Lec 6.

Zero stability

Convergence of LMM. (sketch)

Runge-Kutta.

$$\begin{cases} \dot{u}(t) = 0 \\ u(0) = 0 \end{cases}$$ ← why called zero-stability.

$u_0 = e_0, \cdots, u_{r-1} = e_{r-1}$

$\|e_0\|, \cdots, \|e_{r-1}\|$ arbitrarily small.

$$\|u_N\| \overset{?}{\leq} C(T)\left(\|e_0\| + \cdots + \|e_{r-1}\|\right)$$

Ex. $\begin{cases} u_{n+2} - 3u_{n+1} + 2u_n = 0 \\ u_0 = e_0, \quad u_1 = e_1 \end{cases}$

Difference eq.

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow$ 2 roots $\lambda_1 = 1, \ \lambda_2 = 2.$

General sol.

$u_n = c_1 \cdot \lambda_1^n + c_2 \lambda_2^n = c_1 + c_2 \cdot 2^n$

$= (2e_0 - e_1) + \underbrace{(e_1 - e_0)} \cdot 2^n \to$ exp. increasing when $h \to 0$.

General LMM.

$$\rho(z) = \sum_{j=0}^{r} \alpha_j z^j$$

$$\sum_{j=0}^{r} \alpha_j U_{n+j} = 0. \longrightarrow \rho(z) = 0.$$

(1) special case. r roots are distinct.

$$z_1, \dots, z_r \in \mathbb{C}.$$

$$\rho(z) = \alpha_r (z - z_1) \cdots (z - z_r)$$

Sol.

$$u_j = \sum_{k=1}^{r} c_k z_k^j$$

$$|z_k| \le 1.$$

(2) More generally.

$z_1, \cdots, z_m.$ $m \le r.$

$\mu_1, \cdots, \mu_m \in \mathbb{N}_+$, $\sum_{k=1}^{m} \mu_k = r.$

$$u_j = \sum_{k=1}^{m} \sum_{l=0}^{\mu_k - 1} c_{kl}\, j^l z_k^j$$

a) $|z_k| < 1$. μ_k can be anything.

b) $|z_k| = 1$, $\mu_k = 1$. i.e. simple root.

Def. r- step LMM. is zero-stable if

1) $|z_k| \le 1$.

2) if $|z_k| = 1$. z_k is simple.

root cond.

Ex. $u_{n+2} - 2u_{n+1} + u_n = \frac{h}{2}\left(f_{n+2} - f_n\right)$.

$$\rho(z) - \log z \ \sigma(z) = -\frac{1}{12}\xi^4 + O(\xi^5)$$

$\rho(z) = (z-1)^2 \Rightarrow$ double root $z=1$.

NOT zero stable.

$$\rho(z) - \log z \, \sigma(z) \xrightarrow{z \to 1} \rho(1) = 0.$$

$\Rightarrow$ $z=1$ is always a root.

<u>Thm</u>. Convergence of LMM.

consistency + stability $\to$ Convergence.

LTE (zero)

Sketch :

① consistency : Taylor. $\|\tau_n\| \sim O(h^{p+1})$

② vectorization. For simplicity $d=1$

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = h \sum_{k=0}^{r} \beta_k f_{n+k}. \qquad \alpha_r = 1$$

Convert high order difference eq.

to first " " ".

$$\tilde{u}_n = \begin{bmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+r-1} \end{bmatrix} \in \mathbb{R}^r, \quad \tilde{u}_{n+1} = \begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+r} \end{bmatrix}$$

$$\tilde{f}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=0}^{r} \beta_k f_{n+k} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \\ 0 & \cdots & & 1 \\ -\alpha_0 & \cdots & & -\alpha_{r-1} \end{bmatrix}$$

$$\tilde{u}_{n+1} = A\tilde{u}_n + h\tilde{f}_n$$

$$\tilde{u}(t_{n+1}) = A\tilde{u}(t_n) + h\tilde{f}(t_n) + \tilde{\tau}_n \quad , \quad \tilde{\tau}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tau_n \end{bmatrix}$$

$$\tilde{e}_n = \tilde{u}(t_n) - \tilde{u}_n \quad \in \mathbb{R}^r$$

$$\tilde{e}_{n+1} = A\tilde{e}_n + h\left(\tilde{f}(t_n) - \tilde{f}_n\right) + \tilde{\tau}_n$$

$$= A^{n+1}\tilde{e}_0 + \sum_{j=0}^{n} A^{n-j}\left[h\left(\tilde{f}(t_j) - \tilde{f}_j\right) + \tilde{\tau}_j\right]$$

$$\|\tilde{e}_{n+1}\| \leq \|A^{n+1}\| \cdot \|\tilde{e}_0\| + \sum_{j=0}^{n} \|A^{n-j}\| \left[h \|\tilde{f}(t_j) - \tilde{f}_j\| + \|\tilde{\tau}_j\| \right]$$

③ $\|e_n\| \leq \|\tilde{e}_n\| = \sqrt{\|e_n\|^2 + \cdots + \|e_{n+r-1}\|^2}$

④ Bound $\|A^{n+1}\|$

Jordan decomposition of A

$$A = VJV^{-1}, \quad J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$$

$$J_k = \underbrace{\begin{bmatrix} z_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & z_k \end{bmatrix}}_{\mu_k}$$

$\forall k, \quad \|J_k^n\| < C$ for all n

$\Longleftrightarrow$ root condition.

⑤ Lipschitz continuity

$$\|\tilde{f}(t_n) - \tilde{f}_n\| \longrightarrow \|\tilde{e}_n\|$$

$$\Rightarrow \|\tilde{e}_{n+1}\| \le \sum_{j=0}^{n} \left(* \|\tilde{e}_j\| + * \|\tilde{\tau}_j\| \right)$$

Discrete version of Gronwall's inequality

$$\Rightarrow \|\tilde{e}_N\| \le * \|\tilde{e}_0\| + \frac{*}{h} \max_j \|\tilde{\tau}_j\|$$

$\square$

Runge–Kutta method.

Use several evaluations of f to advance $t_n \to t_{n+1}$.

Ex. $\begin{cases} k_1 = f(u_n, t_n) \\ u_{n+1} = u_n + h k_1 \end{cases}$. Forward Euler.

Ex. $\begin{cases} k_1 = f(u_n, t_n) \\ k_2 = f(u_n + h k_1, t_n + h) \\ u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2) \end{cases}$ modified Euler.

Ex. RK4. c.f. Wikipedia

General r-stage RK.

$$
\begin{cases}
k_1 = f\left(u_n + h\left(a_{11} k_1 + \cdots + a_{1r} k_r\right), t_n + c_1 h\right) \\
\vdots \\
k_r = f\left(u_n + h\left(a_{r1} k_1 + \cdots + a_{rr} k_r\right), t_n + c_r h\right) \\
u_{n+1} = u_n + h\left(b_1 k_1 + \cdots + b_r k_r\right)
\end{cases}
$$

Butcher array

$$
\begin{array}{c|c}
c & A \\
\hline
 & b^T
\end{array}
$$