7310 Lecture notes. Spring 2019

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Typing credits

1. January 15 and 17, 2019 — Yichen Ma

Part I

Construction of measure

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A few completions of of $\mathbb{Q}$:
1) Cauchy sequences
2) Dedekind cuts
3) Axiomated total ordered complete fields
4) Decimal expanation up to equivalence class

Definition 1.1. Dedekind cut is a pair of non-empty sets (A, B) such that it satisfies $\mathbb{Q} = A \sqcup B$ and the following:
1. if $x \in A$, $y < x \implies y \in A$, i.e., A is order-closed downwards.
2. if $x \in B$, $y \geq x$, $\implies y \in B$, i.e., B is order-closed upwards.
3. A does not have a maximum element

Define $\mathbb{R} = \{\text{Dedekind cuts of } \mathbb{Q}\}$. For simplicity, we can use A to represent the pair (A, B). Note we can define an order on $\mathbb{R}$ by set inclusion with respect to A. We can also define addition. subtraction, multiplication, division on $\mathbb{R}$ such that $\mathbb{R}$ is totally ordered, complete field.

Theorem 1.2. $\mathbb{R}$ *is uncountable.*

Proof. Diagonal process. The complete proof is left as an exercise for the reader. $\square$

Question 1: What is the **length** ℓ of a subset of $\mathbb{R}$?
a) $\ell([0, 1]) = 1$;
b) $\ell(\{x\}) = 0$ for any $x \in \mathbb{R}$;
c) $\ell(\{x \in \mathbb{R} : x = 0.{*}{*}3{*}{*}{*}{*}{*}...\}) = \frac{1}{10}$. This is the set of decimal expansions such that there is a "3" in the third place after the dot.

Axioms of length on $\mathbb{R}$:

1. $\ell([0,1]) = 1$. i.e., normalized;
2. for $A \subseteq \mathbb{R}$ such that $A = A_1 \sqcup A_2 \sqcup ...$, $(A) = \sum_{i=1}^{\infty}(A_i)$, i.e., countable additivity;
3. for $A \subseteq \mathbb{R}$, $\alpha \in \mathbb{R}$, $\ell(A+\alpha) = \ell(A)$, i.e., transitivity-invariant.

Theorem 1.3. *Consider the length $\ell : 2^{\mathbb{R}} = P(\mathbb{R}) \longrightarrow [0,\infty]$, such length does not exist.*

Proof. We can write $[0,1) = \bigsqcup_{r\in N}(\mathbb{Q}+r)\cap[0,1)$, where N is the collection of representatives of each equivalence class with respect to the following equivalence relation: $x \sim y \iff x - y \in \mathbb{Q}$. Note that we need Axiom of Choice to define N. Then as length is transitivity-invariant, $\ell(N) = \ell(N+\alpha)\forall\ \alpha$. Hence $1 = \ell([0,1)) = \sum_{x\in N}\ell(N+x)$, but the right hand side is the sum of countably infinite elements with the same value, either equals 0 or infinity. Contradiction. $\square$

Corollary 1.4. *There exist non-measurable sets.*

Definition 1.5. Let X be a set. $\mu\colon 2^X \longrightarrow [0,\infty]$ is called an **outer measure** if

1. $\mu(\varnothing) = 0$;
2. if $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \bigcup_{k=1}^{\infty}\mu(A_k)$ (Hence $\mu(A) \leq \mu(B)$ if $A \subseteq B$).

Definition 1.6. Let X be a set, μ an outer measure defined on X. Then $A \subseteq X$ is called μ-measurable if for all $B \subseteq X$, $\mu(B) = \mu(B\cap A) + \mu(B \cap A^c)$

Definition 1.7. Let X be a set. $\mathcal{F} \subseteq X$ is called σ-algebra if

1. $\varnothing \in \mathcal{F}$
2. $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$
3. if $A_1, A_2, ... \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

We will eventually prove the following important theorem:

Theorem 1.8 (Caratheodory Extension Theorem)**.** *Let X be a set, μ an outer measure on X. Then the class of μ-measurable sets is a σ-algebra.*

Below are some examples of σ-algebras:

- X countable, $\mathcal{F} = 2^X$ is a σ-algebra;
- for any set X, $\mathcal{F} = \{\varnothing, X\}$ is called the trivial σ-algebra;
- for any set X, $A \subseteq X$, $\mathcal{F} = \{\varnothing, A, A^c, X\}$ is a σ-algebra (generated by the set A). That is, we define $\sigma(A) = \{\varnothing, A, A^c, X\} = \bigcap_{\{\mathcal{F}\in S\}}\mathcal{F}$, where $S = \{\mathcal{F}$ is a σ-algebra containing A$\}$.

