

# 7310 Lecture notes. Spring 2019

Instructor: Leonid Petrov

## Typing credits

1. January 15 and 17, 2019 — Yichen Ma
2. January 22 and 24, 2019 — Bennett Rennier

## Contents

<b>I</b>	<b>Construction of measure</b>	<b>2</b>
<b>1</b>	<b>January 15, 2019</b>	<b>2</b>
1.1	Basics . . . . .	2
1.2	Problem of measure . . . . .	2
1.3	Outer measure . . . . .	3
1.4	$\sigma$ -algebras . . . . .	3
<b>2</b>	<b>January 17, 2019</b>	<b>4</b>
2.1	Measures . . . . .	4
2.2	Completeness . . . . .	5
2.3	Caratheodory's theorem. Formulation and first part of proof . .	6
2.4	Definition of a pre-measure . . . . .	6
<b>3</b>	<b>January 22, 2019</b>	<b>7</b>
3.1	From outer measure to measure . . . . .	7
3.2	From pre-measure to outer measure . . . . .	8
3.3	Examples . . . . .	9
3.4	Lebesgue-Stieltjes measures . . . . .	9
<b>4</b>	<b>January 24, 2019</b>	<b>10</b>
4.1	Lebesgue-Stieltjes measures continued . . . . .	10
4.2	From countably additive pre-measure to measure . . . . .	11
4.3	Properties of the Lebesgue and Lebesgue-Stieltjes measures . . .	12
4.4	Lebesgue $\sigma$ -algebra . . . . .	13
4.5	Summary of Part I . . . . .	14

## Part I

## Construction of measure

1 January 15, 2019

## 1.1 Basics

A few completions of  $\mathbb{Q}$ :

- 1) Cauchy sequences
- 2) Dedekind cuts
- 3) Axiomated total ordered complete fields
- 4) Decimal expansion up to equivalence class

**Definition 1.1.** Dedekind cut is a pair of non-empty sets  $(A, B)$  such that it satisfies  $\mathbb{Q} = A \sqcup B$  and the following:

1. if  $x \in A, y < x \implies y \in A$ , i.e.,  $A$  is order-closed downwards.
2. if  $x \in B, y \geq x, \implies y \in B$ , i.e.,  $B$  is order-closed upwards.
3.  $A$  does not have a maximum element

Define  $\mathbb{R} = \{\text{Dedekind cuts of } \mathbb{Q}\}$ . For simplicity, we can use  $A$  to represent the pair  $(A, B)$ . Note we can define an order on  $\mathbb{R}$  by set inclusion with respect to  $A$ . We can also define addition, subtraction, multiplication, division on  $\mathbb{R}$  such that  $\mathbb{R}$  is totally ordered, complete field.

**Theorem 1.2.**  $\mathbb{R}$  is uncountable.

*Proof.* Diagonal process. The complete proof is left as an exercise for the reader.  $\square$

## 1.2 Problem of measure

Question 1: What is the **length**  $\ell$  of a subset of  $\mathbb{R}$ ?

- a)  $\ell([0, 1]) = 1$ ;
- b)  $\ell(\{x\}) = 0$  for any  $x \in \mathbb{R}$ ;
- c)  $\ell(\{x \in \mathbb{R} : x = 0.**3*****\}) = \frac{1}{10}$ . This is the set of decimal expansions such that there is a “3” in the third place after the dot.

**Axioms of length on  $\mathbb{R}$ :**

1.  $\ell([0, 1]) = 1$ . i.e., normalized;
2. for  $A \subseteq \mathbb{R}$  such that  $A = A_1 \sqcup A_2 \sqcup \dots$ ,  $\ell(A) = \sum_{i=1}^{\infty} \ell(A_i)$ , i.e., countable additivity;
3. for  $A \subseteq \mathbb{R}$ ,  $\alpha \in \mathbb{R}$ ,  $\ell(A + \alpha) = \ell(A)$ , i.e., transitivity-invariant.

**Theorem 1.3.** Consider the length  $\ell : 2^{\mathbb{R}} = P(\mathbb{R}) \longrightarrow [0, \infty]$ , such length does not exist.

