Homework 0 Key

1. Express the following complex numbers in exponential form:

   (a) $9(1+i)$

   $= 9\sqrt{2}\exp\left[i\tan^{-1}(1)\right]$

   $=9\sqrt{2}e^{i\pi/4}$

   (b) $1-i$

   $= \sqrt{2}\exp\left[i\tan^{-1}(-1)\right]$

   $= \sqrt{2}e^{-i\pi/4}$

2. Express the following complex numbers in $a+bi$ form:

   (a) $3\exp\left(\frac{\pi i}{2}\right)$

   $= 3\left(\cos\frac{\pi}{2} + i\sin(\frac{\pi}{2})\right)$

   $= 3(0+i)$

   $= 3i$

   (b) $\exp\left(\frac{3\pi i}{2}\right)$

   $= \cos\frac{3\pi}{2} + i\sin(3\frac{\pi}{2})$

   $= 0 + i(-1)$

   $= -i$

   (c) $\exp(4\pi)$

   $= \exp(4\pi)$ because no i

3. Write the complex conjugate of:

   (a) $(1-2i)^{\frac{3}{2}} + \exp(3+4i)$

   Conj: $(1+2i)^{\frac{3}{2}} + \exp(3-4i)$

   (b) $\frac{1}{8\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\frac{r}{a_0}\exp\left(-\frac{r}{2a_0}\right)\sin\theta\exp(i\phi)$

   Conj: $\frac{1}{8\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}}\frac{r}{a_0}\exp\left(-\frac{r}{2a_0}\right)\sin\theta\exp(-i\phi)$

4. Find as many roots as possible for the following equation: $x^5 = 7$

   Roots: $\sqrt[5]{7}, \sqrt[5]{7}e^{i2\pi/5}, \sqrt[5]{7}e^{i4\pi/5}, \sqrt[5]{7}e^{i6\pi/5}, \sqrt[5]{7}e^{i8\pi/5}$

5. Calculate the following derivatives:

   (a) $y(x) = x^x$, $\frac{\mathrm{d}y}{\mathrm{d}x} =?$

   $y(x) = e^{x\ln(x)}$; exponent rule: $a^b = e^{b\ln(a)}$

   $u = x\ln(x)$

   $\mathrm{d}u = (x \cdot \frac{1}{x}) + (1 \cdot \ln x)$

   $\mathrm{d}u = \ln(x) + 1$

   $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{d}u \cdot e^u$

   $\frac{\mathrm{d}y}{\mathrm{d}x} = (\ln x + 1)e^{x\ln x}$

   $\frac{\mathrm{d}y}{\mathrm{d}x} = (\ln x + 1)x^x$

   (b) $y(x) = \frac{\sin(ax)}{\cos(ax)+2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} =?$

   quotient rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

   $\frac{dy}{dx} = \frac{a\cos(ax)\cos((ax)+2)-(-a\sin(ax)\sin(ax))}{(\cos(ax)+2)^2}$

   $\frac{dy}{dx} = \frac{a\cos(ax)\cos((ax)+2)+a\sin^2(ax)}{(\cos(ax)+2)^2}$

   (c) $y(x,z) = x^z + z^x$, $\left(\frac{\partial y}{\partial x}\right)_z =?$

   treat z as constant; exponent rule: $a^b = e^{b\ln(a)}$

   $y(x,z) = x^z + e^{xln(z)}$

   $\left(\frac{\partial y}{\partial x}\right)_z = zx^{z-1} + \ln(z)e^{x\ln(z)}$

   $\left(\frac{\partial y}{\partial x}\right)_z = zx^{z-1} + \ln(z)z^x$

6. Calculate the following integrals:

   (a) $\int x\exp(x)dx$

   integration by parts: $\int u\ dv = uv - \int v\ dx$

   $u = x \quad du = 1 \quad v = \exp(x) \quad dv = \exp(x)$

   $= x\exp(x) - \exp(x) + C$

(b) $\int_{-3}^{3} x \exp\left(-\frac{x^2}{2\pi}\right) dx$

$u = \frac{-x^2}{2\pi}$

$du = -\frac{x}{\pi} dx$

$-\pi du = x dx$

$\int_{-3}^{3} x \exp\left(-\frac{x^2}{2\pi}\right) dx = -\pi \exp\left(-\frac{x^2}{2\pi}\right) |_{-3}^{3}$

