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An Introduction to Inductive Definitions*

PETER ACZEL

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Introduction

Inductive definitions of sets are often informally presented by giving some rules for generating elements of the set and then adding that an object is to be in the set only if it has been generated according to the rules. An equivalent formulation is to characterise the set as the smallest set closed under the rules.

Of course the basic example of an inductive definition is the one generating the natural numbers. But it has long proved a useful device when presenting the syntax of a formal language. Further examples of its informal use may be found in logic and other branches of mathematics. Post [1943] realised that the finitary inductive definitions used in presenting the syntax of any standard formal system could all be put in a canonical form, and the general class of such inductions could be fruitfully studied in abstraction from any specific formal system. Since then inductive definitions have played an important role in the development of ordinary recursion theory and its generalisations. Recent work has tended to present the theory of inductive definitions in abstraction from the original motivating intuitions. Our aim here is to give an introduction to the subject which will connect the informal examples with the recent formulation in terms of iterations of monotone operators. We have in mind a reader familiar with the concept of a formal system and with the elements of ordinary recursion theory.

Most of our exposition will be concerned with monotone induction and its role in extensions of recursion theory. But in 3.5 we review some of the work on non-monotone induction, and outline there the separate motivation that has led to its development. In Section 4 we briefly consider inductive definitions in a more general context. For a detailed development of the theory of positive induction see Moschovakis [1974a]. Several papers on inductive definability may be found in Fenstad and Hinman [1974]. These include a survey, Gandy [1974] and the papers Aanderaa [1974], Cenzer [1974], and Richter and Aczel [1974]. Moschovakis [1976] gives an abstract algebraic approach to the general theory that applies to both monotone and non-monotone induction and also to recursion in higher types.

1. What are inductive definitions?

1.1. Inductive definitions as generalized formal systems

Inductive definitions are used repeatedly when logicians describe the syntax of their languages. For example the terms of a first order language are defined to be the smallest set of expressions containing the variables and constants and closed under the term formation rule:

If $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol of the language then the expression $f(t_1, \ldots, t_n)$ is a term.

Similarily the formulae of a first order language are defined to be the smallest set of expressions containing the atomic formulae that are closed under the various formulae formation rules for the logical symbols $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\exists x$, $\forall x$.

For our purpose, the most useful example to consider will be the definition of the class of theorems of a formal system. Consider the Hilbert-style system **H** for first order logic used in Chapter A.1. The set Th(**H**) of theorems of **H** is there defined in terms of a notion of "proof" for **H**. But Th(**H**) may also be characterized as the smallest set of formulae, containing the axioms, that is closed under the rules of inference. Each instance of a rule of inference has the form:

(*) From the premisses θ for $\theta \in X$, infer the conclusion ψ.

In case of modus ponens X consists of two premisses φ and $(\varphi \rightarrow \psi)$. Instances of the generalization rule only have one premiss. It is convenient to consider an axiom scheme as a special form of rule of inference where in each instance the set of premisses is empty. Using this convention the formal system **H** determines a set $\Phi_{\mathbf{H}}$ of pairs (X, ψ) such that (*) is an instance of a rule of inference of **H**. Then Th(**H**) is simply the smallest set closed under (*) for $(X, \psi) \in \Phi_{\mathbf{H}}$.

Generalizing we obtain the following definitions.

1.1.1. Definition. (i) A *rule* is a pair (X, x) where X is a set, called the set of *premisses* and x is the *conclusion*. The rule will usually be written $X \rightarrow x$.

(ii) If Φ is a set of rules (also called a *rule set* below), then a set A is *Φ-closed* if each rule in Φ whose premisses are in A also has its conclusion in A. We shall write $\Phi : X \rightarrow x$ to denote that the rule $X \rightarrow x$ is in Φ. So A is Φ-closed if $\Phi : X \rightarrow x \;\&\; X \subseteq A$ implies $x \in A$.

(iii) If Φ is a rule set, then $I(\Phi)$, the *set inductively defined by Φ*, is given by $I(\Phi) = \bigcap\{A \mid A \text{ is } \Phi\text{-closed}\}$.

Note. Φ-closed sets exist; e.g. the set of conclusions of rules in Φ. Also, the intersection of any collection of Φ-closed sets is Φ-closed. In particular $I(\Phi)$ is Φ-closed and hence *$I(\Phi)$ is the smallest Φ-closed set.*

Returning to our example we see that $\mathrm{Th}(\mathbf{H}) = I(\Phi_{\mathbf{H}})$. Similarly, rule sets can easily be found for inductively defining the sets of terms and formulae of a first order language.

Note. What is usually called a rule of inference or formation rule corresponds to what we have called a rule set. It is the instances of rules of inference or formation rules that correspond to what we have called a rule.

Perhaps the most familiar example of an inductive definition in mathematics is the one that characterizes the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$ as the smallest set containing 0 and closed under the successor function, i.e., $\omega = I(\Phi_\omega)$ where Φ_ω consists of the rule $\emptyset \to 0$ and the rules $\{n\} \to n+1$ for $n \in \omega$. This characterization justifies the principle of *mathematical induction*: If $\mathcal{P}$ is a property that holds of 0 and holds of $n+1$ whenever it holds of n, then $\mathcal{P}$ holds for all natural numbers, i.e. $\{n \in \omega \mid \mathcal{P}(n)\}$ is Φ_ω-closed implies $\omega \subseteq \{n \in \omega \mid \mathcal{P}(n)\}$.

Generalizing, we see that to each rule set Φ there is a principle of *Φ-induction*: If $\mathcal{P}$ is a property, such that whenever $\Phi : X \to x$ and $\forall y \in X \mathcal{P}(y)$ then $\mathcal{P}(x)$, then $\mathcal{P}(a)$ holds for every $a \in I(\Phi)$.

The above principle is the natural tool to use in proving properties of $I(\Phi)$.

1.1.2. EXAMPLE. To show that every theorem of **H** is universally valid in every structure, it suffices to show that for each structure $\mathfrak{M}$

$$\{\varphi(v_1, \ldots, v_n) \mid \mathfrak{M} \models \forall v_1 \cdots \forall v_n \varphi(v_1 \cdots v_n)\} \quad \text{is } \Phi_{\mathbf{H}}\text{-closed.}$$

So far all our inductive definitions have been finitary, in the sense that each rule has only finitely many premisses. For such Φ we can generalize the standard notion of "proof" for formal systems such as **H**.

1.1.3. DEFINITION. $a_0, \ldots, a_n$ is a *(finite length) Φ-proof of b* if

(i) $a_n = b$,

(ii) for all $m \leq n$ there is an $X \subseteq \{a_i \mid i < m\}$ such that $\Phi : X \to a_m$.

1.1.4. PROPOSITION. *For finitary Φ,*

$$I(\Phi) = \{b \mid b \text{ has a } \Phi\text{-proof}\}.$$

To show that every $b \in I(\Phi)$ has a Φ-proof it suffices to show that the right hand side is Φ-closed and use Φ-induction. For the converse direction if $a_0, \ldots, a_n$ is a Φ-proof it suffices to show by induction on $m \leq n$ that $a_m \in I(\Phi)$.

1.2. The well-founded part of a relation

Let $<$ be a binary relation on a set A. The well-founded part of $<$ is the set $W(<)$ of $a \in A$ such that there is no infinite descending sequence $a > a_0 > a_1 > \cdots$. The relation $<$ is a *well-founded* relation if $A = W(<)$. $W(<)$ can be inductively defined as follows. Let $\Phi_<$ be the set of rules $(<a) \to a$ for $a \in A$, where $(<a) = \{x \in A \mid x < a\}$.

1.2.1. Proposition. $W(<) = I(\Phi_<)$.

Proof. To show that $I(\Phi_<) \subseteq W(<)$ it suffices to show that $W(<)$ is $\Phi_<$-closed and use $\Phi_<$-induction. So assume $(<a) \subseteq W(<)$. Now if $a > a_0 > a_1 > \cdots$, then $a_0 \in (<a) \subseteq W(<)$. But as $a_0 > a_1 > \cdots$, $a_0 \notin W(<)$ which gives a contradiction. Hence $a_0 \in W(<)$.

Conversely, to show $W(<) \subseteq I(\Phi_<)$ let $a \notin I(\Phi_<)$. We shall find $a > a_0 > a_1 > \cdots$ showing that $a \notin W(<)$. As $a \notin I(\Phi_<)$, then $(<a) \not\subseteq I(\Phi_<)$. Hence there is an $a_0 < a$ such that $a_0 \notin I(\Phi_<)$. Repeating we can find $a_1 < a_0$ such that $a_1 \notin I(\Phi_<)$. Repeating indefinitely we obtain $a > a_0 > a_1 > \cdots$. $\square$

Note that a form of the axiom of choice is needed (the axiom of dependent choices).

The principle of $\Phi_<$-induction, becomes, when $<$ is well-founded, the principle of transfinite induction along a well-founded relation. Associated with transfinite induction is the method of definition by transfinite recursion. This enables one to define a unique function f on $W(<)$ so that for $a \in W(<)$, $f(a)$ is defined in terms of $f(x)$ for $x < a$. The uniqueness and existence of such f can be justified by suitable instances of transfinite induction. As an example we may assign to each $a \in W(<)$ an ordinal $|a|_<$ so that

$$|a|_< = \operatorname{Sup}\{|x|_< + 1 \mid x < a\}.$$

The ordinal $|<|$ of the well-founded part of $<$ is defined as

$$|<| = \operatorname{Sup}\{|a|_< + 1 \mid a \in W(<)\}.$$

An inductive definition can often be rephrased in the form $\Phi_<$ for a suitable $<$.

1.2.2. DEFINITION. The rule set Φ is *deterministic* if

$$\Phi : X_1 \rightarrow x \;\&\; \Phi : X_2 \rightarrow x \quad \text{implies} \quad X_1 = X_2.$$

1.2.3. EXAMPLE. $\Phi_<$ is always deterministic. So is Φ_ω. Also the rule sets defining the terms and formulae of first order logic are.

Note that Φ_H is not deterministic.

Now let Φ be deterministic and let A be the set of conclusions of rules in Φ. For $x, y \in A$ let $x < y$ if $\Phi : X \rightarrow y$ for some set X such that $x \in X$ and $X \subseteq A$. Then $\Phi_<$ is the set of rules $X \rightarrow x$ in Φ such that $X \subseteq A$. Hence we get

1.2.4. PROPOSITION. *For deterministic* Φ,

$$I(\Phi) = I(\Phi_<).$$

For deterministic Φ, functions on $I(\Phi)$ can be defined by recursion on the way objects in $I(\Phi)$ are generated, as in transfinite recursion. An example of this from syntax is the operation of substitution. Given an individual variable v and a term t, the function assigning to each formula $\varphi(v)$ the formula $\varphi(t)$ obtained by substituting t for all free occurrences of v in $\varphi(v)$ is naturally defined by a recursion on the way a formula $\varphi(v)$ is generated.

1.3. Inductive definitions as operators

Let $\varphi : \mathrm{Pow}(A) \rightarrow \mathrm{Pow}(A)$ where $\mathrm{Pow}(A)$ denotes the set of all subsets of A. The operator φ is *monotone* if $X \subseteq Y \subseteq A$ implies $\varphi(X) \subseteq \varphi(Y)$. Given φ let Φ_φ be the set of rules $X \rightarrow x$ such that $X \subseteq A$ and $x \in \varphi(X)$. For monotone φ, $X \subseteq A$ is Φ_φ-closed just in case $\varphi(X) \subseteq X$. So $I(\Phi_\varphi) = \bigcap\{X \subseteq A \mid \varphi(X) \subseteq X\}$. Hence it is natural to extend the terminology concerning inductive definitions to monotone operators φ and we write $I(\varphi)$ for $\bigcap\{x \subseteq A \mid \varphi(X) \subseteq X\}$ and call it the set *inductively defined by* φ. All inductive definitions can be obtained using monotone operators. For if Φ is a rule set on A (i.e. $X \cup \{x\} \subseteq A$ whenever $\Phi : X \rightarrow x$) we may define a monotone operator $\varphi : \mathrm{Pow}(A) \rightarrow \mathrm{Pow}(A)$ by

$$\varphi(Y) = \{x \in A \mid \Phi : X \rightarrow x \text{ for some } X \subseteq Y\} \quad \text{for } Y \subseteq A.$$

Then $Y \subseteq A$ is Φ-closed just in case $\varphi(Y) \subseteq Y$ so that $I(\Phi) = I(\varphi)$.

For monotone operators φ there is a useful alternative characterization of $I(\varphi)$ using transfinite iterations φ^λ of φ for ordinals λ. Define $\varphi^\lambda \subseteq A$ by transfinite recursion on the ordinal λ so that

$$\varphi^\lambda = \bigcup_{\mu<\lambda} \varphi^\mu \cup \varphi\left(\bigcup_{\mu<\lambda} \varphi^\mu\right).$$

Also define $\varphi^\infty = \bigcup_\lambda \varphi^\lambda$ where λ ranges over all ordinals.

If we write $\varphi^{<\lambda}$ for $\bigcup_{\mu<\lambda} \varphi^\mu$, then

$$\varphi^\lambda = \varphi^{<\lambda} \cup \varphi(\varphi^{<\lambda}).$$

The sets $\varphi^{<\lambda}$ may be directly defined by the transfinite recursion

$$\varphi^{<\lambda} = \bigcup_{\mu<\lambda} \varphi(\varphi^{<\mu}),$$

or alternatively by

$$\varphi^{<0} = \emptyset, \qquad \varphi^{<\lambda+1} = \varphi^{<\lambda} \cup \varphi(\varphi^{<\lambda}),$$

$$\varphi^{<\lambda} = \bigcup_{\mu<\lambda} \varphi^{<\mu} \quad \text{for limit } \lambda.$$

We may then define $\varphi^\lambda = \varphi^{<\lambda+1}$.

Note. The literature often uses the notation φ^λ for what we have called $\varphi^{<\lambda}$. We have adopted the notation initiated in MOSCHOVAKIS [1974a].

Because $X \subseteq I(\varphi)$ implies $\varphi(X) \subseteq I(\varphi)$, a transfinite induction shows that $\varphi^\lambda \subseteq I(\varphi)$ for all ordinals λ. Hence if we let $\varphi^\infty = \bigcup_\lambda \varphi^\lambda$ then $\varphi^\infty \subseteq I(\varphi)$.

As $\mu < \lambda$ implies $\varphi^{<\mu} \subseteq \varphi^{<\lambda} \subseteq A$, and A is a set there must be an ordinal $\bar{\lambda}$ such that $\varphi^{<\bar{\lambda}+1} = \varphi^{<\bar{\lambda}}$. It follows that $\varphi^\lambda = \varphi^{\bar{\lambda}} = \varphi^{<\bar{\lambda}}$ for all $\lambda \geq \bar{\lambda}$ so that $\varphi^\infty = \varphi^{<\bar{\lambda}}$. Hence $\varphi(\varphi^\infty) = \varphi(\varphi^{<\bar{\lambda}}) \subseteq \varphi^{\bar{\lambda}} = \varphi^\infty$. Hence by φ-induction $I(\varphi) \subseteq \varphi^\infty$. Also for monotone φ, $\mu < \lambda$ implies $\varphi(\varphi^{<\mu}) \subseteq \varphi(\varphi^{<\lambda})$ so that $\varphi^{<\lambda} = \bigcup_{\mu<\lambda} \varphi(\varphi^{<\mu}) \subseteq \varphi(\varphi^{<\lambda})$ and hence $\varphi^\lambda = \varphi(\varphi^{<\lambda})$. It follows that $\varphi^\infty = \varphi(\varphi^\infty)$, φ^∞ is a fixed point of φ (in fact the least one). Thus we have proved:

1.3.1. PROPOSITION. *For monotone* $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$,

(i) $I(\varphi) = \varphi^\infty$, *and*

(ii) $I(\varphi)$ *is the least fixed point of* φ.

