Homework 4 Key

1. *Proof.* $-i\frac{\partial}{\partial x}$ is Hermitian

   $\langle\phi|\hat{p}\psi\rangle = \langle\psi|\hat{p}\phi\rangle^*$

   LHS: $\int \phi^*(-i\hbar\frac{\partial\psi}{\partial x})dx$

   RHS: $\left(\int \psi^*(-i\hbar\frac{\partial\phi}{\partial x})dx\right)^*$

   $\int \psi(i\hbar\frac{\partial\phi^*}{\partial x})dx$

   $i\hbar\int \psi(\frac{\partial\phi^*}{\partial x})dx$

   Using integration by part: $uv - \int uv$

   $\langle\psi|\hat{p}\phi\rangle^* = \phi^*\psi|_{-\infty}^{\infty} - i\hbar\int \phi^*(\frac{\partial\psi}{\partial x})dx$

   Because it says wavefunction tend to 0 when x approaches infinity because for wave functions to be normalized, they must approach to 0 when x approaches infinity

   $\langle\psi|\hat{p}\phi\rangle^* = -i\hbar\int \phi^*(\frac{\partial\psi}{\partial x})dx = \int \phi^*(-i\hbar\frac{\partial\psi}{\partial x})dx$

   $\therefore LHS = RHS \quad \square$

2. If $[x, p_x] = i\hbar$, calculate:

   (a) $[x + ip_x, x + ip_x]$

   $= (x + ip_x)(x + ip_x)\psi - (x + ip_x)(x + ipx)\psi$

   $= \psi(x^2 + 2ip_x + ip_x^2) - \psi(x^2 + 2ip_x + ip_x^2)$

   $= 0$

   (b) $[x - p_x, x + p_x]$

   $= (x - p_x)(x + p_x)\psi - (x + p_x)(x - px)\psi$

   $= \psi(x^2 + xp_x - p_x x - p_x^2) - \psi(x^2 + p_x x - xp_x - p_x^2)$

   $= 2xp_x - 2px_x$

   $= 2(xp_x - p_x x)$

   $= 2i\hbar$

   (c) $[x^2, p_x]$

$= x[x, p_x] + [x, p_x]x$

$= x(-i\hbar) + (-i\hbar)x$

$= 2i\hbar$

3. (a) *Proof.* $[x, p_x^n] = ihnp_x^{n-1}$

$= p_x^{n-1}[x, p_x] + [x, p_x^{n-1}]p_x$

$= p_x^{n-1}[x, p_x] + [x, p_x^{n-2}p_x]p_x$ rewriting

$= p_x^{n-1}[x, p_x] + [x, p_x]p_x^{n-2}p_x + p_x * p_x[x, p_x^{n-2}]$ doing operator identity

$= p_x^{n-1}[x, p_x] + [x, p_x]p_x^{n-1} + p_x^2[x, p_x^{n-3}p_x]$ rewriting

$= p_x^{n-1}[x, p_x] + [x, p_x]p_x^{n-1} + p_x^2 p_x^{n-3}[x, p_x] + p_x^2 p_x[x, p_x^{n-3}]$ identity

$= p_x^{n-1}[x, p_x] + [x, p_x]p_x^{n-1} + p_x^{n-1}[x, p_x] + p_x^3[x, p_x^{n-3}]$ rewriting

This will carry on until $n - n_1 = 0$

$= (-i\hbar)p_x^{n-1} - i\hbar(n-1)p_x^{n-1}$

$= -i\hbar np_x^{n-1}$

□

(b) *Proof.* $[x^n, p_x] = ihnx^{n-1}$

$= x^{n-1}[x, p_x] + [x^{n-1}, p_x]x$

$= x^{n-1}[x, p_x] + [x^{n-2}x, p_x]x$ rewriting

$= x^{n-1}[x, p_x] + [x, p_x]x^{n-2}x + x * x[x^{n-2}, p_x]$ doing operator identity

$= x^{n-1}[x, p_x] + [x, p_x]x^{n-1} + x^2[x^{n-3}x, p_x]$ rewriting

$= x^{n-1}[x, p_x] + [x, p_x]x^{n-1} + x^2 x^{n-3}[x, p_x] + x^2 x[x^{n-3}, p_x]$ identity

$= x^{n-1}[x, p_x] + [x, p_x]x^{n-1} + x^{n-1}[x, p_x] + x^3[x^{n-3}, p_x]$ rewriting

This will carry on until $n - n_1 = 0$

$= (-i\hbar)x^{n-1} - i\hbar(n-1)x^{n-1}$

$= -i\hbar nx^{n-1}$

□

(c) *Proof.* $[x, F(p_x)] = -\frac{\hbar}{i}\frac{\partial}{\partial p_x}(F(p_x))$

turn position operator in momentum operator $\hat{x} = i\hbar\frac{\partial}{\partial p_x}$

$= i\hbar(\frac{\partial}{\partial p_x}F(p)\psi) - F(p)\frac{\partial}{\partial p_x}\psi$
$= i\hbar(\frac{\partial F(p)}{\partial p_x}\psi + F(p)\frac{\partial \psi}{\partial p_x} - F(p)\frac{\partial \psi}{\partial p_x})$
$= i\hbar\frac{\partial F(p)}{\partial p_x}$

$\square$

4. *Proof.* $Tr(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$ if $Tr(A) = \sum_m \langle m|A|m\rangle$

   $Tr(A|\psi\rangle\langle\psi|) = \sum_m \langle m|A|\psi\rangle\langle\psi|m\rangle$

   $Tr(A|\psi\rangle\langle\psi|) = \sum_m \langle\psi|A|m\rangle\langle m|\psi\rangle$ identity operator

   $Tr(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$

   $\square$

5. *Proof.* $|\psi\rangle$ will be a solution if and only if $\langle\psi|(\hat{H} - E)^2|\psi\rangle = 0$

   $\Rightarrow$

   $\langle\psi|(\hat{H}^2 - 2\hat{H}E + E^2)|\psi\rangle = 0$

   $\hat{H}^2\langle\psi|\psi\rangle - 2\hat{H}E\langle\psi|\psi\rangle + E^2\langle\psi|\psi\rangle = 0$

   $\hat{H}^2 - 2\hat{H}E + E^2 = 0$ because $\hat{H}\psi = E\psi$

   $\Leftarrow$

   $if\langle\psi|(\hat{H} - E)^2|\psi\rangle = 0$ is true then

   $\langle\psi|(\hat{H} - E)(\hat{H} - E)|\psi\rangle = 0$ since hermitian, left half = right half

   $(\hat{H} - E)|\psi\rangle = 0$

   $\hat{H}|\psi\rangle = E\psi\rangle$

   $\square$