7310 Lecture notes. Spring 2019

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Contents

Part I

Construction of measure

1   January 15, 2019

1.1   Basics

A few completions of of $\mathbb{Q}$:
  1) Cauchy sequences
  2) Dedekind cuts
  3) Axiomated total ordered complete fields
  4) Decimal expanation up to equivalence class

Definition 1.1. Dedekind cut is a pair of non-empty sets (A, B) such that it satisfies $\mathbb{Q} = A \sqcup B$ and the following:
  1. if $x \in A$, $y < x \implies y \in A$, i.e., A is order-closed downwards.
  2. if $x \in B$, $y \geq x$, $\implies y \in B$, i.e., B is order-closed upwards.
  3. A does not have a maximum element

Define $\mathbb{R} = \{$Dedekind cuts of $\mathbb{Q}\}$. For simplicity, we can use A to represent the pair (A, B). Note we can define an order on $\mathbb{R}$ by set inclusion with respect to A. We can also define addition. subtraction, multiplication, division on $\mathbb{R}$ such that $\mathbb{R}$ is totally ordered, complete field.

Theorem 1.2. $\mathbb{R}$ *is uncountable.*

Proof. Diagonal process. The complete proof is left as an exercise for the reader. $\square$

1.2   Problem of measure

Question 1: What is the **length** ℓ of a subset of $\mathbb{R}$?
  a) $\ell([0, 1]) = 1$;
  b) $\ell(\{x\}) = 0$ for any $x \in \mathbb{R}$;
  c) $\ell(\{x \in \mathbb{R} : x = 0.**3*****...\}) = \frac{1}{10}$. This is the set of decimal expansions such that there is a "3" in the third place after the dot.

  Axioms of length on $\mathbb{R}$:
  1. $\ell([0, 1]) = 1$. i.e., normalized;
  2. for $A \subseteq \mathbb{R}$ such that $A = A_1 \sqcup A_2 \sqcup ...$, $(A) = \sum_{i=1}^{\infty}(A_i)$, i.e., countable additivity;
  3. for $A \subseteq \mathbb{R}$, $\alpha \in \mathbb{R}$, $\ell(A + \alpha) = \ell(A)$, i.e., transitivity-invariant.

Theorem 1.3. *Consider the length* $\ell : 2^{\mathbb{R}} = P(\mathbb{R}) \longrightarrow [0, \infty]$, *such length does not exist.*

Proof. We can write $[0,1) = \bigsqcup_{r \in N}(\mathbb{Q} + r) \cap [0,1)$, where N is the collection of representatives of each equivalence class with respect to the following equivalence relation: $x \sim y \iff x - y \in \mathbb{Q}$. Note that we need Axiom of Choice to define N. Then as length is transitivity-invariant, $\ell(N) = \ell(N + \alpha) \forall \, \alpha$. Hence $1 = \ell([0,1)) = \sum_{x \in N} \ell(N + x)$, but the right hand side is the sum of countably infinite elements with the same value, either equals 0 or infinity. Contradiction. $\square$

Corollary 1.4. *There exist non-measurable sets.*

1.3 Outer measure

Definition 1.5. Let X be a set. $\mu \colon 2^X \longrightarrow [0, \infty]$ is called an **outer measure** if

1. $\mu(\varnothing) = 0$;
2. if $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \bigcup_{k=1}^{\infty} \mu(A_k)$ (Hence $\mu(A) \leq \mu(B)$ if $A \subseteq B$).

Definition 1.6. Let X be a set, μ an outer measure defined on X. Then $A \subseteq X$ is called μ-measurable if for all $B \subseteq X$, $\mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$

1.4 σ-algebras

Definition 1.7. Let X be a set. $\mathcal{F} \subseteq X$ is called σ-algebra if

1. $\varnothing \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. if $A_1, A_2, ... \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

We will eventually prove the following important theorem:

Theorem 1.8 (Caratheodory Extension Theorem). *Let X be a set, μ an outer measure on X. Then the class of μ-measurable sets is a σ-algebra.*

Below are some examples of σ-algebras:

- X countable, $\mathcal{F} = 2^X$ is a σ-algebra;

- for any set X, $\mathcal{F} = \{\varnothing, X\}$ is called the trivial σ-algebra;

- for any set X, $A \subseteq X$, $\mathcal{F} = \{\varnothing, A, A^c, X\}$ is a σ-algebra (generated by the set A). That is, we define $\sigma(A) = \{\varnothing, A, A^c, X\} = \bigcap_{\{\mathcal{F} \in S\}} \mathcal{F}$, where $S = \{\mathcal{F}$ is a σ-algebra containing A$\}$.

