Lec 1.

Basic ODE theory

$\mathbb{R}^n$: n-tuple of real numbers.

$\mathbb{C}^n$: n-tuple of complex numbers.

$\vec{x} \in \mathbb{R}^n \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \cdots, x_n)^T$

Length of vector (vector 2-norm)

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

Matrix $A \in \mathbb{R}^{n \times n}$

Matrix 2-norm

$$\|A\|_2 = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|Ax\|_2$$

By definition

$$\|Ax\|_2 \leq \|A\|_2 \, \|x\|_2$$

$\|x\| := \|x\|_2$

$\|A\| := \|A\|_2$

Other norms.

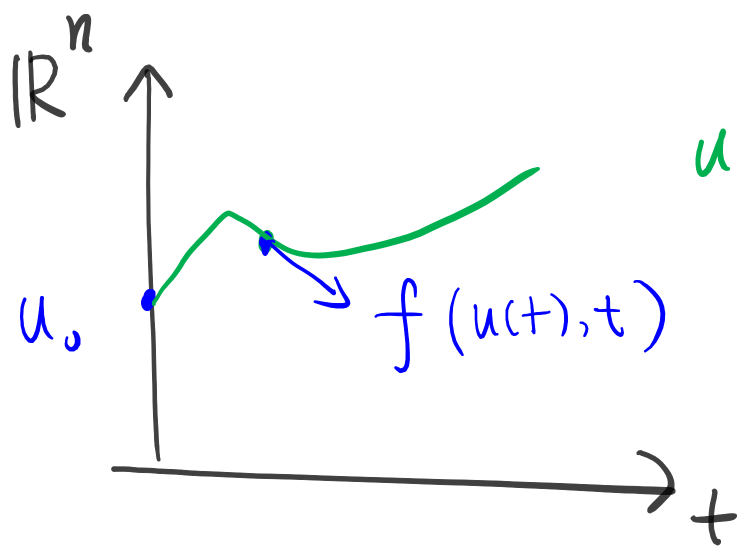
e.g. 1-norm $\quad \|x\|_1 = |x_1| + \cdots + |x_n|$

Derivative $x(t) : \mathbb{R} \to \mathbb{R}^n$

$$\dot{x}(t) \equiv x'(t) \equiv \frac{d}{dt} x(t)$$

Initial value problem (IVP). sys. of ODEs.

$$\begin{cases} u'(t) = f(u(t), t), & t \in \mathcal{I} = [0, T] \\ u(0) = u_0 \end{cases}$$



special case $\quad f(u(t),t) = f(u(t))$

autonomous eq.

"autonomization" $\quad \xi(t) = \begin{pmatrix} u(t) \\ t \end{pmatrix} \in \mathbb{R}^{n+1}$

$$\frac{d}{dt}\xi(t) = \begin{pmatrix} u'(t) \\ t'(t) \end{pmatrix} = \begin{pmatrix} f(\xi(t)) \\ 1 \end{pmatrix}$$

High order time derivatives can be easily handled.

Ex. $\ddot{x}(t) = f(x(t), t)$

Introduce $u(t) = \dot{x}(t)$

$$\begin{cases} \dot{x}(t) = u(t) \\ \dot{u}(t) = f(x(t), t) \end{cases} \quad \text{sys. in } \mathbb{R}^{2n}$$

In general.

$$x^{(n)}(t) + \alpha_{n-1} x^{(n-1)}(t) + \cdots + \alpha_1 \dot{x}(t) + \alpha_0 x(t)$$

$$= f\left(x(t), \dot{x}(t), \cdots, x^{(n-1)}(t), t\right)$$

$$u_1(t) = x(t)$$

$$u_2(t) = \dot{x}(t)$$

$$\vdots$$

$$u_n(t) = x^{(n-1)}(t)$$

$$\frac{d}{dt}u(t) = A\,u(t) + \tilde{f}(u(t), t)$$

Ex. $x : \mathbb{R} \to \mathbb{R}$. extended sys. $\mathbb{R}^n$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdot & 0 \\ & & & & & & \\ 0 & & & & & & 1 \\ -\alpha_0 & -\alpha_1 & & \cdots & & & -\alpha_{n-1} \end{bmatrix} \qquad \tilde{f} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(u(t), t) \end{bmatrix}$$

requirement of $f(u(t),t):\underbrace{\mathbb{R}^n}_{\text{space}}\times\underbrace{\mathbb{R}}_{\text{time}}\longrightarrow\mathbb{R}^n$

