# 7310 Lecture notes. Spring 2019

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## Contents

# Part I
# Construction of measure

## 1  January 15, 2019

### 1.1  Basics

A few completions of of $\mathbb{Q}$:

1) Cauchy sequences
2) Dedekind cuts
3) Axiomated total ordered complete fields
4) Decimal expanation up to equivalence class

**Definition 1.1.** Dedekind cut is a pair of non-empty sets $(A, B)$ such that it satisfies $\mathbb{Q} = A \sqcup B$ and the following:

1. if $x \in A$, $y < x \implies y \in A$, i.e., A is order-closed downwards.
2. if $x \in B$, $y \geq x$, $\implies y \in B$, i.e., B is order-closed upwards.
3. A does not have a maximum element

Define $\mathbb{R} = \{$Dedekind cuts of $\mathbb{Q}\}$. For simplicity, we can use $A$ to represent the pair $(A, B)$. Note we can define an order on $\mathbb{R}$ by set inclusion with respect to $A$. We can also define addition. subtraction, multiplication, division on $\mathbb{R}$ such that $\mathbb{R}$ is totally ordered, complete field.

**Theorem 1.2.** $\mathbb{R}$ *is uncountable.*

*Proof.* Diagonal process. The complete proof is left as an exercise for the reader. $\square$

### 1.2  Problem of measure

Question 1: What is the **length** $\ell$ of a subset of $\mathbb{R}$?

a) $\ell([0, 1]) = 1$;
b) $\ell(\{x\}) = 0$ for any $x \in \mathbb{R}$;
c) $\ell(\{x \in \mathbb{R} : x = 0.**3*****...\}) = \frac{1}{10}$. This is the set of decimal expansions such that there is a "3" in the third place after the dot.

**Axioms of length on $\mathbb{R}$:**

1. $\ell([0, 1]) = 1$. i.e., normalized;
2. for $A \subseteq \mathbb{R}$ such that $A = A_1 \sqcup A_2 \sqcup ...$, $(A) = \sum_{i=1}^{\infty}(A_i)$, i.e., countable additivity;
3. for $A \subseteq \mathbb{R}$, $\alpha \in \mathbb{R}$, $\ell(A + \alpha) = \ell(A)$, i.e., transitivity-invariant.

**Theorem 1.3.** *Consider the length $\ell: 2^{\mathbb{R}} = P(\mathbb{R}) \longrightarrow [0, \infty]$, such length does not exist.*

*Proof.* We can write $[0, 1) = \bigsqcup_{r \in N}(\mathbb{Q} + r) \cap [0, 1)$, where $N$ is the collection of representatives of each equivalence class with respect to the following equivalence relation: $x \sim y \iff x - y \in \mathbb{Q}$. Note that we need Axiom of Choice to define $N$. Then as length is transitivity-invariant, $\ell(N) = \ell(N + \alpha) \forall \ \alpha$. Hence $1 = \ell([0, 1)) = \sum_{x \in N} \ell(N + x)$, but the right hand side is the sum of countably infinite elements with the same value, either equals 0 or infinity. Contradiction. $\square$

**Corollary 1.4.** *There exist non-measurable sets.*

## 1.3   Outer measure

**Definition 1.5.** Let $X$ be a set. $\mu: 2^X \longrightarrow [0, \infty]$ is called an **outer measure** if

    1. $\mu(\varnothing) = 0$;
    2. if $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \bigcup_{k=1}^{\infty} \mu(A_k)$ (Hence $\mu(A) \leq \mu(B)$ if $A \subseteq B$).

**Definition 1.6.** Let $X$ be a set, $\mu$ an outer measure defined on $X$. Then $A \subseteq X$ is called $\mu$-measurable if for all $B \subseteq X$, $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$

## 1.4   $\sigma$-algebras

**Definition 1.7.** Let $X$ be a set. $\mathcal{F} \subseteq X$ is called $\sigma$-algebra if
    1. $\varnothing \in \mathcal{F}$
    2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
    3. if $A_1, A_2, ... \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

We will eventually prove the following important theorem:

**Theorem 1.8** (Caratheodory Extension Theorem)**.** *Let $X$ be a set, $\mu$ an outer measure on $X$. Then the class of $\mu$-measurable sets is a $\sigma$-algebra.*

