BOLETÍN 2

(8) Sea $X_1,\dots,X_n$ una muestra aleatoria simple de una distribución con densidad f_θ. Encuentra el estimador de máxima verosimilitud de θ en los siguientes casos:

a) $f_\theta(x) = \theta(1-x)^{\theta-1} I_{(0,1)}(x) \quad \theta > 1$

$$f_\theta(x) = \begin{cases} \theta(1-x)^{\theta-1} & 0 \le x \le 1 \\ 0 & \text{en otro caso} \end{cases}$$

Se corresponde con una Beta $p=1$ $q=\theta$
(Para identificar fijarnos en la región de ;)

$$E(X) = \int_0^1 \theta x(1-x)^{\theta-1}\,dx \underset{\substack{\uparrow \\ E(X) = \frac{p}{p+q} \\ X\in\beta(p,q)}}{=} \frac{1}{1+\theta}$$

$$\bar{X} = \frac{1}{1+\hat{\theta}} \iff \bar{X} + \hat{\theta}\bar{X} = 1 \iff \hat{\theta} = \frac{1-\bar{X}}{\bar{X}} = g(\bar{X})$$

MÁXIMA VEROSIMILITUD

$$f_\theta(x_1,\dots,x_n) = \theta^n(1-x_1)^{\theta-1}\cdots(1-x_n)^{\theta-1}$$

$$\mathcal{L}(\theta) = f_\theta(x_1,\dots,x_n) = \theta^n \prod_{i\in I}(1-x_i)^{\theta-1}$$

$$\ell(\theta) = \log(\mathcal{L}(\theta)) = \log(\theta^n \prod(1-x_i)^{\theta-1}) = \log(\theta^n) + \log(\prod(1-x_i)^{\theta-1})$$

$$= n\log(\theta) + \sum \log(1-x_i)^{\theta-1} = n\log(\theta) + (\theta-1)\sum_{i=1}^n \log(1-x_i)$$

$$\ell'(\theta) = n\frac{1}{\theta} + \sum_{i=1}^n \log(1-x_i)$$

$$\ell'(\theta) = 0 \iff \frac{n}{\theta} + \sum \log(1-x_i) = 0 \iff \theta = \frac{-n}{\sum\log(1-x_i)}$$

$$\hat{\theta} = \frac{-n}{\sum \log(1-x_i)} \qquad \begin{matrix} 0 \le x_i \le 1 \\ 0 \le 1-x_i \le n \\ \log(1-x_i) < 0 \end{matrix} \qquad ¿\hat{\theta} \text{ distribución?}$$

Para calcular la distribución, calculamos la distribución de $-\log(1-x_i)$

$$Y_i = -\log(1-x_i) \qquad \hat{\theta} = \frac{-n}{\sum\log(1-x_i)} = \frac{+n}{Y_1+\dots+Y_n} = \frac{1}{\bar{Y}}$$

$\bar{Y} = -\log(1-x)$

$G(y) = \mathbb{P}(\bar{Y} \le y) = \mathbb{P}(-\log(1-X) \le y) = \mathbb{P}(\log(1-x) \ge -y) =$

$= 1 - \mathbb{P}(\log(1-x) \le -y) = 1 - \mathbb{P}(1-x \le e^{-y}) = 1 - \mathbb{P}(-x \le e^{-y} - 1) =$

$= 1 - \mathbb{P}(x \ge -e^{-y} + 1) = 1 - (1 - \mathbb{P}(X \le 1 - e^{-y})) = \mathbb{P}(X \le 1 - e^{-y}) = F_\theta(1 - e^{-y})$

$g(y) = f_\theta(1-e^{-y})\, e^{-y} = \theta (e^{-y})^{\theta - 1} e^{-y} = \theta e^{-\theta y}$ $\quad \bar{Y} \sim Exp(\theta) = \Gamma(\theta, 1)$.