The definition of φ^∞ above does not require the operator φ to be monotone. Hence it is natural to extend the notion of an inductive definition to non-monotone operators by calling φ^∞ the *set inductively defined by* φ for *any* operator $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$. It has turned out, perhaps rather suprisingly, that the theory of non-monotone inductive definitions is as rich and interesting as the theory for monotone operators, even though naturally occurring examples of non-monotone induction are harder to come by. Perhaps their main motivation can be seen in terms of systems of notations for ordinals. Associated with any operator $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$ is a function $| \;\; |_\varphi$ mapping φ^∞ into an initial segment of ordinals, given by

$$|a|_\varphi = \text{least } \lambda \text{ such that } a \in \varphi^\lambda \quad \text{for } a \in \varphi^\infty.$$

Let $|\varphi| = \mathrm{Sup}\{|a|_\varphi + 1 \mid a \in \varphi^\infty\}$. Then φ^∞ is a set of notations for the ordinals $< |\varphi|$ via the mapping $| \;\; |_\varphi : \varphi^\infty \to |\varphi|$. (Note that we follow the standard convention of identifying an ordinal with its set of predecessors.) The ordinal may also be characterized as the least ordinal $\bar{\lambda}$ such that $\varphi^{\bar{\lambda}} = \varphi^{<\bar{\lambda}}$. Hence $\varphi^\infty = \varphi^{|\varphi|} = \varphi^{<|\varphi|}$.

1.3.2. EXAMPLE. Let $<$ be a binary relation on the set A and let φ be the monotone operator corresponding to the rule set $\Phi_<$, so that $W(<) = I(\varphi) = \varphi^\infty$. Then it is easily seen that $\varphi^\lambda = \{a \in W(<) \mid |a|_\varphi \leq \lambda\}$, so that $|a|_\varphi = |a|_<$ for $a \in W(<)$ and $|\varphi| = |<|$.

An interesting general problem connected with an operator $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$ is to characterize or estimate the ordinal $|\varphi|$. As $| \;\; |_\varphi : \varphi^\infty \to |\varphi|$ is a surjection and $\varphi^\infty \subseteq A$ the cardinality of $|\varphi|$ must be $\leq$ the cardinality of A.

For monotone operators a better bound can often be found. If κ is a cardinal let us say that φ is *κ-based* if:

$$x \in \varphi(X) \quad \text{implies} \quad x \in \varphi(Y) \quad \text{for some } Y \subseteq X \text{ of cardinality } < \kappa.$$

1.3.3. EXAMPLE. If Φ is a finitary rule set on A, then the monotone operator φ corresponding to it is ω-based. In general if every set of premisses of a rule in Φ has cardinality $< \kappa$, then φ is κ-based.

1.3.4. PROPOSITION. *Let φ be a κ-based monotone operator where κ is regular. Then $|\varphi| \leq \kappa$, so that $I(\varphi) = \varphi^\kappa$.*

PROOF. It suffices to show that $\varphi^\kappa \subseteq \varphi^{<\kappa}$. So let $x \in \varphi^\kappa = \varphi(\varphi^{<\kappa})$. Then $x \in \varphi(X)$ for some $X \subseteq \varphi^{<\kappa}$ of cardinality $<\kappa$. By the regularity of κ, $X \subseteq \varphi^{<\lambda}$ for some $\lambda < \kappa$, so that $x \in \varphi(\varphi^{<\lambda}) = \varphi^\lambda \subseteq \varphi^{<\kappa}$ as required. □

1.3.5. EXAMPLE. If Φ is a finitary rule set with corresponding monotone operator φ, then $|\varphi| \leq \omega$ and $I(\varphi) = \varphi^{<\omega} = \bigcup_{n<\omega} \varphi^{<n}$ where $\varphi^{<0} = \emptyset$ and $\varphi^{<n+1} = \varphi(\varphi^{<n})$ for $n < \omega$.

1.4. Concepts of "proof" for monotone induction

In 1.1.3 we formulated a notion of finite length proof appropriate for finitary rule sets. We now consider a more general notion.

1.4.1. DEFINITION. Let φ be a monotone operator on A. A transfinite sequence $\{a_\mu\}_{\mu\leq\lambda}$ is a *φ-proof of b with length λ* if

(i) $a_\lambda = b$.

(ii) $a_\nu \in \varphi(\{a_\mu \mid \mu < \nu\})$ for all $\nu \leq \lambda$.

As in Proposition 1.1.4 we get:

1.4.2. PROPOSITION. (i) *For any regular cardinal κ, $\varphi^{<\kappa} = \{a \in A \mid a$ has a φ-proof of length $<\kappa\}$.*

(ii) *$I(\varphi) = \{a \in A \mid a$ has a φ-proof$\}$.*

It is sometimes convenient to use an alternative notion of proof that uses well-founded trees instead of transfinite sequences. An example is the notion of derivation for the Gentzen style system **G** of Chapter A.1. There the appropriate rule set is finitary so that the well-founded trees are actually finite.

1.4.3. DEFINITION. A *(well-founded) tree* T is a set of finite sequences of length >0 such that

(i) There is exactly one sequence of length one in T. It is called the *root* (a_T) of the tree.

(ii) If $(a_1, \ldots, a_{n+1}) \in T$, then $(a_1, \ldots, a_n) \in T$.

(iii) T is well-founded in the sense that there is no infinite sequence $a_1, a_2, \ldots$ such that $(a_1, \ldots, a_n) \in T$ for all $n > 0$. Alternatively, the relation $<_T$ is well-founded, where

$$(a_1, \ldots, a_n) <_T (b_1, \ldots, b_n) \quad \text{iff} \quad n = m + 1$$

and $a_i = b_i$ for $i = 1 \cdots m$.

Define the *length* $|T|$ of T to be $|(a_T)|_{<_T}$.

1.4.4. Definition. If Φ is a rule set, a tree T is a *Φ-proof* of a if $a = a_T$ and $\Phi : T_{(a_1,\dots,a_n)} \to a_n$ whenever $(a_1,\dots,a_n) \in T$, where

$$T_{(a_1,\dots,a_n)} = \{a \mid (a_1,\dots,a_n,a) \in T\}.$$

1.4.5. Proposition. (i) $I(\Phi) = \{a \mid a$ *has a tree* Φ*-proof*$\}$.

(ii) *If* Φ *is a rule set on* A *with corresponding monotone operator* $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$, *then for all ordinals* λ,

$$\varphi^\lambda = \{a \in A \mid a \text{ has a tree } \Phi\text{-proof of length } \le \lambda\}.$$

Proof. (i) follows easily from (ii). (ii) will be proved by induction on λ. Let X^λ denote the right-hand side of (ii) and let $X^{<\lambda} = \bigcup_{\mu<\lambda} X^\mu$. By induction hypothesis $\varphi^{<\lambda} = X^{<\lambda}$.

Let $a \in X^\lambda$. Then a has a Φ-proof T with $|T| \le \lambda$. For each $x \in T_{(a)}$ let

$$T^x = \{(x, x_1,\dots,x_n) \mid n \ge 0 \;\&\; (a, x, x_1,\dots,x_n) \in T\}.$$

Then T^x is a Φ-proof of x with $|T^x| < \lambda$. Hence $T_{(a)} \subseteq X^{<\lambda} = \varphi^{<\lambda}$. As $\Phi : T_{(a)} \to a$ it follows that $a \in \varphi(\varphi^{<\lambda}) = \varphi^\lambda$.

Conversely, let $a \in \varphi^\lambda$. Then $a \in \varphi(X^{<\lambda})$. For each $x \in X^{<\lambda}$ let T^x be a Φ-proof of x with $|T^x| < \lambda$. Let

$$T = \{(a)\} \cup \{(a, x, x_1,\dots,x_n) \mid n \ge 0 \;\&\; x \in X^{<\lambda} \;\&\; (x, x_1,\dots,x_n) \in T^x\}.$$

Then T is a Φ-proof of a with $|T| \le \lambda$, so that $a \in X^\lambda$. $\square$

1.5. Monotone induction and games

For those familiar with the elementary concepts associated with games we give a game-theoretic characterization of $I(\Phi)$ for an arbitrary rule set Φ.

For each a we define a game $G(\Phi, a)$ between two players I and II who move alternatively when possible. The play starts by II choosing $a_0 = a$. If after n pairs of moves player II chooses a_n then player I must respond by choosing a set X_n such that $\Phi : X_n \to a_n$ and then II must respond by choosing $a_{n+1} \in X_n$. If either player cannot move then he loses. If the game continues indefinitely, then player I loses.

1.5.1. Proposition. $a \in I(\Phi)$ *iff player* I *has a winning strategy in the game* $G(\Phi, a)$.

Proof. Let W be the set of those a such that the right-hand side holds. Let $\Phi : X \to a$ with $X \subseteq W$. For each $x \in X$ let σ_x be a winning strategy for I in $G(\Phi, x)$. Define the following strategy σ for I in $G(\Phi, a)$. I starts by playing X and then if II chooses $x \in X$ then I continues by using the strategy σ_x. σ is clearly a winning strategy. Hence $a \in W$. Thus W is Φ-closed so that by Φ-induction $I(\Phi) \subseteq W$.

For the converse let $a \in W$. Let σ be a winning strategy for I in $G(\Phi, a)$. Let T be the set of possible finite sequences of moves of II when I follows σ. Then observe that T is a tree Φ-proof of a so that by Proposition 1.4.5, $a \in I(\Phi)$. Thus $W \subseteq I(\Phi)$. □

1.6. Kernels — the dual of an inductive definition

Sometimes inductive definitions present themselves more naturally in a dual form.

If Φ is a rule set let us say that a set X is *Φ-dense* if for every $x \in X$ there is a set $Y \subseteq X$ such that $\Phi : Y \to x$. Define the *kernel* $K(\Phi) = \bigcup\{X \mid X \text{ is } \Phi\text{-dense}\}$. $K(\Phi)$ itself is Φ-dense and is the largest Φ-dense set. If Φ is a rule set on A and $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$ is the monotone operator associated with Φ then $X \subseteq A$ is Φ-dense iff $X \subseteq \varphi(X)$. Hence $K(\Phi) = \bigcup\{X \subseteq A \mid X \subseteq \varphi(X)\}$ and we shall define $K(\varphi) = \bigcup\{X \subseteq A \mid X \subseteq \varphi(X)\}$ for any monotone φ. To make explicit the duality between the kernel construction and induction define the *dual* of an operator φ to be the operator $\breve{\varphi}$ given by $\breve{\varphi}(X) = \neg\varphi(\neg X)$ where $\neg X = A - X$ for $X \subseteq A$. Then $X \subseteq A$ is φ-dense iff $\neg X$ is φ-closed, so that $K(\varphi) = \neg I(\breve{\varphi})$. It follows that $K(\varphi)$ can be defined in terms of transfinite iterations $\varphi^{[\lambda]} = \neg\breve{\varphi}^{\lambda}$, as $K(\varphi) = \bigcap_\lambda \varphi^{[\lambda]}$ where $\varphi^{[\lambda]} = \varphi(\bigcap_{\mu<\lambda} \varphi^{[\mu]})$.

An example of the above is the Cantor–Bendixson construction in general topology. Let E be a subset of a topological space. Let Φ be the set of rules $X \to x$ such that $X \subseteq E$ and $x \in E$ is a limit point of X. The corresponding monotone operator $\varphi : \mathrm{Pow}(E) \to \mathrm{Pow}(E)$ is the closure operation on E. A set $X \subseteq E$ is closed in E just in case X is Φ-closed and is dense in itself just in case it is Φ-dense. Thus $K(\varphi)$ is the largest dense in itself subset of E, called the kernel K of E. When E is a closed subset of a space with a countable basis then $E = K \cup S$ where K is perfect and $S = \neg K = I(\breve{\varphi})$ is a countable set, so that the closure ordinal $|\breve{\varphi}|$ must be countable. This is the Cantor–Bendixson representation of a closed subset of a space with a countable basis.

Another example of the kernel construction comes from the theory of abelian p-groups. These are abelian groups where every element has finite order p^n for some n, where p is a fixed prime. Let G be an abelian p-group. Define $\varphi : \mathrm{Pow}(G) \to \mathrm{Pow}(G)$ by

$$\varphi(X) = pX = \{\overbrace{g + \cdots + g}^{p} \mid g \in X\} \quad \text{for } X \subseteq G.$$

Then φ is a monotone operator mapping subgroups of G to subgroups of G. The φ-dense subgroups of G are just those that are said to be divisible, so that $K(\varphi)$ is the largest divisible subgroup of G. Much of the structure theory of abelian p-groups is concerned with the descending hierarchy of subgroups $\{\varphi^{|\lambda|}\}_\lambda$.

1.7. Some examples of induction in classical mathematics

(1) If X is a subset of a group G then there is a smallest subgroup H of G that contains X. H is inductively defined by the set of rules $\emptyset \rightarrow x$ for $x \in X \cup \{e\}$ and $\{a, b\} \rightarrow ab^{-1}$ for $a, b \in G$. The same notion is used with other algebraic structures such as rings, fields and vector spaces. A slightly different sort of example is the algebraic closure of a subfield of an algebraically closed field. All these examples involve finitary rule sets.

(2) If R is a binary relation on a set A, then the transitive closure of R is the smallest transitive relation extending R. It is inductively defined by the set of rules $\emptyset \rightarrow (a, b)$ if aRb and $\{(a, b), (b, c)\} \rightarrow (a, c)$ for $a, b, c \in A$. Similarily the equivalence relation generated by R is inductively defined by the above rules together with the rules $\emptyset \rightarrow (a, a)$ for $a \in A$ and $\{(a, b)\} \rightarrow (b, a)$ for $a, b \in A$.

(3) For an example of a non-finitary inductive definition we turn to σ-rings and Borel sets. Recall that a set $\mathscr{C} \subseteq \mathrm{Pow}(A)$ is σ-ring if it is closed under complements and unions of countable subsets; i.e., combining these into one $\mathscr{C}$ is a σ-ring if $\mathscr{C}$ is Φ-closed where Φ consists of the rules $\{A_n \mid n \in \omega\} \rightarrow \bigcup_{n<\omega} \neg A_n$ for countable families $\{A_n\}_{n\in\omega}$ with $A_n \subseteq A$. Hence the σ-ring generated by $\mathscr{C}_0 \subseteq \mathrm{Pow}(A)$ is inductively defined by the rule set Φ' consisting of the rules in Φ together with the rules $\emptyset \rightarrow X$ for $X \in \mathscr{C}_0$. As an example the Borel sets of reals are the special case where $A = \mathbb{R}$ and $\mathscr{C}_0$ is the set of open subsets of $\mathbb{R}$. If $\varphi : \mathrm{Pow}(A) \rightarrow \mathrm{Pow}(A)$ is the monotone operator associated with Φ' then φ is $\aleph_1$-based so that by Proposition 1.3.4, $|\varphi| \leq \aleph_1$. The stages φ^λ for $\lambda < \aleph_1$ are just the familiar stages of the Borel hierarchy. φ^0 is the set of open sets and for $\lambda > 0$, the sets in φ^λ are those of the form $\bigcup_{n<\omega} \neg A_n$ where each $A_n \in \varphi^{<\lambda}$.

2. Induction in recursion theory

2.1. Recursively enumerable relations

There are two key results relating the recursively enumerable (r.e.) relations to the finitely presented formal systems such as formal arithmetic:

(I) The theorems of a finitely presented formal system form an r.e. set, when Gödel numbered.

(II) Every r.e. relation can be represented in any sufficiently rich formal system.

These results are important in Gödel's incompleteness theorem. The general notion of representability will be considered in the next section. Here we wish to give a result concerning inductive definitions that yields I.

As we have seen, a formal system will determine a rule set that inductively defines the set of theorems. When expressions are Gödel numbered, a rule set on ω is induced that inductively defines the set of Gödel numbers of theorems. If the formal system is finitely presented the rule set will be a recursive finitary one as defined below.

2.1.1. Definition. Let Φ be a finitary rule set on ω. Φ is *recursive* (*r.e.*) if the relation R_Φ is recursive (r.e.) where R_Φ is the set of pairs $(\langle a_1, \dots, a_n \rangle, b)$ such that $\Phi : \{a_1, \dots, a_n\} \to b$.

