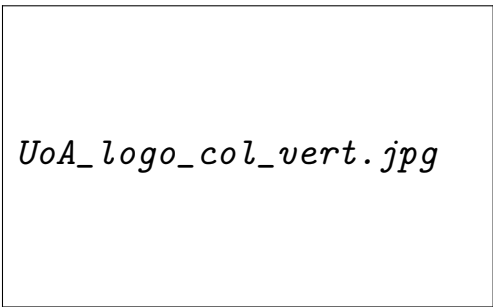


# Computable characterisations of stochastic uncertainty in dynamical systems

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June 8, 2023

*Thesis submitted for the degree of  
Master of Philosophy  
in  
Applied Mathematics  
at The University of Adelaide  
Faculty of Sciences, Engineering and Technology  
School of Computer and Mathematical Sciences*

The logo of the University of Adelaide, featuring a stylized crest with a shield, a cross, and a banner, surrounded by the university's name in a circular border.

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# Acknowledgements



# Dedication



# Abstract

Chapter ?? introduces





# Chapter 1

## Introduction



# Chapter 2

## Background

### 2.1 Results from dynamical systems

$$\frac{dw_t}{dt} = u(w_t, t), \quad w_0 = x \in \Omega, \quad (2.1)$$

where  $u : \Omega \times [0, T] \rightarrow \mathbb{R}^n$  describes the velocity at a point in space and time.

In the mathematical treatment of Lagrangian dynamics, and in particular Lagrangian coherent structures [6], trajectories solving (2.1) are summarised by the flow map. The flow map is an operator mapping

The flow map can be defined formally as follows.

**Definition 2.1.1 (Flow map)** Suppose  $t_1, t_2 \in [0, T]$ . The **flow map**  $F_{t_1}^{t_2} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  from time  $t_1$  to  $t_2$  associated with (2.1) is the unique solution to

$$\frac{\partial F_{t_1}^\tau(x)}{\partial \tau} = u(F_{t_1}^\tau(x), \tau), \quad F_{t_1}^{t_1}(x) = x, \quad (2.2)$$

solved up to time  $\tau = t_2$ . Equivalently,

$$F_{t_1}^{t_2}(x) = x + \int_{t_1}^{t_2} u(F_{t_1}^\tau(x), \tau) d\tau.$$

The flow map satisfies the following properties, under ASSUMPTIONS? For any  $r, s, t \in [0, T]$  and points  $x, w \in \mathbb{R}^n$ ,

1.  $F_s^t$  is invertible with inverse

$$[F_s^t]^{-1}(w) = F_t^s(w).$$

2.  $F_s^t(F_r^s(x)) = F_r^t(x)$

The gradient of the flow map satisfies a useful property; the equation of variations.

**Theorem 2.1.1** *Let  $F_{t_1}^t$  be the flow map corresponding to (2.1). Then, the spatial gradient  $\nabla F_{t_0}^t(x)$  satisfies the equation of variations*

$$\frac{\partial \nabla F_{t_1}^t(x)}{\partial t} = \nabla u(F_{t_1}^t(x), t) \nabla F_{t_1}^t(x). \quad (2.3)$$

**Proof.** Taking the gradient on both sides of (2.2) and using the chain rule gives

$$\nabla \left( \frac{\partial F_{t_1}^t(x)}{\partial t} \right) = \nabla u(F_{t_1}^t(x), t) \nabla F_{t_1}^t(x).$$

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An important inequality for establishing bounds

**Theorem 2.1.2 (Grönwall's inequality)** *Let  $\alpha, \beta, u : [a, b] \rightarrow \mathbb{R}$  be functions such that  $\beta$  and  $u$  are continuous and that the negative part of  $\alpha$  is integrable on every closed and bounded subset of  $[a, b]$ . Then, if  $\beta$  is non-negative and for all  $t \in [a, b]$ ,*

$$u(t) \leq \alpha(t) + \int_a^t \beta(\tau) u(\tau) d\tau$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(\tau) \beta(\tau) \exp \left( \int_\tau^t \beta(s) ds \right) d\tau.$$

Additionally, if  $\alpha$  is non-decreasing, then

$$u(t) \leq \alpha(t) \exp \left( \int_a^t \beta(\tau) d\tau \right)$$

## 2.2 Stochastic differential equations

For an introduction to stochastic differential equations, see Øksendal [34] or Kallianpur and Sundar [26].

**Definition 2.2.1** *The (one-dimensional) canonical Wiener process is a stochastic process  $W_t$  taking values in  $\mathbb{R}$  and satisfying the following properties:*

(i)  $W_0 = 0$ ,

(ii) for every  $s > 0$ , the increments  $W_{s+t} - W_s$  for  $t \geq 0$  are independent of  $W_r$  for all  $r < s$ ,

(iii)  $W_{s+t} - W_t \sim \mathcal{N}(0, s)$  for all  $s, t > 0$ , and

(iv)  $W_t$  is continuous in  $t$  almost surely.

Remarkably, these properties *uniquely* define the Wiener process, with the property that for any  $t > 0$ ,  $W_t \sim \mathcal{N}(0, t)$ , a Gaussian distribution with mean zero and variance  $t$ .

The differential form of an  $n$ -dimensional Itô stochastic differential equation is

$$dy_t = u(y_t, t) dt + \sigma(y_t, t) dW_t, \quad (2.4)$$

where the solution  $y_t$  is a stochastic process taking values in  $\mathbb{R}^n$ ,  $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  is the drift and  $\sigma : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  is the diffusivity. The driving process  $W_t$  is the canonical,  $n$ -dimensional Wiener process as defined in Definition 2.2.1. For a (possibly random) initial condition  $y_0$ , the solution  $y_t$  to (2.4) satisfies

$$y_t = y_0 + \int_0^t u(y_\tau, \tau) d\tau + \int_0^t \sigma(y_\tau, \tau) dW_\tau. \quad (2.5)$$

In the most general case, the drift  $u$  and diffusivity  $\sigma$  are permitted to themselves be random functions [26], but in this thesis we assume that both are deterministic. There is an alternative formulation of a stochastic differential equation, known as the Stratonovich interpretation, which defines the stochastic integral in (2.5) in a way that arises natural from physical considerations. In this thesis, we only consider Itô stochastic differential equations, but there is a conversion between the two interpretations that requires modifying the drift term  $u$  of the SDE.

As with deterministic differential equations, we are interested in knowing precisely when solutions to (2.4) exist, and whether these solutions are unique.

A solution to the stochastic differential equation can be defined rigorously [26].

**Definition 2.2.2** A stochastic process  $\{y_t\}_{t \in [0, T]}$  taking values in  $\mathbb{R}^n$  is said to be a strong solution of (2.4) with initial condition  $y_0 = \xi$  if the following holds:

1. For each  $t$ ,

2.

$$\int_0^T \left( \|u(y_t, t)\| + \|\sigma(y_t, t)\|^2 \right) dt < \infty \text{ a.s.}$$

3. For each  $t \in [0, T]$ ,

$$y_t = \xi + \int_0^t u(y_\tau, \tau) d\tau + \int_0^t \sigma(y_\tau, \tau) dW_\tau \quad \text{a.s.}$$

Solutions are known as *strongly unique* if, for any two strong solutions  $x_t$  and  $y_t$  to (2.4) defined on the same measure space,

$$P(x_t = y_t, \quad \forall t \in [0, T]) = 1$$

That is,  $x_t$  and  $y_t$  are equal almost surely.

We can now state the requirements for (2.4) to have unique solutions.

**Theorem 2.2.1 (Kallianpur and Sundar [26])** *Let  $u_i$  be the  $i$ th component of  $u$  and  $\sigma_{ij}$  the  $(i, j)$ th component of  $\sigma$ , for  $i, j = 1, \dots, n$ . Suppose that for each  $i, j = 1, \dots, n$ ,*

*(i)  $u_i$  is Borel-measurable, and*

*(ii)  $\sigma_{ij}$  is Borel-measurable.*

*Suppose further that for all  $t \in [0, T]$  and  $x, y \in \mathbb{R}^n$ , there is a positive constant  $K$  independent of  $t$  and  $x, y$  such that*

$$(iii) \quad \|u(x, t)\|^2 + \|\sigma(x, t)\|^2 \leq K(1 + \|x\|^2) \quad a.s.$$

$$(iv) \quad \|u(x, t) - u(y, t)\|^2 + \|\sigma(x, t) - \sigma(y, t)\|^2 \leq K\|x - y\|^2 \quad a.s.$$

*Then, (2.4) has a strongly unique strong solution.*

**Proof.** See Theorem 6.2.1 of [26], for instance. □

### 2.2.1 Analytical tools for Itô calculus

There are several tools available for the analytic treatment of Itô integrals and solutions to stochastic differential equations, which we make use of throughout. The first is Itô's isometry, which relates the expectation of an Itô integral to that of a deterministic one and is useful for computing moments.

**Theorem 2.2.2 (Itô's Isometry)** *Let  $f : \Omega \times [0, T] \rightarrow \mathbb{R}$  be an Itô integrable stochastic process. Then, for any  $t \in [0, T]$*

$$\mathbb{E} \left[ \left( \int_0^t f(\omega, \tau) dW_\tau \right)^2 \right] = \mathbb{E} \left[ \int_0^t f(\omega, \tau)^2 d\tau \right]$$

**Proof.** Itô's isometry typically arises in the formal construction of the Itô integral. See ?? for instance. □

Next, we have Itô's Lemma (or the Itô Formula), which is a change-of-variables formula in stochastic calculus and can be thought of as a generalisation of the chain rule from deterministic calculus. We state and use the multidimensional form of the Lemma for solutions to Itô stochastic differential equations, although more general forms exist (e.g. see Theorem 5.4.1 of [26]).

**Theorem 2.2.3 (Itô's Lemma)** *Let  $X_t$  be the strong solution to the stochastic differential equation*

$$dX_t = a(X_t, t) dt + b(X_t, t) dW_t,$$

*where  $a : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n$ ,  $b : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^{n \times p}$  and  $W_t$  is the canonical  $p$ -dimensional Wiener process. If  $f : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^m$  is twice continuously-differentiable, then the stochastic process  $Y_t := f(X_t, t)$  is a strong solution to the stochastic differential equation*

$$dY_t = \left( \frac{\partial f}{\partial t}(X_t, t) + \nabla f(X_t, t) a(X_t, t) + \frac{1}{2} \text{tr} \left[ b(X_t, t)^\top \nabla \nabla f(X_t, t) b(X_t, t) \right] \right) dt + \nabla f(X_t, t) b(X_t, t) dW_t.$$

**Proof.**

□

Our third and final result is the Burkholder-Davis-Gundy inequality, which when applied to stochastic integrals provides bounds on the expected norm.

**Theorem 2.2.4 (Burkholder-Davis-Gundy Inequality)** *Let  $M_t$  be an Itô-integrable stochastic process taking values in  $\mathbb{R}^n$ . Then, for any  $p > 0$  there exists constants  $c_p, C_p > 0$  independent of the stochastic process  $M_t$  such that*

$$c_p \mathbb{E} \left[ \left( \int_0^t \|M_\tau\|^2 d\tau \right)^p \right] \leq \mathbb{E} \left[ \sup_{\tau \in [0, t]} \left\| \int_0^\tau M_s dW_s \right\|^{2p} \right] \leq C_p \mathbb{E} \left[ \left( \int_0^t \|M_\tau\|^2 d\tau \right)^p \right].$$

**Proof.**

□

## 2.2.2 The Fokker-Planck equation

The probability density function  $\rho : \mathbb{R}^n \times [0, T] \rightarrow [0, \infty)$  for the solution to (2.4) at time  $t \in [0, T]$  is the solution to the corresponding Fokker-Planck equation [38]

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \nabla \cdot \nabla \cdot (\rho \sigma \sigma^\top) - \nabla \cdot (\rho u) \quad (2.6)$$

subject to some initial density  $\rho(x, 0)$  given by the initial condition to (2.4). For a fixed and deterministic initial condition  $y_0 = x$ , the corresponding initial condition to (2.6) is the Dirac-delta distribution centred at  $x$ .

For certain choices of the drift  $u$  and diffusivity  $\sigma$ , (2.6) reduces to several other well-known partial differential equations, including:

- When  $\sigma \equiv D$ , a scalar constant, then (2.6) is the convection-diffusion equation with velocity field  $u$  and diffusivity  $D$ .

- The connection between the Fokker-Planck equation and the SDE (2.4) means that the solutions to each of these PDEs can be equivalently thought of as the time-evolution of the probability density of an SDE.

