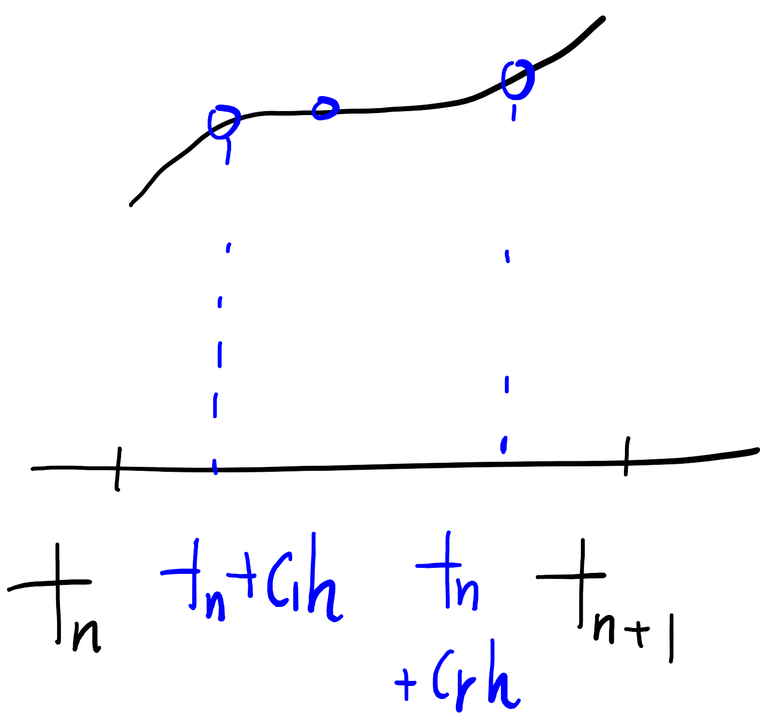
Lec 20.

Collocation method.



$0 \le c_1 < c_2 < \cdots < c_r \le 1$

Lagrange polynomial.

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Discrete

$$\begin{cases} \vec{k} = f \circ (u_n \vec{e} + hA\vec{k},\ t_n \vec{e} + \vec{c}h) \\ u_{n+1} = u_n + h\,\vec{b}^T \vec{k} \end{cases}$$

$v(t)$ satisfies

$$k_i = f(v(t_n + c_i h),\ t_n + c_i h)$$

Continuous

$$\begin{cases} \dot{v}(t) = \tilde{f}(v(t), t) \\ v(t_n) = v_n \end{cases}$$

Linear combination of Lag. poly. using $\{k_i\}_{i=1}^{r}$

Gauss quadrature.

r-pt Gauss-Legendre. algebraic accuracy

$2r-1$

$\Rightarrow x^l$, $0 \le l \le 2r-1$.

quadrature is exact.

G-L: $\begin{cases} \text{nodes: roots of } P_r(x) \subseteq [-1,1] \\ \text{weights: Lag. poly.} \end{cases}$

Any polynomial. $p(x) \in \mathbb{P}_{2r-1}$

$$p(x) = g(x)\,P_r(x) + h(x)$$

$$\begin{array}{cccc} 2r-1 & r-1 & r & r-1 \end{array}$$

exact:

$$\int_{-1}^{1} P(x)\,dx = \underbrace{\cancel{\int_{-1}^{1} g(x)\,P_r(x)\,dx}}_{\substack{\parallel \\ 0 \\ \text{(orthogonality)}}} + \int_{-1}^{1} h(x)\,dx$$

quadrature

$$\sum_{i=1}^{r} p(x_i)\,\omega_i = \sum_{i=1}^{r} \cancel{g(x_i)}\, p_r(x_i)\,\omega_i + \sum_{i=1}^{r} h(x_i)\,\omega_i$$

$\cancel{g(x_i)} = 0$ (roots)