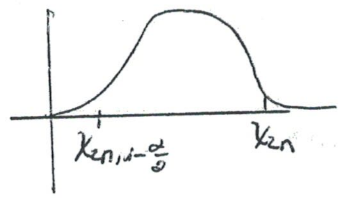
$\sum Y_i \sim \Gamma(\theta, n) \overset{\text{PIVOTE}}{\Longleftrightarrow} \theta \sum Y_i \sim \Gamma(1, n)$

$2\theta \sum Y_i \sim \Gamma\left(\frac{1}{2}, \frac{2n}{2}\right) = \chi^2_{2n}$

$\hat{\theta} = \dfrac{n}{\sum Y_i} \Rightarrow \dfrac{2\theta n}{\hat{\theta}} = 2\theta \sum Y_i$

INTERVALO DE CONFIANZA

$\mathbb{P}\left[\chi^2_{2n,1-\frac{\alpha}{2}} \le \dfrac{2\theta n}{\hat{\theta}} \le \chi^2_{2n,\frac{\alpha}{2}}\right] = 1 - \alpha$



$\mathbb{P}\left[\dfrac{\chi^2_{2n,1-\frac{\alpha}{2}}\,\hat{\theta}}{2n} \le \theta \le \dfrac{\chi^2_{2n,\frac{\alpha}{2}}\,\hat{\theta}}{2n}\right] = 1 - \alpha$

$L = \dfrac{\chi^2_{2n,1-\frac{\alpha}{2}}\,\hat{\theta}}{2n} \qquad U = \dfrac{\chi^2_{2n,\frac{\alpha}{2}}\,\hat{\theta}}{2n}$

e) $f_\theta(x) = \mathbb{I}_{(0,1)}$ si $\theta = 0$ y $f_\theta(x) = (2\sqrt{x})^{-1}\, \mathbb{I}_{(0,1)}(x)$ si $\theta = 1$

$f_\theta(x) = \begin{cases} 1 & \text{si } x \in (0,1) \\ 0 & \text{en otro caso} \end{cases} \qquad f_\theta(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{si } x \in (0,1) \\ 0 & \text{en otro caso} \end{cases}$

$\mathcal{L}(\theta) = \prod f_\theta(x_i)$

$x_1 = 0'96 \quad x_2 = 0'38 \quad x_3 = 0'6 \rightarrow$ $\mathcal{L}(0) = 1$

$\mathcal{L}(1) = \dfrac{1}{8\sqrt{x_1}\sqrt{x_2}\sqrt{x_3}} = 0'27 \Rightarrow \hat{\theta}(x_1, x_2, x_3) = 0$

En general

$\mathcal{L}(0) = 1 \quad \mathcal{L}(1) = \dfrac{1}{2^n \prod \sqrt{x_i}} \quad x_1, \dots x_n \in (0,1)$

$\hat{\theta}(x_1, \dots, x_n) = \begin{cases} 0 & \text{si } \mathcal{L}(0) \ge \mathcal{L}(1) \\ 1 & \text{en otro caso} \end{cases}$

d) $f_\theta(x) = \sigma^{-1} e^{-(x-\mu)/\sigma}\, \mathbb{I}_{(\mu,+\infty)}(x) \quad \theta = (\mu,\sigma) \in \mathbb{R}\times(0,+\infty)$

$$f_\theta(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} & x \geq \mu \\ 0 & \text{en otro caso} \end{cases} \qquad \theta = (\mu,\sigma)\in\mathbb{R}\times(0,+\infty)$$

$$\mathcal{L}(\theta) = \prod f_\theta(x_i) = \frac{1}{\sigma^n} \exp\left\{-\frac{\sum(x_i-\mu)}{\sigma}\right\} \mathbb{I}\{X_{(1)} \geq \mu\}$$

$x_i \geq \mu \quad \exp\left\{\frac{-\sum x_i}{\sigma} + \frac{n\mu}{\sigma}\right\}$ función creciente en μ y alcanza el máximo en el mínimo de los datos.