Below are some examples of $\sigma$-algebras:

- $X$ countable, $\mathcal{F} = 2^X$ is a $\sigma$-algebra;

- for any set $X$, $\mathcal{F} = \{\varnothing, X\}$ is called the trivial $\sigma$-algebra;

- for any set $X$, $A \subseteq X$, $\mathcal{F} = \{\varnothing, A, A^c, X\}$ is a $\sigma$-algebra (generated by the set $A$). That is, we define $\sigma(A) = \{\varnothing, A, A^c, X\} = \bigcap_{\{\mathcal{F} \in S\}} \mathcal{F}$, where $S = \{\mathcal{F} \text{ is a } \sigma\text{-algebra containing A}\}$.

**Proposition 1.9.** *Let $X$ be a set. For any set $\Gamma$ (not necessarily countable), if $\mathcal{F}_x$ is a $\sigma$-algebra on $X$ for all $x \in \Gamma$, then $\bigcap_{x \in \Gamma} \mathcal{F}_x$ is also a $\sigma$-algebra.*

*Proof.* Exercise left for the reader. $\square$

**Definition 1.10.** The Borel $\sigma$-algebra of $\mathbb{R}^d$ (d $\geq$ 1), denoted as $B(\mathbb{R}^d)$, is defined as the $\sigma$-algebra generated by all open sets in $\mathbb{R}^d$.

The Borel $\sigma$-algebra has countable generating sets (e.g. rational rectangles).

# 2    January 17, 2019

## 2.1    Measures

**Definition 2.1.** Let $X$ be a set, $\mathcal{F}$ a $\sigma$-algebra on $X$. Then a **measure** on $\mathcal{F}$ is a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ such that

a) $\mu(\varnothing) = 0$;

b) for $A_1, A_2, ... \in \mathcal{F}$ such that $A_i \cap A_j = \varnothing$ for $i \neq j$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, i.e., $\mu$ is countably additive.

We call $(X, \mathcal{F}, \mu)$ a **measure space**.

Below are some examples of measure spaces. Let $X$ be a set.

- Fix $x \in X$, then for $A \subseteq X$ consider $\delta \colon X \longrightarrow [0, \infty]$ by

$$\delta(A) = \left\{ \begin{array}{ll} 0 & if \ x \notin A \\ 1 & if \ x \in A \end{array} \right.$$

   This is called the $\delta$-measure at $x \in X$.

- Let $\mathcal{F} = 2^X$, for $A \in X$, define $\mu$ such that $\mu(A) = 0$ if $A$ is countable, and $\mu(A) = 1$ if $A$ is co-countable (i.e., $A^c$ is countable). Then $(X, \mathcal{F}, \mu)$ is a measure space.

- For $X$ countable, i.e., $X = \{x_1, x_2, ...\}$. Let $\mathcal{F} = 2^X$. Denote $p(x_i) = $ weight of $x_i$, then for $A \subseteq X$, define $\mu(A) = \sum_{x \in A} p(x)$. $(X, \mathcal{F}, \mu)$ is a measure space.

**Properties of a measure space** $(X, F, \mu)$

1. for $A, B \subseteq X$, $A \subseteq B \implies \mu(A) \leq \mu(B)$;

2. (countably subadditive) if $A \subseteq \bigcup_{i=1}^{\infty} A_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$;

3. (continuity) for a sequence of subsets in $X$, $A_1 \subseteq A_2 \subseteq ...$,

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i);$$

   for a sequence of subsets in $X$, $A_1 \supseteq A_2 \supseteq ...$ such that $\mu(A_i) < \infty \ \forall i$,

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i).$$

A counterexample for 3) if we drop the condition "$\mu(A_i) < \infty \ \forall i$" is $X = \mathbb{R}$, $F = \sigma(\mathbb{R})$, $A_i = (i, \infty)$ for $i \in \mathbb{N}$, then each $\mu(A_i) = \infty$ but $lim_{i \to \infty} \mu(A_i) = 0$.

**Definition 2.2.** Let $(X, F, \mu)$ be a measure space. $\mu$ is called **finite** if $\mu(X) < \infty$, $\mu$ is called $\sigma$-**finite** if $X = \bigcup E_n$ where $\mu(E_n) < \infty \ \forall n$.

