

7310 Lecture notes. Spring 2019

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Contents

I	Construction of measure	1
1	January 15, 2019	2
1.1	Basics	2
1.2	Problem of measure	2
1.3	Outer measure	3
1.4	σ -algebras	3
2	January 17, 2019	4
2.1	Measures	4
2.2	Completeness	5
2.3	Caratheodory's theorem. Formulation and first part of proof . .	6
2.4	Definition of a pre-measure	6
3	January 22, 2019	7
3.1	From outer measure to measure	7
3.2	From pre-measure to outer measure	8

Part I

Construction of measure

1 January 15, 2019

1.1 Basics

A few completions of \mathbb{Q} :

- 1) Cauchy sequences
- 2) Dedekind cuts
- 3) Axiomated total ordered complete fields
- 4) Decimal expansion up to equivalence class

Definition 1.1. Dedekind cut is a pair of non-empty sets (A, B) such that it satisfies $\mathbb{Q} = A \sqcup B$ and the following:

1. if $x \in A, y < x \implies y \in A$, i.e., A is order-closed downwards.
2. if $x \in B, y \geq x, \implies y \in B$, i.e., B is order-closed upwards.
3. A does not have a maximum element

Define $\mathbb{R} = \{\text{Dedekind cuts of } \mathbb{Q}\}$. For simplicity, we can use A to represent the pair (A, B) . Note we can define an order on \mathbb{R} by set inclusion with respect to A . We can also define addition, subtraction, multiplication, division on \mathbb{R} such that \mathbb{R} is totally ordered, complete field.

Theorem 1.2. \mathbb{R} is uncountable.

Proof. Diagonal process. The complete proof is left as an exercise for the reader. \square

1.2 Problem of measure

Question 1: What is the **length** ℓ of a subset of \mathbb{R} ?

- a) $\ell([0, 1]) = 1$;
- b) $\ell(\{x\}) = 0$ for any $x \in \mathbb{R}$;
- c) $\ell(\{x \in \mathbb{R} : x = 0.**3*****\}) = \frac{1}{10}$. This is the set of decimal expansions such that there is a “3” in the third place after the dot.

Axioms of length on \mathbb{R} :

1. $\ell([0, 1]) = 1$. i.e., normalized;
2. for $A \subseteq \mathbb{R}$ such that $A = A_1 \sqcup A_2 \sqcup \dots$, $\ell(A) = \sum_{i=1}^{\infty} \ell(A_i)$, i.e., countable additivity;
3. for $A \subseteq \mathbb{R}$, $\alpha \in \mathbb{R}$, $\ell(A + \alpha) = \ell(A)$, i.e., transitivity-invariant.

Theorem 1.3. Consider the length $\ell : 2^{\mathbb{R}} = P(\mathbb{R}) \longrightarrow [0, \infty]$, such length does not exist.

Proof. We can write $[0, 1) = \bigsqcup_{r \in N} (\mathbb{Q} + r) \cap [0, 1)$, where N is the collection of representatives of each equivalence class with respect to the following equivalence relation: $x \sim y \iff x - y \in \mathbb{Q}$. Note that we need Axiom of Choice to define N . Then as length is transitivity-invariant, $\ell(N) = \ell(N + \alpha) \forall \alpha$. Hence $1 = \ell([0, 1)) = \sum_{x \in N} \ell(N + x)$, but the right hand side is the sum of countably infinite elements with the same value, either equals 0 or infinity. Contradiction. \square

Corollary 1.4. *There exist non-measurable sets.*

1.3 Outer measure

Definition 1.5. Let X be a set. $\mu: 2^X \rightarrow [0, \infty]$ is called an **outer measure** if

1. $\mu(\emptyset) = 0$;
2. if $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \bigcup_{k=1}^{\infty} \mu(A_k)$ (Hence $\mu(A) \leq \mu(B)$ if $A \subseteq B$).

