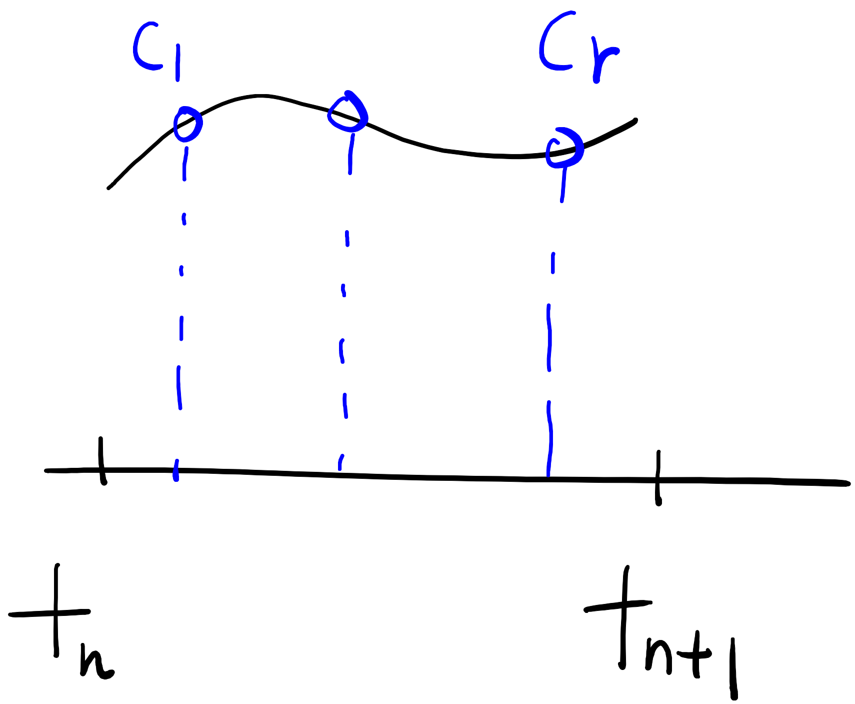
Lec 21.



$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

RK

zero stable.

$\vec{k} = \cdots$

$\rightarrow \vec{k} = \Phi(u_n)$

In collocation. quadrature error

$O(h^{p+1})$.

$\Rightarrow$ LTE is $O(h^{p+1})$

$\Rightarrow$ ODE scheme $O(h^{p})$

$$\begin{cases} \dot{u} = f(u) \\ u(t_n) = u_n \end{cases}$$

v.s.

$$\begin{cases} \dot{v} = f(v) + [\tilde{f}(v) - f(v)] \\ v(t_n) = U_n \end{cases}$$

$$\downarrow$$

$$g(v)$$

$$\tau = \| u_{n+1} - U_{n+1} \|$$

$$g(v(t_n + c_i h)) = 0 .$$

Try simpler (linear) problem

$$\begin{cases} \dot{u} = Au \\ u(t_n) = u_n \end{cases} \quad \text{v.s.} \quad \begin{cases} \dot{v} = Av + g(v) \\ v(t_n) = u_n \end{cases}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \text{ Duhamel}$$

$$u(t_{n+1}) = e^{Ah} u_n$$

$$v(t_{n+1}) = \underbrace{e^{Ah} u_n}_{\to\, u(t_{n+1})} + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)} g(s)\, ds$$

$$\| \tau \| = \left\| \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-s)} g(s)\, ds \right\|$$