**Semicontinuity.** Let $\{A_n : n \in \mathbb{N}\}$ be a collection of sets. Then define

$$\liminf A_n = \{\text{elements in almost all } A_n\} = \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} A_k),$$

$$\limsup A_n = \{\text{elements in } A_n \text{ infinitely often}\} = \bigcap_{n=1}^{\infty} (\bigcup_{k=n}^{\infty} A_k).$$

**Proposition 2.3.** *We have*

$$\mu(\limsup A_n) \geq \limsup \mu(A_n), \qquad \mu(\liminf A_n) \leq \liminf \mu(A_n).$$

*Proof.* Homework exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

## 2.2 Completeness

**Definition 2.4.** Consider $(X, \mathcal{F}, \mu)$, $B \in \mathcal{F}$ is called a **null set** if $\mu(B) = 0$. Note if $B$ is a null set, $A \subseteq B$, then $\mu(A) = 0$ in the sense of outer measure (not necessarily $A \in \mathcal{F}$).

**Definition 2.5.** $(X, \mathcal{F})$ is **complete** if all null sets are contained in $\mathcal{F}$.

Here we introduce a notation "**a.e**", the abbreviation of "almost everywhere". Something happens a.e. if it happens outside a null set. Here are a few examples:

Consider the Dirichlet function $f(x) = \begin{cases} 1 & if \ x \in \mathbb{Q} \\ 0 & if \ x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ . Then $f(x) = 0$ a.e.

Consider the (small) Riemann function $f(x) = \begin{cases} \frac{1}{q} & if \ x = \frac{p}{q} \in \mathbb{Q} \\ 0 & if \ x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$ . Then $f$ is continuous a.e.

**Theorem 2.6.** *There is an unique completion of any measure space $(X, \mathcal{F}, \mu)$. i.e., there exist $\overline{\mathcal{F}}, \overline{\mu}$ with $\overline{\mathcal{F}} \supseteq \mathcal{F}$ and $\overline{\mu}|_{\mathcal{F}} = \mu$. Specifically,*

$$\overline{\mathcal{F}} = \{A \cup B | \ A \in \mathcal{F}, B \subseteq N \text{ for some } N \in \mathcal{F}, \mu(N) = 0\},$$

*which satisfies:*
*1) $\overline{\mathcal{F}}$ is a $\sigma$-algebra;*
*2) there exists unique extension $\overline{\mu}$ of $\mu$, from $\mathcal{F}$ to $\overline{\mathcal{F}}$ such that $\overline{\mu}|_{\mathcal{F}} = \mu$.*

*Proof.* Let $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$. Then it is closed under countable unions and complement, hence so is a$\overline{\mathcal{F}}$. Then set $\overline{\mu}(A \cup B) = \mu(A)$ if $B \subseteq N \in \mathcal{N}$. Then it is left as an exercise to the reader to check:

a) $\overline{\mu}$ is well-defined (i.e., if there is another way of writing $A \cup B = A' \cup B'$, $B' \subseteq N' \in \mathcal{N}$, then the measure does not change);

b) $\overline{\mu}$ is a measure;

c) $\overline{\mu}$ is a unique extension to $\overline{\mathcal{F}}$ defined in the claim. $\qquad\square$

Note that the Borel $\sigma$-algebra is not complete with respect to the length measure $\ell$.

## 2.3 Caratheodory's theorem. Formulation and first part of proof

Let $X$ be a set. Recall the definition of the outer measure $\mu$ on $2^X$.

**Definition 2.7.** $A \subseteq X$ is a $\mu$-measurable set with respect to $\mu$ if $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c)$ $\forall B \in 2^X$.

