Lec 26.

$$\{x_k, r_k, p_k\}$$

$$x_{k+1} = x_k + \alpha_k p_k$$

line search

$$\alpha_k = \frac{r_k^T p_k}{p_k^T A p_k}$$

minimization

$$r_{k+1} = b - A x_{k+1}$$

$$\begin{cases} p_{k+1} = r_{k+1} & \text{SD} \\ p_{k+1} = r_{k+1} + \beta_k p_k & \text{CG} \quad \text{A-ortho} \end{cases}$$

$$\beta_k = -\frac{p_k^T r_{k+1}}{p_k^T A p_k}$$

Connection between

- Practical CG
- Lanczos CG.

$$A V_m = V_{m+1} \tilde{T}_m \quad , \quad \tilde{T}_m = {}_{m+1}\overset{m}{\begin{pmatrix} T_m \\ \hline 0 \cdots t_{m+1,m} \end{pmatrix}}$$

$$T_m y = \beta e_1 \qquad T_m = \begin{pmatrix} T_{11} & T_{12} & & \\ T_{21} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & T_{m,m-1} & T_{mm} \end{pmatrix}$$

$$T_m = L_m U_m = \begin{pmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ & \ddots & \ddots & \\ 0 & & l_{m,m-1} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & u_{m-1,m} \\ 0 & & & u_{mm} \end{pmatrix}$$

$$y = U_m^{-1} L_m^{-1} \beta e_1$$

$$x_{m+1} = x_m + \underbrace{\left(V_m U_m^{-1}\right)}_{P_m}\underbrace{\left(L_m^{-1}\beta e_1\right)}_{w_m}$$

$$P_m = [p_0 \cdots p_{m-1}]$$

$\to$ conjugate directions

Columns of P_m are A-orthogonal.

$$\mathcal{I}_m^T A \mathcal{I}_m = U_m^{-T} \underbrace{V_m^T A V_m}_{T_m = L_m U_m} U_m^{-1}$$

$$= U_m^{-T} L_m$$

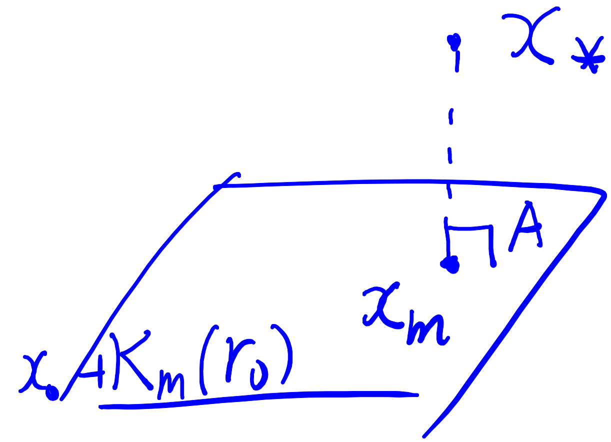
lower triangular × lower " → lower triangular

+ symmetric → diagonal

$\rightarrow$ A-ortho.

Thm. X_m is the sol. of CG at step m.

$$x_m = \underset{\substack{x = x_0 + q(A) r_0 \\ q(A) \in \mathbb{P}_{m-1}}}{\arg\min} \| x - x_* \|_A$$



Pf: Lanczos - CG

$$0 = V_m^T (b - Ax_m)$$

$$= V_m^T A (x_* - x_m)$$

$\square$

Convergence of CG.

