Llista de problemes (4)

2.1 Calculeu $\frac{\partial f}{\partial x}$ i $\frac{\partial f}{\partial y}$ per a (1) $f(x,y) = x\cos x \cos y$ (2) $f(x,y) = (\cancel{x}^2+y^2)\log(x^2+y^2)$

2.2 Calculeu les derivades parcials de $z = \log\sqrt{1+xy}$ en $(1,2)$ i en $(0,0)$

2.3 Estudieu la diferenciabilitat i el caràcter C^1 de les funcions (definint-les adequadament en $(0,0)$ o on calgui)

(a) $f(x,y) = \dfrac{2xy^3}{x^2+y^2}$ (b) $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$

(c) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ (d) $f(x,y) = \dfrac{x^2y}{x^4+y^2}$

2.4 Troben el pla tg a la superfície $z = x^2+y^4$ en $(2,1,5)$

2.5 En quin punt el pla tg a $z = e^x - e^y$ en $(1,1,0)$ talla l'eix z ?

2.6 Per què les gràfiques de $f(x,y) = x^2+y^2$ i $g(x,y) = -x^2-y^2+xy^3$ es poden dir tg en $(0,0)$?

2.7 Calculeu la matriu de $Df(x,y)$ per a

(a) $f(x,y) = xe^{x^2+y^2}$; (b) $f(x,y) = (x\cos y,\ xe^{xy},\ 2xy)$; (c) $f(x,y,z) = (xz,\ y^2+z^2)$

2.8 Sigui $A : \mathbb{R}^n \to \mathbb{R}^m$ lineal.

Proveu que A és diferenciable i calculeu $DA(x)$.

2.9 Calculeu aproximadament els següents valors usant que certes funcions adequades són diferenciables

(a) $\left(0.99\, e^{0.02}\right)^8$

(b) $\sqrt{4.01^2 + 3.98^2 + 2.02^2}$

2.1 (1) $f(x,y) = x\cos x \cos y$

$$\frac{\partial f}{\partial x} = \cos x \cos y - x \sin x \cos y$$

$$\frac{\partial f}{\partial y} = - x \cos x \sin y$$

(2) $f(x,y) = (x^2+y^2)\log(x^2+y^2)$

$$\frac{\partial f}{\partial x} = 2x \log(x^2+y^2) + (x^2+y^2)\frac{1}{x^2+y^2} 2x$$

$$\frac{\partial f}{\partial y} = 2y \log(x^2+y^2) + (x^2+y^2)\frac{1}{x^2+y^2} 2y$$

2.2

Derivades parcials de $z = \log\sqrt{1+xy}$ en $(1,2)$ i en $(0,0)$

Primer usem la definició de derivada

$$\frac{\partial z}{\partial x}(1,2) = \lim_{h\to 0} \frac{z(1+h,2) - z(1,2)}{h} = \lim_{h\to 0} \frac{\log\sqrt{1+(1+h)2} - \log\sqrt{1+1\cdot 2}}{h}$$

$$= \lim_{h\to 0} \frac{\log\sqrt{3+2h} - \log\sqrt{3}}{h} \underset{\text{L'Hôpital}}{=} \lim_{h\to 0} \frac{\frac{1}{\sqrt{3+2h}}\frac{1}{2\sqrt{3+2h}}2 - 0}{1} = \frac{1}{3}$$

$$\frac{\partial z}{\partial y}(1,2) = \lim_{h\to 0} \frac{z(1,2+h) - z(1,2)}{h} = \lim_{h\to 0} \frac{\log\sqrt{1+1\cdot(2+h)} - \log\sqrt{1+1\cdot 2}}{h}$$

$$= \lim_{h\to 0} \frac{\log\sqrt{3+h} - \log\sqrt{3}}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{3+h}}\frac{1}{2\sqrt{3+h}} - 0}{1} = \frac{1}{6}$$

