ESE 670 Convex Optimization Spring 2015

Final Project

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Contents

Part I

Background and Motivation

In many fields, there is a need to draw conclusions from noisy data. In this paper, we look at a method of estimating the minimum number of variables required to describe a process. Knowing only the covariance matrix of noisy observations, we seek to draw conclusions about the underlying number of noiseless variables. One common application of this problem is finding reliable models in econometrics, where we could have a large amount of data but only a small number of variables are actually necessary to describe a process.

For the below problems, $\mathbf{x}$ is a zero mean Gaussian vector of observations, and Σ is its covariance matrix. Assuming that $\mathbf{x}$ can be decomposed into $\mathbf{x} = \hat{\mathbf{x}} + \tilde{\mathbf{x}}$, where $\tilde{\mathbf{x}}$ is the "noise component" (and all entries are essentially, or at least assumed to be, random, so they have no connection to each other or the other variables) and $\hat{\mathbf{x}}$ contains no noise. We also require $E\left(\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T\right) = \tilde{\Sigma}$ and $E\left(\hat{\mathbf{x}}\tilde{\mathbf{x}}^T\right) = 0$, with $\Sigma = \hat{\Sigma} + \tilde{\Sigma}$.

In other words, $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ are zero mean variables, the entries of $\tilde{\mathbf{x}}$ are independent of one another and of $\hat{\mathbf{x}}$ (the noise in one variable does not affect other variables or the noise in other variables), the covariance matrix of $\hat{\mathbf{x}}$ is $\hat{\Sigma}$ and the covariance matrix of $\tilde{\mathbf{x}}$ is $\tilde{\Sigma}$ (which is diagonal), so we have $\Sigma = \hat{\Sigma} + \tilde{\Sigma}$, where Σ is the covariance of $\mathbf{x} = \hat{\mathbf{x}} + \tilde{\mathbf{x}}$.

The goal of the Frisch Problem, (1), is to minimize the rank of $\hat{\Sigma}$, the covariance matrix of the noise free observations, given Σ, see [12]. And, since the rank of $\hat{\Sigma}$ is how many rows of $\hat{\Sigma}$ are linearly dependent, the goal of the Frisch problem is to find a system of variables where the smallest number of them are actually random variables and the greatest number of them are simply correlated to one another (that is, we want to find the minimum number of independent random variables necessary to describe the process).

Problem 1 (The Frisch Problem)

$$\begin{aligned}
&\text{Given } \Sigma \in \mathbb{S}^n_+, \text{ find} \\
&\mathrm{mr}_+\left(\Sigma\right) = \min\left\{ \mathrm{rank}\left(\hat{\Sigma}\right) \middle| \Sigma = \tilde{\Sigma} + \hat{\Sigma} \right. \\
&\quad \left. \text{and } \tilde{\Sigma}, \hat{\Sigma} \geq 0 \text{ and } \tilde{\Sigma} \text{ is diagonal} \right\}
\end{aligned} \tag{1}$$

The Shapiro problem, (2), is a less restricted Frisch problem, in that $\tilde{\Sigma}$ need not be positive definite. Since $\tilde{\Sigma}$ is diagonal, negative definiteness of $\tilde{\Sigma}$ requires a negative number on the diagonal, which makes $\tilde{\Sigma}$ physically meaningless, as a variable must be positively correlated to itself. So we allow some of the diagonal elements in $\tilde{\Sigma}$ to be negative to, as stated in [17], gain insight into potential algebraic relations imposed by the off-diagonal elements of the data covariance matrix.

Problem 2 (The Shapiro Problem)

$$\begin{aligned}
&\text{Given } \Sigma \in \mathbb{S}^n_+, \text{ find} \\
&\mathrm{mr}\left(\Sigma\right) = \min\left\{ \mathrm{rank}\left(\hat{\Sigma}\right) \middle| \Sigma = \tilde{\Sigma} + \hat{\Sigma} \right. \\
&\quad \left. \text{and } \hat{\Sigma} \geq 0 \text{ and } \tilde{\Sigma} \text{ is diagonal} \right\}
\end{aligned} \tag{2}$$

Basically, the Frisch and Shapiro problems, (1) and (2) respectively, identify linear relations of variables when observed through noisy data. A typical linear regression assumes that all variables except one are noisy while the Frisch and Shapiro problems assume that all the variables can be corrupted by noise.