$\sum_{i=1}^{r} h(x_i)\,\omega_i \to$ exact

(Lag. interp. is exact up to $r-1$)

$$\int_a^b f(x)\,dx$$

$$= \int_a^b \sum_{n=0}^{2r-1} \frac{f^{(n)}(a)}{n!}(x-a)^n + O\left((x-a)^{2r}\right) dx$$

exact

after integration

$$(b-a)^{2r+1}$$

3-term recursion.

$$\begin{cases} \ell P_\ell(x) = (2\ell-1)\, x\, P_{\ell-1}(x) - (\ell-1) P_{\ell-2}(x) \\ P_{-1}(x) = 0, \quad P_0(x) = 1 \end{cases}$$

Ortho. poly. $\Longleftrightarrow$ 3 term recursion.

$$\beta_\ell P_\ell(x) = (x - \alpha_\ell) P_{\ell-1}(x) - \gamma_\ell P_{\ell-2}(x)$$

$$+ \sum_{j=0}^{\ell-3} c_j P_j(x)$$

orthogonal $\Rightarrow$ $c_j = 0$ $\Rightarrow$ 3 term

$$\beta_\ell \int_{-1}^{1} P_j(x) P_\ell(x)\, dx = 0$$

$$= \int_{-1}^{1} \underbrace{\left[P_j(x)\,(x-\alpha_\ell)\right] P_{\ell-1}(x)\,dx}_{} = 0$$

$$- \gamma_\ell \int_{-1}^{1} P_j(x)\, P_{\ell-2}(x)\,dx = 0$$

$$+ \sum_{j'=0}^{\ell-3} c_{j'} \int_{-1}^{1} P_j(x)\, P_{j'}(x)\,dx$$

$$\|$$

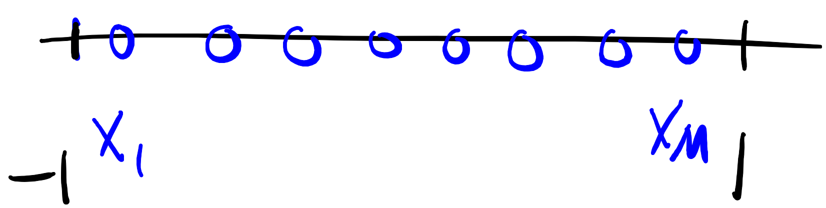
$$c_j \left(\int_{-1}^{1} P_j^2(x)\,dx\right)$$

$$\Rightarrow c_j = 0 \Rightarrow 0 \le j \le \ell - 3.$$

Reformulate:

$$x P_{\ell-1}(x) = \gamma_\ell P_{\ell-2}(x) + \alpha_\ell P_{\ell-1}(x) + \beta_\ell P_\ell(x)$$

$x \in [-1, 1]$ $\qquad$ $\{x_i\}_{i=1}^{M}$



$$P_\ell(x) \longleftrightarrow \begin{array}{cc} M & \begin{bmatrix} | \\ P_\ell(x_i) \\ | \end{bmatrix} \end{array} \leftarrow P_{\ell-1}$$

$$\mathcal{P}_\ell = \begin{bmatrix} | & & | \\ P_0(x_i) & \cdots & P_\ell(x_i) \\ | & & | \end{bmatrix} \in \mathbb{R}^{M \times (\ell+1)}$$

$$A = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_M \end{bmatrix} \in \mathbb{R}^{M \times M}$$

$$A P_{l-1} = \gamma_l P_{l-2} + \alpha_l P_{l-1} + \beta_l P_l$$

T. tridiagonal.

$$A \begin{bmatrix} P_0 & \cdots & P_{l-1} \end{bmatrix} = \begin{bmatrix} P_0 & \cdots & P_l \end{bmatrix} \begin{bmatrix} \alpha_1 & \gamma_2 & 0 & & \\ \beta_1 & \alpha_2 & \gamma_3 & & \\ 0 & \beta_2 & \alpha_3 & \ddots & \\ \vdots & 0 & \beta_3 & \ddots & \gamma_l \\ 0 & & & \ddots & \alpha_l \\ 0 & 0 & \cdots & 0 & \beta_l \end{bmatrix}$$

