2ª prova Càlcul de diverses variables

18 - gener - 2012

1) Definiu integral doble d'una funció en un rectangle.

Calculeu $\int_A (x+y)\,dx\,dy$ on A és el quadrat de vèrtexs

$(0,2)$, $(1,1)$, $(2,2)$, $(1,3)$.

2) Enuncieu el teorema del canvi de variables.

Calculeu

$$\int_D \left(\sqrt{x^2+y^2+z^2} + z\right) dx\,dy\,dz$$

on $D = \{(x,y,z) \mid z \geq 0, \quad x^2+y^2+z^2 \leq 1\}$

3) Trobeu els extrems relatius de la funció

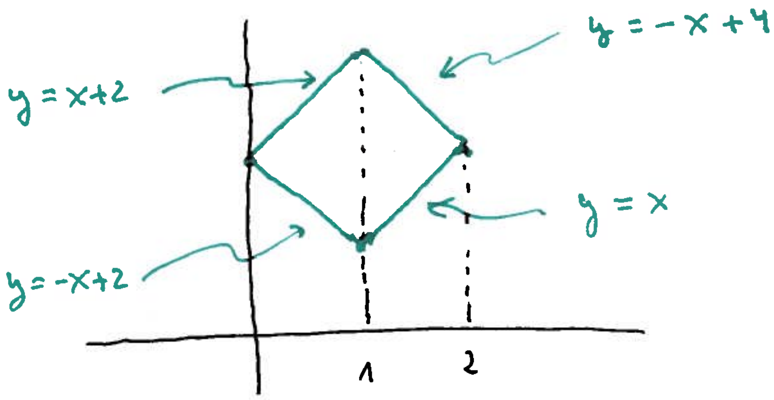
$$f(x,y) = (ax^2+by^2)\,e^{-x^2-y^2}, \qquad 0<a<b$$

4) Determineu els extrems absoluts de

$$f(x,y,z) = x - 2y + 2z$$

sobre l'esfera $x^2+y^2+z^2 = 9$

$$\boxed{I = \int_A (x+y)\,dx\,dy\,, \quad A = \text{quadrat de vertexs } (0,2),\ (1,1),\ (2,2),\ (1,3)}$$

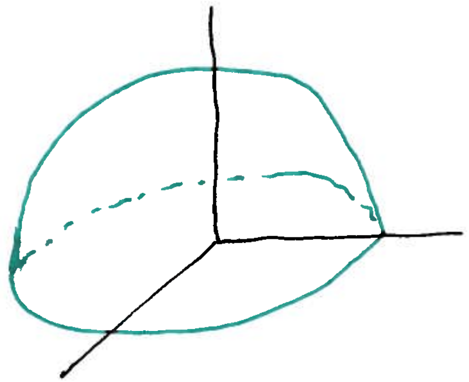


$$I = \int_0^1 \int_{-x+2}^{x+2} (x+y)\,dy\,dx + \int_1^2 \int_x^{-x+4} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_{-x+2}^{x+2} dx + \int_1^2 \left[xy + \frac{y^2}{2}\right]_x^{-x+4} dx$$

$$= \int_0^1 \underbrace{\left[x^2 + \cancel{2x} + x^2 - \cancel{2x} + \frac{1}{2}(x+2)^2 - \frac{1}{2}(-x+2)^2\right]}_{2x^2 + \frac{1}{2}\left(\cancel{x^2} + 4x + \cancel{4} - \cancel{x^2} + 4x - \cancel{4}\right)} dx + \int_1^2 \underbrace{\left[-x^2 + 4x - x^2 + \frac{1}{2}(-x+4)^2 - \frac{x^2}{2}\right]}_{-2x^2 + 4x + \frac{1}{2}\left(\cancel{x^2} - 8x + 16 - \cancel{x^2}\right)} dx$$

$$= \int_0^1 (2x^2 + 4x)\,dx + \int_1^2 (-2x^2 + 8)\,dx = \left[\frac{2}{3}x^3 + 2x^2\right]_0^1 + \left[-\frac{2}{3}x^3 + 8x\right]_1^2 = \frac{8}{3} + \frac{10}{3} = \frac{18}{3} = 6$$