$\forall x \in x_0 + K_m(r_0)$

$$x_* - x = x_* - (x_0 - c_1 r_0 - \cdots - c_m A^{m-1} r_0)$$

$$= A^{-1}(r_0 + c_1 A r_0 + \cdots + c_m A^m r_0)$$

$$= A^{-1} p_m(A)\ r_0$$

$$p_m \in \mathbb{P}_m, \quad p_m(0) = 1$$

one-to-one : $x \longleftrightarrow p_m$

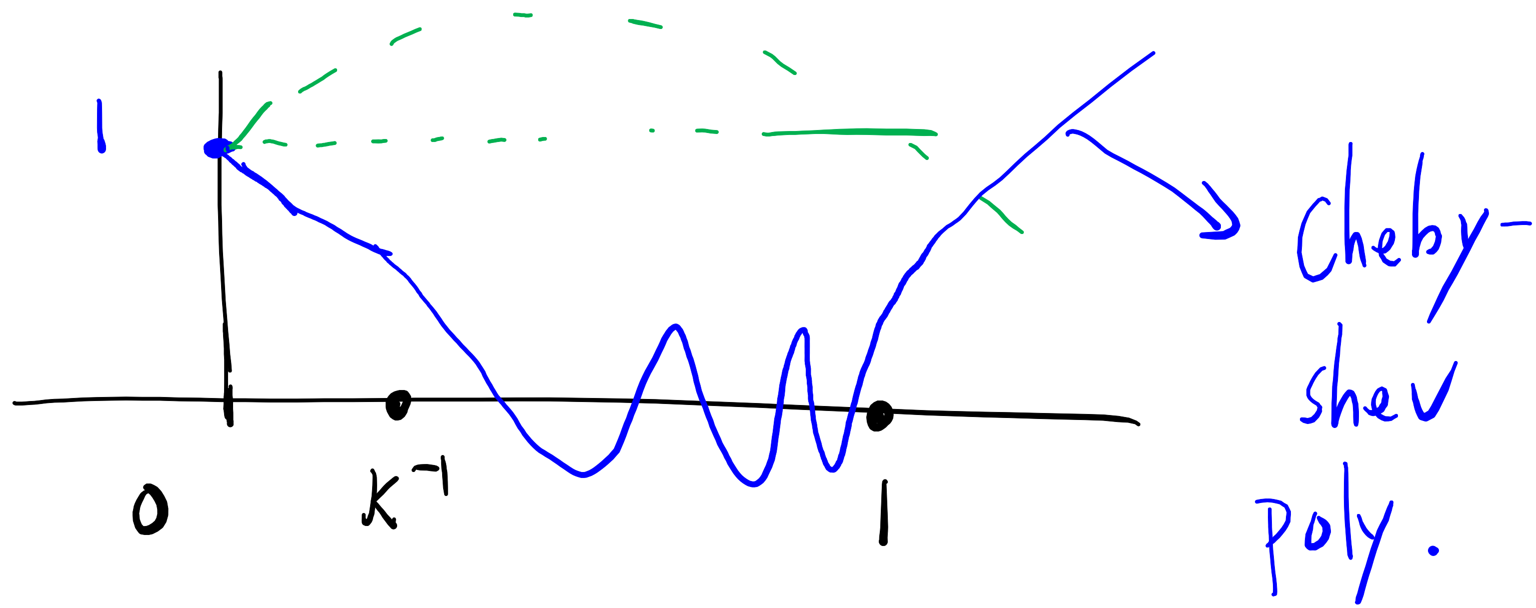
$$\|x_* - x_m\|_A = \min_{x \in x_0 + K_m} \|x_* - x\|_A$$

$$\leq \left(\min_{\substack{p_m \in \mathbb{P}_m \\ p_m(0)=1}} \max_{\lambda_1 \leq \lambda \leq \lambda_N} |p_m(\lambda)| \right) \|A^{-1} r_0\|_A$$

