Orbital Mechanics Mini-Project

Learning Objectives

The main goals of this worksheet are:

- Practice modeling mechanical systems using free body and schematic diagrams, equations of motion, and slope functions.
- Implement a simple one-dimensional model and use it to answer a question about the motion of the Earth under the influence of the gravitational attraction between the Earth and the Sun.
- Implement a simple two-dimensional model using vectors, validate it against the corresponding one-dimensional model, and plot the elliptical trajectory of the Earth's orbit.

Submission Checklist

Please be sure to do the following:

- [x] Write your name here: Lilo
- [x] Write the name(s) of your studio partner(s) here: Alex
- [ ] By midnight on Thursday, November 14: Scan this worksheet and submit it on Canvas. Also submit a link to your CoCalc notebook as a comment.

Then stay tuned for instructions from the NINJAs on how to get checked off. (As with the HIV model, you will need to meet with your studio NINJA and explain your work.)

The Earth Falls into the Sun

The exercise at the end of the Chapter 20 notebook asks:

"If the Earth suddenly stopped orbiting the Sun, I know eventually it would be pulled in by the Sun's gravity and hit it. How long would it take the Earth to hit the Sun?"[1]

[1] From "Ask an Astronomer" at Cornell University.

Your task in this part of the worksheet is to implement a model that answers this question, starting from a blank notebook. For purposes of this worksheet, assume the Sun is unaffected by the gravitational pull of the Earth (or any other celestial bodies).

Preliminaries

1. Draw a free body diagram showing the Earth and the force of gravity on it.

$$\vec{F}_G = \frac{GmM}{r^2}$$

$F = ma$

$\int a = v$

$\int v = r$

$\frac{GM}{r^2} = a$

2. Draw a schematic diagram that shows the Earth and Sun in a one-dimensional coordinate system. How did you decide where to put the origin?

sun stays fixed is a modelling assumption

3. What are the state variables in this system? What are the system variables (constants or parameters)?

system var

- m mass of earth
- M mass of sun
- G gravitational constant

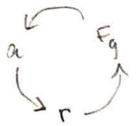
state var

- r dist between sun & earth
- v velocity of earth

4. Write down a mathematical expression for the gravitational force acting on the Earth. What happens when the distance between the Earth and the Sun goes to zero?

$$F_g = \frac{mMG}{r^2}$$

when $r=0$ you get a divide by 0 error but F_g is converging to ∞



5. Write down a pair of first-order differential equations for the system. Check that the signs are correct. How does the mass of the Earth affect its acceleration under the force of gravity?

$$\frac{dv}{dt} = a \cdot \cancel{dt} = \frac{MG}{r^2} \cdot \cancel{dt}$$

$V \mathrel{+}= \frac{dv}{dt}$

$$\frac{kg\left(\frac{m^3}{kg\,s^2}\right)}{m^2} = \frac{m}{s^2}$$

$$\frac{dr}{dt} = v \cdot \cancel{dt}$$

Implementation

You are now ready to implement the model defined above.

As with the HIV model, you and your partner(s) should decide how you are going to work together on this task. We suggest briefly discussing your respective programming experiences in the course so far — what has worked well, and what hasn't?

If pair programming has worked well for you, and/or you would like to try it again, we suggest choosing driver and navigator roles for Tuesday's studio session and reversing them on Wednesday. You could also try a "modified pair programming" approach where each of you works in your own notebook but communicates continually with your partner(s). Regardless of how you organize yourselves, all members of the group are expected to be actively engaged and learning from each other.

You should start from a new notebook that you create. While the immediate goal is to answer the question posed on page 2, you will be using the code you write as a starting point for the next part of the worksheet, so you should find it worthwhile to put some effort into good coding practices (e.g., define variables to hold parameters and constants that are used more than once; encapsulate logical "chunks" of code into functions; test functions separately before calling one function from another).

