Incorporating stochastic sensitivity as an uncertainty measure for data assimilation and parcel trajectories in geophysical flows

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Acknowledgements

Dedication

Abstract

Chapter ?? introduces

Chapter 1

Introduction

Chapter 2

Background

2.1 Notation

For a

2.2 Lagrangian Dynamics

We are typically interested in a spatial domain $\Omega \subseteq \mathbb{R}^n$ and over some finite time interval $[0, T]$ for finite T. Lagrangian trajectories are solutions of the first-order differential equation

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = u\left(x_t, t\right), \qquad x_0 = x \in \Omega, \tag{2.1}$$

where $u : \Omega \times [0, T] \to \mathbb{R}^n$ describes the velocity at a point in space and time.

2.2.1 Flow map

In the mathematical treatment of Lagrangian dynamics, and in particular Lagrangian coherent structures (?), trajectories solving (**??**) are summarised by the flow map. The flow map is an operator mapping
The flow map can be defined more formally as follows.

Definition 2.2.1 (Flow map)
*Suppose $t_1, t_2 \in [0, T]$. The **flow map** $F_{t_1}^{t_2} : \mathbb{R}^n \to \mathbb{R}^n$ from time t_1 to t_2 associated with* (**??**) *is the solution to*

$$\frac{\partial F_{t_1}^{\tau}(x)}{\partial \tau} = u\left(F_{t_1}^{\tau}(x), \tau\right), \qquad F_{t_1}^{t_1}(x) = x,$$

solved up to time $\tau = t_2$. Equivalently,

$$F_{t_1}^{t_2}(x) = x + \int_{t_1}^{t_2} u\left(F_{t_1}^{\tau}(x), \tau\right) \mathrm{d}\tau.$$

The flow map satisfies the following properties
The gradient of the flow map satisfies a useful property; the equation of variations.

Theorem 2.2.1
Let $F_{t_0}^{t}$ be the flow map corresponding to (??). Then, the spatial gradient $\nabla F_{t_0}^{t}(x)$ satisfies the equation of variations

$$\frac{\partial \nabla F_{t_0}^{t}(x)}{\partial t} = \nabla u\left(F_{t_0}^{t}(x), t\right) \nabla F_{t_0}^{t}(x). \tag{2.2}$$

Proof.

□

2.2.2 Lagrangian coherent structures

TODO: Briefly discuss LCSs. Do not need to go into too much detail here. Mention whatever is relevant

2.3 Stochastic Differential Equations

$$\mathrm{d}y_t = u\left(y_t, t\right) \mathrm{d}t + \sigma\left(y_t, t\right) \mathrm{d}W_t. \tag{2.3}$$

A solution to the stochastic differential equation can be defined rigorously **?**.

Definition 2.3.1
A stochastic process $\{y_t\}_{t \in [0,T]}$ taking values in R^n is said to be a ***strong solution*** *of (??) with initial condition $y_0 = \xi$ if the following holds:*

1. *For each t,*
2. $$\int_0^T \left(\left\|u\left(y_t, t\right)\right\| + \left\|\sigma\left(y_t, t\right)\right\|^2\right) \mathrm{d}t < \infty \; a.s.$$
3. *For each $t \in [0, T]$,*

2.4 Stochastic Sensitivity

In most practical situations, the Eulerian velocity data driving ocean and atmospheric models relies upon measurements of estimates obtained on a low resolution spatial discretisation. There are limited tools within the LCS context that explcitly characterise the impact of these uncertainties As such, there is recent interest in addressing this deficiency (??) . ? introduces stochastic sensitivity as a new tool for directly quantifying the impact of Eulerian uncertainty on Lagrangian trajectories. The evolution of Lagrangian trajectories is modelled as solution to a Itô stochastic ordinary differential equation.

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itations here

TODO: Deterministic

The SDE model is

$$\mathrm{d}y_t = u\left(y_t, t\right)\mathrm{d}t + \varepsilon\sigma\left(y_t, t\right)\mathrm{d}W_t, , \tag{2.4}$$

where $0 < \varepsilon \ll 1$ is a parameter quantifying the scale of the noise, $\sigma : \mathbb{R}^2 \times [0, T] \to \mathbb{R}^{2\times 2}$ is the 2×2 diffusion matrix, and W_t is the canonical two-dimensional Wiener process. Since σ can vary by both space and time, the noise is multiplicative.

To quantify uncertainty in a way that is independent of the noise scale ε, ? defined the random variable $z_\varepsilon\left(x, t\right)$ on $\mathbb{R}^2 \times [0, T]$ as

$$z_\varepsilon\left(x, t\right) := \frac{y_t - F_0^t(x)}{\varepsilon}.$$

The main aim is to compute statistics of z_ε at the final time T, so that of $z_\varepsilon\left(x, T\right)$. ? then considers the signed projection of $z_\varepsilon\left(x, T\right)$ onto a ray emanating from the deterministic position $F_0^T(x)$ in a given direction, defining

$$P_\varepsilon\left(x, \theta\right) := \hat{n}^\intercal z_\varepsilon(x, T),$$

where $\theta \in \left[-\pi/2, \pi/2\right)$ and

$$\hat{n}(\theta) = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}.$$

The statistics of $z_\varepsilon\left(x, T\right)$ and $P_\varepsilon(x, \theta)$ are considered in the limit as $\varepsilon \downarrow 0$, which

TODO: something

.

The first result established by ? is that the expected location is deterministic, in the following sense.

Theorem 2.4.1 (?)
For all $x \in \mathbb{R}^2$,

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left[z_\varepsilon(x, T)\right] = 0.$$

The variance of $P_\varepsilon(x,\theta)$ is used to assign a computable scalar measure of uncertainty to the trajectory.

Definition 2.4.1 (?)

a) The ***anisotropic uncertainty*** *is a scalar field* $A : \mathbb{R}^2 \times \left[-\pi/2, \pi/2\right) \to [0,\infty)$ *defined by*

$$A(x,\theta) \coloneqq \sqrt{\lim_{\varepsilon \downarrow 0} \mathbb{V}\left[P_\varepsilon(x,\theta)\right]}.$$

b) The ***stochastic sensitivity*** *is a scalar field* $S : \mathbb{R}^2 \to [0,\infty)$ *defined by*

$$S^2(x) \coloneqq \lim_{\varepsilon \downarrow 0} \sup_\theta \mathbb{V}\left[P_\varepsilon(x,\theta)\right].$$

Theorem 2.4.2 (?)

For $x \in \mathbb{R}^2$ *and* $\theta \in \left[-\pi/2, \pi/2\right)$,

$$A(x,\theta) = \left(\int_0^T \left\| \Lambda\left(F_0^t(x), t\right) J\hat{n}(\theta) \right\| \mathrm{d}t\right)^{1/2}$$

Chapter 3

Extending stochastic sensitivity

There are several shortcomings of **?** that warrant further extension, namely:

1. The tools are restricted to two-dimensional models, and the constructions using projections have no obvious extension to n-dimensions. Extending stochastic sensitivity to n-dimensions will enable application to a much broader class of models beyond the fluid flow context, including high-dimensional climate and ??? models.

2. **?** only computes the expectation and variance of the projections $P_\varepsilon(x, \theta)$, which does not give us the distribution under the limit as ε approaches 0.

3.

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