Here $\langle \quad \rangle : \bigcup_{n \in \omega} \omega^n \to \omega$ is a standard constructive injective coding function for finite sequences of natural numbers. Associated with it are recursive functions $\mathrm{lh} : \omega \to \omega$ and $q : \omega \times \omega \to \omega$ and they satisfy the following:

(i) The range Seq of $\langle \quad \rangle$ is recursive.

(ii) For each $n > 0$ the function $\langle \quad \rangle \restriction \omega^n : \omega^n \to \omega$ is recursive.

(iii) $\mathrm{lh}(\langle x_1, \dots, x_n \rangle) = n$ for $n \geq 0$.

(iv) $q(\langle x_1, \dots, x_n \rangle, i) = x_i$ for $1 \leq i \leq n$.

2.1.2. Proposition. *If Φ is an r.e. finitary rule set on ω, then $I(\Phi)$ is r.e.*

Proof. As Φ is finitary, by Proposition 1.1.4,

$$I(\Phi) = \{a \in \omega \mid \exists y \, \mathrm{Pr}_\Phi(a, y)\}$$

where $\mathrm{Pr}_\Phi(a, b)$ iff $b = \langle a_1, \dots, a_n \rangle$ for some Φ-proof $a_1, \dots, a_n$ of b. Hence to prove the proposition it suffices to show that Pr_Φ is r.e. But this is just a matter of coding:

$$\begin{aligned}\mathrm{Pr}_\Phi(a, b) \Leftrightarrow \mathrm{Seq}(b) \;\&\; & q(b, \mathrm{lh}(b)) = a \\ & \& \; \forall i < \mathrm{lh}(b) \, \exists z \, [R_\Phi(z, q(b, i+1)) \\ & \& \; \forall i < \mathrm{lh}(z)) \, \exists k < i \, (q(z, i+1) = q(x, k+1))].\end{aligned}$$

By hypothesis, using standard closure properties of the r.e. relations we see that the right-hand side is r.e. $\square$

2.2. Π^1_1 relations

When infinitary rules of inference are allowed in a formal system then more general notions of induction are required, e.g. in GRZEGORCZYK, MOSTOWSKI and RYLL-NARDZEWSKI [1958] the ω-rule is added to an otherwise finitely presented formal system for second order arithmetic. The ω-rule allows one to infer $\forall x\,\varphi(x)$ from an infinite set of premisses $\varphi(0), \varphi(1), \ldots$. When Gödel numbered, such a formal system induces a regular arithmetical rule set Φ on ω.

2.2.1. DEFINITION. A rule set Φ on ω is *regular arithmetical* if there are arithmetical relations R and S such that Φ is the set of rules $R_a \to b$ such that $S(a, b)$ where $R_a = \{y \in \omega \mid R(a, y)\}$.

2.2.2. PROPOSITION. (i) *If Φ is a regular arithmetical rule set on ω, then the associated monotone operator $\varphi : \mathrm{Pow}(\omega) \to \mathrm{Pow}(\omega)$ is Π^1_1 (i.e., $\{(X, x) \in \mathrm{Pow}(\omega) \times \omega \mid x \in \varphi(X)\}$ is Π^1_1).*

(ii) *For any monotone Π^1_1 φ the set $I(\varphi)$ is also Π^1_1.*

PROOF. (i) Let Φ be the set of rules $R_a \to b$ such that $S(a, b)$ where R, S are arithmetical. Then the associated monotone operator φ is given by $\varphi(X) = \{b \in \omega \mid \exists a\,[S(a, b)\ \&\ R_a \subseteq X]\}$ for $X \subseteq A$. φ is arithmetical and hence Π^1_1.

(ii) $I(\varphi) = \{a \in \omega \mid \forall X\,[\forall x\,[x \in \varphi(X) \to x \in X] \to a \in X]\}$ so that if φ is Π^1_1 standard quantifier manipulations show that $I(\varphi)$ is also Π^1_1. $\square$

Hence when considering systems with the ω-rule the above proposition suggests that the class of r.e. relations in I and II of 2.1. should be replaced by the class of Π^1_1 relations.

Below we give two further examples of regular arithmetical rule sets.

The first example occurs in the definition of Kleene's system of notations for the recursive ordinals. This may be given as follows. Let $<$ be the smallest transitive relation on ω such that

(i) $a < 2^a$ for $a \in \omega$,

(ii) $\{e\}(n) < 3 \cdot 5^e$ for $e, n \in \omega$ such that $\{e\}(n)$ is defined. (Here $\{e\}$ is the e-th partial recursive function in a standard enumeration.) Then $<$ is easily seen to be an r.e. relation. Now let Φ be the set of rules $\emptyset \to 1$, $\{a\} \to 2^a$ for $a \in \omega$, and $\{\{e\}(n) \mid n < \omega\} \to 3 \cdot 5^e$ for $e \in \omega$ such that $\{e\}(n)$ is defined and $\{e\}(n) < \{e\}(n+1)$ for all $n \in \omega$. Then let $\mathcal{O} = I(\Phi)$. Let $a <_0 b$ iff $a, b \in \mathcal{O}\ \&\ a < b$. As Φ is regular arithmetical, $\mathcal{O}$ and hence $<_0$ are Π^1_1. For $a \in \mathcal{O}$ let $|a|_0 = |a|_\varphi$. Then $|1|_0 = 0$, $|2^a|_0 = |a|_0 + 1$ for $a \in \mathcal{O}$ and $|3 \cdot 5^e|_0 = \lim_{n \in \omega} |\{e\}(n)|_0$ for $3 \cdot 5^e \in \mathcal{O}$. Thus $(\mathcal{O}, <_0, |\ \ |_0)$ is Kleene's recursive analogue of the countable ordinals. The ordinal $|\varphi| = \operatorname{Sup}\{|a|_0 + 1 \mid a \in \mathcal{O}\}$ is the Church–Kleene ordinal ω_1, the first admissible ordinal $> \omega$.

As another example of a regular arithmetical induction we consider the hyperarithmetical hierarchy as formulated in MOSCHOVAKIS [1974a]. This is a recursive analogue of the Borel hierarchy. Call a set $\mathcal{B} \subseteq \operatorname{Pow}(\omega)$ an *effective σ-ring* if $\mathcal{B}$ effectively contains the r.e. sets and is effectively closed under complementation and countable unions. I.e., there is a set $I \subseteq \omega$ and sets $B_i \subseteq \omega$ for $i \in I$ such that $\mathcal{B} = \{B_i \mid i \in I\}$ and:

(i) There is a recursive function $\tau_1 : \omega \to I$ such that $R_e = B_{\tau_1(e)}$ for all $e \in \omega$. (Recall that R_e is the e-th r.e. set in a standard enumeration.)

(ii) There is a recursive function $\tau_2 : \omega \to \omega$ such that if $\{e\}$ is a total function $f : \omega \to I$, then

$$\tau_2(e) \in I \quad \text{and} \quad B_{\tau_2(e)} = \bigcup_{n<\omega} \neg B_{f(n)}.$$

An effective σ-ring may be constructed as follows. Let $\tau_1(n) = 2^n$. Let $\tau_2(e) = 3 \cdot 5^e$. Let Φ be the set of rules $\emptyset \to \tau_1(n)$ for $n \in \omega$ and $\{f(n) \mid n \in \omega\} \to \tau_2(e)$ for $e \in \omega$ such that $\{e\}$ is a totally defined function f. Let $I = I(\Phi)$. Then Φ is a deterministic rule set (see 1.2) so that by recursion we may define $B_e \subseteq \omega$ for $e \in I$ by

$$B_{\tau_1(e)} = R_e \quad \text{for } e \in \omega,$$

$$B_{\tau_2(e)} = \bigcup_{n\in\omega} \neg B_{f(n)} \quad \text{if } \{e\} = f : \omega \to I.$$

Then clearly $\mathcal{B} = \{B_i \mid i \in I\}$ is an effective σ-ring.

As Φ is regular arithmetical, $I = I(\Phi)$ is Π^1_1. Let φ be the monotone operator associated which Φ. $\mathcal{B}$ can be arranged in a hierarchy $\mathcal{B} = \bigcup_{\lambda<\omega_1} \mathcal{B}^\lambda$ where $\mathcal{B}^\lambda = \{B_e \mid e \in \varphi^\lambda\}$ for $\lambda < \omega_1$ $(= |\varphi|)$. So $\mathcal{B}^0$ is the set of r.e. sets, $\mathcal{B}^1$ is the set of Σ^0_2 sets, $\bigcup_{n<\omega} \mathcal{B}^n$ is the set of arithmetical sets. In Section 8E of MOSCHOVAKIS [1974a] there is a proof of the following result.

2.2.3. THEOREM. *$\mathcal{B}$ is the smallest effective σ-ring and coincides with the set of Δ^1_1 subsets of ω.*

2.3. Representability

In this subsection we generalize the approach to the recursively enumerable and Π^1_1 relations on ω in terms of representability to arbitrary structures.

Suppose that we have a set A and a theory T that has individual constants for elements of A as well as individual variables. (The same symbol will be used for a constant and the object it names.)

2.3.1. DEFINITION. $R \subseteq A^n$ is *T-represented by the formula* $\theta(\bar{x})$ if $\bar{x} = x_1, \ldots, x_n$ and

$$R(\bar{a}) \Leftrightarrow T \vdash \theta(\bar{a}) \quad \text{for } \bar{a} \in A^n.$$

If $A = \omega$ and T is a finitely presented theory such as formal arithmetic, then the set $\ulcorner T \urcorner$ of Gödel numbers $\ulcorner \theta \urcorner$ of theorems θ of T forms an r.e. set and hence every T-representable relation is also r.e. For if R is T-represented by $\theta(\bar{x})$, then $R(\bar{a}) \Leftrightarrow f(\bar{a}) \in \ulcorner T \urcorner$ for $\bar{a} \in \omega^n$, where f is the recursive function given by $f(\bar{a}) = \ulcorner \theta(\bar{a}) \urcorner$ for $\bar{a} \in \omega^n$. Result II of 2.1 gives a converse of this result for sufficiently rich T. Hence for suitable systems T, the T-representable relations are exactly the r.e. relations.

Ordinary recursion theory can be developed from scratch by making a suitable choice of T. For example Post's canonical systems (see POST [1943]) make one such choice, which is further refined in SMULLYAN [1961]. Kleene's systems of equations for representing the partial recursive functions (see KLEENE [1952]) give another approach.

In GRZEGORCZYK, MOSTOWSKI and RYLL-NARDZEWSKI [1958] it is shown that in second order formal arithmetic with the ω-rule exactly the Π^1_1 relations are representable. Hence the notion of representability gives a way of uniformly treating the r.e. relations and the Π^1_1 relations. Below we give such a uniform treatment for arbitrary structures $\mathfrak{A}$ that gives these classes of relations on ω when $\mathfrak{A}$ is the structure $\mathfrak{N} = \langle \omega, S, P \rangle$ of arithmetic where S and P are the graphs of addition and multiplication.

First we need to give some definitions. Given a set A we introduce the full first order language L^A over A. L^A has individual constants for the elements of A and related constants for the relations on A. There are also individual variables and n-ary relation variables for each $n > 0$.

The *elementary* (i.e. first order) formulae of L^A are built up in the usual way using the connectives and the individual quantifiers $\forall x$, $\exists x$. The second order formulae are obtained by allowing quantifiers $\forall X^n$, $\exists X^n$ where X^n is an n-place relation variable. All elementary or second order sentences of L^A are either true or false in the standard interpretation. We shall be interested in various subclasses of formulae. The *existential* formulae are built up from atomic formulae and their negations using $\vee$, $\wedge$ and $\exists x$. Given a binary relation $<$ on A we may introduce the restricted quantifiers $\forall x < y$, $\exists x < y$ abbreviating $\forall x\,(x < y \rightarrow$ and $\exists x\,(x < y \wedge$. Then we may define the restricted elementary formulae as those built up using only restricted quantifiers. We may define the Σ_n^0 and Π_n^0 prenex formulae, where we allow the matrix to be restricted, in the usual way, e.g. Σ_2^0 formulae have the form $\exists x_1 \cdots \exists x_n \forall y_1 \cdots \forall y_m \theta$, where $n, m \geq 0$ and θ is restricted.

We will also be interested in classifications of second order formulae. A formula is Π_1^1 if it has the form $\forall X_1 \cdots \forall X_m \theta$ where $m \geq 0$ and θ is elementary. Similarily we define the Π_n^1 and Σ_n^1 formulae for $n > 0$, by counting the number of alternating blocks of relation quantifiers.

Given a structure $\mathfrak{A} = \langle A, R_1, \ldots, R_l \rangle$ $\mathrm{L}(\mathfrak{A})$ is the sublanguage of L^A that only allows relation constants for equality and the relations $R_1, \ldots, R_l$. If $\mathcal{F}$ is a collection of formulae of L^A, then $R \subseteq A^n$ is *$\mathcal{F}$-definable over* $\mathfrak{A}$ if there is a formula $\theta(\bar{x})$ of $\mathrm{L}(\mathfrak{A})$ in $\mathcal{F}$ such that $\bar{x} = x_1, \ldots, x_n$ includes all the free variables of $\theta(\bar{x})$ and

$$R(\bar{a}) \Leftrightarrow \theta(\bar{a}) \text{ is true} \quad \text{for } \bar{a} \in A^n.$$

Many constructions in ordinary recursion theory make use of some coding apparatus, e.g. in Gödel numbering. To extend such constructions to $\mathfrak{A}$ we shall need such apparatus to be definable on $\mathfrak{A}$ in a suitable way.

2.3.2. DEFINITION. A *coding scheme* for A is a triple $\mathcal{C} = \langle N, \leq, \langle\ \ \rangle\rangle$ where:

(i) $N \subseteq A$ and $\leq$ is a binary relation on N such that $\langle N, \leq \rangle \cong \langle \omega, \leq \rangle$. We shall identify N with $\omega = \{0, 1, \ldots\}$.

(ii) $\langle\ \ \rangle : \bigcup_{n \in \omega} A^n \to A$ is an injective function. Associated with $\mathcal{C}$ are the following.

(iii) Seq, the set of codes of finite sequences is the range of $\langle\ \ \rangle$.

(iv) $\mathrm{lh} : \mathrm{Seq} \to N$ is given by $\mathrm{lh}(\langle x_1, \ldots, x_n \rangle) = n$.

(v) $q : \mathrm{Seq} \times N \to A$ is given by

$$q(\langle x_1, \ldots, x_n \rangle, i) = \begin{cases} x_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

(vi) $s : N \to N$ is given by $s(n) = n + 1$.

If $\mathcal{F}$ is a class of formulae we say that *$\mathcal{C}$ is $\mathcal{F}$-definable over* $\mathfrak{A}$ if N, $\leq$, Seq and the graphs of lh, q and s are $\mathcal{F}$-definable over $\mathfrak{A}$.

$\mathfrak{A}$ is *$\mathcal{F}$-acceptable* if there is a coding scheme $\mathcal{C}$ over A that is $\mathcal{F}$-definable over $\mathfrak{A}$. In case $\mathcal{F}$ is the class of elementary formulae we just write *acceptable* for $\mathcal{F}$-acceptable.

Given a coding scheme $\mathcal{C}$ over A and a structure $\mathfrak{A}$, formulae θ of $\mathrm{L}(\mathfrak{A})$ can be assigned "Gödel numbers" $\ulcorner\theta\urcorner \in A$ in a standard way. We shall not go into the details of this here but shall occasionally need to make use of certain facts about such a Gödel numbering.

Given a structure $\mathfrak{A}$ let $T(\mathfrak{A})$ be the formal system, in the first order language $\mathrm{L}(\mathfrak{A})$, that has a standard system of axioms and rules of inference for first order logic with equality and has the diagram of $\mathfrak{A}$ as a set of non-logical axioms. So every true sentence of $\mathrm{L}(\mathfrak{A})$, that is atomic or the negation of an atomic sentence, is an axiom of $T(\mathfrak{A})$.

