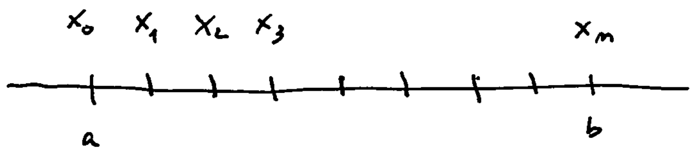
Integrals dobles = integrals de funcions de dues variables

Recordem que si $f: I = [a,b] \longrightarrow \mathbb{R}$ def. que f és integrable de la següent manera:

Considerem particions de $[a,b]$ del tipus $a = x_0 < x_1 < \dots < x_n = b$, $x_{j+1} - x_j = \frac{b-a}{n}$



Prenem $c_j \in [x_j, x_{j+1}]$, $\forall j$

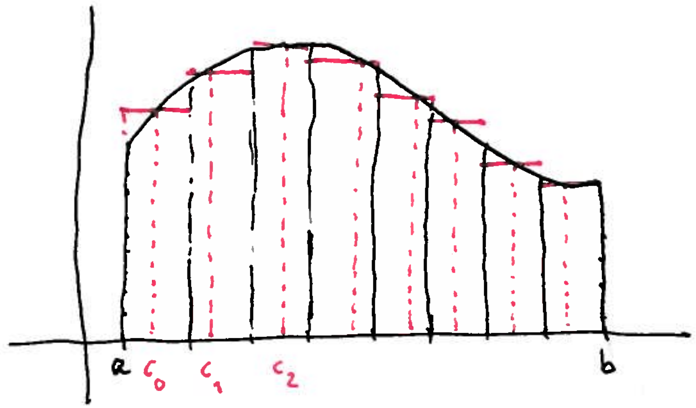
Considerem $S_n = \sum_{j=0}^{n-1} f(c_j)\,\Delta x$, $\Delta x = \frac{b-a}{n}$

f és integrable en $[a,b]$ si

$$\lim_{\Delta x \to 0} \sum_{j=0}^{n-1} f(c_j)\,\Delta x$$ existeix i pren un valor

independentment de l'elecció dels c_j.

El límit es diu integral de f en $[a,b]$, $\int_a^b f = \int_a^b f(x)\,dx$



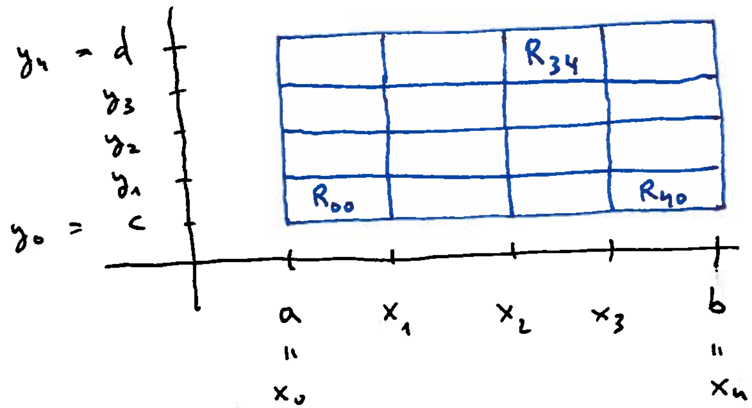
Integral doble en un rectangle

Siguin $R = [a,b] \times [c,d]$ un rectangle i $f : R \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

Considerem una partició de R en rectangles $R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$ on

$$a = x_0 < x_1 < \dots < x_m = b \,, \qquad c = y_0 < y_1 < \dots < y_m = d$$

$$x_{j+1} - x_j = \frac{b-a}{m} = \Delta x \qquad y_{k+1} - y_k = \frac{d-c}{m} = \Delta y$$