In general, the solution to a stochastic differential equation cannot be expressed analytically, either as an explicit expression involving the Wiener process  $W_t$  or as a probability measure or density function. At best, most solutions can be written in terms of an Itô integral which can otherwise not be simplified. However, there are several simple examples for which a solution can be written, and even time-marginal probability density functions can be derived. Here, we list several examples which are used to validate theory and test algorithms throughout this thesis.

**Example 2.2.2 (Benê's SDE)** *The 1-dimensional stochastic differential equation*

is known as Benê's stochastic differential equation [41]. The probability density function of a weak solution of (2.7) can be derived using an appropriate change of measure with Girsanov's theorem. A proof of this, and the derivation of a weak solution to (2.7) are provided in Section 7.3 of [41]. The probability density function  $p : \mathbb{R} \times (0, \infty) \rightarrow [0, \infty)$  for the solution  $x_t$  at time  $t > 0$  is given by

where  $x_0 \in \mathbb{R}$  is a fixed initial condition. This probability density function is plotted, for the initial condition  $x_0 = 1/2$  and various times, in Figure 2.1. The resulting density is not symmetric and bimodal, with the two modes moving apart in the positive and negative  $x$ -directions respectively as  $t$  increases. For fixed  $t$ , the probability density function can be expressed as the mixture of two Gaussians with respective means  $x_0 + t$  and  $x_0 - t$ , with details provided in Appendix A.1. This expression allows easy calculation of the mean and expectation of  $x_t$ , as

and

$$\mathbb{V}[x_t] = \mathbb{V}, \mathbb{Z}, \mathbb{Z}, \mathbb{R}, \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{N}, \mathbb{C}, \mathbb{Q}, \mathbb{N}, \mathbf{R}, \mathbf{R}, \mathbf{N}, \mathbf{N}, \mathcal{R}, \mathfrak{R}, \mathbb{1}$$



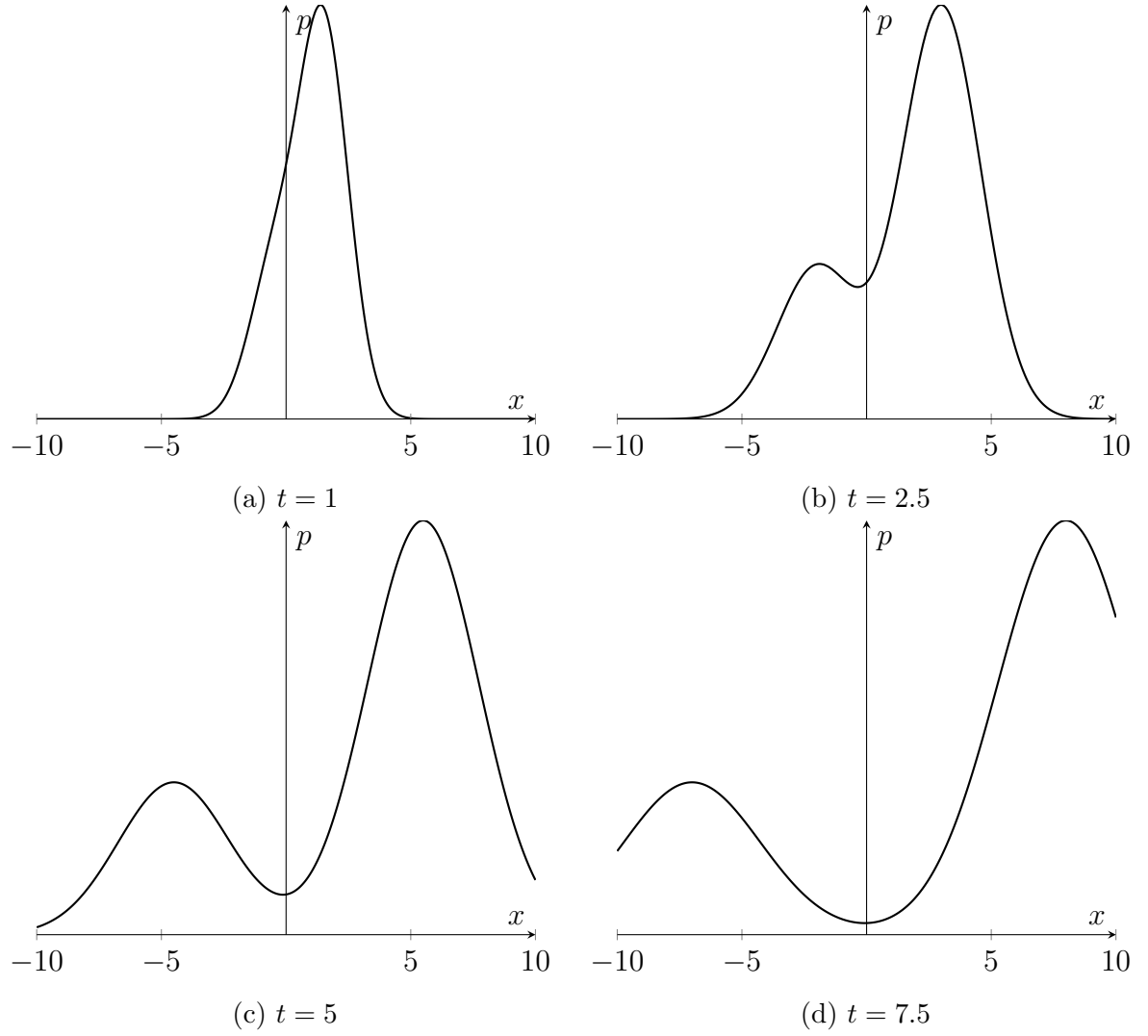


Figure 2.1: The probability density function (2.8) for the time-marginal solution of Benê's SDE (2.7), for the initial condition  $x_0 = 1/2$  at various times. The density function consists of two distinct modes that move further apart as  $t$  increases.

## 2.3 Stochastic sensitivity

Following the nomenclature of fluid dynamics, given unsteady Eulerian velocity data  $u : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}^2$ , the evolution of Lagrangian trajectories is governed by the ordinary differential equation

$$\frac{dx_t}{dt} = u(x_t, t).$$

Solutions can be summarised by the flow map  $F_{t_1}^{t_2}$ , as in ?? . In most practical situations, the Eulerian velocity data driving ocean and atmospheric models relies upon measurements of estimates obtained on a low resolution spatial discretisation. Balasuriya [4] introduces stochastic sensitivity as a new tool for directly quantifying the impact of Eulerian uncertainty on Lagrangian trajectories. The evolution of Lagrangian trajectories is modelled as solution to a Itô stochastic ordinary differential equation.

To directly account for these unresolved sources of uncertainty, the “true” Lagrangian trajectories evolve as solution to the stochastic differential equation

$$dy_t = u(y_t, t) dt + \varepsilon \sigma(y_t, t) dW_t, \quad (2.9)$$

where  $0 < \varepsilon \ll 1$  is a parameter quantifying the scale of the noise,  $\sigma : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}^{2 \times 2}$  is the  $2 \times 2$  diffusion matrix, and  $W_t$  is the canonical two-dimensional Wiener process. In the original formulation [4],  $\varepsilon$  is a dimensionless parameter and  $\sigma$  is dimensional, but an alternative scaling technique relates  $\varepsilon$  to spatial and velocity uncertainty scales in the data (see the follow-up work [3, 5, 20]) Since  $\sigma$  can vary by both space and time, the noise is multiplicative. The diffusion matrix  $\sigma$  is specified *a priori*, based on any knowledge of how uncertainty varies with space and time, e.g. from experimental considerations, observation error estimates. If no such prior information is known, then  $\sigma \equiv I$ , the  $2 \times 2$  identity matrix is the default choice.

To quantify uncertainty in a way that is independent of the noise scale  $\varepsilon$ , Balasuriya [4] defined the random variable  $z_\varepsilon(x, t)$  on  $\mathbb{R}^2 \times [0, T]$  as

$$z_\varepsilon(x, t) := \frac{y_t - F_0^t(x)}{\varepsilon}.$$

The main aim is to compute statistics of  $z_\varepsilon$  at the final time  $T$ , so that of  $z_\varepsilon(x, T)$ . Balasuriya [4] then considers the signed projection of  $z_\varepsilon(x, T)$  onto a ray emanating from the deterministic position  $F_0^T(x)$  in a given direction, defining

$$P_\varepsilon(x, \theta) := \hat{n}^\top z_\varepsilon(x, T),$$

where  $\theta \in [-\pi/2, \pi/2)$  and

$$\hat{n}(\theta) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}.$$

The statistics of  $z_\varepsilon(x, T)$  and  $P_\varepsilon(x, \theta)$  are considered in the limit as  $\varepsilon \downarrow 0$ , which provides a characterisation of the uncertainty of the model *independently* of the scale of the noise. Balasuriya [4] provided computable expressions for the mean and variance of  $P_\varepsilon(x, \theta)$  in this limit of small noise, which we summarise here. For proofs of these results, see the appendices of [4].

The first result established by Balasuriya [4] is that the expected location is deterministic, in the following sense.

**Theorem 2.3.1 ([4])** *For all  $x \in \mathbb{R}^2$ ,*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[z_\varepsilon(x, T)] = 0.$$

The variance of  $P_\varepsilon(x, \theta)$  is used to assign a computable scalar measure of uncertainty to the trajectory.

**Definition 2.3.1 ([4])** *a) The **anisotropic uncertainty** is a scalar field  $A : \mathbb{R}^2 \times [-\pi/2, \pi/2) \rightarrow [0, \infty)$  defined by*

$$A(x, \theta) := \sqrt{\lim_{\varepsilon \downarrow 0} \mathbb{V}[P_\varepsilon(x, \theta)]}.$$

*b) The **stochastic sensitivity** is a scalar field  $S : \mathbb{R}^2 \rightarrow [0, \infty)$  defined by*

$$S^2(x) := \lim_{\varepsilon \downarrow 0} \sup_{\theta} \mathbb{V}[P_\varepsilon(x, \theta)].$$

By employing techniques from both deterministic and stochastic calculus (i.e. Grönwall's inequality, the Burkholder-Davis-Gundy inequality, Itô's Lemma), Balasuriya further established expressions for both the anisotropic uncertainty and the stochastic sensitivity that are computable given only the flow map and velocity data.

**Theorem 2.3.2 ([4])** *For  $x \in \mathbb{R}^2$ , set  $w := F_0^t(x)$ . Then, for any  $\theta \in [-\pi/2, \pi/2)$ ,*

$$A(x, \theta) = \left( \int_0^T \|\Lambda(x, t, T) J \hat{n}(\theta)\| dt \right)^{1/2},$$

where

$$\Lambda(x, t, T) := e^{\int_t^T [\nabla \cdot u](F_0^\xi(x), \xi) d\xi} \sigma(F_0^t(x), t)^\top J \nabla_w F_T^t(w),$$

with the gradients  $\nabla_w$  of the flow map taken with respect to the mapped position  $w$ , and

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Additionally, stochastic sensitivity is computed as

$$S^2(x) = P(x) + N(x),$$

with

$$\begin{aligned} L(x) &:= \frac{1}{2} \sum_{i=1}^2 \int_0^T \left[ \Lambda_{i2}(x, t, T)^2 - \Lambda_{i1}(x, t, T)^2 \right] dt \\ M(x) &:= \sum_{i=1}^2 \int_0^T \Lambda_{i1}(x, t, T) \Lambda_{i2}(x, t, T) dt \\ N(x) &:= \sqrt{L^2(x) + M^2(x)} \\ P(x) &:= \left| \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 \int_0^T \Lambda_{ij}(x, t, T)^2 dt \right|, \end{aligned}$$

where  $\Lambda_{ij}$  is the  $(i, j)$ -element of  $\Lambda$ .

### 2.3.1 Current applications & shortcomings

Since stochastic sensitivity is only a recent development, it has only been applied in a limited number of places so far. Here, we briefly review the literature in which the original formulation stochastic sensitivity by [4] has been applied.

- Balasuriya [5] uses stochastic sensitivity to compute an error bound for the finite-time Lyapunov computation.
- Fang et al. [20]
- Badza et al. [3] investigate the impact of velocity uncertainty on Lagrangian coherent structures (e.g. see the reviews by Balasuriya et al. [6] and Hadjighasem et al. [22]) extracted as robust sets with stochastic sensitivity. The stochastic model (2.9) is used to generate realisations of Lagrangian trajectories subject to noise on the velocity field. By directly capturing such uncertainty as a means of coherent set Badza et al. [3] showed that robust sets extracted with stochastic sensitivity do SOMETHING.