Note that we require $\tilde{\Sigma}$ to be diagonal because we assume the noise is random (and not related to the any of the other variables).

The trace minimization problems, (4) and (5), which are convex relaxations of (1), are useful for finding a lower bound for the Frisch Problem, (1). Similarly, the dual problem (3) is useful for finding an upper bound for the Frisch Problem, (1).
Problem 3 (The dual Frisch Problem)

$$\begin{aligned}&\text{Given } S \in \mathbb{S}^n_{++}, \text{ find}\\ &\mathrm{mr}_{\text{dual}}(S) = \min\left\{\operatorname{rank}\left(\hat{S}\right) \middle| S = E - \hat{S}\right.\\ &\quad\left.\text{and } \hat{S}, E \geq 0 \text{ and } E \text{ is diagonal}\right\}\end{aligned} \tag{3}$$

Problem 4 (The relaxed Frisch problem)

$$\begin{aligned}&\text{Given } \Sigma \in S^n_{++}, \text{ find}\\ &\min_D \{\operatorname{trace}(\Sigma - D) | \, \Sigma \geq D \geq 0\\ &\quad \text{and } D \text{ is diagonal}\}\end{aligned} \tag{4}$$

Problem 5 (The weighted relaxed Frisch problem)

$$\begin{aligned}&\text{Given } \Sigma \in S^n_{++} \text{ and diagonal } W > 0, \text{ find}\\ &\min\left\{\operatorname{trace}\left(W\hat{\Sigma}\right) \middle| \Sigma = \hat{\Sigma} + D\right.\\ &\quad\left.\text{and } \hat{\Sigma} \geq 0, D \geq 0 \text{ and } D \text{ is diagonal}\right\}\end{aligned} \tag{5}$$

Part II
Dual Problem

1 Matrix Inversion Lemma

Lemma: Let $\Sigma = D + FF^T$ with $\Sigma, D \in \mathbb{S}^n_+$ where $\Sigma, D > 0$ and $F \in \mathbb{R}^{n\times r}$, then we have

$S = \Sigma^{-1} = D^{-1} - \left(D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)\left(D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)^T$

$= D^{-1} - \left(D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)\left(\left(\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)^T F^T\left(D^{-1}\right)^T\right) = D^{-1} - GG^T,$

where $G = \left(D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)$.

Proof: Using the Woodbury Matrix Identity , see [21], $(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1}VA^{-1}$
$\Sigma^{-1} = \left(D + FF^T\right)^{-1} = D^{-1} - D^{-1}F\left(I + F^TD^{-1}F\right)^{-1}F^TD^{-1}$, by the Woodbury Matrix Identity
$= D^{-1} - D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}F^TD^{-1}$
$= D^{-1} - \left(D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)\left(\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}F^TD^{-1}\right)$
$= D^{-1} - \left(D^{-1}F\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)\left(D^{-1}\left(\left(I + F^TD^{-1}F\right)^{-\frac{1}{2}}\right)^T F\right)^T$

$$= D^{-1} - \left(D^{-1}F\left(I+F^TD^{-1}F\right)^{-\frac{1}{2}}\right)\left(D^{-1}\left(\left(I+F^TD^{-1}F\right)^{-\frac{1}{2}}\right)^T F\right)^T, \ I+F^TD^{-1}F \text{ is symmetric}$$

$$= D^{-1} - \left(D^{-1}F\left(I+F^TD^{-1}F\right)^{-\frac{1}{2}}\right)\left(D^{-1}\left(\left(I+F^TD^{-1}F\right)^{-\frac{1}{2}}\right) F\right)^T = D^{-1} - GG^T$$

Note, the converse holds as well: if we have $\Sigma^{-1} = E - GG^T$, then let $E^{-1} = D$ and $F = \left(E^{-1}G\left(I - G^TE^{-1}G\right)^{-\frac{1}{2}}\right)$, so $\Sigma = D + FF^T$, again, by the Woodbury Matrix Identity.