We now begin the proof of the Caratheodory's extension theorem:

**Theorem 2.8** (Caratheodory)**.** *Let $\mu$ be an outer measure on a set $X$, then the following statements are true:*

*1) $\mathcal{F} = \{A \colon A$ is a $\mu$-measurable set$\}$ is a $\sigma$-algebra;*

*2) $\mu|_{\mathcal{F}}$ is a complete measure.*

*Proof.* For 1), first of all, $\mathcal{F}$ is closed under complement by the definition of the outer measure and the measurable sets. Then for $A, B \in \mathcal{F}$, take any $S \subseteq X$, then

$$\mu(S) = \mu(S \cap A) + \mu(S \cap A^c)$$
$$= \mu(S \cap A \cap B) + \mu(S \cap A \cap B^c) + \mu(S \cap A^c \cap B) + \mu(S \cap A^c \cap B^c).$$

Note $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, then by subadditivity,

$$\mu(S \cap (A \cap B)) + \mu(S \cap (A \cap B^c)) + \mu(S \cap (A^c \cap B)) \geq \mu(S \cap (A \cup B)).$$

Hence $\mu(S) \geq \mu(E \cap (A \cup B)) + \mu(E \cap (A \cup B)^c)$. Hence $A \cup B \in \mathcal{F}$, so $\mathcal{F}$ is an algebra.

Also, if $A, B \in \mathcal{F}$ with $A \cap B = \varnothing$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \cap A^c) = \mu(A) + \mu(B).$$

This shows finite additivity of $\mu$ on $\mathcal{F}$.

(other parts will be proven next time.) $\qquad\square$

## 2.4 Definition of a pre-measure

We will also work with outer measures constructed using pre-measures:

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**Definition 2.9.** Let $X$ be a set, $\Gamma \subseteq 2^X$, $\nu : \Gamma \longrightarrow [0, \infty]$ be any function. Then the **pre-measure** $\mu$ on $2^X$ is defined as for $F \in 2^X$,

$$\mu(F) = \inf\left\{ \sum_{n=1}^{\infty} \nu(A_n) : A_n \in \Gamma, F \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

By agreement, $\inf \varnothing = 0$.

**Proposition 2.10.** *With the same set up as above, $\mu$ is an outer measure.*

*Proof.* This will also be proven next time. $\qquad\square$

# 3  January 22, 2019

## 3.1  From outer measure to measure

We are on a path of constructing the measure. Last time, we defined pre-measures, and this definition will be used later. Here we show how an outer measure leads to a measure, i.e., finish the proof of the Caratheodory theorem.

Recall that if $\mu$ is an outer measure, we say that a set $A$ is $\mu$-measurable if for all $E \subseteq X$,
$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

Note that in the proofs we need only to check $\geq$ since $\leq$ follows directly from subadditivity. Denote by $\mathcal{F}$ the set of all $\mu$-measurable subsets.

*Rest of the proof of Caratheodory theorem (Theorem 2.8).* We have shown in Lecture 2 that $\mathcal{F}$ is an algebra (closed under finite unions, intersections, and complements) and that $\mu$ is finitely additive on $\mathcal{F}$.

Now we need to show that $\mathcal{F}$ is a $\sigma$-algebra. Let $A_1, A_2, \ldots \in \mathcal{F}$ be disjoint, and define $B_n = \bigsqcup_{j=1}^{n} A_j$. We know that $B_n \in \mathcal{F}$, so we need to show that $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

First, for all $E \subseteq X$, we have

$$\mu(E \cap B_n) = \mu(E \cap B_n \cap A_n) + \mu(E \cap B_n \cap A_n^c) = \mu(E \cap A_n) + \mu(E_\cap B_{n-1}).$$

Continuing this, we see that

$$\mu(E \cap B_n) = \sum_{i=1}^{n} \mu(E \cap A_i).$$

Thus,

$$\mu(E) = \mu(E \cap B_n) + \mu(E \cap B_n^c) \geq \sum_{i=1}^{n} \mu(E \cap A_i) + \mu(E \cap B^c).$$

The last inequality is due to the fact that in the second summand we have passed to the smaller set $E \cap B$. In the previous inequality we can now pass to $n \to \infty$, and all the sums in the right-hand side are bounded:

$$\mu(E) \geq \sum_{i=1}^{\infty} \mu(E \cap A_i) + \mu(E \cap B^c).$$

By subadditivity, this is

$$\geq \mu \left( \bigcup_{i=1}^{\infty} (E \cap A_i) \right) + \mu(E \cap B^c) = \mu(E \cap B) + \mu(E \cap B^c).$$

This shows that $B \in \mathcal{F}$, so $\mathcal{F}$ is a $\sigma$-algebra.

