Homework 3 Key

1. Which of the following functions are eigenvectors of the operators $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$?

   λ is the eigenvalue of the operators.

   (a) $\exp(ax)$

   $\frac{d}{dx}\exp(ax) = a\exp(ax) = \lambda\exp(ax)$ Yes, an eigenvectors of $\frac{d}{dx}$

   $\frac{d^2}{dx^2}\exp(ax) = a^2\exp(ax) = \lambda\exp(ax)$ Yes, an eigenvectors of $\frac{d^2}{dx^2}$

   (b) $\exp(ax^2)$

   $\frac{d}{dx}\exp(ax^2) = 2ax\exp(ax^2) \neq \lambda\exp(ax^2)$ Not an eigenvectors of $\frac{d}{dx}$

   $\frac{d^2}{dx^2}\exp(ax^2) = 2a(2ae^{ax^2}x^2 + e^{ax^2}) \neq \lambda\exp(ax^2)$ Not an eigenvectors of $\frac{d^2}{dx^2}$

   (c) x

   $\frac{d}{dx}x = 1 \neq \lambda x$ Not an eigenvectors of $\frac{d}{dx}$

   $\frac{d^2}{dx^2}x = 0 \neq \lambda x$ Not an eigenvectors of $\frac{d^2}{dx^2}$

   (d) x^2

   $\frac{d}{dx}x^2 = 2x \neq \lambda x^2$ Not an eigenvectors of $\frac{d}{dx}$

   $\frac{d^2}{dx^2}x^2 = 2 \neq \lambda(x^2)$ Not an eigenvectors of $\frac{d^2}{dx^2}$

   (e) x^2

   $\frac{d}{dx}\sin(ax) = a\cos(ax) \neq \lambda\sin(ax)$ Not an eigenvectors of $\frac{d}{dx}$

   $\frac{d^2}{dx^2}\sin(ax) = -a^2\sin(ax) = \lambda\sin(ax)$ Yes, an eigenvectors of $\frac{d^2}{dx^2}$

2. Show that any linear combination of the functions $\exp(i2x)$ and $\exp(-i2x)$ is an eigenfunction of the operator $\frac{d^2}{dx^2}$.

   Proof. $aV_1 + bV_2 \in \mathbb{V}; \forall a, b \in \mathbb{C}$

   $\frac{d^2}{dx^2}\Big(a\exp(i2x) + b\exp(-i2x)\Big) = -4\Big(a\exp(i2x) + b\exp(-i2x)\Big)$

   So yes, an eigenfunction with eigenvalue of -4. □

3. If the operators A and B are Hermitian, will the operator $C = A + iB$ also be Hermitian? Explain.

   $\hat{A}g_i(x) = a_i g_i(x)$

   $\int g_j^*(x)\hat{A}g_i(x)dx = \int g_i(x)\hat{A}^* g_j^*(x)dx$

   $\int g_j^*(x)\hat{B}g_i(x)dx = \int g_i(x)\hat{B}^* g_j^*(x)dx$

   For C to be Hermitian:

   $\int g_j^*(x)(\hat{A} + i\hat{B})g_i(x)dx = \int g_i(x)(\hat{A} + i\hat{B})^* g_j^*(x)dx$

   $\int \hat{A}g_j^*(x)g_i(x) + i\int \hat{B}g_j^*(x)g_i(x)dx = \int \hat{A}g_i(x)g_j^*(x) + i\int \hat{B}^* g_i(x)g_j^*(x)dx$

   C is Hermitian if $\hat{A} + i\hat{B} = \hat{A} - i\hat{B}$.

4. Under which conditions will the function $\exp(-ax^2)$ be an eigenvector of the operator $A = \frac{d^2}{dx^2} - bx^2$.

   λ is the eigenvalue of the operators.

