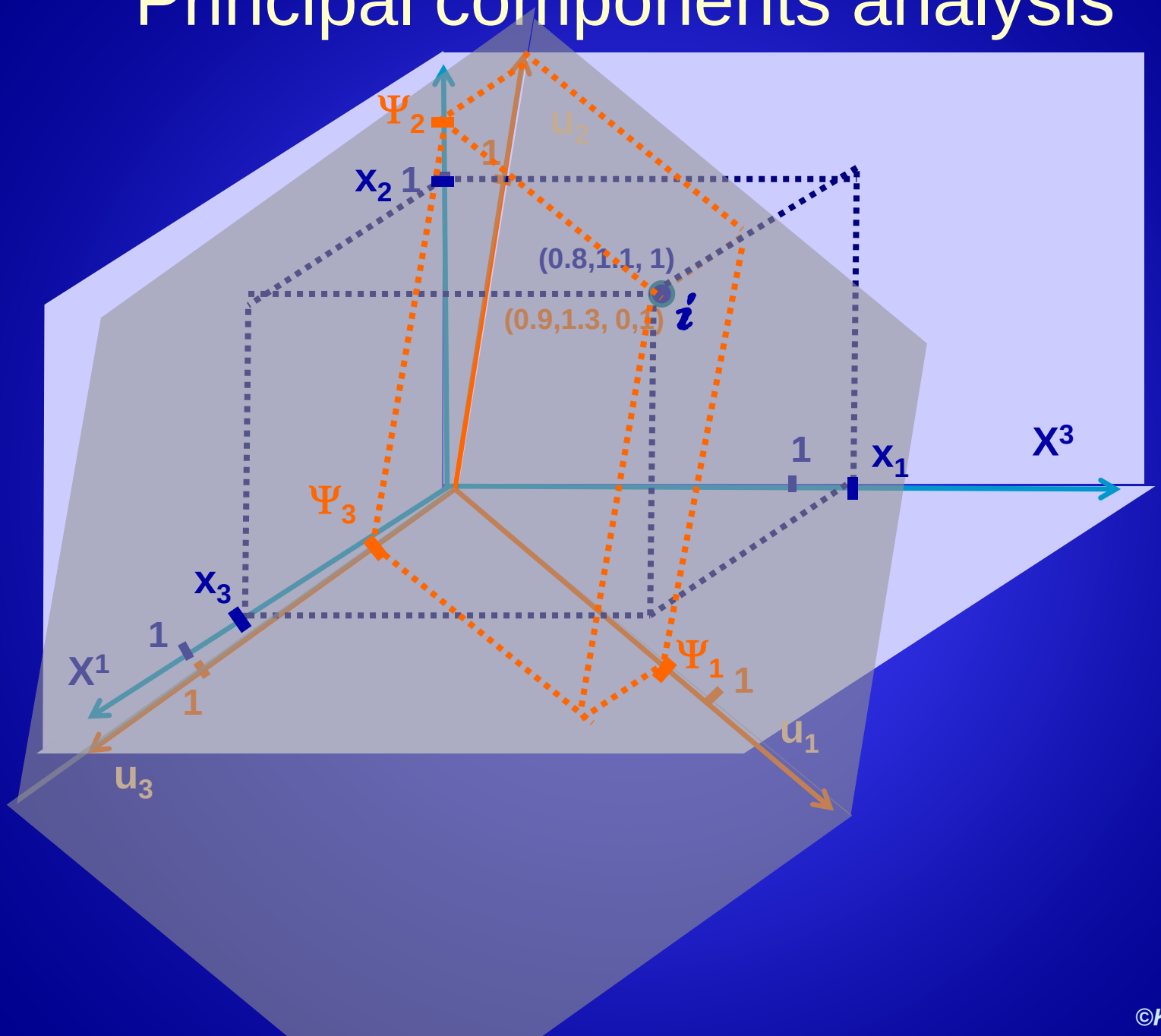
eigen values λ_α *(quantity of information converved by factor* α*)*

(Projected inertia)

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \geq \lambda_r$

$\Sigma_{\forall \alpha} \ \lambda_\alpha$ = Total inertia of X (information of data)

Principal components analysis



Factorial Methods

- Given <X,M,D>

$$i = (x_{1i}, \ldots, x_{pi})$$

Points in projected space: $i = (\Psi_{1i}, \ldots, \Psi_{\alpha i}, .., \Psi_{ri})$ (often r=p)

$$\Psi_{\alpha i} = x_{1i}u_{1\alpha} + x_{2i}u_{2\alpha} + \cdots + x_{pi}u_{p\alpha} \qquad \psi_\alpha = Xu_\alpha$$