The $\sigma$-additivity of $\mu$ on $\mathcal{F}$ is straightforward, take $B = E$ in the previous argument, which leads to $\mu(B) = \sum_{i=1}^{\infty} \mu(A_i)$.

Finally, to show completeness, let $A \subseteq X$ be with the outer measure $\mu(A) = 0$. We want to show that $A \in \mathcal{F}$. We have for all $E \subseteq X$:

$$\mu(E) \leq \mu(E \cap A) + \mu(E \cap A^c) = \mu(E \cap A^c) \leq \mu(E)$$

because the outer measure is subadditive and so $\mu(E \cap A) = 0$. The last inequality is also due to subadditivity. This implies that $\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$, so $E \in \mathcal{F}$. □

## 3.2 From pre-measure to outer measure

Recall the definition of a pre-measure:

**Definition 3.1** (Pre-measure). Let $X$ be a set and let $\mathcal{E}$ be a set of subsets of $X$. Then any function $\nu : \mathcal{E} \to [0, \infty]$ is called a *pre-measure*.

**Proposition 3.2.** *If $\nu$ is a pre-measure, then $\mu$ defined by*

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \nu(A_i) : A_i \in \mathcal{E}, E \subseteq \cup_{i=1}^{\infty} A_i \right\}$$

*is an outer measure on $\mathcal{E}$.*

*Proof.* We need only prove the subadditivity of $\mu$. Let $F \subseteq \cup_{i=1}^{\infty} F_i$ and let $\varepsilon > 0$. By construction of $\mu$, there are sets $A_i^n$ such that $F_i \subseteq \cup_{n=1}^{\infty} A_i^n$ and

$$\mu(F_i) + \frac{\varepsilon}{2^i} \geq \sum_{n=1}^{\infty} \nu(A_i^n).$$

We also note that

$$F \subseteq \bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_i^n.$$

This means

$$\mu(F) \le \mu\left(\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty} A_i^n\right) \le \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \mu(A_i^n) \le \sum_{i=1}^{\infty}\left(\mu(F_i) + \frac{\varepsilon}{2^i}\right).$$

As we can make $\varepsilon$ as small as we'd like, this implies that $\mu(F) \le \sum_{i=1}^{\infty} \mu(F_i)$ as we wanted. $\qquad\square$

## 3.3  Examples

Now, pre-measures are an easy way of constructing an outer measure, but as we will see, the resulting outer measure may not agree with our original pre-measure. For example, let $\mathcal{E} = \{\varnothing, A, A^c, X\}$ where $\nu(\varnothing) = 0$, $\nu(A) = a$, $\nu(A^c) = b$, and $\nu(X) = 1$. Then for the corresponding outer measure $\mu$ we have that $\mu(\varnothing) = 0$ and $\mu(X) = \min(a+b, 1)$. Furthermore, if $B \subseteq A$, then $\mu(B) = a$ and if $B \subseteq A^c$, then $\mu(B) = b$. However, is $A$ even $\mu$-measurable? We see that $A$ is $\mu$-measurable only if

$$\min(a + b, 1) = \mu(X) = \mu(E \cap A) + \mu(E \cap A^c) = \mu(A) + \mu(A^c) = a + b.$$

In other words, if $a + b > 1$, then $A$ will not be one of our $\mu$-measurable sets. This should serve as a warning that the sets in $\mathcal{E}$ may not be $\mu$-measurable.

For a second example, let $\mathcal{E}$ be the set of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$. We let $\nu(\varnothing) = 0$ and $\nu(A) = \infty$ for $\varnothing \ne A \in \mathcal{E}$. We note that in this case, there is more than one possible $\mu$ which extends $\nu$. For example, we can assign all points measure zero, or measure 1, or measure 2, or measure $\infty$.

## 3.4  Lebesgue-Stieltjes measures

The *Borel $\sigma$-algebra* is the $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ generated by all open subsets of $\mathbb{R}$. A measure $\mu$ on $\mathcal{B}(\mathbb{R})$ is called a *Borel measure*. (Sometimes we will assume that it's $\sigma$-finite; moreover, that bounded sets have finite measure.)

**Remark 3.3.** The cardinality of the Borel $\sigma$-algebra is continuum. This can be proven using transfinite induction.

