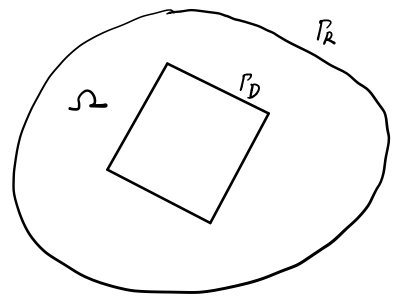
For the Helmholtz equation (two dimensional)

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega$$
$$\frac{\partial u}{\partial n} - iku = g \quad \text{on } \Gamma_R$$
$$u = 0 \quad \text{on } \Gamma_D$$



We introduce the physical flux vector $\vec{p} = \nabla u$, so we can develope the FOSLS formulation:

$$\left.\begin{aligned} \vec{p} - \nabla u &= \vec{0} \\ \nabla\cdot\vec{p} + k^2 u &= 0 \\ \nabla \times \vec{p} &= \vec{0} \end{aligned}\right\} \text{ in } \Omega$$

$$\left.\begin{aligned} u &= 0 \\ \vec{n}\times\vec{p} &= \vec{0} \end{aligned}\right\} \text{ on } \Gamma_D \qquad (1)$$

$$\vec{p}\cdot n - iku = g \quad \text{on } \Gamma_R$$

$\Rightarrow$

$$\left.\begin{aligned} u_2 - \frac{\partial}{\partial x} u_1 &= 0 \\ u_3 - \frac{\partial}{\partial y} u_1 &= 0 \\ \frac{\partial}{\partial x} u_2 + \frac{\partial}{\partial y} u_3 + k^2 u_1 &= 0 \\ \frac{\partial}{\partial x} u_3 - \frac{\partial}{\partial y} u_2 &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\left.\begin{aligned} u_1 &= 0 \\ n_1^D u_3 - n_2^D u_2 &= 0 \end{aligned}\right\} \text{ on } \Gamma_D$$

$$n_1^R u_2 + n_2^R u_3 - iku_1 = g \quad \text{on } \Gamma_R$$

Let $U = (u_1, u_2, u_3)^T$ and $u = u_1$, $p = (u_2, u_3)^T$

Then (1) can be written as

$$LU = AU + BU_x + CU_y = 0 \quad \text{in } \Omega$$
$$DU = 0 \quad \text{on } \Gamma_D$$
$$RU = g \quad \text{on } \Gamma_R$$

where $A, B, C \in \mathbb{R}^{4\times3}$, $D \in \mathbb{R}^{2\times3}$, $R \in \mathbb{R}^{1\times3}$:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -n_2^D & n_1^D \end{pmatrix}, \qquad R = \begin{pmatrix} -ik & n_1^R & n_2^R \end{pmatrix}$$

where $n^D = (n_1^D, n_2^D)$ and $n^R = (n_1^R, n_2^R)$ are the outward directed normal of Γ_D and Γ_R respectively. and $U_x = \frac{\partial}{\partial x}U$, $U_y = \frac{\partial}{\partial y}U$

Define the space $V = [H^1(\Omega)]^3$ and a quadratic form

$$J(U;g) = \int_\Omega \|LU\|^2 d\Omega + \int_{\Gamma_D} \|DU\|^2 ds + \int_{\Gamma_R} |RU - g|^2 ds, \quad U \in V, \quad (2)$$

The solution of (1) minimizes (2), and take the variation for (2), we have

seek $U_1 \in V$, such that $a(U_1, U_2) = F(U_2)$. $\forall U_2 \in V$

where $a(U_1, U_2) = \int_\Omega LU_1 \cdot \overline{LU_2}\, d\Omega + \int_{\Gamma_D} DU_1 \cdot \overline{DU_2}\, ds + \int_{\Gamma_R} RU_1 \cdot \overline{RU_2}\, ds$

$$F(U) = \frac{1}{2} \int_{\Gamma_R} RU \cdot \bar{g} + \overline{RU} \cdot g \, ds$$

If we have a finite dimensional space $W^L \subset V$, we can define a discrete problem : find $U^L \in W^L$, such that

$$J(U^L; g) = \min_{U \in W^L} J(U; g)$$

the variation form is seek $U^L \in W^L$, such that

$$a(U^L, V^L) = F(V^L) \quad \text{for all } V^L \in W^L \quad (3)$$

suppose the basis of W^L are $\{b_i^L\}_{i=1}^{N_L}$, and $U = \sum_{i=1}^{N_L} \alpha_i b_i^L$, ($\dim(W^L) = N^L$)

we can have a matrix form :

$$K^L \alpha = f^L \quad (4)$$

where $\alpha = (\alpha_1, \cdots, \alpha_{N_L})^T$, $f = (F(b_1^L), \cdots, F(b_{N_L}^L))^T$, $K_{ij} = a(b_i^L, b_j^L)$

Minimization of the functional (2) can be used as a guiding principle.

Relaxation:

suppose U is a initial solution,

relaxation sweep can be defined as:

For $i = 1, 2, \cdots, N_L$, $s = \arg\min_s J(U + s b_i^L)$, set $U \leftarrow U + s b_i^L$

In multigrid method, we have a hierarchy of nested meshes $T_0 < T_1 < \cdots T_L$ and a set of finite dimensional spaces $\{W^\ell\}_{\ell=0}^L$

$$W^L = [S_1^0(T_L)]^3$$

$$W^\ell = [\mathrm{span}\{ b_p^\ell(x) \exp(i k d_t^\ell \cdot \vec{x})\}]^3$$

In correction schema, the coarser grid provides a correction for the fine grid approximation U^ℓ. $U^\ell \leftarrow U^\ell + V^{\ell-1}$, and we choose the correction in the sense that it minimizes the functional over all possible choices

$$V^{\ell-1} = \arg\min_{V^{\ell-1} \in W^{\ell-1}} J(U^\ell + V^{\ell-1}) \qquad (4)$$

In order to add the correction to U^ℓ, we need an operator $I_{\ell-1}^\ell$ that maps the $V^{\ell-1}$ to the W^ℓ space, since the finite element spaces are not nested ($W^{\ell-1} \not\subset W^\ell$), we employ the "energy projection" ($I_{\ell-1}^\ell(V) - V \perp W^\ell$)

$$I_{\ell-1}^\ell : W^{\ell-1} \to W^\ell$$

$$a(I_{\ell-1}^\ell(V), b_i^\ell) = a(V, b_i^\ell), \text{ for all } b^\ell$$

assume $\quad I_{\ell-1}^\ell V = \sum \varphi_j b_j^\ell$

then $\quad \sum_i a(b_i^\ell, b_j^\ell)\varphi_i = a(V, b_j^\ell)$,

matrix form: $\quad [a(b_i^\ell, b_j^\ell)]_{ij}\varphi = [a(V, b_j^\ell)]_j$

Coarser grid correction problem

$$\begin{aligned} V^{\ell-1} &= \arg\min_v J(U^\ell + v) \\ &= \arg\min_v \tfrac{1}{2} a(U^\ell+v, U^\ell+v) - F(U^\ell+v) \\ &= \arg\min_v \tfrac{1}{2} a(v,v) - F(v) + \tfrac{1}{2} a(U^\ell, v) + \tfrac{1}{2} a(v, U^\ell) \\ &\hat{=} \arg\min_v J^h(v) \end{aligned}$$