NUMERICAL COMPUTATIONS FOR AN EFFECTIVE MODEL OF TWISTED BILAYER GRAPHENE

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1. STANDARD MONOLAYER

We choose for the microscopic lattice, the orientation

$$a_1 = a \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad a_2 = a \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$a_1^* = \frac{2\pi}{a} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} = \frac{4\pi}{a\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \qquad a_2^* = \frac{2\pi}{a} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -1 \end{pmatrix} = \frac{4\pi}{a\sqrt{3}} \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \quad (1)$$

and for the Macroscopic lattice, we choose the orientation

$$b_1 = b \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \qquad b_2 = b \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad b_1^* = \frac{4\pi}{b\sqrt{3}} \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad b_2^* = \frac{4\pi}{b\sqrt{3}} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$(2)$$

so $-Jb_j^* = \frac{a}{b}a_j^*$ and $Jb_j = \frac{b}{a}a_j$ and

$$\mathcal{M}_b := \begin{pmatrix} b_1 & b_2 \end{pmatrix} = \frac{b}{2} \begin{pmatrix} -1 & 1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix}$$

In reduced coordinates, with

$\mathcal{M} : \mathbb{T}^2 \simeq [0,1]^2 \to \Omega$,

$$\mathcal{M} := \frac{a}{2} \begin{pmatrix} \sqrt{3} & \sqrt{3} \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \end{pmatrix}, \qquad \mathcal{M}^{-1} = \frac{1}{a} \begin{pmatrix} \frac{1}{\sqrt{3}} & 1 \\ \frac{1}{\sqrt{3}} & -1 \end{pmatrix}$$

and

$$2\pi \left(\mathcal{M}^{-1} \right)^* = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} = \frac{2\pi}{a} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & -1 \end{pmatrix} =: S$$

1.1. Dirac point. We have

$$K = \frac{-a_1^* + a_2^*}{3}, \qquad a_1^* \cdot a_2^* = -\frac{|a_j^*|^2}{2}, \qquad |K| = \frac{|a_j^*|}{\sqrt{3}}$$

1.2. From q to m_q.
Suppose you know q in cartesian coordinates and you want to compute m^q, its reduced coordinates, that is $m^q a = q$, then since $m^q a = (a_1^* a_2^*) \begin{pmatrix} m_1^q \\ m_2^q \end{pmatrix} = 2\pi \left(\mathcal{M}^{-1} \right)^* \begin{pmatrix} m_1^q \\ m_2^q \end{pmatrix}$,

$$\begin{pmatrix} m_1^q \\ m_2^q \end{pmatrix} = \frac{1}{2\pi} \mathcal{M}^* q \qquad (3)$$

1.3. **Fourier conventions.** We will manipulate functions which are Ω-periodic in $\mathbf{x}$, but not in z, our Fourier transform conventions will be

$$(\mathcal{F}f)_m (k_z) := \frac{1}{2\pi |\Omega|} \int_{\Omega \times \mathbb{R}} e^{-i(ma^*\mathbf{x}+k_z z)} f(\mathbf{x}, z) \mathrm{d}\mathbf{x}\mathrm{d}z$$

hence any function can be decomposed as

$$f(\mathbf{x}, z) = \sum_{m \in \mathbb{Z}^d} \int_{\mathbb{R}} e^{i(ma^*\mathbf{x}+k_z z)} f_{\mathbf{G}}(k_z) \mathrm{d}k_z$$

We also recall that $\int_{\mathbb{R}} e^{ipz} \mathrm{d}z = 2\pi\delta(p)$.

Now we consider that f and g are L-periodic in z, and $\int_{\mathbb{R}} \mathrm{d}z \simeq \int_{[0,L]} \mathrm{d}z$ so the Fourier transform is

$$\boxed{(\mathcal{F}f)_{m,m_z} := \frac{1}{\Gamma} \int_{\Omega \times [0,L]} e^{-i\left(ma^*\mathbf{x}+m_z \frac{2\pi}{L} z\right)} f(\mathbf{x}, z) \mathrm{d}\mathbf{x}\mathrm{d}z}$$

where $\Gamma := \sqrt{L |\Omega|}$ and the reconstruction formula is

$$\boxed{f(\mathbf{x}, z) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2 \\ m_z\in\mathbb{Z}}} \frac{e^{i\left(\mathbf{m}a^*\cdot\mathbf{x}+m_z \frac{2\pi}{L} z\right)}}{\Gamma} \widehat{f}_{\mathbf{m},m_z}} \tag{4}$$

We define the scalar product

$$\langle f, g \rangle := \int_{\Omega \times [0,L]} \overline{f} g$$

and compute Plancherel's formula

$$\langle f, g \rangle = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2 \\ m_z\in\mathbb{Z}}} \overline{\widehat{f}_{\mathbf{m},m_z}} \widehat{g}_{\mathbf{m},m_z}. \tag{5}$$

Hence, as a verification, we test that the normalization of the $\widehat{u}_j$'s is the right one by checking that $\|u_j\|_{L^2}^2 = 1$ via (5).

We implement the Fourier transforms

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myfft(a,B) = fft(a)*sqrt(B)/length(a)
myifft(a) = ifft(a)*length(a)/sqrt(B)
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where $B = \Gamma^2 = L |\Omega|$ in 3d, $B = L$ in 1d in z, and $B = |\Omega|$ in 2d in (x, y). If $a_i = f(x_i)$ are the actual values of the functions, then $myfft(a)[m] \simeq (\mathcal{F}f)_{m-1}$ up to Riemann series errors.

1.4. **Rotation action.** We know that $R_{\frac{2\pi}{3}} (ma^*) = \left(R_{\frac{2\pi}{3}}^{\mathrm{red}} m\right) a^*$ where

$$R_{\frac{2\pi}{3}}^{\mathrm{red}} = S^{-1} R_{\frac{2\pi}{3}} S = \mathcal{M}^* R_{\frac{2\pi}{3}} (\mathcal{M}^*)^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad R_{-\frac{2\pi}{3}}^{\mathrm{red}} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

and

$$\mathcal{R}_{\frac{2\pi}{3}} f(x) = \sum_m f_m e^{i\left(R_{\frac{2\pi}{3}}^{\mathrm{red}} m\right) a^*\cdot x} = \sum_m f_{R_{-\frac{2\pi}{3}}^{\mathrm{red}} m} e^{ima^*\cdot x}$$

Similarly, $R_{\frac{\pi}{2}}\left(ma^*\right) = \left(R_{\frac{\pi}{2}}^{\mathrm{red}}m\right)a^*$ where

$$R_{\frac{\pi}{2}}^{\mathrm{red}} = S^{-1}R_{\frac{\pi}{2}}S = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}, \qquad R_{-\frac{\pi}{2}}^{\mathrm{red}} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix} =: \frac{1}{\sqrt{3}}M$$

and

$$\mathcal{R}_{\frac{\pi}{2}}f(x) = \sum_m f_m e^{i\left(R_{\frac{\pi}{2}}^{\mathrm{red}}m\right)a^*\cdot x} = \sum_m f_{Mm}e^{i\frac{1}{\sqrt{3}}ma^*\cdot x} = \mathcal{L}f\left(\frac{x}{\sqrt{3}}\right)$$

where $\mathcal{L}$ is the action of M on the Fourier coefficients of f.