There are several limitations to the work as presented in [4],

1. The tools are restricted to two-dimensional models, and the constructions using projections have no obvious extension to  $n$ -dimensions. Extending stochastic sensitivity to  $n$ -dimensions will enable application to a much broader class of models beyond the fluid flow context, including high-dimensional climate models.

2. Balasuriya [4] only computes the expectation and variance of the projections  $P_\varepsilon(x, \theta)$ , which does not give us the distribution under the limit as  $\varepsilon$  approaches 0.
3. The computational formula for the anisotropic uncertainty and stochastic sensitivity, as described in Theorem 2.3.2, require knowledge of the divergence  $\nabla \cdot u$  of the velocity field, and computation of four integrals.

An alternative approach to uncertainty quantification in Lagrangian dynamics was recently introduced by Branicki and Uda [12], extending results from their earlier work in [11]. However, the divergence approach does not provide any insight into the underlying probability distribution. Moreover, stochastic sensitivity



## Chapter 3

# Publication: Gaussian characterisation of model uncertainty

The following is a copy of the published article by Blake et al. [10]. Sections 3.2 to 3.7 are as presented in [10]. The appendices include more details than in the published version.

### 3.1 Statement of Authorship

### 3.2 Abstract

Prediction via deterministic continuous-time models will always be subject to model error, for example due to unexplainable phenomena, uncertainties in any data driving the model, or discretisation/resolution issues. A standard method for uncertainty quantification in such instances is to introduce noise into the system, and use stochastic simulations to empirically obtain error distributions. To supplement this computationally expensive approach, we develop an explicit and computable time-evolving uncertainty distribution for stochastic differential equations with small multiplicative noise. For any initial condition, we rigorously establish convergence bounds for all moments of the deviation of the stochastic solution from its linearised counterpart. This result extends previous work, that showed the convergence of the Kullback-Leibler divergence. We provide a characterisation of the Gaussian distribution that is the solution to the linearised equation, expressed explicitly in terms of solutions to a reference deterministic model. This characterisation provides a practical framework for quantifying uncertainty in deterministic differential equation models, with applications including oceanographic and atmospheric modelling, data assimilation and Lagrangian coherent structure extraction

### 3.3 Introduction

Many phenomena across geophysical, biological and socio-economic applications can be modelled using a continuous-time dynamical system, i.e., an ordinary differential equation [13, 44, 45, e.g.]. Given initial values of a multi-dimensional state variable, such equations can be solved numerically to predict the state at future times. The governing dynamics may be specified using existing phenomenological models, but in modern applications these are usually supplemented or driven by observed data. Standard examples include the modelling of weather using available data [29, 37], and predicting concentrations of, for instance, temperature, pollutants or phytoplankton in the ocean using observed current velocity data [1, 19].

All methods using this approach have uncertainties in the model specification arising from a variety of sources [20]: the model not capturing all phenomena because of the inevitable lack of a complete understanding of all processes involved, errors in measured data, information only available on spatio-temporal grids (resolution error), etc. In the absence of any other understanding of these multitudinous issues, a well-established way of tackling such uncertainties in the model is to think of these as stochastic [7, 34]. Running many realisations of stochastic perturbations to the deterministic model can generate statistics to improve predictions and estimate their associated uncertainties [3, 16, e.g.]. However, in practice a very large number of simulations is necessary to generate convergent statistics [21, 30]. Thus, numerically solving such stochastic systems – potentially with data-based terms – is often computationally expensive, and does not necessarily provide conceptual insight into how the model uncertainties affect predictions.

Clearly, possessing a broader theoretical understanding of how stochastic terms impact continuous dynamical systems would be valuable. Stochastic differential equations (SDEs) provide a natural framework for introducing uncertainty, as a noise process, into the continuous time evolution of a variable. Generally, in modelling situations the dynamics are highly nonlinear and one expects the noise to be multiplicative (i.e. vary with both state and time), e.g. in atmospheric [42, 43] and oceanic [27] systems and from experimental and observational considerations. Such SDEs are intractable to solve analytically [34] and computationally expensive to simulate accurately [33]. Having a data-based model—that is, possessing terms in the equations which are driven by data rather than by explicitly specified functions—renders additional problems in obtaining a theoretical understanding of the prediction error.

A common intuitive approach to characterising the uncertainty arising from an otherwise analytically intractable nonlinear SDE is via a multivariate Gaussian approximation, which is used across a diversity of literature. For instance, one can formally “linearise” the SDE in some sense to obtain a Gaussian density, and this approach is used in filtering theory [24]. Other approaches first assume a Gaussian distribution and obtain formal computations for its mean and covariance [41]. However, both approaches lack rigorous justification and a precise understanding of *how* the Gaussian distribution arises from the



nonlinear SDE. This is particularly the case when the noise is multiplicative, which is a situation that is often ignored but necessary in practice. Sanz-Alonso and Stuart [40] partially addressed these issues, by showing that the Kullback-Leibler (KL) divergence between the solutions of autonomous SDEs with additive noise and a linearised equivalent can be bounded by the scale of the noise. In this manuscript, we relax the hypotheses of [40] to cater for time-dependent coefficients and for multiplicative noise. Furthermore, our result explicitly establishes the convergence rate of all moments of the deviation considered in [40], which cannot be inferred from the KL divergence alone.

In this paper, we remedy this deficiency by proving rigorously that the noise-scaled deviation between the SDE solution and a reference deterministic solution converges towards a multivariate Gaussian distribution, in the limit of small noise. We consider a general class of SDEs with fully non-autonomous terms and multiplicative noise. The Gaussian distribution arises as the solution to a formal linearisation of the SDE about a deterministic trajectory (in the absence of noise). By bounding all raw moments of the difference between the SDE and the linearised solutions by the noise scale (see Theorem 3.4.1), we show that the stochastic deviation converges in distribution to a multivariate Gaussian random variable (see Theorem 3.4.2). The covariance matrix characterising this Gaussian can be explicitly written in terms of the flow map of the underlying deterministic system and the (potentially spatio-temporally varying) diffusion matrix, and is available even if the deterministic model is only available via data. The Gaussian distribution is consistent with that seen in other literature [24, 40, 41], while we additionally show convergence of *all* the moments of the deviation distribution. The results hold independently of the initial condition and for all finite times; the uncertainty evolution of any deterministic trajectory with time is therefore encapsulated in our results.

The quantification of prediction uncertainty that we present here was originally motivated by the “stochastic sensitivity” approach of Balasuriya [4]. In the context of two-dimensional, unsteady fluid flow, stochastic sensitivity works with Eulerian velocity data as the underlying deterministic model, and seeks to quantify the uncertainty in an eventual Lagrangian trajectory location. This methodology was developed as a tool for determining Lagrangian coherent structures (LCS) [6, 22] in fluid flows, in that clusters of trajectories which have small uncertainty may be thought of as more “coherent” than others. In particular, [4] derived the limiting mean and variance of the noise-scaled deviation, and provided computable expressions in terms of the deterministic flow map and velocity field. However, this was restricted to two-dimensional systems and did not characterise the limiting distribution itself.

The contributions of this work are:

- In Section 3.4, we prove rigorously that all moments of the noise-scaled solution to a multidimensional stochastic differential equation with non-autonomous coefficients and multiplicative noise converges towards those of a linearised SDE, in the limit of small noise. The Gaussian distribution solving the linearised SDE appears in

other literature and applications but often lacks justification [24, 41]. On the other hand, when the linearisation is justified, this is disregarding time-dependence in the coefficients and multiplicative noise [40].

- We present characterisations of the limiting Gaussian distribution in terms of gradients of *either* the velocity field (as an ODE consistent with that arising elsewhere [24, 40, 41]) or the deterministic flow map. The latter is an alternative characterisation that allows the Gaussian distribution to be computed *entirely from the solution dynamics* of a deterministic model and specification of any multiplicative noise effects known prior.
- In Section 3.5, we generalise the two-dimensional stochastic sensitivity approach of [4] to arbitrary dimensions. Our expressions enable the computation of stochastic sensitivity in any dimension, as a scalar measure of uncertainty about any solution trajectory of the deterministic model. This also extends stochastic sensitivity as a means of Lagrangian coherent structure extraction to fluid flows of arbitrary dimension.
- In Section 3.6, we validate the results of Section 3.4 using stochastic simulations from a 2-dimensional model. In particular, we demonstrate that the first four moments of the distance between the realisations and the Gaussian limit follow the predicted bound. We also illustrate a key prediction from Section 3.5; that the computable covariance matrix of the Gaussian limit captures the time-evolution of uncertainty, even when the noise is multiplicative.

This work is relevant to the well-known “stochastic parameterisation” approach in weather and climate modelling, in which stochastic components are introduced to account for unresolved subgrid effects [7, 31, 35]. Since this work is fundamental, in establishing a convergence result for a general class stochastic differential equations, we do not explicitly describe how to construct an appropriate stochastic parameterisation (e.g. specification of the coefficients of the SDE). Instead, we expect that this result will be useful in the analysis of stochastic parameterisations, and to convert otherwise computationally expensive schemes into an efficient approximations, a goal explicitly identified in [31]. We also expect that this work will find application in data assimilation [15, 24, 29, 37], as a means of characterising forecast uncertainty. The original stochastic sensitivity tools have been applied to identify Lagrangian coherent structures (LCSs) in 2-dimensional fluid flow [3, 4]. By extending the theory of these tools into arbitrary dimensions, our results can also be used to extract coherent structures in  $n$ -dimensional flows. These potential applications are discussed in ??.

### 3.4 The Gaussian limit

Suppose we are interested in the evolution of a  $\mathbb{R}^n$ -valued state variable  $y_t$  over a finite time interval  $[0, T]$ . Our model, accounting for uncertainties arising from a range of sources, for the evolution of this variable is the Itô stochastic differential equation

$$dy_t^{(\varepsilon)} = u(y_t^{(\varepsilon)}, t) dt + \varepsilon \sigma(y_t^{(\varepsilon)}, t) dW_t, \quad (3.1)$$

where  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$  is the governing reference vector field, and can be inferred from underlying physics or available data, for instance. The canonical  $n$ -dimensional Wiener process  $W_t$  is a continuous white-noise stochastic process with independent Gaussian increments. The scale of the noise is parameterised as  $0 < \varepsilon \ll 1$  and  $\sigma : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times n}$  is a deterministic diffusion matrix. The noise in (3.1) is multiplicative, in that the diffusion matrix  $\sigma$  can vary with both state and time. We assume that  $\sigma$  is specified *a priori*, or if no such information is known, then  $\sigma \equiv I_n$ , the  $n \times n$  identity matrix, is a default choice. We assume certain generic smoothness and boundedness conditions on the various functions outlined; these are stated explicitly in Hypothesis H in Section 3.8 and ensure the existence of unique solutions to (3.1). The stochastic solution  $y_t^{(\varepsilon)}$  to (3.1) describes the evolution of the state variable through time, accounting for ongoing uncertainty with noise-scale  $\varepsilon$ .

In the absence of any uncertainty (i.e.  $\varepsilon = 0$ ), (3.1) reduces to the ordinary differential equation

$$\frac{dw_t}{dt} = u(w_t, t). \quad (3.2)$$

Let the flow map  $F_0^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the function which evolves an initial condition from time 0 to time  $t$  according to the flow of (3.2). We refer to (3.2) as the *reference* deterministic model associated with (3.1) in that it either demonstrates the dominant physics (as would be the case if we think of the noise in (3.1) as capturing stochastic parameterisation) or is the best-available model (for example if  $u$  is available from data, and (3.1) represents the uncertainty of such data). Solutions to the reference deterministic model are more readily available, e.g. in terms of computational efficiency when solving numerically, than those of the stochastic model, but do not account for inevitable uncertainty. Here, we establish a Gaussian characterisation and approximation of the solution to (3.1) constructed from the deterministic flow map, thereby taking advantage of the easily available solutions to (3.1) and avoiding the need for computationally expensive stochastic simulation.

To characterise the uncertainty, we fix the *identical* initial condition  $x \in \mathbb{R}^n$  to *both* the stochastic model (3.1) and the reference deterministic model (3.2), and consider their evolution in time. We will show that the stochastic deviation between solutions of these can be characterised exactly in terms of a Gaussian in the limit of small noise, i.e.  $\varepsilon \rightarrow 0$ . This quantifies the time-evolving uncertainty of predictions from the deterministic model

(3.2). We provide explicit analytical expressions for the limiting distribution, written in terms of the flow map of the deterministic system and  $\sigma$ , thereby providing strong theoretical insight while nullifying the need to perform expensive SDE simulations in approximating such a distribution.