Proposition 1.9. *Let X be a set. For any set Γ (not necessarily countable), if $\mathcal{F}_x$ is a σ-algebra on X for all $x \in \Gamma$, then $\bigcap_{x \in \Gamma} \mathcal{F}_x$ is also a σ-algebra.*

Proof. Exercise left for the reader. $\square$

Definition 1.10. The Borel σ-algebra of $\mathbb{R}^d$ (d ≥ 1), denoted as $B(\mathbb{R}^d)$, is defined as the σ-algebra generated by all open sets in $\mathbb{R}^d$.

The Borel σ-algebra has countable generating sets (e.g. rational rectangles).

2 January 17, 2019

2.1 Measures

Definition 2.1. Let X be a set, $\mathcal{F}$ a σ-algebra on X. Then a **measure** on $\mathcal{F}$ is a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ such that

    a) $\mu(\varnothing) = 0$;

    b) for $A_1, A_2, \ldots \in \mathcal{F}$ such that $A_i \cap A_j = \varnothing$ for $i \neq j$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, i.e., μ is countably additive.

    We call $(X, \mathcal{F}, \mu)$ a **measure space**.

Below are some examples of measure spaces. Let X be a set.

- Fix $x \in X$, then for $A \subseteq X$ consider $\delta \colon X \longrightarrow [0, \infty]$ by

$$\delta(A) = \begin{cases} 0 & if \ x \notin A \\ 1 & if \ x \in A \end{cases}$$

  This is called the δ-measure at $x \in X$.

- Let $\mathcal{F} = 2^X$, for $A \in X$, define μ such that $\mu(A) = 0$ if A is countable, and $\mu(A) = 1$ if A is co-countable (i.e., A^c is countable). Then $(X, \mathcal{F}, \mu)$ is a measure space.

- For X countable, i.e., $X = \{x_1, x_2, \ldots\}$. Let $\mathcal{F} = 2^X$. Denote $p(x_i) =$ weight of x_i, then for $A \subseteq X$, define $\mu(A) = \sum_{x \in A} p(x)$. $(X, \mathcal{F}, \mu)$ is a measure space.

Properties of a measure space (X, F, μ)

1. for $A, B \subseteq X$, $A \subseteq B \implies \mu(A) \leq \mu(B)$;

2. (countably subadditive) if $A \subseteq \bigcup_{i=1}^{\infty} A_i \implies \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$;

3. (continuity) for a sequence of subsets in X, $A_1 \subseteq A_2 \subseteq \ldots$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i);$$

   for a sequence of subsets in X, $A_1 \supseteq A_2 \supseteq \ldots$ such that $\mu(A_i) < \infty \ \forall i$,

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i).$$

A counterexample for 3) if we drop the condition "$\mu(A_i) < \infty \ \forall i$" is $X = \mathbb{R}$, $F = \sigma(\mathbb{R})$, $A_i = (i, \infty)$ for $i \in \mathbb{N}$, then each $\mu(A_i) = \infty$ but $\lim_{i \to \infty} \mu(A_i) = 0$.

Definition 2.2. Let (X, F, μ) be a measure space. μ is called **finite** if $\mu(X) < \infty$, μ is called σ-**finite** if $X = \bigcup E_n$ where $\mu(E_n) < \infty \ \forall n$.

Semicontinuity. Let $\{A_n : n \in \mathbb{N}\}$ be a collection of sets. Then define

$$\liminf A_n = \{\text{elements in almost all } A_n\} = \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} A_k),$$

$$\limsup A_n = \{\text{elements in } A_n \text{ infinitely often}\} = \bigcap_{n=1}^{\infty} (\bigcup_{k=n}^{\infty} A_k).$$

Proposition 2.3. *We have*

$$\mu(\limsup A_n) \geq \limsup \mu(A_n), \qquad \mu(\liminf A_n) \leq \liminf \mu(A_n).$$

Proof. Homework exercise. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.2 Completeness

Definition 2.4. Consider $(X, \mathcal{F}, \mu)$, $B \in \mathcal{F}$ is called a **null set** if $\mu(B) = 0$. Note if B is a null set, $A \subseteq B$, then $\mu(A) = 0$ in the sense of outer measure (not necessarily $A \in \mathcal{F}$).

Definition 2.5. $(X, \mathcal{F})$ is **complete** if all null sets are contained in $\mathcal{F}$.