*Proof.* We can write  $[0, 1) = \bigsqcup_{r \in N} (\mathbb{Q} + r) \cap [0, 1)$ , where  $N$  is the collection of representatives of each equivalence class with respect to the following equivalence relation:  $x \sim y \iff x - y \in \mathbb{Q}$ . Note that we need Axiom of Choice to define  $N$ . Then as length is transitivity-invariant,  $\ell(N) = \ell(N + \alpha) \forall \alpha$ . Hence  $1 = \ell([0, 1)) = \sum_{x \in N} \ell(N + x)$ , but the right hand side is the sum of countably infinite elements with the same value, either equals 0 or infinity. Contradiction.  $\square$

**Corollary 1.4.** There exist non-measurable sets.

### 1.3 Outer measure

**Definition 1.5.** Let  $X$  be a set.  $\mu : 2^X \longrightarrow [0, \infty]$  is called an **outer measure** if

1.  $\mu(\emptyset) = 0$ ;
2. if  $A \subseteq \bigcup_{k=1}^{\infty} A_k$ , then  $\mu(A) \leq \bigcup_{k=1}^{\infty} \mu(A_k)$  (Hence  $\mu(A) \leq \mu(B)$  if  $A \subseteq B$ ).

**Definition 1.6.** Let  $X$  be a set,  $\mu$  an outer measure defined on  $X$ . Then  $A \subseteq X$  is called  $\mu$ -measurable if for all  $B \subseteq X$ ,  $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$

### 1.4 $\sigma$ -algebras

**Definition 1.7.** Let  $X$  be a set.  $\mathcal{F} \subseteq 2^X$  is called  $\sigma$ -algebra if

1.  $\emptyset \in \mathcal{F}$
2.  $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. if  $A_1, A_2, \dots \in \mathcal{F}$ , then  $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

We will eventually prove the following important theorem:

**Theorem 1.8** (Caratheodory Extension Theorem). Let  $X$  be a set,  $\mu$  an outer measure on  $X$ . Then the class of  $\mu$ -measurable sets is a  $\sigma$ -algebra.

Below are some examples of  $\sigma$ -algebras:

- $X$  countable,  $\mathcal{F} = 2^X$  is a  $\sigma$ -algebra;
- for any set  $X$ ,  $\mathcal{F} = \{\emptyset, X\}$  is called the trivial  $\sigma$ -algebra;
- for any set  $X$ ,  $A \subseteq X$ ,  $\mathcal{F} = \{\emptyset, A, A^c, X\}$  is a  $\sigma$ -algebra (generated by the set  $A$ ). That is, we define  $\sigma(A) = \{\emptyset, A, A^c, X\} = \bigcap_{\{\mathcal{F} \in \mathcal{S}\}} \mathcal{F}$ , where  $\mathcal{S} = \{\mathcal{F} \text{ is a } \sigma\text{-algebra containing } A\}$ .

**Proposition 1.9.** Let  $X$  be a set. For any set  $\Gamma$  (not necessarily countable), if  $\mathcal{F}_x$  is a  $\sigma$ -algebra on  $X$  for all  $x \in \Gamma$ , then  $\bigcap_{x \in \Gamma} \mathcal{F}_x$  is also a  $\sigma$ -algebra.

*Proof.* Exercise left for the reader.  $\square$

**Definition 1.10.** The Borel  $\sigma$ -algebra of  $\mathbb{R}^d$  ( $d \geq 1$ ), denoted as  $B(\mathbb{R}^d)$ , is defined as the  $\sigma$ -algebra generated by all open sets in  $\mathbb{R}^d$ .

The Borel  $\sigma$ -algebra has countable generating sets (e.g. rational rectangles).

## 2 January 17, 2019

### 2.1 Measures

**Definition 2.1.** Let  $X$  be a set,  $\mathcal{F}$  a  $\sigma$ -algebra on  $X$ . Then a **measure** on  $\mathcal{F}$  is a function  $\mu : \mathcal{F} \rightarrow [0, \infty]$  such that

- a)  $\mu(\emptyset) = 0$ ;
- b) for  $A_1, A_2, \dots \in \mathcal{F}$  such that  $A_i \cap A_j = \emptyset$  for  $i \neq j$ ,  $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ , i.e.,  $\mu$  is countably additive.

We call  $(X, \mathcal{F}, \mu)$  a **measure space**.

Below are some examples of measure spaces. Let  $X$  be a set.

- Fix  $x \in X$ , then for  $A \subseteq X$  consider  $\delta : X \rightarrow [0, \infty]$  by

$$\delta(A) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

This is called the  $\delta$ -measure at  $x \in X$ .

- Let  $\mathcal{F} = 2^X$ , for  $A \in \mathcal{F}$ , define  $\mu$  such that  $\mu(A) = 0$  if  $A$  is countable, and  $\mu(A) = 1$  if  $A$  is co-countable (i.e.,  $A^c$  is countable). Then  $(X, \mathcal{F}, \mu)$  is a measure space.
- For  $X$  countable, i.e.,  $X = \{x_1, x_2, \dots\}$ . Let  $\mathcal{F} = 2^X$ . Denote  $p(x_i) =$  weight of  $x_i$ , then for  $A \subseteq X$ , define  $\mu(A) = \sum_{x \in A} p(x)$ .  $(X, \mathcal{F}, \mu)$  is a measure space.