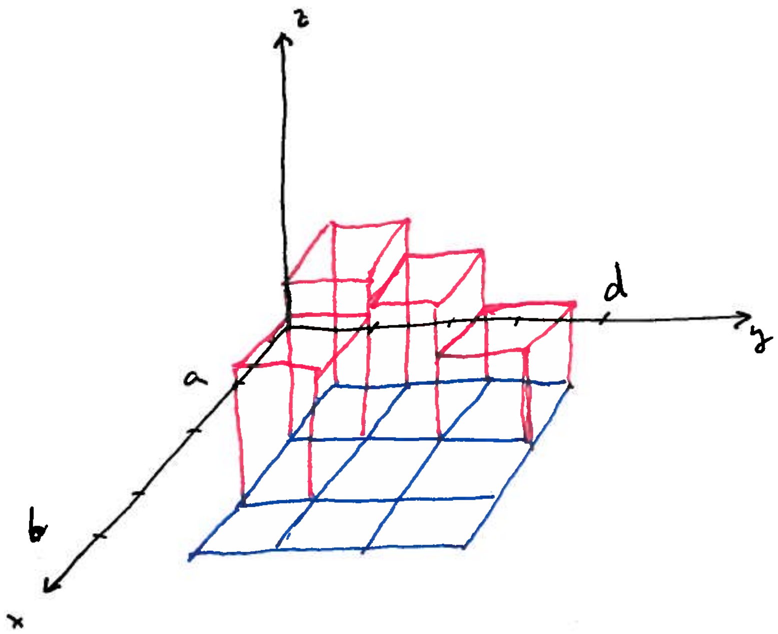
Prenem $c_{jk} \in R_{jk}$

i considerem

$$S_m = \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} f(c_{jk})\, \Delta x\, \Delta y$$

↑ ↑ depenen de m

(suma de Riemann)



Les sumes de Riemann $S_m = \sum_{j,k=0}^{m-1} f(c_{jk})\,\Delta x \,\Delta y$ representen la suma dels volums dels paral·lelepípides de la figura.

Si f és positiva les sumes de Riemann aproximen el volum limitat per la gràfica de f, $z=0$, $x=a$, $x=b$, $y=c$, $y=d$

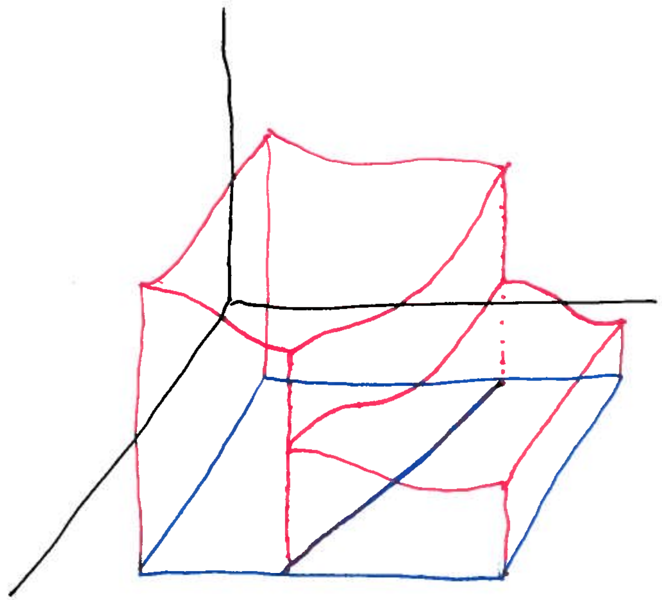
<u>Definició</u> $f : R \to \mathbb{R}$ és integrable si $\exists \lim_{m\to\infty} S_m$ i és independent de l'elecció dels punts $c_{jk} \in R_{jk}$.

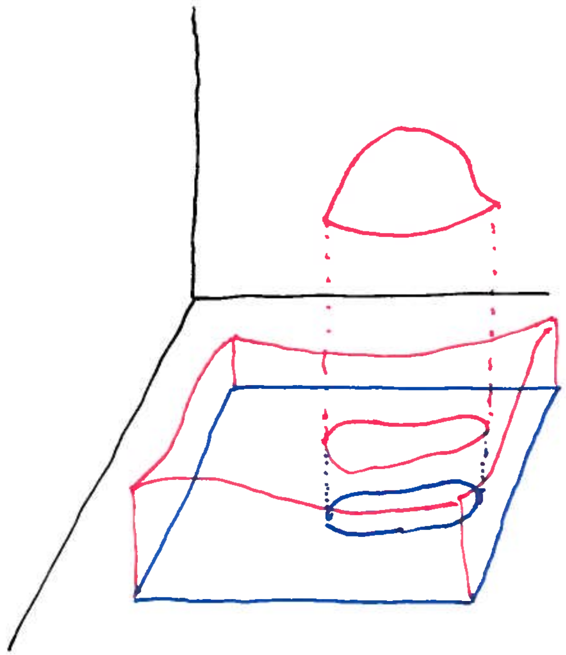
Quan f és integrable, $\lim_{m\to\infty} S_m$ es diu integral de f en R i es representa per

$$\int_R f\,, \quad \int_R f(x,y)\,dx\,dy\,, \quad \iint_R f(x,y)\,dx\,dy\,, \quad \int_R f(x,y)\,dA$$