Here we introduce a notation "**a.e**", the abbreviation of "almost everywhere". Something happens a.e. if it happens outside a null set. Here are a few examples:

Consider the Dirichlet function $f(x) = \begin{cases} 1 & if \ x \in \mathbb{Q} \\ 0 & if \ x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$. Then $f(x) = 0$ a.e.

Consider the (small) Riemann function $f(x) = \begin{cases} \frac{1}{q} & if \ x = \frac{p}{q} \in \mathbb{Q} \\ 0 & if \ x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$. Then f is continuous a.e.

Theorem 2.6. *There is an unique completion of any measure space $(X, \mathcal{F}, \mu)$. i.e., there exist $\overline{\mathcal{F}}, \overline{\mu}$ with $\overline{\mathcal{F}} \supseteq \mathcal{F}$ and $\overline{\mu}|_{\mathcal{F}} = \mu$. Specifically,*

$$\overline{\mathcal{F}} = \{A \cup B| \ A \in \mathcal{F}, B \subseteq N \text{ for some } N \in \mathcal{F}, \mu(N) = 0\},$$

which satisfies:
1) $\overline{\mathcal{F}}$ is a σ-algebra;
2) there exists unique extension $\overline{\mu}$ of μ, from $\mathcal{F}$ to $\overline{\mathcal{F}}$ such that $\overline{\mu}|_{\mathcal{F}} = \mu$.

Proof. Let $\mathcal{N} = \{N \in \mathcal{F} : \mu(N) = 0\}$. Then it is closed under countable unions and complement, hence so is a $\overline{\mathcal{F}}$. Then set $\overline{\mu}(A \cup B) = \mu(A)$ if $B \subseteq N \in \mathcal{N}$. Then it is left as an exercise to the reader to check:
a) $\overline{\mu}$ is well-defined (i.e., if there is another way of writing $A \cup B = A' \cup B'$, $B' \subseteq N' \in \mathcal{N}$, then the measure does not change);
b) $\overline{\mu}$ is a measure;
c) $\overline{\mu}$ is a unique extension to $\overline{\mathcal{F}}$ defined in the claim. $\qquad\qquad\qquad$ $\square$

Note that the Borel σ-algebra is not complete with respect to the length measure ℓ.

2.3 Caratheodory's theorem. Formulation and first part of proof

Let X be a set. Recall the definition of the outer measure μ on 2^X.

Definition 2.7. $A \subseteq X$ is a μ-measurable set with respect to μ if $\mu(A) = \mu(A \cap B) + \mu(A \cap B^c) \ \forall B \in 2^X$.

We now begin the proof of the Caratheodory's extension theorem:

Theorem 2.8 (Caratheodory). *Let μ be an outer measure on a set X, then the following statements are true:*
  1) $\mathcal{F} = \{A \colon A$ is a μ-measurable set$\}$ is a σ-algebra;
  2) $\mu|_{\mathcal{F}}$ is a complete measure.

Proof. For 1), first of all, $\mathcal{F}$ is closed under complement by the definition of the outer measure and the measurable sets. Then for $A, B \in \mathcal{F}$, take any $S \subseteq X$, then

$$\mu(S) = \mu(S \cap A) + \mu(S \cap A^c)$$
$$= \mu(S \cap A \cap B) + \mu(S \cap A \cap B^c) + \mu(S \cap A^c \cap B) + \mu(S \cap A^c \cap B^c).$$

Note $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, then by subadditivity,

$$\mu(S \cap (A \cap B)) + \mu(S \cap (A \cap B^c)) + \mu(S \cap (A^c \cap B)) \geq \mu(S \cap (A \cup B)).$$

Hence $\mu(S) \geq \mu(E \cap (A \cup B)) + \mu(E \cap (A \cup B)^c)$. Hence $A \cup B \in \mathcal{F}$, so $\mathcal{F}$ is an algebra.

Also, if $A, B \in \mathcal{F}$ with $A \cap B = \varnothing$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \cap A^c) = \mu(A) + \mu(B).$$

This shows finite additivity of μ on $\mathcal{F}$.

(other parts will be proven next time.) $\qquad\square$

2.4 Definition of a pre-measure

We will also work with outer measures constructed using pre-measures:

Definition 2.9. Let X be a set, $\Gamma \subseteq 2^X$, $\nu : \Gamma \longrightarrow [0, \infty]$ be any function. Then the **pre-measure** μ on 2^X is defined as for $F \in 2^X$,

$$\mu(F) = \inf\left\{ \sum_{n=1}^{\infty} \nu(A_n) : A_n \in \Gamma, F \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

By agreement, $\inf \varnothing = 0$.