Our generalization of the class of r.e. relations on ω is the class of $T(\mathfrak{A})$-representable relations on A. For this to be a good notion we shall require that $\mathfrak{A}$ is existentially acceptable with an existentially definable coding scheme $\mathscr{C}$. Let $<$ be the relation on A that is the strict ordering relation of the copy of the natural numbers on A given by $\mathscr{C}$. Let $\Sigma_1^0(\mathfrak{A})$ be the class of relations on A that are Σ_1^0 definable over $\mathfrak{A}$, where the above $<$ relation is used in defining restricted formulae. As on ω, we can show that the finitary rule set inductively defining the theorems of A, induces a $\Sigma_1^0(\mathfrak{A})$ finitary rule set on A.

2.3.3. Definition. A finitary rule set Φ on A is $\Sigma_1^0(\mathfrak{A})$ if the relation R_Φ is $\Sigma_1^0(\mathfrak{A})$ where R_Φ is the set of pairs $(\langle a_1, \ldots, a_n\rangle, b)$ such that $\Phi : \{a_1, \ldots, a_n\} \to b$.

The following is proved exactly as Proposition 2.2.2.

2.3.4. Proposition. *If $\mathfrak{A}$ is existentially acceptable and Φ is a $\Sigma_1^0(\mathfrak{A})$ finitary rule set, then $I(\Phi)$ is also $\Sigma_1^0(\mathfrak{A})$.*

2.3.5. Corollary. *If $\mathfrak{A}$ is existentially acceptable, then $\ulcorner T(\mathfrak{A})\urcorner$ is $\Sigma_1^0(\mathfrak{A})$ and hence every $T(\mathfrak{A})$-respresentable relation is $\Sigma_1^0(\mathfrak{A})$.*

To prove the last part we need the fact that if $\theta(\bar{x})$ is a formula and $f(\bar{a}) = \ulcorner\theta(\bar{a})\urcorner$ for $\bar{a} \in A^n$, then the graph of f is in $\Sigma_1^0(\mathfrak{A})$. The converse to this will be given in the next section.

Let us now generalize the ω-rule. The system $T_\infty(\mathfrak{A})$ is obtained from $T(\mathfrak{A})$ by adding the following infinitary rule

A-rule: From $\theta(a)$ for $a \in A$ infer $\forall x\, \theta(x)$.

Let us call an elementary formula $\varphi(X_1, \ldots, X_m, x_1 \cdots x_n)$ of $\mathrm{L}(\mathfrak{A})$ *universally true* if the Π_1^1 sentence

$$\forall X_1 \cdots \forall X_m \, \forall x_1 \cdots \forall x_n \, \varphi(X_1, \ldots, X_m, x_1, \ldots, x_n)$$

is true. It is easily seen that the axioms of $T_\infty(\mathfrak{A})$ are all universally true and all the rules of inference of $T_\infty(\mathfrak{A})$ preserve universal truth. Hence we have:

2.3.6. PROPOSITION. *$T_\infty(\mathfrak{A}) \vdash \varphi$ implies φ is universally true.*

In case $\mathfrak{A}$ is countable a completeness theorem holds that gives the converse to Proposition 2.3.6. It can be proved using the omitting types theorem for countable first order languages (see CHANG and KEISLER [1973]) or else it can be proved by a direct Henkin construction, as in GRILLIOT [1974].

2.3.7. THEOREM. *For countable $\mathfrak{A}$,*

$$T_\infty(\mathfrak{A}) \vdash \varphi \quad \textit{iff} \quad \varphi \textit{ is universally valid.}$$

2.3.8. COROLLARY. *For countable $\mathfrak{A}$ a relation on A is $T_\infty(\mathfrak{A})$-representable iff it is $\Pi^1_1(\mathfrak{A})$ (i.e. Π^1_1-definable over $\mathfrak{A}$).*

PROOF. By Theorem 2.3.7, $\theta(\bar{x})$ $T_\infty(\mathfrak{A})$-represents R iff $\forall X_1 \cdots \forall X_m \, \theta(\bar{x})$ defines R, where $X_1, \ldots, X_m$ are the relation variables occurring in the elementary formula $\theta(\bar{x})$. This result includes the special case of arithmetic when $\mathfrak{A} = \mathfrak{N}$. $\square$

As before, if we assume given a coding scheme $\mathscr{C}$ over $\mathfrak{A}$, the rule set inductively defining the theorems of $T_\infty(\mathfrak{A})$ when Gödel numbered induces a rule set on A that has the following property when $\mathscr{C}$ is elementary over $\mathfrak{A}$.

2.3.9. DEFINITION. A rule set Φ on A is *regular elementary over* $\mathfrak{A}$ if there are elementary relations R, S such that Φ is the set of rules $R_a \to b$ such that $S(a, b)$.

2.3.10. PROPOSITION. *If $\mathfrak{A}$ is an acceptable structure, then there is a regular elementary rule set Φ such that $\ulcorner T_\infty(\mathfrak{A}) \urcorner = I(\Phi)$.*

This result will be needed in the next section.

Finally we consider the notion of truth for (elementary) sentences of $L(\mathfrak{A})$. It is easily seen that every true sentence can be proved in $T_\infty(\mathfrak{A})$ and hence we have the characterization. If φ is a sentence of $L(\mathfrak{A})$ φ is true iff

$T_\infty(\mathfrak{A}) \vdash \varphi$. Alternatively we may give the usual inductive definition for truth that follows the way φ is built up. This can be expressed as follows. Let us call expressions of the form $+\varphi$ or $-\varphi$ where φ is an sentence of $\mathrm{L}(\mathfrak{A})$, *labelled* sentences. Now let Φ be the following set of rules on labelled sentences.

$$\emptyset \to +\theta \quad \text{for each true atomic sentence } \theta,$$

$$\emptyset \to -\theta \quad \text{for each false atomic sentence } \theta;$$

$$\{+\varphi, +\psi\} \to +(\varphi \wedge \psi) \quad \text{for sentences } \varphi, \psi,$$

$$\{-\varphi\} \to -(\varphi \wedge \psi) \quad \text{for sentences } \varphi, \psi,$$

$$\{-\psi\} \to -(\varphi \wedge \psi) \quad \text{for sentences } \varphi, \psi;$$

similar rules for $\neg$, $\vee$, $\to$;

$$\{+\varphi(a) \mid a \in A\} \to +\forall x\, \varphi(x) \quad \text{for sentence } \forall x\, \varphi(x),$$

$$\{-\varphi(a)\} \to -\forall x\, \varphi(x) \quad \text{for } a \in A;$$

and similar rules for $\exists x$.

Then if φ is a sentence,

$$\varphi \text{ is true} \quad \text{iff} \quad +\varphi \in I(\Phi),$$

$$\varphi \text{ is false} \quad \text{iff} \quad -\varphi \in I(\Phi).$$

Note again, that when Gödel numbered, using an elementary coding scheme, the above rule set Φ induces a regular elementary rule set on A.

3. Classes of inductive definitions

3.1. The general framework

Much recent work on inductive definitions falls under the following general framework. Assume given an infinite set A. Let $\mathscr{C}$ be a class of operators, each of the form $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ for some $n > 0$.

3.1.1. Definition. (i) $R \subseteq A^n$ is *$\mathscr{C}$-inductive* if there is $\varphi : \mathrm{Pow}(A^m) \to \mathrm{Pow}(A^m)$ in $\mathscr{C}$ with $m \geq n$ and $\bar{b} \in A^{m-n}$ such that for $\bar{a} \in A^n$,

$$R(a) \Leftrightarrow (\bar{b}, \bar{a}) \in \varphi^\infty.$$

(ii) $\mathrm{IND}(\mathscr{C})$ is the set of $\mathscr{C}$-inductive relations.

(iii) $|\mathscr{C}| = \mathrm{Sup}\{|\varphi| \mid \varphi \in \mathscr{C}\}$.

Why not define IND($\mathscr{C}$) to be $\{\varphi^\infty \mid \varphi \in \mathscr{C}\}$? This turns out not to be natural, as we will want IND($\mathscr{C}$) to have certain closure properties, e.g. we generally want IND($\mathscr{C}$) to be closed under intersections and there seem to be no reasonable conditions on $\mathscr{C}$ that will ensure this for $\{\varphi^\infty \mid \varphi \in \mathscr{C}\}$. We give a general result below that gives closure properties of IND($\mathscr{C}$) given suitable assumptions on $\mathscr{C}$.

Call $\tau : A^n \to A^m$ a *section map* if $m \geq n$ and for some $\bar{b} \in A^{m-n}$,

$$\tau(\bar{a}) = (\bar{b}, \bar{a}) \quad \text{for all } \bar{a} \in A^n.$$

Then $\mathrm{IND}(\mathscr{C}) = \{\tau^{-1}\varphi^\infty \mid \varphi \in \mathscr{C} \;\&\; \tau \text{ is a section map}\}$.

Section maps are used to code several inductive definitions into one. The following is the key to this.

3.1.2. LEMMA. *Let $n_1, \ldots, n_k > 0$ and $m \geq \max(n_1, \ldots, n_k) + 1$. Then there are section maps $\tau_1 : A^{n_1} \to A^m, \ldots, \tau_k : A^{n_k} \to A^m$ that have pairwise disjoint ranges.*

PROOF. Choose pairwise distinct elements $c_1, \ldots, c_k \in A$ and define

$$\tau_i(\bar{a}) = (\overbrace{c_i, \ldots, c_i}^{m-n_i}, \bar{a}) \in A^m \quad \text{for } \bar{a} \in A^{n_i}. \quad \square$$

3.1.3. DEFINITION. An operator

$$\theta : \mathrm{Pow}(A^{n_1}) \times \cdots \times \mathrm{Pow}(A^{n_k}) \to \mathrm{Pow}(A^n)$$

is *section codable* in $\mathscr{C}$ if for section maps $\tau_1 : A^{n_1} \to A^m, \ldots, \tau_k : A^{n_k} \to A^m$, $\tau : A^n \to A^m$ $\varphi \in \mathscr{C}$ where $\varphi : \mathrm{Pow}(A^m) \to \mathrm{Pow}(A^m)$ is given by

$$\varphi(S) = \tau\theta(\tau_1^{-1}S, \ldots, \tau_k^{-1}S) \quad \text{for } S \subseteq A^m.$$

3.1.4. PROPOSITION. *If* (i) *$\mathscr{C}$ is closed under unions (i.e. $\varphi, \psi \in \mathscr{C}$ implies $\varphi \cup \psi \in \mathscr{C}$, where $\varphi \cup \psi(S) = \varphi(S) \cup \psi(S)$),*

(ii) *every operator in $\mathscr{C}$ is section codable in $\mathscr{C}$,*

then IND($\mathscr{C}$) *is closed under every operator* $\theta : \mathrm{Pow}(A^{n_1}) \times \cdots \times \mathrm{Pow}(A^{n_k}) \to \mathrm{Pow}(A^n)$ *that is section codable in $\mathscr{C}$ and is monotone in each argument.*

PROOF. Let θ be as above, section codable in $\mathscr{C}$ and monotone in each argument. Let $R_i = \sigma_i^{-1}\varphi_i^\infty$ where $\sigma_i : A^{n_i} \to A^{m_i}$ is a section map and $\varphi_i : \mathrm{Pow}(A^{m_i}) \to \mathrm{Pow}(A^{m_i})$ is in $\mathscr{C}$ for $i = 1, \ldots, k$. We wish to show that $\theta(R_1, \ldots, R_k) = \tau^{-1}\varphi^\infty$ for some section map τ and some $\varphi \in \mathscr{C}$.

Let $m = \max(m_1, \dots, m_k, n) + 1$, and choose section maps $\tau_1 : A^{m_1} \to A^m, \dots, \tau_k : A^{m_k} \to A^m$, $\tau : A^n \to A^m$ with pairwise disjoint ranges. Let $\varphi_i'(S) = \tau_i\varphi_i(\tau_i^{-1}S)$ for $i = 1, \dots, k$ and $S \subseteq A^m$. Let $\theta'(S) = \tau\theta((\tau_1\sigma_1)^{-1}S, \dots, (\tau_k\sigma_k)^{-1}S)$ for $S \subseteq A^m$. Finally let $\varphi(S) = \varphi_1'(S) \cup \cdots \cup \varphi_k'(S) \cup \theta'(S)$ for $S \subseteq A^m$. By assumption (ii) each $\varphi_i' \in \mathscr{C}$. As θ is section codable in $\mathscr{C}$, $\theta' \in \mathscr{C}$. Hence by assumption (i) $\varphi \in \mathscr{C}$.

Now it is easy to see that $\varphi_i^\infty = \tau_i^{-1}\varphi^\infty$ so that $R_i = (\tau_i\sigma_i)^{-1}\varphi^\infty$ for $i = 1, \dots, k$. Also, $\tau^{-1}\varphi^\lambda = \theta((\tau_1\sigma_1)^{-1}\varphi^{<\lambda}, \dots, (\tau_k\sigma_k)^{-1}\varphi^{<\lambda})$. Hence as θ is monotone,

$$\begin{aligned}
\tau^{-1}\varphi^\infty &= \bigcup_\lambda \tau^{-1}\varphi^\lambda \\
&= \bigcup_\lambda \theta((\tau_1\sigma_1)^{-1}\varphi^{<\lambda}, \dots, (\tau_k\sigma_k)^{-1}\varphi^{<\lambda}) \\
&= \theta((\tau_1\sigma_1)^{-1}\varphi^\infty, \dots, (\tau_k\sigma_k)^{-1}\varphi^\infty) \\
&= \theta(R_1, \dots, R_k). \quad \square
\end{aligned}$$

Note. Monotonicity of θ is essential in the above theorem. In general IND($\mathscr{C}$) will not be closed under complementation.

3.1.5. EXAMPLE. The basic first order monotone operators are $\vee^n$, $\wedge^n$, $\exists^n$ and $\forall^n$ for $n > 0$.

$$\begin{aligned}
\vee^n(R, S) &= R \cup S \quad \text{for } R, S \subseteq A^n, \\
\wedge^n(R, S) &= R \cap S \quad \text{for } R, S \subseteq A^n; \\
\exists^n(R) &= \{\bar{a} \in A^n \mid \exists x\, R(x, \bar{a})\} \quad \text{for } R \subseteq A^{n+1}, \\
\forall^n(R) &= \{\bar{a} \in A^n \mid \forall x\, R(x, \bar{a})\} \quad \text{for } R \subset A^{n+1}.
\end{aligned}$$

Usually the class $\mathscr{C}$ of operators is specified by definability conditions.

Let $\varphi(X, \bar{x})$ be a formula of L^A having free at most the n-ary relation variable X and the individual variables $\bar{x} = x_1, \dots, x_n$. We say that $\varphi(X, \bar{x})$ *defines the operator* $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ if

$$\varphi(S) = \{\bar{a} \in A^n \mid \varphi(S, \bar{a}) \text{ is true}\} \quad \text{for } S \subseteq A^m.$$

Given a class $\mathscr{F}$ of formulae of L^A and a structure $\mathfrak{A} = \langle A, R_1, \dots, R_l\rangle$ let $\mathscr{F}(\mathfrak{A})$ denote the class of operators definable by a formula of $\mathscr{F}$ in $\mathrm{L}(\mathfrak{A})$. Let mon-$\mathscr{F}(\mathfrak{A})$ denote the subclass of monotone operators in $\mathscr{F}(\mathfrak{A})$. Finally let pos-$\mathscr{F}(\mathfrak{A})$ denote the class of operators definable by a formula $\varphi(X, \bar{x})$ of $\mathscr{F}$ in $\mathrm{L}(\mathfrak{A})$ in which X occurs only positively, i.e. $\varphi(X, \bar{x})$ is built up from formulae not involving X and atomic formulae involving X using only the positive connectives $\vee$, $\wedge$ and the quantifiers. Operators in pos-$\mathscr{F}(\mathfrak{A})$ are automatically monotone. So we have

$$\text{pos-}\mathscr{F}(\mathfrak{A}) \subseteq \text{mon-}\mathscr{F}(\mathfrak{A}) \subseteq \mathscr{F}(\mathfrak{A})$$

and hence

$$\text{IND}(\text{pos-}\mathscr{F}(\mathfrak{A})) \subseteq \text{IND}(\text{mon-}\mathscr{F}(\mathfrak{A})) \subseteq \text{IND}(\mathscr{F}(\mathfrak{A}))$$

and $|\text{pos-}\mathscr{F}(\mathfrak{A})| \leq |\text{mon-}\mathscr{F}(\mathfrak{A})| \leq |\mathscr{F}(\mathfrak{A})|$.

