Lec 3.

convergence order

2nd order Taylor method.

Adams - Bashforth method.

$$\|e_{n+1}\| \le (1+hL)^{n+1}\|e_0\| + \left[(1+hL)^{n}\|\tau_0\| + \cdots + \|\tau_n\|\right]$$

$$\le (1+hL)^{n+1}\|e_0\| + \frac{M h^{\cancel{2}}}{2}\cdot\frac{(1+hL)^{n+1}-1}{\cancel{h}L}$$

$$\left(\text{use} \quad 1+x \le e^{x}\right)$$

$$(1+hL)^{n} \le e^{hLn}, \qquad T = Nh$$

$$\Rightarrow \|e_N\| \le \underbrace{e^{LT}}_{\text{exp. amp. fac.}}\left(\underbrace{\|e_0\|}_{\text{initial error}} + \underbrace{\frac{Mh}{2L}}_{\text{LTE } O(h^{p})}\right) \longrightarrow \text{a priori analysis.}$$

order of method

Stability consistency.

$$\tau \sim O(h^{p+1}) \longrightarrow \|e_N\| \sim O(h^p)$$

stability + consistency $\Rightarrow$ convergence.

check convergence order numerically.

$u(t)$ $\quad$ $u_h(t_n) \equiv u_n$

exact sol $\quad$ numer. sol.

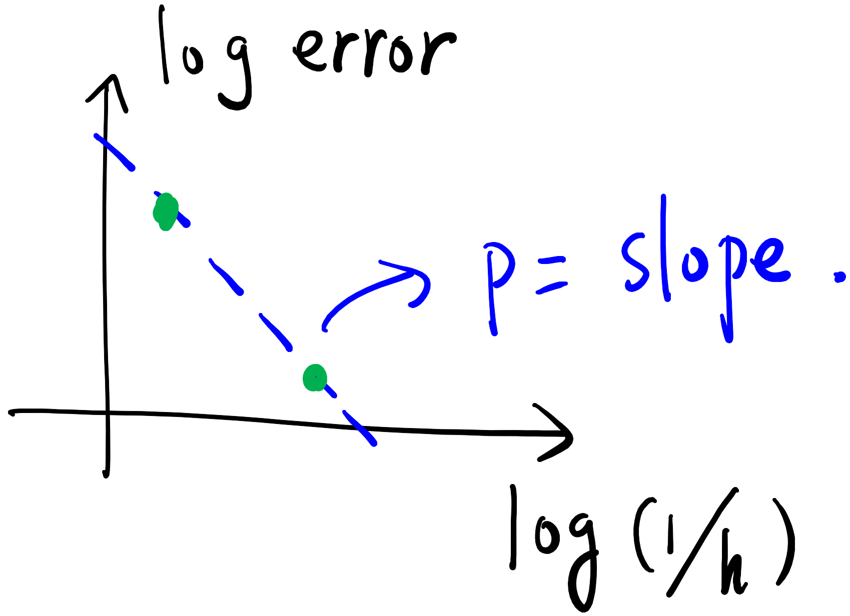
$t \in [0, T]$ $\quad$ $t_n = nh$

$n = 0, \cdots, N$

$$u(T) = u_h(T) + v(T)h^p + \cancel{O(h^{p+1})}$$

$$\|u(T) - u_h(T)\| \approx C h^p, \quad C = \|v(T)\|$$

$$\log \|u(T) - u_h(T)\| \approx \log C - p \log(1/h)$$



u(T) not known?

$$u(T) \approx u_h(T) + v(T)\, h^p$$

$$\approx u_{\frac{h}{2}}(T) + v(T)\frac{h^p}{2^p}$$

$$\approx u_{\frac{h}{4}}(T) + v(T)\frac{h^p}{4^p}$$

$$\| u_h(T) - u_{\frac{h}{2}}(T) \| \approx C h^p \left(1 - \frac{1}{2^p}\right)$$

$$\| u_{\frac{h}{2}}(T) - u_{\frac{h}{4}}(T) \| \approx C h^p \left(\frac{1}{2^p} - \frac{1}{4^p}\right)$$

$$\frac{\| u_h(T) - u_{\frac{h}{2}}(T) \|}{\| u_{\frac{h}{2}}(T) - u_{\frac{h}{4}}(T) \|} \approx 2^p$$

everything is computable !

