Derivades d'ordre superior

Recordem que f és de classe C^1 si existeixen les funcions derivades parcials i són contínues. Per ex. si $f: \mathbb{R}^3 \to \mathbb{R}$, existeixen $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ i són contínues.

Si existeixen les funcions derivades parcials de $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ i són contínues diem que f és de classe C^2.

Escrivim $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y \partial z}$, etc.

En general, si $f: \mathbb{R}^n \to \mathbb{R}^m$ hi ha $n \cdot m$ derivades parcials primeres

i $n^2 \cdot m$ derivades parcials segones.

Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x^2 y, \; x e^y)$

$$\frac{\partial f_1}{\partial x} = 2xy, \quad \frac{\partial f_1}{\partial y} = x^2, \quad \frac{\partial f_2}{\partial x} = e^y, \quad \frac{\partial f_2}{\partial y} = x e^y$$

$$\frac{\partial^2 f_1}{\partial x^2} = 2y, \quad \frac{\partial^2 f_1}{\partial y \partial x} = 2x, \quad \frac{\partial^2 f_1}{\partial x \partial y} = 2x, \quad \frac{\partial^2 f_1}{\partial y^2} = 0, \quad \frac{\partial^2 f_2}{\partial x^2} = 0, \quad \frac{\partial^2 f_2}{\partial y \partial x} = e^y, \quad \text{etc.}$$

Si existeixen les funcions derivades parcials de les derivades parcials segones i són contínues diem que <u>f és de classe C^3</u>. Escrivim $\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}$, etc.

En general hi ha $n^3 \cdot m$ derivades parcials terceres.

Anàlogament es defineixen les funcions de classe C^4, C^5, ...

Diem que f és de classe C^∞ si és de classe C^k $\forall k \geq 0$.

<u>Ex</u> La funció $f(x,y) = e^x \cos y$ és de classe C^∞.

$$\frac{\partial f}{\partial x} = e^x \cos y, \qquad \frac{\partial f}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y, \quad \frac{\partial^2 f}{\partial y \partial x} = -e^x \sin y, \quad \frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y, \quad \frac{\partial^2 f}{\partial y^2} = -e^x \cos y$$

$$\frac{\partial^3 f}{\partial x^3} = e^x \cos y, \quad \frac{\partial^3 f}{\partial y \partial x^2} = -e^x \sin y, \quad \text{etc}$$

<u>Teorema</u> Si f és de classe C^2, les derivades parcials segones creuades

són iguals:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}, \qquad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

<u>Conseqüència</u>: si f és C^3, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ són C^2

$$\underbrace{\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial x}\right)}_{\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}} = \underbrace{\frac{\partial^2}{\partial y \partial x}\left(\frac{\partial f}{\partial x}\right)}_{\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial f}{\partial x}} \longrightarrow \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial x}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial f}{\partial z}\right) \longrightarrow \frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial y \partial x \partial z}$$

Fórmula de Taylor

Si $f: I \subset \mathbb{R} \to \mathbb{R}$, $a \in I$: f és k vegades derivable en a llavors

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \dots + \frac{1}{k!} f^{(k)}(a)(x-a)^k + R_k(x,a)$$

$$\text{amb} \quad \lim_{x \to a} \frac{R_k(x,a)}{(x-a)^k} = 0$$

També ho podem escriure

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2!} f''(a) h^2 + \dots + \frac{1}{k!} f^{(k)}(a) h^k + \tilde{R}_k(a,h)$$

$$\text{amb} \quad \lim_{h \to 0} \frac{\tilde{R}_k(a,h)}{h^k} = 0$$

Per a funcions de diverses variables sabem que si f és diferenciable en $a \in \mathcal{U} \subset \mathbb{R}^m$

$f: \mathcal{U} \subset \mathbb{R}^m \longrightarrow \mathbb{R}$

$$f(a+h) = f(a) + Df(a)h + R_1(a,h)$$

amb $$\lim_{h \to 0} \frac{R_1(a,h)}{\|h\|} = 0$$

i $$Df(a)h = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_m}(a)\right) \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(a)\, h_i$$

<u>Teorema</u> Si $f: \mathcal{U} \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ és de classe C^k, $a \in \mathcal{U} \subset \mathbb{R}^m$,

$$f(a+h) = f(a) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(a)\, h_i + \frac{1}{2!} \sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\, h_i h_j + \frac{1}{3!} \sum_{i,j,l=1}^{m} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_l}(a)\, h_i h_j h_l$$

$$+ \dots + \frac{1}{k!} \sum_{i_1, i_2, \dots, i_k = 1}^{m} \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}(a)\, h_{i_1} h_{i_2} \cdots h_{i_k} + R_k(a,h)$$

amb $$\lim_{h \to 0} \frac{R_k(a,h)}{\|h\|^k} = 0$$

Si escrivim $a+h=x \longrightarrow h = x-a$ i $h_i = x_i - a_i$

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\,(x_i - a_i) + \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\,(x_i - a_i)(x_j - a_j) + \dots$$

$$+ \frac{1}{k!}\sum_{i_1, i_2, \dots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_k}}(a)\,(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2})\cdots(x_{i_k} - a_{i_k}) + R_k(a,x)$$

amb $\displaystyle\lim_{x\to a} \frac{R_k(a,x)}{\|x-a\|^k} = 0$

$R_K(a,h)$ rep el nom de residu d'ordre K

Si f és de classe C^{K+1} podem escriure

$$R_K(a,h) = \frac{1}{(K+1)!}\int_0^1 (1-t)^K \sum_{i_1,\dots,i_{K+1}=1}^{m} \frac{\partial^{K+1} f}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_{K+1}}}(a+th)\, h_{i_1} h_{i_2}\cdots h_{i_{K+1}}\, dt$$

<u>Ex</u> Escriure la fórmula de Taylor fins a ordre 2 de $f(x,y) = (x+y)^2$ en $(1,2)$

$f(1,2) = (1+2)^2 = 9$

$\frac{\partial f}{\partial x} = 2(x+y)$, $\frac{\partial f}{\partial y} = 2(x+y)$ $\longrightarrow$ $\frac{\partial f}{\partial x}(1,2) = 6$, $\frac{\partial f}{\partial y}(1,2) = 6$

$\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = 2$, $\frac{\partial^2 f}{\partial y^2} = 2$

$$f(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2) + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(1,2)(x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,2)(x-1)(y-2) + \frac{\partial^2 f}{\partial y^2}(1,2)(y-2)^2\right] + R_2$$

$$f(x,y) = 9 + 6(x-1) + 6(y-2) + \frac{1}{2}\left[2(x-1)^2 + 4(x-1)(y-2) + 2(y-2)^2\right] + R_2$$