Teorema Sigui $f : R \to \mathbb{R}$ acotada i contínua en $R = [a,b] \times [c,d]$ excepte potser sobre un subconjunt format per una unió finita de gràfics de funcions contínues.

Llavors f és integrable.





Propietats bàsiques

Siguin R un rectangle, f, g integrables en R i $c \in \mathbb{R}$

(i) $f+g$ és integrable en R i $\int_R (f+g) = \int_R f + \int_R g$

(ii) cf és integrable en R i $\int_R (cf) = c \int_R f$

(iii) si $f(x,y) \geq g(x,y)$ $\forall (x,y) \in R$, $\int_R f \geq \int_R g$

(iv) siguin $R_1, \dots, R_m$ rectangles t.q. (a) $R_i \cap R_j$ només conté punts frontera si $i \neq j$

(b) $Q = R_1 \cup \dots \cup R_m$ és un rectangle

si f és integrable en R_i, $\forall i$, llavors

f és integrable en Q i $\int_Q f = \sum_{i=1}^{m} \int_{R_i} f$

Consequència important

Si f és integrable en R, $\qquad \left| \int_R f \right| \leq \int_R |f|$

prova: tenim que

$$-|f| \leq f \leq |f|$$

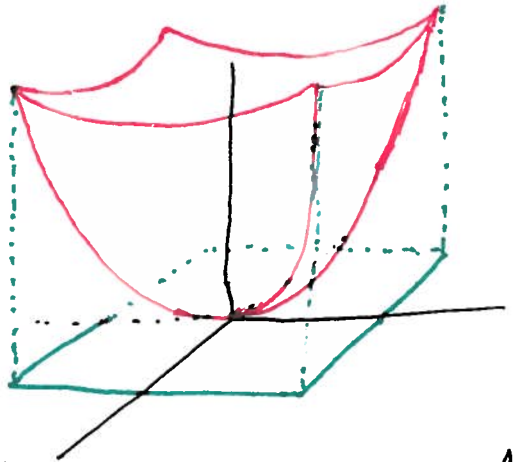
$$\Rightarrow \quad \underbrace{\int_R (-|f|)}_{\displaystyle = -\int_R |f|} \leq \int_R f \leq \int_R |f| \qquad \Rightarrow \qquad \left| \int_R f \right| \leq \int_R |f|$$

Mètode per calcular integrals dobles: teorema de Fubini

<u>Teorema</u> (versió simple) Sigui $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ contínua. $R = [a,b] \times [c,d]$

$$\int_R f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

<u>Ex. d'aplicació</u>



$R = [-1,1] \times [-1,1]$, $f(x,y) = x^2 + y^2$

$$\int_R (x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left(\int_{-1}^{1} (x^2+y^2)\,dy\right) dx = \int_{-1}^{1} \left(x^2 y + \frac{y^3}{3} \right)\Big|_{-1}^{1} dx = \int_{-1}^{1} \left(x^2 + \frac{1}{3} + x^2 + \frac{1}{3}\right) dx = 2\int_{-1}^{1} \left(x^2 + \frac{1}{3}\right) dx$$

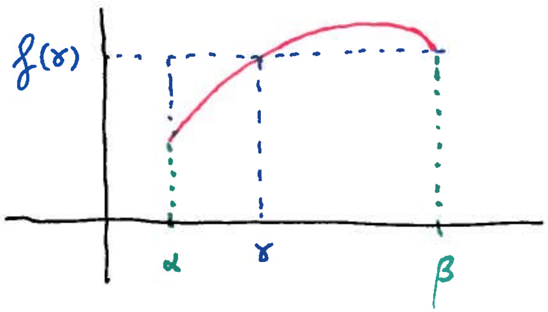
$$= 2\left(\frac{x^3}{3} + \frac{x}{3}\right)\Big]_{-1}^{1} = 2\left(\frac{2}{3} + \frac{2}{3}\right) = \frac{8}{3}.$$

$$\int_R (x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left(\int_{-1}^{1} (x^2+y^2)\,dx\right) dy = \int_{-1}^{1} \left(\frac{x^3}{3} + y^2 x \right)\Big|_{-1}^{1} dy = \int_{-1}^{1} \left(\frac{1}{3} + y^2 + \frac{1}{3} + y^2\right) dy = 2\int_{-1}^{1} \left(y^2 + \frac{1}{3}\right) dy = \frac{8}{3}$$