To express our results, we define the noise-scaled deviation

$$z_t^{(\varepsilon)}(x) := \frac{y_t^{(\varepsilon)} - F_0^t(x)}{\varepsilon}, \quad z_0^{(\varepsilon)}(x) = 0, \quad (3.3)$$

where  $x \in \mathbb{R}^n$  is fixed and certain, and satisfies  $y_0^{(\varepsilon)} = x$ . We wish to understand the limiting behaviour of  $z_t^{(\varepsilon)}(x)$  as  $\varepsilon$  approaches zero.

**Theorem 3.4.1 (All moments are bounded)** *Fix  $x \in \mathbb{R}^n$  and let  $z_t(x)$  be the solution to the linearised SDE*

$$dz_t(x) = \nabla u(F_0^t(x), t) z_t(x) dt + \sigma(F_0^t(x), t) dW_t, \quad z_0(x) = 0, \quad (3.4)$$

where  $W_t$  is the same Wiener process driving (3.1). Then for any  $r \geq 1$  and  $t \in [0, T]$ , there exists a  $D_r(t) \in [0, \infty)$  independent of  $x$  such that for all  $\varepsilon > 0$ ,

$$\mathbb{E} \left[ \left\| z_t^{(\varepsilon)}(x) - z_t(x) \right\|^r \right] \leq D_r(t) \varepsilon^r, \quad (3.5)$$

where  $\|\cdot\|$  is the Euclidean norm.

**Proof.** See Section 3.9. Showing the result employs the Burkholder-Davis-Gundy inequality, Grönwall's inequality, Taylor's theorem and the bounds placed on the SDE coefficients, to explicitly construct the bounding coefficient  $D_r(t)$ .  $\square$

Taking the limit as  $\varepsilon$  approaches 0 in (3.5) shows that  $z_t^{(\varepsilon)}(x)$  converges in  $r$ th moment to  $z_t(x)$ , which in turn implies convergence in probability and convergence in distribution (or weak convergence). It is important to note that the stochastic differential equation (3.1) and the linearised equation (3.4) must be defined with the *same* Wiener process  $W_t$  for Theorem 3.4.1 to hold as stated. However, by weakening the convergence we can think of  $z_t^{(\varepsilon)}(x)$  as converging to a Gaussian distribution (the distribution of the linearised solution) with no reference to the driving Wiener process.

**Theorem 3.4.2 (Explicit Gaussian limit)** *For any  $x \in \mathbb{R}^n$  and  $t \in [0, T]$ ,*

$$z_t^{(\varepsilon)}(x) \xrightarrow{d} \mathcal{N}(0, \Sigma(x, t)) \quad \text{as } \varepsilon \rightarrow 0,$$

where the covariance matrix  $\Sigma$  is given by

$$\Sigma(x, t) = \int_0^t L(x, t, \tau) L(x, t, \tau)^\top d\tau, \quad (3.6)$$

with

$$L(x, t, \tau) := \nabla F_0^t(x) [\nabla F_0^\tau(x)]^{-1} \sigma(F_0^\tau(x), \tau). \quad (3.7)$$

Moreover, the covariance matrix  $\Sigma$  is the matrix solution to the ordinary differential equation

$$\frac{d\Sigma}{dt} = \left[ \nabla u(F_0^t(x), t) \right] \Sigma + \Sigma \left[ \nabla u(F_0^t(x), t) \right]^\top + \sigma(F_0^t(x), t) \sigma(F_0^t(x), t)^\top, \quad (3.8)$$

subject to  $\Sigma(x, 0) = O$ , the  $n \times n$  zero matrix.

**Proof.** See Section 3.10. The Gaussianity of the limiting process  $z_t(x)$ , and therefore the limit in distribution of  $z_t^{(\varepsilon)}(x)$ , is first established, and then the explicit expression for the covariance matrix  $\Sigma$  is derived by employing Itô's isometry and properties of the flow map.  $\square$

The covariance matrix  $\Sigma$  uniquely characterises the limiting Gaussian distribution in Theorem 3.4.2, and captures the impact of both the deterministic dynamics of the model (through the flow map gradients) and multiplicative noise (by evaluating the diffusion matrix  $\sigma$  along the deterministic trajectory). Through (3.6) and (3.7), the Gaussian distribution can be computed entirely from flow map data and specification of  $\sigma$ , without any reference to the governing vector field  $u$  in (3.2). Alternatively, if the velocity gradients  $\nabla u$  are available, then  $\Sigma$  can be computed as the solution to (3.8). Solving (3.2) and (3.8) jointly describes the scheme for computing the Gaussian approximation seen elsewhere, e.g. Algorithm 9.4 of [41] or Equations (1.2) and (1.3) of [40]. Moreover, for a fixed time  $t \in [0, T]$ , Theorem 3.4.2 justifies the approximation

$$y_t^{(\varepsilon)} \sim \mathcal{N}(F_0^t(x), \varepsilon^2 \Sigma(x, t)), \quad (3.9)$$

for small values of  $\varepsilon$ .

The theoretical results in this section, and the computability of the limiting distribution will be verified with numerical simulation of an example in Section 3.6. This theory has many applications and extensions which are discussed in ??.

## 3.5 Extending stochastic sensitivity

The covariance matrix  $\Sigma$  provides a direct extension of the stochastic sensitivity tools first introduced by Balasuriya [4] for the fluid flow context. Here, the deterministic model (3.2) is seen as a “best-available” model for the evolution of trajectories, and the driving vector field  $u$  is the Eulerian velocity of the fluid. Stochastic sensitivity ascribes a scalar value to each deterministic trajectory by computing the maximum variance of the scaled deviations, when projected onto rays emanating from the origin [4]. The natural restating of this original definition of stochastic sensitivity [4] in the  $n$ -dimensional setting is as follows:

**Definition 3.5.1 (Stochastic sensitivity in  $\mathbb{R}^n$ )** *The stochastic sensitivity is a scalar field  $S^2 : \mathbb{R}^n \times [0, T] \rightarrow [0, \infty)$  given by*

$$S^2(x, t) := \limsup_{\varepsilon \downarrow 0} \left\{ \mathbb{V} \left[ p^\top z_t^{(\varepsilon)}(x) \right] : p \in \mathbb{R}^n, \|p\| = 1 \right\}.$$

Definition 3.5.1 is in the spirit of principal components analysis [25], performing a dimension reduction by projecting onto the direction in which the variance is maximised, thus capturing the most uncertainty in the data with a scalar value.

The anisotropic uncertainty in two-dimensions [4] is the direction-dependent projection (prior to optimising over all directions in Definition 3.5.1). Explicit theoretical expressions for both the stochastic sensitivity and the anisotropic sensitivity in two dimensions were obtained by Balasuriya [4]; these allowed for quantifying certainty in eventual trajectory locations without having to perform stochastic simulations. We show here that our results in  $n$ -dimensions are a generalisation of the two-dimensional ones in [4], which moreover establish Gaussianity as well as an explicit expression for the uncertainty measure. A theoretically pleasing and computable expression for the stochastic sensitivity is obtainable: ■

**Theorem 3.5.1 (Computation of  $S^2$ )** *For any  $x \in \mathbb{R}^n$  and  $t \in [0, T]$ ,*

$$S^2(x, t) = \|\Sigma(x, t)\|, \quad (3.10)$$

where the covariance matrix  $\Sigma$  is defined in (3.6) and  $\|\cdot\|$  denotes here the spectral norm induced by the Euclidean norm. Equivalently,  $S^2(x, t)$  is given by the maximum eigenvalue of  $\Sigma(x, t)$ .

**Proof.** See Section 3.11. This result uses Theorem 3.4.1 to establish the convergence of the covariance matrices, and then the properties of the spectral norm to establish (3.10). □

The stochastic sensitivity field can be calculated given any velocity data  $u$ , and through the explicit expression (3.6) for  $\Sigma$  can even be computed from only flow map data. Computation does not require knowledge of the noise scale  $\varepsilon$ , so the  $S^2$  field is intrinsic in capturing the impact of the model dynamics on uncertainty, and any specified non-uniform diffusivity.

It has already been shown that, in the fluid flow context, stochastic sensitivity can identify coherent regions in two-dimensions [3, 4]. A simple approach is to define robust sets, which are those initial conditions for which the corresponding  $S^2$  value, i.e., the uncertainty in eventual location, are below some specified threshold. This threshold can be defined precisely in terms of a spatial lengthscale of interest and the advective and diffusive characteristics of the flow, as Definition 2.9 of [4]. Such a definition extends

to the  $n$ -dimensional case as presented here, moreover establishing an easily computable method for determining coherent sets by using the covariance matrix  $\Sigma$ .

Independent of the fluid mechanics context, Theorem 3.5.1 indicates that even for general systems, the matrix norm of  $\Sigma(x, t)$ , i.e., the stochastic sensitivity  $S^2(x, t)$ , can be used as *one* number which encapsulates the uncertainty of an initial state  $x$  after  $t$  time units.

## 3.6 Numerical validation and applications

This section will validate the theory presented in Section 3.4. Following the example in Chapter 5 of [39], we consider an unsteady meandering jet in two dimensions, which may serve as an idealised model of geophysical Rossby waves. The velocity field for  $y \equiv (y_1, y_2)$  is given by [39]

$$u(y, t) = \begin{bmatrix} c - A \sin(Ky_1) \cos(y_2) + \epsilon_{mj} l_1 \sin(k_1(y_1 - c_1 t)) \cos(l_1 y_2) \\ AK \cos(Ky_1) \sin(y_2) + \epsilon_{mj} k_1 \cos(k_1(y_1 - c_1 t)) \sin(l_1 y_2) \end{bmatrix}. \quad (3.11)$$

The velocity field describes a kinematic travelling wave with deterministic oscillatory perturbations in a co-moving frame. Here,  $A$  is the amplitude and  $c$  is the phase speed of the primary wave, and  $K$  is the wavenumber in the  $y_1$ -direction. The oscillatory perturbation has amplitude  $\epsilon_{mj}$ , phase speed  $c_1$  (in the co-moving frame), and wavenumbers  $k_1$  and  $l_1$  in the  $y_1$ - and  $y_2$ -directions respectively. Throughout, we take the parameter values  $c = 0.5$ ,  $A = 1$ ,  $K = 4$ ,  $l_1 = 2$ ,  $k_1 = 1$ ,  $c_1 = \pi$ , and  $\epsilon_{mj} = 0.3$ . For these values, the flow consists of a meandering jet with vortex structures within the meanders, and a chaotic zone which influences the fluid transfer between the jet and the vortices. All necessary flow map data is obtained by directly solving (3.2) numerically, with the standard Euler scheme. The flow map gradients required for computing the covariance matrix with (3.6) are calculated via a star-grid finite-difference approximation, using a spatial resolution of 0.001.

All simulations in this section were generated using the Julia programming language [8], with implementations of the ordinary and stochastic differential equation solvers provided by the DifferentialEquations.jl package [36]. All figures were created using the Makie.jl package [18]. The code is available as open source<sup>1</sup>.

### 3.6.1 Validation of Theorem 3.4.1

This section will validate the bound in Theorem 3.4.1 directly, and illustrate the convergence of the SDE solution towards the expected Gaussian distribution described in Theorem 3.4.2. For each value of  $\varepsilon$  considered, we use the Euler-Maruyama method [28] to generate

<sup>1</sup>at <https://github.com/liamblake/explicit-gaussian-characterisation-uncertainty>.

$N = 10000$  independent realisations of the solutions to (3.1) and (3.4). A step size of  $\delta t = 10^{-4}$  is used to ensure that numerical error does not dominate over the theoretical predictions. These solution samples are generated with the *same* realisations of the Wiener process increments  $W_{t+\delta t} - W_t \sim \mathcal{N}(0, \delta t I_n)$ . We consider the initial condition  $x = (0, 1)$  and the prediction of the model at time  $t = 1$ . For each realisation of  $y_t^{(\varepsilon)}$ , a corresponding realisation of the scaled deviation  $z_t^{(\varepsilon)}(x)$  is computed. In the following, let  $\hat{z}_1^{(\varepsilon)}, \dots, \hat{z}_N^{(\varepsilon)}$  and  $\hat{z}_1, \dots, \hat{z}_N$  denote the  $N$  realisations of  $z_t^{(\varepsilon)}(x)$  and  $z_t(x)$  respectively.

