rep / Order	w/	w/out
matters	$_nP_r$	n^r
doesn't	$_nC_r$	$_{n-1+r}C_r$

$_nP_n = n!$ $\quad$ $_nP_1 = n$

$_nC_n = 1$ $\quad$ $_nC_1 = n$

$_nP_r = {_nC_r} \cdot r!$

$\binom{n}{r} = \binom{n}{n-r}$

$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

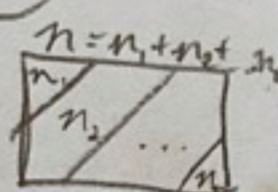
n objects of 2 types

e.g. $\underbrace{a_1 a_2 \dots a_r}_{r}$ $\underbrace{B_1 B_2 \dots B_{n-r}}_{n-r}$ $\quad \dfrac{n!}{\underbrace{r!}_{a}\underbrace{(n-r)!}_{B}} = \binom{n}{r}$

n objects of k types

e.g. $\underbrace{a_1 a_1 \dots a_1}_{n_1}$ $\underbrace{a_2 \dots a_2}_{n_2}$ $\dots$ $\underbrace{a_k a_k \dots a_k}_{n_k}$

$\dfrac{n!}{n_1!\, n_2! \dots n_k} = \binom{n}{n_1 n_2 \dots n_k}$



De Morgan's Laws

$(A \cap B)^c = A^c \cup B^c$

$(A \cup B^c) = A^c \cap B^c$

Inclusion Exclusion

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Partition

① $A \cap B = \emptyset$ M.E

② $A \cup B = S$ exhaustive

Distributive

$A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

multiplication rules

$n(E_1) = n_1$ $n(E_2) = n_2$

$n(E_1 E_2) = n_1 n_2$

Law of Total Prob: given $\{B_1, B_2, \dots B_n\}$ partition of S for event A

$P(A) = \sum_{i=1}^{n} P(A \cap B_n) = P(A \cap B_1) + P(A \cap B_2) + \dots (P(A \cap B_n))$

$= \sum_{i=1}^{n} P(A|B_n)P(B_n) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \dots + P(A|B_n)P(B_n)$ where $P(B_i) \neq 0$

Bayes Theorem: given $\{B_1, B_2, \dots B_n\}$ partition of S

$P(B_k|A) = \dfrac{P(A|B_k)P(B_k)}{\sum_{i=1} P(A|B_n)P(B_n)}$

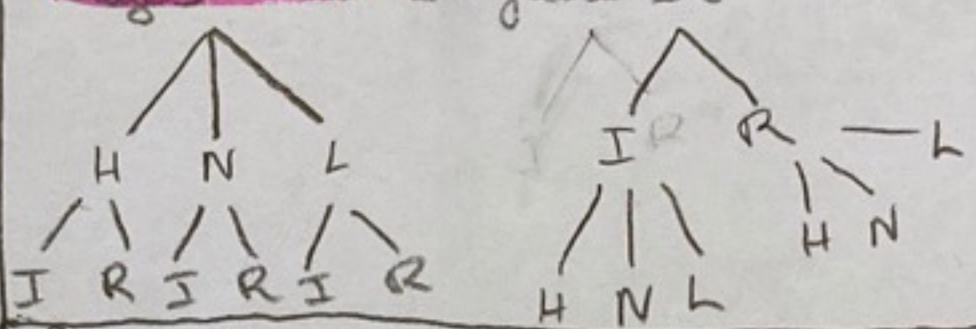
multiplication rule: when $P(A) \neq 0$, $P(B) \neq 0$

$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$

Bayes Thm, special case: 2 sets B & B^c

$P(B|A) = \dfrac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$

Bayes Tree Diagram ex



Random Variable of X

$X: S \to R$, assigns # to outcomes of set

$A \subset B$ set

element $a \in A$ set

Discrete Distributions

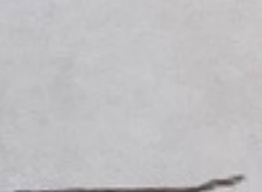
$S = \{s_1, s_2, \dots s_n\}$ $\quad X(s_1) = $ a real # $= x$