d_m

$$\|A^{-1} r_0\|_A = \| x_0 - x_* \|_A$$

W.L.O.G. $\lambda_1 = \kappa^{-1}, \quad \lambda_N = 1$



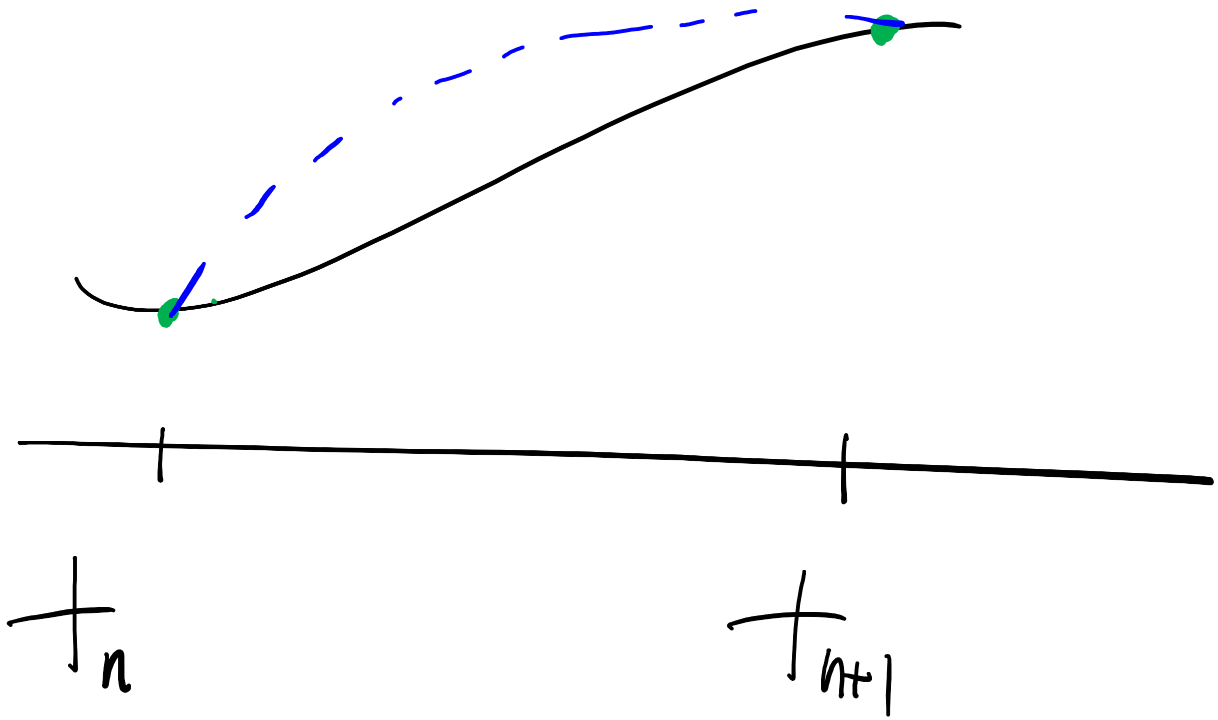
$$d_m \le 2 \left(\frac{\sqrt{k} - 1}{\sqrt{k} + 1} \right)^m$$

Preconditioning

"no free lunch"

use fixed pt iteration

to solve implicit eqs. $\rightarrow$ no saving.



① non-trivial nonlinear solver.

Newton. Broyden ...

② non-trivial linear solver.

CG. GMRES...

sparse direct.

③ preconditioning.

$A > 0$. $\quad Ax = b$. $\quad \kappa(A) \gg 1$

$M > 0 \quad M^{-1}Ax = M^{-1}b$.

$$\kappa(M^{-1}A) \ll \kappa(A)$$

"ideal" preconditioner : $M = A$.

$M^{-1}A$ is NOT symmetric w.r.t. Euclidean inner product

A symmetric . $\mathbb{R}^{N\times N}$

$\forall\ x, y \in \mathbb{R}^N$. $(x, y) := x^T y$

$$(x, Ay) = (Bx, y)$$

we define $A^T := B$

$$x^T A y = x^T B^T y \implies B^T = A$$

Claim. $M^{-1}A$ is symmetric w.r.t. M-inner product.

$$(x, M^{-1}Ay)_M = x^T M M^{-1} A y$$

$$= x^T A M^{-1} M y$$

$$= (M^{-1}Ax, y)_M$$

$\square$.