Then $\Psi_\alpha^{'} D\Psi_\alpha = \lambda_\alpha$

Illustrative points z also projectable

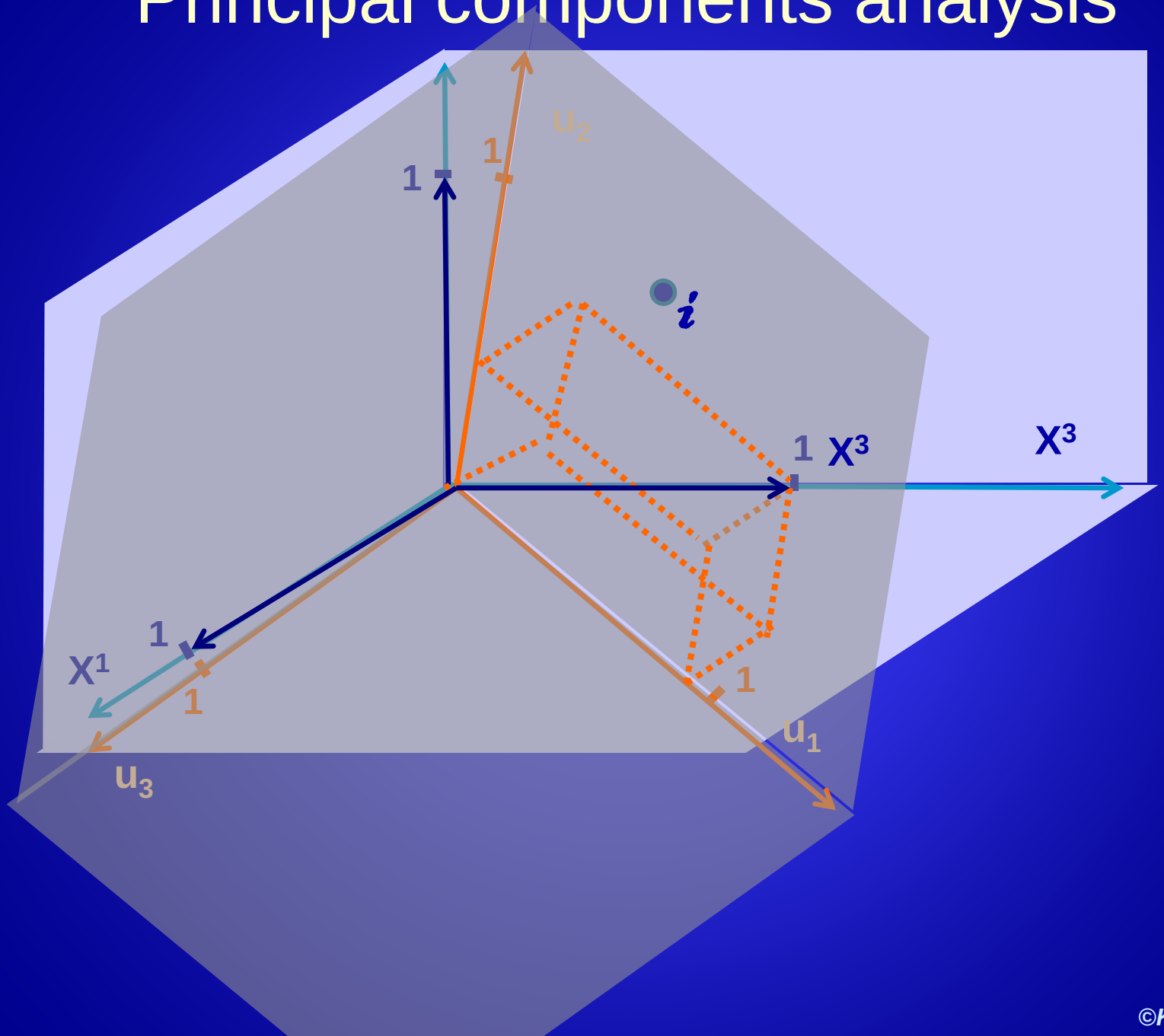
$$\Psi_{\alpha z} = x_{1z}u_{1\alpha} + x_{2z}u_{2\alpha} + \cdots + x_{pz}u_{p\alpha}$$

Factors are linear combinations of original variables

Original variables project as VECTORS over factorial space

angle and lenght important

Principal components analysis



Factorial Methods

- Principal Components Analysis
  - Output: K factors rotating original X variables
  - Factors: Linear combinations of original variables

Several uses:

- As an associative data mining method to analyze relationships among variables
  Project variables and modalities and find associations

- As a preprocessing method for elicitation of latent variables
  Project active and illustrative variables/individuals on first/second factorial plane and interpret factors (find latent variables)

- As a preprocessing method for multidimensionality reduction

  Select more informative factors $\kappa << p$ *(accumulate 80% inertia)*
  Reduce data matrix to selected factors
  Alternative, keep variables mainly contributing to selected factors (smaller angles with factorial axis)

Factorial Methods

- Given <X,M,D>

Diagonalize Covariance matrix X'DX

Get r eigen values λ_α *and sort decreasingly*

$$\{\lambda_\alpha\}_{\alpha=1:r} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots\ldots \geq \lambda_r$$

Corresponding eigenvectors $u_\alpha = (u_{\alpha 1} \ldots\ldots u_{\alpha p})$

for $M = \mathbb{I}_p$: $u^*_\alpha = u_\alpha$; *for* $M \neq \mathbb{I}_p$: $u^*_\alpha = M^{-1/2} u_\alpha$