Figure 3.1 shows the resulting simulations of  $y_t^{(\varepsilon)}$  for four different values of  $\varepsilon$ . The realisations are binned as a histogram and bin counts are normalised, to provide an empirical estimate of the probability density function of  $y_t^{(\varepsilon)}$ . Superimposed (in solid black) are the first, second and third standard-deviation contours of the probability density function of the Gaussian approximation (3.9). The first three standard-deviation levels of the  $2 \times 2$  sample covariance matrix of the realisations, are also overlaid (in dashed blue). As  $\varepsilon$  decreases towards 0, the samples increasingly resemble a Gaussian distribution, and both the mean and covariance coincide with the corresponding limits.

To directly validate Theorem 3.4.1 for  $r \geq 1$ , define the error metric

$$\Gamma_z^{(r)}(\varepsilon) := \frac{1}{N} \sum_{i=1}^N \left\| \hat{z}_i^{(\varepsilon)} - \hat{z}_i \right\|^r,$$

which is an estimator of the right-hand side of (3.5). For  $r = 1, 2, 3, 4$ ,  $\Gamma_z^{(r)}(\varepsilon)$  is shown (in a logarithmic scale) for decreasing values of  $\varepsilon$  in Figure 3.2. Theorem 3.4.1 predicts that  $\log_{10} \left( \Gamma_z^{(r)}(\varepsilon) \right)$  should decay linearly with a slope greater than  $r$  as  $\varepsilon$  decreases to zero. The lines of best fit for each value of  $r$  in Figure 3.2 show this behaviour, and are therefore consistent with Theorem 3.4.1.

### 3.6.2 The evolution of $\Sigma(x, t)$ through time

Here we shall illustrate that the limiting covariance matrix  $\Sigma$  captures the time-evolution of model uncertainty. Consider the same meandering jet model in (3.11), with the parameter values used in the previous subsection. We fix the noise scale parameter at  $\varepsilon = 0.03$  and consider the evolution of the stochastic solution to (3.1) and the limiting Gaussian distribution (3.9) for times  $t$  in the interval  $[0, 1]$ . We also consider two different choices of the diffusion matrix  $\sigma$ : the identity as before, and

$$\sigma_M(x) := \begin{bmatrix} x_1 & 0.5 \\ x_1 & 0.5(x_1 + x_2) \end{bmatrix}, \quad (3.12)$$

to include both multiplicative and non-diagonal noise which grow for larger values of the coordinates.



paper1\_gaussian\_limit/figures/y\_histogram.pdf

Figure 3.1: Histograms of  $y_t^{(\varepsilon)}$  from direct simulation of (3.1), for four different  $\varepsilon$  values. Overlaid in black are contours of the Gaussian limit (3.9), which correspond to the first three standard deviation levels centred at the limiting mean  $F_0^t(x)$ . In dashed blue are corresponding contours computed from the sample covariance matrix of the realisations.

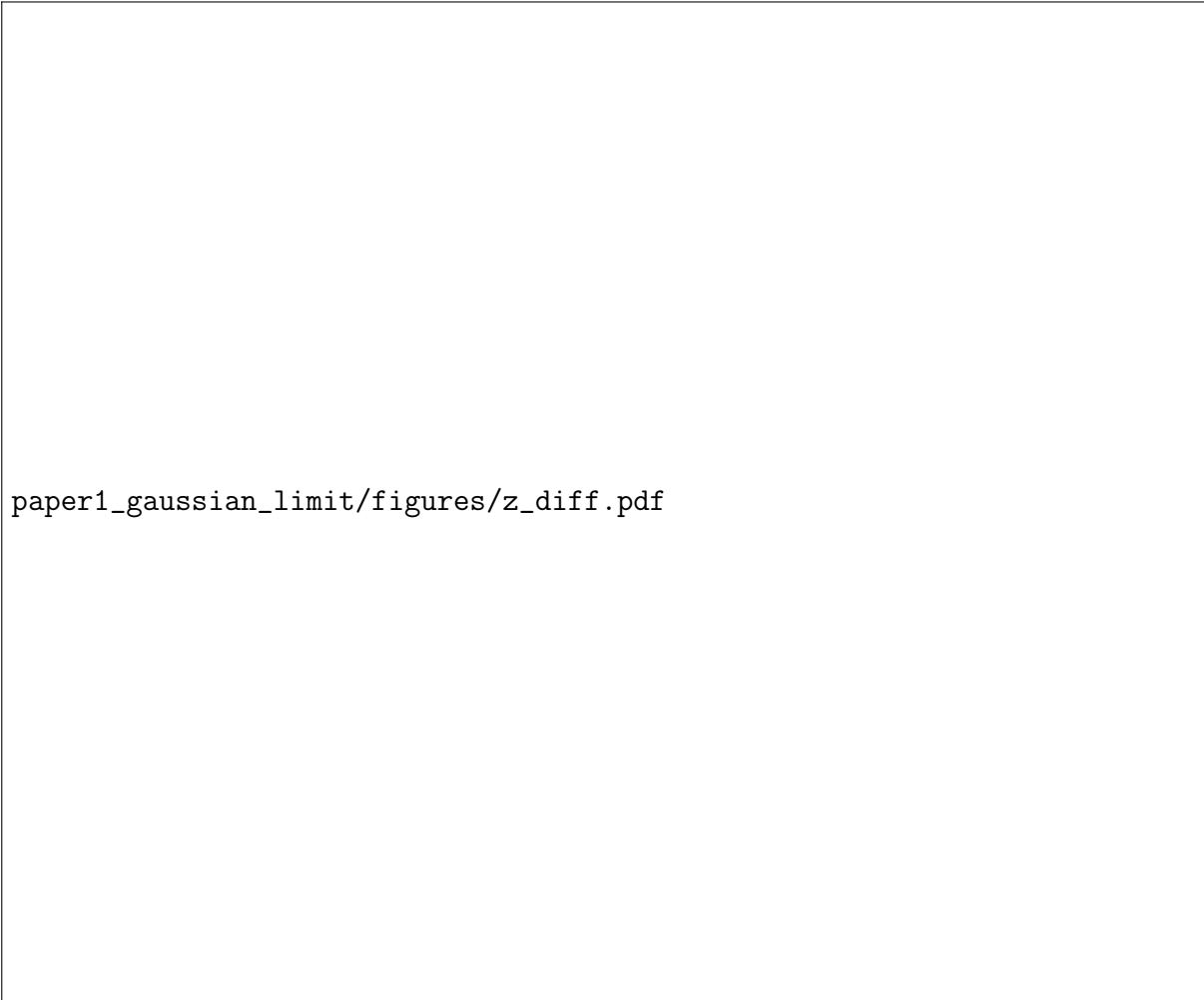


Figure 3.2: Validation of Theorem 3.4.1, by plotting the sample  $r$ th raw moment distance (the error metric  $\Gamma_z^{(r)}(\varepsilon)$ ) between 10000 realisations of  $z_t^{(\varepsilon)}(x)$  and  $z_t(x)$ , for decreasing values of  $\varepsilon$ . A line of best fit (in red) is placed on each, and the resulting slope indicated.

paper1\_gaussian\_limit/figures/through\_time.pdf

Figure 3.3: Histograms of  $y_t^{(0.03)}$  for (from left to right)  $t = 0.2, 0.4, 0.6, 0.8, 1.0$ , with the time-evolution of the deterministic trajectory in grey and contours of the limiting covariance matrix (3.6) for each time. The right figure uses the diffusion matrix  $\sigma_M(x)$  as defined in (3.12).

Figure 3.3 plots histograms of realisations of the solution to (3.1) at several different times, evolving from the same initial condition  $x = (0, 1)$ , and the time-evolution of the corresponding deterministic trajectory solving (3.2). Overlaid on each histogram are the contours of the limiting Gaussian distributions, computed entirely from covariance matrix (3.6). Although each distribution is non-Gaussian, the evolution of the uncertainty distribution through time is captured by  $\Sigma$ . For examples, features of the error distributions, such as stretching and rotation, are reflected in  $\Sigma$ . This remains the case even when the noise is multiplicative with  $\sigma = \sigma_M$ . The computation of  $\Sigma$  circumvents the need for expensive Monte-Carlo simulation to draw conclusions about such evolution of uncertainty. ■

## 3.7 Discussion

This paper has contributed a rigorous justification, in terms of error bounds and a small-noise limit, for an easily computable linearisation approximation to the solution of nonlinear stochastic differential equations, as seen across diverse places in the literature [24, 40, 41, e.g.]. The theory applies to fully non-autonomous SDEs with multiplicative noise. This result extends the convergence bound on the Kullback-Leibler divergence by Sanz-Alonso and Stuart [40] to an explicit bound on the convergence of all moments of the difference between the exact SDE solution and the approximation, and further establishes the exact Gaussian distribution in the small-noise limit. While it is known

that convergence of the KL divergence leads to convergence of the moments [32], this manuscript provides the exact rate of that convergence. Our bound is verified numerically by plotting the first four raw moments of the distance between the true noise-scaled solution and the linearised solution (see Figure 3.2). The results, plotted across three orders of magnitude of the small noise parameter, match our theoretical prediction exactly.

In addition, we described a framework in which uncertainty in deterministic models can be ascribed without the need for expensive stochastic simulation, and purely from the deterministic solution dynamics. We illustrated how the Gaussian limit reflects the time-evolution of uncertainty (see Figure 3.3), even when the true uncertainty distributions are themselves non-Gaussian.

A powerful advantage of this framework is that the diffusivity matrix  $\sigma$  is permitted to vary spatio-temporally, allowing for multiplicative noise. Multiplicative noise is often ignored in practice, due to difficulties in working with analytically (see, for instance, the prior lack of rigorous justification of linearisations when the noise is multiplicative) and generating numerical realisations efficiently (e.g. the review in [33]). It has also been shown that multiplicative noise on linear dynamics can model departures from Gaussianity observed in climate statistics [43], as opposed to nonlinear dynamics with only additive noise. The spatio-temporal dependence of  $\sigma$  can also capture experimental and observational considerations that are otherwise ignored in the deterministic model, such as cloud cover when using satellite measurements or nonuniform uncertainty across the field of view of a camera. We therefore present a highly flexible framework that can capture any prior knowledge of non-uniform uncertainty that arises from modelling or experimental considerations.

This paper also supplied theoretical and computational extension to the “stochastic sensitivity” tools introduced by Balasuriya [4]. Stochastic sensitivity was hitherto derived as the variance of an unknown limiting distribution and could only be computed in two spatial dimensions: we established that stochastic sensitivity, in any number of dimensions, is computable as the operator norm of the covariance matrix of our limiting SDE. We have also established that the limiting distribution in question is Gaussian, which may provide insight into properties of stochastic sensitivity as a means of uncertainty quantification in any model (not just in the fluids context) where an  $n$ -dimensional state variable evolves according to a “best available” model.

The Gaussian approximation presented here arises as the leading order term in a power series expansion of the SDE solution in terms of the noise scale parameter  $\varepsilon$  [9]. A further extension would be to explore the higher-order terms in such an expansion, which could lead to a practical framework for constructing higher-order characterisations and approximations of the stochastic solution. However, the higher-order terms are known to be individually non-Markovian, and satisfy non-linear SDEs for which the solution is not expected to be analytically available [9].

In this paper, we have assumed throughout that the initial condition  $x$ , from which

both the stochastic differential equation and the deterministic flow map evolves from, is *certain* (i.e. not a random quantity). However, in practice there is uncertainty associated with the initial state which should also be accounted for. The bound in Theorem 3.4.1 is independent of the initial condition, suggesting that the required extension of the theoretical result is straightforward. This extension will broaden our framework, allowing for uncertainties in *both* the initial state and the time-evolution of the model to be characterised at once in a precise sense.

Similarly, we assume that the reference deterministic model (3.2) for the evolution of the state variable is “correct” and known exactly, in that in the absence of any noise (i.e.  $\varepsilon = 0$ ), the SDE model (3.1) reduces to the deterministic (3.2). The Gaussian characterisation is computed from knowledge of either the driving vector field or the solution data itself, i.e. the flow map. However, these components of the deterministic model may not be known exactly, e.g. from solving (3.2) numerically, interpolation error, etc. There is a need to extend the theory presented here to account for this case; to, for instance, establish a bound in the error between the SDE solution and the limiting Gaussian, as in Theorem 3.4.1, if the Gaussian distribution is constructed from an “incorrect” deterministic model. Both of these theoretical extensions, to uncertain initial conditions and incorrect deterministic dynamics, are currently being pursued.

### 3.7.1 Applications

Here, we briefly discuss some anticipated applications of this work across a wide range of fields, including climate and ocean modelling, data assimilation and Lagrangian coherent structures.