1.5. **Action of mirror.** We define $M := \mathrm{diag}\ (-1,1,-1)$, we have

$$\mathbb{M}u(x) := u(Mx)$$

With the lattice a defined in (1), we obtain

2. COMPARISION WITH EXISTING RESULTS

From [2], we verified that with $T = 0$, we have Fig 3(a), with the right energies

2.1. **Reduction of Fourier coefficients in** $2d$ **to** $1d$**.** This is used to compute V_{int}. We take a function f and define its average

$$g(z) := \frac{1}{|\Omega|}\int_\Omega f$$

and since

$$\widehat{f}_{0,m_z} = \frac{1}{\sqrt{L\,|\Omega|}}\int_\Omega f(x,z)e^{-i\frac{2\pi}{L}m_z z}\mathrm{d}x\mathrm{d}z$$

then

$$\widehat{g}_{m_z} = \frac{1}{|\Omega|\,\sqrt{L}}\int_{\Omega\times[0,L]} f(x,z)e^{-i\frac{2\pi}{L}m_z z}\mathrm{d}x\mathrm{d}z = \frac{\widehat{f}_{0,m_z}}{\sqrt{|\Omega|}}$$

3. COMPUTATION OF V_{int}

For $\mathbf{s} \in \Omega := [0,1]\mathbf{a}_1 + [0,1]\mathbf{a}_2$, we denote by $V_{\mathbf{s}}^{(2)}$ the true Kohn-Sham mean-field potential for the configuration where the two sheets are aligned (no angle), but with the upper one shifted by a vector $\mathbf{s}$. We set

$$\begin{aligned}
V_{\mathrm{int},\mathbf{s}}(z) &:= \frac{1}{|\Omega|}\int_\Omega \left(V_{\mathbf{s}}^{(2)}(\mathbf{x},z) - V(\mathbf{x},z+\tfrac{d}{2}) - V(\mathbf{x}-\mathbf{s},z-\tfrac{d}{2})\right)\mathrm{d}\mathbf{x} \\
&= \frac{1}{|\Omega|}\int_\Omega \left(V_{\mathbf{s}}^{(2)}(\mathbf{x},z) - V(\mathbf{x},z+\tfrac{d}{2}) - V(\mathbf{x},z-\tfrac{d}{2})\right)\mathrm{d}\mathbf{x} \\
&= \frac{1}{|\Omega|^{\frac{3}{2}}}\sum_{\substack{\mathbf{m}\in\mathbb{Z}^2 \\ m_z\in\mathbb{Z}}} \left(\left(\widehat{V_{\mathbf{s}}^{(2)}}\right)_{\mathbf{m},m_z} - \widehat{V}_{\mathbf{m},m_z}e^{im_z\frac{2\pi}{L}\frac{d}{2}} - \widehat{V}_{\mathbf{m},m_z}e^{-im_z\frac{2\pi}{L}\frac{d}{2}}\right) \\
&\qquad\qquad \times \int_\Omega e^{i\left(\mathbf{m}a^*\cdot\mathbf{x}+m_z\frac{2\pi}{L}z\right)}\mathrm{d}\mathbf{x} \\
&= \frac{1}{\sqrt{|\Omega|}}\sum_{m_z\in\mathbb{Z}} e^{im_z\frac{2\pi}{L}z}\left(\left(\widehat{V_{\mathbf{s}}^{(2)}}\right)_{0,m_z} - 2\widehat{V}_{0,m_z}\cos\left(m_z\frac{\pi d}{L}\right)\right)
\end{aligned}$$

and we obtain the Fourier coefficients

$$\left(\widehat{V_{\text{int},\mathbf{s}}}\right)_{m_z} = \frac{1}{\sqrt{|\Omega|}} \left(\left(\widehat{V_{\mathbf{s}}^{(2)}}\right)_{0,m_z} - 2\widehat{V}_{0,m_z} \cos\left(m_z \frac{\pi d}{L}\right) \right)$$

We then compute

$$V_{\text{int}}(z) := \frac{1}{|\Omega|} \int_{\Omega} V_{\text{int},\mathbf{s}}(z) \mathrm{d}\mathbf{s} = \frac{1}{N_{\text{int}}^2} \sum_{s_x, s_y \in [\![1, N_{\text{int}}]\!]} V_{\text{int},(\text{s}_\text{x},\text{s}_\text{y})}^{\text{array}}(z)$$

and finally obtain the Fourier coefficients

$$\boxed{\left(\widehat{V_{\text{int}}}\right)_{m_z} = \frac{1}{N_{\text{int}}^2} \sum_{s_x, s_y \in [\![1, N_{\text{int}}]\!]} \left(\widehat{V_{\text{int},\mathbf{s}}}\right)_{m_z}}$$

and we expect $V_{\text{int},\mathbf{s}}$ not to depend too much on $\mathbf{s}$, that is we expect that

$$\delta_{V_{\text{int}}} := \frac{\int_{\Omega \times \mathbb{R}} |V_{\text{int},\mathbf{s}}(z) - V_{\text{int}}(z)|^2 \, \mathrm{d}\mathbf{s}\mathrm{d}z}{|\Omega| \int_{\mathbb{R}} V_{\text{int}}(z)^2 \mathrm{d}z}$$

$$= \frac{\sum_{m_z} \int_{\Omega} \left| \left(\widehat{V_{\text{int},\mathbf{s}}}\right)_{m_z} - \left(\widehat{V_{\text{int}}}\right)_{m_z} \right|^2 \mathrm{d}\mathbf{s}}{|\Omega| \sum_{m_z} \left(\widehat{V_{\text{int}}}\right)_{m_z}^2}$$

$$= \frac{\sum_{s_x, s_y, m_z} \left| \left(\widehat{V_{\text{int},(\text{s}_\text{x},\text{s}_\text{y})}}\right)_{m_z} - \left(\widehat{V_{\text{int}}}\right)_{m_z} \right|^2}{N_{\text{int}}^2 \sum_{m_z} \left(\widehat{V_{\text{int}}}\right)_{m_z}^2}$$

is small. We also verify that $V_{\text{int}}(-z) = V_{\text{int}}(z)$.

4. Effective potentials

We defined

$$((f, g))^{\eta, \eta'}(\mathbf{X}) := \int_{\Omega \times \mathbb{R}} \overline{f}\left(x - \eta J\mathbf{X}, z - \eta \frac{d}{2}\right) g\left(x - \eta' J\mathbf{X}, z - \eta' \frac{d}{2}\right) \mathrm{d}\mathbf{x}\mathrm{d}z$$

and

$$\langle\!\langle f, g \rangle\!\rangle^{\eta, \eta'}(\mathbf{X})$$
$$:= e^{i(\eta - \eta')\mathbf{K} \cdot J\mathbf{X}} \int_{\Omega \times \mathbb{R}} \overline{f}\left(x - \eta J\mathbf{X}, z - \eta \frac{d}{2}\right) g\left(x - \eta' J\mathbf{X}, z - \eta' \frac{d}{2}\right) \mathrm{d}\mathbf{x}\mathrm{d}z$$

so $\langle\!\langle f, g \rangle\!\rangle^{\eta, \eta'} = e^{i(\eta - \eta')\mathbf{K} \cdot J\mathbf{X}} ((f, g))^{\eta, \eta'}$. Now we make the approximation

$$\int_{\Omega \times \mathbb{R}} \simeq \int_{\Omega \times [0, L]}$$

The situation is drawn on Figure **??**. The functions are defined on $[-L/2, L/2]$ but we need to integrate on the common segment, which is $[-\frac{L-d}{2}, \frac{L-d}{2}]$, so on $[-L/2, L/2]$ to recover the initial domain.

Firstly, using the Fourier decomposition (4),

$$\left(\!\left(f,g \right)\!\right)^{\eta,\eta'} = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{i(\eta-\eta')ma^*\cdot J\mathbf{X}} \sum_{m_z\in\mathbb{Z}} e^{i(\eta-\eta')\frac{2\pi}{L}m_z\frac{d}{2}} \overline{\widehat{f}_{m,m_z}} \widehat{g}_{m,m_z}$$

$$= \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{i(\eta-\eta')ma^*\cdot J\mathbf{X}} C_{\mathbf{m}}$$

where

$$\boxed{C_{\mathbf{m}} := \sum_{m_z\in\mathbb{Z}} e^{i(\eta-\eta')\frac{d\pi}{L}m_z} \overline{\widehat{f}_{m,m_z}} \widehat{g}_{m,m_z}}.$$

Remark that for any $\eta,\eta' \in \{\pm\}$, $\left(\!\left(f,g \right)\!\right)^{\eta,\eta'}$ is $2J\mathbb{L}$-periodic. We also define

$$C_{\mathbf{m}}^{\eta} := \sum_{m_z\in\mathbb{Z}} e^{\eta i2\frac{d\pi}{L}m_z} \overline{\widehat{f}_{m,m_z}} \widehat{g}_{m,m_z}$$

Then,

$$\left\langle\!\left\langle f,g \right\rangle\!\right\rangle^{\eta,\eta'} = e^{i(\eta-\eta')\mathbf{K}\cdot J\mathbf{X}} \left(\!\left(f,g \right)\!\right)^{\eta,\eta'} = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{i(\eta-\eta')(m+m_K)a^*\cdot J\mathbf{X}} C_{\mathbf{m}}$$

Hence

$$\boxed{\left(\!\left(f,g \right)\!\right)^{+-} \left(-\tfrac{3}{2}J\mathbf{X}\right) = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{i3ma^*\cdot\mathbf{X}} C_{\mathbf{m}}^{+} = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{ima^*\cdot\mathbf{X}} C_{\frac{\mathbf{m}}{3}}^{+},}$$

and

$$\boxed{\left\langle\!\left\langle f,g \right\rangle\!\right\rangle^{+-} \left(-\tfrac{3}{2}J\mathbf{X}\right) = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{i3(m+m_k)a^*\cdot\mathbf{X}} C_{\mathbf{m}}^{+} = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{ima^*\cdot\mathbf{X}} C_{\frac{\mathbf{m}-3\mathbf{m}_K}{3}}^{+}}$$

where $C_{\frac{\mathbf{m}}{n}} := 0$ if n does not divide m_1 and m_2. Numerically, there is no loss of information since all C_m's are taken into account if the "ecut" is large enough.