Assuming that $\mu$ is a finite Borel measure, we can define the function $F(x) = \mu((-\infty, x])$. We see that $F$ is weakly increasing and is right-continuous.[1] On the other hand, having a right-continuous and weakly increasing function, we can define a pre-measure $\nu$ for which $\nu((a, b]) = F(b) - F(a)$. Measures constructed from such pre-measures are called *Lebesgue-Stieltjes measures*. In particular, if we take $F(x) = x$ as our pre-measure function, then the resulting measure is called the Lebesgue measure (note that it is not finite). In the next lecture we discuss how this works exactly.

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[1]It is right-continuous because $\mu\left((-\infty, a]\right) = \mu\left(\bigcap_{n=1}^{\infty}(-\infty, a_n]\right) = \lim_{n \to +\infty} \mu\left((-\infty, a_n]\right)$ by the continuity of measure, where $a_n$ decrease to $a$.

# 4 January 24, 2019

## 4.1 Lebesgue-Stieltjes measures continued

The goal now is to finish the construction of measure from pre-measures, and apply this to Lebesgue-Stieltjes measures. There are two main pieces of the construction that are still missing:

1. If the pre-measure is countably additive on $\mathcal{E}$, then the resulting measure (coming from the corresponding outer measure by the Caratheodory theorem) agrees with the pre-measure on $\mathcal{E}$, and the sets from $\mathcal{E}$ are measurable.

2. The pre-measure constructed from a function $F$ is countably additive.

The first part is an abstract statement; and the second part is a concrete result that applies on $\mathbb{R}$ and in the Lebesgue-Stieltjes context. We start with the second one.

Let $\mathcal{A}$ be the set of finite disjoint unions of intervals of the form $(a, b]$, where $-\infty \leq a \leq b \leq \infty$. This is an algebra (i.e. is closed under finite intersections, finite unions, and complements). Let $F$ be a weakly increasing right-continuous function. We construct a pre-measure $\nu$ on $\mathcal{A}$ where for disjoint unions,

$$\nu\left(\bigsqcup_{i=1}^{n}(a_i, b_i]\right) = \sum_{i=1}^{n}(F(b_i) - F(a_i))$$

and we'll use this pre-measure to construct our measure $\mu$. There are some potential issues in this definition (that $\nu$ is well-defined) that are resolved in the next proposition:

**Proposition 4.1.**    • *$\nu$ is well-defined;*

- *$\nu$ is finitely additive;*

- *$\nu$ is countably additive.*

*Proof.* Let $A = \cup I_i = \cup J_j$ where $I_i, J_j$ are intervals. Then we see that

$$\nu(A) = \sum_i \sum_j \nu(I_i \cap J_j)$$

the latter of which is equal to both $\sum_i \nu(I_i)$ and $\sum_j \nu(J_j)$, proving that $\nu$ is well-defined. The fact that $\nu$ is finitely additive on $\mathcal{A}$ is evident from the definition.

We will now prove that $\nu$ is countably additive on $\mathcal{A}$. Let $I_i \in \mathcal{A}$ and $I = \cup_{i=1}^{\infty} I_i$. We can assume without loss of generality that $I$ is an interval (we can apply the rest of our reasoning to the individual components of $I$). We will

further assume that $I$ has finite measure (the infinite measure case is left as an exercise). Now what we want to show is that $\nu(\cup_{i=1}^{\infty} I_i) = \sum_{i=1}^{\infty} \nu(I_i)$. We have

$$\nu(I) = \nu(\cup_{i=1}^{n} I_i) + \nu(I \setminus \cup_{i=1}^{n} I_i) \geq \sum_{i=1}^{n} \nu(I_i),$$

and then we can take the limit as $n \to \infty$ to see that $\nu(\cup I_i) \geq \sum \nu(I_i)$.

It remains to establish the reverse inequality. Now, let

$$I = (a, b] = \cup_{j=1}^{\infty} (a_j, b_j].$$

For all $\varepsilon > 0$, there exists $\delta > 0$ such that $F(a + \delta) - F(a) < \varepsilon$ and there exists $\delta_j$'s such that $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j}$. Here we used the right continuity of $F$.