This work fits in with recent interest in stochastic parameterisation as a means to account for unresolved subgrid effects in climate modelling [7, 31, 35]. In particular, the recent review [31] concludes, “The aim of current and future developments in stochastic representations of model uncertainty is to develop schemes that are computationally highly efficient and contribute only moderately to the overall computational cost...”. This paper provides one method to convert a stochastic parameterisation (formulated as a SDE) to a computationally cheaper set of coupled ODEs for the mean and variance of an approximate Gaussian, together with a convergence proof and error estimates.

To ascribe uncertainties directly onto the deterministic model, we assume that the diffusivity matrix  $\sigma$  is specified *a priori*, to capture any known multiplicative noise effects. There are methods for estimating  $\sigma$  directly from observed data, e.g. the Bayesian inference approach of [46] or via statistical estimation as in [17], which can be used in our framework. In particular, [46] relies upon computationally expensive numerical approximations to compute the likelihood of each trajectory, whereas from this paper we have a potentially more efficient computation, using the analytically available Gaussian limit. Coupling these approaches with the approximation here could provide a complete and practical framework to characterise the uncertainty in the flow by efficiently

estimating the (multiplicative) diffusion from observed trajectory data.

Data assimilation is a framework for improving uncertainties in predictions by combining model forecasts with observational data, accounting for error in both, and uncertainty quantification refers to the broader goal of capturing the uncertainty inherent in prediction [15, 24, 29, 37]. The Gaussian limit here provides a characterisation of model uncertainty, and may therefore be useful in data assimilation and uncertainty quantification. The linearisation of the stochastic differential equation (3.1) used to construct the Gaussian approximation has been employed in data assimilation, e.g. in the continuous time continuous state-space extended Kalman filter [24, §9]. The convergence analysis of this paper could contribute a new term, estimating the error due to linearisation, to the *forecast uncertainty* covariance matrix employed in these extended Kalman filters.

Stochastic sensitivity provides a novel method for extracting Lagrangian coherent structures (LCSs) [6, 22] from fluid flow, by considering regions with uncertainty (as measured by the stochastic sensitivity field) below a prescribed threshold. Whereas the original formulation in [4] was restricted to two-dimensional flows, here we have an extension of the LCS extraction scheme to arbitrary dimensions.

Moreover, most traditional LCS measures are completely deterministic measures, not accounting for any uncertainty in the driving velocity field, and the sensitivity of these methods to such uncertainty has not been investigated in detail. The robustness of several LCS methods to stochastic noise has recently been explored in [3], but via stochastic simulation and summary statistics. In this paper we have presented a theoretical result for characterising Lagrangian trajectory uncertainty, which can be used to perform a purely theoretical analysis of such sensitivity in LCS computations. An initial study into the impact of uncertainty of one such method – the finite-time Lyapunov exponent – has already been performed using stochastic sensitivity [5], albeit in only two-dimensions and without knowledge that the limiting distribution is Gaussian.

### 3.8 Preliminaries for proofs

Throughout, we use the norm symbol  $\|\cdot\|$  to denote (i) for a vector, the standard Euclidean vector norm, (ii) for a matrix, the spectral norm induced by the Euclidean norm, and (iii) for a 3rd-order tensor, the spectral norm induced by the matrix norm. The gradient symbol  $\nabla$  generically refers to derivatives with respect to the state variable. We write  $W_t = \left(W_t^{(1)}, \dots, W_t^{(n)}\right)^\top$  as the components of the canonical  $n$ -dimensional Wiener process, where each  $W_t^{(i)}$  are mutually independent 1-dimensional Wiener processes. The flow map  $F_0^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$  summarises solutions of the deterministic model (3.2), given by

$$F_0^t(x) = x + \int_0^t u(F_0^\tau(x), \tau) d\tau. \quad (3.13)$$

for an initial condition  $x \in \mathbb{R}^n$ . The spatial gradient (with respect to the initial condition) of the flow map solves the equation of variations associated with (3.2), i.e.

$$\frac{\partial}{\partial t} \nabla F_0^t(x) = \nabla u(F_0^t(x), t) \nabla F_0^t(x). \quad (3.14)$$

Hypothesis H summarises the minimum requirements of smoothness, measurability and boundedness in the coefficients of the main stochastic differential equation (3.1).

**Hypothesis H** *Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$  and  $\sigma : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times n}$  be such that:*

(H.1) *For each  $t \in [0, T]$ , the function  $u(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $u(x, t)$  is twice continuously differentiable on  $\mathbb{R}^n$ , and each component of the function  $\sigma(\cdot, t) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  given by  $\sigma(x, t)$  is differentiable on  $\mathbb{R}^n$ .*

(H.2) *For each  $x \in \mathbb{R}^n$ , the function  $u(x, \cdot) : [0, T] \rightarrow \mathbb{R}^n$  and each component of the function  $\sigma(x, \cdot) : [0, T] \rightarrow \mathbb{R}^{n \times n}$  are Borel-measurable on  $[0, T]$ .*

(H.3) *There exists a constant  $K_u > 0$  such that for any  $t \in [0, T]$  and  $x \in \mathbb{R}^n$ ,*

$$\max \left\{ \|u(x, t)\|, \|\nabla u(x, t)\|, \|\nabla \nabla u(x, t)\| \right\} \leq K_u.$$

(H.4) *There exists a constant  $K_\sigma > 0$  such that for any  $t \in [0, T]$  and  $x \in \mathbb{R}^n$ ,*

$$\max \left\{ \|\sigma(x, t)\|, \|\nabla \sigma(x, t)\| \right\} \leq K_\sigma.$$

The conditions in Hypothesis H imply that  $u$  and  $\sigma$  are globally Lipschitz and guarantee that for each value of  $\varepsilon$ , there exists a unique strong solution to (3.1) [26]. Specifically, for every  $x, y \in \mathbb{R}^n$  and  $t \in [0, T]$ ,

$$\|u(x, t) - u(y, t)\| \leq K_u \|x - y\|, \quad (3.15)$$

and if  $\sigma_{i\cdot}$  denotes the  $i$ th row of  $\sigma$ , then

$$\|\sigma_{i\cdot}(x, t) - \sigma_{i\cdot}(y, t)\| \leq K_\sigma \|x - y\|. \quad (3.16)$$

The following inequalities are used several times throughout. For any real numbers  $x_1, \dots, x_m \geq 0$  and  $r \geq 1$ ,

$$\left( \sum_{i=1}^m x_i \right)^r \leq m^{r-1} \sum_{i=1}^m x_i^r. \quad (3.17)$$

This results from an application of the finite form of Jensen's inequality. An implication of the equivalence of the  $L^1$  and Euclidean norms and (3.17) is that for any  $z \in \mathbb{R}^n$  and  $r \geq 1$ ,

$$\|z\|^r \leq \left( \sum_{i=1}^n |z_i| \right)^r \leq n^{r-1} \sum_{i=1}^n |z_i|^r, \quad (3.18)$$

where  $z_i$  denotes the  $i$ th component of  $z$ . If each component  $z_i$  of a vector  $z$  is bounded by a constant  $K$ , then

$$\|z\| \leq \sqrt{n}K. \quad (3.19)$$

Similarly, if  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  is a vector-valued function such that each component of  $f$  is integrable over an interval  $[0, t]$ , then for all  $r \geq 1$ ,

$$\left\| \int_0^t f(\tau) d\tau \right\|^r \leq t^{r-1} \int_0^t \|f(\tau)\|^r d\tau. \quad (3.20)$$

This inequality results from an application of Hölder's inequality.

### 3.9 Proof of Theorem 3.4.1

To prove the main result, we first require several lemmas. The first lemma is an extension of [4, Lem. A.1] to the  $n$ -dimensional case.

**Lemma 3.9.1** *Let  $y_t^{(\varepsilon)}$  denote the strong solution to (3.1) for  $t \in [0, T]$ , then for any  $p > 0$  there exists a constant  $G_p > 0$  such that*

$$\mathbb{E} \left[ \left\| \int_0^t \sigma \left( y_\tau^{(\varepsilon)}, \tau \right) dW_\tau \right\|^{2p} \right] \leq n^{3p} K_\sigma^{2p} G_p t^p.$$

**Proof.** For  $i \in \{1, \dots, n\}$ , let  $\sigma_{i\cdot}$  denote the  $i$ th row of  $\sigma$ . Define the stochastic process

$$M_\tau^{(i)} := \sigma_{i\cdot} \left( y_\tau^{(\varepsilon)}, \tau \right)$$

for  $\tau \in [0, t]$ , so that

$$\left[ \int_0^t \sigma \left( y_\tau^{(\varepsilon)}, \tau \right) dW_\tau \right]_i = \int_0^t M_\tau^{(i)} dW_\tau.$$

Then, from H.4 and using (3.19),

$$\int_0^t \|M_\tau^{(i)}\|^2 d\tau \leq \int_0^t n K_\sigma^2 d\tau < \infty,$$



so we can apply the Burkholder-Davis-Gundy inequality (e.g. [26, Thm. 5.6.3]) to  $M_\tau$ , which asserts that there exists a constant  $G_p > 0$  depending only on  $p$  such that

$$\begin{aligned} \mathbb{E} \left[ \left| \int_0^t M_\tau^{(i)} dW_\tau \right|^{2p} \right] &\leq G_p \mathbb{E} \left[ \left( \int_0^t \left\| \sigma_{i \cdot} \left( y_\tau^{(\varepsilon)}, \tau \right) \right\|^2 d\tau \right)^p \right] \\ &\leq G_p n^p K_\sigma^{2p} t^p. \end{aligned} \quad (3.21)$$

where the second inequality uses H.4 and (3.19). Then, we can write

$$\mathbb{E} \left[ \left\| \int_0^t \sigma \left( y_\tau^{(\varepsilon)}, \tau \right) dW_\tau \right\|^{2p} \right] \leq n^{3p} K_\sigma^{2p} G_p t^p,$$

using (3.18) and then (3.21). □

The next Lemma is a similar extension of [4, Lem. 2.2].

**Lemma 3.9.2** *Let  $q \geq 1$ , then for all  $\varepsilon > 0$  and  $\tau \in [0, T]$*

$$\mathbb{E} \left[ \int_0^t \left\| z_\tau^{(\varepsilon)}(x) \right\|^q d\tau \right] \leq H_q(t),$$

where

$$H_q(t) := 2^{q-1} n^{3q/2} K_\sigma^q G_{q/2} t^{q/2+1} \exp \left( 2^{q-1} K_u^q t^q \right).$$

**Proof.** Consider the integral form of (3.1),

$$y_t^{(\varepsilon)} = x + \int_0^t u \left( y_\tau^{(\varepsilon)}, \tau \right) d\tau + \varepsilon \int_0^t \sigma \left( y_\tau^{(\varepsilon)}, \tau \right) dW_\tau.$$

Using (3.13),

$$y_t^{(\varepsilon)} - F_0^t(x) = \int_0^t \left( u \left( y_\tau^{(\varepsilon)}, \tau \right) - u \left( F_0^\tau(x), \tau \right) \right) d\tau + \varepsilon \int_0^t \sigma \left( y_\tau^{(\varepsilon)}, \tau \right) dW_\tau,$$

and so

$$\left\| y_t^{(\varepsilon)} - F_0^t(x) \right\|^q \leq 2^{q-1} \left( t^{q-1} \int_0^t \left\| u \left( y_\tau^{(\varepsilon)}, \tau \right) - u \left( F_0^\tau(x), \tau \right) \right\|^q d\tau + \varepsilon^q \left\| \int_0^t \sigma \left( y_\tau^{(\varepsilon)}, \tau \right) dW_\tau \right\|^q \right), \quad \blacksquare$$

using (3.17) and then (3.20). Taking the expectation and applying Lemma 3.9.1 with  $p = q/2$  gives

$$\begin{aligned} \mathbb{E} \left[ \left\| y_t^{(\varepsilon)} - F_0^t(x) \right\|^q \right] &\leq 2^{q-1} t^{q-1} \mathbb{E} \left[ \int_0^t \left\| u \left( y_\tau^{(\varepsilon)}, \tau \right) - u \left( F_0^\tau(x), \tau \right) \right\|^q d\tau \right] \\ &\quad + 2^{q-1} \varepsilon^q n^{3q/2} K_\sigma^q G_{q/2} t^{q/2}. \end{aligned} \quad (3.22)$$

We note that  $\mathbb{E} \left[ \left\| y_t^{(\varepsilon)} - F_0^t(x) \right\|^q \right] < \infty$  from H.3, so by Tonelli's theorem (e.g. [14, Thm. 2.3.9]),

$$\mathbb{E} \left[ \int_0^t \left\| y_\tau^{(\varepsilon)} - F_0^\tau(x) \right\|^q d\tau \right] = \int_0^t \mathbb{E} \left[ \left\| y_\tau^{(\varepsilon)} - F_0^\tau(x) \right\|^q \right] d\tau.$$