Proposition 2.10. *With the same set up as above, μ is an outer measure.*

Proof. This will also be proven next time. $\qquad\square$

3   January 22, 2019

3.1   From outer measure to measure

We are on a path of constructing the measure. Last time, we defined pre-measures, and this definition will be used later. Here we show how an outer measure leads to a measure, i.e., finish the proof of the Caratheodory theorem.

Recall that if μ is an outer measure, we say that a set A is μ-measurable if for all $E \subseteq X$,

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c).$$

Note that in the proofs we need only to check $\geq$ since $\leq$ follows directly from subadditivity. Denote by $\mathcal{F}$ the set of all μ-measurable subsets.

Rest of the proof of Caratheodory theorem (Theorem 2.8). We have shown in Lecture 2 that $\mathcal{F}$ is an algebra (closed under finite unions, intersections, and complements) and that μ is finitely additive on $\mathcal{F}$.

Now we need to show that $\mathcal{F}$ is a σ-algebra. Let $A_1, A_2, \ldots \in \mathcal{F}$ be disjoint, and define $B_n = \bigsqcup_{j=1}^{n} A_j$. We know that $B_n \in \mathcal{F}$, so we need to show that $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

First, for all $E \subseteq X$, we have

$$\mu(E \cap B_n) = \mu(E \cap B_n \cap A_n) + \mu(E \cap B_n \cap A_n^c) = \mu(E \cap A_n) + \mu(E \cap B_{n-1}).$$

Continuing this, we see that

$$\mu(E \cap B_n) = \sum_{i=1}^{n} \mu(E \cap A_i).$$

Thus,

$$\mu(E) = \mu(E \cap B_n) + \mu(E \cap B_n^c) \geq \sum_{i=1}^{n} \mu(E \cap A_i) + \mu(E \cap B^c).$$

The last inequality is due to the fact that in the second summand we have passed to the smaller set $E \cap B$. In the previous inequality we can now pass to $n \to \infty$, and all the sums in the right-hand side are bounded:

$$\mu(E) \geq \sum_{i=1}^{\infty} \mu(E \cap A_i) + \mu(E \cap B^c).$$

By subadditivity, this is

$$\geq \mu\left(\bigcup_{i=1}^{\infty}(E \cap A_i)\right) + \mu(E \cap B^c) = \mu(E \cap B) + \mu(E \cap B^c).$$

This shows that $B \in \mathcal{F}$, so $\mathcal{F}$ is a σ-algebra.

The σ-additivity of μ on $\mathcal{F}$ is straightforward, take $B = E$ in the previous argument, which leads to $\mu(B) = \sum_{i=1}^{\infty} \mu(A_i)$.

Finally, to show completeness, let $A \subseteq X$ be with the outer measure $\mu(A) = 0$. We want to show that $A \in \mathcal{F}$. We have for all $E \subseteq X$:

$$\mu(E) \leq \mu(E \cap A) + \mu(E \cap A^c) = \mu(E \cap A^c) \leq \mu(E)$$

because the outer measure is subadditive and so $\mu(E \cap A) = 0$. The last inequality is also due to subadditivity. This implies that $\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$, so $E \in \mathcal{F}$. $\qquad \square$

3.2 From pre-measure to outer measure

Recall the definition of a pre-measure:

Definition 3.1 (Pre-measure)**.** Let X be a set and let $\mathcal{E}$ be a set of subsets of X. Then any function $\nu : \mathcal{E} \to [0, \infty]$ is called a *pre-measure*.

Proposition 3.2. *If ν is a pre-measure, then μ defined by*

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \nu(A_i) : A_i \in \mathcal{E}, E \subseteq \cup_{i=1}^{\infty} A_i \right\}$$

is an outer measure on $\mathcal{E}$.

Proof. We need only prove the subadditivity of μ. Let $F \subseteq \cup_{i=1}^{\infty} F_i$ and let $\varepsilon > 0$. By construction of μ, there are sets A_i^n such that $F_i \subseteq \cup_{n=1}^{\infty} A_i^n$ and

$$\mu(F_i) + \frac{\varepsilon}{2^i} \geq \sum_{n=1}^{\infty} \nu(A_i^n).$$

We also note that

$$F \subseteq \bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_i^n.$$

This means

$$\mu(F) \leq \mu \left(\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_i^n \right) \leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu(A_i^n) \leq \sum_{i=1}^{\infty} \left(\mu(F_i) + \frac{\varepsilon}{2^i} \right).$$

As we can make ε as small as we'd like, this implies that $\mu(F) \leq \sum_{i=1}^{\infty} \mu(F_i)$ as we wanted. $\qquad \square$