$$\hat{\mu} = X_{(1)}$$

Respecto a σ

$$\mathcal{L}(\theta) = \prod f_\theta(x_i) = \frac{1}{\sigma^n} \exp\left\{\frac{-\sum(x_i-\mu)}{\sigma}\right\}$$

$$l(\theta) = \log(\mathcal{L}(\theta)) = \log\left(\frac{1}{\sigma^n}\right) - \frac{\sum(x_i-\mu)}{\sigma} = -n\log(\sigma) - \frac{\sum(x_i-\mu)}{\sigma}$$

$$l'(\theta) = \frac{-n}{\sigma} + \frac{\sum(x_i-\mu)}{\sigma^2} = 0 \Leftrightarrow \frac{-n\sigma + \sum(x_i-\mu)}{\sigma^2} = 0 \Leftrightarrow -n\sigma + \sum(x_i-\mu) = 0$$

$$\sigma = \frac{\sum(x_i-\mu)}{n}$$

b) $f_\theta(x) = \theta^x(1-\theta)^{1-x}\, \mathbb{I}_{\{0,1\}}(x) \quad \theta\in\left[\frac{1}{2},\frac{3}{4}\right]$

$$f_\theta(x) = \begin{cases} \theta^x(1-\theta)^{1-x} & \text{si } x=0 \text{ ó } x=1 \\ 0 & \text{en otro caso} \end{cases}$$

$$\mathcal{L}(\theta) = f_\theta(x_1,\dots,x_n) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$l(\theta) = \log(\mathcal{L}(\theta)) = \sum x_i \log(\theta) + (n-\sum x_i)\log(1-\theta)$$

$$\frac{\partial}{\partial\theta}\log(f_\theta(x_1,\dots,x_n)) = \frac{\sum x_i}{\theta} - \frac{(n-\sum x_i)}{1-\theta} = 0 \Rightarrow \sum x_i - \cancel{\theta\sum x_i} - n\theta + \cancel{\sum x_i\theta} = 0$$

$$\Rightarrow \sum x_i - n\theta = 0 \Rightarrow \hat{\theta} = \frac{\sum x_i}{n}$$

$$\hat{\theta} = \begin{cases} \frac{\sum x_i}{n} & \text{si } \frac{\sum x_i}{n} \in \left[\frac{1}{2},\frac{3}{4}\right] \\ 1/2 & \text{si } \frac{\sum x_i}{n} < 1/2 \\ 3/4 & \text{si } \frac{\sum x_i}{n} > 3/4 \end{cases}$$

Si $\theta > \bar{x}$ $\mathcal{L}(\theta)$ monótona decreciente

Si $\theta < \bar{x}$ $\mathcal{L}(\theta)$ monótona creciente

(11) Sea X una variable aleatoria con función de densidad

$$f_\theta(x) = \frac{x}{\theta} e^{-x^2/2\theta} \quad x \in \mathbb{R}^+$$

siendo θ un parámetro positivo desconocido. Dada una muestra aleatoria simple de X calcula el estimador de máxima verosimilitud de θ.
Calcula su distribución límite y comprueba si es asintóticamente eficiente

EMV

$$\mathcal{L}(\theta) = f_\theta(x_1,\dots,x_n) = \frac{\prod x_i}{\theta^n} e^{-\frac{\sum x_i^2}{2\theta}}$$

$$\ell(\theta) = \log(\mathcal{L}(\theta)) = \log\left(\frac{\prod x_i}{\theta^n}\right) + -\frac{\sum x_i^2}{2\theta} = \log(\prod x_i) - n\log(\theta) - \frac{\sum x_i^2}{2\theta}$$

$$\ell'(\theta) = -\frac{n}{\theta} + \frac{\sum x_i^2}{\theta^2} = \frac{-n\theta + \sum x_i^2}{\theta^2}$$

$$\ell'(\theta) = 0 \Leftrightarrow -n\theta + \sum x_i^2 = 0 \Leftrightarrow \boxed{\hat{\theta} = \frac{\sum x_i^2}{2n}}$$

• DISTRIB. ASINTÓTICA

$$Y = \frac{X^2}{2} \quad \hat{\theta} = \frac{\sum x_i^2}{2n} = \frac{Y_1+\dots+Y_n}{n} = \bar{Y} \quad \text{PARA PODER USAR EL TCL}$$