Recordo el teorema del valor mig per a integrals:

<u>Teorema</u>

Si $f: [\alpha, \beta] \longrightarrow \mathbb{R}$ és contínua $\exists\, \gamma \in [\alpha, \beta]$ t.q. $\displaystyle\int_\alpha^\beta f(x)\, dx = f(\gamma)(\beta - \alpha)$

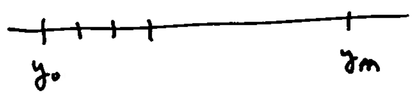


<u>Esquema de la dem. del teorema de Fubini</u>

Provarem que $$\int_R f(x,y)\, dx\, dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Definim $$F(x) = \int_c^d f(x,y)\, dy \longrightarrow \int_R f(x,y)\, dx\, dy = \int_a^b F(x)\, dx$$

Considerem la partició $c = y_0 < y_1 < \dots < y_m = d$

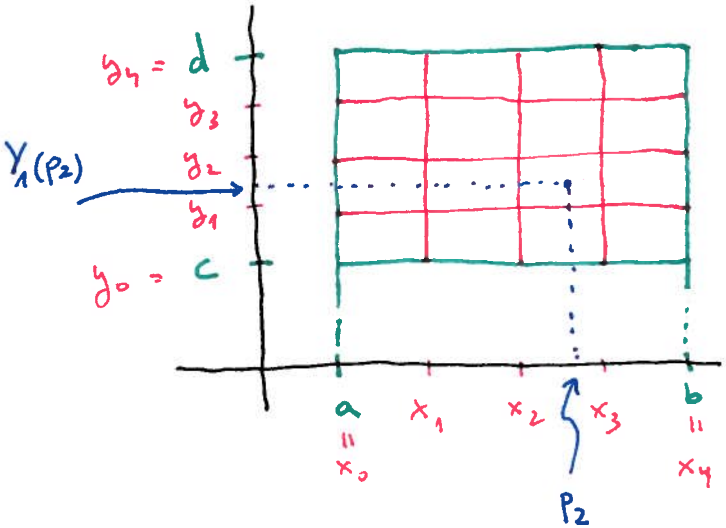


, $y_{k+1} - y_k = \Delta y$

$$\int_{y_k}^{y_{k+1}} f(x,y)\, dy = f(x, Y_k(x))\,(y_{k+1} - y_k) \quad \text{per cada } k, \quad Y_k(x) \in [y_k, y_{k+1}]$$

$$F(x) = \int_c^d f(x,y)\,dy = \sum_{k=0}^{m-1} \int_{y_k}^{y_{k+1}} f(x,y)\,dy = \sum_{k=0}^{m-1} f(x, Y_k(x))\,\Delta y .$$

Considerem la partició $a = x_0 < x_1 < \dots < x_m = b$, $\quad x_{j+1} - x_j = \Delta x = \dfrac{b-a}{m}$



$$R_{jk} = [x_j, x_{j+1}] \times [y_k, y_{k+1}]$$

Escollim $c_{jk} \in R_{jk}$ de la forma

$$c_{jk} = (p_j, Y_k(p_j))$$

Per la def. d'integral $\displaystyle\int_R f(x,y)\,dx\,dy = \lim_{m\to\infty} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} f(c_{jk})\,\Delta x\,\Delta y$

$$= \lim_{m\to\infty} \sum_{j=0}^{m-1} \Big(\sum_{k=0}^{m-1} \underbrace{f(c_{jk})}_{f(p_j, Y_k(p_j))} \Delta y \Big) \Delta x = \lim_{m\to\infty} \sum_{j=0}^{m-1} F(p_j)\,\Delta x = \int_a^b F(x)\,dx$$

Teorema de Fubini (versió més general)

Sigui $f: R \longrightarrow \mathbb{R}$ acotada, $R = [a,b] \times [c,d]$ i suposem que f és contínua excepte en un subconjunt format per la unió finita de corbes contínues.