#### Properties of a measure space $(X, \mathcal{F}, \mu)$

1. for  $A, B \subseteq X$ ,  $A \subseteq B \implies \mu(A) \leq \mu(B)$ ;
2. (countably subadditive) if  $A \subseteq \bigcup_{i=1}^{\infty} A_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ ;
3. (continuity) for a sequence of subsets in  $X$ ,  $A_1 \subseteq A_2 \subseteq \dots$ ,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i);$$

for a sequence of subsets in  $X$ ,  $A_1 \supseteq A_2 \supseteq \dots$  such that  $\mu(A_i) < \infty \forall i$ ,

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

A counterexample for 3) if we drop the condition “ $\mu(A_i) < \infty \forall i$ ” is  $X = \mathbb{R}$ ,  $F = \sigma(\mathbb{R})$ ,  $A_i = (i, \infty)$  for  $i \in \mathbb{N}$ , then each  $\mu(A_i) = \infty$  but  $\lim_{i \rightarrow \infty} \mu(A_i) = 0$ .

**Definition 2.2.** Let  $(X, F, \mu)$  be a measure space.  $\mu$  is called **finite** if  $\mu(X) < \infty$ ,  $\mu$  is called  **$\sigma$ -finite** if  $X = \bigcup E_n$  where  $\mu(E_n) < \infty \forall n$ .

**Semicontinuity.** Let  $\{A_n : n \in \mathbb{N}\}$  be a collection of sets. Then define

$$\begin{aligned} \liminf A_n &= \{\text{elements in almost all } A_n\} = \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} A_k \right), \\ \limsup A_n &= \{\text{elements in } A_n \text{ infinitely often}\} = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} A_k \right). \end{aligned}$$

**Proposition 2.3.** We have

$$\mu(\limsup A_n) \geq \limsup \mu(A_n), \quad \mu(\liminf A_n) \leq \liminf \mu(A_n).$$

*Proof.* Homework exercise. □

## 2.2 Completeness

**Definition 2.4.** Consider  $(X, \mathcal{F}, \mu)$ ,  $B \in \mathcal{F}$  is called a **null set** if  $\mu(B) = 0$ . Note if  $B$  is a null set,  $A \subseteq B$ , then  $\mu(A) = 0$  in the sense of outer measure (not necessarily  $A \in \mathcal{F}$ ).

**Definition 2.5.**  $(X, \mathcal{F})$  is **complete** if all null sets are contained in  $\mathcal{F}$ .

Here we introduce a notation “**a.e.**”, the abbreviation of “almost everywhere”. Something happens a.e. if it happens outside a null set. Here are a few examples:

Consider the Dirichlet function  $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ . Then  $f(x) = 0$  a.e.

Consider the (small) Riemann function  $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ . Then  $f$  is continuous a.e.

**Theorem 2.6.** There is an unique completion of any measure space  $(X, \mathcal{F}, \mu)$ . i.e., there exist  $\overline{\mathcal{F}}, \overline{\mu}$  with  $\overline{\mathcal{F}} \supseteq \mathcal{F}$  and  $\overline{\mu}|_{\mathcal{F}} = \mu$ . Specifically,

$$\overline{\mathcal{F}} = \{A \cup B \mid A \in \mathcal{F}, B \subseteq N \text{ for some } N \in \mathcal{F}, \mu(N) = 0\},$$

which satisfies:

- 1)  $\overline{\mathcal{F}}$  is a  $\sigma$ -algebra;
- 2) there exists unique extension  $\overline{\mu}$  of  $\mu$ , from  $\mathcal{F}$  to  $\overline{\mathcal{F}}$  such that  $\overline{\mu}|_{\mathcal{F}} = \mu$ .

*Proof.* Let  $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$ . Then it is closed under countable unions and complement, hence so is  $\overline{\mathcal{F}}$ . Then set  $\overline{\mu}(A \cup B) = \mu(A)$  if  $B \subseteq N \in \mathcal{N}$ . Then it is left as an exercise to the reader to check:

- a)  $\bar{\mu}$  is well-defined (i.e., if there is another way of writing  $A \cup B = A' \cup B'$ ,  $B' \subseteq N' \in \mathcal{N}$ , then the measure does not change);
- b)  $\bar{\mu}$  is a measure;
- c)  $\bar{\mu}$  is a unique extension to  $\bar{\mathcal{F}}$  defined in the claim. □

Note that the Borel  $\sigma$ -algebra is not complete with respect to the length measure  $\ell$ .