Similarly

$$\left(\!\left(f,g \right)\!\right)^{-+} \left(-\tfrac{3}{2}J\mathbf{X}\right) = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{-i3ma^*\cdot\mathbf{X}} C_{\mathbf{m}}^{-} = \sum_{\mathbf{m}\in\mathbb{Z}^2} e^{ima^*\cdot\mathbf{X}} C_{-\frac{\mathbf{m}}{3}}^{-},$$

For the potentials, we finally need to implement

$$\mathbb{W}_{j,j'}^{+} = \left(\!\left(\overline{u}_j u_{j'}, V \right)\!\right)^{+-}, \qquad \mathbb{W}_{j,j'}^{-} = \left(\!\left(\overline{u}_j u_{j'}, V \right)\!\right)^{-+},$$

$$\mathbb{V}_{j,j'} = \left\langle\!\left\langle (V+V_{\mathrm{int}})\, u_j, u_{j'} \right\rangle\!\right\rangle^{+-}$$

If $f(z) = \varepsilon f(-z)$, then $\widehat{f}_{-m_z} = \varepsilon \widehat{f}_{m_z}$, from this we see that $\overline{C_{\mathbf{m}}^{u_{j'},u_j}} = C_{\mathbf{m}}^{u_j,u_{j'}}$ and hence $\mathbb{V}(-X)^* = \mathbb{V}(X)$

4.1. Magnetic term.

As for the magnetic term, we have

$$(-i\nabla_{\mathbf{x}} + \mathbf{K})\, g = \sum_{\mathbf{m},m_z} (\mathbf{m}+\mathbf{m}_K)\, \mathbf{a}^* e^{i\left(\mathbf{m}a^*\cdot\mathbf{x}+m_z\frac{2\pi}{L}z\right)} \widehat{f}_{\mathbf{m},m_z}$$

so

$$\left\langle\!\left\langle f, (-i\nabla_{\mathbf{x}} + \mathbf{K})g \right\rangle\!\right\rangle^{+-}(\mathbf{X}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} (\mathbf{m}+\mathbf{m}_K)\, \mathbf{a}^*\, C_{\mathbf{m}} e^{2i(\mathbf{m}+\mathbf{m}_K)\mathbf{a}^*\cdot J\mathbf{X}}$$

and

$$\langle\!\langle f, (-i\nabla_{\mathbf{x}} + \mathbf{K})g\rangle\!\rangle^{+-}\left(-\tfrac{3}{2}J\mathbf{X}\right) = \sum_{\mathbf{m}\in\mathbb{Z}^2}(\mathbf{m} + \mathbf{m}_K)\,\mathbf{a}^*\,C_{\mathbf{m}}e^{i3(\mathbf{m}+\mathbf{m}_K)\mathbf{a}^*\cdot\mathbf{X}}$$

so

$$\boxed{\langle\!\langle f, (-i\nabla_{\mathbf{x}} + \mathbf{K})g\rangle\!\rangle^{+-}\left(-\tfrac{3}{2}J\mathbf{X}\right) = \frac{1}{3}\sum_{\mathbf{m}\in\mathbb{Z}^2}\mathbf{ma}^*\,C_{\frac{\mathbf{m}-3\mathbf{m}_K}{3}}e^{i\mathbf{ma}^*\cdot\mathbf{X}}}$$

so we can implement

$$\boldsymbol{\mathcal{A}}_{j,j'}\left(-\tfrac{3}{2}J\mathbf{X}\right) = \langle\!\langle u_j, (-i\nabla_{\mathbf{x}} + \mathbf{K})u_{j'}\rangle\!\rangle^{+-}\left(-\tfrac{3}{2}J\mathbf{X}\right)$$

4.2. $\mathbb{W}$'s $V_{\mathbf{int}}$ term. We write $V_{\mathrm{int}}(z) = \frac{1}{\sqrt{L}}\sum_{m_z\in\mathbb{Z}}\widehat{V}_{\mathrm{int}}^{m_z}e^{i\frac{2\pi}{L}m_z z}$ hence

$$\langle u_j, V_{\mathrm{int}}u_{j'}\rangle = \frac{1}{L^{\frac{3}{2}}}\sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ m_z,m_z',M_z\in\mathbb{Z}}}\left(\overline{\widehat{u}_j}\right)_{\mathbf{m},m_z}\left(\widehat{u}_{j'}\right)_{\mathbf{m},m_z'}\left(\widehat{V_{\mathrm{int}}}\right)_{M_z}\int_z e^{iz\frac{2\pi}{L}(M_z+m_z'-m_z)}$$

$$= \frac{1}{\sqrt{L}}\sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ m_z,m_z'\in\mathbb{Z}}}\left(\overline{\widehat{u}_j}\right)_{\mathbf{m},m_z}\left(\widehat{u}_{j'}\right)_{\mathbf{m},m_z'}\left(\widehat{V_{\mathrm{int}}}\right)_{m_z-m_z'}$$

and the matrix $M_{j,j'} := \langle u_j, V_{\mathrm{int}}u_{j'}\rangle$ is such that $M^* = M$ and $M_{11} = M_{22}$.

In the function $\mathbb{V}(X) = \langle u_j, Vu_i\rangle(X)$, when $V \to V + V_{\mathrm{int}}$, we have

$$\widetilde{\mathbb{V}}(X) = \langle u_j, (V + V_{\mathrm{int}})u_i\rangle(X) = \mathbb{V}(X) + \langle u_j, V_{\mathrm{int}}u_i\rangle$$

but at the level of Fourier coefficients,

$$\widehat{\widetilde{\mathbb{V}}}_0 = \widehat{\mathbb{V}}_0 + \frac{\langle u_j, V_{\mathrm{int}}u_i\rangle}{\sqrt{|\Omega|}}$$

so when we add it to the Fourier Hamiltonian, we should not forget to divide by $\sqrt{|\Omega|}$

5. BM CONFIGURATION

From [1], the BM Hamiltonian is

$$H = \begin{pmatrix} -i\sigma\nabla & T^c(x) \\ T^c(x)^* & -i\sigma\nabla \end{pmatrix},$$

where

$$\boxed{T_1 = \begin{pmatrix} w_0 & w_1 \\ w_1 & w_0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} w_0 & w_1 e^{-i\phi} \\ w_1 e^{i\phi} & w_0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} w_0 & w_1 e^{i\phi} \\ w_1 e^{-i\phi} & w_0 \end{pmatrix}}$$

and where, for $x \in \mathbb{R}^2$,

$$T^c(x) := \sum_{j=1}^3 T_j e^{-iq_j^c\cdot x} = \sum_{j=1}^3 T_j e^{iq_j a^*\cdot x}, \qquad \widehat{T}_p = \sum_{j=1}^3 T_j \delta_{p,q_j^c}$$

and

$$q_1^c = \frac{4\pi}{a\sqrt{3}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_1^* + a_2^*,$$

$$q_2^c = \frac{4\pi}{a\sqrt{3}}\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = -a_2^*, \qquad q_3^c = \frac{4\pi}{a\sqrt{3}}\begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = -a_1^*,$$

where we took rotated q_j^c's by J with respect to [1], and with a rescaling of $\frac{4\pi}{a\sqrt{3}}$.

