"Accuracy Properties of the wave-ray multigrid algorithm for Helmholtz equations"

1.

Each characteristic component can be represented as a product of a lattice principal component and a smooth envelope function. Hence, a general representation of the error that cannot be efficiently reduced by a regular multigrid cycle can be given by

$$v(x,y) = \sum_{\ell=1}^{L} \hat{v}^{\ell}(x,y) e^{i(k_1^{\ell}x + k_2^{\ell}y)}, \tag{1.4}$$

where $\hat{v}^{\ell}(x,y)$ are smooth functions. Note that the functions $\hat{v}^{\ell}(x,y)$ are not uniquely defined: Fourier components of v with frequencies between two lattice points can be assigned to these points with arbitrary weights, yet still yielding the desired smoothness. Considered in the corresponding rotated coordinates, the scale of smoothness of $\hat{v}^{\ell}(x,y)$ is $O(L^2/k)$ in the propagation direction ξ^{ℓ}, and $O(L/k)$ in the η^{ℓ} direction. (In other words, the function $v(x,y)$ essentially consists of Fourier components with frequencies in a ring of width $O(k/L^2)$ around the principal circle.)

We call the functions $\hat{v}^{\ell}(x,y)$ the *ray functions*; the equations we derive for them, the *ray equations*. The residual ray equation for $\hat{v}^{\ell}(x,y)$ is

$$\Delta \hat{v}^{\ell}(x,y) + 2ik_1^{\ell}\frac{\partial}{\partial x}\hat{v}^{\ell}(x,y) + 2ik_2^{\ell}\frac{\partial}{\partial y}\hat{v}^{\ell}(x,y) = \hat{r}^{\ell}(x,y). \tag{1.5}$$

What is the scale of smoothness?
How to derive it?

The error for which the usual multigrid cycle is not effective can efficiently be reduced by a multigrid procedure which on coarse grids represents the smooth ray functions $\hat{v}^{\ell}(x,y)$ rather than the nonsmooth function $v(x,y)$ itself. The coarser the grid, the larger the scale of smoothness of $\hat{v}^{\ell}(x,y)$ represented on it. This requires a larger L, i.e., a finer lattice of principal components.

Why larger L on coarser grid?

2.

2.1.1. Wave phase error. We define a *wave discrete principal component* $e^{i(k_1^h x + k_2^h y)}$ as any one that satisfies the discrete homogeneous equations (1.2),

$$L^h e^{i(k_1^h x + k_2^h y)} = 0. \tag{2.1}$$

By Taylor expansion in (1.2) it can be shown that, for $kh \ll 1$, the value of the corresponding discrete wave number $|k^h| = \sqrt{(k_1^h)^2 + (k_2^h)^2}$ can be approximated by

$$|k^h| \approx k\left(1 + \frac{k^2h^2}{\gamma}\right), \qquad 24 \le \gamma \le 48, \tag{2.2}$$

where the exact value of γ depends on the ratio $k_1^h : k_2^h$. The accumulated value of the relative phase error (the phase error over the period $2\pi/k$) propagating through the $kd/2\pi$ wavelengths of Ω can therefore be estimated as

$$E(kd, kh) \approx kd\frac{k^2h^2}{2\pi\gamma}. \tag{2.3}$$

I'm not familiar with this phase error analysis.
Where is γ from?

3. How to better understand Radiation boundary conditions?

"Wave-Ray multigrid method for standing wave equations"

1.

11. Appendix A. Here we present a more detailed description of the separation process used in our model algorithm to approximate the ray residual functions $\hat{r}_l^1$, $l = 1, \ldots, 8$, on the finest ray level 1 with mesh-size $(4/k_0, 2/k_0)$, where $k_0 = 2^j$, and $j \in Z$ is chosen so that $\overline{k} = k/k_0 \in [0.66, 1.33]$.

The separation starts with the wave function r, defined on the wave level M_r with mesh-size $(1/k_0, 1/k_0)$ as follows

$$r = I_{M_r+1}^{M_r}[\ldots[I_{M_0}^{M_0-1}R^{M_0}]\ldots],$$

where I_j^{j-1}, $j = M_0, \ldots, M_r + 1$ are full-weighting operators; $R^{M_0} = f^{M_0} - L^{M_0}u^{M_0}$ is the wave residual function; u^{M_0} is the current solution approximation; f^{M_0} is the FAS right-hand-side and L^{M_0} is the operator (2.2) on the wave level $m = M_0$. This function is than interpolated to the rotated coordinates (ξ, η) and multiplied by $e^{-ik\xi}$, giving as a result the function $r_0(\xi, \eta)$.

The first separation operator W_0 is applied to $r_0(\xi, \eta)$. W_0 is a tensor product of two perpendicular "diagonal" one-dimensional weighting operators with the frequency parameter (see Sec. 4) taken equal to 2. The resulting function $r_1(\xi, \eta)$, defined on the grid with mesh-size $(h_\xi, h_\eta) = (2/k_0, 2/k_0)$, is given by

$$r_1(\xi, \eta) = \min(1, \overline{k}) \times [W_0(r_0(\xi, \eta)].$$

The next separation operator W_1, applied to $r_1(\xi, \eta)$ is a tensor product of a weighting operator in the ξ direction defined by the frequency parameter $\max(1, \overline{k})$, and a weighting operator in the η direction defined by the frequency parameter $.85 \max(1, \overline{k})$.