As with the HIV model, this is a good opportunity to practice the QMRI format of a computational essay; if you use this notebook as a starting point for Project 3, you will have a head start. Keep in mind that you and your partner(s) will each need to explain your work to your studio NINA during your respective check-off meetings — try to make your notebook readable both by your future selves and by someone who hasn't worked with you this week.

r_E r_S r_{init}

You are welcome to implement the model in any way you like, and very welcome to "cut and paste" code you've already worked with in the chapter notebooks. Here are some additional suggestions to consider:

$r_{init} - (r_E + r_S)$

1. While it is perfectly reasonable to simulate the system using Euler's method (as we did in the HIV model and Project 2), we encourage you to use the ModSimPy ODE solver (like most of the examples in the book from Chapter 18 onward).

2. As with your simulations of objects thrown off tall buildings, you will probably want to use an event function to stop the simulation when some condition is met. In the Chapter 20 notebook, Allen suggests using the condition that the surface of the Earth touches the surface of the Sun. (Why?) ← end condition: $r_{earth} + r_{sun} == r$

3. Your results will probably be expressed in seconds and meters, which are hard to relate to on astronomical scales. Practice converting to more human-friendly units (e.g., days and millions of kilometers), and check that the results make sense.

4. Even though you are only asked for a numerical answer, try graphing the position of the Earth over time (again, in human-friendly units) to verify that the simulation is behaving as you expect. You should label your graph, of course.

$\frac{MG}{r^2} \cdot dt$

So . . . how long does it take for the Earth to hit the Sun?

$\frac{dv}{dt}$ $\frac{kg\left(\frac{m^3}{kg\,s^2}\right) s}{m^2} \cdot \frac{m}{s}$

$\frac{MG}{r^2} \cdot dt$

The Earth Orbits the Sun

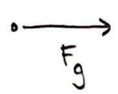
While it might be fun to imagine the Earth screeching to a halt in its orbit and then careening toward the Sun, most of the time we are interested in celestial bodies that are in motion around each other.

Your task in this part of the worksheet is to extend the model you built in the first part to allow the Earth to move in two dimensions with a non-zero initial velocity. You can continue to assume that the only force on the Earth is the Sun's gravitational pull (and continue to neglect the Earth's pull on the Sun).

Preliminaries

$F_c = \frac{mv^2}{r}$

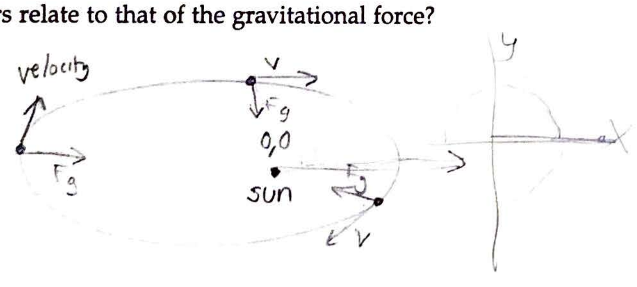
1. Draw another free body diagram showing the Earth and the force of gravity on it. Has anything changed?



$$\frac{G\cancel{m}M}{r^{\cancel{2}}} = \frac{\cancel{m}v^2}{\cancel{r}} \quad \boxed{v = \sqrt{\frac{GM}{r}}}$$

2. Draw a schematic diagram that shows the Earth and Sun in a two-dimensional coordinate system. Include vectors that represent the position and velocity of the Earth. How does the direction of these vectors relate to that of the gravitational force?

$r = \sqrt{x^2+y^2}$



3. What are the state variables in this system? Are there any new system variables (constants or parameters)?

system

M mass of Sun

G Gravitational constant

m mass of Earth ?

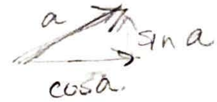
v_0 initial velocity

state

r dist between Earth & Sun

v velocity of Earth

4. Write down a mathematical expression for the gravitational force acting on the Earth, using vector notation. How does it relate to the scalar expression you wrote earlier?



$$\frac{d\vec{r}}{dt} = \hat{V}$$

$$\frac{d\vec{v}}{dt} = \frac{MG}{|\vec{r}|^2} \cdot \hat{r}$$

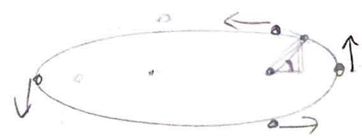
$$F(\vec{r}) = \frac{G \cancel{m} M}{|\vec{r}|^2} \cdot \hat{r} = \cancel{m}\vec{a}$$

$$\vec{v} = \vec{a}\,dt + \vec{V}_0$$

wait but $\hat{v} != \hat{r}$?