2 Dual Formulation

Now we arrive at the dual problem, (3). When $S = \Sigma^{-1} = E - GG^T$, then $E = \Sigma^{-1} + GG^T > 0$ (because $S = \Sigma^{-1}$ and GG^T are both symmetric positive definite). By the converse of the above lemma, when we have $\Sigma^{-1} = E - GG^T$, we can find $\Sigma = D + FF^T$. Since $E = D^{-1}$ and $D > 0$, we have that $E > 0$.

Therefore when D is invertible, we have $\Sigma = D + FF^T$ and $\Sigma^{-1} = D^{-1} - GG^T$. Therefore we have (6).

$$\text{mr}_+(\Sigma) = \text{mr}_+\left(D + FF^T\right) = \text{mr}_{\text{dual}}\left(D^{-1} - GG^T\right) = \text{mr}_{\text{dual}}\left(\Sigma^{-1}\right) \tag{6}$$

However, if D is not invertible, then we have (7).

$$\text{mr}_+(\Sigma) \leq \text{mr}_{\text{dual}}\left(\Sigma^{-1}\right) \tag{7}$$

For example, if we have

$$\Sigma = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \hat{\Sigma} + \tilde{\Sigma} = \hat{\Sigma} + D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} - GG^T$$

Therefore, we have $\text{rank}\left(FF^T\right) = 1$ (that is, $\text{mr}_+(\Sigma) = 1$) and we have

$$GG^T = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} e_1 - 1 & 0 & 1 \\ 0 & e_2 - 1 & 1 \\ 1 & 1 & e_3 - 3 \end{bmatrix}$$

$$\text{rank}\left(GG^T\right) = \text{rank}\left(\begin{bmatrix} e_1 - 1 & 0 & 1 \\ 0 & e_2 - 1 & 1 \\ 1 & 1 & e_3 - 3 \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} e_1 - 1 & 0 & 1 \\ 0 & e_2 - 1 & 1 \\ 1 & 1 & e_3 - 3 \end{bmatrix}\right)$$

Where $E = \text{diag}\{e_1, e_2, e_3\} \geq 0$ and $GG^T \geq 0$. When $E = \text{diag}\{1, 1, 5\}$, we have

$$\text{rank}\left(GG^T\right) = \text{rank}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}\right) = 2$$

Therefore, the minimum rank is 1 or 2. Now to show that the minimum rank is always greater than 1, we prove by contradiction. First assume that the every row is a linear multiple of the first row, that is $GG^T_{1,j} = \alpha_i GG^T_{i,j}$ for $i = 2, 3$, which is the definition of $\text{rank}\left(GG^T\right) = 1$. So we have $e_1 - 1 = \alpha_2 \cdot 0$

and $0 = \alpha_2 (e_2 - 1)$ and $1 = \alpha \cdot 1$. Therefore, $e_1 = e_2 = 1$, so the rank of the matrix is 2, a contradiction. Therefore, the minimum rank of the matrix GG^T is 2.

Therefore $\mathrm{mr}_+ (\Sigma) = \mathrm{rank}\left(FF^T\right) = 1 < \mathrm{mr}_{\text{dual}} \left(\Sigma^{-1}\right) = \mathrm{rank}\left(GG^T\right) = 2$.

Part III
Trace Minimization

3 Motivation

The relaxed Frisch problem, that is, the trace minimization problem, (4), is a common heuristic for estimating the minimum rank of a matrix. Note that $\mathrm{tr}(X) = \sum_{i=1}^{n} \lambda_i (X) = \|\lambda(X)\|_1$ for $X \geq 0$. As seen within the context of penalty functions, minimizing the l_1-norm of the vector containing the eigenvalues of X, results in many zero values, see [8, 11], and $\mathrm{rank}(X) = \sum_{\lambda_i(X)>0} 1$ (the rank of a matrix is the number of non-zero eigenvalues for symmetric matrices). Because (1) seems to have no analytic solution, see [4, 8, 7], the convex relaxation, (4), is useful as a lower bound for (1), see [6].

This convex optimization problem, (4), is easily formulated and solved in CVX. Note that we require $\Sigma = \hat{\Sigma} + \tilde{\Sigma}$, where $\Sigma, \hat{\Sigma} \in \mathbb{S}_+^n$ and $\tilde{\Sigma}$ is diagonal, see [15, 17].

