Llista de problemes (7)

3.1 Calculeu les següents integrals en $R = [0,1] \times [0,1]$

a) $\int_R (xy)^2 \cos x^3 \, dx\, dy$, b) $\int_R (ax+by+c)\, dx\, dy$, c) $\int_R \sin(x+y)\, dx\, dy$

d) $\int_R y e^{xy}\, dx\, dy$

3.2 Calculeu el volum de la regió limitada pels plans xz, yz, xy, $x=1$, $y=1$ i la superfície $z = x^2 + y^4$.

3.3 Siguin $f : [a,b] \longrightarrow \mathbb{R}$, $g : [c,d] \longrightarrow \mathbb{R}$ contínues. Si $R = [a,b] \times [c,d]$ proveu que

$$\int_R f(x)\, g(y)\, dx\, dy = \left(\int_a^b f(x)\, dx \right) \left(\int_c^d g(y)\, dy \right)$$

3.4 Calculeu

$$\int_R x(y^2 - 6x)\, dx\, dy, \qquad R = [-1,3] \times [1,2]$$

3.5 Sigui f contínua, $f \geqslant 0$, definida en un rectangle R. Proveu que $\int_R f = 0$ implica $f = 0$

3.6 Calculeu

$$\int_R x \sin y \, dx \, dy \quad , \quad \text{on} \quad R = [0,2] \times [0, 2\pi] .$$

3.1

a) $$\int_R (xy)^2 \cos x^3 \, dx\, dy = \int_0^1 \left(\int_0^1 x^2 y^2 \cos x^3 \, dx \right) dy = \int_0^1 \left(y^2 \left[\frac{\sin x^3}{3} \right]_0^1 \right) dy$$

$$= \int_0^1 y^2 \frac{\sin 1}{3} \, dy = \frac{\sin 1}{3} \left[\frac{y^3}{3} \right]_0^1 = \frac{\sin 1}{9}$$

b) $$\int_R (ax+by+c)\, dx\, dy = \int_0^1 \left(\int_0^1 (ax+by+c)\, dx \right) dy = \int_0^1 \left[a\frac{x^2}{2} + byx + cx \right]_0^1 dy$$

$$= \int_0^1 \left(\frac{a}{2} + by + c - 0 \right) dy = \left[\frac{a}{2} y + b\frac{y^2}{2} + cy \right]_0^1 = \frac{a}{2} + \frac{b}{2} + c$$

c) $$\int_R \sin(x+y)\, dx\, dy = \int_0^1 \left(\int_0^1 \sin(x+y)\, dx \right) dy = \int_0^1 - \cos(x+y) \Big]_0^1 dy$$

$$= \int_0^1 \left(-\cos(1+y) + \cos y \right) dy = \Big[-\sin(1+y) + \sin y \Big]_0^1 = -\sin 2 + \sin 1 + \sin 1 - \sin 0$$

$$= 2 \sin 1 - \sin 2$$

3.1 (2)

d) $$\int_R y e^{xy}\,dx\,dy = \int_0^1 \left(\int_0^1 y e^{xy}\,dx \right) dy = \int_0^1 \left. e^{xy} \right]_0^1 dy$$

$$= \int_0^1 (e^y - e^0)\,dy = \left[e^y - y \right]_0^1 = e - 1 - (e^0 - 0) = e - 2$$

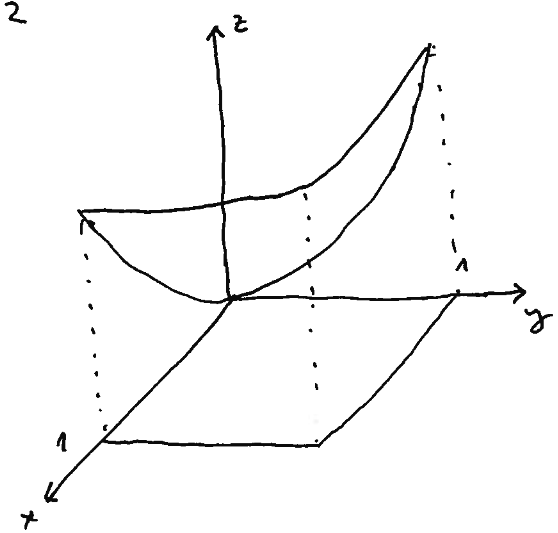
Ara calculem la integral amb l'altre ordre

$$\int_0^1 \left(\int_0^1 y e^{xy}\,dy \right) dx \underset{*}{=} \int_0^1 \left[y \frac{e^{xy}}{x} - \frac{e^{xy}}{x^2} \right]_0^1 dx = \int_0^1 \left(\frac{e^x}{x} - \frac{e^x}{x^2} + \frac{1}{x^2} \right) dx$$