Now, using the Lipschitz condition (3.15) on (3.22) and interchanging the expectation and integral,

$$\mathbb{E} \left[ \left\| y_t^{(\varepsilon)} - F_0^t(x) \right\|^q \right] \leq 2^{q-1} K_u^q t^{q-1} \int_0^t \mathbb{E} \left[ \left\| y_\tau^{(\varepsilon)} - F_0^\tau(x) \right\|^q \right] d\tau + 2^{q-1} \varepsilon^q n^{3q/2} K_\sigma^q G_{q/2} t^{q/2}.$$

Applying Grönwall's inequality then gives

$$\mathbb{E} \left[ \left\| y_t^{(\varepsilon)} - F_0^t(x) \right\|^q \right] \leq 2^{q-1} \varepsilon^q n^{3q/2} K_\sigma^q G_{q/2} t^{q/2} \exp \left( 2^{q-1} K_u^q t^q \right).$$

Hence, for  $\tau \leq t$  we have

$$\mathbb{E} \left[ \left\| z_\tau^{(\varepsilon)}(x) \right\|^q \right] \leq 2^{q-1} n^{3q/2} K_\sigma^q G_{q/2} \tau^{q/2} \exp \left( 2^{q-1} K_u^q \tau^q \right) < \infty,$$

and so Tonelli's theorem allows us to again interchange the expectation and integral so that

$$\begin{aligned} \mathbb{E} \left[ \int_0^t \left\| z_\tau^{(\varepsilon)}(x) \right\|^q d\tau \right] &= \int_0^t \mathbb{E} \left[ \left\| z_\tau^{(\varepsilon)}(x) \right\|^q \right] d\tau \\ &\leq 2^{q-1} n^{3q/2} K_\sigma^q G_{q/2} t^{q/2+1} \exp \left( 2^{q-1} K_u^q t^q \right), \end{aligned}$$

as desired.  $\square$

We now have all the tools to prove Theorem 3.4.1. By rearranging the integral forms of (3.1) and using (3.13) and (3.3),  $z_t^{(\varepsilon)}(x)$  satisfies the integral equation

$$z_t^{(\varepsilon)}(x) = \int_0^t v^{(\varepsilon)} \left( z_\tau^{(\varepsilon)}(x), \tau \right) d\tau + \int_0^t \sigma \left( \varepsilon z_\tau^{(\varepsilon)}(x) + F_0^\tau(x), \tau \right) dW_\tau, \quad (3.23)$$

where

$$v^{(\varepsilon)} \left( z_\tau^{(\varepsilon)}, \tau \right) := \frac{1}{\varepsilon} \left[ u \left( \varepsilon z_\tau^{(\varepsilon)} + F_0^\tau(x), \tau \right) - u \left( F_0^\tau(x), \tau \right) \right].$$

Subtracting the integral form of (3.4) from (3.23) gives

$$\begin{aligned} z_t^{(\varepsilon)}(x) - z_t(x) &= \int_0^t \left[ v^{(\varepsilon)} \left( z_\tau^{(\varepsilon)}(x), \tau \right) - \nabla u \left( F_0^\tau(x), \tau \right) z_\tau(x) \right] d\tau \\ &\quad + \int_0^t \left[ \sigma \left( \varepsilon z_\tau^{(\varepsilon)} + F_0^\tau(x), \tau \right) - \sigma \left( F_0^\tau(x), \tau \right) \right] dW_\tau \\ &= A(t) + B(t) + C(t), \end{aligned}$$

where

$$\begin{aligned} A(t) &:= \int_0^t \left[ v^{(\varepsilon)} \left( z_\tau^{(\varepsilon)}(x), \tau \right) - \nabla u \left( F_0^\tau(x), \tau \right) z_\tau^{(\varepsilon)}(x) \right] d\tau \\ B(t) &:= \int_0^t \nabla u \left( F_0^\tau(x), \tau \right) \left[ z_\tau^{(\varepsilon)}(x) - z_\tau(x) \right] d\tau \\ C(t) &:= \int_0^t \left[ \sigma \left( \varepsilon z_\tau^{(\varepsilon)}(x) + F_0^\tau(x), \tau \right) - \sigma \left( F_0^\tau(x), \tau \right) \right] dW_\tau. \end{aligned}$$

Then, using (3.17) and taking expectation,

$$\mathbb{E} \left[ \left\| z_t^{(\varepsilon)}(x) - z_t(x) \right\|^r \right] \leq 3^{r-1} \left( \mathbb{E} \left[ \left\| A(t) \right\|^r \right] + \mathbb{E} \left[ \left\| B(t) \right\|^r \right] + \mathbb{E} \left[ \left\| C(t) \right\|^r \right] \right) \quad (3.24)$$

First consider  $A(t)$ . Since for any  $t \in [0, T]$ ,  $u(\cdot, t)$  is twice continuously differentiable under H.1, for each  $i = 1, \dots, n$  there exists by Taylor's theorem (e.g. see [23, Cor. A9.3.]) a function  $R_i : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$  such that

$$u_i \left( \varepsilon z + F_0^\tau(x), \tau \right) = u_i \left( F_0^\tau(x), \tau \right) + \varepsilon \left[ \nabla u_i \left( F_0^\tau(x), \tau \right) \right] z + R_i(\varepsilon z, \tau) \quad (3.25)$$

for any  $z \in \mathbb{R}^n$ , where  $u_i$  denotes the  $i$ th component of  $u$ . The function  $R_i$  satisfies

$$|R_i(\varepsilon z, \tau)| \leq \frac{1}{2} \left\| \nabla \nabla u_i \left( F_0^\tau(x), \tau \right) \right\| \|\varepsilon z\|^2 \leq \frac{\varepsilon^2 K_u}{2} \|z\|^2. \quad (3.26)$$

Rearranging (3.25),

$$v_i^{(\varepsilon)}(z, \tau) - \left[ \nabla u_i \left( F_0^\tau(x), \tau \right) \right] z = \frac{1}{\varepsilon} R_i(\varepsilon z, \tau),$$

where  $v_i^{(\varepsilon)}$  is the  $i$ th component of  $v^{(\varepsilon)}$ . Let  $R(\varepsilon z, \tau) := (R_1(\varepsilon z, \tau), \dots, R_n(\varepsilon z, \tau))^\top$ , then

$$A(t) = \frac{1}{\varepsilon} \int_0^t R \left( \varepsilon z_\tau^{(\varepsilon)}(x), \tau \right) d\tau,$$

and since each component of  $R$  is bounded as in (3.26), using (3.19)

$$\|R(\varepsilon z, \tau)\| \leq \frac{\sqrt{n} \varepsilon^2 K_u}{2} \|z\|^2.$$

Taking the norm and expectation then gives

$$\begin{aligned}
\mathbb{E}[\|A(t)\|^r] &= \frac{1}{\varepsilon^r} \mathbb{E} \left[ \left\| \int_0^t R \left( \varepsilon z_\tau^{(\varepsilon)}(x), \tau \right) d\tau \right\|^r \right] \\
&\leq \frac{t^{r-1}}{\varepsilon^r} \mathbb{E} \left[ \int_0^t \left\| R \left( \varepsilon z_\tau^{(\varepsilon)}(x), \tau \right) \right\|^r d\tau \right] \\
&\leq \frac{t^{r-1} n^{r/2} \varepsilon^{2r} K_u^r}{2^r \varepsilon^r} \mathbb{E} \left[ \int_0^t \left\| z_\tau^{(\varepsilon)}(x) \right\|^{2r} d\tau \right] \\
&\leq \frac{t^{r-1} n^{r/2} \varepsilon^r K_u^r H_{2r}(t)}{2^r}, \tag{3.27}
\end{aligned}$$

where the second inequality uses (3.20), and  $H_{2r}(t)$  is obtained from Lemma 3.9.2 with  $q = 2r$ .

Next, consider  $B(t)$ , for which

$$\mathbb{E}[\|B(t)\|^r] \leq \int_0^t t^{r-1} K_u^r \mathbb{E} \left[ \left\| z_\tau^{(\varepsilon)}(x) - z_\tau(x) \right\|^r \right] d\tau. \tag{3.28}$$

using (3.20) and then H.3, and interchanging the expectation and the integral uses the fact that  $\mathbb{E}[\|z_\tau^{(\varepsilon)}(x)\|] < \infty$  and  $\mathbb{E}[\|z_\tau(x)\|] < \infty$ .

Finally, consider  $C(t)$ . For each  $i \in \{1, \dots, n\}$ , define the stochastic process

$$N_t^{(i)} := \sigma_i \cdot \left( \varepsilon z_\tau^{(\varepsilon)}(x) + F_0^\tau(x), \tau \right) - \sigma_i \cdot \left( F_0^\tau(x), \tau \right).$$

Then, the  $i$ th component of  $C(t)$  is

$$[C(t)]_i = \int_0^t N_\tau^{(i)} dW_\tau.$$

From H.4 and using (3.19),

$$\int_0^t \left\| N_\tau^{(i)} \right\|^2 d\tau \leq \int_0^t 4n K_\sigma^2 d\tau < \infty,$$

so we can apply the Burkholder-Davis-Gundy inequality on  $N_\tau^{(i)}$  to write

$$\begin{aligned}
\mathbb{E} \left[ \left| [C(t)]_i \right|^r \right] &\leq G_{r/2} \mathbb{E} \left[ \left( \int_0^t \left\| \sigma_{i \cdot} \left( \varepsilon z_\tau^{(\varepsilon)}(x) + F_0^\tau(x), \tau \right) - \sigma_{i \cdot} \left( F_0^\tau(x), \tau \right) \right\|^2 d\tau \right)^{r/2} \right] \\
&\leq G_{r/2} \mathbb{E} \left[ \left( \int_0^t K_\sigma^2 \left\| \varepsilon z_\tau^{(\varepsilon)}(x) \right\|^2 d\tau \right)^{r/2} \right] \\
&\leq G_{r/2} K_\sigma^r \varepsilon^r t^{r/2-1} \mathbb{E} \left[ \int_0^t \left\| z_\tau^{(\varepsilon)}(x) \right\|^r d\tau \right] \\
&\leq G_{r/2} K_\sigma^r \varepsilon^r t^{r/2-1} H_r(t),
\end{aligned} \tag{3.29}$$

where the second inequality uses the Lipschitz condition on  $\sigma$  in (3.16), the third inequality uses (3.20), and the fourth inequality uses Lemma 3.9.2 with  $q = r$ . Then, we have

$$\mathbb{E} \left[ \left\| C(t) \right\|^r \right] \leq n^r G_{r/2} K_u^r \varepsilon^r t^{r/2-1} H_r(t), \tag{3.30}$$

using (3.18), and then (3.29).

Combining (3.27), (3.28) and (3.30) into (3.24), we have

$$\begin{aligned}
\mathbb{E} \left[ \left\| z_t^{(\varepsilon)}(x) - z_t(x) \right\|^r \right] &\leq 3^{r-1} \left( \frac{t^{r-1} n^{r/2} \varepsilon^r K_u^r H_{2r}(t)}{2^r} + n^r G_{r/2} K_u^r \varepsilon^r t^{r/2-1} H_r(t) \right) \\
&\quad + \int_0^t 3^{r-1} t^{r-1} K_u^r \mathbb{E} \left[ \left\| z_\tau^{(\varepsilon)}(x) - z_\tau(x) \right\|^r \right] d\tau.
\end{aligned}$$

Applying Grönwall's inequality, we have

$$\mathbb{E} \left[ \left\| z_t^{(\varepsilon)}(x) - z_t(x) \right\|^r \right] = 3^{r-1} K_u^r \left( \frac{t^{r-1} n^{r/2} H_{2r}(t)}{2^r} + n^r G_{r/2} t^{r/2-1} H_r(t) \right) \exp \left( 3^{r-1} t^r K_u^r \right) \varepsilon^r.$$

Define

$$D_r(t) := 3^{r-1} K_u^r \left( \frac{t^{r-1} n^{r/2} H_{2r}(t)}{2^r} + n^r G_{r/2} t^{r/2-1} H_r(t) \right) \exp \left( 3^{r-1} t^r K_u^r \right).$$

then we have shown (3.5), the desired result.

## 3.10 Proof of Theorem 3.4.2

In this appendix, we establish the exact Gaussian distribution of the solution to the linearised SDE.