While $\mathrm{mr}(\Sigma)$ is invariant under scaling, because rank is invariant under scaling, that is $\mathrm{mr}(\Sigma) = \mathrm{mr}(2 \cdot \Sigma)$, trace does vary under scaling, that is $\mathrm{tr}(2A) = 2 \cdot \mathrm{tr}(A)$, therefore motivating the weighted trace minimization, (5), see [17, 10]. One suitable choice of weight, W, is Σ^{-1},

$$\begin{aligned}
\mathrm{rank}(\Sigma) &= \mathrm{trace}\left(\hat{\Sigma}^\dagger \hat{\Sigma}\right) \\
&\geq \mathrm{trace}\left(\left(\hat{\Sigma} + D\right)^{-1} \hat{\Sigma}\right) \\
&= \mathrm{trace}\left(\Sigma^{-1} \hat{\Sigma}\right) \\
&\geq \mathrm{trace}\left(\left(\hat{\Sigma} + \epsilon I_n\right)^{-1} \hat{\Sigma}\right) \qquad (8)
\end{aligned}$$

Note that $\hat{\Sigma}^\dagger$ is the Moore-Penrose pseudo inverse and $\epsilon > 0$, and see [3] for proof that $\mathrm{rank}(\Sigma) = \mathrm{trace}\left(\hat{\Sigma}^\dagger \hat{\Sigma}\right)$.

4 Numerical Estimate

To create numerical results with the rank of $\hat{\Sigma}$ predetermined as r, we create a matrix of observations $X = [x_1, x_2, ..., x_T] \in \mathbb{R}^{n \times T}$ and $x_i \in \mathbb{R}^{n \times 1}$ and $i = 1, 2, ..., T$ (that is, we make T observations of n variables) using the MATLAB code (note that randn gives normally distributed pseudorandom numbers):

```
X=zeros(m,T)
for i=1:T

    X(:.i)=randn(m,1);

end
for i=r+1:m

    X(i,:)=x(i-r,:);
```

```
end
Sigma_hat = X*X';
Sigma = Sigma_hat + diag(rand(m,1));
```

This creates a positive definite $\hat{\Sigma}$ matrix that has rank r and a positive definite Σ and $\tilde{\Sigma}$, where $\tilde{\Sigma}$ is diagonal ($\tilde{\Sigma} = \text{diag(rand(m,1))}$). A physical intuition of these inputs and the goal is: given a covariance matrix of noisy data, $\Sigma \in \mathbb{R}^{n \times n}$, where the process that the uncorrupted covariance matrix, $\hat{\Sigma} \in \mathbb{R}^{n \times n}$, describes requires exactly r independent random variables, we attempt to find r.

Using $W = \Sigma^{-1}$, we obtain the rank estimates in tables 1, 2 and 3 (as the average of three trials). Using this weight, when the number of observations, T, is increased, the estimate of the rank actually worsens. This is solved in MATLAB, using CVX, as follows

```
[n,~] = size(Sigma);
W = inv(Sigma);
cvx_begin
    variables D(n) Sigma_h(n,n)
    minimize trace(W*Sigma_h)
    subject to
        diag(D) == semidefinite(n);
        Sigma_h == semidefinite(n);
        Sigma == Sigma_h + diag(D);
cvx_end
trace_min = trace(W*Sigma_h);
```

A more accurate, though more computationally intensive, weight, however, involves an iterative reweighting process. Specifically, we use $W_k = (\Sigma - D_k + \epsilon I_n)$ and $W_0 = (\Sigma + \epsilon I_n)^{-1}$, that is $D_0 = 0$, where we find D_k by solving the convex optimization problem (9). That is, we simultaneously optimize over D and $\hat{\Sigma}$, and use the minimizer D to find the next weight.

$$\left(D_{k+1}, \hat{\Sigma}_{k+1}\right) = \arg\min_{D,\hat{\Sigma}} \left\{ \text{tr}\left(W \cdot \hat{\Sigma}\right) \middle| D \text{ is diagonal, } D \geq 0,\ \hat{\Sigma} \geq 0,\ \Sigma = \hat{\Sigma} + \mathrm{D} \right\} \tag{9}$$