Ara usem la funció derivada

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1+xy}}\frac{1}{2\sqrt{1+xy}}y \longrightarrow \frac{\partial z}{\partial x}(1,2) = \frac{1}{\sqrt{3}}\frac{1}{2\sqrt{3}}2 = \frac{1}{3} \quad \Big\| \quad \frac{\partial z}{\partial x}(0,0) = 0$$

$$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{1+xy}}\frac{1}{2\sqrt{1+xy}}x \longrightarrow \frac{\partial z}{\partial y}(1,2) = \frac{1}{\sqrt{3}}\frac{1}{2\sqrt{3}}\cdot 1 = \frac{1}{6} \quad \Big\| \quad \frac{\partial z}{\partial y}(0,0) = 0$$

2.3 (a) $f(x,y) = \dfrac{2xy^3}{x^2+y^2}$

Continuïtat en (0,0) ?

$$0 \leq \left|\frac{2xy^3}{x^2+y^2}\right| \leq \left|\frac{2xy^3}{y^2}\right| = |2xy| \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

$$\underset{0}{\downarrow} \qquad\qquad\qquad \underset{0}{\downarrow}$$

Si $f(0,0)=0$, f és contínua en $(0,0)$.

Derivades parcials:

$$\frac{\partial f}{\partial x} = \frac{2y^3(x^2+y^2) - 2xy^3 \cdot 2x}{(x^2+y^2)^2} = \frac{2y^5 - 2x^2y^3}{(x^2+y^2)^2}, \quad \text{si } (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y} = \frac{6xy^2(x^2+y^2) - 2xy^3 2y}{(x^2+y^2)^2} = \frac{6x^3y^2 + 2xy^4}{(x^2+y^2)^2}, \quad \text{si } (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Les derivades parcials són contínues en $\mathbb{R}^2 - \{(0,0)\} \Rightarrow f$ és C^1 en $\mathbb{R}^2 - \{(0,0)\}$

$\Rightarrow f$ és diferenciable en $\mathbb{R}^2 - \{(0,0)\}$

2.3 (2)

<u>Diferenciabilitat en (0,0)</u>. La derivada, si $\exists$, ha de ser $Df(0,0) = (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)\begin{pmatrix}x\\y\end{pmatrix}}{\|(x,y)-(0,0)\|} = \lim \frac{\frac{2xy^3}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim \frac{2xy^3}{(x^2+y^2)^{3/2}}$$

$$\underset{\substack{\downarrow\\0}}{0} \leq \left|\frac{2xy^3}{(x^2+y^2)^{3/2}}\right| \leq \left|\frac{2xy^3}{(y^2)^{3/2}}\right| = \underset{\substack{\downarrow\\0}}{|2x|} \Rightarrow \text{el límit } \exists \text{ i val } 0$$

Per tant f és diferenciable en $(0,0)$ i $Df(0,0) = (0,0)$

<u>Continuïtat de $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ en $(0,0)$</u> ?

$$\frac{\partial f}{\partial x} = \frac{2y^5 - 2x^2y^3}{(x^2+y^2)^2}$$

$$0 \leq \left|\frac{2y^5}{(x^2+y^2)^2} - \frac{2x^2y^3}{(x^2+y^2)^2}\right| \leq \left|\frac{2y^5}{(x^2+y^2)^2}\right| + \left|\frac{2x^2y^3}{(x^2+y^2)^2}\right| \leq \left|\frac{2y^5}{(y^2)^2}\right| + \left|\frac{x^2y^2}{(x^2+y^2)^2}\cdot 2y\right|$$

$$\leq |2y| + \left|\frac{1}{4}2y\right| \longrightarrow 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(0,0)$$

$$\frac{\partial f}{\partial y} = \frac{6x^3y^2 + 2xy^4}{(x^2+y^2)^2}$$

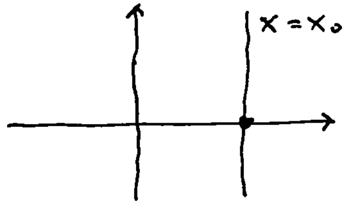
$$0 \leq \left|\frac{6x^3y^2}{(x^2+y^2)^2}\right| + \left|\frac{2xy^4}{(x^2+y^2)^2}\right| \leq \left|6x\frac{1}{4}\right| + \left|\frac{2xy^4}{(y^2)^2}\right| = \frac{3}{2}|x| + 2|x| \longrightarrow 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(0,0)$$