**Lemma 3.10.1** *The strong solution (e.g. [26, Def. 6.1.1]) to the linearised SDE (3.4) can be written as the Itô integral.*

$$z_t(x) = \int_0^t \nabla F_0^t(x) [\nabla F_0^\tau(x)]^{-1} \sigma(F_0^\tau(x), \tau) dW_\tau. \quad (3.31)$$

**Proof.** The equation (3.4) is linear and homogeneous, so the solution method is well-known, e.g. see [41, §4.3]. Let  $\Psi = \Psi(t)$  be the invertible fundamental matrix solution of the differential equation

$$\frac{d\Psi(t)}{dt} = \nabla u(F_0^t(x), t) \Psi(t), \quad \Psi(0) = I. \quad (3.32)$$

The equation (3.32) is exactly the equation of variations (3.14) for the deterministic system, so  $\Psi(t) = \nabla F_0^t(x)$ , noting that  $\Psi(0) = \nabla F_0^0(x) = \nabla x = I$ . Using Itô's Lemma (e.g. see [26, Thm. 5.5.1]) on  $M(t) := \Psi(t)^{-1} z_t(x)$ , it follows that

$$z_t(x) = \int_0^t \nabla F_0^t(x) [\nabla F_0^\tau(x)]^{-1} \sigma(F_0^\tau(x), \tau) dW_\tau$$

satisfies (3.4) and is therefore a strong solution.  $\square$

We next establish that the Itô integral of a matrix-valued deterministic function with respect to a multidimensional Wiener process is a multidimensional Gaussian process.

**Lemma 3.10.2** *Let  $a, b \in \mathbb{R}$  and let  $g : [a, b] \rightarrow \mathbb{R}^{n \times n}$  be a matrix-valued deterministic function such that each element of  $g$  is Itô-integrable. Consider the Itô integral*

$$\mathcal{I}[g] := \int_a^b g(t) dW_t,$$

*Then, the integral  $\mathcal{I}[g]$  is a  $n$ -dimensional multivariate Gaussian random variable.*

**Proof.** For  $i, j \in \{1, \dots, n\}$ , let  $g_{ij} : [a, b] \rightarrow \mathbb{R}$  be the  $(i, j)$ th element of  $g$ . Then, let

$$\mathcal{I}[g_{ij}] := \int_a^b g_{ij}(t) dW_t^{(i)},$$

so that the  $i$ th element of  $\mathcal{I}[g]$  is

$$\mathcal{I}[g]_i = \sum_{j=1}^n \mathcal{I}[g_{ij}].$$

Each  $\mathcal{I}[g_{ij}]$  is an Itô integral of a deterministic, scalar-valued function with respect to a one-dimensional Brownian motion, which is well-known to be a Gaussian process (e.g.

see [2, Lem. 4.3.11]). Moreover, each element of  $\mathcal{I}[g]$  is the sum of independent Gaussian random variables and is therefore itself Gaussian. Hence,  $\mathcal{I}[g]$  follows a multivariate Gaussian distribution.  $\square$

We can now prove Theorem 3.4.2. We have that

$$z_t(x) = \int_0^t \nabla F_0^t(x) [\nabla F_0^\tau(x)]^{-1} \sigma(F_0^\tau(x), \tau) dW_\tau = \int_0^t L(x, t, \tau) dW_\tau,$$

where  $L$  is as defined in (3.7). For any fixed  $t \in [0, T]$ ,  $z_t(x)$  is the Itô integral of a deterministic, matrix-valued function, and so  $z_t(x)$  is an  $n$ -dimensional Gaussian random variable by Lemma 3.10.2. Moreover [26],  $\mathbb{E}[z_t(x)] = 0$ . The variance of  $z_t(x)$  is then

$$\mathbb{V}[z_t(x)] = \mathbb{E} \left[ \left( z_t(x) - \mathbb{E}[z_t(x)] \right) \left( z_t(x) - \mathbb{E}[z_t(x)] \right)^\top \right] = \mathbb{E} \left[ z_t(x) z_t(x)^\top \right].$$

Let  $L_{ij}$  denote the  $(i, j)$ th element of  $L$  for  $i, j \in \{1, \dots, n\}$ . Then the  $(i, j)$ th element of  $\mathbb{V}[z_t(x)]$  is

$$\begin{aligned} \left[ \mathbb{V}[z_t(x)] \right]_{ij} &= \mathbb{E} \left[ \left( \sum_{k=1}^n \int_0^t L_{ik}(x, t, \tau) dW_\tau^{(k)} \right) \left( \sum_{l=1}^n \int_0^t L_{il}(x, t, \tau) dW_\tau^{(l)} \right) \right] \\ &= \sum_{k=1}^n \sum_{l=1}^n \mathbb{E} \left[ \left( \int_0^t L_{ik}(x, t, \tau) dW_\tau^{(k)} \right) \left( \int_0^t L_{il}(x, t, \tau) dW_\tau^{(l)} \right) \right] \\ &= \sum_{k=1}^n \int_0^t L_{ik}(x, t, \tau) L_{jk}(x, t, \tau) d\tau \\ &= [\Sigma(x, t)]_{ij} \end{aligned}$$

where the third equality uses Itô's isometry [26] and the fact that  $W_t^{(k)}$  is independent of  $W_t^{(l)}$  for  $k \neq l$ . So  $\mathbb{V}[z_t(x)] = \Sigma(x, t)$  and  $z_t(x) \sim \mathcal{N}(0, \Sigma(x, t))$ .

Taking the limit as  $\varepsilon \rightarrow 0$  in (3.5),  $z_t^{(\varepsilon)}(x)$  converges in  $r$ th moment to  $z_t(x)$ , which implies that  $z_t^{(\varepsilon)}(x)$  converges in distribution to  $z_t(x)$  also. Since  $z_t(x)$  is known to follow a Gaussian distribution, we can write

$$z_t^{(\varepsilon)}(x) \xrightarrow{d} \mathcal{N}(0, \Sigma(x, t)).$$

Finally, we show that  $\Sigma(x, t)$  is the solution to the matrix differential equation (3.8). We clearly have that  $\Sigma(x, 0) = O$ , the  $n \times n$  zero matrix. Differentiating the expression

(3.6) with the Leibniz integral rule

$$\begin{aligned}
\frac{d\Sigma(x, t)}{dt} &= L(x, t, t) L(x, t, t)^\top + \int_0^t \frac{\partial}{\partial t} \left[ L(x, t, \tau) L(x, t, \tau)^\top \right] d\tau \\
&= \sigma(F_0^t(x), t) \sigma(F_0^t(x), t)^\top + \int_0^t \frac{\partial L(x, t, \tau)}{\partial t} L(x, t, \tau)^\top d\tau \\
&\quad + \int_0^t L(x, t, \tau) \frac{\partial L(x, t, \tau)^\top}{\partial t} d\tau.
\end{aligned} \tag{3.33}$$

Now, using the equation of variations (3.14),

$$\frac{\partial L(x, t, \tau)}{\partial t} = \left[ \nabla u(F_0^t(x), t) \right] L(x, t, \tau). \tag{3.34}$$

Using (3.34) in (3.33),

$$\begin{aligned}
\frac{d\Sigma(x, t)}{dt} &= \sigma(F_0^t(x), t) \sigma(F_0^t(x), t)^\top + \left[ \nabla u(F_0^t(x), t) \right] \int_0^t L(x, t, \tau) L(x, t, \tau)^\top d\tau \\
&\quad + \int_0^t L(x, t, \tau) L(x, t, \tau)^\top d\tau \left[ \nabla u(F_0^t(x), t) \right]^\top \\
&= \sigma(F_0^t(x), t) \sigma(F_0^t(x), t)^\top + \left[ \nabla u(F_0^t(x), t) \right] \Sigma(x, t) + \Sigma(x, t) \left[ \nabla u(F_0^t(x), t) \right]^\top.
\end{aligned}$$

### 3.11 Proof of Theorem 3.5.1

Let  $x \in \mathbb{R}^n$  and  $t \in [0, T]$  be fixed. We first establish that the  $\varepsilon$ -limit and the variance operator can be exchanged. Using the fact that  $\mathbb{E}[z_t(x)] = 0$  and properties of expectation, we have

$$\left\| \mathbb{E}[z_t^{(\varepsilon)}(x)] \right\| = \left\| \mathbb{E}[z_t^{(\varepsilon)}(x) - z_t(x)] \right\| \leq \mathbb{E} \left[ \left\| z_t^{(\varepsilon)}(x) - z_t(x) \right\| \right] \leq D_1(t)\varepsilon,$$

so

$$\lim_{\varepsilon \downarrow 0} \left\| \mathbb{E}[z_t^{(\varepsilon)}(x)] \right\| = 0.$$



Using the fact that  $\rho(z) = \left\| \mathbb{E}[zz^\top] \right\|^{1/2}$  defines a norm on the space of  $\mathbb{R}^n$ -valued random vectors and the reverse triangle inequality,

$$\begin{aligned} \left| \left\| \mathbb{V}[z_t^{(\varepsilon)}(x)] \right\|^{1/2} - \left\| \Sigma(x, t) \right\|^{1/2} \right| &= \left| \left\| \mathbb{E}[z_t^{(\varepsilon)}(x)z_t^{(\varepsilon)}(x)^\top] - \mathbb{E}[z_t^{(\varepsilon)}(x)] \mathbb{E}[z_t^{(\varepsilon)}(x)^\top] \right\|^{1/2} \right. \\ &\quad \left. - \left\| \mathbb{E}[z_t(x)z_t(x)^\top] \right\|^{1/2} \right| \\ &\leq \mathbb{E} \left[ \left\| z_t^{(\varepsilon)}(x) - z_t(x) \right\|^2 \right]^{1/2} + \left\| \mathbb{E}[z_t^{(\varepsilon)}(x)] \mathbb{E}[z_t^{(\varepsilon)}(x)^\top] \right\|^{1/2} \\ &\leq D_2(t)^{1/2} \varepsilon + \left\| \mathbb{E}[z_t^{(\varepsilon)}(x)] \right\|, \end{aligned}$$

and so taking the limit as  $\varepsilon \downarrow 0$  and squaring both sides, we have

$$\lim_{\varepsilon \downarrow 0} \left\| \mathbb{V}[z_t^{(\varepsilon)}(x)] \right\| = \left\| \Sigma(x, t) \right\|. \quad (3.35)$$

For  $\varepsilon > 0$ , define

$$S_{(\varepsilon)}^2(x, t) := \sup \left\{ \mathbb{V}[p^\top z_t^{(\varepsilon)}(x)] \mid p \in \mathbb{R}^n, \|p\| = 1 \right\} = \sup \left\{ p^\top \mathbb{V}[z_t^{(\varepsilon)}(x)] p \mid p \in \mathbb{R}^n, \|p\| = 1 \right\}$$

Since  $\mathbb{V}[z_t^{(\varepsilon)}(x)]$  is symmetric and positive definite, the Cholesky decomposition provides an  $n \times n$  matrix  $\Pi^{(\varepsilon)}(x, t)$  such that

$$\mathbb{V}[z_t^{(\varepsilon)}(x)] = \left[ \Pi^{(\varepsilon)}(x, t) \right]^\top \Pi^{(\varepsilon)}(x, t),$$

allowing us to write

$$S_{(\varepsilon)}^2(x, t) = \left\| \Pi^{(\varepsilon)}(x, t) \right\|^2 = \left\| \mathbb{V}[z_t^{(\varepsilon)}(x)] \right\|.$$

using properties of the spectral norm. Taking the limit as  $\varepsilon$  approaches zero and using (3.35),

$$S^2(x, t) = \lim_{\varepsilon \downarrow 0} S_{(\varepsilon)}^2(x, t) = \lim_{\varepsilon \downarrow 0} \left\| \mathbb{V}[z_t^{(\varepsilon)}(x)] \right\| = \left\| \Sigma(x, t) \right\|.$$

Since  $\Sigma(x, t)$  is symmetric and positive definite, the operator norm, and therefore  $S^2(x, t)$ , is given by the largest eigenvalue of  $\Sigma(x, t)$ .



## Chapter 4

### A Gaussian mixture model



# Chapter 5

## Applications in oceanography

### 5.1 Limitations of stochastic simulation



# Chapter 6

## Future outlook and conclusions

In this chapter, we briefly discuss the implications of the work presented in this thesis, and highlight several avenues for further extensions and applications.

### 6.1 Further theoretical developments

### 6.2 Bayesian inference and data assimilation

### 6.3 Lagrangian coherent structures

### 6.4 Implications for the Fokker-Planck equation





# Appendix A

## Derivation of analytical SDE solutions

### A.1 Benê's SDE



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