$\begin{bmatrix} P_0 & \cdots & P_{l-1} \end{bmatrix} \in \mathbb{R}^{m \times l}$

$$\begin{bmatrix} T \\ \hline 0 \cdots 0 \; \beta_l \end{bmatrix} \in \mathbb{R}^{(l+1) \times l}$$

$$A \mathcal{P}_{l-1} = \mathcal{P}_{l-1} T + \beta_l p_l e_l^T$$

Lanczos procedure.

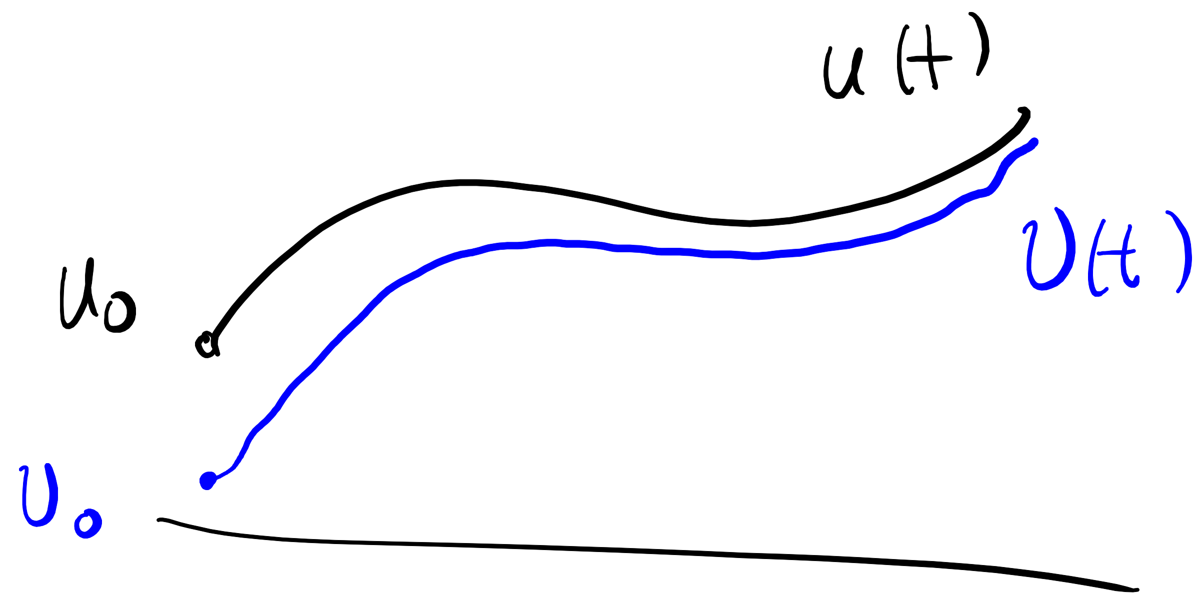
$$e_l^T = [0, \cdots, 0, 1]$$

$\uparrow$

l-th position

<u>Thm</u>. All G-L are B-stable.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases} \qquad \begin{cases} \dot{v}(t) = f(v(t), t) \\ v(0) = v_0 \end{cases}$$



$$E(t) = \|u(t) - v(t)\|^2$$

$$E'(t) \leq 0, \quad \forall t$$

$$\left(u - v,\ f(u) - f(v)\right) \leq 0$$

$\forall u, v$

With collocation method.

$$\dot{\tilde{u}}(t) = \tilde{f}(\tilde{u}(t))$$

Lag interp.