$\{u^*_\alpha\}_{\alpha=1:r}$ *orthonormal base for individuals*

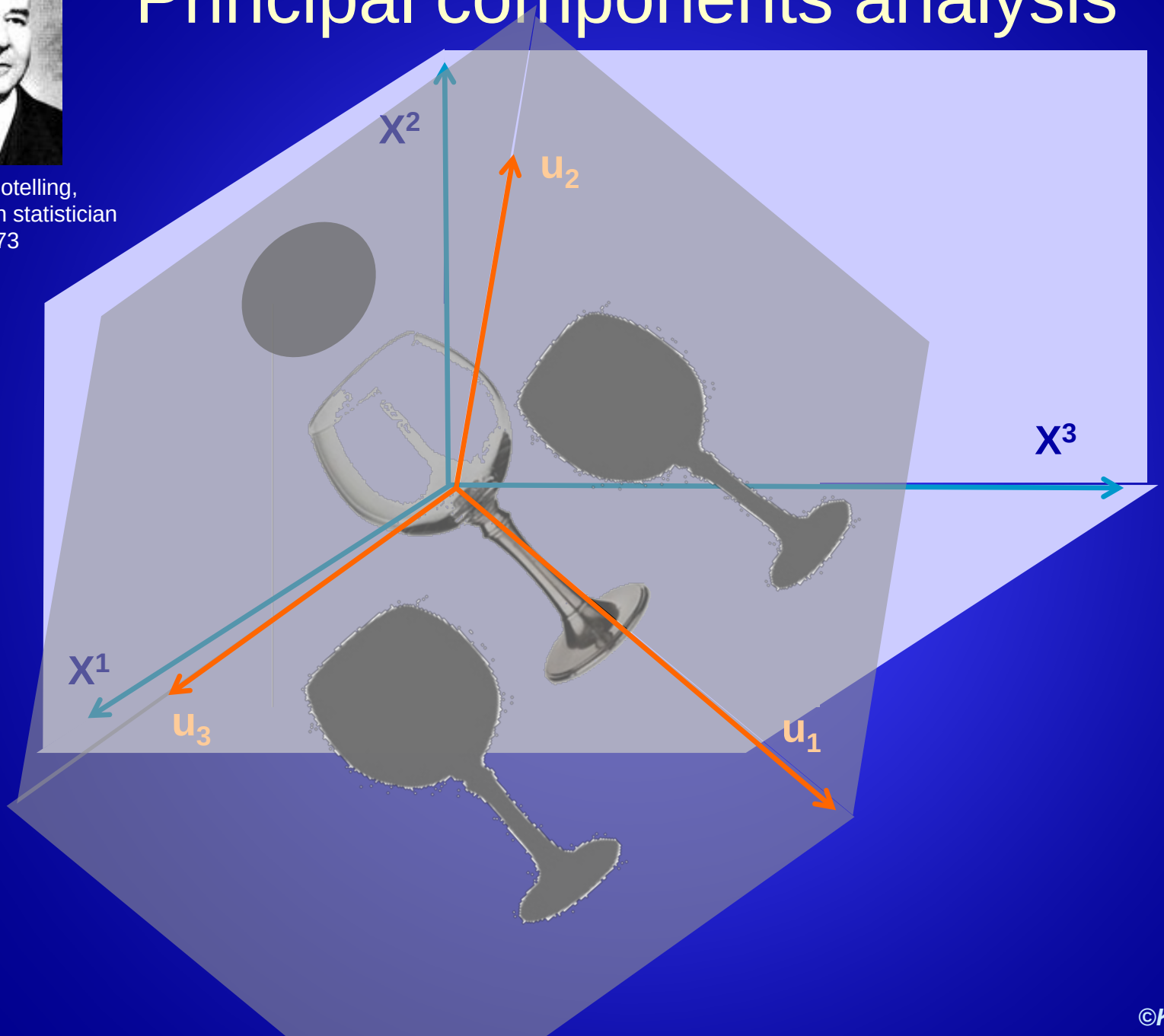
u^*_α *are the principal factors of X : good rotation directions*

$U^* = ([u^*_1]\ [u^*_2] \ldots\ldots\ [u^*_r])$ *is the basis for the projection space*

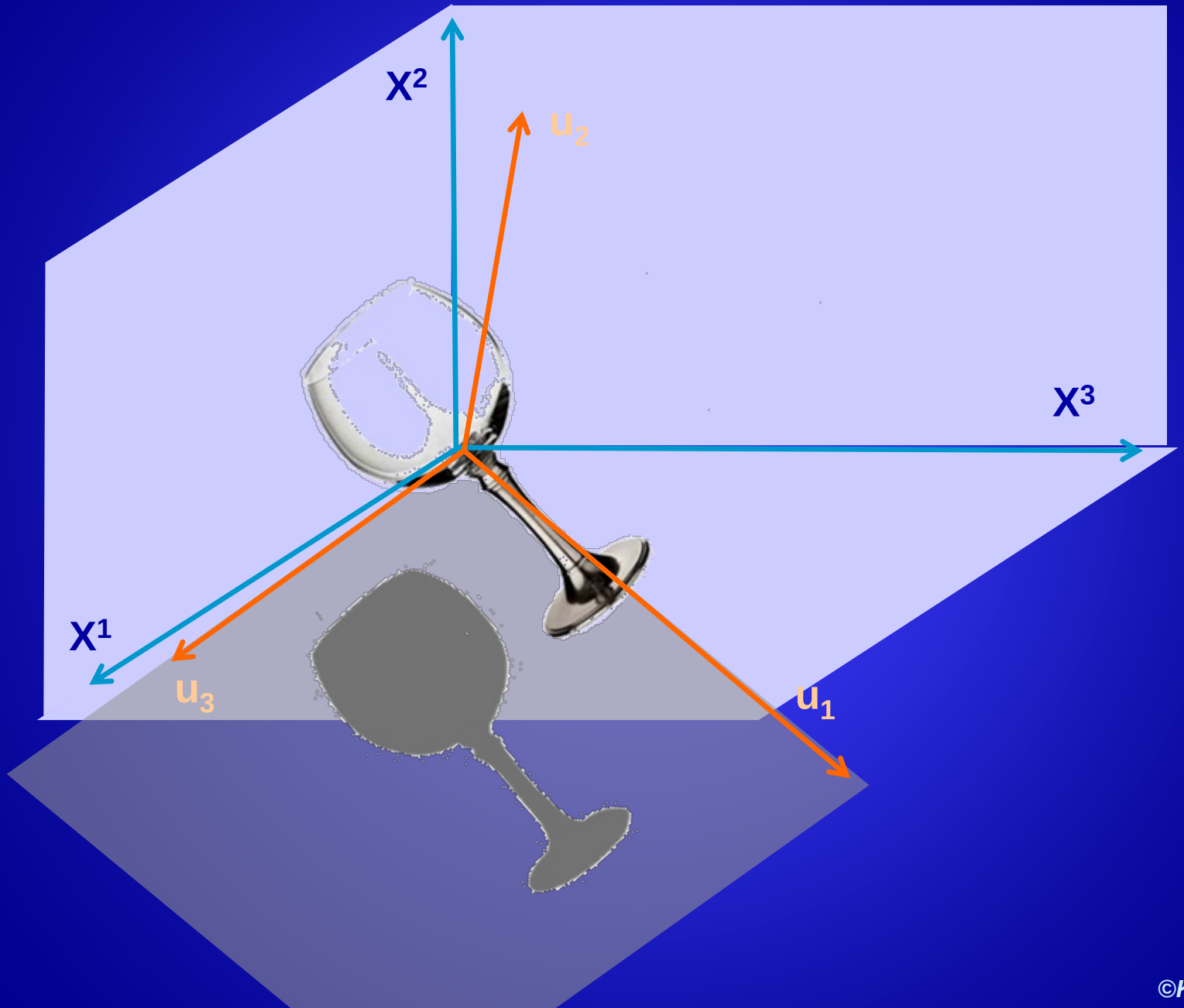
$U^*_k = ([u^*_1]\ [u^*_2] \ldots\ldots\ [u^*k])$ *is the basis for projecting in first k dimensions(k<r)*