$X_1,\dots,X_n$

$\Leftrightarrow Y_1 = \frac{X_1^2}{2}, \dots, Y_n = \frac{X_n^2}{2}$ idénticamente distribuidas $Y = \frac{X^2}{2}$

$$\frac{\sqrt{n}[\bar{Y} - E[Y]]}{\sqrt{Var(Y)}} \xrightarrow{d} N(0,1) \Leftrightarrow \bar{Y} = AN\left(E(Y), \frac{Var(Y)}{n}\right)$$

$$G(y) = P(Y \le y) = P\left(\frac{X^2}{2} \le y\right) = P(X^2 \le 2y) = P(X \le \pm\sqrt{2y}) \underset{X \text{ positiva}}{=} P(X \le \sqrt{2y}) = F_\theta(\sqrt{2y})$$

$$g(y) = f_\theta(\sqrt{2y}) \sqrt{2}\frac{1}{2}y^{-1/2} = \frac{\sqrt{2y}}{\theta} e^{-\frac{2y}{2\theta}} \sqrt{2}\frac{1}{2}y^{-1/2} = \frac{1}{\theta} e^{-\frac{y}{\theta}} \quad y>0$$

$$Y \sim Exp\left(\frac{1}{\theta}\right) = \Gamma\left(\frac{1}{\theta}, 1\right) \quad E[Y] = \theta \quad Var(Y) = \theta^2$$

$$\bar{Y} = AN\left(\theta, \frac{\theta^2}{n}\right)$$

EFICIENCIA

Cota F-C-R $Var(T) \geq C_n = \frac{(g'(\theta))^2}{I_n(\theta)} = \frac{1}{I_n(\theta)}$

$E(T) = g(\theta)$

$g(\theta) = \theta \quad g'(\theta) = 1$

$\overline{Y} = AN(\theta, \underbrace{\boxed{\frac{\theta^2}{n}}}_{= \sigma_n^2(\theta)})$

Eficiencia $= \frac{C_n}{\sigma_n^2}$ $\begin{cases} = 1 ? \text{ Eficiencia} \\ \xrightarrow{n\to\infty} 1 \text{ Asintóticamente eficiente} \end{cases}$

$$C_n = \frac{(g'(\theta))^2}{I_n(\theta)} = \frac{1}{I_n(\theta)} = \frac{1}{n/\theta^2} = \frac{\theta^2}{n}$$

Calculamos $I_n(\theta)$

$$I_n(\theta) = E_\theta\left[\left(\frac{\partial}{\partial\theta} \log f_\theta(x_1, \dots, x_n)\right)^2\right]$$

De los apartados anteriores sabemos que $\frac{\partial}{\partial\theta} \log(f_\theta(x_1,\dots,x_n)) = \frac{-2n\theta + \Sigma x_i^2}{2\theta^2}$

$$\frac{\partial^2}{\partial\theta^2} \log(f_\theta(x_1,\dots,x_n)) = \frac{-2n(2\theta^2) - 4\theta(-2n\theta + \Sigma x_i^2)}{4\theta^4} = \frac{-n\theta + 2n\theta - \Sigma x_i^2}{\theta^3} = \frac{n\theta - \Sigma x_i^2}{\theta^3}$$

$$E[\Sigma x_i^2] = \Sigma E[x_i^2] = \Sigma\, 2\theta = 2n\theta$$

Anterior $\Rightarrow Y = \frac{X^2}{2} \Rightarrow X^2 = 2Y \Rightarrow E[X^2] = 2E[Y] = 2\theta$

$$I_n(\theta) = E\left[-\frac{\partial^2}{\partial\theta^2} \log f_\theta(x_1, \dots x_n)\right] = E\left[\frac{-n\theta + \Sigma x_i^2}{\theta^3}\right] = \frac{-n}{\theta^2} + \frac{E[\Sigma x_i^2]}{\theta^3} =$$

$$= \frac{-n}{\theta^2} + \frac{2n\theta}{\theta^3} = \frac{-n+2n}{\theta^2} = \frac{n}{\theta^2}$$

ASISTÓTICAMENTE EFICIENTE

Una sucesión de estimadores T_n asintóticamente normal y consistente para θ

si $\lim_{n\to\infty} \frac{\text{Cota F-C-R}}{\sigma_n(\theta)^2} = \lim_{n\to\infty} \frac{1/I_n(\theta)}{\sigma_n(\theta)^2} = 1$

$$\lim_{n\to\infty} \frac{1/I_n(\theta)}{\sigma_n(\theta)^2} = \lim_{n\to\infty} \frac{1/n/\theta^2}{\theta^2/n} = \lim_{n\to\infty} \frac{\theta^2/n}{\theta^2/n} = 1$$

$\hat{\theta}$ es asintóticamente eficiente.

