Lec 24.

General matrix

FOM.

$$A V_K = V_K \tilde{H}_K \qquad \tilde{H}_K = \begin{bmatrix} H_K \\ 0 \cdots h_{k+1,k} \end{bmatrix}$$

$$V_K \perp r \Rightarrow H_K y = \beta e_1$$

GMRES

$$\min \|r\|_2 \Rightarrow \min_y \|\tilde{H}_K y - \beta \tilde{e}_1\|_2$$

$\left\{ \begin{array}{l} \text{Givens rotation} \\ \text{residual: efficient computation} \end{array} \right.$

real symmetric / Complex Hermitian

(complex symmetric)

$$AV = VH \qquad H = \begin{pmatrix} h_{11} & \cdots & \cdots & h_{1N} \\ h_{21} & & & \vdots \\ & \ddots & & \vdots \\ 0 & & h_{N,N-1} & h_{N,N} \end{pmatrix}$$

$$V^T V = I_N$$

$$\Rightarrow H = V^T A V \leftarrow \text{symmetric}$$

↳ symmetric + upper Heisenberg ⇒ tridiagonal.

$$A V_k = V_{k+1} \tilde{T}_k .$$

$$\tilde{T}_k = \begin{pmatrix} T_k \\ \hline 0 \cdots h_{k+1,k} \end{pmatrix}$$

$$T_k = \begin{pmatrix} h_{11} & h_{12} & & 0 \\ h_{21} & h_{22} & h_{23} & \\ & \ddots & \ddots & \ddots \\ 0 & & h_{k,k-1} & h_{kk} \end{pmatrix} \rightarrow \text{Lanczos}$$

3 term recurrence $\rightarrow$ lower memory footprint.

FOM-like:

$$A V_k = V_{k+1} \tilde{T}_k$$

$$T_k y = \beta e_1 \Rightarrow x = x_0 + V_k y$$

$\Rightarrow$ conjugate gradient. (CG)

convergence $\begin{cases} \text{positive definite.} & \text{CG never fails} \\ \text{indefinite.} & \text{CG may fail (just like FOM)} \end{cases}$

GMRES-like [Saad, Schultz. 86]

$$\min_y \|\tilde{T}_k y - \beta \tilde{e}_1\|_2 \Rightarrow \text{Minimal residual (MINRES)}$$

use : symmetric indefinite [Paige & Sanders. 75]

Bi orthogonality. General matrix

$$A V_k = V_{k+1} \tilde{H}_k , \qquad V_k^T V_k = I_k$$

symmetric

$$A V_k = V_{k+1} \tilde{T}_k , \qquad V_k^T V_k = I_k .$$

Trade orthogonality for tridiagonality?

Lanczos biorthogonalization method.

For simplicity.

$$AV = VD$$

$$\underset{\substack{\parallel \\ W^T}}{V^{-1}}\, V = I. \quad \Rightarrow \quad W^T V = I$$

W is orthogonal to V!

$$AV = VT$$ ← tridiagonal.

$$\Rightarrow W^T A V = T = \begin{pmatrix} \alpha_1 & \beta_2 & & 0 \\ \delta_2 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_N \\ 0 & & \delta_N & \alpha_N \end{pmatrix}$$

$$A v_k = v_{k-1} \beta_k + v_k \alpha_k + v_{k+1} \delta_{k+1}$$

NOT a set of closed eqns.

Need w_k's to proceed.

$$W^T A = T W^T \xrightarrow{\text{transpose}} A^T W = W T^T$$

$$A^T w_k = w_{k-1} \delta_k + w_k \alpha_k + w_{k+1} \beta_{k+1}$$

$$\begin{cases} w_i^T A v_j = T_{ij} & \text{tridiagonality} \\ w_i^T v_j = \delta_{ij} & \text{biorthogonality.} \end{cases}$$

How to compute $\alpha_k, \beta_k, \delta_k$'s?

$$\tilde{v}_{k+1} = A v_k - v_k \alpha_k - v_{k-1} \beta_k$$

$$\tilde{w}_{k+1} = A^T w_k - w_k \alpha_k - w_{k-1} \delta_k$$

$$\Rightarrow \quad \tilde{w}_{k+1}^T \tilde{v}_{k+1} = \beta_{k+1} \delta_{k+1}$$

Pick a convention

$$\delta_{k+1} = \sqrt{\left|\tilde{w}_{k+1}^T \tilde{v}_{k+1}\right|}$$

$$\Rightarrow \beta_{k+1} = \frac{\tilde{w}_{k+1}^T \tilde{v}_{k+1}}{\delta_{k+1}}.$$

$$\left.\begin{aligned} v_{k+1} &= \tilde{v}_{k+1} / \delta_{k+1} \\ w_{k+1} &= \tilde{w}_{k+1} / \beta_{k+1} \end{aligned}\right\} \text{exer.}$$

$$\alpha_{k+1} = w_{k+1}^T A v_{k+1} \,)$$

FOM-like

$$A V_k = V_{k+1} \tilde{T}_k$$

$$(A^T W_k = W_{k+1} T_k^T) \leftarrow \text{discard}$$

$$T_k y = \beta e_1 \Rightarrow \text{Bi-CG}$$

GMRES-like.

$$\min_y \| \tilde{T}_k y - \beta \tilde{e}_1 \|_2$$

NOT a real residual

due to loss of ortho.

Quasi-minimal residual

(QMR)

Summary

Solve \ Krylov	Arnoldi	Lanczos	Bi-Lanczos
square $Hy = \beta e_1$	FOM non-sym apply A once high mem	CG sym. pos. def. apply A once low mem	Bi-CG non-sym apply A, A^T low mem
rectangular $\min \lVert \tilde{H} y - \tilde{\beta e_1} \rVert_2$	GMRES	MINRES sym. indef	QMR