$= -\pi \exp\left(-\frac{3^2}{2\pi}\right) - \left(-\pi \exp\left(-\frac{(-3)^2}{2\pi}\right)\right)$

$= 0$

7. What is the dimension of the vector spaces spanned by the following vectors? In each case, propose a minimal orthonormal set of vectors that could span these spaces:

using dot product

(a) $\begin{pmatrix} -14 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -1.5 \end{pmatrix}$ one dimension; first 2x second $\begin{pmatrix} \frac{-14\sqrt{205}}{205} \\ \frac{3\sqrt{205}}{205} \end{pmatrix}$

(b) $\begin{pmatrix} -14 \\ 3 \end{pmatrix}, \begin{pmatrix} -7 \\ -1.5 \end{pmatrix}$ two dimension;linear independent $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} -14 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 1.5 \end{pmatrix}$ two dimension;linear independent $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

8. Calculate the determinant of the following matrices

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 \cdot \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \cdot \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \cdot \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

$= 1 \cdot (45 - 48) - 2 \cdot (36 - 42) + 3 \cdot (32 - 35)$

$= -3 + 12 - 9$

$= 0$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{pmatrix} = 1 \cdot \begin{pmatrix} 10 & 12 \\ 8 & 9 \end{pmatrix} - 2 \cdot \begin{pmatrix} 8 & 12 \\ 7 & 9 \end{pmatrix} + 3 \cdot \begin{pmatrix} 8 & 10 \\ 7 & 8 \end{pmatrix}$

$= 1 \cdot (90 - 96) - 2 \cdot (72 - 84) + 3 \cdot (64 - 70)$

$= -6 + 24 - 18$

$= 0$

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix} = 1 \cdot \begin{pmatrix} 8 & 9 \\ 5 & 6 \end{pmatrix} - 2 \cdot \begin{pmatrix} 7 & 9 \\ 4 & 6 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 & 8 \\ 4 & 5 \end{pmatrix}$

$= 1 \cdot (48 - 45) - 2 \cdot (42 - 36) + 3 \cdot (35 - 32)$

$= 3 - 12 + 9$

$= 0$

9. The function $g(x, y)$ is defined over the circumference of a circle of radius 2 centered in the origin. Calculate the maximum of $g(x, y)$ if:

   (a) $g(x, y) = 2x + 2y$

   Use the Lagrangian multipliers, $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, where you want to maximaze your value $f(x)$ in a constraint of $g(x)$. In this case, the constraint is defined to be $x^2 + y^2 = 4$, which is the function that encompasses the circumference of a circle with $r = 2$.

   Your three equations are:

   $2 = \lambda 2x;$ $\quad 2 = 2y;$ $\quad x^2 + y^2 = 4$

   $\frac{1}{\lambda} = x;$ $\quad \frac{1}{\lambda} = y;$ $\quad x^2 + y^2 = 4$

   $\frac{1}{\lambda}^2 + \frac{1}{\lambda}^2 = 4$

   $\lambda = \pm \frac{1}{\sqrt{2}}$

   plug λs into $x^2 + y^2 = 4$, the max coordinate is $(\sqrt{2}, \sqrt{2})$

   g(x,y) max is $4\sqrt{2}$

   (b) $g(x, y) = x^2 + y^2$

   Your three equations are:

   $2x = \lambda 2x;$ $\quad 2y = 2y;$ $\quad x^2 + y^2 = 4$

   $2x - \lambda 2x = 0;$ $\quad 2y - \lambda 2y = 0;$ $\quad x^2 + y^2 = 4$

$2x(1-\lambda) = 0;$ $\quad$ $2y(1-\lambda) = 0;$ $\quad$ $x^2 + y^2 = 4$

$x = 0$ *or* $\lambda = 1$ $\quad$ $y = 0$ *or* $\lambda = 1$

plug x and y into $x^2 + y^2 = 4$ to get coordinate, cannot do anything about λ because it is not in terms of x or y.

g(x,y) max is 4