- Si $\forall x \in [a,b]$ $\quad \exists \displaystyle\int_c^d f(x,y)\,dy$

  llavors $\quad \exists \displaystyle\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_R f(x,y)\,dx\,dy$

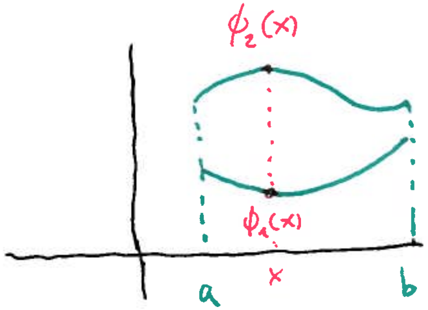
- Si $\forall y \in [c,d]$ $\quad \exists \displaystyle\int_a^b f(x,y)\,dx$

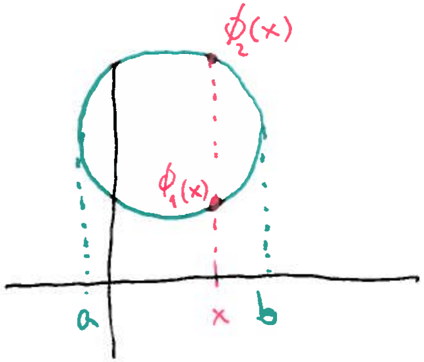
  llavors $\quad \exists \displaystyle\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_R f(x,y)\,dx\,dy$

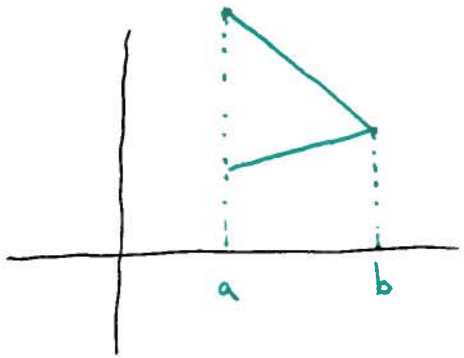
Integrals dobles

Introduïm un tipus de regions que direm elementals

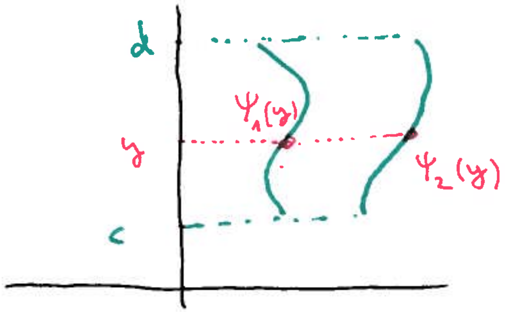
tipus 1 $\quad D = \{ (x,y) \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x) \}$, ϕ_1, ϕ_2 contínues

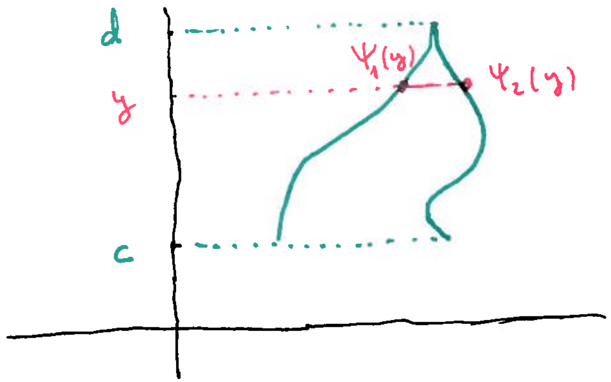






tipus 2 $\quad D = \{ (x,y) \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y) \}$,





