Lec 22.

$$Ax = b.$$

$$A \in \mathbb{R}^{N\times N}. \qquad A \succ 0$$

$$\inf_{x\in\mathbb{R}^N} \varphi(x) = \frac{1}{2}x^T A x - b^T x.$$

$$\inf_{x \in \mathbb{R}^N} \|Ax-b\|_2^2 = x^T A^T A x - b^T A x - x^T A^T b + b^T b$$

$$\Rightarrow \quad A^T A x = A^T b .$$

$$\Rightarrow \quad Ax = b .$$

$$\kappa(A) = \|A\|_2 \, \|A^{-1}\|_2$$

$$A = USV^T$$

$$S = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_N \end{bmatrix} . \quad \sigma_i \geq 0 .$$

$$\sigma_1 \geq \cdots \geq \sigma_N > 0$$

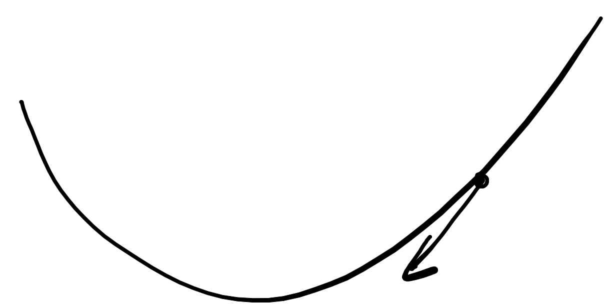
$A^{-1} = V S^{-1} U^T$. $\quad \kappa(A) = \dfrac{\sigma_1}{\sigma_N} \geq 1$

$\kappa(A^T A)$ $\quad A^T A = V S^2 V^T$

$$\kappa(A^T A) = \frac{\sigma_1^2}{\sigma_N^2}$$

SD.

$\nabla_x \varphi(x) = Ax - b$



$$p = -\nabla_x \varphi(x) = b - Ax := r$$

residual.

$$\inf_{\alpha \in \mathbb{R}} \varphi(x + \alpha p)$$

$$\|$$

$$\frac{1}{2}(x+\alpha p)^T A (x+\alpha p) - b^T (x+\alpha p)$$

$$\Rightarrow \quad \alpha = \frac{p^T(b-Ax)}{p^T A p} = \frac{p^T p}{p^T A p}$$

Alg. Steepest descent.

$x^{(0)}$.

for $k = 0, 1, \cdots$

$$r^{(k)} = b - \underbrace{Ax^{(k)}}$$ → check convergence

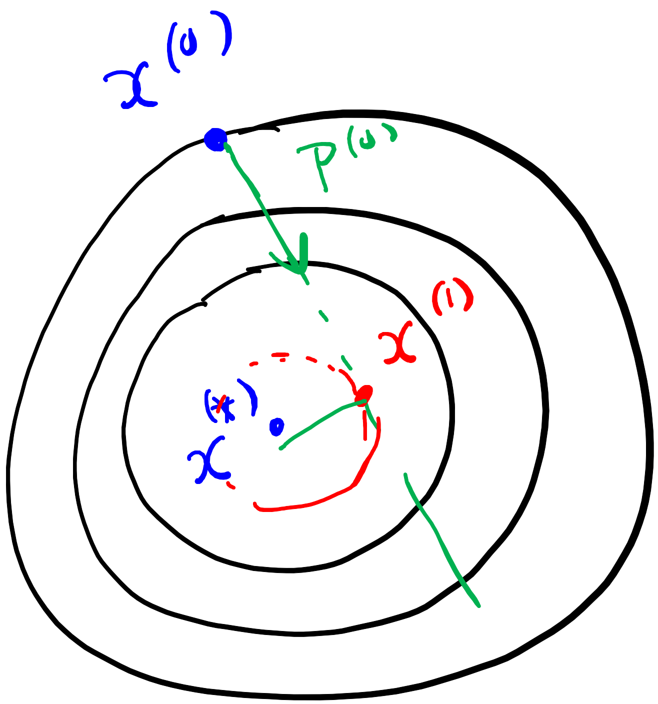
$$\alpha^{(k)} = \frac{(r^{(k)})^T (r^{(k)})}{(r^{(k)})^T \underbrace{A(r^{(k)})}}$$