### 2.3 Caratheodory's theorem. Formulation and first part of proof

Let  $X$  be a set. Recall the definition of the outer measure  $\mu$  on  $2^X$ .

**Definition 2.7.**  $A \subseteq X$  is a  $\mu$ -measurable set with respect to  $\mu$  if  $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c) \forall B \in 2^X$ .

We now begin the proof of the Caratheodory's extension theorem:

**Theorem 2.8** (Caratheodory). *Let  $\mu$  be an outer measure on a set  $X$ , then the following statements are true:*

- 1)  $\mathcal{F} = \{A: A \text{ is a } \mu\text{-measurable set}\}$  is a  $\sigma$ -algebra;
- 2)  $\mu|_{\mathcal{F}}$  is a complete measure.

*Proof.* For 1), first of all,  $\mathcal{F}$  is closed under complement by the definition of the outer measure and the measurable sets. Then for  $A, B \in \mathcal{F}$ , take any  $S \subseteq X$ , then

$$\begin{aligned} \mu(S) &= \mu(S \cap A) + \mu(S \cap A^c) \\ &= \mu(S \cap A \cap B) + \mu(S \cap A \cap B^c) + \mu(S \cap A^c \cap B) + \mu(S \cap A^c \cap B^c). \end{aligned}$$

Note  $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$ , then by subadditivity,

$$\mu(S \cap (A \cap B)) + \mu(S \cap (A \cap B^c)) + \mu(S \cap (A^c \cap B)) \geq \mu(S \cap (A \cup B)).$$

Hence  $\mu(S) \geq \mu(E \cap (A \cup B)) + \mu(E \cap (A \cup B)^c)$ . Hence  $A \cup B \in \mathcal{F}$ , so  $\mathcal{F}$  is an algebra.

Also, if  $A, B \in \mathcal{F}$  with  $A \cap B = \emptyset$ ,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \cap A^c) = \mu(A) + \mu(B).$$

This shows finite additivity of  $\mu$  on  $\mathcal{F}$ .

(other parts will be proven next time.) □

### 2.4 Definition of a pre-measure

We will also work with outer measures constructed using pre-measures:

**Definition 2.9.** Let  $X$  be a set,  $\Gamma \subseteq 2^X$ ,  $\nu : \Gamma \rightarrow [0, \infty]$  be any function. Then the **pre-measure**  $\mu$  on  $2^X$  is defined as for  $F \in 2^X$ ,

$$\mu(F) = \inf \left\{ \sum_{n=1}^{\infty} \nu(A_n) : A_n \in \Gamma, F \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

By agreement,  $\inf \emptyset = 0$ .

**Proposition 2.10.** *With the same set up as above,  $\mu$  is an outer measure.*

*Proof.* This will also be proven next time.  $\square$

### 3 January 22, 2019

#### 3.1 From outer measure to measure

We are on a path of constructing the measure. Last time, we defined pre-measures, and this definition will be used later. Here we show how an outer measure leads to a measure, i.e., finish the proof of the Caratheodory theorem.

Recall that if  $\mu$  is an outer measure, we say that a set  $A$  is  $\mu$ -measurable if for all  $E \subseteq X$ ,

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

Note that in the proofs we need only to check  $\geq$  since  $\leq$  follows directly from subadditivity. Denote by  $\mathcal{F}$  the set of all  $\mu$ -measurable subsets.

*Rest of the proof of Caratheodory theorem (Theorem 2.8).* We have shown in Lecture 2 that  $\mathcal{F}$  is an algebra (closed under finite unions, intersections, and complements) and that  $\mu$  is finitely additive on  $\mathcal{F}$ .

Now we need to show that  $\mathcal{F}$  is a  $\sigma$ -algebra. Let  $A_1, A_2, \dots \in \mathcal{F}$  be disjoint, and define  $B_n = \bigsqcup_{j=1}^n A_j$ . We know that  $B_n \in \mathcal{F}$ , so we need to show that  $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$ .