Taylor expansion.

$$u(t_{n+1}) = u(t_n) + \dot{u}(t_n)\, h + \frac{1}{2} \ddot{u}(t_n)\, h^2 + \cancel{\cdots}$$

$$\dot{u}(t_n) = f\big(u(t_n), t_n\big), \qquad u(t) \in \mathbb{R}^m$$

$$\ddot{u}_i(t_n) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j}\big(u(t_n), t_n\big) \cdot \dot{u}_j(t_n) = f_j\big(u(t_n), t_n\big)$$

$$+ \frac{\partial f_i}{\partial t}\big(u(t_n), t_n\big)$$

$$(D_u f)_{ij} = \frac{\partial f_i}{\partial u_j} \quad \text{Jacobian matrix}$$

$$\Rightarrow \ddot{u}(t_n) = D_u f\big(u(t_n), t_n\big) \cdot f\big(u(t_n), t_n\big)$$

$$+ D_t f\big(u(t_n), t_n\big)$$

2nd order Taylor scheme.

$$u_{n+1} = u_n + f(u_n, t_n)\,h + \frac{h^2}{2}\left[\underline{D_u f(u_n, t_n)} \cdot \underline{f(u_n, t_n)} + \underline{D_t f(u_n, t_n)}\right]$$

almost NEVER used.

① Computation of Jacobian.

② Higher order method is impossible.

Idea: combine a few evaluations of f to approximate directional derivative.

$$f\left(u_n + h f(u_n, t_n), t_{n+1}\right)$$

$$\approx f(u_n, t_n) + h\left[D_u f(u_n, t_n) \cdot f(u_n, t_n) + D_t f(u_n, t_n)\right]$$

$$+ O(h^2)$$

Define $\bar{u} = u_n + h f(u_n, t_n)$

$$u_{n+1} = u_n + h f(u_n, t_n) + \frac{h}{2}\Big[f(\bar{u}, t_{n+1}) - f(u_n, t_n) + O(h^2) \Big]$$

$$= u_n + \frac{h}{2} f(u_n, t_n) + \frac{h}{2} f(\bar{u}, t_{n+1}) + O(\cancel{h^3})$$

Alg. $\bar{u} = u_n + h f(u_n, t_n)$ ← Forward Euler

$$u_{n+1} = u_n + \frac{h}{2}\Big(f(u_n, t_n) + f(\bar{u}, t_{n+1}) \Big)$$

modified Euler's method.

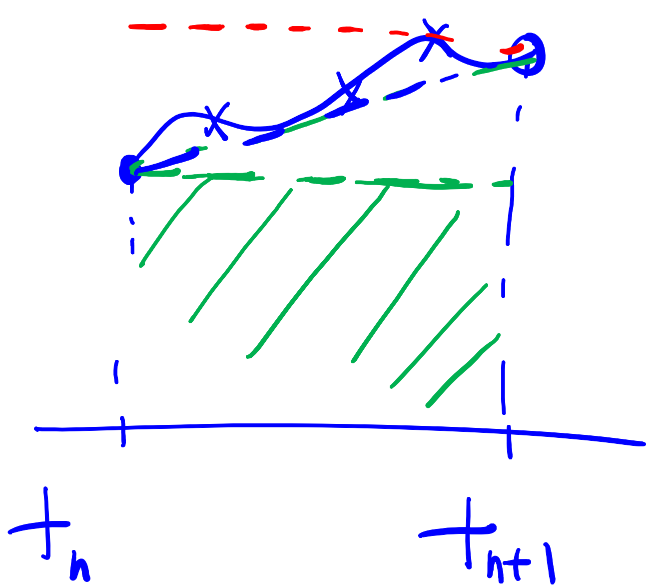
→ one type of Runge-Kutta method.

Integral representation

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 , \end{cases} \Rightarrow u(t) = u_0 + \int_0^t f(u(s), s)\, ds$$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f(u(s), s)\, ds .$$

replace by quadrature.



$$\int_{t_n}^{t_{n+1}} f(u(s), s)\, ds \approx f(u_n, t_n) \cdot h$$

forward Euler

$$\approx f(u_{n+1}, t_{n+1})\, h$$

backward Euler $\rightarrow$ implicit

$$\approx \frac{h}{2}\left[f(u_n, t_n) + f(u_{n+1}, t_{n+1}) \right]$$

trapezoidal rule

$\rightarrow$ implicit.