Principal components analysis

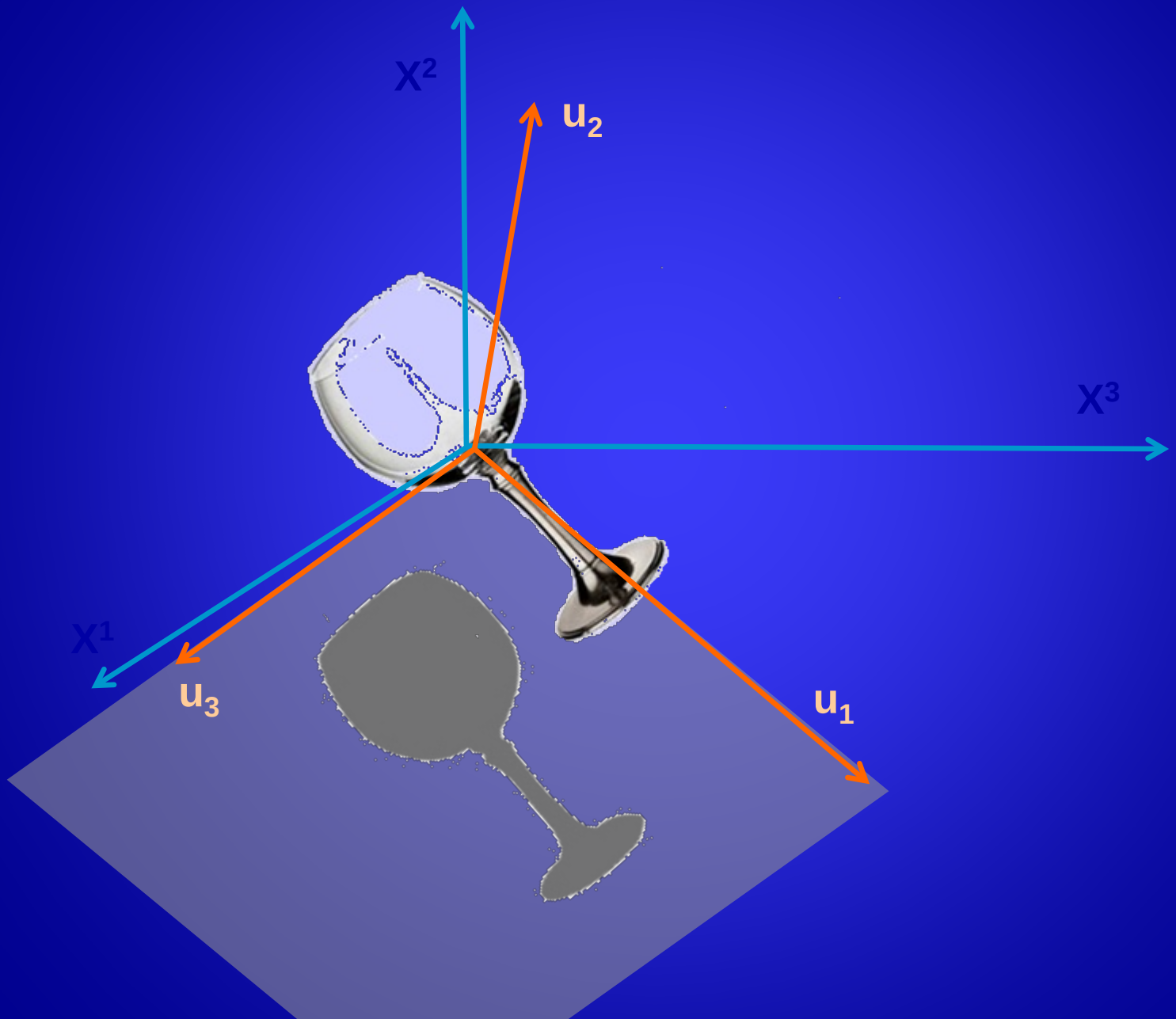
Harold Hotelling,
American statistician
1895-1973