We define the reduced dual vectors $q_j := -\mathcal{M}^* q_j^c / 2\pi$ so

$$T(x) = T^c(\mathcal{M}x) = \sum_{j=1}^{3} T_j e^{-ix \cdot \mathcal{M}^* q_j^c} = \sum_{j=1}^{3} T_j e^{i2\pi x \cdot q_j}$$

and we compute

$$\boxed{q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad q_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad q_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$$

Or

$$\boxed{T(x) = \sum_{j=1}^{3} T_j e^{iq_j a^* \cdot x}}$$

Since $T_j^* = T_j$, then $T(-x)^* = T(x)$

6. Operators in basis

6.1. **Goal.** Our goal is to study the eigenvalue equation

$$\boxed{\mathcal{H}\psi = \varepsilon_\theta \mathcal{S}E\psi}$$

remark that energies have to be rescaled by ε_θ ! The operator $\mathcal{S}$ is Hermitian and positive and

$$\boxed{\mathcal{H} := \frac{1}{\varepsilon_\theta}\mathcal{V} + c_\theta T + \varepsilon_\theta T^{(1)}}$$

where

$$T := v_{\mathrm{F}}\begin{pmatrix} \boldsymbol{\sigma} \cdot (-i\nabla) & \boldsymbol{\mathcal{A}} \cdot (-i\nabla) \\ \boldsymbol{\mathcal{A}}^* \cdot (-i\nabla) & \boldsymbol{\sigma} \cdot (-i\nabla) \end{pmatrix},$$

$$T^{(1)} := v_{\mathrm{F}}\begin{pmatrix} -\boldsymbol{\sigma} \cdot J(-i\nabla) - \frac{1}{2}\Delta & \boldsymbol{\mathcal{A}} \cdot J(-i\nabla) - \frac{1}{2}\Sigma\Delta \\ \boldsymbol{\mathcal{A}}^* \cdot J(-i\nabla) - \frac{1}{2}\Sigma^*\Delta & \boldsymbol{\sigma} \cdot J(-i\nabla) - \frac{1}{2}\Delta \end{pmatrix},$$

$$\mathcal{V} := \begin{pmatrix} \mathbb{W} & \mathbb{V} \\ \mathbb{V}^* & \mathbb{W} \end{pmatrix},$$

and their Bloch transform becomes

$$T_k := v_{\mathrm{F}}\begin{pmatrix} \boldsymbol{\sigma} \cdot (-i\nabla + k) & \boldsymbol{\mathcal{A}} \cdot (-i\nabla + k) \\ \boldsymbol{\mathcal{A}}^* \cdot (-i\nabla + k) & \boldsymbol{\sigma} \cdot (-i\nabla + k) \end{pmatrix},$$

$$T_k^{(1)} := v_{\mathrm{F}}\begin{pmatrix} -\boldsymbol{\sigma} \cdot J(-i\nabla + k) + \frac{1}{2}(-i\nabla + k)^2 & \boldsymbol{\mathcal{A}} \cdot J(-i\nabla + k) + \frac{1}{2}\Sigma(-i\nabla + k)^2 \\ \boldsymbol{\mathcal{A}}^* \cdot J(-i\nabla + k) + \frac{1}{2}\Sigma^*(-i\nabla + k)^2 & \boldsymbol{\sigma} \cdot J(-i\nabla + k) + \frac{1}{2}(-i\nabla + k)^2 \end{pmatrix}$$

and we want the middle of the spectrum of

$$\mathcal{H}_k := \mathcal{S}^{-\frac{1}{2}} \left(\frac{1}{\varepsilon_\theta} \mathcal{V} + c_\theta T_k + \varepsilon_\theta T_k^{(1)} \right) \mathcal{S}^{-\frac{1}{2}}$$

6.2. Basis. We define $e_m := \frac{1}{\sqrt{|\Omega|}} e^{ima^* \cdot x}$, and

$$e_{\alpha,m} := e_\alpha \otimes e_m = e_\alpha \frac{e^{ima^* \cdot x}}{\sqrt{|\Omega|}}, \qquad \text{where } e_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots$$

6.3. Multiplication-derivation operators. For $A = (A_1, A_2)$ and $A_j = \sum_\ell \left(\widehat{A_j} \right)_\ell e^{i\ell a^* \cdot x}$, we have

$$\langle e_n, A \cdot (-i\nabla + k)e_m \rangle = \sum_\ell \left(\widehat{A_1} \right)_\ell (ma^* + k)_1 \left\langle e_n, e^{i\ell a^* \cdot x} e_m \right\rangle$$

$$+ \left(\widehat{A_2} \right)_\ell (ma^* + k)_2 \left\langle e_n, e^{i\ell a^* \cdot x} e_m \right\rangle$$

$$= \left(\widehat{A_1} \right)_{n-m} (ma^* + k)_1 + \left(\widehat{A_2} \right)_{n-m} (ma^* + k)_2 = \widehat{A}_{n-m} \cdot (ma^* + k)$$

For $V = \sum_\ell \widehat{V}_\ell e^{i\ell a^* x}$, we have $\langle e_n, V e_m \rangle = \widehat{V}_{n-m}$ and

$$\left\langle e_n, V(-i\nabla + k)^2 e_m \right\rangle = (ma^* + k)^2 \widehat{V}_{n-m}$$

6.4. On-diagonal potential. For a general $W^\pm = \sum_m W_m^\pm e^{ima^* \cdot x}$, we have

$$\left\langle e_{\alpha,n}, \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix} e_{\beta,m} \right\rangle = \delta_{\alpha\in\{1,2\}}^{\beta\in\{1,2\}} \left(W_{n-m}^+ \right)_{\alpha_1\beta_1} + \delta_{\alpha\in\{3,4\}}^{\beta\in\{3,4\}} \left(W_{n-m}^- \right)_{\alpha_2\beta_2}$$

6.5. Off-diagonal potential. For a general $V = \sum_m V_m e^{ima^* \cdot x}$, we have $V^* = \sum_m V_m^* e^{-ima^* \cdot x}$ and

$$M_{IJ} := \left\langle e_{\alpha,n}, \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} e_{\beta,m} \right\rangle$$

$$= \sum_k \left(\delta_{\alpha\in\{1,2\}}^{\beta\in\{3,4\}} \delta_{m+k-n} \langle e_{\alpha_1}, V_k e_{\beta_2} \rangle + \delta_{\alpha\in\{3,4\}}^{\beta\in\{1,2\}} \delta_{m-k-n} \langle e_{\alpha_2}, V_k^* e_{\beta_1} \rangle \right)$$

$$= \delta_{\alpha\in\{1,2\}}^{\beta\in\{3,4\}} \langle e_{\alpha_1}, V_{n-m} e_{\beta_2} \rangle + \delta_{\alpha\in\{3,4\}}^{\beta\in\{1,2\}} \langle e_{\alpha_2}, V_{m-n}^* e_{\beta_1} \rangle$$

$$= \delta_{\alpha\in\{1,2\}}^{\beta\in\{3,4\}} \left(V_{n-m} \right)_{\alpha_1\beta_2} + \delta_{\alpha\in\{3,4\}}^{\beta\in\{1,2\}} \overline{\left(V_{m-n} \right)_{\beta_1\alpha_2}}$$

and M is also Hermitian.

