

Lec 16.

Hamiltonian system.

$$H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$$

↑ ↑
momentum space .

P q

$$\begin{cases} \dot{P} = -\frac{\partial H(P, q)}{\partial q} \\ \dot{q} = \frac{\partial H(P, q)}{\partial P} \end{cases}$$

$$H(p, q) = T(p) + V(q) \quad \text{separable}$$

Ex. N - particles interacting in a potential field $V(q_1, \dots, q_N)$

$$H(p, q) = \sum_{i=1}^N \frac{1}{2m_i} p_i^2 + V(q_1, \dots, q_N)$$

$$\begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}}_{J^{-1}} \begin{pmatrix} \frac{\partial H(p(t), q(t))}{\partial p} \\ \frac{\partial H(p(t), q(t))}{\partial q} \end{pmatrix} \rightarrow \nabla_u H(u(t))$$

$$\varphi_t(p_0, q_0) = (p(t), q(t)) := u(t)$$

$$\boxed{\frac{du(t)}{dt} = J^{-1} \nabla_u H(u(t))}$$

$$J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \quad \boxed{J^{-1} = -J = J^T}$$

Area preserving \Rightarrow Consider

Jacobian

$$\bar{\Phi}_t(p_0, q_0) = \begin{bmatrix} \frac{\partial p(t)}{\partial p_0} & \frac{\partial q(t)}{\partial p_0} \\ \frac{\partial p(t)}{\partial q_0} & \frac{\partial q(t)}{\partial q_0} \end{bmatrix}$$

$$\bar{\Phi}_t(u_0) = \frac{\partial u(t)}{\partial u_0}$$

Want to show: $|\bar{\Phi}_t(u_0)| = 1$.

Thm (Poincaré) Hamiltonian dynamics

$$\underline{\Phi}_t^T J \underline{\Phi}_t = J$$

symplectic

$$u(t) = \varphi_t(u_0)$$

Symplectic \Rightarrow area preserving

$$\det(\underline{\Phi}_t^T J \underline{\Phi}_t) = \det(\underline{\Phi}_t)^2 \det(J) = \det(J)$$

$$\Rightarrow \det(\underline{\Phi}_t) = \pm 1.$$

Pf: $\underline{\Phi}_0 = I$ $\underline{\Phi}_0^T J \underline{\Phi}_0 = J$. $t=0$ ✓

$$\frac{d}{dt} \underline{\Phi}_t = \frac{d}{dt} D_{u_0} \varphi_t(u_0) = D_{u_0} \frac{d}{dt} \varphi_t(u_0)$$

$$= D_{u_0} \left(J^{-1} D_u H(\varphi_t(u_0)) \right)$$

$$= J^{-1} \underbrace{D_u^2 H(u(t))}_{\text{Hessian . symmetric}} \cdot \underline{\Phi}_t$$

Hessian . symmetric

$$\frac{d}{dt} \left(\underline{\Phi}_t^T J \bar{\Phi}_t \right)$$

$$= \underline{\Phi}_t^T D_u^2 H (-J^{-1}) J \underline{\Phi}_t$$

$$+ \underline{\Phi}_t^T J J^{-1} D_u^2 H \bar{\Phi}_t = 0 \quad \square.$$

Symplectic scheme.

Discrete flow map

$$\varphi_n : u_0 \mapsto u_n(u_0)$$

Jacobian $\Phi_n := \frac{\partial \varphi_n(u_0)}{\partial u_0}$

$$\Phi_n^T J \Phi_n = J$$

Lemma: $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$. $f \in C^1(\mathbb{R}^n)$

$Df(u)$ is symmetric i.e.

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_j}{\partial u_i}$$

Then $\exists H: \mathbb{R}^n \rightarrow \mathbb{R}$.

s.t. $f(u) = D_u H(u)$.

$$Pf: \quad H(u) = \int_0^1 u \cdot f(t+u) dt$$

$$\frac{\partial H(u)}{\partial u_j} = \int_0^1 f_j(t+u) dt$$

$$+ \int_0^1 \sum_{i=1}^N u_i \frac{\partial f_i(t+u)}{\partial u_j} dt$$

$$= \int_0^1 f_j(t+u) + \sum_{i=1}^N u_i \cdot \frac{\partial f_j(t+u)}{\partial u_i} t dt$$

$$= \int_0^1 \frac{d}{dt} \left(+ f_j (+u) \right) dt$$

$$= f_j (u) \quad \square .$$

Thm. (Inverse of Poincaré)

$$f: \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}, \quad f \in C^1(\mathbb{R}^{2d})$$

flow map of $\dot{u} = f(u)$

is symplectic . Then $\exists H$

$$f = J^{-1} D_u H .$$