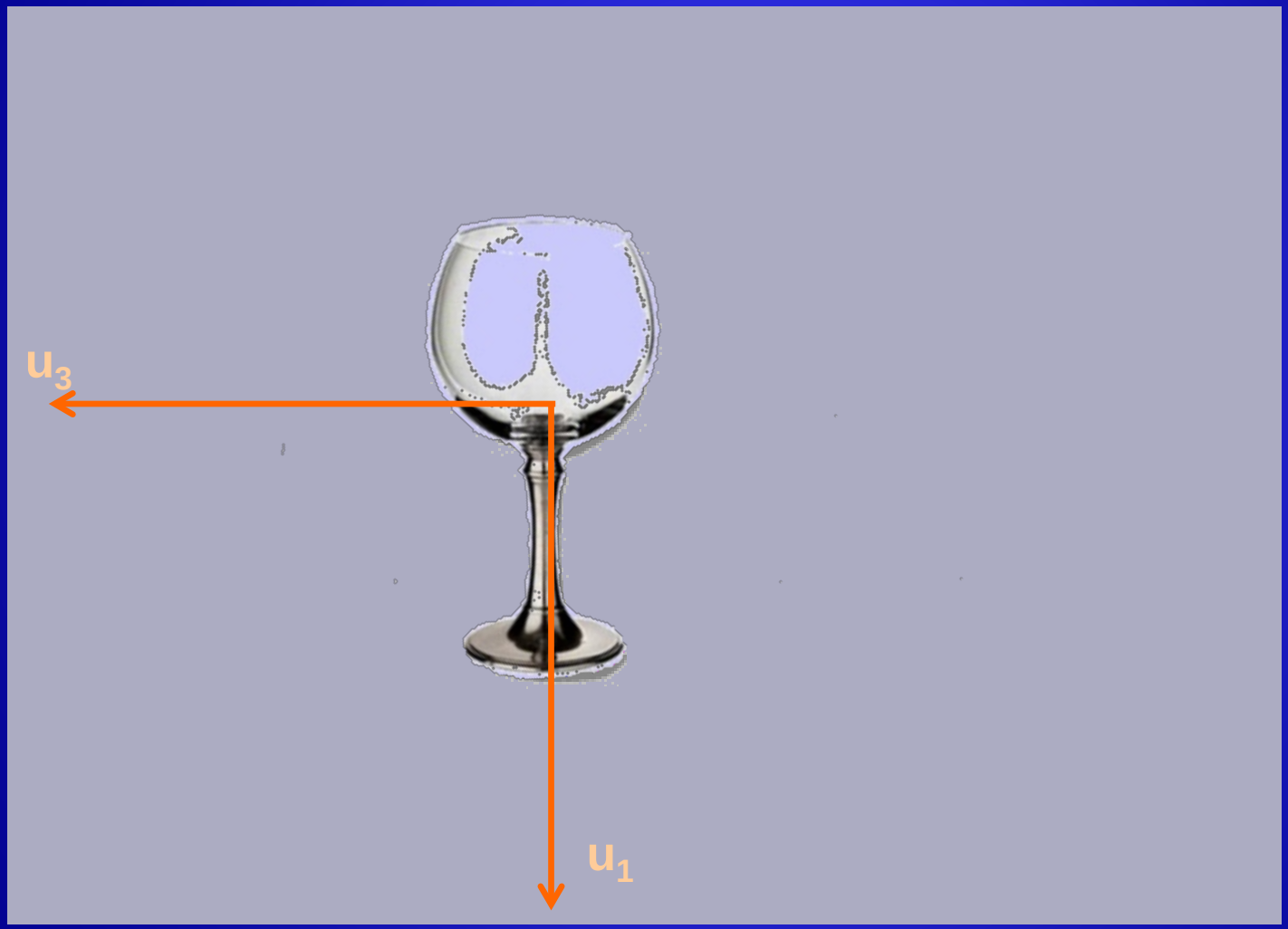
Principal components analysis



Principal components analysis



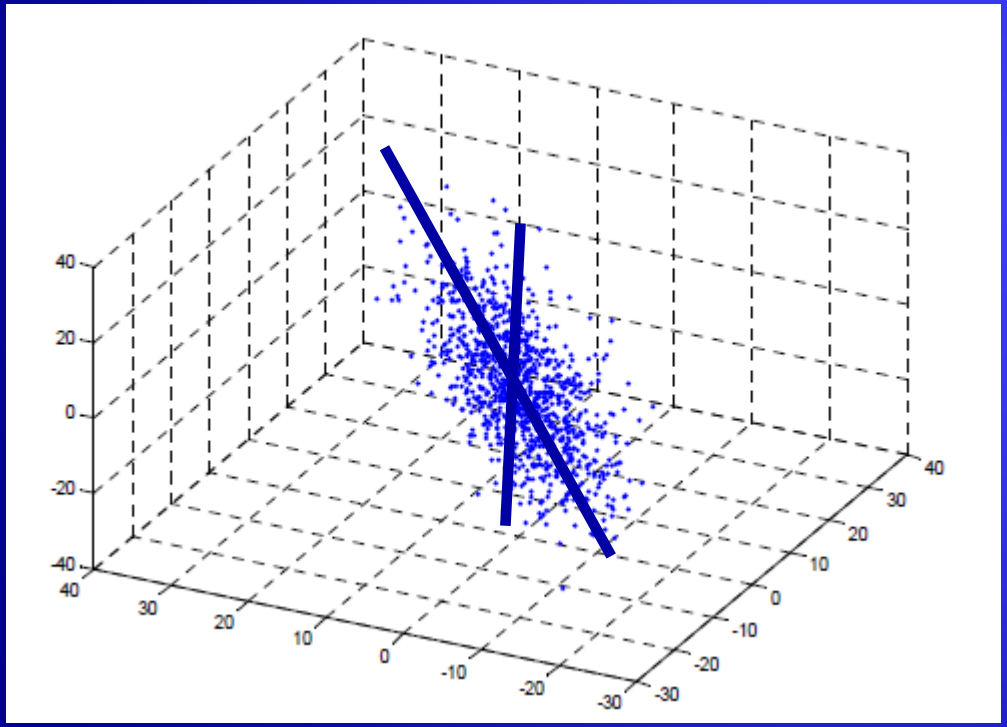
Principal components analysis

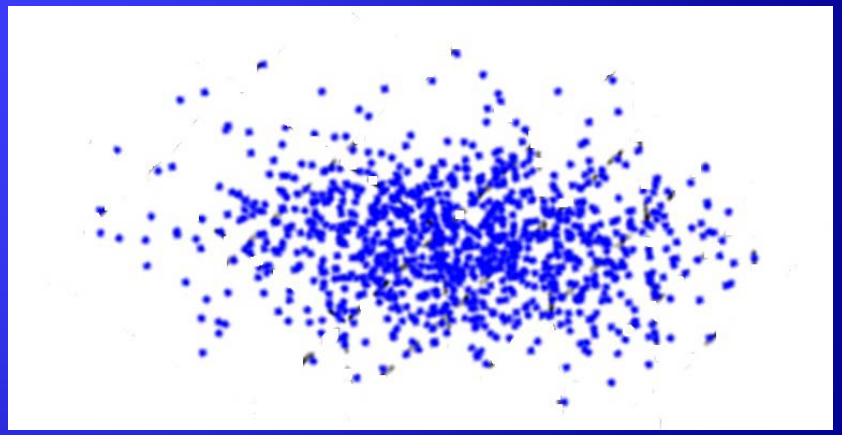


Principal components analysis

- Find the most informative projection planes of data cloud

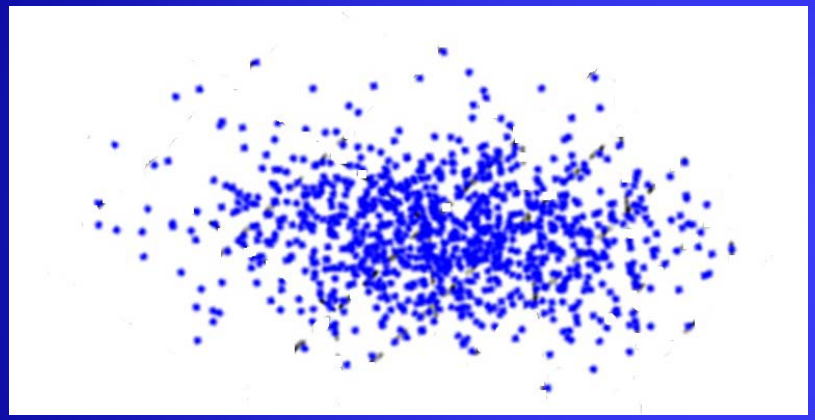
  (factorial planes)