2.3 (3) $\quad f(x,y) = \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{x^2+y^2}{xy}$

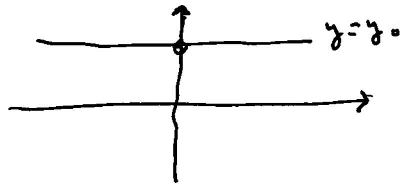
En $\{(x,y) \mid x=0\} \cup \{(x,y) \mid y=0\}$ f no està definida

Si $x_0 \neq 0$, $\displaystyle\lim_{(x,y)\to(x_0,0)} \frac{x^2+y^2}{xy} = ?$



Calculem lim segons la recta $x = x_0$

$$\lim_{y\to 0} \frac{x_0^2+y^2}{x_0 y} = \frac{x_0^2}{0} = \text{“}\infty\text{”}$$



Si $y_0 \neq 0$, $\displaystyle\lim_{(x,y)\to(0,y_0)} \frac{x^2+y^2}{xy} = ?$

Calculem lim segons la recta $y = y_0$:

$$\lim_{x\to 0} \frac{x^2+y_0^2}{x y_0} = \frac{y_0^2}{0} = \text{“}\infty\text{”}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = ?$$

Calculem lim segons la recta $y = mx$, $m \neq 0$

$$\lim_{x\to 0} \frac{x^2+m^2x^2}{x\,m\,x} = \lim_{x\to 0} \frac{1+m^2}{m} = \frac{1+m^2}{m} \quad \Rightarrow \quad \text{el límit no } \exists$$

2.3 (4) El domini de f més gran on pot ser contínua és

$$\mathcal{U} = \{(x,y) \mid x \neq 0, \ y \neq 0\}$$

<u>Derivades parcials en $\mathcal{U}$</u>

$$\frac{\partial f}{\partial x} = \frac{1}{y} - \frac{y}{x^2}$$

$$\frac{\partial f}{\partial y} = \frac{-x}{y^2} + \frac{1}{x}$$

Les derivades parcials són contínues en $\mathcal{U}$ $\Rightarrow$ f és C^1 en $\mathcal{U}$

$\Rightarrow$ f és diferenciable en $\mathcal{U}$

2.3 (5) (c) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

f és contínua en $U = \mathbb{R}^2 \setminus \{(0,0)\}$

$\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = ?$ $\quad 0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq \left|\dfrac{xy}{\sqrt{x^2}}\right| = |y| \Rightarrow$ lim $\exists$ i val 0

$\underset{0}{\downarrow}$ $\qquad \underset{0}{\downarrow}$

Si estenem el domini de f a $\mathbb{R}^2$ definint $f(0,0)=0$, f és contínua en $\mathbb{R}^2$

<u>Derivades parcials</u> (calculades a teoria ~~com~~ exemple)

$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{3/2}} \quad \text{si} \quad (x,y) \neq (0,0), \qquad \frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{3/2}} \quad \text{si} \quad (x,y) \neq (0,0), \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

<u>Continuïtat de les derivades en (0,0)</u>

$\lim_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial x} = ?$ Límit segons $y = mx$, $\lim_{x\to 0} \dfrac{m^3x^3}{(x^2+m^2x^2)^{3/2}} = \lim_{x\to 0} \dfrac{m^3}{(1+m^2)^{3/2}} \dfrac{x^3}{(x^2)^{3/2}}$ no $\exists$

$\lim_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial y} = ?$ Límit segons $y = mx$, $\lim_{x\to 0} \dfrac{x^3}{(x^2+m^2x^2)^{3/2}} = \lim_{x\to 0} \dfrac{1}{(1+m^2)^{3/2}} \dfrac{x^3}{(x^2)^{3/2}}$ no $\exists$

2.3 (b)