Definition 1.6. Let X be a set, μ an outer measure defined on X . Then $A \subseteq X$ is called μ -measurable if for all $B \subseteq X$, $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$

1.4 σ -algebras

Definition 1.7. Let X be a set. $\mathcal{F} \subseteq 2^X$ is called σ -algebra if

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

We will eventually prove the following important theorem:

Theorem 1.8 (Caratheodory Extension Theorem). *Let X be a set, μ an outer measure on X . Then the class of μ -measurable sets is a σ -algebra.*

Below are some examples of σ -algebras:

- X countable, $\mathcal{F} = 2^X$ is a σ -algebra;
- for any set X , $\mathcal{F} = \{\emptyset, X\}$ is called the trivial σ -algebra;
- for any set X , $A \subseteq X$, $\mathcal{F} = \{\emptyset, A, A^c, X\}$ is a σ -algebra (generated by the set A). That is, we define $\sigma(A) = \{\emptyset, A, A^c, X\} = \bigcap_{\{\mathcal{F} \in S\}} \mathcal{F}$, where $S = \{\mathcal{F} \text{ is a } \sigma\text{-algebra containing } A\}$.

Proposition 1.9. *Let X be a set. For any set Γ (not necessarily countable), if \mathcal{F}_x is a σ -algebra on X for all $x \in \Gamma$, then $\bigcap_{x \in \Gamma} \mathcal{F}_x$ is also a σ -algebra.*

Proof. Exercise left for the reader. \square

Definition 1.10. The Borel σ -algebra of \mathbb{R}^d ($d \geq 1$), denoted as $B(\mathbb{R}^d)$, is defined as the σ -algebra generated by all open sets in \mathbb{R}^d .

The Borel σ -algebra has countable generating sets (e.g. rational rectangles).

2 January 17, 2019

2.1 Measures

Definition 2.1. Let X be a set, \mathcal{F} a σ -algebra on X . Then a **measure** on \mathcal{F} is a function $\mu : \mathcal{F} \rightarrow [0, \infty]$ such that

- a) $\mu(\emptyset) = 0$;
- b) for $A_1, A_2, \dots \in \mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, i.e., μ is countably additive.

We call (X, \mathcal{F}, μ) a **measure space**.

Below are some examples of measure spaces. Let X be a set.

- Fix $x \in X$, then for $A \subseteq X$ consider $\delta : X \rightarrow [0, \infty]$ by

$$\delta(A) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

This is called the δ -measure at $x \in X$.

- Let $\mathcal{F} = 2^X$, for $A \in \mathcal{F}$, define μ such that $\mu(A) = 0$ if A is countable, and $\mu(A) = 1$ if A is co-countable (i.e., A^c is countable). Then (X, \mathcal{F}, μ) is a measure space.
- For X countable, i.e., $X = \{x_1, x_2, \dots\}$. Let $\mathcal{F} = 2^X$. Denote $p(x_i) =$ weight of x_i , then for $A \subseteq X$, define $\mu(A) = \sum_{x \in A} p(x)$. (X, \mathcal{F}, μ) is a measure space.

Properties of a measure space (X, \mathcal{F}, μ)

1. for $A, B \subseteq X$, $A \subseteq B \implies \mu(A) \leq \mu(B)$;
2. (countably subadditive) if $A \subseteq \bigcup_{i=1}^{\infty} A_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$;
3. (continuity) for a sequence of subsets in X , $A_1 \subseteq A_2 \subseteq \dots$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i);$$

for a sequence of subsets in X , $A_1 \supseteq A_2 \supseteq \dots$ such that $\mu(A_i) < \infty \forall i$,

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i).$$

A counterexample for 3) if we drop the condition " $\mu(A_i) < \infty \forall i$ " is $X = \mathbb{R}$, $\mathcal{F} = \sigma(\mathbb{R})$, $A_i = (i, \infty)$ for $i \in \mathbb{N}$, then each $\mu(A_i) = \infty$ but $\lim_{i \rightarrow \infty} \mu(A_i) = 0$.

Definition 2.2. Let (X, \mathcal{F}, μ) be a measure space. μ is called **finite** if $\mu(X) < \infty$, μ is called **σ -finite** if $X = \bigcup E_n$ where $\mu(E_n) < \infty \forall n$.