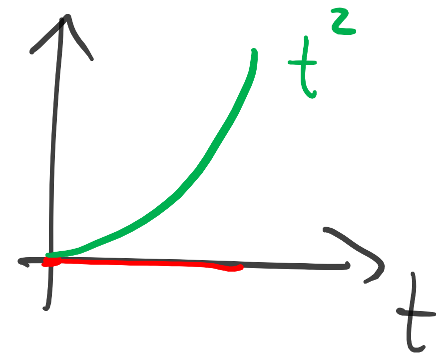
f continuous. w.r.t u, t.

Thm. (Cauchy-Peano)

f continuous. sol. to IVP. exists.

Ref. [Hai] I. 7.

Ex. $\begin{cases} u'(t) = 2\sqrt{u(t)} \\ u(0) = 0. \end{cases}$



Guess 1: $u(t) \equiv 0$

Guess 2: $u(t) = \begin{cases} t^2, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Uniqueness. Lipschitz Continuity on f.

<u>Def</u>. $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lip. cont.

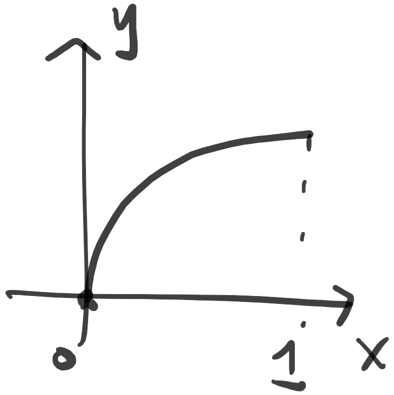
if $\exists\, L > 0$ s.t. $\forall\, x, y \in \mathbb{R}^n$

$$\|f(x) - f(y)\|_2 \leq L \|x-y\|_2$$

2-norm in $\mathbb{R}^m$

2-norm in $\mathbb{R}^n$

Ex.



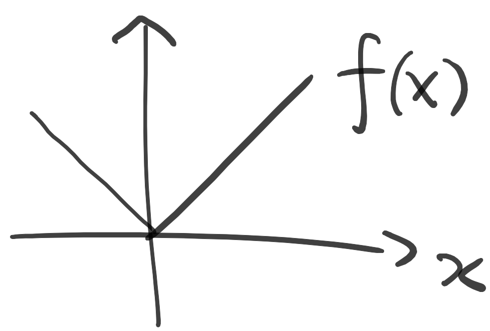
$f(x) = 2\sqrt{x}$

$y > 0$

$$\frac{|f(y) - f(0)|}{|y - 0|} = \frac{2\sqrt{y}}{y} = \frac{2}{\sqrt{y}} \quad \text{unbounded.}$$

is NOT Lip cont. on $[0,1]$

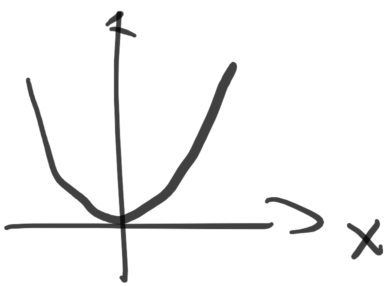
Ex. $f(x) = |x|$



$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{\big||x|-|y|\big|}{|x-y|} \leq 1.$$

is Lip. cont. $L=1$.

Ex. $f(x)=x^2$. on $[-1,1]$.



$$\frac{|f(x)-f(y)|}{|x-y|} = |x+y| \leq 2.$$

is Lip. cont. $L=2$

is NOT Lip. cont. on $\mathbb{R}$.