ψ_1, ψ_2 contínues

Def Si D és una regió elemental prenem R rectangle t.q. $D \subset R$

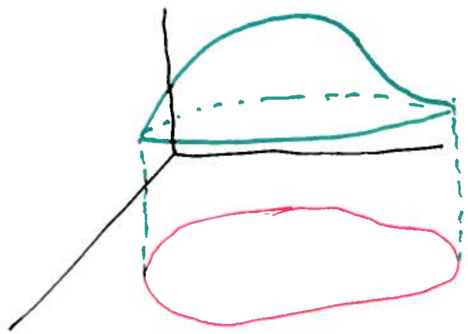
Si $f: D \to \mathbb{R}$ és contínua definim

$$\int_D f(x,y)\,dx\,dy = \int_R f^*(x,y)\,dx\,dy$$

on

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \setminus D \end{cases}$$

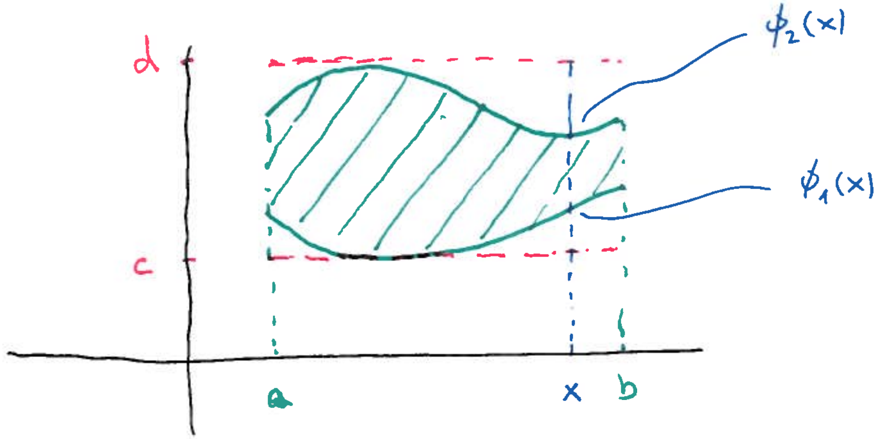
Si $f \geq 0$, $\int_D f$ representa el volum entre D en el ple x-y i la gràfica de f



Càlcul efectiu de la integral

tipus 1 $\qquad D = \{ (x,y) \mid \ a \leq x \leq b \ , \ \ \phi_1(x) \leq y \leq \phi_2(x) \}$

$$R = [a,b] \times [c,d] \supset D$$



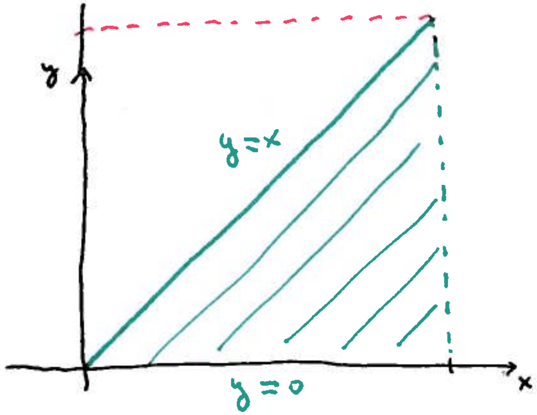
$$\int_D f(x,y)\,dx\,dy = \int_R f^*(x,y)\,dx\,dy$$

$$= \int_a^b \left(\int_c^d f^*(x,y)\,dy \right) dx = \int_a^b \left(\int_c^{\phi_1(x)} f^*(x,y)\,dy + \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy + \int_{\phi_2(x)}^{d} f^*(x,y)\,dy \right) dx$$

$$= \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

Cas particular $\quad f(x,y) = 1$: $\qquad \int_D f = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} 1 \, dy \right) dx = \int_a^b \left[\phi_2(x) - \phi_1(x) \right] dx = \text{àrea}\,(D).$

Ex Càlcul de $\int_T (x^2y^3+x)\,dx\,dy$ on T és el triangle de vèrtexs $(0,0)$, $(0,1)$, $(1,1)$