Using this iterative method and $\epsilon = 5 \cdot 10^{-3}$, we obtain the rank estimates in tables 4, 5 and 6 (as the average of three trials). For this iterative method, when the number of observations, T, is increased, the lower bound tightens. This is solved in MATLAB, using CVX, as follows

```
epsilon = 5e-3;
[n,~] = size(Sigma);
W = inv(Sigma+epsilon*eye(n,n)); W_old = 10*ones(n,n);
D = zeros(n,1); D_old = ones(n,1); trace_min_old = 0; trace_min = 1;
while ( norm(W_old - W) > 1e-2 && norm(D_old - D) > 1e-2 &&
abs(trace_min - trace_min_old) > 1e-3)
    W_old = W; D_old = D;
    trace_min_old = trace_min;
    cvx_begin
        variables D(n) Sigma_h(n,n)
        minimize trace(W*Sigma_h)
        subject to
            diag(D) == semidefinite(n);
            Sigma_h == semidefinite(n);
            Sigma == Sigma_h + diag(D);
        cvx_end
    trace_min =  trace(W*Sigma_h);
    W = inv(Sigma - diag(D) + epsilon*eye(n,n));
end
```

5 Relationship to Log-Det

Another common heuristic for the rank minimization problem is $\log\det(\Sigma - D)$, this is explored in more detail in [10, 11].

The rank minimization problem with $X \in R^{n\times n}$ where $X \geq 0$ can be estimated by (10), where $\mathcal{C}$ is the constraint set. Note that log denotes the natural logarithm.

$$\begin{aligned} &\text{minimize} \log\det(X + \epsilon I_n) \\ &\text{subject to } X \in \mathcal{C} \end{aligned} \tag{10}$$

(10) can be used to estimate the Frisch problem, (1), as (11).

$$\begin{aligned} &\text{Given } \Sigma \in \mathbb{S}_+^n, \text{ find} \\ &\min\left\{ \log\det\left(\hat{\Sigma} + \epsilon I_n\right) \middle| \Sigma = \tilde{\Sigma} + \hat{\Sigma} \right. \\ &\left. \quad \text{and } \tilde{\Sigma}, \hat{\Sigma} \geq 0 \text{ and } \tilde{\Sigma} \text{ is diagonal} \right\} \end{aligned} \tag{11}$$

Note that the objective function, $\log\det\left(\hat{\Sigma} + \epsilon I_n\right)$ is, in fact, concave (so minimizing it is not a convex problem). But since it is smooth on the positive definite cone, it can be minimized locally, see [8]. Using a Taylor series centered at $\hat{\Sigma}_k$, we can estimate $\log\det\left(\hat{\Sigma} + \epsilon I_n\right) \approx \log\det\left(\hat{\Sigma}_k + \epsilon I_n\right) + \operatorname{tr}\left(\left(\hat{\Sigma}_k + \epsilon I_n\right)^{-1}\left(\hat{\Sigma} - \hat{\Sigma}_k\right)\right)$ (because $\nabla \log\det(X) = X^{-1}$).

$$W_{k+1} = \arg\min_{\hat{\Sigma}} \left[\operatorname{tr}\left((W_k + \epsilon I_n)^{-1}\hat{\Sigma}\right)\right] \tag{12}$$

Therefore, we use the iterative method given in (12), which is a simple version of (9). So, as pointed out in [17, 10], the iterative method for trace minimization actually corresponds to minimizing $\log\det(\Sigma - D + \epsilon I_n) = \log\det\left(\hat{\Sigma} + \epsilon I_n\right) = \log\det(W + \epsilon I_n)$ by local linearization, as the objective function in (11) is equivalent to $W_k + \epsilon I_n = \hat{\Sigma}_k + \epsilon I_n = \Sigma - D + \epsilon I_n$.