Exer. f cont. C^1 $\|\nabla f\|_2$ bounded on I

$\Rightarrow f$ Lip. cont. on I.

Def. $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is Lip. cont.

if $\exists L > 0$. $\forall t \in I$, $\forall x, y \in \mathbb{R}^n$

$$\|f(x,t) - f(y,t)\|_2 \leq L\|x-y\|_2 .$$

Thm. (Picard-Lindelöf)

$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Lip. cont.

$\Rightarrow$ sol. exist. unique.

Ref. [Hai] I. 8.

Forward Euler.

$$0 = t_0 < t_1 < \cdots < t_N = T$$

Assume uniform step size $t_n = n\,h$, $h = t_2 - t_1$

$u(t_n)$: true sol. at t_n

$u_n \equiv u_h(t_n)$: numer sol at t_n.

$$\dot{u}(t_n) \approx \frac{u_{n+1} - u_n}{h} = f(u_n, t_n)$$

f_n

$$u_{n+1} = u_n + h f(u_n, t_n)$$

Linear multistep method (LMM)

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = h \sum_{k=0}^{r} \beta_k \underbrace{f(u_{n+k}, t_{n+k})}_{f_{n+k}}$$

Forward Euler.

$$r = 1, \quad \alpha_1 = 1, \quad \alpha_0 = -1$$

$$\beta_1 = 0, \quad \beta_0 = 1.$$

Convergence

error $e_n = u(t_n) - u_n$

Goal :

$h \to 0$. $\max_{0 \le t_n \le T} \| e_n \| \to 0$.

Def. A LMM is convergent if for all IVPs.

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases} \qquad 0 \le t \le T.$$

f sufficiently smooth.

$$\| u(t) - u_h(t) \| \to 0 \text{ as } h \to 0, \quad \forall\, 0 \le t = nh \le T.$$

whenever the starting values satisfy.

$$\| u(t_n) - u_h(t_n) \| \to 0 \text{ as } h \to 0.$$

for $n = 0, \cdots, K$.

(K const. indep. of h)

<u>Def</u> A LMM is convergent of order p.

if $\exists\, h_0 > 0$, $C > 0$. s.t.

$$\|u(t) - u_h(t)\| \le C h^p, \quad 0 < h \le h_0, \quad 0 \le t = nh \le T.$$

whenever initial data satisfy.

$$\|u(t_n) - u_h(t_n)\| \le \tilde{C} h^p, \quad 0 < h \le h_0$$

$n = 0, \cdots, k$. K indep. of h.

<u>Thm</u>. Forward Euler is convergent of order 1.

<u>Sketch</u>

$$u_{n+1} = u_n + f(u_n, t_n)\, h$$

$$u(t_{n+1}) = u(t_n) + f(u(t_n), t_n)\, h + \tau_n$$

local truncation error (LTE)

$$e_{n+1} = e_n + h\left(f(u(t_n), t_n) - f(u_n, t_n)\right) + \tau_n$$

Lip. cont.

$$\|e_{n+1}\| \leq \|e_n\| + hL\|e_n\| + \|\tau_n\| = (1+hL)\|e_n\| + \|\tau_n\|$$

$$\leq (1+hL)^2\|e_{n-1}\| + (1+hL)\|\tau_{n-1}\| + \|\tau_n\|$$

$$\leq (1+hL)^{n+1}\|e_0\| + \left[(1+hL)^{n}\|\tau_0\| + \cdots + \|\tau_n\|\right]$$