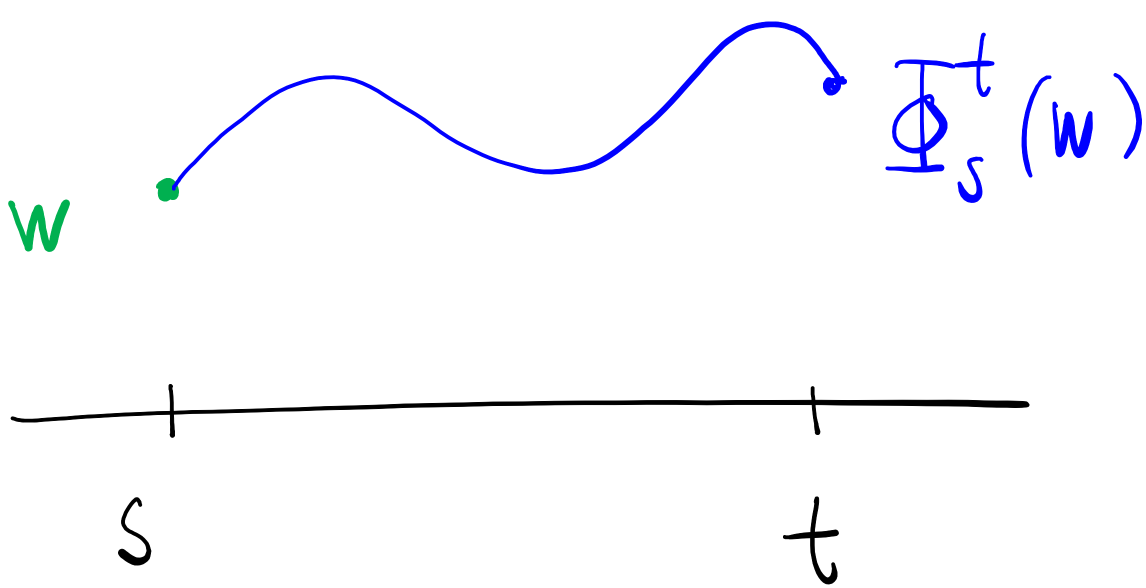
$$\xrightarrow{\text{quadrature}} \left\| \sum_{i=1}^{r} e^{A(t_{n+1}-(t_n+c_i h))} g(t_n+c_i h)\, \omega_i \right\|$$

$$+ O(h^{p+1})$$

Revisit Duhamel.

$$\begin{cases} \dot{\Phi}_s^t(w) = A\, \Phi_s^t(w) \\ \Phi_s^s(w) = w \end{cases}$$

flow map



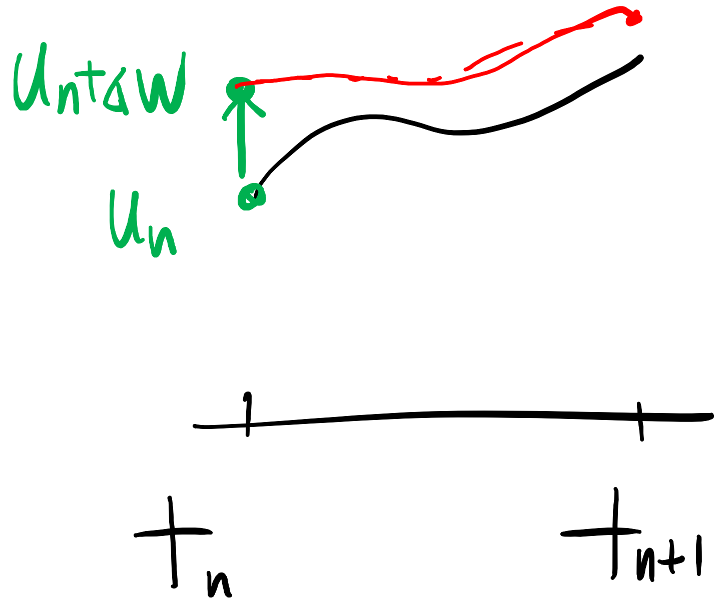
$$u(t_{n+1}) = \Phi_{t_n}^{t_{n+1}}(u_n)$$

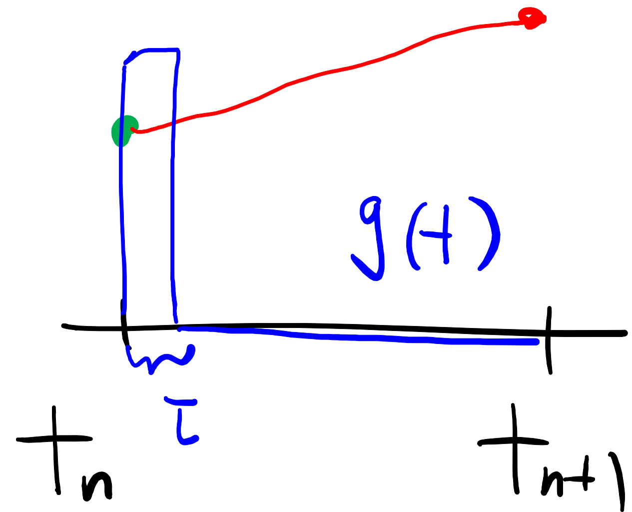
perturb initial data.

$$u_n \longrightarrow u_n + \Delta w$$

$$u(t_{n+1}) \longrightarrow u(t_{n+1}) + \Delta u(t_{n+1})$$

$$\| \quad \frac{\partial \Phi_{t_n}^{t_{n+1}}}{\partial w}(u_n)\,\Delta w + O(\Delta w^2)$$





$$\begin{cases} \dot{u} = Au \\ u(t_n) = u_n + \Delta w \end{cases}$$

$$\begin{cases} \dot{u} = Au + g \\ u(t_n) = u_n \end{cases}$$

$$g(t) \sim \begin{cases} \dfrac{\Delta w}{\tau}, & t_n \le t \le t_n + \tau \\ 0, & \text{otherwise} \end{cases}$$