6 Results

Table 1: $\operatorname{tr}\left(W \cdot \hat{\Sigma}\right)$, using $W = \Sigma^{-1}$, $T = 1.1 \cdot n$

$n \setminus r$	$\frac{n}{10}$	$\frac{2n}{10}$	$\frac{3n}{10}$	$\frac{4n}{10}$	$\frac{5n}{10}$	$\frac{6n}{10}$	$\frac{7n}{10}$	$\frac{8n}{10}$	$\frac{9n}{10}$	n
10	0.997415	1.98677	2.95115	3.88208	4.80425	4.29605	4.46468	4.87742	5.66255	6.31088
100	9.99722	19.9866	29.9592	39.9116	49.8277	50.7169	56.8788	65.1098	76.0441	86.594
200	19.998	39.9852	59.9611	79.9013	99.8051	101.852	117.096	135.965	156.745	179.205
300	29.9977	59.9868	89.9614	119.899	149.817	154.962	177.705	206.111	239.448	274.205

Table 2: tr $\left(W \cdot \hat{\Sigma}\right)$, using $W = \Sigma^{-1}$, $T = 3 \cdot n$

$n \setminus r$	$\frac{n}{10}$	$\frac{2n}{10}$	$\frac{3n}{10}$	$\frac{4n}{10}$	$\frac{5n}{10}$	$\frac{6n}{10}$	$\frac{7n}{10}$	$\frac{8n}{10}$	$\frac{9n}{10}$	n
10	0.999291	1.99637	2.98709	3.96035	4.95926	4.25974	3.96884	4.03463	4.34167	4.83579
100	9.99915	19.9954	29.9884	39.9726	49.9538	46.2231	47.4324	51.0231	57.3616	63.1037
200	19.9995	39.9961	59.9885	79.9727	99.9542	93.6033	97.5926	106.178	117.211	131.251
300	30.0005	59.9969	89.9877	119.974	149.955	140.87	147.843	161.81	179.263	200.323

Table 3: tr $\left(W \cdot \hat{\Sigma}\right)$, using $W = \Sigma^{-1}$, $T = \frac{n}{2}$

$n \setminus r$	$\frac{n}{10}$	$\frac{2n}{10}$	$\frac{3n}{10}$	$\frac{4n}{10}$	$\frac{5n}{10}$	$\frac{6n}{10}$	$\frac{7n}{10}$	$\frac{8n}{10}$	$\frac{9n}{10}$	n
10	0.995561	1.94843	2.82642	3.19124	3.90922	3.93197	4.23633	4.4865	4.59367	4.65837
100	9.99434	19.9619	29.8353	39.2694	46.5853	48.8119	49.1322	49.43	49.4983	49.5946
200	19.9934	39.957	59.8412	79.3693	95.1028	98.7647	99.2296	99.3968	99.5403	99.5744
300	29.9933	59.9557	89.8425	119.323	144.524	148.64	149.196	149.418	149.55	149.622

Table 4: tr $\left(W_k \cdot \hat{\Sigma}\right)$, using $W_{k+1} = \left(\Sigma - D_{k+1} + \epsilon I_n\right)^{-1}$, $T = 1.1 \cdot n$

$n \setminus r$	$\frac{n}{10}$	$\frac{2n}{10}$	$\frac{3n}{10}$	$\frac{4n}{10}$	$\frac{5n}{10}$	$\frac{6n}{10}$	$\frac{7n}{10}$	$\frac{8n}{10}$	$\frac{9n}{10}$	n
10	0.999871	1.99938	2.9983	4.00202	5.02453	4.99145	5.6524	5.97134	6.95009	7.29806
100	10.0002	20.0001	30.001	40.0608	50.1841	56.612	65.6351	73.9709	83.9397	93.4888
200	20.0016	40.0027	59.9997	80.0779	100.168	116.275	133.956	152.258	171.89	190.943
300	30.0085	60.002	90.002	120.1	150.161	174.974	202.283	231.589	259.912	289.204

Table 5: tr $\left(W_k \cdot \hat{\Sigma}\right)$, using $W_{k+1} = \left(\Sigma - D_{k+1} + \epsilon I_n\right)^{-1}$, $T = 3 \cdot n$

$n \setminus r$	$\frac{n}{10}$	$\frac{2n}{10}$	$\frac{3n}{10}$	$\frac{4n}{10}$	$\frac{5n}{10}$	$\frac{6n}{10}$	$\frac{7n}{10}$	$\frac{8n}{10}$	$\frac{9n}{10}$	n
10	0.999974	1.99981	2.99973	4.00141	5.04483	5.07452	5.66327	6.65688	7.60361	7.63237
100	10.0052	20.0023	30.0044	40.0266	50.0714	56.984	65.6697	75.3125	84.3126	93.9329
200	20.028	40.0209	60.0075	80.0589	100.076	115.998	134.978	152.643	171.947	191.238
300	30.0108	60.0114	90.0227	120.113	150.11	175.325	203.637	231.976	260.302	289.301