LTE

$$\tau_n = u(t_{n+1}) - u(t_n) - \int_{t_n}^{t_{n+1}} f(u(t_n), t_n)\, ds$$

$$= \left[\cancel{u(t_n)} + \cancel{u'(t_n)\, h} + \int_{t_n}^{t_{n+1}} (t_{n+1}-s)\, u''(s)\, ds\right]$$

$$- \cancel{u(t_n)} - \cancel{u'(t_n)\, h}$$

$$M := \sup_{0\leq t\leq T} \|u''(t)\|$$

$$\|\tau_n\| \leq \left[\int_{t_n}^{t_{n+1}} (t_{n+1}-s)\, ds\right] M = \frac{h^2 M}{2}$$

$$\|e_n\| \le (1+hL)^n \|e_0\| + \left(\frac{1}{2}Mh^{\cancel{2}}\right)\frac{(1+hL)^n-1}{\cancel{h}L}$$

sum of geo. series.

$$1+x \le e^x, \quad x \ge 0. \quad (T=nh)$$

$$(1+hL)^n \le e^{hLn} = e^{LT}$$

$$\|e_n\| \le \underbrace{e^{LT}}_{\text{amp. fac}} \underbrace{\|e_0\|}_{\text{initial error}} + \underbrace{\left(\frac{1}{2}M\,\frac{e^{LT}-1}{L}\right)}_{\text{amp. fac.}} \underbrace{h}_{\text{LTE.}} \qquad \square$$

stability.

consistency.

Stability + consistency $\Longrightarrow$ Convergence.

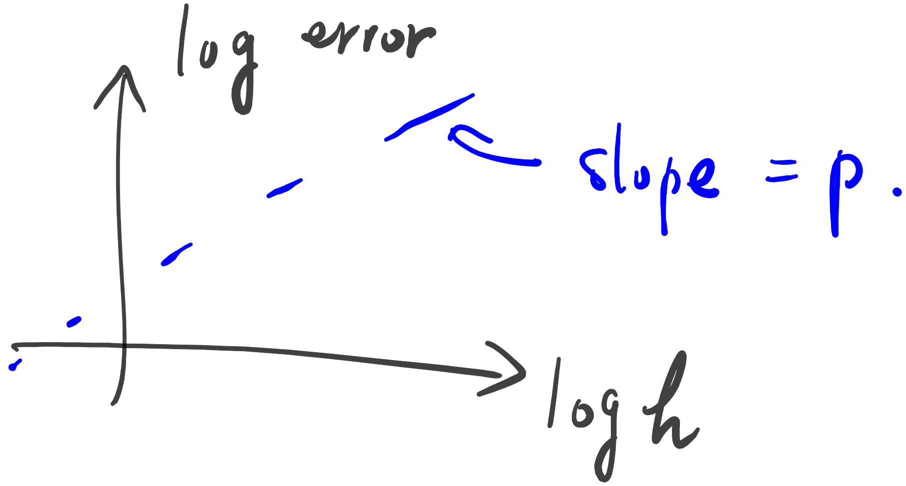
Numerical test for convergence order.

$$u(T) = u_h(T) + C h^p + O(h^{p+1})$$

exact value

$$\| u(T) - u_h(T) \| \approx C h^p$$

$$\log \| u(T) - u_h(T) \| \approx \log C + p \log h .$$



What if $u(T)$ is NOT known?

Introduce another eq. at a diff. step size.

$$u(T) \approx U_h(T) + C h^p$$

$$\approx U_{\frac{h}{2}}(T) + C \left(\frac{h}{2}\right)^p$$

$$\approx U_{\frac{h}{4}}(T) + C \left(\frac{h}{4}\right)^p$$

$$\| -u_h(T) + u_{\frac{h}{2}}(T) \| \approx C\left[1-\left(\frac{1}{2}\right)^p\right] h^p$$

$$\| -u_{\frac{h}{2}}(T) + u_{\frac{h}{4}}(T) \| \approx C\left[1-\left(\frac{1}{2}\right)^p\right] h^p \cdot \left(\frac{1}{2}\right)^p$$

$$\Rightarrow \quad p = \frac{\log \dfrac{\|u_h(T) - u_{\frac{h}{2}}(T)\|}{\|u_{\frac{h}{2}}(T) - u_{\frac{h}{4}}(T)\|}}{\log 2}.$$