Proposition 1.9. *Let X be a set. For any set Γ (not necessarily countable), if $\mathcal{F}_x$ is a σ-algebra on X for all $x \in \Gamma$, then $\bigcap_{x\in\Gamma}\mathcal{F}_x$ is also a σ-algebra.*

Proof. Exercise left for the reader. $\square$

Definition 1.10. The Borel σ-algebra of $\mathbb{R}^d$ (d ≥ 1), denoted as $B(\mathbb{R}^d)$, is defined as the σ-algebra generated by all open sets in $\mathbb{R}^d$.

The Borel σ-algebra has countable generating sets (e.g. rational rectangles).

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Definition 2.1. Let X be a set, $\mathcal{F}$ a σ-algebra on X. Then a **measure** on $\mathcal{F}$ is a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ such that

a) $\mu(\varnothing) = 0$;

b) for $A_1, A_2, ... \in \mathcal{F}$ such that $A_i \cap A_j = \varnothing$ for $i \neq j$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, i.e., μ is countably additive.

We call $(X, \mathcal{F}, \mu)$ a **measure space**.

Below are some examples of measure spaces. Let X be a set.

- Fix $x \in X$, then for $A \subseteq X$ consider δ: $X \longrightarrow [0, \infty]$ by

$$\delta(A) = \begin{cases} 0 & if\ x \notin A \\ 1 & if\ x \in A \end{cases}$$

  This is called the δ-measure at $x \in X$.

- Let $\mathcal{F} = 2^X$, for $A \in X$, define μ such that $\mu(A) = 0$ if A is countable, and $\mu(A) = 1$ if A is co-countable (i.e., A^c is countable). Then $(X, \mathcal{F}, \mu)$ is a measure space.

- For X countable, i.e., $X = \{x_1, x_2, ...\}$. Let $\mathcal{F} = 2^X$. Denote $p(x_i) =$ weight of x_i, then for $A \subseteq X$, define $\mu(A) = \sum_{x \in A} p(x)$. $(X, \mathcal{F}, \mu)$ is a measure space.

Properties of a measure space (X, F, μ)

1. for $A, B \subseteq X$, $A \subseteq B \Longrightarrow \mu(A) \leq \mu(B)$;
2. (countably subadditive) if $A \subseteq \bigcup_{i=1}^{\infty} A_i \Longrightarrow \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$;
3. (continuity) for a sequence of subsets in X, $A_1 \subseteq A_2 \subseteq ...$,

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i);$$

   for a sequence of subsets in X, $A_1 \supseteq A_2 \supseteq ...$ such that $\mu(A_i) < \infty$ $\forall i$,

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i).$$

A counterexample for 3) if we drop the condition "$\mu(A_i) < \infty\ \forall i$" is $X = \mathbb{R}$, $F = \sigma(\mathbb{R})$, $A_i = (i, \infty)$ for $i \in \mathbb{N}$, then each $\mu(A_i) = \infty$ but $lim_{i\to\infty}\mu(A_i) = 0$.

Definition 2.2. Let (X, F, μ) be a measure space. μ is called **finite** if $\mu(X) < \infty$, μ is called **σ-finite** if $X = \bigcup E_n$ where $\mu(E_n) < \infty\ \forall n$.

Semicontinuity. Let $\{A_n : n \in \mathbb{N}\}$ be a collection of sets. Then define

$$\liminf A_n = \{\text{elements in almost all } A_n\} = \bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_k),$$

$$\limsup A_n = \{\text{elements in } A_n \text{ infinitely often}\} = \bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k).$$

Proposition 2.3. *We have*

$$\mu(\limsup A_n) \geq \limsup \mu(A_n), \qquad \mu(\liminf A_n) \leq \liminf \mu(A_n).$$

Proof. Homework exercise. □

Definition 2.4. Consider $(X, \mathcal{F}, \mu)$, $B \in \mathcal{F}$ is called a **null set** if $\mu(B) = 0$. Note if B is a null set, $A \subseteq B$, then $\mu(A) = 0$ in the sense of outer measure (not necessarily $A \in \mathcal{F}$).

Definition 2.5. $(X, \mathcal{F})$ is **complete** if all null sets are contained in $\mathcal{F}$.