En $\mathbb{R}^2 \setminus \{(0,0)\}$ les derivades parcials són contínues

$\Rightarrow$ f és diferenciable en $\mathbb{R}^2 - \{(0,0)\}$

$$Df(x,y) = \left(\frac{y^3}{(x^2+y^2)^{3/2}} , \frac{x^3}{(x^2+y^2)^{3/2}} \right)$$

<u>Diferenciabilitat en (0,0)</u>

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - (0,0)\begin{pmatrix} x \\ y \end{pmatrix}}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \quad \text{no } \exists$$

$\Rightarrow$ f no és diferenciable en $(0,0)$

2.3 (7)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2} = ?$$

Límit segons $y = mx$, $\lim_{x\to 0} \frac{x^2 m x}{x^4+m^2x^2} = \lim_{x\to 0} \frac{m x}{x^2+m^2} = \frac{0}{m^2} = 0$ si $m \neq 0$

" $y=0$, $\lim_{x\to 0} \frac{0}{x^4+0} = 0$

" $x=0$, $\lim_{y\to 0} \frac{0}{0+y^2} = 0$

Límit segons $y = x^2$, $\lim_{x\to 0} \frac{x^2 \cdot x^2}{x^4+(x^2)^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$ $\Longrightarrow \nexists \lim$

f és contínua en $\mathbb{R}^2 - \{(0,0)\}$

f és diferenciable en $\mathbb{R}^2 - \{(0,0)\}$

$$\frac{\partial f}{\partial x} = \frac{2xy(x^4+y^2) - x^2y\,4x^3}{(x^4+y^2)^2} = \frac{2xy^3 - 2x^5y}{(x^4+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^4+y^2) - x^2y\,2y}{(x^4+y^2)^2} = \frac{x^6 - x^2y^2}{(x^4+y^2)^2}$$

2.3 (8)

Si definissim f en $(0,0)$ per $f(0,0)=0$, f no seria contínua en $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

f no seria diferenciable en $(0,0)$

2.4

La superfície $z = x^2 + y^4$ és la gràfica de la funció $f(x,y) = x^2 + y^4$

El pla tg en un punt $(x_0, y_0, f(x_0,y_0))$ ve donat per

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0)$$

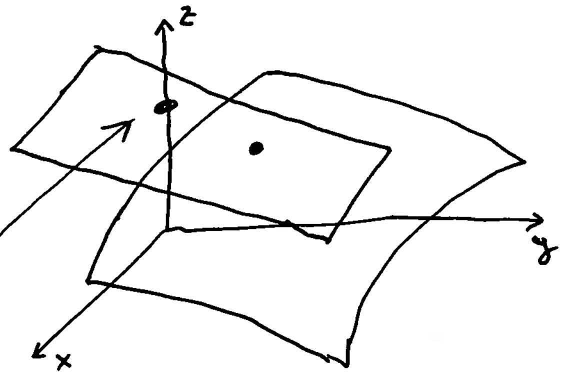
$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial x}(2,1) = 4$

$\frac{\partial f}{\partial y} = 4y^3$, $\frac{\partial f}{\partial y}(2,1) = 4$

$$z = 5 + 4(x-2) + 4(y-1)$$

$$\boxed{z = 4x + 4y - 7}$$

2.5 $z = e^x - e^y$ $(x_0, y_0, z_0) = (1, 1, 0)$



punt de tall

$f(x,y) = e^x - e^y$

$\frac{\partial f}{\partial x} = e^x$

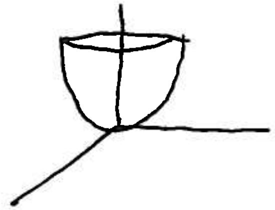
$\frac{\partial f}{\partial y} = -e^y$

Eq. pla : $z = 0 + e(x-1) - e(y-1) = e(x-y)$

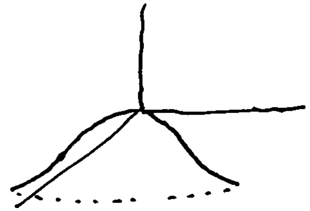
Eix z : $\begin{cases} x = 0 \\ y = 0 \end{cases}$