No té primitiva

$$* \quad \int \underbrace{y}_{u} \underbrace{e^{xy}}_{v'}\,dy = uv - \int u'v\,dy = y \frac{e^{xy}}{x} - \int \frac{e^{xy}}{x}\,dy = y \frac{e^{xy}}{x} - \frac{e^{xy}}{x^2}$$

3.2



$$\text{volum} = \int_{[0,1]\times[0,1]} (x^2+y^4)\,dx\,dy = \int_0^1 \left(\int_0^1 (x^2+y^4)\,dx \right) dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^4 x \right]_0^1 dy = \int_0^1 \left(\frac{1}{3} + y^4 \right) dy = \left[\frac{1}{3} y + \frac{y^5}{5} \right]_0^1 = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

3.3

$$\int_R f(x)\,g(y)\,dx\,dy = \int_a^b \left(\int_c^d f(x)\,g(y)\,dy \right) dx$$

$$= \int_a^b f(x) \left(\int_c^d g(y)\,dy \right) dx = \int_c^d g(y)\,dy \int_a^b f(x)\,dx$$

3.4 $\quad R = [-1,3] \times [1,2]$

$$\int_R x(y^2-6x)\,dx\,dy = \int_{-1}^{3}\left(\int_1^2 (xy^2-6x^2)\,dy\right)dx$$

$$= \int_{-1}^{3}\left[x\frac{y^3}{3} - 6x^2y\right]_1^2 dx = \int_{-1}^{3}\left(x\frac{8}{3} - 12x^2 - x\frac{1}{3} + 6x^2\right)dx$$

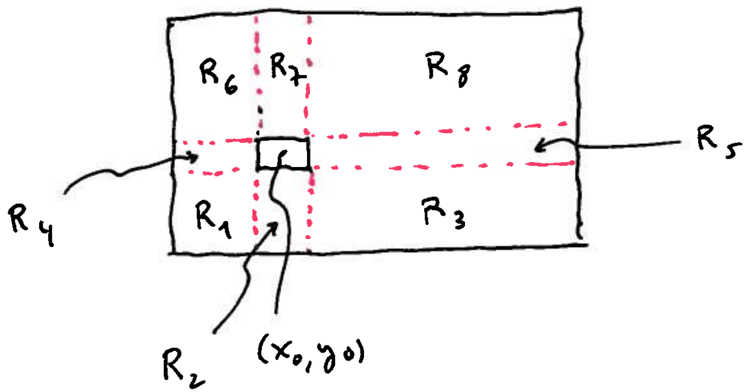
$$= \left[\frac{4}{3}x^2 - 4x^3 - \frac{x^2}{6} + 2x^3\right]_{-1}^{3} = \left[\frac{7}{6}x^2 - 2x^3\right]_{-1}^{3}$$

$$= \frac{7}{6}9 - 2\cdot 27 - \left(\frac{7}{6}+2\right) = \frac{7}{6}8 - 2\cdot 28 = -\frac{140}{3}$$

3.5 f contínua, $f \geq 0$, $f: R \to \mathbb{R}$, $\int_R f = 0$

Suposem que $\exists\, (x_0, y_0) \in R$ t.q. $f(x_0, y_0) = a > 0$.

f contínua $\Rightarrow \exists\, R^*$ t.q. $f(x,y) > \frac{a}{2}$ $\forall (x,y) \in R^*$



$$R = R^* \cup \left(\bigcup_{i=1}^{8} R_i \right)$$

$$\int_{R^*} f \geq \int_{R^*} \frac{a}{2} = \frac{a}{2}\, \text{àrea}(R^*) > 0$$

$$\int_{R_i} f \geq \int_{R_i} 0 = 0, \quad \forall i$$

$$\int_R f = \int_{R^*} f + \sum_{i=1}^{8} \int_{R_i} f > 0$$

(contradicció)

3.6

$$\int_R x \sin y \, dx\, dy = \int_0^2 \left(\int_0^{2\pi} x \sin y \, dy \right) dx = \int_0^2 \Big[-x \cos y \Big]_0^{2\pi} dx$$

$$= \int_0^2 (-x \cos 2\pi + x \cos 0) \, dx = \int_0^2 (-x + x) \, dx$$

$$= \int_0^2 0 \, dx = 0$$