Semicontinuity. Let $\{A_n : n \in \mathbb{N}\}$ be a collection of sets. Then define

$$\begin{aligned}\liminf A_n &= \{\text{elements in almost all } A_n\} = \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right), \\ \limsup A_n &= \{\text{elements in } A_n \text{ infinitely often}\} = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right).\end{aligned}$$

Proposition 2.3. We have

$$\mu(\limsup A_n) \geq \limsup \mu(A_n), \quad \mu(\liminf A_n) \leq \liminf \mu(A_n).$$

Proof. Homework exercise. □

2.2 Completeness

Definition 2.4. Consider (X, \mathcal{F}, μ) , $B \in \mathcal{F}$ is called a **null set** if $\mu(B) = 0$. Note if B is a null set, $A \subseteq B$, then $\mu(A) = 0$ in the sense of outer measure (not necessarily $A \in \mathcal{F}$).

Definition 2.5. (X, \mathcal{F}) is **complete** if all null sets are contained in \mathcal{F} .

Here we introduce a notation “**a.e.**”, the abbreviation of “almost everywhere”. Something happens a.e. if it happens outside a null set. Here are a few examples:

Consider the Dirichlet function $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then $f(x) = 0$ a.e.

Consider the (small) Riemann function $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then f is continuous a.e.

Theorem 2.6. There is an unique completion of any measure space (X, \mathcal{F}, μ) . i.e., there exist $\overline{\mathcal{F}}, \overline{\mu}$ with $\overline{\mathcal{F}} \supseteq \mathcal{F}$ and $\overline{\mu}|_{\mathcal{F}} = \mu$. Specifically,

$$\overline{\mathcal{F}} = \{A \cup B \mid A \in \mathcal{F}, B \subseteq N \text{ for some } N \in \mathcal{F}, \mu(N) = 0\},$$

which satisfies:

- 1) $\overline{\mathcal{F}}$ is a σ -algebra;
- 2) there exists unique extension $\overline{\mu}$ of μ , from \mathcal{F} to $\overline{\mathcal{F}}$ such that $\overline{\mu}|_{\mathcal{F}} = \mu$.

Proof. Let $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$. Then it is closed under countable unions and complement, hence so is $\overline{\mathcal{F}}$. Then set $\overline{\mu}(A \cup B) = \mu(A)$ if $B \subseteq N \in \mathcal{N}$. Then it is left as an exercise to the reader to check:

- a) $\overline{\mu}$ is well-defined (i.e., if there is another way of writing $A \cup B = A' \cup B'$, $B' \subseteq N' \in \mathcal{N}$, then the measure does not change);
- b) $\overline{\mu}$ is a measure;
- c) $\overline{\mu}$ is a unique extension to $\overline{\mathcal{F}}$ defined in the claim. □

Note that the Borel σ -algebra is not complete with respect to the length measure ℓ .

2.3 Caratheodory's theorem. Formulation and first part of proof

Let X be a set. Recall the definition of the outer measure μ on 2^X .

Definition 2.7. $A \subseteq X$ is a μ -measurable set with respect to μ if $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c) \forall B \in 2^X$.

We now begin the proof of the Caratheodory's extension theorem:

Theorem 2.8 (Caratheodory). *Let μ be an outer measure on a set X , then the following statements are true:*

- 1) $\mathcal{F} = \{A : A \text{ is a } \mu\text{-measurable set}\}$ is a σ -algebra;
- 2) $\mu|_{\mathcal{F}}$ is a complete measure.

Proof. For 1), first of all, \mathcal{F} is closed under complement by the definition of the outer measure and the measurable sets. Then for $A, B \in \mathcal{F}$, take any $S \subseteq X$, then

$$\begin{aligned} \mu(S) &= \mu(S \cap A) + \mu(S \cap A^c) \\ &= \mu(S \cap A \cap B) + \mu(S \cap A \cap B^c) + \mu(S \cap A^c \cap B) + \mu(S \cap A^c \cap B^c). \end{aligned}$$

Note $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, then by subadditivity,

$$\mu(S \cap (A \cap B)) + \mu(S \cap (A \cap B^c)) + \mu(S \cap (A^c \cap B)) \geq \mu(S \cap (A \cup B)).$$

Hence $\mu(S) \geq \mu(E \cap (A \cup B)) + \mu(E \cap (A \cup B)^c)$. Hence $A \cup B \in \mathcal{F}$, so \mathcal{F} is an algebra.