Intersecció pla amb eix z $z = 0$ $\rightarrow$ $\boxed{(0,0,0)}$

2.6 $f(x,y) = x^2 + y^2$



$g(x,y) = -x^2 - y^2 + xy^3$



Pla tg a graf f en $(0,0)$

$$z = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) = 0 \rightarrow \boxed{z=0}$$

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 2y$$

Pla tg a graf g en $(0,0)$

$$z = g(0,0) + \frac{\partial g}{\partial x}(0,0)(x-0) + \frac{\partial g}{\partial y}(0,0)(y-0) = 0 \rightarrow \boxed{z=0}$$

$$\frac{\partial g}{\partial x} = -2x + y^3 \quad , \quad \frac{\partial g}{\partial y} = -2y + 3xy^2$$

2.7 (a) $f(x,y) = x e^{x^2+y^2}$

$Df(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$Df(x,y) = \left(e^{x^2+y^2} + x e^{x^2+y^2} \cdot 2x \ , \ x e^{x^2+y^2} \cdot 2y \right)$$

(b) $f(x,y) = (x \cos y, \ x e^{xy} , \ 2xy)$

$Df(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$Df(x,y) = \begin{pmatrix} \cos y & -x \sin y \\ e^{xy} + xy e^{xy} & x^2 e^{xy} \\ 2y & 2x \end{pmatrix}$$

(c) $f(x,y,z) = (xz, \ y^2+z^2)$

$Df(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$Df(x,y,z) = \begin{pmatrix} z & 0 & x \\ 0 & 2y & 2z \end{pmatrix}$$

2.8 Sigui $A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ lineal. Proveu que A és diferenciable i calculeu $DA(x)$.

Hem de veure que $\exists T$ lineal t.q.

$$\lim_{x \to x_0} \frac{Ax - Ax_0 - T(x-x_0)}{\|x-x_0\|} = 0$$

Si prenem $T = A$, $Ax - Ax_0 - A(x-x_0) = 0$ per linealitat

i per tant el límit és $\lim_{x \to x_0} \frac{0}{\|x-x_0\|} = 0$

<u>Conclusió</u>: A és diferenciable en x_0 i $DA(x_0) = A$

L'aplicació
$$\begin{aligned} \mathbb{R}^n &\longrightarrow L(\mathbb{R}^n, \mathbb{R}^m) \\ x &\longmapsto DA(x) \end{aligned}$$
és constant

2.9 (a) $\left(0.99\, e^{0.02}\right)^8 = 1.0828509\ldots$

Considerem $f(x,y) = (x\, e^{y})^8$

$0.99 \approx 1$, $0.02 \approx 0$

Prenem $(x_0, y_0) = (1, 0)$

$$f(x,y) \simeq f(x_0,y_0) + Df(x_0,y_0)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

$$Df(x,y) = \left(8\,(x e^{y})^7 e^{y} \,,\; 8\,(x e^{y})^7\, x e^{y} \right)$$

$$f(0.99, 0.02) \simeq f(1,0) + Df(1,0)\begin{pmatrix} -0.01 \\ 0.02 \end{pmatrix} = 1 + (8, 8)\begin{pmatrix} -0.01 \\ 0.02 \end{pmatrix} = 1 - 0.08 + 0.16 = 1.08$$

2.9 (2)

(b) $\sqrt{4.01^2 + 3.98^2 + 2.02^2} = 6.000075\ldots$

Considerem $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

Prenem $(x_0, y_0, z_0) = (4, 4, 2)$

$$Df(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right)$$

$$f(4.01, 3.98, 2.02) \simeq f(4,4,2) + Df(4,4,2) \begin{pmatrix} 0.01 \\ -0.02 \\ 0.02 \end{pmatrix}$$

$$= 6 + \left(\frac{4}{6}, \frac{4}{6}, \frac{2}{6} \right) \begin{pmatrix} 0.01 \\ -0.02 \\ 0.02 \end{pmatrix}$$

$$= 6 + \frac{1}{100} \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right) \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 6.$$