$$\dot{\tilde{u}}(t_n + c_i h) = f(\tilde{u}(t_n + c_i h))$$

$$\tilde{u}(t_n) = u_n, \quad \tilde{u}(t_{n+1}) = u_{n+1}.$$

$\{u_n\}$, $\{v_n\}$

$$E_n = \| u_n - v_n \|^2 \qquad \cdot \quad E_{n+1} \leq E_n$$

$$E_{n+1} = \| u_{n+1} - v_{n+1} \|^2$$

$$\tilde{E}(t) = \| \tilde{u}(t) - \tilde{v}(t) \|^2 \in \mathbb{P}_{2r}(t)$$

$$\tilde{E}(t_n) = E_n, \quad \tilde{E}(t_{n+1}) = E_{n+1}$$

$$E_{n+1} = E_n + \int_{t_n}^{t_{n+1}} \dot{\tilde{E}}(t)\, dt \quad \in \mathbb{P}_{2r-1}(t)$$

Gauss quad is exact

$$= E_n + \sum_{i=1}^{r} \dot{\tilde{E}}(t_n + c_i h)\, w_i$$

$$\dot{\tilde{E}}(t_n+c_ih) = 2\left(\tilde{u}(t_n+c_ih) - \tilde{v}(t_n+c_ih),\right.$$

$$f(\tilde{u}(t_n+c_i h)) - f(\tilde{v}(t_n+c_i h)))\leq 0$$

$\square$.

Thm. Any r-step GL.

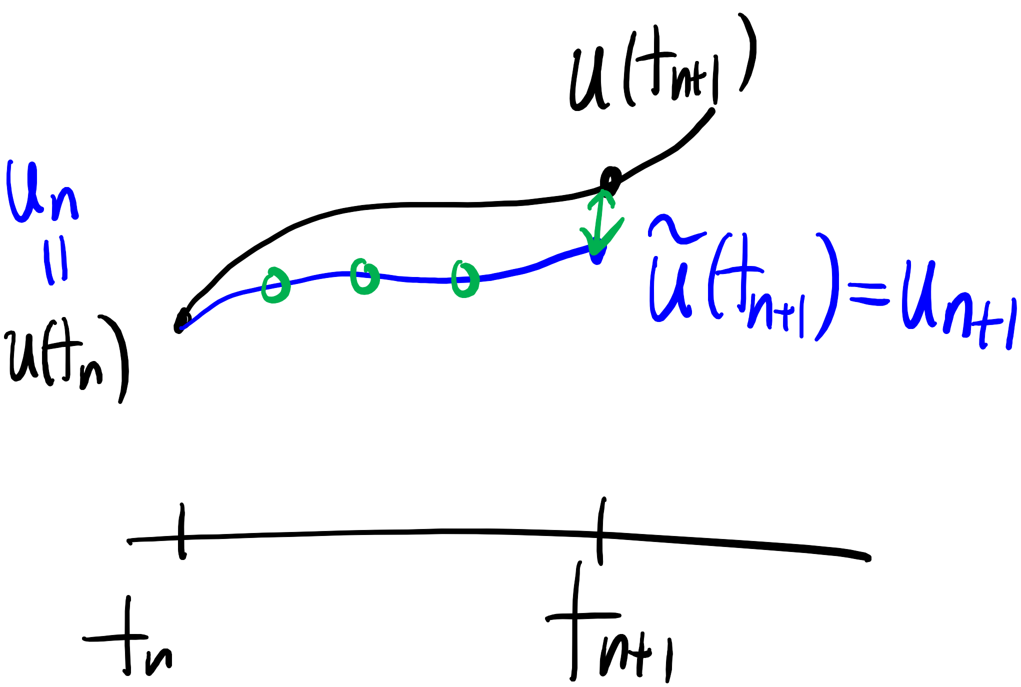
(i) A-stable

(ii) B-stable

(iii) symplectic.

Continuous perspective

$$\dot{\tilde{u}} = f(\tilde{u}) + \underbrace{\left[\tilde{f}(\tilde{u}) - f(\tilde{u})\right]}_{g(\tilde{u})}$$



We know $\quad g(\tilde{u}(t_n + c_i h)) = 0$