6.6. Off-diagonal magnetic term.

For a general $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, $A_j = \sum_\ell (A_j)_\ell \, e^{i\ell a^* \cdot x}$, we have $A_j^* = \sum_\ell (A_j)_\ell^* \, e^{-i\ell a^* \cdot x}$ and we compute

$$
\left\langle e_{\alpha,n}, \begin{pmatrix} 0 & A \cdot (-i\nabla + k) \\ A^* \cdot (-i\nabla + k) & 0 \end{pmatrix} e_{\beta,m} \right\rangle
$$
$$
= \delta_{\alpha \in \{1,2\}}^{\beta \in \{3,4\}} \left((ma^* + k)_1 \left((A_1)_{n-m}\right)_{\alpha_1 \beta_2} + (ma^* + k)_2 \left((A_2)_{n-m}\right)_{\alpha_1 \beta_2} \right)
$$
$$
+ \delta_{\alpha \in \{3,4\}}^{\beta \in \{1,2\}} \left((ma^* + k)_1 \overline{\left((A_1)_{m-n}\right)_{\beta_1 \alpha_2}} + (ma^* + k)_2 \overline{\left((A_2)_{m-n}\right)_{\beta_1 \alpha_2}} \right)
$$

6.7. Dirac operator.

We have

$$
\sigma \cdot (-i\nabla + k) = \sigma_1 \left(-i\partial_1 + k_1\right) + \sigma_2 \left(-i\partial_2 + k_2\right)
$$
$$
= \begin{pmatrix} 0 & -i\left(\partial_1 - i\partial_2\right) + \overline{k_\mathbb{C}} \\ -i\left(\partial_1 + i\partial_2\right) + k_\mathbb{C} & 0 \end{pmatrix}
$$

where

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
$$

so, with $k_\mathbb{C} := k_1 + ik_2$,

$$
\sigma \cdot (-i\nabla + k) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e_m = (ma^* + k)_\mathbb{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_m
$$
$$
\sigma \cdot (-i\nabla + k) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_m = \overline{(ma^* + k)_\mathbb{C}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e_m
$$

Then

$$
\begin{pmatrix} \sigma \cdot (-i\nabla + k) & 0 \\ 0 & \sigma \cdot (-i\nabla + k) \end{pmatrix} e_{1,m} = (ma^* + k)_\mathbb{C} \, e_{2,m}
$$
$$
\begin{pmatrix} \sigma \cdot (-i\nabla + k) & 0 \\ 0 & \sigma \cdot (-i\nabla + k) \end{pmatrix} e_{2,m} = \overline{(ma^* + k)_\mathbb{C}} \, e_{1,m}
$$
$$
\begin{pmatrix} \sigma \cdot (-i\nabla + k) & 0 \\ 0 & \sigma \cdot (-i\nabla + k) \end{pmatrix} e_{3,m} = (ma^* + k)_\mathbb{C} \, e_{4,m}
$$
$$
\begin{pmatrix} \sigma \cdot (-i\nabla + k) & 0 \\ 0 & \sigma \cdot (-i\nabla + k) \end{pmatrix} e_{4,m} = \overline{(ma^* + k)_\mathbb{C}} \, e_{3,m}
$$

We know that $e^{-ikx} \left(-i\nabla\right) e^{ikx} \cdot = -i\nabla + k$ hence

$$
e^{-ikx} \left(-\frac{1}{2}\Delta\right) e^{ikx} \cdot = \frac{1}{2}\left(-i\nabla + k\right)^2
$$

and with $f(x) = \sum_m \widehat{f}_m e^{ima^* x}$

$$
(-i\nabla + k) f = \sum_m (ma^* + k) \widehat{f}_m e^{ima^* x},
$$

so

$$
\frac{1}{2}\left(-i\nabla + k\right)^2 f = \sum_m \frac{1}{2}\left(ma^* + k\right)^2 \widehat{f}_m e^{ima^* x}
$$

We have

$$\left\langle e_{\alpha,n}, \frac{1}{2}\left(-i\nabla + k\right)^2 e_{\beta,m}\right\rangle = \frac{1}{2}\left(ma^* + k\right)^2 \delta_{\alpha,\beta}\delta_{m-n}$$

We have

$$\sigma \cdot k = \begin{pmatrix} 0 & \overline{k_{\mathbb{C}}} \\ k_{\mathbb{C}} & 0 \end{pmatrix}, \qquad (Jk)_{\mathbb{C}} = ik_{\mathbb{C}}, \qquad \sigma \cdot Jk = \begin{pmatrix} 0 & -i\overline{k_{\mathbb{C}}} \\ ik_{\mathbb{C}} & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} -\sigma \cdot J\left(-i\nabla + k\right) & 0 \\ 0 & \sigma \cdot J\left(-i\nabla + k\right) \end{pmatrix} e_{1,m} = -i\left(ma^* + k\right)_{\mathbb{C}} e_{2,m}$$

$$\begin{pmatrix} -\sigma \cdot J\left(-i\nabla + k\right) & 0 \\ 0 & \sigma \cdot J\left(-i\nabla + k\right) \end{pmatrix} e_{2,m} = i\overline{\left(ma^* + k\right)_{\mathbb{C}}} e_{1,m}$$

$$\begin{pmatrix} -\sigma \cdot J\left(-i\nabla + k\right) & 0 \\ 0 & \sigma \cdot J\left(-i\nabla + k\right) \end{pmatrix} e_{3,m} = i\left(ma^* + k\right)_{\mathbb{C}} e_{4,m}$$

$$\begin{pmatrix} -\sigma \cdot J\left(-i\nabla + k\right) & 0 \\ 0 & \sigma \cdot J\left(-i\nabla + k\right) \end{pmatrix} e_{4,m} = -i\overline{\left(ma^* + k\right)_{\mathbb{C}}} e_{3,m}$$

For a general $V = \sum_m \widehat{V}_m e^{ima^* \cdot x}$, we have $V^* = \sum_m \widehat{V}_m^* e^{-ima^* \cdot x}$ and we compute

$$\left\langle e_{\alpha,n}, \begin{pmatrix} 0 & V\left(-i\nabla + k\right)^2 \\ V^*\left(-i\nabla + k\right)^2 & 0 \end{pmatrix} e_{\beta,m}\right\rangle$$

$$= (ma^* + k)^2 \left(\delta_{\alpha\in\{1,2\}}^{\beta\in\{3,4\}}\left(\widehat{V}_{n-m}\right)_{\alpha_1\beta_2} + \delta_{\alpha\in\{3,4\}}^{\beta\in\{1,2\}}\overline{\left(\widehat{V}_{m-n}\right)_{\beta_1\alpha_2}}\right)$$

7. Renormalization of the equation

We know that

$$\left(-i\nabla + k + A(x)\left(-i\nabla\right) + v(x)\right)\psi = E\psi$$

with $x = \lambda y$, we define $\phi(y) := \psi(\lambda y)$ and

$$\left(\left(\left(-i\nabla + k\right) + A(\lambda y)\left(-i\nabla\right) + v(\lambda y)\right)\psi\right)(\lambda y) = E\psi(\lambda y)$$

but $(\nabla\psi)(\lambda y) = \frac{1}{\lambda}\nabla\phi(y)$, so

$$\left(\frac{-i\nabla}{\lambda} + k + \frac{A(\lambda y)}{\lambda}\left(-i\nabla\right) + v(\lambda y)\right)\phi = E\phi$$

We enter $V\left(\frac{3}{2}JX\right)$ for each potential V, hence we need to apply a coefficient $\frac{2}{3}$ to each derivation operator.

8. Symmetries

8.1. **Particle-hole.** We define

$$\mathcal{S}u(x) := i\begin{pmatrix} 0 & -\mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} u(-x)$$

We have

$$\mathcal{S}\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}\mathcal{S} = -\begin{pmatrix} 0 & B^*(-x) \\ B(-x) & 0 \end{pmatrix}$$

We have $T(-x)^* = T(x)$ hence we should have that

$$\mathcal{S}H\mathcal{S} = -H$$

We compute

$$\mathcal{S}_{IJ} = \langle e_{\alpha,n}, \mathcal{S}e_{\beta,m}\rangle = i\left\langle e_{\alpha,n}, \begin{pmatrix} -e_{\beta_2,-m} \\ e_{\beta_1,-m} \end{pmatrix}\right\rangle$$

$$= i\delta_{m+n}\left(\delta^{\beta\in\{1,2\}}_{\alpha\in\{3,4\}}\delta_{\beta_1-\alpha_2} - \delta^{\beta\in\{3,4\}}_{\alpha\in\{1,2\}}\delta_{\beta_2-\alpha_1}\right)$$