1) Sea X una variable aleatoria de Bernoulli de parámetro θ. Calcula el estimador de máxima verosimilitud de θ, su error cuadrático medio y su distribución asintótica. Calcula la cota de Frechet-Cramer-Rao y comprueba que es asintóticamente eficiente.

$$f_\theta(x) = \theta^x (1-\theta)^{1-x}$$

$$f_\theta(x_1, \dots, x_n) = \theta^{\sum x_i}(1-\theta)^{n-\sum x}$$

→ESTIMADOR DE MAXIMA VEROSIMILITUD

$$\mathcal{L}(\theta) = f_\theta(x_1, \dots, x_n) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

$$\ell(\theta) = \log(\mathcal{L}(\theta)) = \sum x_i \log(\theta) + (n - \sum x_i)\log(1-\theta)$$

$$\ell'(\theta) = \sum x_i \cdot \frac{1}{\theta} + (n - \sum x_i)\frac{-1}{1-\theta} = \frac{\sum x_i - \cancel{\sum x_i \theta} - n\theta + \cancel{\theta \sum x_i}}{\theta(1-\theta)} = \frac{\sum x_i - n\theta}{\theta(1-\theta)}$$

$$\ell'(\theta) = 0 \iff \sum x_i - n\theta = 0 \Rightarrow \theta = \frac{\sum x_i}{n}$$

$$EMV \Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \bar{X}$$

→ERROR CUADRÁTICO MEDIO

$$ECM(\hat{\theta}) = Sesgo^2 + Var^2(\hat{\theta}) = \frac{\theta^2(1-\theta)^2}{n^2}$$

$$Sesgo = E(\hat{\theta}) - \theta = E[\bar{X}] - \theta = 0$$

$$Var^2(\hat{\theta}) = Var\left(\frac{\sum x_i}{n}\right) = \frac{1}{n^2} Var(\sum x_i) = \frac{1}{n^2} n\theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

→DISTRIBUCIÓN ASINTÓTICA

Empleando el Teorema Central del Límite como $X_1, \dots, X_n$ son variables aleatorias simples de X con $E(X) = \theta$ y $Var(X) = \theta(1-\theta)$ se tiene que

$$\frac{\sqrt{n}\,[\bar{X} - E(\bar{X})]}{\sqrt{Var(\bar{X})}} \xrightarrow{d} N(0,1) \iff \bar{X} = AN\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$

° COTA F-C-R

$$C_n = \frac{(g'(\theta))^2}{I_n(\theta)} \rightarrow C_n = \frac{1}{\frac{n}{\theta(1-\theta)}} = \frac{\theta(1-\theta)}{n}$$

$$E(\hat{\theta}) = g(\theta) = \theta \Rightarrow g'(\theta) = 1$$

Cálculo de $I_n(\theta)$

De los apartados anteriores tenemos que

$$\frac{\partial}{\partial\theta} \log(f_\theta(x_1,\dots,x_n)) = \frac{\sum x_i - n\theta}{\theta(1-\theta)}$$

$$\frac{\partial^2}{\partial\theta^2} \log(f_\theta(x_1,\dots,x_n)) = \frac{-n(\theta(1-\theta))-(\sum x_i - n\theta)(1-2\theta)}{\theta^2(1-\theta)^2} = \frac{\cancel{-n\theta} + n\theta^2 - \sum x_i + 2\theta\sum x_i + \cancel{n\theta} - \dots}{\theta^2(1-\theta)^2}$$