Also, if $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \cap A^c) = \mu(A) + \mu(B).$$

This shows finite additivity of μ on \mathcal{F} .

(other parts will be proven next time.) □

2.4 Definition of a pre-measure

We will also work with outer measures constructed using pre-measures:

Definition 2.9. Let X be a set, $\Gamma \subseteq 2^X$, $\nu : \Gamma \rightarrow [0, \infty]$ be any function. Then the **pre-measure** μ on 2^X is defined as for $F \in 2^X$,

$$\mu(F) = \inf \left\{ \sum_{n=1}^{\infty} \nu(A_n) : A_n \in \Gamma, F \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

By agreement, $\inf \emptyset = 0$.

Proposition 2.10. *With the same set up as above, μ is an outer measure.*

Proof. This will also be proven next time. □

3 January 22, 2019

3.1 From outer measure to measure

We are on a path of constructing the measure. Last time, we defined pre-measures, and this definition will be used later. Here we show how an outer measure leads to a measure, i.e., finish the proof of the Caratheodory theorem.

Recall that if μ is an outer measure, we say that a set A is μ -measurable if for all $E \subseteq X$,

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

Note that in the proofs we need only to check \geq since \leq follows directly from subadditivity. Denote by \mathcal{F} the set of all μ -measurable subsets.

Rest of the proof of Caratheodory theorem (Theorem 2.8). We have shown in Lecture 2 that \mathcal{F} is an algebra (closed under finite unions, intersections, and complements) and that μ is finitely additive on \mathcal{F} .

Now we need to show that \mathcal{F} is a σ -algebra. Let $A_1, A_2, \dots \in \mathcal{F}$ be disjoint, and define $B_n = \bigsqcup_{j=1}^n A_j$. We know that $B_n \in \mathcal{F}$, so we need to show that $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

First, for all $E \subseteq X$, we have

$$\mu(E \cap B_n) = \mu(E \cap B_n \cap A_n) + \mu(E \cap B_n \cap A_n^c) = \mu(E \cap A_n) + \mu(E \cap B_{n-1}).$$

Continuing this, we see that

$$\mu(E \cap B_n) = \sum_{i=1}^n \mu(E \cap A_i).$$

Thus,

$$\mu(E) = \mu(E \cap B_n) + \mu(E \cap B_n^c) \geq \sum_{i=1}^n \mu(E \cap A_i) + \mu(E \cap B^c).$$

The last inequality is due to the fact that in the second summand we have passed to the smaller set $E \cap B$. In the previous inequality we can now pass to $n \rightarrow \infty$, and all the sums in the right-hand side are bounded:

$$\mu(E) \geq \sum_{i=1}^{\infty} \mu(E \cap A_i) + \mu(E \cap B^c).$$

By subadditivity, this is

$$\geq \mu\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu(E \cap B^c) = \mu(E \cap B) + \mu(E \cap B^c).$$

This shows that $B \in \mathcal{F}$, so \mathcal{F} is a σ -algebra.

The σ -additivity of μ on \mathcal{F} is straightforward, take $B = E$ in the previous argument, which leads to $\mu(B) = \sum_{i=1}^{\infty} \mu(A_i)$.

Finally, to show completeness, let $A \subseteq X$ be with the outer measure $\mu(A) = 0$. We want to show that $A \in \mathcal{F}$. We have for all $E \subseteq X$:

$$\mu(E) \leq \mu(E \cap A) + \mu(E \cap A^c) = \mu(E \cap A^c) \leq \mu(E)$$

because the outer measure is subadditive and so $\mu(E \cap A) = 0$. The last inequality is also due to subadditivity. This implies that $\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$, so $E \in \mathcal{F}$. \square

3.2 From pre-measure to outer measure