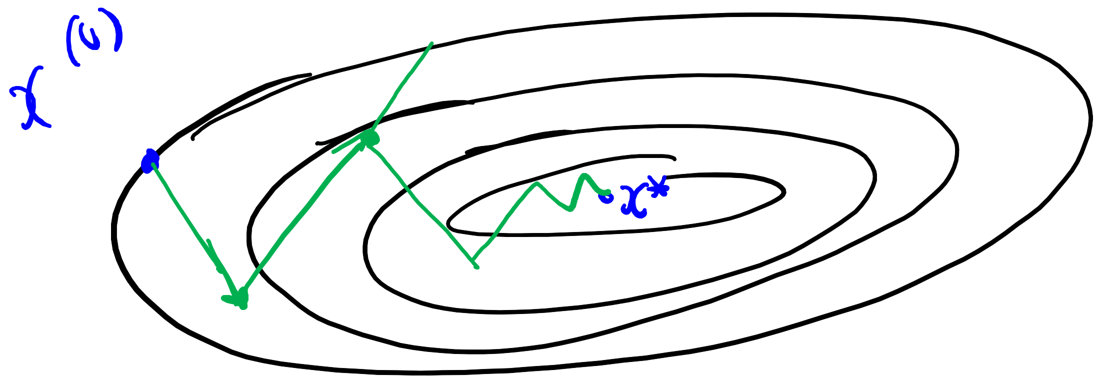
2 mat-vec

$$x^{(k+1)} \leftarrow x^{(k)} + \alpha^{(k)} r^{(k)}$$

(exer) Reduce to 1 mat-vec per iter.

2D $\varphi(x_1, x_2)$





$x^{(k)}$, $r^{(k)}$. Solve linear system

$\text{span}\{r^{(k)}\}$.

Krylov method.

$$x\ , \qquad r = b - Ax$$

$$K_m(r) = \operatorname{span}\{r, Ar, \cdots, A^{m-1}r\}.$$

$$y_m \in \mathbb{R}^m$$

$$x' \leftarrow x + [r, Ar, \cdots, A^{m-1}r]\, y_m$$

Arnoldi procedure.

$r \quad . \quad Ar$

$$v_1 = \frac{r}{\beta}, \quad \beta = \|r\|_2$$

$$\begin{cases} \tilde{v}_2 = A v_1 - \gamma_1 v_1, \quad \tilde{v}_2^T v_1 = 0 \\ v_2 = \dfrac{\tilde{v}_2}{\|\tilde{v}_2\|} \end{cases}$$

$$\rightarrow \quad v_2 = \alpha_1 A u_1 - \tilde{\gamma}_1 v_1$$

$$\tilde{v}_3 = A v_2 - \gamma_1 v_1 - \gamma_2 v_2$$

$$v_3 = \frac{\tilde{v}_3}{\|\tilde{v}_3\|} \qquad \cdots$$

$$A \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_{k+1} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{k} \\ h_{21} & h_{22} & & \\ 0 & h_{32} & \cdots & h_{kk} \\ \vdots & & & \\ \hline 0 & & & h_{k+1,k} \end{bmatrix} \in \mathbb{R}^{(k+1)\times k} \leftarrow \tilde{H}_k$$

$$\widetilde{H}_k = \begin{bmatrix} H_k \\ \hdashline h_{k+1,k}\, e_k^T \end{bmatrix}$$

Upper Hessenberg matrix.

$$A \underbrace{[v_1 \cdots v_k]}_{V_k} = [v_1 \cdots v_k] H_k + h_{k+1,k}\, v_{k+1}\, e_k^T$$

Full orthogonalization method (FOM)

$$x = x_0 + V_k\, y_k$$

$$
\begin{aligned}
r &= b - Ax \\
&= b - Ax_0 - A\bar{V}_k y_k \\
&= r_0 - \bar{V}_{k+1} \tilde{H}_k y_k \\
&= \beta \bar{V}_k e_1 - \bar{V}_{k+1} \tilde{H}_k y_k
\end{aligned}
$$

FOM requires

$$r \perp K_k(r_0)$$

$$\Longleftrightarrow \quad \bar{V}_k^T r = 0$$

$$0 = \beta e_1 - \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} H_k \\ h_{k+1,k} e_k^T \end{bmatrix} y_k$$

$$= \beta e_1 - H_k y_k$$

$$y_k = \beta H_k^{-1} e_1$$

$$x \leftarrow x_0 + \bar{V}_k y_k$$

You can use FOM

1. No restart $\rightarrow$ memory intensive

2. restart $\rightarrow$ at most k copies of vectors in V_k