Lec 11.

Absolute stability

LMM: $\rho(\omega) - z\,\sigma(\omega)$

check root condition w.r.t.

poly. of ω

Boundary locus.

$\downarrow$

$|\omega| = 1$ at least for one root

$$\omega = e^{i\theta}, \quad \theta \in [0, 2\pi)$$

$$\rho(e^{i\theta}) - z\,\sigma(e^{i\theta}) = 0$$

$$\left\{ z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, \quad 0 \le \theta < 2\pi \right\}.$$

RAS for RK.

$$\vec{k} = f(u_n \vec{e} + hA\vec{k})$$

$$= \lambda u_n \vec{e} + \underbrace{\lambda h}_{z} A \vec{k}$$

$$\Rightarrow \vec{k} = \lambda (I - zA)^{-1} \vec{e} \, u_n$$

$$u_{n+1} = u_n + h\,\vec{b}^T \vec{k}$$

$$= u_n + z\,\vec{b}^T (I - zA)^{-1} \vec{e}\, u_n$$

$$= \Big(\underbrace{1 + z\,\vec{b}^T (I - zA)^{-1} \vec{e}}_{R(z)}\Big) u_n$$

<u>Def</u>. RAS for RK

$$\{ z \in \mathbb{C} : |R(z)| \le 1 \}.$$

Ex. $\begin{array}{c|c} & \frac{1}{2} \\ \hline & 1 \end{array}$ $\quad \begin{cases} k = f\left(u_n + \frac{1}{2} h k\right) \\ u_{n+1} = u_n + h k \end{cases}$

$$R(z) = 1 + z\left(1 - z\tfrac{1}{2}\right)^{-1}$$

$$= \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$$

$|R(z)| \leq 1 \iff \mathrm{Re}\, z \leq 0.$ A-stable

Ex.
$$\begin{array}{c|cc} & 0 & \\ & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \quad \left\{ \begin{aligned} k_1 &= f(u_n) \\ k_2 &= f\left(u_n + \frac{h}{2} k_1\right) \\ u_{n+1} &= u_n + h k_2 \end{aligned} \right.$$

explicit midpoint rule.

$$R(z) = 1 + z \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{z}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{z}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

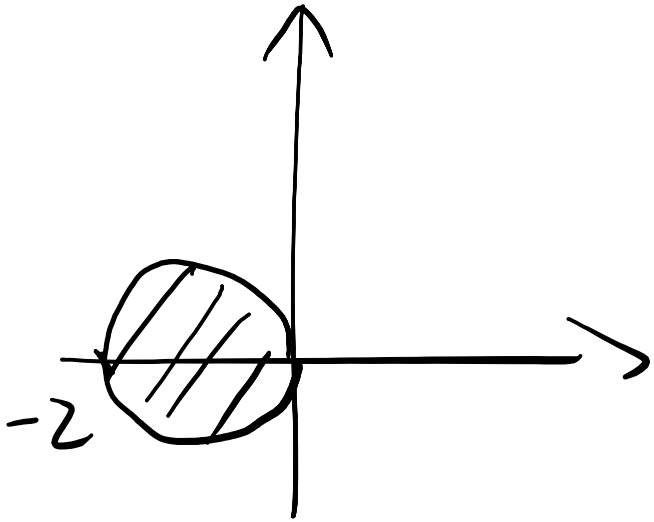
$$= 1 + z\left(\frac{z}{2} + 1\right)$$

$$= \underbrace{1 + z + \frac{z^2}{2}}_{\text{Taylor } e^z}$$

Boundary locus.

$$1 + z + \frac{1}{2}z^2 = e^{i\theta}, \quad \theta \in [0, 2\pi)$$

$\longrightarrow$ solve quadratic eq.



<u>Thm</u>. No explicit RK can be A-stable

using $A^r = 0$

Pf: $$(I - zA)^{-1} = \sum_{i=0}^{r-1} (zA)^i$$

$$R(z) = 1 + z\,\vec{b}^T \left(\sum_{i=0}^{r-1} z^i A^i \right) \vec{e}$$

$$\in P_r(z)$$

as $\operatorname{Re} z \to -\infty$,

$|R(z)| \sim a_r |z|^r$ unbounded.

$\mathrm{RAS} \not\supseteq \{\operatorname{Re} z \le 0\}$. $\square$.

$$|R(z) - e^z| \sim O(z^{p+1})$$

If true. Then $\operatorname{Re} z < 0$.

$R(z) \approx e^z \to$ small.

$\to |R(z)| \leq 1$. (at least around $z=0$)

For LMM. high order

$\Rightarrow$ Smaller RAS.

For RK high order

$\Rightarrow$ larger RAS.