$\{x : X(s) = x, s \in S\}$ $\quad X(S) = S_x = \{$all possible values of $X\}$

CDF

$F_x(x) = P(X \leq x)$

- non decreasing
- $\lim_{x \to +\infty} F_x(x) = 1$
- $\lim x \to -\infty$ $F_x(x) = 0$

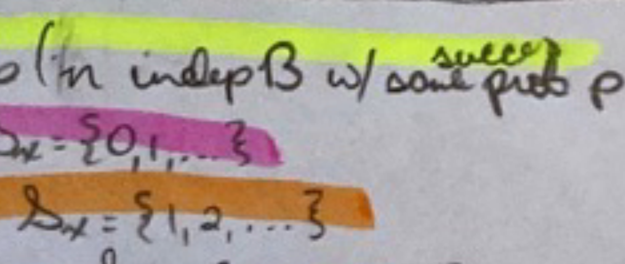


PMF $P(X = x)$

(a) $f(x) > 0$, $x \in S$

(b) $\sum_{x \in S} f(x) = 1$

(c) $P(X \in A) = \sum_{x \in A} f(x)$, where $A \subset S$



Calculator Keys $\quad$ $e = 2.718281828416\dots$

$e^x \to$ $\boxed{2nd}$ $\boxed{LN}$ $\quad$ $_nC_r \to$ $\boxed{2nd}$ $\boxed{8}$ $\quad$ log or ln: $x \leq 0$

$\sqrt[x]{y} \to$ $\boxed{2^{nd}}$ $\boxed{y^x}$ $\quad$ $_nP_r \to$ $\boxed{2nd}$ $\boxed{9}$

$x! \to$ $\boxed{2^{nd}}$ $\boxed{3}$ $\quad$ $x^3 \to$ $\boxed{2^{nd}}$ $\boxed{1}$

$\boxed{a\ b/c}$ → mixed frac $\quad$ $\boxed{2nd}$ $\boxed{d/c}$ → mixed # to mixed frac

$\boxed{2^{nd}}$ $\boxed{\log}$ $\to [10^x] \Rightarrow 2\,[10^x] = 100$, $\log_{10} 100 = 2$

Bernoulli $0<p<1$ $X \sim Ber(p)$ $S_x = \{0, 1\}$

Binomial $0<p<1$ $X \sim Bin(n, p)$ # of successes in n fixed trials (n indep B w/ same success prob p, X = # succes

Poisson $\lambda > 0$ $X \sim Poisson(\lambda)$ # of arrivals in fixed period $S_x = \{0, 1, \dots\}$

Geometric $0<p<1$ $X \sim Geom(p)$ # of trials through 1^{st} success $S_x = \{1, 2, \dots\}$

Negative Binomial $X \sim NB(r, p)$ # of trials through r^{th} success $S_x = \{r, r+1, \dots\}$

Hypergeometric $X \sim HG(N_1, N_2, n)$ # of marked individuals in sample taken w/out rep $\dfrac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}$

Discrete Uniform outcomes that are equally likely $f(x) = \frac{1}{m}$, m positive int, $k = 1, 2, \dots m-1$

$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ \frac{k}{m}, & k \leq x < k+1 \\ 1, & m \leq x \end{cases}$ $\quad$ ex. rolling a die, flipping coin

Geom dist

$f(x) = P(X = x) = p(1-p)^{x-1}$

1^{st} trial $\rightarrow$ $x-1^{th}$ trial | x^{th} trial

failures $(1-p)^{x-1}$ | 1^{st} success p

Ex: X = # of selections until 2 W + B

$p = P(2W, 2B) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{18}{35}$ $X \sim Geom(18/35)$

Ex X = # of trials until defective item $X \sim Geom(0.01)$

$P(X \geq 100) = 1 - P(X < 100) = 1 - P(X \leq 99)$

Bin dist

X = # of heads in 10 trial w/ success $p = 1/2$

$X \sim Bin(10, 1/2)$

Hypergeo

X = # of tagged fish $X \sim HG(10, 40, 7)$ $n = 7$

$P(X=2) = f_X(2) = \frac{\binom{10}{2}\binom{40}{5}}{\binom{50}{7}}$

Negative Bin (Note $NB(1,p) \overset{dis}{=} Geom(p)$)