The resulting function $r_2(\xi, \eta)$ is defined on the grid with mesh-size $(4/k_0, 2/k_0)$ and is given by

$$r_2(\xi, \eta) = W_1(r_1(\xi, \eta)).$$

Finally, a weighting operator in the η direction defined by the frequency parameter $.75 \max(1, \overline{k}^2)$ is applied to r_2, yielding the target function $\hat{r}(\xi, \eta)$ on the same grid with $(h_\xi, h_\eta) = (4/k_0, 2/k_0)$.

I can't figure out the detail of separation procedure.
How to determine these frequency parameter?
What's the purpose of coarsing?

2. In chapter 7

$$\Delta u(x,y) + k^2 u(x,y) = f(x,y), \qquad (x,y) \in \Re^2, \tag{2.1}$$

$$\frac{u^h_{i-1,j} - 2u^h_{i,j} + u^h_{i+1,j}}{h^2} + \frac{u^h_{i,j-1} - 2u^h_{i,j} + u^h_{i,j+1}}{h^2} + k^2 u^h_{i,j} = f_{i,j}, \tag{2.2}$$

The symbol of (2.1) for $e^{i(\omega_1 x + \omega_2 y)}$ is given by $-\omega^2 + k^2$, while the symbol of the difference operator (2.2) is $k^2 + 2h^{-2}(\cos(\omega_1 h) + \cos(\omega_2 h) - 2)$. Hence, the relative error for such a component is

$$E^h(\omega_1, \omega_2) = \left| \frac{2h^{-2}(\cos(\omega_1 h) + \cos(\omega_2 h) - 2) + \omega^2}{-\omega^2 + k^2} \right|. \quad (k^2)$$

What is the "symbol"?
Why E^h stands for relative error?

3. In section 8. Ray cycle

Following relaxation, a switch is made to a coarser level, and the following transfers are made. If n is even, meaning that the number of ray functions remains the same on the coarser grid, the residual transfer is done as in a regular V-cycle with coarsening only in the propagation direction (no separation is applied). Otherwise, the next coarser level employs twice as many ray functions; half of them correspond to the same principal lattice components that already appeared on the coarser lattice (corresponding to the *finer* ray level), and another half are represented only on the finer lattice (*coarser* ray level). If $n-1$ and n are the finer and the coarser levels, correspondingly, then the coarse-grid residuals $\hat{r}^n$ are evaluated by the following two formulae:

$$\hat{r}^n_{2l+1} = W^n_{n-1} R^{n-1}_l, \tag{8.1}$$

$$\hat{r}^n_{2l} = W^n_{n-1}[e^{n-1}_l I^{2l}_{2l-1}(R^{n-1}_l) + e^{n-1}_{l+1} I^{2l}_{2l+1}(R^{n-1}_{l+1})]/e^n_{2l}, \tag{8.2}$$

In transferring ray residuals, why first multiply the exponent and then divide one in (8.2)?
In my understanding, all $\hat{r}^n_i$ are in the rotated coordinate (ξ^n_i, η^n_i), shouldn't we transfer them to (x, y) coordinate first then do the process like (8.2) and then transfer to rotated coordinate?

"The partition of unity method"

proof of Lemma 1

Proof. Because the spaces V_j^1 contain the fundamental system $\{\sinh kx, \cosh kx\}$, it is enough to approximate a particular solution to

$$-u'' + k^2 u = f \quad \text{on } \Omega_j \cap \Omega$$

By Taylor's theorem, on $\Omega_j \cap \Omega$, $f(x) = l(x) + r(x)$ where $l(x)$ is linear and $|r(x)| \leqslant (2h)^2 \|f''\|_{L^\infty(\Omega)}$ (note that $\operatorname{diam} \Omega_j \leqslant 2h$). A particular solution to the problem with the right-hand side $r(x)$ is given by the solution u_r to

$$-u_r'' + k^2 u_r = r \quad \text{on } \Omega_j \cap \Omega$$
$$u = 0 \quad \text{on } \partial(\Omega_j \cap \Omega)$$

Thus,

$$\|u_r'\|^2_{L^2(\Omega_j \cap \Omega)} + k^2 \|u_r\|^2_{L^2(\Omega_j \cap \Omega)} \leqslant \frac{2h}{k^2} \|r\|^2_{L^\infty(\Omega_j \cap \Omega)} \qquad (1)$$

from whence

$$\|u_r'\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^{1/2} \frac{h^2}{k} \|f''\|_{L^\infty(\Omega)}$$

$$\|u_r\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^{1/2} \frac{h^2}{k} \min(h, k^{-1}) \|f''\|_{L^\infty(\Omega)}$$

with $C > 0$ independent of h, k, and f. Finally, a particular solution to the problem with the right-hand side $l(x)$ is given by $u_l(x) = k^{-2} l(x)$ which can be approximated in V_j^1 such that

$$\|u_l - v_j\|_{L^2(\Omega_j \cap \Omega)} + h\|(u_l - v_j)'\|_{L^2(\Omega_j \cap \Omega)} \leqslant Ch^3 h^{1/2} \min(1, (kh)^{-2}) \|f'\|_{L^\infty(\Omega)}, \qquad (2)$$

where $C > 0$ is independent of h, k, and f. The assertion of the lemma follows. □

In the proof of Lemma 1, I think (1) should be

$$\|u_r'\|^2_{L^2(\Omega_j \cap \Omega)} + \frac{h^2}{2} \|u_r\|^2_{L^2(\Omega_j \cap \Omega)} \leqslant \frac{2h}{k^2} \|r\|^2_{L^\infty(\Omega_j \cap \Omega)}$$

In the first line of the proof, why it has the conclusion that it is enough to approximate a particular solution?

And how to derive (2)?