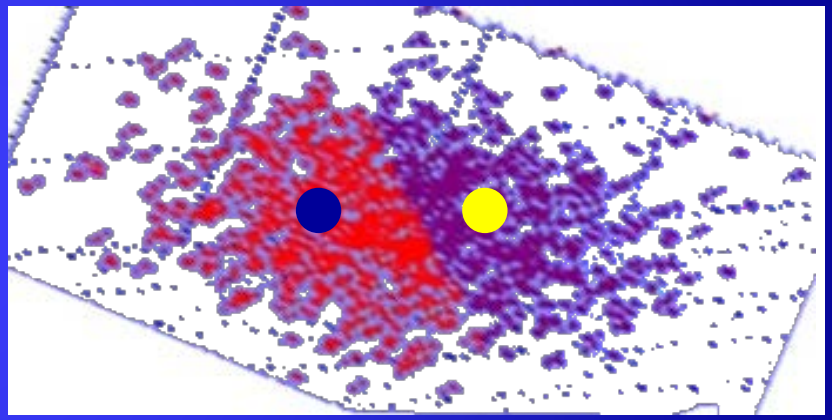




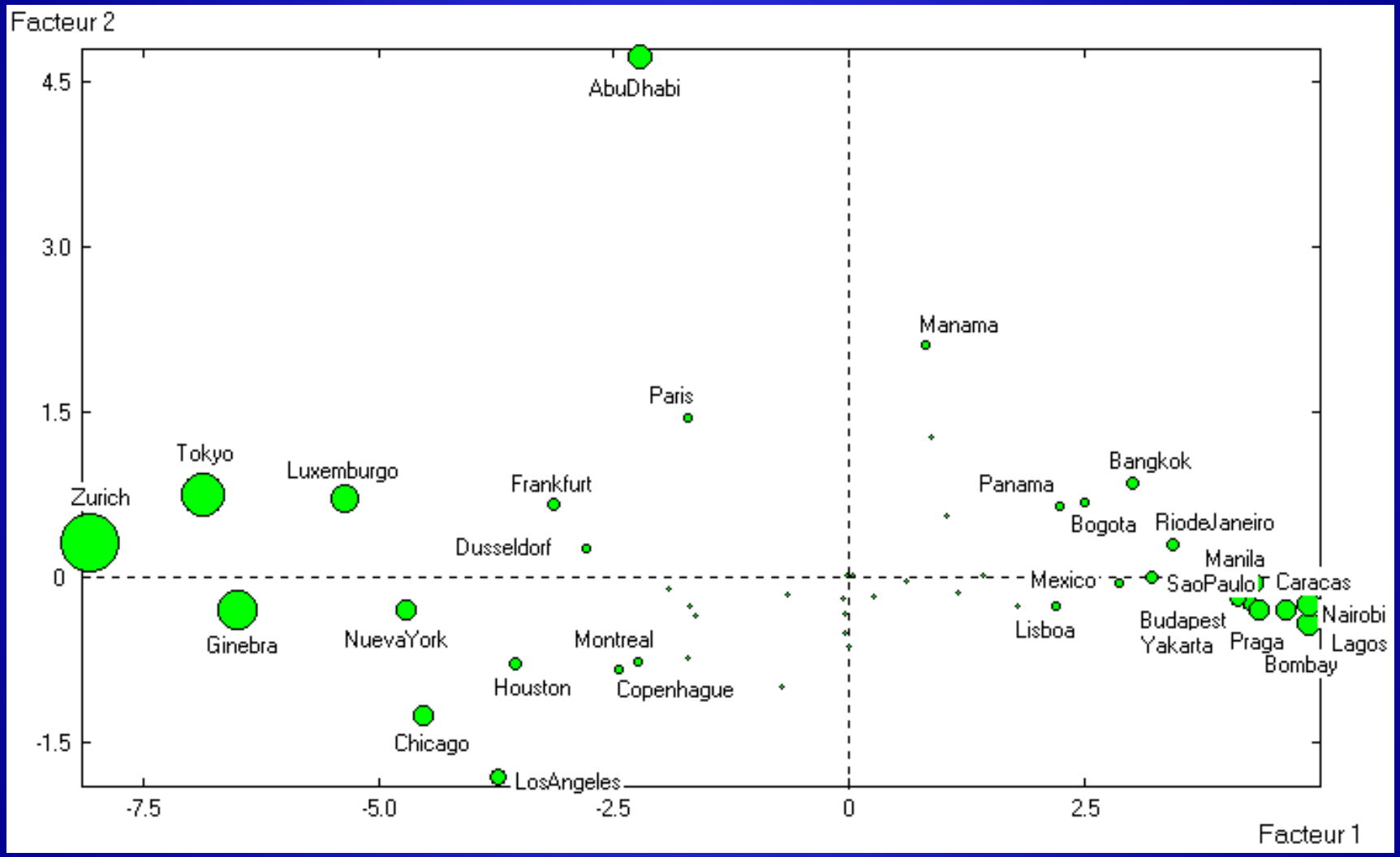
Principal components analysis

- Introduce qualitative information (projecting modalities)