$f(x) = P(X = x) = \binom{x-1}{r-1} \cdot p^{r-1}(1-p)^{x-r} \cdot p$

$\underbrace{O\,O \ldots O}_{}$ $x-1^{th}$ trial | O x^{th} trial, r^{th} success

$r-1$ successes

$x-r$ failures

ex: X = # of free throws to make 10 "success shots"

80% success prob of free throw

$X \sim NB(10, 0.8)$

Poisson

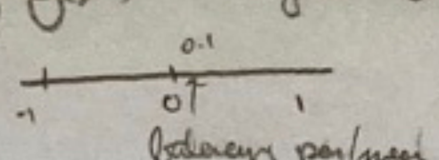
X = # of calls in 9 min, $X \sim Poisson(6)$

unit: 3 min avg: 2 calls

$P(X \geq 5) = 1 - P(X < 5) = 1 - P(X \leq 4) = 1 - F_X(4)$

X a discrete r.v. w/ pmf $f(x)$ weighted avg of #'s in S_x

$\mu = E(X) = \sum_{x \in S} x \cdot f(x)$



eg

x	$f(x)$
-1	0.4
0	0.1
1	0.5

$S_X = \{-1, 0, 1\}$

$E(X) = \sum_{x \in S} x \cdot f(x)$

$E(\log X) = \sum_{x \in S} \log x \cdot f_X(x)$ given $f_X(x) = 1/3$, $x \in \{-1, 0, 1\}$

$E(X^2) = \sum_{x \in S} x^2 \cdot f_X(x) = (-1)^2(\frac{1}{3}) + (0)^2(\frac{1}{3}) + (1)^2(\frac{1}{3})$

$E(e^X) = \sum_{x \in S} e^x \cdot f_X(x) = e^{-1}(\frac{1}{3}) + e^0(\frac{1}{3}) + e^1(\frac{1}{3})$

$E(\frac{1}{\log X}) = \sum_{x \in S_X} \frac{1}{\log x} \cdot f(x)$

$$\boxed{M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = E(X^k) \text{ } k^{th} \text{ moment}}$$

Thm

(a) $E(c) = c$

(b) $E[c\,u(X)] = c\,E[u(X)]$

(c) $E[c_1 u_1(X) + c_2 u_2(X)] = c_1 E[u_1(X)] + c_2 E[u_2(X)]$

MGF

$M_X(t) = E(e^{tX}) < \infty$, around $t = 0$

$= \sum_{x \in S} e^{tx} \cdot f(x)$

raw moments: $E(X), E(X^2), \ldots$

$M_X(0) = E(e^{0X}) = E(e^0) = E(1) = 1$, exists for any r.v.

uniqueness

X

$M_X(t) = M_Y(t)$ $\forall t$

then $X \overset{d}{=} Y$

$Var(X) = \sigma^2 = E[(X-\mu)^2]$ where $\mu = E(X)$

$SD(X) = \sqrt{\sigma^2}$, $\sigma \geq 0$

$Var(c) = 0$

$Var(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2 = a^2 Var(X)$

$= a^2 E(X^2) - a^2[E(X)]^2$

$E(X^k) \overset{def}{=} \sum_{x \in S} x^k f_X(x)$, k^{th} moment about constant b

$E[(X-b)^k] \overset{def}{=} \sum_{x \in S} (x-b)^k f_X(x)$

$b = \mu = E(X)$, $E[(X-\mu)^k]$ "central moment"

$k = 2$, 2^{nd} central moment is variance

$k = 3$, index of skewness

$\gamma \overset{def}{=} \frac{E[(X-\mu)^3]}{\sigma^3} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$

$\gamma = 0$ symmetric $\gamma > 0$ right $\gamma < 0$ left

ex

x	$f(x)$
1	1/6
2	2/6
3	3/6

$E(X) = 1 \cdot 1/6 + 2 \cdot 2/6 + 3 \cdot 3/6 = 7/3$

$Var = (1)^2(1/6) + (2)^2(2/6) + (3)^2(3/6) - (7/3)^2 = 5/9$

$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3} = \frac{E[(X - 7/3)^3]}{(\sqrt{5/9})^3}$

$\gamma = \frac{E(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$