Here we introduce a notation "**a.e**", the abbreviation of "almost everywhere". Something happens a.e. if it happens outside a null set. Here are a few examples:

Consider the Dirichlet function $f(x) = \begin{cases} 1 & if\ x \in \mathbb{Q} \\ 0 & if\ x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$. Then $f(x) = 0$ a.e.

Consider the (small) Riemann function $f(x) = \begin{cases} \frac{1}{q} & if\ x = \frac{p}{q} \in \mathbb{Q} \\ 0 & if\ x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$. Then f is continuous a.e.

Theorem 2.6. *There is an unique completion of any measure space $(X, \mathcal{F}, \mu)$. i.e., there exist $\overline{\mathcal{F}}, \overline{\mu}$ with $\overline{\mathcal{F}} \supseteq \mathcal{F}$ and $\overline{\mu}|_{\mathcal{F}} = \mu$. Specifically,*

$$\overline{\mathcal{F}} = \{A \cup B |\ A \in \mathcal{F}, B \subseteq N \text{ for some } N \in \mathcal{F}, \mu(N) = 0\},$$

which satisfies:

1) $\overline{\mathcal{F}}$ is a σ-algebra;

2) there exists unique extension $\overline{\mu}$ of μ, from $\mathcal{F}$ to $\overline{\mathcal{F}}$ such that $\overline{\mu}|_{\mathcal{F}} = \mu$.

Proof. Let $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$. Then it is closed under countable unions and complement, hence so is a$\overline{\mathcal{F}}$. Then set $\overline{\mu}(A \cup B) = \mu(A)$ if $B \subseteq N \in \mathcal{N}$. Then it is left as an exercise to the reader to check:

a) $\overline{\mu}$ is well-defined (i.e., if there is another way of writing $A \cup B = A' \cup B'$, $B' \subseteq N' \in \mathcal{N}$, then the measure does not change);

b) $\overline{\mu}$ is a measure;

c) $\overline{\mu}$ is a unique extension to $\overline{\mathcal{F}}$ defined in the claim. □

Note that the Borel σ-algebra is not complete with respect to the length measure ℓ.

Let X be a set. Recall the definition of the outer measure μ on 2^X.

Definition 2.7. $A \subseteq X$ is a μ-measurable set with respect to μ if $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c)\ \forall B \in 2^X$.

We now begin the proof of the Caratheodory's extension theorem:

Theorem 2.8 (Caratheodory)**.** *Let μ be an outer measure on a set X, then the following statements are true:*

1) $\mathcal{F} = \{A \colon A \text{ is a } \mu\text{-measurable set}\}$ *is a σ-algebra;*

2) $\mu|_{\mathcal{F}}$ *is a complete measure.*

Proof. For 1), first of all, $\mathcal{F}$ is closed under complement by the definition of the outer measure and the measurable sets. Then for $A, B \in \mathcal{F}$, take any $S \subseteq X$, then

$$\begin{aligned}\mu(S) &= \mu(S \cap A) + \mu(S \cap A^c)\\ &= \mu(S \cap A \cap B) + \mu(S \cap A \cap B^c) + \mu(S \cap A^c \cap B) + \mu(S \cap A^c \cap B^c).\end{aligned}$$

Note $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, then by subadditivity,

$$\mu(S \cap (A \cap B)) + \mu(S \cap (A \cap B^c)) + \mu(S \cap (A^c \cap B)) \geq \mu(S \cap (A \cup B)).$$

Hence $\mu(S) \geq \mu(E \cap (A \cup B)) + \mu(E \cap (A \cup B)^c)$. Hence $A \cup B \in \mathcal{F}$, so $\mathcal{F}$ is an algebra.

Also, if $A, B \in \mathcal{F}$ with $A \cap B = \varnothing$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \cap A^c) = \mu(A) + \mu(B).$$

This shows finite additivity of μ on $\mathcal{F}$.

(other parts will be proven next time.) □

We will also work with outer measures constructed using pre-measures:

Definition 2.9. Let X be a set, $\Gamma \subseteq 2^X$, $\nu : \Gamma \longrightarrow [0, \infty]$ be any function. Then the **pre-measure** μ on 2^X is defined as for $F \in 2^X$,

$$\mu(F) = \inf\left\{\sum_{n=1}^{\infty} \nu(A_n) : A_n \in \Gamma, F \subseteq \bigcup_{n=1}^{\infty} A_n\right\}.$$

By agreement, $\inf \varnothing = 0$.

Proposition 2.10. *With the same set up as above, μ is an outer measure.*

Proof. This will also be proven next time. □