Now since $[a + \delta, b]$ is compact and covered by the open sets $(a_i, b_i + \delta_i)$, there is a finite number of $(a_i, b_i + \delta_i)$ covers $[a + \delta, b]$ for $i \in \{1, \ldots, N\}$. This means

$$\nu(I) \leq 2\varepsilon + \sum_{i=1}^{\infty} (F(b_i) - F(a_i)),$$

and since $\varepsilon$ was arbitrarily small, we are done. $\qquad \square$

## 4.2  From countably additive pre-measure to measure

Let us now focus on properties of measures which come from a countably additive pre-measure. This will complete the construction of the measure.

**Proposition 4.2.** *If $\mu$ is a measure constructed by the pre-measure $\nu$, then $\mu\big|_{\mathcal{A}} = \nu\big|_{\mathcal{A}}$.*

*Proof.* Take an $E \in \mathcal{A}$ and cover it with sets $I_i \in \mathcal{A}$, i.e. $E \subseteq \cup_{j=1}^{\infty} I_j$. Now let $B_n = E \cap (I_n \setminus \cup_{j=1}^{n-1} I_j)$ which disjointly cover $E$. We then get

$$\nu(E) = \sum_{i=1}^{\infty} \nu(B_i) \leq \sum_{i=1}^{\infty} \nu(I_i).$$

If we take the infimum over any such sets $I_j$, we have that $\nu(E) \leq \mu(E)$.

Since $E \subseteq E$, we already have that $\mu(E) \leq \nu(E)$. Thus, $\nu\big|_{\mathcal{A}} = \mu\big|_{\mathcal{A}}$. $\qquad \square$

**Proposition 4.3.** *Let $\mathcal{A}$ be the set of finite unions of half-open intervals. Then for any $\mu$, all the sets in $\mathcal{A}$ are $\mu$-measurable.*

*Proof.* Let $B \in \mathcal{A}$ and $E \subseteq X$. For any $\varepsilon > 0$, we can find $A_i \in \mathcal{A}$ such that $E \subseteq \cup_{i=1}^{\infty} A_i$ and $\mu(E) + \varepsilon \geq \sum_{i=1}^{\infty} \nu(A_i)$. We then have that

$$\mu(E) + \varepsilon \geq \sum_{i=1}^{\infty} \nu(A_i)$$
$$= \sum_{i=1}^{\infty} (\nu(A_i \cap B) + \nu(A_i \cap B^c))$$
$$\geq \mu(E \cap B) + \mu(E \cap B^c),$$

which proves that $B$ is $\mu$-measurable. $\square$

**Proposition 4.4.** *Let $\nu$ be a $\sigma$-finite pre-measure and $\mu$ is the induced measure. If $\rho$ is extension of $\nu$ (i.e. $\rho\big|_{\mathcal{A}} = \nu$), then $\mu = \rho$.*

*Proof.* Let $E \in \mathcal{F}_\mu$, $E \subseteq \cup_{i=1}^\infty A_i$, $A_i \in \mathcal{A}$. Then

$$\rho(E) \leq \sum_{i=1}^\infty \rho(A_i) = \sum_{i=1}^\infty \nu(A_i)$$

so $\rho(E) \leq \mu(E)$.

Since $\rho$ is $\sigma$-finite, we need only to prove that they concide on finitely-measurably sets. As such, let $\mu(E) < \infty$. Now let $A_i \in \mathcal{A}$ and $E \subseteq A = \cup_{i=1}^\infty A_i$ where $\mu(A \Delta E) < \varepsilon$. We first see that

$$\rho(A) = \lim_{n \to \infty} \rho(\cup_{i=1}^n A_i) = \lim_{n \to \infty} \mu(\cup_{i=1}^n A_i) = \mu(A).$$

Since $A \setminus E \subseteq A \Delta E$, we get that

$$\mu(E) \leq \mu(A) = \rho(A) = \rho(E) + \rho(A \setminus E) = \rho(E) + \mu(A \setminus E) = \rho(E) + \varepsilon.$$

This proves that $\mu(E) \leq \rho(E) + \varepsilon$ as we wanted. $\square$

## 4.3 Properties of the Lebesgue and Lebesgue-Stieltjes measures

**Definition 4.5.** Any measure that is constructed from $F$ is called a *Lebesgue-Stieltjes measure*. If we use the particular function $F(x) = x$, the measure is called the *Lebesgue measure* and is denoted by $\ell$ instead of $\mu$.