$$= \frac{\sum x_i(2\theta-1) - n\theta^2}{\theta^2(1-\theta)^2}$$

$$I_n(\theta) = E\left[-\frac{\partial^2}{\partial\theta^2} \log(f_\theta(x_1,\dots,x_n))\right] = E\left[\frac{-\sum x_i(2\theta-1)+n\theta^2}{\theta^2(1-\theta)^2}\right] =$$

$$= \frac{-E[\sum x_i](2\theta-1)+n\theta^2}{\theta^2(1-\theta)^2} = \frac{-\sum E[x_i](2\theta-1)+n\theta^2}{\theta^2(1-\theta)^2} =$$

$$= \frac{-n\theta(2\theta-1)+n\theta^2}{\theta^2(1-\theta)^2} = \frac{-2n\theta^2+n\theta+n\theta^2}{\theta^2(1-\theta)^2} = \frac{-n\theta^2+n\theta}{\theta^2(1-\theta)^2} = \frac{n\theta\cancel{(1-\theta)}}{\theta^2(1-\theta)^2} = \frac{n}{\theta(1-\theta)}$$

ASINTÓTICAMENTE EFICIENTE

Una sucesión de estimadores T_n , asintóticamente normal y consistente para θ es asintóticamente eficiente si

$$\lim_{n\to\infty} \frac{\text{Cota F-C-R}}{\sigma_n(\theta)^2} = 1$$

$$\lim_{n\to\infty} \frac{\text{Cota FCR}}{\sigma_n(\theta)^2} = \lim_{n\to\infty} \frac{\frac{\theta(1-\theta)}{n}}{\frac{\theta(1-\theta)}{n}} = 1 \Rightarrow \hat{\theta} \text{ asintóticamente eficiente}$$

(2) Sea X una variable aleatoria de Poisson de parámetro λ. Calcula el estimador de máxima verosimilitud de λ, su error cuadrático medio y su distribución asintótica. Calcula la cota de Frechet-Cramer-Rao y comprueba que es asintóticamente eficiente.

$X \in Pois(\lambda) \quad E(X)=\lambda \quad Var(X)=\lambda$

$f_\theta(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} \qquad \lambda=\theta$

→ ESTIMADOR DE MÁXIMA VEROSIMILITUD

$\mathcal{L}(\theta) = f_\theta(x_1,\dots,x_n) = \dfrac{e^{-n\theta}\theta^{\sum x_i}}{x_1!\cdots x_n!} = K e^{-n\theta}\theta^{\sum x_i} \qquad K = \dfrac{1}{x_1!\cdots x_n!}$

$\ell(\theta) = \log(f_\theta(x_1,\dots,x_n)) = -n\theta + \sum x_i \log(\theta) - \log(x_1!\cdots x_n!)$

$\ell'(\theta) = -n + \dfrac{\sum x_i}{\theta}$

$\ell'(\theta)=0 \Rightarrow \boxed{\hat{\theta} = \dfrac{\sum x_i}{n} = \bar{X}}$ Estimador de máxima verosimilitud

→ ERROR CUADRÁTICO MEDIO

$ECM(\hat{\theta}) = Sesgo^2 + Var(\hat{\theta})^2 = \dfrac{\theta^2}{n^2}$

$Sesgo = E(\hat{\theta}) - \theta = E[\bar{X}] - \theta = \dfrac{\sum E(x_i)}{n} - \theta = \dfrac{n\theta}{n} - \theta = 0$

$Var(\hat{\theta}) = Var[\bar{X}] = Var\left[\dfrac{\sum x_i}{n}\right] = \dfrac{1}{n^2} Var[\sum x_i] = \dfrac{1}{n^2}\sum Var[x_i] = \dfrac{n\theta}{n^2} = \dfrac{\theta}{n}$

→ DISTRIBUCIÓN ASINTÓTICA

Por el Teorema Central del Límite $X_1,\dots,X_n$ m.a.s idénticamente distribuidas que $X \in Pois(\theta)$ $E[X]=\theta$ $Var(X)=\theta<+\infty$

$\dfrac{\sqrt{n}\,[\bar{X} - E[X]]}{\sqrt{Var(X)}} \xrightarrow{d} N(0,1) \Rightarrow \bar{X} = AN\left(\theta, \dfrac{\theta}{n}\right)$