For any function B and any vector function $\boldsymbol{A}$, we have

$$\mathcal{S}\begin{pmatrix} 0 & B(\mathbf{X}) \\ B^*(\mathbf{X}) & 0 \end{pmatrix}\mathcal{S} = -\begin{pmatrix} 0 & B^*(-\mathbf{X}) \\ B(-\mathbf{X}) & 0 \end{pmatrix}$$

$$\mathcal{S}\begin{pmatrix} 0 & B(\mathbf{X})\Delta \\ B^*(\mathbf{X})\Delta & 0 \end{pmatrix}\mathcal{S} = -\begin{pmatrix} 0 & B^*(-\mathbf{X})\Delta \\ B(-\mathbf{X})\Delta & 0 \end{pmatrix}$$

$$\mathcal{S}\begin{pmatrix} 0 & i\boldsymbol{A}(\mathbf{X})\cdot\nabla \\ i\boldsymbol{A}(\mathbf{X})^*\cdot\nabla & 0 \end{pmatrix}\mathcal{S} = \begin{pmatrix} 0 & i\boldsymbol{A}(-\mathbf{X})^*\cdot\nabla \\ i\boldsymbol{A}(-\mathbf{X})\cdot\nabla & 0 \end{pmatrix},$$

we also compute that

$$\mathcal{S}\begin{pmatrix} \sigma\cdot\nabla & 0 \\ 0 & \sigma\cdot\nabla \end{pmatrix}\mathcal{S} = -\begin{pmatrix} \sigma\cdot\nabla & 0 \\ 0 & \sigma\cdot\nabla \end{pmatrix},$$

hence if the operator Γ is a linear combination of the terms

$$\begin{pmatrix} \sigma\cdot(-i\nabla) & 0 \\ 0 & \sigma\cdot(-i\nabla) \end{pmatrix}, \begin{pmatrix} \sigma\cdot J(-i\nabla) & 0 \\ 0 & \sigma\cdot J(-i\nabla) \end{pmatrix},$$

$$\begin{pmatrix} 0 & \mathbb{V} \\ \mathbb{V}^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Sigma\Delta \\ \Sigma^*\Delta & 0 \end{pmatrix}$$

it satisfies the symmetry $\mathcal{S}\Gamma\mathcal{S} = -\Gamma$, and those are the particle-hole symmetric terms of our effective Hamiltonian. However, if Γ is a linear combination of the operators

$$\begin{pmatrix} 0 & \boldsymbol{\mathcal{A}}\cdot(-i\nabla) \\ \boldsymbol{\mathcal{A}}^*\cdot(-i\nabla) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \boldsymbol{\mathcal{A}}\cdot J(-i\nabla) \\ \boldsymbol{\mathcal{A}}^*\cdot J(-i\nabla) & 0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{2}\Delta & 0 \\ 0 & -\frac{1}{2}\Delta \end{pmatrix}, \begin{pmatrix} \mathbb{W} & 0 \\ 0 & \mathbb{W}^* \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix}$$

of the effective Hamiltonian $\mathcal{H}_{d,\theta}$, it satisfies $\mathcal{S}\Gamma\mathcal{S} = \Gamma$ and hence break the particle-hole symmetry.

But now we also compute that

$$\mathcal{S}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\mathcal{S} = k,$$

$$\mathcal{S}\begin{pmatrix} \sigma(-i\nabla+k) & 0 \\ 0 & \sigma(-i\nabla+k) \end{pmatrix}\mathcal{S} = -\begin{pmatrix} \sigma(-i\nabla-k) & 0 \\ 0 & \sigma(-i\nabla-k) \end{pmatrix}$$

8.2. **Mirror.** First, for any function B, we have $\sigma_1 B^*\sigma_1 = \begin{pmatrix} \overline{B_{22}} & \overline{B_{12}} \\ \overline{B_{21}} & \overline{B_{11}} \end{pmatrix}$.

The mirror operator for the BM Hamiltonian is

$$\mathcal{M}u(\mathbf{X}) := \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}u(\overline{\mathbf{X}})$$

where $\overline{\mathbf{X}} := (X_1, -X_2) =: M\mathbf{X}$, it satisfies $\mathcal{M} = \mathcal{M}^{-1} = \mathcal{M}^*$.

Next,

$$\mathcal{M} \begin{pmatrix} 0 & B(\mathbf{X}) \\ B(\mathbf{X})^* & 0 \end{pmatrix} \mathcal{M} = \begin{pmatrix} 0 & \sigma_1 B^*(\overline{\mathbf{X}})\sigma_1 \\ \sigma_1 B(\overline{\mathbf{X}})\sigma_1 & 0 \end{pmatrix}$$

In cartesian coordinates, we have

$$T(M\mathbf{X}) = \sum_{j=1}^{3} T_j e^{ix \cdot M^* q_j^c} = \sum_{j=1}^{3} T_j e^{ix \cdot M q_j^c}$$

because $M^* = M$. But

$$\sigma_1 T^*(M\mathbf{X})\sigma_1 = \begin{pmatrix} w_0 \left(\sum_{j=1}^{3} e^{ix \cdot M q_j} \right) & w_1 \left(e^{ix \cdot M q_1} + e^{i\phi} e^{ixM q_2} + e^{i2\phi} e^{ix \cdot M q_3} \right) \\ . & . \end{pmatrix}$$

$$= \begin{pmatrix} w_0 \left(\sum_{j=1}^{3} e^{ix \cdot q_j} \right) & w_1 \left(e^{ix \cdot q_1} + e^{-i\phi} e^{ixq_2} + e^{-i2\phi} e^{ix \cdot q_3} \right) \\ . & . \end{pmatrix} = T(\mathbf{X})$$

where we used that $Mq_1^c = q_1^c$, $Mq_2^c = q_3^c$ and $Mq_3^c = q_2^c$.

We search the action on reduced Fourier coefficients. We have

$$f(Mx) = \sum_m e^{ix \cdot M(ma^*)} = \sum_m e^{ix \cdot (M^r m)a^*}$$

where $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$M^r = S^{-1} M S = \mathcal{M}^* M (\mathcal{M}^*)^{-1} = \sigma_1$$

9. Non Local term

From the theoretical investigations, we have

$$F_0^{\eta,j,s}(\mathbf{X}) := \int_{\mathbb{R}^3} \overline{\varphi_{\mathrm{Bl},s}(\mathbf{y}, z)} \Phi_j \left(\mathbf{y} + \mathbf{a}_s - 2\eta J\mathbf{X}, z - \eta d \right) \mathrm{d}\mathbf{y}\mathrm{d}z$$

and

$$\mathbb{W}_{\mathrm{nl},-1}^{\eta}(\mathbf{X})_{jj'} := \frac{v_0}{|\Omega|} \sum_{s \in \{1,2\}} \overline{F_0^{\eta,j,s}(\mathbf{X})} F_0^{\eta,j',s}(\mathbf{X}).$$

Since $\varphi_{\mathrm{Bl},s}$ is localized, we periodize it and we make the approximation

$$F_0^{\eta,j,s}(\mathbf{X}) \simeq \int_{\Omega \times [0,L]} \overline{\varphi_{\mathrm{Bl},s}(\mathbf{y}, z)} \Phi_j \left(\mathbf{y} + \mathbf{a}_s - 2\eta J\mathbf{X}, z - \eta d \right) \mathrm{d}\mathbf{y}\mathrm{d}z$$

$$= \int_{\Omega \times [0,L]} \overline{\varphi_s(\mathbf{y}, z)} u_j \left(\mathbf{y} + \mathbf{a}_s - 2\eta J\mathbf{X}, z - \eta d \right) \mathrm{d}\mathbf{y}\mathrm{d}z$$

and we define φ such that $\varphi_{\mathrm{Bl},s} = e^{i\mathbf{K}\mathbf{y}} \varphi_s$, because it is $\widehat{\varphi_s}$ which is stored by DFTK, so

$$\varphi_s(\mathbf{y}, z) = \sum_{\mathbf{G}, G_z} \frac{e^{i(\mathbf{G}\mathbf{y} + G_z z)}}{\Gamma} \widehat{\varphi}_{s,\mathbf{G},G_z}, \qquad u_j(\mathbf{y}, z) = \sum_{\mathbf{G}, G_z} \frac{e^{i(\mathbf{G}\mathbf{y} + G_z z)}}{\Gamma} \widehat{(u_j)}_{\mathbf{G},G_z}$$

where $\mathbf{K}$ is the Dirac point, thus

$$F_0^{\eta,j,s}(\mathbf{X}) = \sum_{\mathbf{G},G_z} e^{i(\mathbf{G}(\mathbf{a}_s - 2\eta J\mathbf{X}) - \eta G_z d)} \overline{\widehat{\varphi}}_{s,\mathbf{G},G_z} \widehat{(u_j)}_{\mathbf{G},G_z}$$

$$F^{\eta,j,s}(\mathbf{X}) := F_0^{\eta,j,s}\left(-\tfrac{3}{2}J\mathbf{X}\right)$$