3.2. Positive existential induction

In this subsection we look at perhaps the simplest example of the general framework, i.e. we consider the class of operators pos-$\mathscr{F}(\mathfrak{A})$, where $\mathscr{F}$ is the class of existential formulae. We shall write IND$(\exists - \mathfrak{A})$ and $|\exists - \mathfrak{A}|$ for IND(pos-$\mathscr{F}(\mathfrak{A})$) and $|\text{pos-}\mathscr{F}(\mathfrak{A})|$, when $\mathscr{F}$ is this class. We will see that for existentially acceptable $\mathfrak{A}$ the class IND$(\exists - \mathfrak{A})$ coincides with the $T(\mathfrak{A})$-representable relations and gives a good generalization of the r.e. relations on ω.

3.2.1. PROPOSITION. *If $\mathfrak{A}$ is infinite, then*

$$|\exists - \mathfrak{A}| = \omega.$$

PROOF. $|\exists - \mathfrak{A}| \geq \omega$, as $|\varphi_n| = n$ for each $n \in \omega$ where

$$\varphi_n(X) = \left\{x \in A \mid \bigvee_{i<n} \left[\bigwedge_{j<i} X(a_j) \wedge x = a_i\right]\right\} \quad \text{for } X \subseteq A,$$

where $a_0, \ldots, a_{n-1}$ are pairwise distinct elements of A. Clearly each φ_n is positive existential, $\varphi_n^i = \{a_j \mid j < i\}$ for $i \leq n$ and hence $I(\varphi) = \{a_j \mid j < n\}$.

To show that $|\exists - \mathfrak{A}| \leq \omega$, by Proposition 1.3.4 it suffices to show that each positive existential operator φ is ω-based. For this it suffices to show that, for each existential formula $\theta(X)$ positive in the relation variable X and containing no other free variables, $\theta(R)$ is true implies $\theta(S)$ is true for some finite $S \subseteq R$. This is easily proved by induction on the way such formulae $\theta(X)$ are built up.

The next result gives closure properties for IND$(\exists - \mathfrak{A})$.

3.2.2. PROPOSITION. (i) *If $\theta : \text{Pow}(A^{n}) \times \cdots \times \text{Pow}(A^{n_k}) \to \text{Pow}(A^{n})$ is positive existential over $\mathfrak{A}$ (or equivalently is section codable in the class of positive existential operators), then* IND$(\exists - \mathfrak{A})$ *is closed under θ. Hence the relations $=$, $R_1, \ldots, R_l$ and their complements are* IND$(\exists - \mathfrak{A})$, *as the appropriate constant operators are positive existential over $\mathfrak{A}$. Also*

IND($\exists - \mathfrak{A}$) *is closed under* $\vee^n$, $\wedge^n$, $\exists^n$ *for* $n > 0$ *as these are positive existential over* $\mathfrak{A}$.

(ii) *If* $N \subseteq A$, $<$ *is a relation on* A, *and* $S : N \to N$ *such that* $\langle N, <, S\rangle \cong \langle \omega, <, S\rangle$ *and* N, $<$ *and the graph of* S *are existentially definable over* $\mathfrak{A}$, *then* IND($\exists - \mathfrak{A}$) *is* $\forall^n_<$*-closed for* $n > 0$ *where*

$$\forall^n_<(R) = \{(a, \bar{b}) \in A^{n+1} \mid a \in N \;\&\; \forall x\,(x < a \to R(x, \bar{b}))\} \quad \textit{for } R \subseteq A^{n+1}.$$

PROOF. (i) This is an application of Proposition 3.1.4.

(ii) Let $R = \tau^{-1}\varphi^\infty$ where $\tau : A^{n+1} \to A^m$ is a section map and $\varphi : \mathrm{Pow}(A^m) \to \mathrm{Pow}(A^m)$ is positive existential over $\mathfrak{A}$. We show that $\forall^n_<(R)$ is in IND($\exists - \mathfrak{A}$). First identify N with $\omega = \{0, 1, \ldots\}$. Let $\tau_i : A^m \to A^{m+1}$ be the section maps $\tau_i(\bar{x}) = (i, \bar{x})$ for $\bar{x} \in A^m$ where $i = 0, 1$. Let $\psi(X) = \tau_0\varphi(\tau_0^{-1}X) \cup (\tau_1\tau)\theta((\tau_0\tau)^{-1}X, (\tau_1\tau)^{-1}X)$ for $X \subseteq A^m$, where

$$\theta(Y, Z) = \{(x, \bar{x}) \in A^{n+1} \mid x = 0 \vee \exists y\,(x = s(y) \;\&\; Y(y, \bar{x}) \;\&\; Z(y, \bar{x}))\}$$

$$\text{for } Y, Z \subseteq A^{n+1}.$$

Then ψ is positive existential over $\mathfrak{A}$. Clearly $\tau_0^{-1}\psi^\infty = \varphi^\infty$. So $R = (\tau_0\tau)^{-1}\psi^\infty$. Hence if $S = (\tau_1\tau)^{-1}\psi^\infty$, then $S(x, \bar{x}) \Leftrightarrow x = 0$ or $\exists y\,(x = s(y) \;\&\; R(y, \bar{x}) \;\&\; S(y, \bar{x}))$. This can only hold if $S = \forall^n_<(R)$. Hence $\forall^n_<(R) = (\tau_1\tau)^{-1}\psi^\infty$ is in IND($\exists - \mathfrak{A}$). $\square$

3.2.3. PROPOSITION. *If* $\varphi : \mathrm{Pow}(A^m) \to \mathrm{Pow}(A^m)$ *is positive existential over* $\mathfrak{A}$, *then* $I(\varphi)$ *is* $T(\mathfrak{A})$*-represented by* $\psi(X) \to X(\bar{x})$, *where* $\psi(X)$ *is* $\forall \bar{y}\,(\varphi(X, \bar{y}) \to X(\bar{y}))$ *where* $\varphi(X, \bar{x})$ *is an existential formula of* L($\mathfrak{A}$) *positive in* X *that defines the operator.*

PROOF. Let T be the relation represented by $\psi(X) \to X(\bar{x})$. First note that if $\bar{a} \in T$, then by Proposition 2.3.6, $\psi(X) \to X(\bar{a})$ is universally valid, i.e. $a \in \bigcap\{S \subseteq A^m \mid \varphi(S) \subseteq S\} = I(\varphi)$. Thus $T \subseteq I(\varphi)$. To show $I(\varphi) \subseteq T$ it suffices to show that $\varphi(T) \subseteq T$. We need the following.

Claim. Let $\theta(X)$ be an existential formula of L($\mathfrak{A}$) containing x only positively and containing no other variables free. Then

$$\theta(T) \text{ is true} \quad \text{implies} \quad T(\mathfrak{A}) \vdash \psi(X) \to \theta(X).$$

This claim is proved by an easy induction on the number of logical symbols in $\theta(X)$.

Now if $\bar{a} \in \varphi(T)$, then $\varphi(T, \bar{a})$ is true, and hence by the claim $T(\mathfrak{A}) \vdash \psi(X) \to \varphi(x, \bar{a})$. Recalling that $\psi(X)$ is $\forall \bar{y}\, (\varphi(X, \bar{y}) \to X(\bar{y}))$ it follows that $T(\mathfrak{A}) \vdash \psi(X) \to X(\bar{a})$ and hence $\bar{a} \in T$. Thus $\varphi(T) \subseteq T$ as required. □

3.2.4. THEOREM. *Let $\mathfrak{A}$ be an existentially acceptable structure. Then the following are equivalent for a relation R on A.*

(i) $R \in \mathrm{IND}(\exists - \mathfrak{A})$,

(ii) *R is $T(\mathfrak{A})$-representable,*

(iii) *R is $\Sigma_1^0(\mathfrak{A})$-definable.*

Note. In (iii), $\Sigma_1^0(\mathfrak{A})$ is defined relative to the copy $\langle N, < \rangle$ of $\langle \omega, < \rangle$ where $\mathscr{C} = \langle N, <, \langle\ \ \rangle\rangle$ is an existentially definable coding scheme over $\mathfrak{A}$. It follows from the theorem that $\Sigma_1^0(\mathfrak{A})$ is essentially independent of the coding scheme $\mathscr{C}$ used.

PROOF. (i)$\to$(ii). Let $R = \tau^{-1}\varphi^{\infty}$ where τ is a section map $\tau(\bar{a}) = (\bar{b}, \bar{a})$ for $\bar{a} \in A^n$, and φ is a positive existential operator over $\mathfrak{A}$. Then by Proposition 3.2.3, φ^{∞} is $T(\mathfrak{A})$-representable by a formula $\theta(y)$ say. Then

$$R(\bar{a}) \Leftrightarrow \tau(\bar{a}) \in \varphi^{\infty} \Leftrightarrow T(\mathfrak{A}) \vdash \theta(\bar{b}, \bar{a}) \quad \text{for } \bar{a} \in A^n,$$

Hence R is $T(\mathfrak{A})$-represented by $\theta(\bar{b}, \bar{x})$.

(ii)$\to$(iii). This is just Corollary 2.3.5.

(iii)$\to$(i). This follows from Proposition 3.2.2. □

The following property holds for the r.e. relations on ω and is one of the basic structural properties used in recursion theory and its generalizations.

3.2.5. DEFINITION. A class Γ of relations on A has the *parametrization property* if for each $n > 0$ there is a relation $U \subseteq A^{n+1}$ such that $U \in \Gamma$ and for each $R \subseteq A^n$ that is in Γ there is an $a \in A$ such that

$$R = U_a = \{\bar{a} \in A^n \mid U(a, \bar{a})\}.$$

3.2.6. THEOREM. *If $\mathfrak{A}$ is existentially acceptable, then* $\mathrm{IND}(\exists - \mathfrak{A})$ *has the parametrization property.*

PROOF. By Corollary 2.3.5 and Theorem 3.2.4 the set $\ulcorner T(\mathfrak{A}) \urcorner$ is in $\mathrm{IND}(\exists - \mathfrak{A})$. We also need the following fact about coding the syntax of $\ulcorner T(\mathfrak{A}) \urcorner$.

The relation Sub is in $\Sigma_1^0(\mathfrak{A})$ and hence in $\mathrm{IND}(\exists - \mathfrak{A})$, where for $a, b, c \in A$ $\mathrm{Sub}(a, b, c) \Leftrightarrow$ there is an elementary formula $\theta(\bar{x})$ in $\mathrm{L}(\mathfrak{A})$ with $a = \ulcorner\theta(\bar{x})\urcorner$ and there is $\bar{b}$ such that $b = \langle\bar{b}\rangle$ and $c = \ulcorner\theta(\bar{b})\urcorner$.

Now, given $n > 0$ let $U \subseteq A^{n+1}$ be given by

$$U(a, \bar{a}) \Leftrightarrow \exists x\, [\mathrm{Sub}(a, \langle\bar{a}\rangle, x)\ \&\ x \in \ulcorner T(\mathfrak{A})\urcorner].$$

Using the closure properties of $\mathrm{IND}(\exists - \mathfrak{A})$ we see that $U \in \mathrm{IND}(\exists - \mathfrak{A})$. Now if $R \subseteq A^n$ is in $\mathrm{IND}(\exists - \mathfrak{A})$ then by 3.2.4, it is $T(\mathfrak{A})$-represented by a formula $\theta(\bar{x})$ say. Let $a = \ulcorner\theta(\bar{x})\urcorner$. Then for $\bar{a} \in A^n$

$$\begin{aligned} R(\bar{a}) &\Leftrightarrow \ulcorner\theta(\bar{a})\urcorner \in T(\mathfrak{A}) \\ &\Leftrightarrow U(a, \bar{a}). \end{aligned}$$

Hence $R = U_a$. □

3.3. Positive elementary induction

In this subsection we will outline some of the properties of positive elementary induction. The theory has been presented in MOSCHOVAKIS [1974a] and readers should look there for a detailed development of the subject.

If $\mathscr{F}$ is the class of elementary formulae of L^A, then we shall write $\mathrm{IND}(\mathfrak{A})$ for $\mathrm{IND}(\text{pos-}\mathscr{F}(\mathfrak{A}))$ and $\kappa(\mathfrak{A})$ for $|\text{pos-}\mathscr{F}(\mathfrak{A})|$.

3.3.1. PROPOSITION. $\mathrm{IND}(\mathfrak{A})$ *is positive elementary closed over* $\mathfrak{A}$ *(i.e. if θ is section codable in the class of positive elementary operations, then* $\mathrm{IND}(\mathfrak{A})$ *is closed under θ). Hence the relations* $=, R_1, \ldots, R_l$ *and their complements are in* $\mathrm{IND}(\mathfrak{A})$ *and* $\mathrm{IND}(\mathfrak{A})$ *is closed under* $\vee^n, \wedge^n, \exists^n$ *and* $\forall^n$ *for* $n > 0$.

This is an application of Proposition 3.1.4.

3.3.2. PROPOSITION. *Let Φ be a rule set on A that is regular elementary over* $\mathfrak{A}$. *Let* $\varphi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$ *be the associated monotone operator. Then φ is positive elementary over $\mathfrak{A}$ and hence* $I(\Phi) = I(\varphi) \in \mathrm{IND}(\mathfrak{A})$.

PROOF. Let Φ be the set of rules $R_a \to b$ such that $S(a, b)$. Then for $X \subseteq A$,

$$\varphi(X) = \{b \in A \mid \exists y\, [S(y, b) \wedge \forall x\, (R(a, x) \to X(x))]\}.$$

If R and S are replaced by their elementary definitions then we obtain an elementary definition of φ. □

3.3.3. COROLLARY. *If $\mathfrak{A}$ is acceptable, then* $\ulcorner T_\infty(\mathfrak{A})\urcorner \in \mathrm{IND}(\mathfrak{A})$ *and hence every $T_\infty(\mathfrak{A})$-representable relation is in* $\mathrm{IND}(\mathfrak{A})$.

PROOF. The first part follows from Propositions 3.3.2 and 2.3.10. The last part is proved as in the last part of Corollary 2.3.5 using some of the closure properties of $\mathrm{IND}(\mathfrak{A})$ given in 3.3.1. □

3.3.4. PROPOSITION. *If $\varphi : \mathrm{Pow}(A^m) \to \mathrm{Pow}(A^m)$ is positive elementary over $\mathfrak{A}$, then $I(\varphi)$ is $T_\infty(\mathfrak{A})$-representable.*

PROOF. This result is proved as in the proof of Proposition 3.2.3. The claim used there must be modified by replacing $T(\mathfrak{A})$ by $T_\infty(\mathfrak{A})$ and allowing $\theta(X)$ to be any elementary formula of $\mathrm{L}(\mathfrak{A})$. The A-rule of $T_\infty(\mathfrak{A})$ is just what is needed in order to take care of the proof for the case that $\theta(X)$ is a universal quantification. □

As a consequence of the previous two results we have:

3.3.5. THEOREM. *For acceptable $\mathfrak{A}$ and relation R on A, $R \in \mathrm{IND}(\mathfrak{A})$ iff R is $T_\infty(\mathfrak{A})$-representable.*

As in the proof of Theorem 3.2.6, we have:

3.3.6. COROLLARY. *If $\mathfrak{A}$ is acceptable, then* $\mathrm{IND}(\mathfrak{A})$ *has the parametrization property.*

The following result is called the Abstract Kleene Theorem in MOSCHOVAKIS [1974a]. The proof given there uses a quite different method to the one we use.