5. Similarly, write down a pair of vector-valued first-order differential equations for the system. How do they compare to their scalar counterparts?

6. What is the initial position of the Earth in your coordinate system? What behavior do you expect if the initial velocity is zero? If you wanted to start the Earth moving at its actual orbital speed in a direction that is perpendicular to the gravitational force, write down an initial velocity vector that would achieve this.[2]



[2] Feel free to estimate or look up the orbital speed of the Earth.

init pos $\measuredangle 0^\circ$, dist $r > 0$

if init vel = 0 it will crash into the sun (in 65 days)

$$\hat{V} = V_0 \hat{y}$$

$$V_0 = 29770 \frac{m}{s} = 2.977e4 \frac{m}{s}$$

Implementation

You are now ready to adapt your original model to implement the model defined above.

We strongly recommend using the same notebook you created for the one-dimensional version of the model. Here are some additional suggestions to consider:

1. If you haven't yet worked through the Chapter 22 notebook (on vectors and projectile motion), you might want to do that before attempting this part of the worksheet. Be sure you understand how to use the ModSimPy ODE solver with vector-valued slope functions.[3]

2. If you create a new function that is similar to an existing function, give it a different name (e.g., `slope_func_2d`). You might want to use a `Params` object to hold the parameters of your model so you can easily vary them (e.g., to try different initial conditions).

3. You may need to play around with the end time (`t_end`) and step size (`dt`) of your simulation. If you are using an event function and the event doesn't get triggered, try reducing `dt` (and think about why this might be necessary).

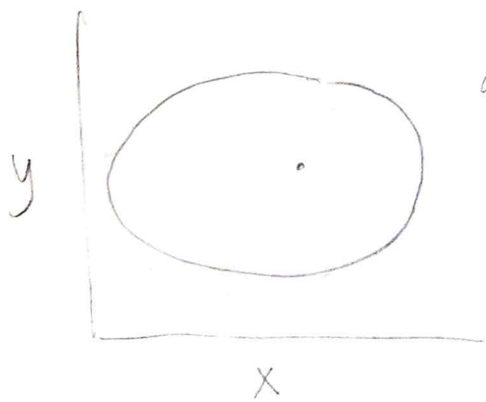
4. Try plotting and animating the trajectory of the Earth, using the code in the Chapter 22 notebook as a starting point.

5. As always, test your code as you go, and seek help if you need it.

[3] You could also break the vectors into x and y components, and keep track of each component as a separate state variable. This is how we used to have to do things until Allen wrote an ODE solver that understands vectors. Thanks, Allen!

Can your two-dimensional model replicate the behavior of the Earth falling into the Sun? Try it!

What happens if you run the simulation with initial conditions similar to the Earth's actual orbit? If you run the simulation for a full year, does the Earth complete a full orbit? Draw a picture of the trajectory you obtain.



an ellipse
sort of

Reflection Questions

1. What problems did you encounter while implementing these two models of orbital mechanics? How did you overcome them? What strategies do you want to remember for Project 3?

using the ode solver seemed to break our code and we think it was due to our use of vectors. It warned us divide by 0 errors happened in one component of the position vector. To fix this, we manually coded Euler's method to check if it works, and printed output to see how the vectors were changing.

2. How does the approach to mechanical systems we are taking in this course compare to the approach(es) you encountered in high school physics? What are the most noticeable similarities and differences? What do you see as the main benefits and drawbacks of the ModSim approach?

The modsim approach uses quantities and numbers, unlike in high school where we used equations with abstract variables most of the time. Modsim derives things numerically in code whereas physics class derived things mathematically. Both are good, but ModSim is more real-world applicable.