$$= \sum_{\mathbf{G},G_z} e^{i(\mathbf{G}(\mathbf{a}_s - 3\eta\mathbf{X}) - \eta G_z d)} \overline{\widehat{\varphi}}_{s,\mathbf{G},G_z} \widehat{(u_j)}_{\mathbf{G},G_z}$$

$$= \sum_{\mathbf{G},G_z} e^{i\left(\mathbf{G}\left(-\eta\frac{\mathbf{a}_s}{3} + \mathbf{X}\right) - \eta G_z d\right)} \overline{\widehat{\varphi}}_{s,-\frac{\eta}{3}\mathbf{G},G_z} \widehat{(u_j)}_{-\frac{\eta}{3}\mathbf{G},G_z}$$

has Fourier coefficients

$$\widehat{(F^{\eta,j,s})}_{\mathbf{G}} = \sum_{G_z} e^{-i\eta\left(\mathbf{G}\frac{\mathbf{a}_s}{3} + G_z d\right)} \overline{\widehat{\varphi}}_{s,-\frac{\eta}{3}\mathbf{G},G_z} \widehat{(u_j)}_{-\frac{\eta}{3}\mathbf{G},G_z}$$

$$\widehat{(F^{\eta,j,s})}_m = \sum_{m_z} e^{-i\eta\left(ma^*\frac{\mathbf{a}_s}{3} + \frac{2\pi}{L}m_z d\right)} \overline{\widehat{\varphi}}_{s,-\frac{\eta}{3}m,m_z} \widehat{(u_j)}_{-\frac{\eta}{3}m,m_z}$$

$$= \sum_{m_z} e^{-i\eta\left(2\pi m\frac{\mathbf{a}_s^{\mathrm{red}}}{3} + \frac{2\pi}{L}m_z d\right)} \overline{\widehat{\varphi}}_{s,-\frac{\eta}{3}m,m_z} \widehat{(u_j)}_{-\frac{\eta}{3}m,m_z}$$

On the functions given by DFTK, we remark that $\varphi_s[m]$ given is periodic and that

$$\mathcal{R}_{\frac{2\pi}{3}} \varphi_{\mathrm{Bl},s} = \tau^s \varphi_{\mathrm{Bl},s}.$$

9.1. Symmetries. We have

$$\mathcal{R}_{\frac{2\pi}{3}} F_0^{\eta,j,s} = \int_{\mathbb{R}^3} \overline{\varphi_{\mathrm{Bl},s}(\mathbf{y},z)} \Phi_j \left(R_{-\frac{2\pi}{3}}\left(R_{\frac{2\pi}{3}}\mathbf{y} + R_{\frac{2\pi}{3}}\mathbf{a}_s - 2\eta J\mathbf{X}\right), z - \eta d\right) \mathrm{d}\mathbf{y}\mathrm{d}z$$

$$= \int_{\mathbb{R}^3} \overline{\mathcal{R}_{\frac{2\pi}{3}}\varphi_{\mathrm{Bl},s}(\mathbf{y},z)} \left(\mathcal{R}_{\frac{2\pi}{3}}\Phi_j\right)\left(\mathbf{y} + R_{\frac{2\pi}{3}}\mathbf{a}_s - 2\eta J\mathbf{X}, z - \eta d\right) \mathrm{d}\mathbf{y}\mathrm{d}z$$

$$= \tau^{j-s} \int_{\mathbb{R}^3} \overline{\varphi_{\mathrm{Bl},s}(\mathbf{y},z)} \Phi_j \left(\mathbf{y} + R_{\frac{2\pi}{3}}\mathbf{a}_s - 2\eta J\mathbf{X}, z - \eta d\right) \mathrm{d}\mathbf{y}\mathrm{d}z$$

and if $\varphi_{\mathrm{Bl},s}(y + R_{\frac{2\pi}{3}}a_s) = \varphi_{\mathrm{Bl},s}(y + a_s)$, then

$$\mathcal{R}_{\frac{2\pi}{3}}\left(\overline{F_0^{\eta,j,s}} F_0^{\eta,j',s}\right) = \tau^{j'-j} \overline{F_0^{\eta,j,s}} F_0^{\eta,j',s}$$

10. Change of basis for getting $\Phi_j \in L^2_{\tau,\overline{\tau}}$

Numerically, DFTK gives

$$\phi, \psi \in \mathrm{Ker}\left(\mathcal{R}_{\frac{2\pi}{3}} - \tau\right) + \mathrm{Ker}\left(\mathcal{R}_{\frac{2\pi}{3}} - \overline{\tau}\right)$$

but we want to separate the spaces and obtain $\phi_1 \in \mathrm{Ker}\left(\mathcal{R}_{\frac{2\pi}{3}} - \tau\right)$ so that $\phi_2(x,z) := \overline{\phi_1}(-x,z) \in \mathrm{Ker}\left(\mathcal{R}_{\frac{2\pi}{3}} - \overline{\tau}\right)$, which existence is ensured by [3].

First we define

$$c := \left\|\left(\mathcal{R}_{\frac{2\pi}{3}} - \tau\right)\phi_a\right\|_{L^2}^2, \qquad s := \left\langle\left(\mathcal{R}_{\frac{2\pi}{3}} - \tau\right)\phi_a, \left(\mathcal{R}_{\frac{2\pi}{3}} - \tau\right)\phi_b\right\rangle.$$

Then we parametrize

$$\phi_1 = e^{i\alpha} \left(\frac{s}{|s|} \cos\theta \phi_a + e^{i\beta} \sin\theta \phi_b \right)$$

and we want $\left(\mathcal{R}_{\frac{2\pi}{3}} - \tau \right) \phi_1 = 0$ which is equivalent to

$$\frac{s}{|s|} \cos\theta \left(\mathcal{R}_{\frac{2\pi}{3}} - \tau \right) \phi_a + e^{i\beta} \sin\theta \left(\mathcal{R}_{\frac{2\pi}{3}} - \tau \right) \phi_b = 0$$

and we take the scalar product with $\left(\mathcal{R}_{\frac{2\pi}{3}} - \tau \right) \phi_a$ so that

$$\frac{c}{|s|} \cos\theta + e^{i\beta} \sin\theta = 0$$

Now we necessarily have $e^{i\beta} = \pm$ so $\cos\theta = \mp\frac{|s|}{c} \sin\theta$ and finally using $\cos^2 + \sin^2 = 1$,

$$|\cos\theta| = \frac{1}{\sqrt{1 + \left(\frac{c}{|s|}\right)^2}}, \qquad |\sin\theta| = \frac{1}{\sqrt{1 + \left(\frac{|s|}{c}\right)^2}},$$

and also choosing $\alpha = 0$ if $\cos\theta \geqslant 0$ and π otherwise, which does not change anything, we have

$$\phi_1 = \frac{s}{|s|} \frac{1}{\sqrt{1 + \left(\frac{c}{|s|}\right)^2}} \phi_a \pm \frac{1}{\sqrt{1 + \left(\frac{|s|}{c}\right)^2}} \phi_b$$

and $\phi_2(x) = \overline{\phi_1(-x)}$. By multiplying by e^{-iKx}, we also obtain

$$\boxed{u_1 = \frac{s}{|s|} \frac{1}{\sqrt{1 + \left(\frac{c}{|s|}\right)^2}} u_a \pm \frac{1}{\sqrt{1 + \left(\frac{|s|}{c}\right)^2}} u_b}$$

and $u_2(x) = \overline{u_1(-x)}$.

11. THE 1/3 SCALING OF COORDINATES

Taken from [5, Appendix G.3, G.4] for instance, the moiré lattice vectors are

$$a_1 = \frac{2\pi}{3k_\theta} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \qquad a_2 = \frac{2\pi}{3k_\theta} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$$

and $T(x) = \sum_{j=1}^3 T_j e^{-iq_j x}$ has

$$q_1 = k_\theta \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad q_{2,3} = \frac{k_\theta}{2} \begin{pmatrix} \pm\sqrt{3} \\ 1 \end{pmatrix}$$

and we remark that $a_1 \cdot q_1 = -\frac{2\pi}{3}$ so actually $q_j \notin \mathbb{L}^*$ but $3q_j \in \mathbb{L}^*$.