First, for all  $E \subseteq X$ , we have

$$\mu(E \cap B_n) = \mu(E \cap B_n \cap A_n) + \mu(E \cap B_n \cap A_n^c) = \mu(E \cap A_n) + \mu(E \cap B_{n-1}).$$

Continuing this, we see that

$$\mu(E \cap B_n) = \sum_{i=1}^n \mu(E \cap A_i).$$

Thus,

$$\mu(E) = \mu(E \cap B_n) + \mu(E \cap B_n^c) \geq \sum_{i=1}^n \mu(E \cap A_i) + \mu(E \cap B^c).$$

The last inequality is due to the fact that in the second summand we have passed to the smaller set  $E \cap B$ . In the previous inequality we can now pass to  $n \rightarrow \infty$ , and all the sums in the right-hand side are bounded:

$$\mu(E) \geq \sum_{i=1}^{\infty} \mu(E \cap A_i) + \mu(E \cap B^c).$$

By subadditivity, this is

$$\geq \mu\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu(E \cap B^c) = \mu(E \cap B) + \mu(E \cap B^c).$$

This shows that  $B \in \mathcal{F}$ , so  $\mathcal{F}$  is a  $\sigma$ -algebra.

The  $\sigma$ -additivity of  $\mu$  on  $\mathcal{F}$  is straightforward, take  $B = E$  in the previous argument, which leads to  $\mu(B) = \sum_{i=1}^{\infty} \mu(A_i)$ .

Finally, to show completeness, let  $A \subseteq X$  be with the outer measure  $\mu(A) = 0$ . We want to show that  $A \in \mathcal{F}$ . We have for all  $E \subseteq X$ :

$$\mu(E) \leq \mu(E \cap A) + \mu(E \cap A^c) = \mu(E \cap A^c) \leq \mu(E)$$

because the outer measure is subadditive and so  $\mu(E \cap A) = 0$ . The last inequality is also due to subadditivity. This implies that  $\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$ , so  $E \in \mathcal{F}$ .  $\square$

### 3.2 From pre-measure to outer measure

Recall the definition of a pre-measure:

**Definition 3.1** (Pre-measure). Let  $X$  be a set and let  $\mathcal{E}$  be a set of subsets of  $X$ . Then any function  $\nu : \mathcal{E} \rightarrow [0, \infty]$  is called a *pre-measure*.

**Proposition 3.2.** If  $\nu$  is a pre-measure, then  $\mu$  defined by

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \nu(A_i) : A_i \in \mathcal{E}, E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$$

is an outer measure on  $\mathcal{E}$ .

*Proof.* We need only prove the subadditivity of  $\mu$ . Let  $F \subseteq \bigcup_{i=1}^{\infty} F_i$  and let  $\varepsilon > 0$ . By construction of  $\mu$ , there are sets  $A_i^n$  such that  $F_i \subseteq \bigcup_{n=1}^{\infty} A_i^n$  and

$$\mu(F_i) + \frac{\varepsilon}{2^i} \geq \sum_{n=1}^{\infty} \nu(A_i^n).$$

We also note that

$$F \subseteq \bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_i^n.$$



This means

$$\mu(F) \leq \mu\left(\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_i^n\right) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A_i^n) \leq \sum_{i=1}^{\infty} \left(\mu(F_i) + \frac{\varepsilon}{2^i}\right).$$

As we can make  $\varepsilon$  as small as we'd like, this implies that  $\mu(F) \leq \sum_{i=1}^{\infty} \mu(F_i)$  as we wanted.  $\square$

### 3.3 Examples

Now, pre-measures are an easy way of constructing an outer measure, but as we will see, the resulting outer measure may not agree with our original pre-measure. For example, let  $\mathcal{E} = \{\emptyset, A, A^c, X\}$  where  $\nu(\emptyset) = 0$ ,  $\nu(A) = a$ ,  $\nu(A^c) = b$ , and  $\nu(X) = 1$ . Then for the corresponding outer measure  $\mu$  we have that  $\mu(\emptyset) = 0$  and  $\mu(X) = \min(a+b, 1)$ . Furthermore, if  $B \subseteq A$ , then  $\mu(B) = a$  and if  $B \subseteq A^c$ , then  $\mu(B) = b$ . However, is  $A$  even  $\mu$ -measurable? We see that  $A$  is  $\mu$ -measurable only if

$$\min(a+b, 1) = \mu(X) = \mu(E \cap A) + \mu(E \cap A^c) = \mu(A) + \mu(A^c) = a + b.$$

In other words, if  $a+b > 1$ , then  $A$  will not be one of our  $\mu$ -measurable sets. This should serve as a warning that the sets in  $\mathcal{E}$  may not be  $\mu$ -measurable.