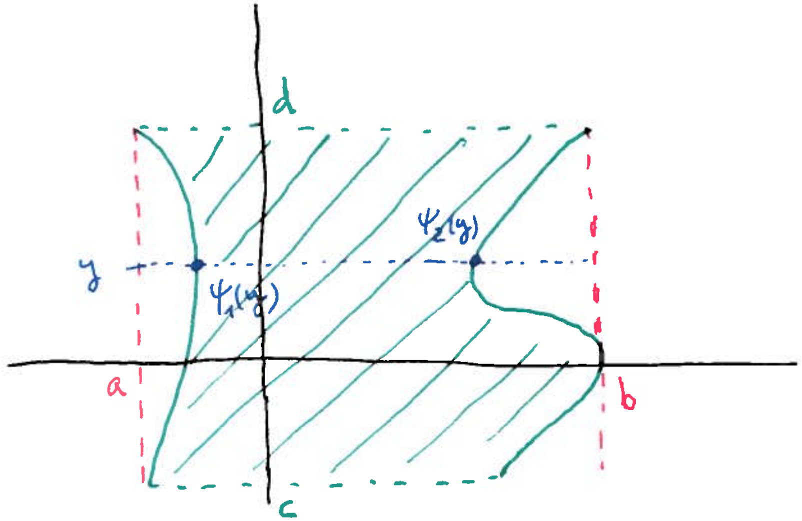
$$\int_T (x^2y^3+x)\,dx\,dy = \int_0^1\left[\int_0^x (x^2y^3+x)\,dy\right]dx = \int_0^1\left[x^2\frac{y^4}{4}+xy\right]_0^x dx$$

$$= \int_0^1\left(\frac{x^6}{4}+x^2\right)dx = \left.\frac{x^7}{4\cdot 7}+\frac{x^3}{3}\right]_0^1 = \frac{1}{28}+\frac{1}{3} = \frac{3+28}{4\cdot 7\cdot 3} = \frac{31}{84}$$

<u>Càlcul de la integral</u> (tipus 2)

$$D = \{ (x,y) \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y) \}$$

$$R = [a,b] \times [c,d] \supset D$$

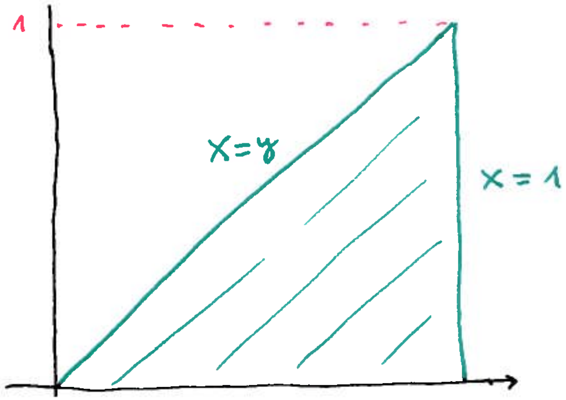


$$\int_T f(x,y)\, dx\, dy = \int_R f^*(x,y)\, dx\, dy$$

$$= \int_c^d \left(\int_a^b f^*(x,y)\, dx \right) dy = \int_c^d \left(\int_a^{\psi_1(y)} f^*(x,y)\, dx + \int_{\psi_1(y)}^{\psi_2(y)} f^*(x,y)\, dx + \int_{\psi_2(y)}^{b} f^*(x,y)\, dx \right) dy$$

$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx \right) dy$$

Ex Càlcul de $\int_T (x^2y^3+x)\,dx\,dy$, T triangle de vèrtexs (0,0), (0,1), (1,1)

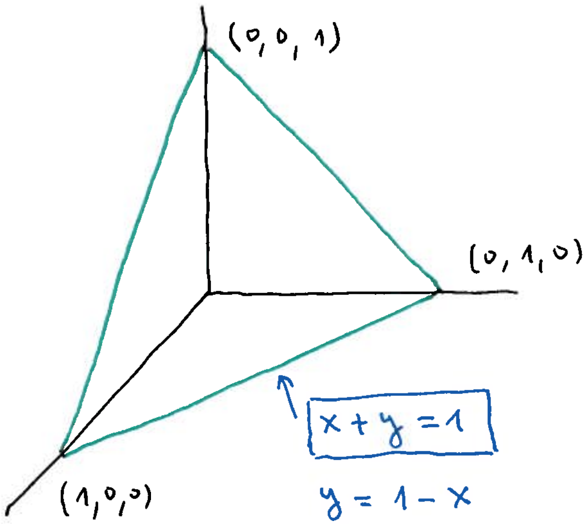


$$\int_T (x^2y^3+x)\,dx\,dy = \int_0^1 \left(\int_y^1 (x^2y^3+x)\,dx \right) dy = \int_0^1 \left[\frac{x^3y^3}{3} + \frac{x^2}{2} \right]_y^1 dy$$

$$= \int_0^1 \left[\frac{y^3}{3} + \frac{1}{2} - \frac{y^6}{3} - \frac{y^2}{2} \right] dy = \left[\frac{y^4}{12} + \frac{y}{2} - \frac{y^7}{21} - \frac{y^3}{6} \right]_0^1$$

$$= \frac{1}{12} + \frac{1}{2} - \frac{1}{21} - \frac{1}{6} = \frac{1+6-2}{12} - \frac{1}{21} = \frac{5}{12} - \frac{1}{21} = \frac{35-4}{3\cdot 4\cdot 7} = \frac{31}{84}$$