12. Change of gauge on the phasis of wavefunctions

When we change $\Phi_1 \to \Phi_1 e^{i\theta}$, then $u_1 \to u_1 e^{i\theta}$, $u_2 \to u_2 e^{-i\theta}$ because $u_2(x) = \overline{u_1(-x)}$, and hence

$$\boxed{\overline{u_1}u_2 \to \overline{u_1}u_2 e^{-2i\theta}}$$

We define

$$\mathcal{U} := \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

have

$$\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \mathbb{W}^+ & \mathbb{V} \\ \mathbb{V}^* & \mathbb{W}^- \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} = \begin{pmatrix} U\mathbb{W}^+U^* & U\mathbb{V}U^* \\ U\mathbb{V}^*U^* & U\mathbb{W}^-U^* \end{pmatrix}$$

and with $U := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, we have

$$U \begin{pmatrix} B^+ & B \\ B^* & B^- \end{pmatrix} U^* = \begin{pmatrix} B^+ & Be^{2i\theta} \\ B^*e^{-2i\theta} & B^- \end{pmatrix}$$

hence if we define H_θ to be H with $u_1 \to u_1 e^{i\theta}$, we have that

$$\mathcal{U}H_\theta\mathcal{U}^*$$

is constant in θ.

13. Comparision between BM and our model

13.1. **Rescaling.** The BM Hamiltonian is

$$\begin{pmatrix} -iv_0\sigma\nabla & w_1 T^{\mathrm{TKV}}(k_\theta x) \\ w_1 T^{*,\mathrm{TKV}}(k_\theta x) & -iv_0\sigma\nabla \end{pmatrix}.$$

We consider the rescaling

$$Su(x) := u\left(\frac{x}{k_\theta}\right), \qquad S^*u(y) = k_\theta^2 u\left(k_\theta y\right), \qquad SS^* = k_\theta^2$$

where we defined S^* as $\int_\Omega \overline{f}\, Sg = \int_{L\Omega/k_\theta} g\, \overline{S^*f}$. We have $\nabla S^* = k_\theta S^*\nabla$ so $S\nabla S^* = k_\theta^3\nabla$ and $SfS^* = k_\theta^2 f\left(\frac{x}{k_\theta}\right)$ so when $x = yk_\theta$ is the microscopic scale

$$\frac{1}{k_\theta^3 v_0} S\left(\begin{pmatrix} -iv_0\sigma\nabla & w_1 T^{\mathrm{TKV}}(k_\theta x) \\ w_1 T^{*,\mathrm{TKV}}(k_\theta x) & -iv_0\sigma\nabla \end{pmatrix} - E \right) S^*$$

$$= \begin{pmatrix} -i\sigma\nabla & \alpha T^{\mathrm{TKV}}(x) \\ \alpha T^{*,\mathrm{TKV}}(x) & -i\sigma\nabla \end{pmatrix} - \varepsilon$$

where $\alpha := \frac{w_1}{k_\theta v_0}$ and where $\varepsilon = \frac{E}{v_0 k_\theta}$ is the unit of [4, Fig 1] defined in the caption.

13.2. **Rotation and reduced coordinates of q.** In [4], the orientation of the lattice (one of the equations below (6)), is with reciprocal vectors

$$b_{1,2}^{*,\mathrm{TKV}} = \sqrt{3} \begin{pmatrix} \pm 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

and to compare with our lattice defined in (2), we have

$$-Jb_1^{*,\mathrm{TKV}} = b_1^*, \qquad -Jb_2^{*,\mathrm{TKV}} = -b_2^*, \qquad b = \frac{4\pi}{3}$$

corresponding to the direct lattice

$$b_1 = b \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad b_2 = b \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad \mathcal{M}_b = \frac{b}{2} \begin{pmatrix} -1 & 1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix}$$

The q_j's are

$$q_{2,3}^{\mathrm{TKV}} = \begin{pmatrix} \pm\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad q_1^{\mathrm{TKV}} = -q_2^{\mathrm{TKV}} - q_3^{\mathrm{TKV}}$$

so

$$\sqrt{3} J q_{2,3}^{\mathrm{TKV}} = \pm b_{2,1}^{*,\mathrm{TKV}}$$

and we do a rotation, $q_j := -J q_j^{\mathrm{TKV}}$,

$$q_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad q_{2,3} = \begin{pmatrix} \frac{1}{2} \\ \mp\frac{\sqrt{3}}{2} \end{pmatrix}, \qquad q_1 = -q_2 - q_3$$

We have

$$T(x) := T^{\mathrm{TKV}}(Jx) = \sum_j T_j e^{-iq_j x} \underset{(3)}{=} \sum_j T_j e^{i\widetilde{m}_{q_j} b^* x}$$

where $\widetilde{m}_{q_j} = -\frac{1}{2\pi} \mathcal{M}_b^* q_j$, that is

$$\widetilde{m}_{q_1} = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \widetilde{m}_{q_2} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \widetilde{m}_{q_3} = \frac{1}{3} \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

and we redefine

$$m_{q_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad m_{q_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad m_{q_3} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \tag{6}$$

so

$$T(x) = \sum_j T_j e^{i m_{q_j} \frac{b^*}{3} x} \tag{7}$$

We conjugate again and get

$$\mathcal{R}_{-\frac{\pi}{2}} \left(\begin{pmatrix} -i\sigma\nabla & \alpha T^{\mathrm{TKV}}(x) \\ \alpha T^{*,\mathrm{TKV}}(x) & -i\sigma\nabla \end{pmatrix} - \varepsilon \right) \mathcal{R}_{\frac{\pi}{2}}$$

$$= \begin{pmatrix} -i\sigma \cdot J\nabla & \alpha T(x) \\ \alpha T^*(x) & -i\sigma \cdot J\nabla \end{pmatrix} - \varepsilon$$

the action of J corresponding to the rotation of the dual lattice vectors, so if we write ∇ in our new lattice b, we have

$$\begin{pmatrix} -i\sigma\nabla & \alpha T(x) \\ \alpha T^*(x) & -i\sigma\nabla \end{pmatrix} - \varepsilon$$

13.3. **Rescaling again.** To write the Fourier coefficients of T, we need to rescale, so ne define $Su(x) := u(3x)$ and as previously, doing "$k_\theta = 1/3$", we have $SS^* = 1/9$, $S\nabla S^* = \left(1/3^3\right)\nabla$

$$3^2 S\left(\begin{pmatrix} -i\sigma\nabla & \alpha T\left(x\right) \\ \alpha T^*(x) & -i\sigma\nabla \end{pmatrix} - \varepsilon\right)S^* = \begin{pmatrix} -\frac{1}{3}i\sigma\nabla & \alpha T\left(3x\right) \\ \alpha T^*(3x) & -\frac{1}{3}i\sigma\nabla \end{pmatrix} - \varepsilon$$

and now we can implement the Fourier coefficients of $T(3\cdot)$, given by (6), because

$$T(3x) = \sum_j T_j e^{im_{q_j}b^*x}$$

13.4. **Relation to our model.** We compute, for $j \in \{1,2,3\}$,

$$v_\theta^m(x) = v_m e^{ima^*\left(\cos\frac{\theta}{2}x + \sin\frac{\theta}{2}Jx\right)} + v_m e^{ima^*\left(\cos\frac{\theta}{2}x - \sin\frac{\theta}{2}Jx\right)}$$

$$= 2iv_m e^{ima^*\cos\frac{\theta}{2}x}\sin ma^*\sin\frac{\theta}{2}Jx$$

$$= 2iv_m e^{ima^*\cos\frac{\theta}{2}x}\sin m\frac{a^*}{2k_D}k_\theta Jx$$

and

$$\frac{a_1^*}{2k_D} = \frac{\sqrt{3}}{2}\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = -\frac{\sqrt{3}}{2}Jq_2, \qquad \frac{a_2^*}{2k_D} = \frac{\sqrt{3}}{2}\begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = -\frac{\sqrt{3}}{2}Jq_3$$

We deduce that

$$v_\theta^m(x) = -2iv_m e^{ima^*\cos\frac{\theta}{2}x}\sin k_\theta\frac{\sqrt{3}}{2}mb^*\cdot x$$

where $b_1^* := q_2$, $b_2^* := q_3$. We define $m_2 = (1,0)$, $m_3 = (0,1)$, $m_1 = (-1,-1)$, so the three modes are

$$m_j b^* = q_j$$

and

$$v_\theta^{m_j}(x) = \cdot\sin k_\theta\frac{\sqrt{3}}{2}q_j\cdot x$$

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