3.3.7. THEOREM. *If $\mathfrak{A}$ is a countable acceptable structure and R is a relation on A, then $R \in \mathrm{IND}(\mathfrak{A})$ iff R is $\Pi^1_1(\mathfrak{A})$.*

This theorem is an immediate consequence of Corolary 2.3.8 and Theorem 3.3.5.

3.3.8. DEFINITION. A *norm* on a set R is a map $\sigma : R \twoheadrightarrow \lambda$ of R onto an ordinal λ. λ is called the *length* of σ.

If $R \subseteq A^n$ then associated with σ are the $2n$-ary relations $<^*_\sigma$ and $\leq^*_\sigma$ given by

$$\bar{a} <^*_\sigma \bar{b} \Leftrightarrow R(\bar{a})\ \&\ (R(\bar{b}) \Rightarrow \sigma(\bar{a}) < \sigma(\bar{b})),$$

$$\bar{a} \leq^*_\sigma \bar{b} \Leftrightarrow R(\bar{a})\ \&\ (R(\bar{b}) \Rightarrow \sigma(\bar{a}) \leq \sigma(\bar{b})),$$

for $\bar{a}, \bar{b} \in A^n$.

3.3.9. EXAMPLE. If $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$, then $|\ \ |_\varphi : \varphi^\infty \to |\varphi|$ is a norm on φ^∞. We write $<^*_\varphi$ and $\leq^*_\varphi$ rather than $<^*_\sigma$ and $\leq^*_\sigma$ when $\sigma = |\ \ |_\varphi$.

3.3.10. DEFINITION. Let Γ be a class of relations on A. If R is a relation on A, then a norm σ on R is a *Γ-norm* if $<^*_\sigma$ and $\leq^*_\sigma$ are in Γ. Γ is *normed* if every relation in Γ has a Γ-norm. For normed Γ let $\mathrm{o}(\Gamma)$ be the supremum of the lengths of the Γ-norms.

3.3.11. PROPOSITION. $\mathrm{IND}(\mathfrak{A})$ *is normed.*

PROOF. Let $R = \tau^{-1}\varphi^\infty$ where τ is a section map and φ is positive elementary over $\mathfrak{A}$. Let $\sigma_0(\bar{a}) = |\tau(\bar{a})|_\varphi$ for $\bar{a} \in R$. $\sigma_0 : R \to |\varphi|$ may not be a norm as its range may not be an initial segment of ordinals. But there is a unique order preserving function f mapping the range of σ_0 onto an ordinal λ. Then if $\sigma(\bar{a}) = f(\sigma_0(\bar{a}))$ for $\bar{a} \in R$, $\sigma : R \twoheadrightarrow \lambda$ is a norm on R of length $\lambda \leq |\varphi|$.

Note that for $\bar{a}, \bar{b} \in A^n$, $\bar{a} <^*_\sigma \bar{b}$ iff $\tau(\bar{a}) <^*_\varphi \tau(\bar{b})$ and similarly for $\leq^*_\sigma$. Hence to prove the proposition it suffices to show that $<^*_\varphi$ and $\leq^*_\varphi$ are in $\mathrm{IND}(\mathfrak{A})$. This follows from:

3.3.12. FIRST STAGE COMPARISON THEOREM. *Let $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ be a positive elementary operator. Then there are positive elementary operators $\varphi_<, \varphi_\leq : \mathrm{Pow}(A^{2n}) \to \mathrm{Pow}(A^{2n})$ such that*

$$<^*_\varphi = I(\varphi_<) \quad \textit{and} \quad \leq^*_\varphi = I(\varphi_\leq).$$

Moreover, $\varphi_<(X) = \{(\bar{a}, \bar{b}) \in A^{2n} \mid (\bar{b}, \bar{a}) \in \varphi_\leq(X)\}$ for $X \subseteq A^{2n}$.

PROOF. $\varphi_<$ and $\varphi_\leq$ are defined as follows. Let

$$\theta(X) = \{(\bar{a}, \bar{b}) \in A^{2n} \mid \bar{a} \in \varphi'(\{x \in A^n \mid (\bar{b}, \bar{x}) \in X\})\} \quad \text{for } X \subseteq A^{2n},$$

where $\varphi'(Y) = \varphi(Y) \cup Y$ for $Y \subseteq A^n$. Then let $\varphi_\leq(X) = \theta(\check{\theta}(x))$ for $X \subseteq A^{2n}$ and define $\varphi_<$ as required by the theorem. Then observe that θ is positive elementary and hence that $\varphi_\leq$ and $\varphi_<$ are also. The proof that these operators inductively define $\leq^*_\varphi$ and $<^*_\varphi$ follows easily from the lemma below. $\square$

3.3.13. LEMMA. *For all ordinals* λ,

$$\bar{a} <^*_\varphi \bar{b} \;\&\; \bar{a} \in \varphi^\lambda \Leftrightarrow (\bar{a}, \bar{b}) \in \varphi^\lambda_<,$$

$$\bar{a} \leq^*_\varphi \bar{b} \;\&\; \bar{a} \in \varphi^\lambda \Leftrightarrow (\bar{a}, \bar{b}) \in \varphi^\lambda_\leq.$$

PROOF. This lemma is proved by induction on λ. □

3.3.14. DEFINITION. A relation R on A such that both R and $\neg R$ are in IND($\mathfrak{A}$) is called *hyperelementary over* $\mathfrak{A}$. We write HYP($\mathfrak{A}$) for the class of such relations.

This class generalizes the class of hyperarithmetical relations on ω.

3.3.15. COROLLARY. *If* $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ *is positive elementary over* $\mathfrak{A}$, *then* $\varphi^\lambda \in \mathrm{HYP}(\mathfrak{A})$ *for all* $\lambda < |\varphi|$.

PROOF. If $\lambda < |\varphi|$, then $\lambda = |\bar{a}|_\varphi$ for some $\bar{a} \in I(\varphi)$. So

$$\varphi^\lambda = \{x \in A^n \mid x \leq^*_\varphi a\} = \{x \in A^n \mid (x, a) \in \varphi^\infty_\leq\} = \{\bar{x} \in A^n \mid (\bar{a}, \bar{x}) \in \check{\varphi}^\infty_<\},$$

$$\neg \varphi^\lambda = \{\bar{x} \in A^n \mid \bar{a} <^*_\varphi \bar{x}\} = \{\bar{x} \in A^n \mid (\bar{a}, \bar{x}) \in \varphi^\infty_<\}.$$

So if $\tau(\bar{x}) = (\bar{a}, \bar{x})$ for $\bar{x} \in A^n$, then

$$\varphi^\lambda = \tau^{-1}\check{\varphi}^\infty_< \quad \text{and} \quad \neg \varphi^\lambda = \tau^{-1}\varphi^\infty_<.$$

Hence $\varphi^\lambda \in \mathrm{HYP}(\mathfrak{A})$. □

The basic properties of the Π^1_1 relations on ω, and more generally of IND($\mathfrak{A}$) for $\mathfrak{A}$ an acceptable structure, are incorporated in the following important definition first introduced in MOSCHOVAKIS [1974a].

3.3.16. DEFINITION. A class Γ of relations on a set A is a *Spector class over* $\mathfrak{A}$ if

(i) Γ is positive elementary closed over $\mathfrak{A}$ (see Proposition 3.3.1),

(ii) Γ contains a coding scheme $\mathscr{C}$ on A (i.e. N, $\leq$, Seq, the graphs of lh, q and s and the complements of all these are in Γ where these are defined in Definition 2.3.2),

(iii) Γ has the parametrization property (see Definition 3.2.5),

(iv) Γ is normed.

3.3.17. THEOREM. *If* $\mathfrak{A}$ *is an acceptable structure, then* IND($\mathfrak{A}$) *is the smallest Spector class over* $\mathfrak{A}$.

This result also holds under the slightly weaker hypothesis that $\mathfrak{A}$ is *almost acceptable*, where this means that IND($\mathfrak{A}$) contains a coding scheme on A. The fact that IND($\mathfrak{A}$) is a Spector class over $\mathfrak{A}$ is obtained by combining Proposition 3.3.1 and 3.3.12, and Corollary 3.3.6. To see that it is the smallest one requires the following result about Spector classes.

3.3.18. THEOREM. *Let Γ be a Spector class over $\mathfrak{A}$. Let $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ be positive elementary over $\mathfrak{A}$. Then $I(\varphi) \in \Gamma$.*

PROOF. We sketch a proof of this. The details may be found in 9A of MOSCHOVAKIS [1974a]. By the parametrization property of Γ choose $U \subseteq A^{n+2}$ in Γ to parametrize the $(n+1)$-ary relations in Γ. Let $\tau : U \twoheadrightarrow \kappa$ be a Γ-norm on U. Let $Q(t, \bar{a}) \Leftrightarrow \bar{a} \in \varphi(\{\bar{x} \in A^n \mid (t, t, \bar{x}) <^*_\tau (t, t, \bar{a})\})$. Then as Γ is positive elementary closed $Q \in \Gamma$ and hence $Q = U_a$ for some $a \in A$.

Let $P(\bar{a}) \Leftrightarrow Q(a, \bar{a})$ for $\bar{a} \in A^n$. Let $\sigma_0 : P \to \kappa$ be given by $\sigma_0(\bar{a}) = \tau(a, a, \bar{a})$ for $a \in P$. σ_0 may not be surjective, but by composing with an order preserving mapping of the range of σ_0 onto an initial segment of ordinals we obtain a norm σ on P. Then for $a \in A^n$,

$$P(\bar{a}) \Leftrightarrow \bar{a} \in \varphi(\{\bar{x} \in A^n \mid \bar{x} <^*_\sigma \bar{a}\}).$$

But it may easily be shown that any P with a norm satisfying this equivalence must be identical with $I(\varphi)$. Hence as $P \in \Gamma$, $I(\varphi) \in \Gamma$. □

The ordinal $\kappa(\mathfrak{A})$ may be characterised in terms of IND($\mathfrak{A}$) as the sup of the lengths of the positive elementary inductive norms over $\mathfrak{A}$, i.e.:

3.3.19. THEOREM. $\kappa(\mathfrak{A}) = \mathrm{o}(\mathrm{IND}(\mathfrak{A}))$.

The next result generalizes the Spector–Gandy theorem for the Π^1_1 relations on ω.

3.3.20. ABSTRACT SPECTOR–GANDY THEOREM. *Let $\mathfrak{A}$ be an acceptable structure. If $R \subseteq A^n$, then $R \in \mathrm{IND}(\mathfrak{A})$ iff there is an elementary second order relation $\mathscr{R}$ such that for $\bar{a} \in A^n$*

$$R(\bar{a}) \Leftrightarrow \exists X \in \mathrm{HYP}(\mathfrak{A})\, \mathscr{R}(X, \bar{a}).$$

This result was first stated and proved in MOSCHOVAKIS [1974a]. A simplification of that proof was given in ACZEL [1972]. It used a new proof of Moschovakis's second stage comparison theorem (7 C.1. of MOSCHOVAKIS [1974a]). The following consequence is the main fact needed.

3.3.21. PROPOSITION. *Let* $\varphi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ *be positive elementary over* $\mathfrak{A}$. *Then there is a second order relation* $\mathcal{Q}$ *that is elementary over* $\mathfrak{A}$ *such that if* $\bar{a} \in I(\varphi)$ *or* $\bar{b} \in I(\varphi)$, *then the following are equivalent*

(i) $\bar{a} \leq^*_\varphi \bar{b}$,

(ii) $\exists X \mathcal{Q}(X, \bar{a}, \bar{b})$,

(iii) $\exists X \in \mathrm{Hyp}(\mathfrak{A}) \mathcal{Q}(X, \bar{a}, \bar{b})$.

PROOF. Let $\varphi_\leq : \mathrm{Pow}(A^{2n}) \to \mathrm{Pow}(A^{2n})$ be the operator used in the First Stage Comparison Theorem 3.3.12. Let $\mathcal{Q}(X, \bar{a}) \Leftrightarrow \bar{a} \in X \mathbin{\&} X \subseteq \varphi_\leq(X)$, for $X \subseteq A^{2n}$ and $\bar{a} \in A^{2n}$. Then $\mathcal{Q}$ is elementary over $\mathfrak{A}$. To see that (i) and (ii) are equivalent,

$$\begin{aligned} \bar{a} \leq^*_\varphi \bar{b} \Leftrightarrow \neg\, \bar{b} <^*_\varphi \bar{a} \Leftrightarrow (\bar{b}, \bar{a}) \notin \varphi^\infty_< \Leftrightarrow (\bar{a}, \bar{b}) \notin (\check{\varphi}_\leq)^\infty \\ \Leftrightarrow \neg \forall Y [\check{\varphi}_\leq(Y) \subseteq Y \Rightarrow Y(\bar{a}, \bar{b})] \\ \Leftrightarrow \exists X [\check{\varphi}_\leq(\neg X) \subseteq \neg X \mathbin{\&} X(\bar{a}, \bar{b})] \Leftrightarrow \exists X \mathcal{Q}(X, \bar{a}, \bar{b}). \end{aligned}$$

As (iii) $\Rightarrow$ (ii) is trivial it only remains to show that (i) $\Rightarrow$ (iii). So let $\bar{a} \leq^*_\varphi \bar{b}$. Then for some $\lambda < |\varphi|$, $\bar{a} \in \varphi^\lambda$ and hence by Definition 3.3.13, $(\bar{a}, \bar{b}) \in \varphi^\lambda_\leq$. As $\varphi^\lambda_\leq \subseteq \varphi(\varphi^\lambda_\leq)$ it follows that $\mathcal{Q}(\varphi^\lambda_\leq, \bar{a}, \bar{b})$. As $\lambda < |\varphi| = |\varphi_\leq|$ it follows from Corollary 3.3.15 that $\varphi^\lambda_\leq | \in \mathrm{HYP}(\mathfrak{A})$. Hence $\exists X \in \mathrm{HYP}(\mathfrak{A}) \mathcal{Q}(X, \bar{a}, \bar{b})$. □

We end this outline of the theory of positive elementary induction by giving, without proof, Moschovakis's normal form theorem for an acceptable structure. For this we need to introduce the game quantifier.

In general, by a *quantifier* on a set A we mean a set $\mathsf{Q} \subseteq \mathrm{Pow}(A)$. If $R \subseteq A$ we write $\mathsf{Q} z R(z)$ instead of $R \in \mathsf{Q}$. For example the existential and universal quantifiers on A are $\exists = \mathrm{Pow}(A) - \{\emptyset\}$ and $\forall = \{A\}$.

Given an elementary coding scheme for a structure $\mathfrak{A}$ the *game quantifier* G on A is the set of $R \subseteq A$ such that

$$(*) \qquad \{\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots\} \bigvee_{m \in \omega} R(\langle x_1, y_1, \ldots, x_m, y_m \rangle).$$

Eq. $(*)$ is interpreted in the usual way in terms of a game $\mathcal{G}(R)$ between two players $\exists$ and $\forall$ who alternately choose elements of A, starting with $\forall$, to produce a sequence $x_1, y_1, x_2, y_2, \ldots$. If for some m, $\langle x_1, y_1, \ldots, x_m, y_m \rangle \in R$, then player $\exists$ wins. If this never happens then player $\forall$ wins. Now $(*)$ is interpreted to mean that player $\exists$ has a winning strategy for the game $\mathcal{G}(R)$.