Table 6: tr $\left(W_k \cdot \hat{\Sigma}\right)$, using $W_{k+1} = \left(\Sigma - D_{k+1} + \epsilon I_n\right)^{-1}$, $T = \frac{n}{2}$

$n \setminus r$	$\frac{n}{10}$	$\frac{2n}{10}$	$\frac{3n}{10}$	$\frac{4n}{10}$	$\frac{5n}{10}$	$\frac{6n}{10}$	$\frac{7n}{10}$	$\frac{8n}{10}$	$\frac{9n}{10}$	n
10	0.999663	1.99941	2.98814	3.99362	4.95287	4.97333	4.97821	5.05931	4.99414	4.99342
100	9.99585	19.999	29.9934	39.9983	49.0247	49.9821	49.9899	49.9928	49.9918	49.9925
200	19.9776	39.9738	59.999	80.0051	99.6709	99.9817	99.9955	99.9943	99.9949	99.9921
300	30.0023	59.9984	90.0042	120.024	148.801	149.99	149.995	149.995	149.994	149.981

7 Analysis

Both methods have some trouble with ranks larger than $\frac{n}{2}$, but for the iterative method, simply increasing the number of observations can improve the estimate. However, increasing the number of observations for the simple inverse method severely loosens the estimates for ranks larger than $\frac{n}{2}$, although it gives a nearly exact (or very slightly over) estimate for ranks less than $\frac{n}{2}$. This makes the iterative method more attractive for a system with a suspected large rank or a large number of observations. The simple inverse method is only more attractive if rank is less than half the suspected number of variables. While both methods certainly give a lower bound on the rank, the iterative method gives unquestionably better results, at the cost of a greater computational complexity.

For example, for $m = 300$, $T = 330$ and $r = 100$ (the most computationally demanding case), the time to run the simple inverse weighting method is about 15 seconds, while the iterative method takes about 85 seconds (about five or six iterations) (using a late 2013 quad core macbook pro). The iterative method can easily be sped up by loosening any of the stopping conditions, but there is no ideal stopping condition for every set of conditions. For small enough matrices, strict stopping conditions are appropriate, but for larger matrices, loose stopping conditions avoid unnecessary iterations, as the estimate can oscillate around a certain value, delaying convergence.

While the iterative method is dependent on ϵ, increasing ϵ has mixed results. For larger values of ϵ, low ranks tend to be more tightly bound, at the expense of higher ranks, and for smaller values of ϵ, higher ranks tend to be more tightly bound at the expense of overestimating lower ranks. The slightly overestimated results in the results are a function of this value of ϵ chosen for the numerical trials $(5 \cdot 10^{-3})$, but the overestimate is very slight and attributable to round-off error.

For example, for $m = 100$, $T = 110$ and $r = 100$, when $\epsilon = 10^{-5}$, the estimate is 95.023, but when $r = 40$, the estimate is 41.913. And when $\epsilon = 0.5$, the estimate for $r = 100$ is 90.700 and the estimate for $r = 40$ is 39.881. The best option is probably to choose ϵ based on the expected rank of a system, or to run a few trials, some for small ϵ and some for large ϵ.

As an exploration of both methods, we investigate the case when $m = 600$, $T = 660$ and $r = 100$ and $r = 600$. Using the simple inverse method, which takes about 84 and 110 seconds (respectively), we obtain the estimates 99.992 and 559.478, for $r = 100$ and $r = 600$, respectively. Using the iterative method, which takes about 163 seconds and 68 minutes (respectively), we obtain the estimates 100.054 and 584.917 for $r = 100$ and $r = 600$ respectively. Similarly, for $m = 800$, $T = 880$ and $r = 100$ and $r = 800$, using the simple inverse method, which takes about 381 and 413 seconds (respectively), we obtain the estimates 100.0033 and 746.9087, respectively. And for the iterative method, which takes about 702 seconds and 191 minutes (respectively), we obtain the estimates 100.4239 and 782.93, respectively (note that the iterative method was near 783 for more than half of the iterations, but the strict stopping conditions were not met).