The Lebesgue measure has some particularly nice properties:

i) It is shift-invariant (this is because $F(x) = x$ is shift-invariant)

ii) For any $s \in \mathbb{R}$, $\ell(sE) = |s|\ell(E)$

iii) Any measure defined on the Borel set with the two above properties must in fact be the Lebesgue measure.

HW Exercise: Let $\mu$ be a Lebesgue-Stieltjes measure. Show that

i) For all $E \in \mathcal{F}_\mu$, $\mu(E) = \inf\{\sum_{i=1}^\infty \mu(a_i, b_i) : E \subseteq \cup_{i=1}^\infty (a_i, b_i)\}$.

ii) For all $E \in \mathcal{F}_\mu$, $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \, \text{open}\}$.

iii) For all $E \in \mathcal{F}_\mu$, $\mu(E) = \sup\{\mu(K) : E \subseteq K, K \, \text{compact}\}$.

However, measurable sets should not scare you. For example, we can approximate measurable sets by some tamer looking sets, which should help calm your worries.

**Proposition 4.6.** *The following are equivalent:*

a) *$E$ is $\mu$-measurable*

b) *$E = V \setminus N$ where $N$ is a null set and $V$ is some countable intersection of open sets*

c) *$E = H \cup N$ where $N$ is a null set and $H$ is some countable union of closed sets*

In fact, we also have that

**Proposition 4.7.** *If $E$ is $\mu$-measurable and $\mu(E) < \infty$, then for any $\varepsilon > 0$, there are open intervals $(a_i, b_i)$ such that*

$$\mu(E \Delta \cup_{i=1}^{n} (a_i, b_i)) < \varepsilon.$$

HW Exercise: There exists an open set $U \subseteq (0,1)$ which is dense in $(0,1)$ and $\ell(U) < \varepsilon$. Simiarly, $K = [0,1] \setminus U$, where $K$ is closed, nowhere dense such that $\ell(K) > 1 - \varepsilon$.

How do arbitrary Lebesgue-Stieltjes measures look like? Well, here are some examples:

1) The Lebesgue measure itself

2) Discrete, weighted, atomic measures like $\mu = \sum_{i=1}^{N} a_i \delta_{x_i}$ where $a_i > 0$

3) Singular measures where $F$ is continuous but almost everywhere constant. For an example, the Cantor function, see below.

If $f$ is the Cantor function and $x$ is not in the Cantor set, it's pretty easy to see what $f(x)$ is. Now for the $x$ that do lie in the Cantor set, $f(x)$ can be found as the result of writing $x$ in ternary and then replacing every 2 with 1 and intrepreting this new string as binary. More explicity, if $x$ is in the Cantor set it's ternary expansion will look like

$$\sum_{i=1}^{\infty} b_i 3^{-i} \text{ where } b_i \in \{0,2\},$$

which means $f(x)$ will be

$$f(x) = \sum_{i=1}^{\infty} \frac{b_i}{2} 2^{-i}.$$

## 4.4 Lebesgue $\sigma$-algebra

The $\sigma$-algebra $\mathcal{L}$ of Lebesgue measurable sets is complete, it is a completion of the Borel $\sigma$-algebra. As such, it contains all Borel sets. The cardinality of the Lebesgue $\sigma$-algebra is hypercontinuum $2^{\mathfrak{c}}$, since there are uncountable Borel sets of Lebesgue measure zero (like the Cantor set), and each subset of them belongs to $\mathcal{L}$.

### 4.5 Summary of Part I

We have discussed measures and construction of measures, focusing on Lebesgue-Stieltjes measures. The latter construction works as follows:

1) Take an $F$ which is right-continuous

2) Use this $F$ to construct a pre-measure $\nu$

3) Use this $\nu$ to construct an outer measure $\mu$

4) Restrict this outer measure to form the measure we want (Carathéodory's theorem)

There is a number of statements we proved about general measures, and other facts were related to the particular Lebesgue-Stieltjes context.

# Part II
# Lebesgue integral

## 5  March 19, 2019