3.3.22. THEOREM (5C of MOSCHOVAKIS [1974a]). *Let $\mathfrak{A}$ be an acceptable structure. Let* G *be the game quantifier on A relative to an elementary coding scheme for $\mathfrak{A}$. Then for $R \subseteq A^n$, $R \in \mathrm{IND}(\mathfrak{A})$ iff there is an elementary relation $S \subseteq A^{n+1}$ such that for $\bar{a} \in A^n$,*

$$R(\bar{a}) \Leftrightarrow \mathsf{G} z\, S(z, \bar{a}).$$

3.4. Relativisation to a non-trivial monotone quantifier

In this subsection we indicate how the notions and results of the last section can be generalised by relativising to a non-trivial monotone quantifier Q on A. $\mathrm{Q} \subseteq \mathrm{Pow}(A)$ is *non-trivial* if $\mathrm{Q} \neq \emptyset$ and $\mathrm{Q} \neq \mathrm{Pow}(A)$. Q is *monotone* if $A \supseteq X \supseteq Y \in \mathrm{Q}$ implies $X \in \mathrm{Q}$. The quantifier $\check{\mathrm{Q}}$ *dual* to Q is given by $\check{\mathrm{Q}} z\, R(z) \Leftrightarrow \neg \mathrm{Q} z \neg R(z)$. The language $\mathrm{L}^A(\mathrm{Q})$ is defined like L^A except that formulae $\mathrm{Q} z\, \phi(z)$ and $\check{\mathrm{Q}} z\, \phi(z)$ are allowed for any formula $\phi(z)$.

The standard interpretation of L^A is extended to sentences of $\mathrm{L}^A(\mathrm{Q})$ by requiring that

$$\mathrm{Q} x\, \phi(x) \text{ is true} \quad \text{iff} \quad \{a \in A \mid \phi(a) \text{ is true}\} \in \mathrm{Q},$$

and similarly for $\check{\mathrm{Q}} x\, \phi(x)$.

The language $\mathrm{L}(\mathfrak{A}, \mathrm{Q})$ is the sublanguage of $\mathrm{L}^A(\mathrm{Q})$ corresponding to the sublanguage $\mathrm{L}(\mathfrak{A})$ of L^A. Positive and negative occurrences of relations in a formula of $\mathrm{L}^A(\mathrm{Q})$ are defined by treating Q and $\check{\mathrm{Q}}$ in the same way as the ordinary quantifiers. Define the Q-*elementary* formulae of $\mathrm{L}^A(\mathrm{Q})$ like the elementary formulae of L^A except that $\mathrm{Q}x$ and $\check{\mathrm{Q}}x$ are allowed. Now define $\mathrm{IND}(\mathfrak{A}, \mathrm{Q})$ as in the definition of $\mathrm{IND}(\mathfrak{A})$ except using positive Q-elementary operators instead of the positive elementary ones.

Now the result of 3.3 concerning $\mathrm{IND}(\mathfrak{A})$ will carry over to $\mathrm{IND}(\mathfrak{A}, \mathrm{Q})$ with suitable changes, i.e. "elementary" should be replaced by "Q-elementary". Where $\mathfrak{A}$ is required to be acceptable, it is sufficient here that $\mathfrak{A}$ is Q-acceptable, i.e. there is a Q-elementary coding scheme. Among the positive Q-elementary operators there are the Q^n and $\check{\mathrm{Q}}^n$ defined like $\forall^n$ and $\exists^n$. The theory $T_\infty(\mathfrak{A})$ needs to be replaced by the theory $T_\infty(\mathfrak{A}, \mathrm{Q})$. This is obtained from $T_\infty(\mathfrak{A})$ by allowing Q-elementary formulae and adding the following possibly infinitary rules.

Q-rule: From $\theta(a)$ for $a \in X$ infer $\mathrm{Q} x\, \theta(x)$ if $X \in \mathrm{Q}$.

$\check{\mathrm{Q}}$-rule: From $\theta(a)$ for $a \in X$ infer $\check{\mathrm{Q}} x\, \theta(x)$ if $X \in \check{\mathrm{Q}}$.

(See ACZEL [1970] where such rules are considered.) Using $T_\infty(\mathfrak{A}, \mathsf{Q})$, results 3.3.3–3.3.6 carry over to IND($\mathfrak{A}$, Q). There does not seem to be any analogue of Theorem 3.3.7. Theorem 3.3.17 carries over to yield that if $\mathfrak{A}$ is Q-acceptable then IND($\mathfrak{A}$, Q) is the smallest Spector class over $\mathfrak{A}$ that is closed under each Q^n and $\check{\mathsf{Q}}^n$.

The Abstract Spector–Gandy Theorem 3.3.20 also carries over to IND($\mathfrak{A}$, Q). Of course HYP($\mathfrak{A}$, Q) must be used instead of HYP($\mathfrak{A}$).

The Normal Form Theorem 3.3.22 also carries over, but for this it is necessary to generalise the game quantifier. This has been carried out in ACZEL [1975] where it is shown how to interpret any infinite string $\mathsf{Q}_1x_1\,\mathsf{Q}_2x_2\cdots$ of non-trivial monotone quantifiers as the following alternating string of ordinary quantifiers:

$$\exists X_1 \in \mathsf{Q}_1\,\forall x_1 \in X_1\,\exists X_2 \in \mathsf{Q}_2\,\forall x_2 \in X_2 \cdots .$$

This can be interpreted in terms of a game between two players as in the definition of the game quantifier. Now given a Q-elementary coding scheme for the structure $\mathfrak{A}$ we define the relativisation Q^+ of the game quantifier as the set of $R \subseteq A$ such that

$$\{\mathsf{Q}x_1\,\check{\mathsf{Q}}y_1\,\forall x_2\,\exists y_2\,\mathsf{Q}x_3\,\check{\mathsf{Q}}y_3\cdots\}\bigvee_{m\in\omega} R(\langle x_1, y_1, \ldots, x_m, y_m\rangle).$$

Note that $\forall^+$ is just the game quantifier G. Theorem 3.3.22 carries over to IND($\mathfrak{A}$, Q) if G is replaced by Q^+.

We end this section by stating an important recently obtained result.

3.4.1. THEOREM (HARRINGTON [1975]). *Let $\mathfrak{A}$ be an acceptable structure. Every Spector class over $\mathfrak{A}$ has the form* IND($\mathfrak{A}$, Q) *for some non-trivial monotone quantifier* Q *on A.*

3.5. Non-monotone induction

The first examples of non-monotone inductive definitions were those that appeared in the construction of various systems of notations for larger and larger segments of the countable ordinals. The first such systems were those of Church–Kleene for the recursive ordinals. One such example is Kleene's $\mathcal{O}$ considered in 2.2. The first ordinal not having a notation in $\mathcal{O}$ is the Church–Kleene ordinal ω_1, that is a recursive analogue of the first uncountable ordinal. The inductive definition of $\mathcal{O}$ uses a monotone operator. But attempts to extend $\mathcal{O}$ to systems of notations for recursive analogues of the higher number classes soon lead to non-monotone operators. For example let $\phi : \mathrm{Pow}(\omega) \to \mathrm{Pow}(\omega)$ be the monotone operator associated with the regular arithmetical rule set generating Kleene's $\mathcal{O}$ in 2.2. We may extend 0 by adding a notation 7 (say) for the ordinal ω, and then continuing as before, i.e. we define a non-monotone operator $\phi_1 : \mathrm{Pow}(\omega) \to \mathrm{Pow}(\omega)$ given by

$$\phi_1(X) = \begin{cases} \phi(X) & \text{if } \phi(X) \not\subseteq X, \\ \{7\} & \text{if } \phi(X) \subseteq X \end{cases} \qquad \text{for } X \subseteq \omega.$$

Then for $\lambda < |\phi| = \omega_1$, $\phi_1^\lambda = \phi^\lambda$ so that $\phi_1^{\omega_1} = \phi^{<\omega_1} \cup \{7\} = \mathcal{O} \cup \{7\}$. Hence $|7|_{\phi_1} = \omega_1$. It is not hard to see that $|\phi_1| = \omega_1 + \omega_1$ so that ϕ_1^∞ is a set of notations for the ordinals $< \omega_1 + \omega_1$. The above may be continued very much further and leads to notation systems for much larger ordinals such as recursive analogues of the finite and transfinite number classes. (See Addison and Kleene [1957], Kreider and Rogers [1961], Putnam [1961] and Richter [1965, 1967, 1968].)

The above work was concerned with the details of specific notation systems. In Putnam [1964] a more general approach was taken where it was shown that arbitrary Δ_2^1 inductive definitions can only give notation systems for an initial segment of the Δ_2^1 ordinals (i.e. the order types of Δ_2^1 well-orderings of natural numbers). Further work has shifted the interest from the study of specific inductively defined notation systems to the study of classes $\mathscr{C}$ of inductive definitions on ω, and the associated ordinals $|\mathscr{C}|$. This shift, encouraged by Gandy, led to the work of Richter [1971], Aczel and Richter [1972], Richter and Aczel [1974], Aanderaa [1974], Cenzer [1974] and Richter [1976]. This work gives characterisations of $|\mathscr{C}|$, for various classes $\mathscr{C}$, in terms of recursive analogues of large cardinals. Such analogues were defined using Kripke's theory of recursion on admissible ordinals. The starting point is to take the admissible ordinals as the recursive analogue of the regular ordinals. It turns out that the first admissible $> \omega$ is the Church–Kleene ordinal ω_1 so that the two approaches to recursive analogues agree. Other analogues are the recursive inaccessibles (admissible limits of admissible ordinals) and the recursively Mahlo ordinals (admissibles α such that for every α-recursive $f : \alpha \to \alpha$ there is an admissible $\beta < \alpha$ closed under f). For recursive analogues of even larger cardinals reflection properties were introduced in Richter and Aczel [1974]. Examples are the Π_{n+2}^0-reflecting ordinals which give a recursive analogue for the Π_n^1-indescribable cardinals for $n > 0$.

Below we shall state some of the results obtained. But first we need a construction invented in Richter [1971]. Given $\phi, \psi : \mathrm{Pow}(\omega) \to \mathrm{Pow}(\omega)$ define $[\phi, \psi] : \mathrm{Pow}(\omega) \to \mathrm{Pow}(\omega)$ by

$$[\phi, \psi](X) = \begin{cases} \phi(X) & \text{if } \phi(X) \not\subseteq X, \\ \psi(X) & \text{if } \phi(X) \subseteq X \end{cases} \quad \text{for } X \subseteq \omega.$$

Note that ϕ_1 introduced above is $[\phi, \psi]$ where $\psi(X) = \{7\}$ for all $X \subseteq \omega$. If $\mathscr{C}_1$, $\mathscr{C}_2$ are classes of operators on ω let $[\mathscr{C}_1, \mathscr{C}_2] = \{[\phi_1, \phi_2] \mid \phi_1 \in \mathscr{C}_1$ and $\phi_2 \in \mathscr{C}_2\}$.

3.5.1. PROPOSITION. (i) $|\Pi^0_0| = \omega$.

(ii) (Gandy, unpublished) $|\Pi^0_1| = \omega_1$.

(iii) (PUTNAM [1964]) $|\Delta^1_2| = \delta^1_2$, *the first non* Δ^1_2 *ordinal.*

(iv) (RICHTER [1971]) $|[\Pi^0_0, \Pi^0_0]| = \omega^2$, $|[\Pi^0_1, \Pi^0_0]| =$ *first recursive inaccessible*, $|[\Pi^0_1, \Pi^0_1]| =$ *first recursively Mahlo ordinal.*

(v) (ACZEL and RICHTER [1972]) $|\Pi^0_n| = |\Sigma^0_{n+1}| =$ *first* Π^0_{n+1} *reflecting ordinal.*

(vi) (ACZEL and RICHTER [1972]) $|\Pi^1_1| =$ *first* Π^1_1 *reflecting ordinal,* $|\Sigma^1_1| =$ *first* Σ^1_1 *reflecting ordinal.*

(vii) (RICHTER [1976]) $|\Pi^1_2| =$ *first* Π^1_2 *reflecting ordinal.*

(viii) (AANDERAA [1974]) $|\Pi^1_1| < |\Sigma^1_1|$ *and* $|\Sigma^1_2| < |\Pi^1_2|$.

Many more results may be found in the above-mentioned papers. It is interesting to compare these results with corresponding results for classes of positive and monotone operators on ω.

3.5.2. PROPOSITION. (i) $|\text{pos-}\Sigma^0_1| = |\text{mon-}\Sigma^0_1| = \omega$.

(ii) (SPECTOR [1961]) $|\text{pos-}\Pi^0_1| = |\text{mon-}\Pi^1_1| = \omega_1$.

(iii) (Grilliot, unpublished) $|\text{pos-}\Sigma^1_1| = |\text{mon-}\Sigma^1_1| = |\Sigma^1_1|$.

(vi) (Gandy, unpublished) $|\text{pos-}\Sigma^1_2| = |\text{mon-}\Sigma^1_2| = \delta^1_2$.

Moschovakis has recently raised the problem of characteristing $|\text{pos-}\Pi^1_2| = |\text{mon-}\Pi^1_2|$. It is not even known whether this ordinal is admissible.

In the above we have only stated results about $|\mathscr{C}|$. Of course the class IND($\mathscr{C}$) is also of interest. Under general conditions on $\mathscr{C}$ the ordinal $|\mathscr{C}|$ is admissible and IND($\mathscr{C}$) is a Spector class with $|\mathscr{C}| = \text{o(IND}(\mathscr{C}))$. Moreover in many cases considered in RICHTER and AZCEL [1974], IND($\mathscr{C}$) may be characterized, in terms of α-recursion, as the class of α-r.e. relations on ω where $\alpha = |\mathscr{C}|$.

The theory of non-monotone induction on ω has been most elegantly generalised to abstract structures in MOSCHOVAKIS [1974b]. The central notion of that paper is that of a *typical, non-monotone* class of second order relations over an abstract structure $\mathfrak{A}$. Examples are the classes of Σ_k^m or Π_k^m definable second order relations over $\mathfrak{A}$ when $m, k \geq 1$ or $m = 0$ and $k \geq 2$. In case $m = 0$ these classes are defined relative to a hyper-elementary coding scheme over $\mathfrak{A}$, as in 2.3.

3.5.3. THEOREM (MOSCHOVAKIS [1974b]). *Let $\mathcal{T}$ be a typical, non-monotone class of second order relations on $\mathfrak{A}$. Let $\mathscr{E}$ be the set of operators $\phi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ such that $\{(\bar{x}, X) \in A^n \times \mathrm{Pow}(A^n) \mid \bar{x} \in \phi(X)\}$ is in $\mathcal{T}$. Then $\mathrm{IND}(\mathscr{E})$ is a Spector class over $\mathfrak{A}$ with $|\mathscr{E}| = \mathrm{o}(\mathrm{IND}(\mathscr{E}))$.*

Moschovakis goes on to characterise $\mathrm{IND}(\mathscr{E})$ as the smallest Spector class over $\mathfrak{A}$ satisfying certain additional conditions.

We end this subsection by stating a special case of an interesting general result of HARRINGTON and KECHRIS [1975].

Let $\mathfrak{A}$ be an acceptable structure, and let $\mathscr{E}$ (pos-$\mathscr{E}$, mon-$\mathscr{E}$) be the class of elementary (positive elementary, monotone elementary) operators over $\mathfrak{A}$. Let $\mathrm{WF} = \{S \subseteq A^2 \mid S \text{ is well-founded}\}$.

3.5.4. THEOREM (HARRINGTON and KECHRIS [1975]). *If* WF *is elementary over $\mathfrak{A}$, then $\mathrm{IND}(\mathscr{E}) = \mathrm{IND}(\text{mon-}\mathscr{E})$.*

This result is in contrast to the situation for countable $\mathfrak{A}$ when $\mathrm{IND}(\text{pos-}\mathscr{E}) = \Pi_1^1(\mathfrak{A}) = \mathrm{IND}(\text{mon-}\mathscr{E})$. ($\mathrm{IND}(\mathscr{E})$ is always very much larger than $\mathrm{IND}(\text{pos-}\mathscr{E})$.)

3.6. Induction and admissible sets

In this subsection we will assume some familiarity with the notion of an admissible set (see Chapter A.7).