For a second example, let  $\mathcal{E}$  be the set of finite unions of sets of the form  $(a, b] \cap \mathbb{Q}$ . We let  $\nu(\emptyset) = 0$  and  $\nu(A) = \infty$  for  $\emptyset \neq A \in \mathcal{E}$ . We note that in this case, there is more than one possible  $\mu$  which extends  $\nu$ . For example, we can assign all points measure zero, or measure 1, or measure 2, or measure  $\infty$ .

### 3.4 Lebesgue-Stieltjes measures

The *Borel  $\sigma$ -algebra* is the  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  generated by all open subsets of  $\mathbb{R}$ . A measure  $\mu$  on  $\mathcal{B}(\mathbb{R})$  is called a *Borel measure*. (Sometimes we will assume that it's  $\sigma$ -finite; moreover, that bounded sets have finite measure.)

**Remark 3.3.** The cardinality of the Borel  $\sigma$ -algebra is continuum. This can be proven using transfinite induction.

Assuming that  $\mu$  is a finite Borel measure, we can define the function  $F(x) = \mu((-\infty, x])$ . We see that  $F$  is weakly increasing and is right-continuous.<sup>1</sup> On the other hand, having a right-continuous and weakly increasing function, we can define a pre-measure  $\nu$  for which  $\nu((a, b]) = F(b) - F(a)$ . Measures constructed from such pre-measures are called *Lebesgue-Stieltjes measures*. In particular, if we take  $F(x) = x$  as our pre-measure function, then the resulting measure is called the Lebesgue measure (note that it is not finite). In the next lecture we discuss how this works exactly.

<sup>1</sup>It is right-continuous because  $\mu((-\infty, a]) = \mu(\bigcap_{n=1}^{\infty} (-\infty, a_n]) = \lim_{n \rightarrow +\infty} \mu((-\infty, a_n])$  by the continuity of measure, where  $a_n$  decrease to  $a$ .

## 4 January 24, 2019

### 4.1 Lebesgue-Stieltjes measures continued

The goal now is to finish the construction of measure from pre-measures, and apply this to Lebesgue-Stieltjes measures. There are two main pieces of the construction that are still missing:

1. If the pre-measure is countably additive on  $\mathcal{E}$ , then the resulting measure (coming from the corresponding outer measure by the Caratheodory theorem) agrees with the pre-measure on  $\mathcal{E}$ , and the sets from  $\mathcal{E}$  are measurable.
2. The pre-measure constructed from a function  $F$  is countably additive.

The first part is an abstract statement; and the second part is a concrete result that applies on  $\mathbb{R}$  and in the Lebesgue-Stieltjes context. We start with the second one.

Let  $\mathcal{A}$  be the set of finite disjoint unions of intervals of the form  $(a, b]$ , where  $-\infty \leq a \leq b \leq \infty$ . This is an algebra (i.e. is closed under finite intersections, finite unions, and complements). Let  $F$  be a weakly increasing right-continuous function. We construct a pre-measure  $\nu$  on  $\mathcal{A}$  where for disjoint unions,

$$\nu\left(\bigsqcup_{i=1}^n (a_i, b_i]\right) = \sum_{i=1}^n (F(b_i) - F(a_i))$$

and we'll use this pre-measure to construct our measure  $\mu$ . There are some potential issues in this definition (that  $\nu$  is well-defined) that are resolved in the next proposition:

**Proposition 4.1.**    •  $\nu$  is well-defined;

- $\nu$  is finitely additive;
- $\nu$  is countably additive.

*Proof.* Let  $A = \cup I_i = \cup J_j$  where  $I_i, J_j$  are intervals. Then we see that

$$\nu(A) = \sum_i \sum_j \nu(I_i \cap J_j)$$

the latter of which is equal to both  $\sum_i \nu(I_i)$  and  $\sum_j \nu(J_j)$ , proving that  $\nu$  is well-defined. The fact that  $\nu$  is finitely additive on  $\mathcal{A}$  is evident from the definition.

We will now prove that  $\nu$  is countably additive on  $\mathcal{A}$ . Let  $I_i \in \mathcal{A}$  and  $I = \cup_{i=1}^{\infty} I_i$ . We can assume without loss of generality that  $I$  is an interval (we can apply the rest of our reasoning to the individual components of  $I$ ). We will

further assume that  $I$  has finite measure (the infinite measure case is left as an exercise). Now what we want to show is that  $\nu(\cup_{i=1}^{\infty} I_i) = \sum_{i=1}^{\infty} \nu(I_i)$ . We have

$$\nu(I) = \nu(\cup_{i=1}^n I_i) + \nu(I \setminus \cup_{i=1}^n I_i) \geq \sum_{i=1}^n \nu(I_i),$$

and then we can take the limit as  $n \rightarrow \infty$  to see that  $\nu(\cup I_i) \geq \sum \nu(I_i)$ .