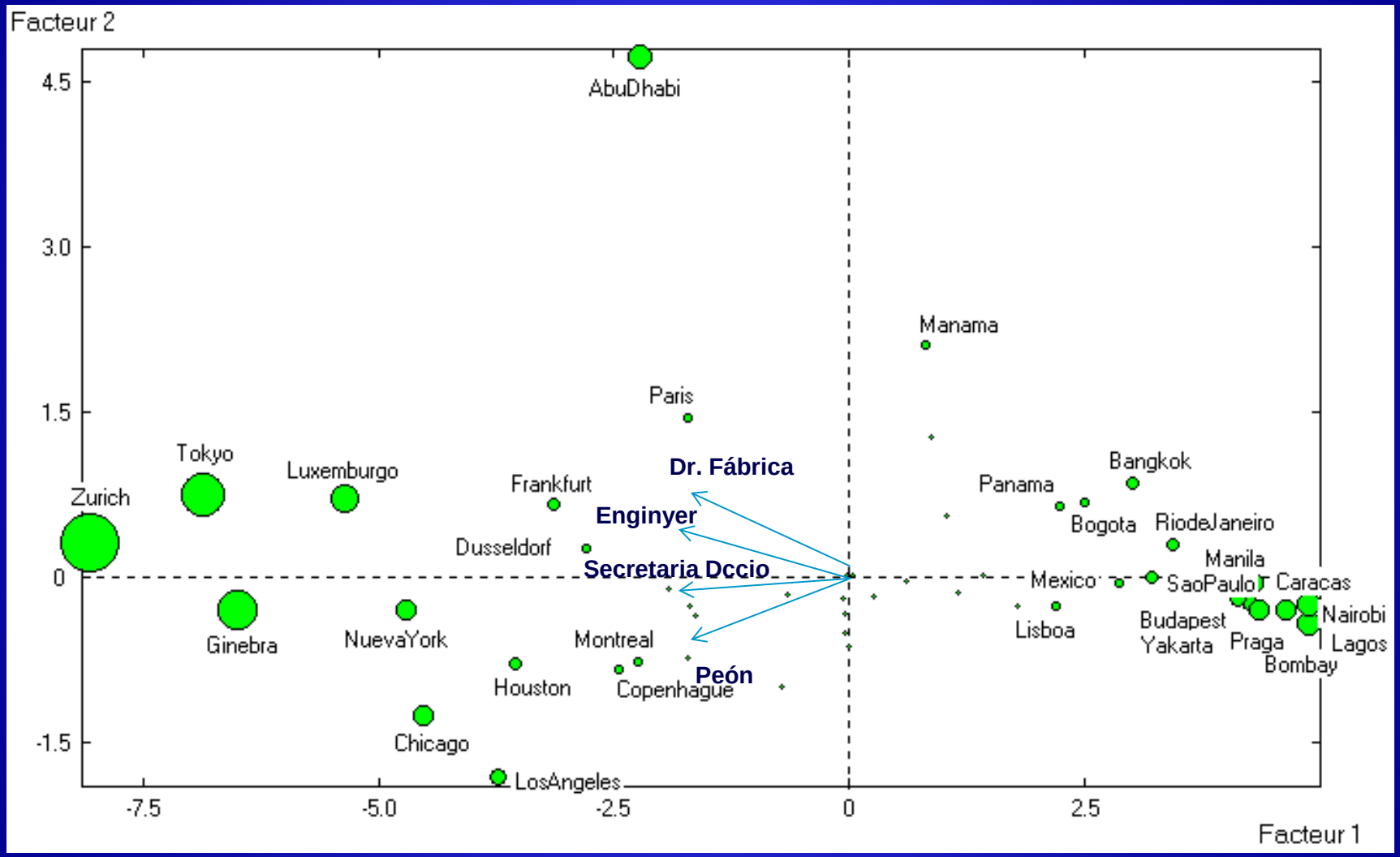


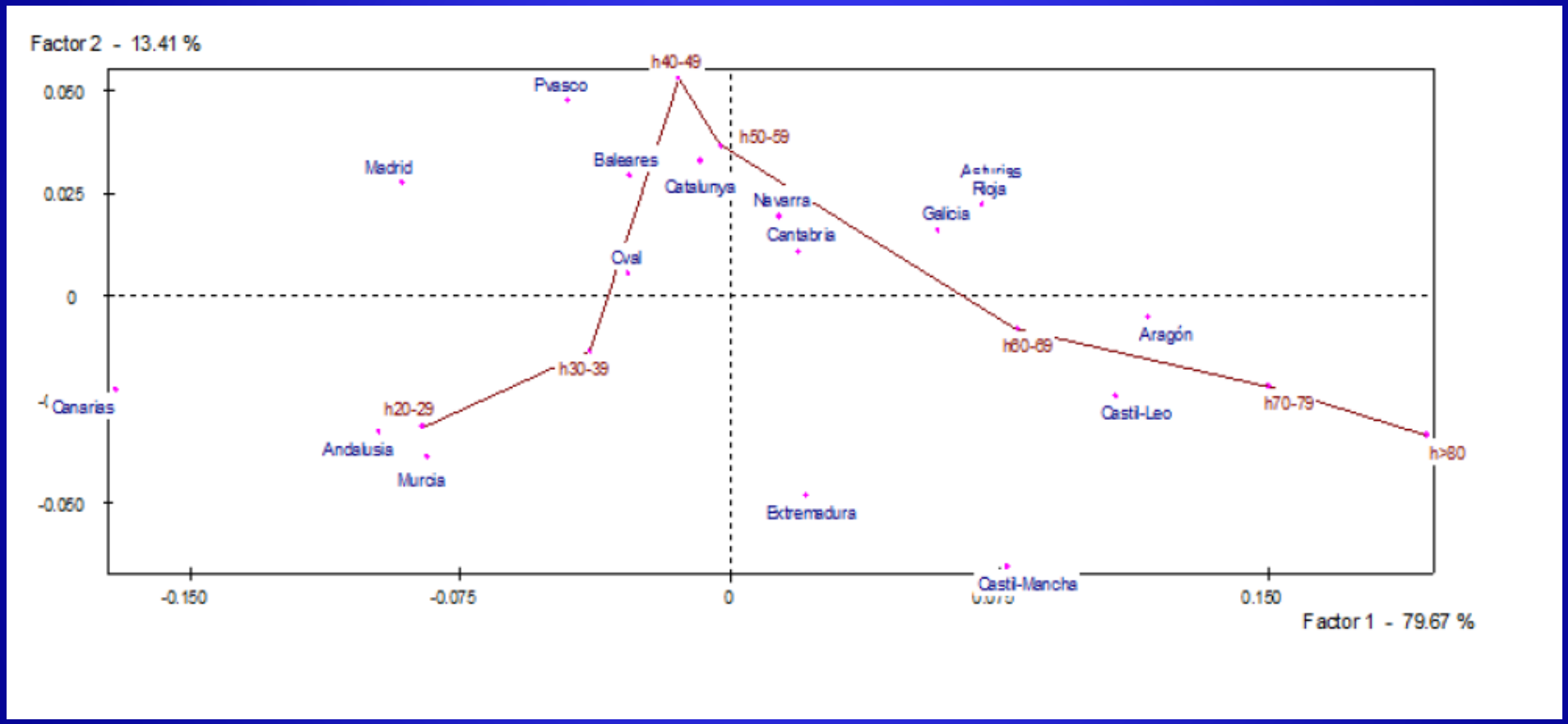


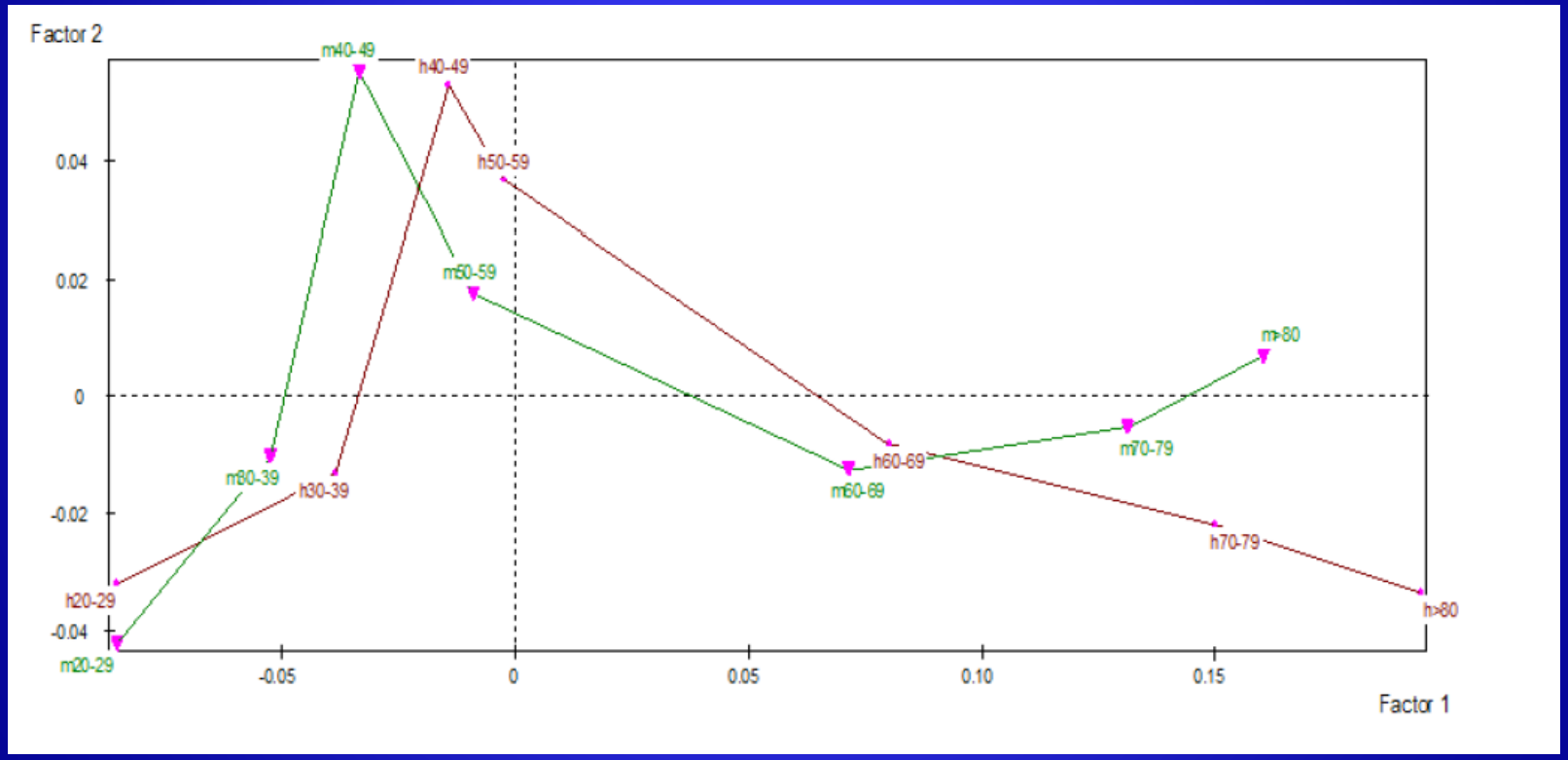
Visualisation of international cities according their salaries. USB 1994.



Visualisation of international cities according their salaries. USB 1994.




Factor 2 - 13.41 %
0.050
0.025
0
-0.050
h40-49
Pvasco
h50-59
Madrid
Baleares
Catalunya
Navarra
Asturias
Rioja
Galicia
Cantabria
Oval
h60-69
Aragón
h30-39
Canarias
h20-29
h70-79
Castil-Leo
Andalusia
h>80
Murcia
Extremadura
Castil-Mancha
-0.150
-0.075
0
0.150
Factor 1 - 79.67 %


Factor 2
m40-49
h40-49
h50-59
0.04
0.02
m50-59
0
m>80
m70-79
m30-39
h60-69
m60-69
-0.02
h70-79
h20-29
h>80
-0.04
m20-29
-0.05
0
0.05
0.10
0.15
Factor 1