While too computationally demanding to be investigated in detail, these results imply that both methods behave similarly for small and large values of m.

Interestingly, but not unexpectedly, when the number of observations is less than the actual rank, the maximum rank for both estimates is the number of observations, as seen in tables 3 and 6.

In conclusion, minimizing the weighted trace, a convex relaxation of the rank minimization problem, is very reliable as a lower bound for the minimum rank and nearly exact when the rank is less than half the size of the covariance matrix. For the iterative weighting method, since there are many parameters that can tweak the performance and accuracy of the trace minimization problem, it can help to tweak these parameters, such as ϵ and the stopping criteria, at and before runtime. Even the simple inverse weighting method is very accurate for ranks less than half the size of the covariance matrix, though the iterative method should be chosen if the rank could be greater.

References

[1] A. Agarwal, S. Negahban, and M. Wainwright. Noisy matrix decomposition via convex relaxation: Optimal rates in high dimensions. *The Annals of Statistics*, (40):1171–1197, 2012.

[2] T.W. Anderson. Estimating linear statistical relationships. *The Annals of Statistics*, pages 1–45, 1984.

[3] O. Baksalarya, D. Bernsteinb, and G Trenklerc. On the equality between rank and trace of an idempotent matrix. *Applied Mathematics and Computation*, 217:4076–4080, 2010.

[4] S. Bittanti and P. Colaneri. *Period Systems: Filtering and Control*. Springer, 2009.

[5] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[6] E. Candès and T. Tao. The power of convex relaxation: Near-optimal matrix completion. In *Information Theory: IEEE Transactions on*, volume 56.5, pages 2053–2080, 2010.

[7] J. David and B. Moor. "the opposite of analytic centering for solving minimum rank problems in control and identification.". In *Decision and Control, 1993., Proceedings of the 32nd IEEE Conference on*. IEEE, 1993.

[8] M. Fazel. *Matrix Rank Minimization With Applications*. PhD thesis, Stanford University, 2002.

[9] M. Fazel, H. Hindi, and S. Boyd. A rank minimization heuristic with application to minimum order system approximation. In *2001 American Control Conference*, volume 6, 2001.

[10] M. Fazel, H. Hindi, and S. Boyd. Log-det heuristic for matrix rank minimization with applications to hankel and euclidean distance matrices. In *2003 American Control Conference*, volume 3, 2003.

[11] M. Fazel, H. Hindi, and S. Boyd. "rank minimization and applications in system theory.". In *American Control Conference, 2004. Proceedings of the 2004*, volume 4. IEEE, 2004.

[12] R. Frisch. La resolution des problèms de programme linéaire par la méthode du potential logarithmique. *Cahiers diiy Séminaire d'Econometrie*, 4:7–20.

[13] M. Grant, S. Boyd, and Y. Yinyu. *CVX: MATLAB Software for Disciplined Convex Programming*, 2008.

[14] Y. Kim and M. Mesbahi. On the rank minimization problem. In *2004 American Control Conference*, 2004.

[15] C. A. Los. Identification of a linear system from inexact data: A three-variable example. *Computers Math. Applic.*, 17(8/9):1285–1304, 1989.

[16] L. Ning and T. Georgiou. The wasserstein metric in factor analysis. In *SIAM Conf. on Control and its Applications*, 2013.

[17] L. Ning, T. Georgiou, A. Tannenbaum, and S. Boyd. Linear models based on noisy data and the frisch scheme. *CoRR*, abs/1304.3877, 2013.

[18] B. Recht, M. Fazel, and P. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Review 52.3*, (471-501), 2010.

[19] A. Shapiro. Rank-reducibility of a symmetric matrix and sampling theory of minimum trace factor analysis. *Psychometrika*, 47(2), 1982.

[20] A. Shapiro and J. Berge. Statistical inference of minimum rank factor analysis. *Psychometrika*, 67(1):79–94, 2002.

[21] E. Weisstein. Mathworld–a wolfram web resource. *http://mathworld.wolfram.com/WoodburyFormula.html*, April 6, 2015.

[22] K. Woodgate. On computing the maximum corank in the frisch scheme. Citeseer; Pre-print 4 pages, (2007).