It remains to establish the reverse inequality. Now, let

$$I = (a, b] = \cup_{j=1}^{\infty} (a_j, b_j].$$

For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $F(a + \delta) - F(a) < \varepsilon$  and there exists  $\delta_j$ 's such that  $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j}$ . Here we used the right continuity of  $F$ .

Now since  $[a + \delta, b]$  is compact and covered by the open sets  $(a_i, b_i + \delta_i)$ , there is a finite number of  $(a_i, b_i + \delta_i)$  covers  $[a + \delta, b]$  for  $i \in \{1, \dots, N\}$ . This means

$$\nu(I) \leq 2\varepsilon + \sum_{i=1}^{\infty} (F(b_i) - F(a_i)),$$

and since  $\varepsilon$  was arbitrarily small, we are done.  $\square$

## 4.2 From countably additive pre-measure to measure

Let us now focus on properties of measures which come from a countably additive pre-measure. This will complete the construction of the measure.

If  $\mu$  is a measure constructed by the pre-measure  $\nu$ , then  $\mu_{\mathcal{A}} = \nu_{\mathcal{A}}$ .

*Proof.* Take an  $E \in \mathcal{A}$  and cover it with sets  $I_i \in \mathcal{A}$ , i.e.  $E \subseteq \cup_{j=1}^{\infty} I_j$ . Now let  $B_n = E \cap (I_n \setminus \cup_{j=1}^{n-1} I_j)$  which disjointly cover  $E$ . We then get

$$\nu(E) = \sum_{i=1}^{\infty} \nu(B_i) \leq \sum_{i=1}^{\infty} \nu(I_i).$$

If we take the infimum over any such sets  $I_j$ , we have that  $\nu(E) \leq \mu(E)$ .

Since  $E \subseteq E$ , we already have that  $\mu(E) \leq \nu(E)$ . Thus,  $\nu_{\mathcal{A}} = \mu_{\mathcal{A}}$ .  $\square$

Let  $\mathcal{A}$  be the set of finite unions of half-open intervals. Then for any  $\mu$ , all the sets in  $\mathcal{A}$  are  $\mu$ -measurable.

*Proof.* Let  $B \in \mathcal{A}$  and  $E \subseteq X$ . For any  $\varepsilon > 0$ , we can find  $A_i \in \mathcal{A}$  such that  $E \subseteq \cup_{i=1}^{\infty} A_i$  and  $\mu(E) + \varepsilon \geq \sum_{i=1}^{\infty} \nu(A_i)$ . We then have that

$$\begin{aligned} \mu(E) + \varepsilon &\geq \sum_{i=1}^{\infty} \nu(A_i) \\ &= \sum_{i=1}^{\infty} (\nu(A_i \cap B) + \nu(A_i \cap B^c)) \\ &\geq \mu(E \cap B) + \mu(E \cap B^c), \end{aligned}$$

which proves that  $B$  is  $\mu$ -measurable.  $\square$

Let  $\nu$  be a  $\sigma$ -finite pre-measure and  $\mu$  is the induced measure. If  $\rho$  is extension of  $\nu$  (i.e.  $\rho_{\mathcal{A}} = \nu$ ), then  $\mu = \rho$ .

*Proof.* Let  $E \in \mathcal{F}_\mu$ ,  $E \subseteq \cup_{i=1}^\infty A_i$ ,  $A_i \in \mathcal{A}$ . Then

$$\rho(E) \leq \sum_{i=1}^\infty \rho(A_i) = \sum_{i=1}^\infty \nu(A_i)$$

so  $\rho(E) \leq \mu(E)$ .

Since  $\rho$  is  $\sigma$ -finite, we need only to prove that they coincide on finitely-measurable sets. As such, let  $\mu(E) < \infty$ . Now let  $A_i \in \mathcal{A}$  and  $E \subseteq A = \cup_{i=1}^\infty A_i$  where  $\mu(A \Delta E) < \varepsilon$ . We first see that

$$\rho(A) = \lim_{n \rightarrow \infty} \rho(\cup_{i=1}^n A_i) = \lim_{n \rightarrow \infty} \mu(\cup_{i=1}^n A_i) = \mu(A).$$

Since  $A \setminus E \subseteq A \Delta E$ , we get that

$$\mu(E) \leq \mu(A) = \rho(A) = \rho(E) + \rho(A \setminus E) = \rho(E) + \mu(A \setminus E) = \rho(E) + \varepsilon.$$

This proves that  $\mu(E) \leq \rho(E) + \varepsilon$  as we wanted.  $\square$

### 4.3 Properties of the Lebesgue and Lebesgue-Stieltjes measures

**Definition 4.2.** Any measure that is constructed from  $F$  is called a *Lebesgue-Stieltjes measure*. If we use the particular function  $F(x) = x$ , the measure is called the *Lebesgue measure* and is denoted by  $\ell$  instead of  $\mu$ .