Given an admissible set A, let $\mathfrak{A} = \langle A, \in \restriction A \rangle$. The class of Σ_1 *formulae of* L^A are built up from the atomic formulae and their negations using $\vee$, $\wedge$, $\exists x$ and the restricted universal quantifier $\forall x \in y$. As in 2.3 the Σ_1 relations over $\mathfrak{A}$ are those relations on A definable over A by a Σ_1 formula of $\mathrm{L}(\mathfrak{A})$. (Note that this means that constants for elements of A are allowed.) As in 3.1 we define the class of operators $\phi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ that are positive Σ_1 over $\mathfrak{A}$, the ordinal $\mathrm{o}(A)$ of A is the smallest ordinal not in A. Finally, recall that a relation $R \subseteq A^n$ is *A-finite* if $R \in A$. (Note that, as A is closed under pairing, $A^n \subseteq A$ for $n > 0$. So every relation on A is a subset of A.) Much of the interest in admissible sets centres around the infinitary languages L_A that are associated with them. Recall that these are extensions of first order languages where the formulae are set-theoretically represented as elements of A, and infinite conjunctions and disjunctions $\bigwedge \Phi$ and $\bigvee \Phi$ are allowed as long as Φ is an A-finite set of formulae. Just as the syntax of first order languages is full of effectively presented finitary inductive definitions, the syntax of L_A uses the following. Below let A be a fixed admissible set.

3.6.1. DEFINITION. (i) A rule set Φ on A is *A-finitary* if the set of premisses of every rule in Φ is A-finite.

(ii) An A-finitary rule set Φ is Σ_1 *over* $\mathfrak{A}$ if it is so as a binary relation on A.

The following result essentially generalises Proposition 2.1.2 (in fact 2.1.2 is essentially the special case when A is the set HF of hereditarily finite sets).

3.6.2. PROPOSITION. *If Φ is an A-finitary rule set that is Σ_1 over $\mathfrak{A}$, then $I(\Phi)$ is Σ_1 over $\mathfrak{A}$.*

PROOF. Let $\phi : \mathrm{Pow}(A) \to \mathrm{Pow}(A)$ be the monotone operator associated with Φ. Let $\mathrm{Pr}_\Phi(a, b)$ if $a \in A$ is a ϕ-proof of b (see Definition 1.4.1). Then it follows easily from the assumptions that Pr_Φ is Σ_1 over $\mathfrak{A}$. Hence it would suffice to show that $I(\Phi) = \{b \in A \mid \exists a \in A\ \mathrm{Pr}_\Phi(a, b)\}$. But the natural proof of this will only work if $\mathfrak{A}$ satisfies "every set can be well-ordered".

Hence we use a modification of the notion of ϕ-proof. We say that $\{a_\mu\}_{\mu \le \lambda}$ is a *ϕ-quasi-proof of b* if

(i) $a_\lambda = \{b\}$, and

(ii) $a_\nu \subseteq \phi(\bigcup_{\mu < \nu} a_\mu)$ for all $\nu \le \lambda$.

As in the proof of Proposition 1.4.2 we can show that for an arbitrary monotone operator

$$(*) \qquad I(\phi) = \{a \in A \mid a \text{ has a } \phi\text{-quasi-proof}\}.$$

But the proof of $(*)$ requires no form of the axiom of choice, in contrast to the proof of 1.4.2.

Now let $\mathrm{qPr}_\Phi(a, b)$ if $a \in A$ is a ϕ-quasi-proof of b. As qPr_Φ is easily seen to be Σ_1 over $\mathfrak{A}$ it suffices to prove the

Claim. $I(\Phi) = \{b \in A \mid \exists a \in A\ \mathrm{qPr}_\Phi(a, b)\}$.

The inclusion $\supseteq$ follows from (*) above. For the other inclusion it is sufficient to show that the right-hand side is Φ-closed. So let $\Phi : X \to b$ such that every element of X has a ϕ-quasi-proof in A, i.e.,

$$\forall x \in X \, \exists a \in A \; \mathrm{qPr}_{\Phi}(a, x).$$

We must find a ϕ-quasi-proof of b in A. As qPr_{Φ} is Σ_1 over $\mathfrak{A}$ we may use strong Σ_1 collection to find a set $Y \in A$ such that

$$\forall x \in X \, \exists a \in Y \, \mathrm{qPr}_{\Phi}(a, x)$$

and

$$\forall a \in Y \, \exists x \in X \, \mathrm{qPr}_{\Phi}(a, x).$$

It follows that each $a \in Y$ is a sequence $\{a_\mu\}_{\mu \le \lambda_a}$. Let λ be the least ordinal such that $\lambda > \lambda_a$ for all $a \in Y$, $c_\lambda = \{b\}$ and $c_\mu = \bigcup\{a_\mu \mid a \in Y$ and $\mu \le \lambda_a\}$ for $\mu < \lambda$. Then as A is an admissible set $\lambda < \mathrm{o}(A)$ and $c \in A$. Also, by construction c is a ϕ-quasi-proof of b. Hence $\exists a \in A \; \mathrm{qPr}_{\Phi}(a, b)$. $\square$

Note that it follows from the above claim that only ϕ-quasi-proofs $\{a_\mu\}_{\mu \le \lambda}$ of length $\lambda < \mathrm{o}(A)$ are needed. Hence $|\phi| \le \mathrm{o}(A)$.

Remark. The same proof shows that if $\mathfrak{A}$ is a model of ZF, then $I(\Phi)$ is first order definable over $\mathfrak{A}$ whenever Φ is an A-finitary rule set that is first order definable over $\mathfrak{A}$.

There is an approach to Barwise's compactness theorem for L_A, when A is a countable admissible set, that makes use of the class of positive Σ_1 over $\mathfrak{A}$ inductive definitions. This idea was first suggested by Gandy. A version of this approach may be found in ACZEL [1973]. Here we will just consider the following result.

3.6.3. THEOREM (Gandy). *Let $\phi : \mathrm{Pow}(A^n) \to \mathrm{Pow}(A^n)$ be positive Σ_1 over $\mathfrak{A}$. Then*

(i) *if $R \subseteq A^n$ is Σ_1 over $\mathfrak{A}$, then $\bar{a} \in \phi(R)$ implies $\bar{a} \in \phi(R')$ for some A-finite $R' \subseteq R$.*

(ii) *$I(\phi)$ is Σ_1 over $\mathfrak{A}$ and $|\phi| \le \mathrm{o}(A)$.*

PROOF. (i) It suffices to show that for each Σ_1 formula $\theta(X)$ of $L(\mathfrak{A})$, containing positive occurrences of the n-ary relation variables X, but no other free variables, if $R \subseteq A^n$ is Σ_1 over $\mathfrak{A}$, then $\theta(R)$ implies $\theta(R')$ for some A-finite $R' \subseteq R$. This may be proved by a straightforward induction on the way $\theta(X)$ is built up. The key case, when $\theta(X)$ is a restricted universal quantification, requires an application of Σ_1-collection.

(ii) Let Φ be the set of rules $X \to \bar{a}$ such that $X \subseteq A^n$ is A-finite and $\bar{a} \in \phi(X)$. Then clearly Φ is a Σ_1 over $\mathfrak{A}$, A-finitary rule set. Hence by Proposition 3.6.2, $I(\Phi)$ is Σ_1 over $\mathfrak{A}$, so that it suffices to show that $I(\phi) = I(\Phi)$. Note that ϕ is not necessarily the monotone operator associated with Φ, but by (i) it is so on Σ_1 over $\mathfrak{A}$ relations, and this will suffice. To show that $I(\phi) \subseteq I(\Phi)$ it suffices to show that $I(\Phi)$ is ϕ-closed. So let $\bar{x} \in \phi(I(\Phi))$. As $I(\Phi)$ is Σ_1 over $\mathfrak{A}$, by (i) there is an A-finite $X \subseteq I(\Phi)$ such that $\bar{x} \in \phi(X)$. Hence $\Phi : X \to \bar{x}$ so that $\bar{x} \in I(\Phi)$. To show that $I(\Phi) \subseteq I(\phi)$ it suffices to show that $I(\phi)$ is Φ-closed. So let $\Phi : X \to \bar{x}$ where $X \subseteq I(\phi)$. Then $\bar{x} \in \phi(X) \subseteq \phi(I(\phi)) = I(\phi)$ as required. Finally, $|\phi| \leq \mathrm{o}(A)$ may be seen to follow from the note at the end of the proof of Proposition 3.6.2. $\square$

We end this subsection by stating the central result of BARWISE, GANDY and MOSCHOVAKIS [1971].

3.6.4. DEFINITION. Let A be an admissible set.

(i) $\pi : D \twoheadrightarrow A$ is a *projection of* x *on* A if $D \subseteq x \in A$ and π is a surjection onto A.

$\mathfrak{A}$ is *projectible to* x if A admits a projection on x that is Δ_1 over $\mathfrak{A}$ (i.e. the graph of the projection and its complement are both Σ_1 over $\mathfrak{A}$).

(ii) $\tau : \mathrm{o}(A) \to A$ is a *resolution* of A if $A = \bigcup_{\alpha < \mathrm{o}(A)} \tau(\alpha)$.

$\mathfrak{A}$ is *resolvable* if A admits a resolution that is Δ_1 over $\mathfrak{A}$.

3.6.5. THEOREM (BARWISE, GANDY and MOSCHOVAKIS [1971]). *Let A be any transitive set closed under pairing. Let*

$$A^+ = \bigcap\{B \mid A \in B \text{ and } B \text{ is admissible}\}.$$

Then

(i) *A^+ is admissible,*

(ii) $\mathrm{HYP}(\mathfrak{A})$ *is the set of A^+-finite relations on A and* $\kappa(\mathfrak{A}) = \mathrm{o}(A^+)$, *and if* $\mathfrak{A}^+ = \langle A^+, \in \restriction A^+ \rangle$, *then*

(iii) *$\mathfrak{A}^+$ is resolvable and projectible to A and* $\mathrm{IND}(\mathfrak{A})$ *is the set of Σ_1 over $\mathfrak{A}^+$ relations on A.*

Remarks. The above result has been generalised in two directions. Firstly, $\mathrm{IND}(\mathfrak{A})$ can be replaced by an arbitrary Spector class Γ over $\mathfrak{A} =$

$\langle A, \in \restriction A \rangle$ for any transitive set A. Then A^+ must be replaced by $M(\Delta)$ where $\Delta = \Gamma \cap \neg\Gamma$ and $M(\Delta) = \bigcap\{B \mid \Delta \subseteq B$ and B is admissible$\}$.

In (ii), HYP($\mathfrak{A}$) and $\kappa(\mathfrak{A})$ must be replaced by Δ and o(Γ) respectively. (iii) is replaced by (iii)′:

(iii)′ There is a relation R on $M(\Delta)$ such that $M(\Delta)$ is admissible relative to R and $\mathcal{M}(\Delta) = \langle M(\Delta), \in \restriction M(\Delta), R \rangle$ is resolvable and projectible to A. Moreover Γ is the set of relations on A that are Σ_1 over $\mathcal{M}(\Delta)$.

Although R and hence $\mathcal{M}(\Delta)$ are not unique the class of relations on $M(\Delta)$ that are Σ_1 over $\mathcal{M}(\Delta)$ is independent of the choice of R and is called the companion of Γ. This generalization is 9E.1. of MOSCHOVAKIS [1974a]. The second generalization is to allow $\mathfrak{A}$ to be an arbitrary abstract structure. This requires the notion of an admissible structure with set A of urelemente that has been introduced in BARWISE [1974, 1975]. The details have been worked out in ENNIS [1975]. There, Ennis shows that the result still holds when (ii) in the definition of a Spector class is weakened to (ii)′:

(ii)′ Γ contains the graph of a pairing function on A.

4. Induction in foundations

In this final section we shall briefly consider the role of inductive definitions in the context of foundations. So far we have taken for granted the standard framework of modern mathematics, formalisable in ZF set theory. But the concept of an inductive definition can also be considered within the other conceptual frameworks that have arisen in work on foundations (e.g. finitist, predicative or intuitionistic mathematics). Within a non-classical framework there arises the question of which inductive definitions can be justified.

It will be helpful to make the distinction between *fundamental* and *non-fundamental* inductive definitions, following the terminology of §53 of KLEENE [1952]. This is a distinction concerning the context in which an inductive definition is presented. The inductive definition of the domain $\mathbf{N}$ of natural numbers is most naturally presented as a fundamental definition of a new sort of object. But in the context of ZF set theory the natural numbers $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$ are objects that exist independently of the inductive definition of ω. So the latter is a non-fundamental inductive definition. Another example is the inductive definition of the set of even numbers as the smallest set of natural numbers containing 0, and $n + 2$ whenever it contains n.

The standard model of ZF set theory is given by a fundamental inductive definition of the universe of (well-founded) sets. This consists of the collection of rules: If s is a set of objects of the universe, then s itself is an object of the universe.

While the fundamental domains associated with a conceptual framework will usually be explicit, it may require further analysis to decide which non-fundamental inductive definitions are justifiable. For example, given a collection of rules Φ on the domain $\mathbb{N}$ of natual numbers, we have defined $I(\Phi)$ as $\bigcap[X \subseteq N \mid X$ is Φ-closed}, i.e.,

$$n \in I(\Phi) \Leftrightarrow (\forall X \subseteq \mathbb{N})[X \text{ is } \Phi\text{-closed} \rightarrow n \in X].$$

But this is an impredicative instance of the comprehension scheme (a subset of $\mathbb{N}$ is defined using quantification over all subsets of $\mathbb{N}$). If impredicative definitions are not allowed we must look for other possible definitions of $I(\Phi)$. In case Φ is finitary we can use Proposition 1.1.4. In general $I(\Phi)$ could be defined using transfinite iterations of operators, as in Proposition 1.3.1 or transfinite proofs, as in Proposition 1.4.2. But this would require a suitable notion of ordinal which itself would need an inductive definition. An alternative approach is to take induction as a primitive method of definition, not needing justification in terms of other methods.

An example of a formal system treating induction as primitive is the system $\mathbf{ID}_1$ of FEFERMAN [1970] (see also FRIEDMAN [1970], ZUCKER [1973] and MARTIN-LÖF [1971]). The language of $\mathbf{ID}_1$ is obtained from the language of formal arithmetic by adding a new n-ary relation symbol P_ϕ for each arithmetical formula $\phi = \phi(X, \bar{x})$ containing only positive occurrences of the n-ary relation variable X and at most the free individual variables $\bar{x} = x_1, \ldots, x_n$. The axioms for $\mathbf{ID}_1$ are obtained from the axioms for formal arithmetic by extending the mathematical induction scheme to all formulae of $\mathbf{ID}_1$ and adding the following axiom and axiom scheme for each P_ϕ.

(i) $\forall \bar{x}\, (\phi(P_\phi, \bar{x}) \rightarrow P_\phi(\bar{x}))$,

(ii) $\forall \bar{x}\, (\phi(\{\bar{z} \mid \psi(\bar{z})\}, \bar{x}) \rightarrow \psi(\bar{x})) \rightarrow \forall \bar{x}\, (P_\phi(\bar{x}) \rightarrow \psi(\bar{x}))$,

for all formulae $\psi(\bar{z})$ of $\mathbf{ID}_1$.

In (ii), $\phi(\{\bar{z} \mid \psi(z)\}, \bar{x})$ denotes the result of replacing each occurrence of $X(\bar{t})$ in $\phi(X, \bar{x})$ by $\psi(\bar{t})$, where $\bar{t} = t_1, \ldots, t_n$ is a sequence of terms, and bound variables are changed as required by the usual conventions.

On the standard model of arithmetic these axioms express that P_ϕ is inductively defined by the positive arithmetical operator defined by $\phi(X, \bar{x})$.

The procedure for constructing $\mathbf{ID}_1$ from formal arithmetic may be repeated to obtain $\mathbf{ID}_2, \mathbf{ID}_3, \ldots$. But the resulting systems are still much weaker than fully impredicative systems such as second order arithmetic. On the other hand $\mathbf{ID}_1$ is already stronger than the systems of predicative mathematics of FEFERMAN [1968]. Thus inductive definability is a notion intermediate in strength between predicative and fully impredicative definability.

It would be interesting to formulate a coherent conceptual framework that made induction the principal notion. There are suggestions of this in the literature, but the possibility has not yet been fully explored.

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