③ PUM-Based Wave-Ray Multigrid

↳ PU $\hat{=}$ partition of unity

We consider the following Helmholtz boundary value problem posed on a two-dimensional domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} \Delta u + k^2 &= 0 \quad \text{in } \Omega \,, \\ \frac{\partial u}{\partial \boldsymbol{n}} - iku &= g \quad \text{on } \Gamma_R \,, \\ u &= 0 \quad \text{on } \Gamma_D \,. \end{aligned} \tag{4.1}$$

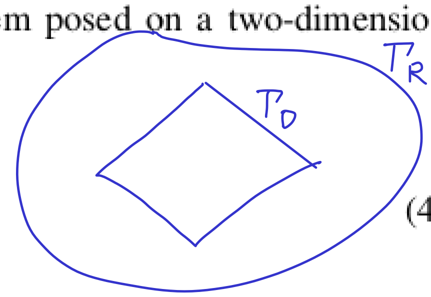

Fig. 4.1.

Here, $k > 0$ is the wavenumber, and Γ_D and Γ_R are two well separated parts of the boundary, $\Gamma_R \cup \Gamma_D = \partial\Omega$, see Figure 4.1. The boundary value problem (4.1) has a variational formulation in $H^1_{\Gamma_D}(\Omega)$: $\mathsf{a}(u, v) = \ell(v)$ for all $v \in H^1_{\Gamma_D}(\Omega)$.

Discretization on a fine triangular mesh $\mathcal{T}_L$ is based on piecewise linear Lagrangian finite elements (space $\mathcal{S}^0_1(\mathcal{T}_L)$) with zero boundary conditions on Γ_D). Further we assume a hierarchy of nested meshes $\mathcal{T}_0 \prec \mathcal{T}_1 \prec \cdots \prec \mathcal{T}_L$ created by uniform, regular refinement.

Let $\mathcal{V}_l$ denote the set of vertices of $\mathcal{T}_l$ not lying on Γ_D. Write $b^l_{\boldsymbol{p}}$ for the piecewise linear nodal basis function ("tent function") associated with vertex $\boldsymbol{p} \in \mathcal{V}_l$. Write

$$\boldsymbol{d}^m_k = \begin{pmatrix} \cos \frac{2\pi}{m} k \\ \sin \frac{2\pi}{m} k \end{pmatrix} , \quad k = 0, \dots, m-1 \,, \tag{4.2}$$

and define the wave modulated partition of unity space according to

$$\begin{aligned} W_L &:= \mathcal{S}^0_1(\mathcal{T}_L) \,, \\ W_l &:= \operatorname{span}\left\{ b^l_{\boldsymbol{p}}(\boldsymbol{x}) \underbrace{\exp(ik\boldsymbol{d}^{2^{L+1-l}}_k \cdot \boldsymbol{x})}_{\hookrightarrow \text{ plane wave !}} : \boldsymbol{p} \in \mathcal{V}_l,\ k = 0, \dots, 2^{L+1-l} - 1 \right\} , \\ & \quad l = 0, \dots, L-1 \,. \end{aligned} \tag{4.3}$$

The following two level algorithm (correction scheme) is proposed for solving the variational problem on W_l, $l \geq 1$:

1. Conduct a directional Gauss-Seidel relaxation.
2. Solve the residual equation on W_{l-1} to obtain a correction.
3. Perform another directional Gauss-Seidel relaxation.

The directional Gauss-Seidel relaxation is based on the following ordering of the degrees of freedom:

- D.o.f. are first ordered according to the direction $\boldsymbol{d}$ they are associated with
- D.o.d. belonging to the same direction are partially ordered in *upstream fashion* with respect to that direction.

▷ Since the spaces W_ℓ are not nested suitable transfer operators for residuals and corrections are not straightforward.
Idea: "Adjacent directions"

Implementation in LehrFEM++

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PUM:

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