Pf: LTE for $\dot{u} = \lambda u$

$$\tau_n = u(t_{n+1}) - R(\lambda h)\, u(t_n)$$

$$= \left[e^{\lambda h} - R(\lambda h) \right] u(t_n)$$

$$\sim O\Big((\underbrace{\lambda h}_{z})^{p+1} \Big) \qquad \square$$

whether $R(z) = e^{z} + O(z^{p+1})$

$\cancel{\Rightarrow}$ order p ?

$$R(z) = 1 + z\vec{b}^{T}(I - zA)^{-1}\vec{e}$$

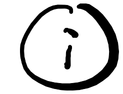
$$= 1 + \sum_{i=0}^{\infty} \vec{b}^{T} A^{i} \vec{e} \cdot z^{i+1}$$

$$= \sum_{i=0}^{p} \frac{1}{i!} z^{i} + O(z^{p+1})$$

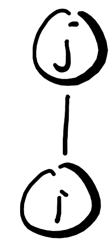
$\Rightarrow$ p conditions

$$\vec{b}^T A^{i-1} \vec{e} = \frac{1}{i!}, \quad 1 \le i \le p.$$

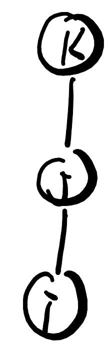
$i=1$ $\quad \vec{b}^T \vec{e} = \sum_i b_i = 1.$



$i=2$ $\quad \vec{b}^T A \vec{e} = \sum_{ij} b_i a_{ij} = \frac{1}{2}$



$i=3,$ $\quad \vec{b}^T A^2 \vec{e} = \sum_{ijk} b_i a_{ij} a_{jk} = \frac{1}{3!}$



$\dots$

The condition $|R(z) - e^z| \sim O(z^{p+1})$

only checks <u>line diagrams</u>.

Summary

	accuracy	zero	abs
LMM	$\rho(z) - \log z\, \sigma(z)$	$\rho(z)$	$\rho(w) - z\sigma(w)$
RK	diagram	uncond.	$R(z)$

How does RAS matter in general?

$$\begin{cases} \dot{u} = f(u) \\ u(t_n) = \dots \end{cases} \Rightarrow u(t_{n+1})$$

Linearization

$$f : \mathbb{R}^d \to \mathbb{R}^d$$

$$f(u(t)) \approx f(u_n) + \underbrace{(Df)(u(t_n))}_{A} \cdot (u(t) - u(t_n))$$

diagonalize A $\qquad \lambda_i$

$\mathrm{Re}\,\lambda_i > 0$: exp. growing.

$\mathrm{Re}\,\lambda_i = 0$. oscillating.

$\mathrm{Re}\,\lambda_i \ll 0$ Stiff

Prothero-Robinson example.

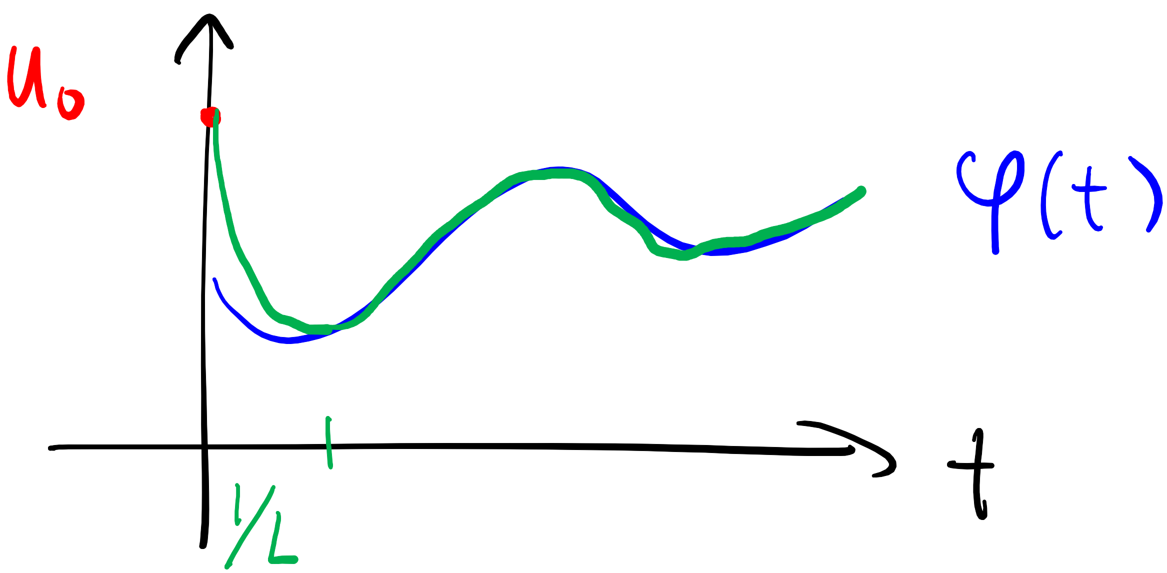
$$\begin{cases} \dot{u}(t) = -L\left(u(t) - \varphi(t)\right) + \dot{\varphi}(t) \\ u(0) = u_0 \end{cases}$$

$L \gg 0$. $\varphi(t)$ smooth.

Duhamel's principle.

$$u(t) = e^{-Lt} u_0 + \int_0^t e^{-(L-s)} \left(L\varphi(s) + \dot{\varphi}(s) \right) ds$$

$$= e^{-Lt} \left(u_0 - \varphi(0) \right) + \varphi(t)$$



Theoretical challenge. error analysis.

Backward Euler.

① $Lh < \frac{1}{2} \Rightarrow \frac{1}{1-Lh}$

② $\max_{0 \le nh \le T} \|e_n\| \le e^{LT}\|e_0\| + \frac{\max \|\tau_n\|}{h} e^{LT}$