Volum del tetraedre de vèrtexs $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

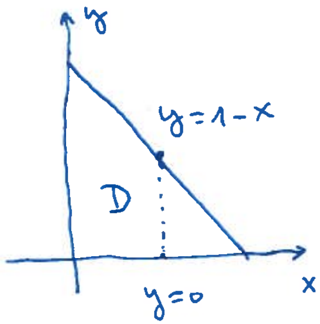


Eq. pla.

$$\text{vector normal} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, 1, 1)$$

$$(x-1) + (y-0) + (z-0) = 0$$

$$\boxed{z = -x - y + 1}$$



D = triangle de vèrtexs $(0,0)$, $(1,0)$, $(0,1)$ en el pla x-y

$$\text{volum} = \int_D (-x-y+1)\, dx\, dy = \int_0^1 \left(\int_0^{1-x} (-x-y+1)\, dy \right) dx$$

$$= \int_0^1 \left[-xy - \frac{y^2}{2} + y \right]_0^{1-x} dx = \int_0^1 \left[-x(1-x) - \frac{(1-x)^2}{2} + (1-x) \right] dx = \int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2} \right) dx$$

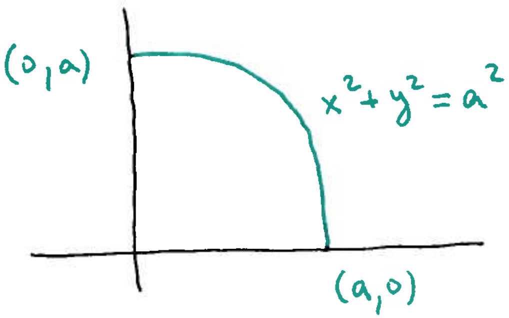
$$= \int_0^1 \frac{(1-x)^2}{2}\, dx = -\frac{(1-x)^3}{6} \Bigg]_0^1 = 0 + \frac{1}{6}$$

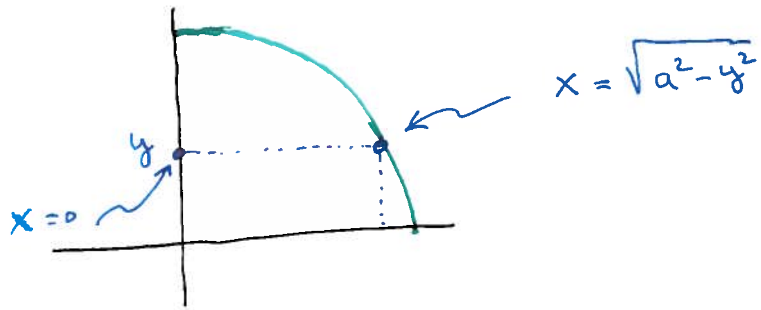
Ex Canvi d'ordre d'integració en $I = \int_0^a \left(\int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2} \, dy \right) dx$

1r hem d'identificar el domini

$0 \leq x \leq a$

donat x $\quad 0 \leq y \leq (a^2-x^2)^{1/2}$. $\quad \left\{ \begin{array}{l} y = (a^2-x^2)^{1/2} \rightarrow y^2 = a^2 - x^2 \\ \rightarrow x^2+y^2 = a^2 \end{array} \right.$





$$I = \int_0^a \left(\int_0^{\sqrt{a^2-y^2}} (a^2-y^2)^{1/2} \, dx \right) dy = \int_0^a \left[x \, (a^2-y^2)^{1/2} \right]_0^{\sqrt{a^2-y^2}} dy = \int_0^a (a^2-y^2) \, dy$$

$$= \left[a^2 y - \frac{y^3}{3} \right]_0^a = a^3 - \frac{a^3}{3} = \frac{2}{3} a^3$$