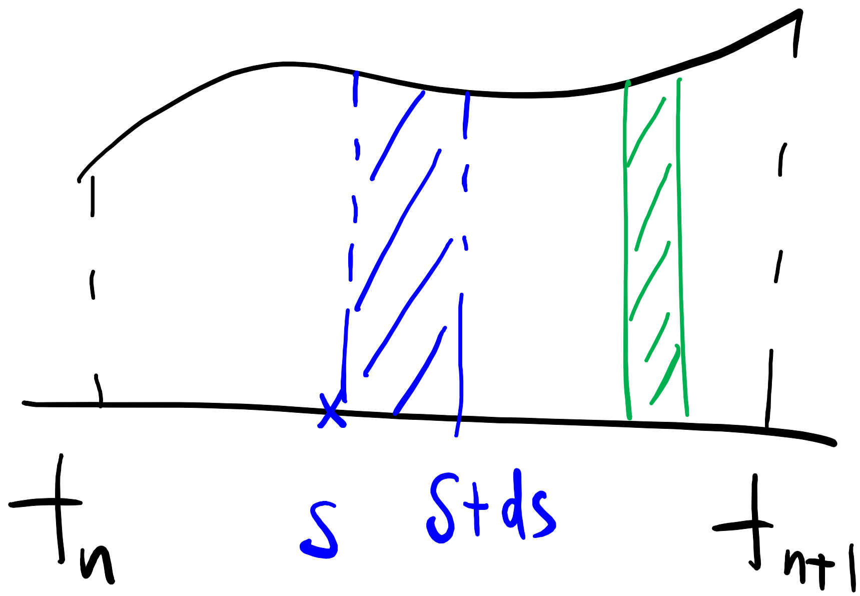
Linear case

$$\Phi_s^t(w) = e^{A(t-s)} w$$

$$\frac{\partial \Phi_s^t(w)}{\partial w} = e^{A(t-s)} \rightarrow \text{independent of } w$$

Duhamel

$$v(t) = u(t) + \int_{t_n}^{t} \frac{\partial \Phi_s^t}{\partial w} g(s)\, ds$$



$$\Delta W = g(s)\,ds$$

Nonlinear case.

$$\begin{cases} \dot{u} = f(u) \\ u(t_n) = u_n \end{cases} \longrightarrow \begin{cases} \dot{\Phi}_s^t(w) = f\left(\Phi_s^t(w)\right). \\ \Phi_s^s(w) = w. \end{cases}$$

$$\begin{cases} \dot{v} = f(v) + g(v) \\ v(t_n) = u_n \end{cases}$$

$$v(t) = u(t) + \int_{t_n}^{t} \frac{\partial \Phi_s^t}{\partial w}\big(v(s)\big)\, g(s)\, ds$$

Alekseev-Gröbner thm.

Collocation

$$\|\tau\| = \|v(t_{n+1}) - u(t_{n+1})\|$$

$$= \left\| \int_{t_n}^{t_{n+1}} \frac{\partial \Phi_s^{t_{n+1}}}{\partial w}(v(s))\, g(s)\, ds \right\|$$

$$= \underset{\substack{\uparrow \\ \text{quadrature} \\ \text{node}}}{0} + O(h^{p+1}) \qquad \square.$$

Iterative method for sparse linear systems.

$$Ax = b. \rightarrow Ax - b = 0$$

① $F(x) = 0$

nonlinear solver.

② classical splitting $\begin{cases} \text{Jacobi} \\ \text{Gauss-Seidel. SOR} \end{cases}$

Steepest descent.

Assume black-box access

$$x \longrightarrow Ax$$

① sparse

② dense but fast application.

a) FFT. b) sparse + low rank

c) hierarchically low-rank ...

What's the objective function.

$A \in \mathbb{R}^{N \times N}$, symmetric

positive definite

$A \succ 0$

$$\varphi(x) = \frac{1}{2} x^T A x - b^T x.$$

$$\nabla \varphi(x) = Ax - b = 0$$

$$\nabla^2 \varphi = A > 0$$

$$\Rightarrow \quad x^* = A^{-1} b$$ global minima.

Think: $\varphi(x) = \|Ax - b\|_2^2$