   $\left(\frac{d^2}{dx^2} - bx^2\right)(e^{-ax^2}) = \lambda(e^{-ax^2})$

   $\frac{d^2}{dx^2}e^{-ax^2} - bx^2 e^{ax^2} = \lambda e^{-ax^2}$

   $-2a(-2ae^{-ax^2}x^2 + e^{-ax^2}) - bx^2 e^{-ax^2} = \lambda e^{ax^2}$

   $4a^2x^2e^{-ax^2} - 2ae^{-ax^2} - bx^2e^{-ax^2} = \lambda e^{ax^2}$

   $4a^2x^2e^{-ax^2} - 2ae^{-ax^2} - 4a^2x^2e^{-ax^2} = \lambda e^{ax^2}$

   if $b = 4a^2$ or $\lambda = -2a$

5. Let $\hat{A}$ be an Hermitian operator, let $\{|a_k\rangle\}$ be the set of its eigenvectors. That is: $\forall k; \hat{A}|a_k\rangle = \lambda_k|a_k\rangle$

   $\forall|a_k'\rangle \in \mathbb{V}$; $|a_k'\rangle = \sum_k c_k|a_k\rangle$; linear combination of an arbitrary vector in this set

   $\hat{A}|a_k'\rangle = \sum_k c_k\hat{A}|a_k\rangle$

   $\hat{A}|a_k'\rangle = \sum_k c_k\lambda_k|a_k\rangle$; Applying Hermitian property $\hat{A}|a_k\rangle = \lambda_k|a_k\rangle$

   Proof. : $\hat{A} = \sum_k \lambda|a_k\rangle\langle a_k|$

   $\hat{A} = \sum_k \lambda|a_k\rangle\langle a_k|$

$\hat{A}|a'_k\rangle = \sum_k \lambda|a_k\rangle\langle a_k|\ |a'_k\rangle$

$\hat{A}|a'_k\rangle = \sum_k \lambda|a_k\rangle\langle a_k|\sum_k c_k|a_k\rangle$

$\hat{A}|a'_k\rangle = \sum_k \lambda c_k|a_k\rangle$; Applying identity operator and this is equal to line 3 so $\square$

6. Let $\hat{A}$ be an operator such that $\hat{A}^2 = \hat{A}$. Proof eigenvalues are only 0 and 1.

   λ is eigenvalue of the operators.

   $\hat{A}v = \lambda v$

   $\hat{A}^2 v = \hat{A}(\lambda v) = \lambda^2 v$

   $\lambda = \lambda^2$; $\quad \lambda$ can only be 0 or 1.

7. Let $\hat{A}, \hat{B}, \hat{C}$ be three arbitrary operators. Proof that:

   Proof. $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$

   $\hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) - (\hat{B}\hat{C} - \hat{C}\hat{B})\hat{A} + \hat{B}(\hat{C}\hat{A} - \hat{A}\hat{C}) - (\hat{C}\hat{A} - \hat{A}\hat{C})\hat{B} + \hat{C}(\hat{A}\hat{B} - \hat{B}\hat{A}) - (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} = 0$

   $\hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} + \hat{B}\hat{C}\hat{A} - \hat{B}\hat{A}\hat{C} - \hat{C}\hat{A}\hat{B} + \hat{A}\hat{C}\hat{B} + \hat{C}\hat{A}\hat{B} - \hat{C}\hat{B}\hat{A} - \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C} = 0$ $\square$

8. Optional: Let $\hat{A}, \hat{B}$ be two arbitrary operators, and let $\{|m\rangle\}$ be an arbitrary orthonormal basis.

   Proof. $\sum_m \langle m|\hat{A}\hat{B}|m\rangle = \sum_m \langle m|\hat{B}\hat{A}|m\rangle$

   $\exists\ m, m' \in \{|m\rangle\}$

   $\sum_{mm'} \langle m|\hat{A}|m'\rangle\langle m'|\hat{B}|m\rangle$; inserting identity operator

   $\sum_{mm'} \langle m'|\hat{B}|m\rangle\langle m|\hat{A}|m'\rangle$; flipping the order

   $\sum_{m'} \langle m'|\hat{B}\hat{A}|m'\rangle$; simplify because identity operator $\square$