The Lebesgue measure has some particularly nice properties:

- i) It is shift-invariant (this is because  $F(x) = x$  is shift-invariant)
- ii) For any  $s \in \mathbb{R}$ ,  $\ell(sE) = |s|\ell(E)$
- iii) Any measure defined on the Borel set with the two above properties must in fact be the Lebesgue measure.

HW Exercise: Let  $\mu$  be a Lebesgue-Stieltjes measure. Show that

- i) For all  $E \in \mathcal{F}_\mu$ ,  $\mu(E) = \inf\{\sum_{i=1}^\infty \mu(a_i, b_i) : E \subseteq \cup_{i=1}^\infty (a_i, b_i)\}$ .
- ii) For all  $E \in \mathcal{F}_\mu$ ,  $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$ .
- iii) For all  $E \in \mathcal{F}_\mu$ ,  $\mu(E) = \sup\{\mu(K) : E \subseteq K, K \text{ compact}\}$ .

However, measurable sets should not scare you. For example, we can approximate measurable sets by some tamer looking sets, which should help calm your worries.

**Proposition 4.3.** *The following are equivalent:*

- a)  $E$  is  $\mu$ -measurable
- b)  $E = V \setminus N$  where  $N$  is a null set and  $V$  is some countable intersection of open sets
- c)  $E = H \cup N$  where  $N$  is a null set and  $H$  is some countable union of closed sets

In fact, we also have that

**Proposition 4.4.** *If  $E$  is  $\mu$ -measurable and  $\mu(E) < \infty$ , then for any  $\varepsilon > 0$ , there are open intervals  $(a_i, b_i)$  such that*

$$\mu(E \Delta \cup_{i=1}^n (a_i, b_i)) < \varepsilon.$$

HW Exercise: There exists an open set  $U \subseteq (0, 1)$  which is dense in  $(0, 1)$  and  $\ell(U) < \varepsilon$ . Similarly,  $K = [0, 1] \setminus U$ , where  $K$  is closed, nowhere dense such that  $\ell(K) > 1 - \varepsilon$ .

How do arbitrary Lebesgue-Stieltjes measures look like? Well, here are some examples:

- 1) The Lebesgue measure itself
- 2) Discrete, weighted, atomic measures like  $\mu = \sum_{i=1}^N a_i \delta_{x_i}$  where  $a_i > 0$
- 3) Singular measures where  $F$  is continuous but almost everywhere constant. For an example, the Cantor function, see below.

If  $f$  is the Cantor function and  $x$  is not in the Cantor set, it's pretty easy to see what  $f(x)$  is. Now for the  $x$  that do lie in the Cantor set,  $f(x)$  can be found as the result of writing  $x$  in ternary and then replacing every 2 with 1 and interpreting this new string as binary. More explicitly, if  $x$  is in the Cantor set its ternary expansion will look like

$$\sum_{i=1}^{\infty} b_i 3^{-i} \text{ where } b_i \in \{0, 2\},$$

which means  $f(x)$  will be

$$f(x) = \sum_{i=1}^{\infty} \frac{b_i}{2} 2^{-i}.$$

## 4.4 Lebesgue $\sigma$ -algebra

The  $\sigma$ -algebra  $\mathcal{L}$  of Lebesgue measurable sets is complete, it is a completion of the Borel  $\sigma$ -algebra. As such, it contains all Borel sets. The cardinality of the Lebesgue  $\sigma$ -algebra is hypercontinuum  $2^{\mathfrak{c}}$ , since there are uncountable Borel sets of Lebesgue measure zero (like the Cantor set), and each subset of them belongs to  $\mathcal{L}$ .

## 4.5 Summary of Part I

We have discussed measures and construction of measures, focusing on Lebesgue-Stieltjes measures. The latter construction works as follows:

- 1) Take an  $F$  which is right-continuous
- 2) Use this  $F$  to construct a pre-measure  $\nu$
- 3) Use this  $\nu$  to construct an outer measure  $\mu$
- 4) Restrict this outer measure to form the measure we want (Carathéodory's theorem)

There is a number of statements we proved about general measures, and other facts were related to the particular Lebesgue-Stieltjes context.

## Part II

# Lebesgue integral