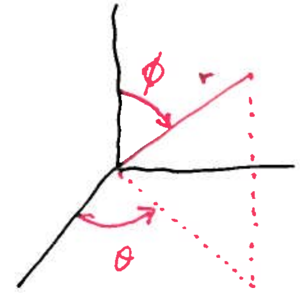
$$\boxed{I = \iint_D \left(\sqrt{x^2+y^2+z^2} + z \right) dx\,dy\,dz \,, \quad \text{on} \quad D = \{ z \geq 0,\ x^2+y^2+z^2 \leq 1 \}}$$



$$x = r \sin\phi \cos\theta$$

$$y = r \sin\phi \sin\theta$$

$$z = r \cos\phi$$



$$I = \iint_A \left(\sqrt{r^2} + r\cos\phi \right) r^2 \sin\phi = \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \int_0^1 dr \quad r^3 \left(\sin\phi + \sin\phi\cos\phi \right)$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \left(\sin\phi + \sin\phi\cos\phi \right) d\phi \int_0^1 r^3 dr = 2\pi \left[-\cos\phi + \frac{\sin^2\phi}{2} \right] \Big|_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^1$$

$$= 2\pi \left(1 + \frac{1}{2} \right) \frac{1}{4} = \frac{3\pi}{4}$$

Extrems relatius de

$$f(x,y) = (ax^2+by^2)\, e^{-x^2-y^2} \quad , \quad 0 < a < b \qquad e \text{ representarà } e^{-x^2-y^2}$$

$$\frac{\partial f}{\partial x} = 2ax\,e + (ax^2+by^2)\,e\,(-2x) = 2x\left[a - ax^2 - by^2\right] e = 0 \quad \begin{matrix} \nearrow\ x = 0 \\ \searrow\ a = ax^2+by^2 \end{matrix}$$

$$\frac{\partial f}{\partial y} = 2by\,e + (ax^2+by^2)\,e\,(-2y) = 2y\left[b - ax^2 - by^2\right] e = 0 \quad \begin{matrix} \nearrow\ y = 0 \\ \searrow\ b = ax^2+by^2 \end{matrix}$$

$P_1 = (0,0)$, $P_2 = (0,1)$, $P_3 = (0,-1)$, $P_4 = (1,0)$, $P_5 = (-1,0)$

$$\frac{\partial^2 f}{\partial x^2} = 2\left[a - ax^2 - by^2\right] e + 2x(-2ax)\,e + 2x\left[a - ax^2 - by^2\right](-2x)\,e$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x(-2by)\,e + 2x\left(a - ax^2 - by^2\right) e\,(-2y)$$

$$\frac{\partial^2 f}{\partial y^2} = 2\left[b - ax^2 - by^2\right] e + 2y(-2by)\,e + 2y\left[b - ax^2 - by^2\right] e\,(-2y)$$

$$Hf(0,0) = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} \quad \boxed{\text{Mínim}} \quad ; \quad Hf(\pm 1, 0) = \begin{pmatrix} -4ae^{-1} & 0 \\ 0 & 2(b-a)e^{-1} \end{pmatrix} \quad \boxed{\text{p. sella}}$$

$$Hf(0,\pm 1) = \begin{pmatrix} 2(a-b)e^{-1} & 0 \\ 0 & -4be^{-1} \end{pmatrix} \quad \boxed{\text{Màxims}}$$

$$\boxed{\text{Extrems de } f(x,y,z) = x - 2y + 2z \text{ amb la condició } x^2+y^2+z^2 = 9}$$

$$1 = \lambda 2x \qquad x = \frac{1}{2\lambda}$$

$$-2 = \lambda 2y \qquad y = -\frac{1}{\lambda}$$

$$2 = \lambda 2z \qquad z = \frac{1}{\lambda}$$

$$x^2+y^2+z^2 = 9 \longrightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 9$$

$$\frac{1}{\lambda^2}\left(\frac{1}{4} + 1 + 1\right) = 9$$

$$\frac{1}{\lambda^2}\,\frac{9}{4} = 9 \longrightarrow \lambda = \pm\frac{1}{2}$$

Dos candidats

$P_1 = (1, -2, 2)$ màxim

$P_2 = (-1, 2, -2)$ mínim