Lec 4.

Adams - Bashforth

Adams - Moulton.

convergence of trapezoidal rule

LMM

$$\sum_{j=0}^{r} \alpha_j U_{n+j} = h \sum_{k=0}^{r} \beta_k f(U_{n+k}, t_{n+k}). \quad \alpha_r \neq 0$$

$$\beta_r \begin{cases} = 0 . & \text{explicit} \\ \neq 0 . & \text{implicit} \end{cases}$$

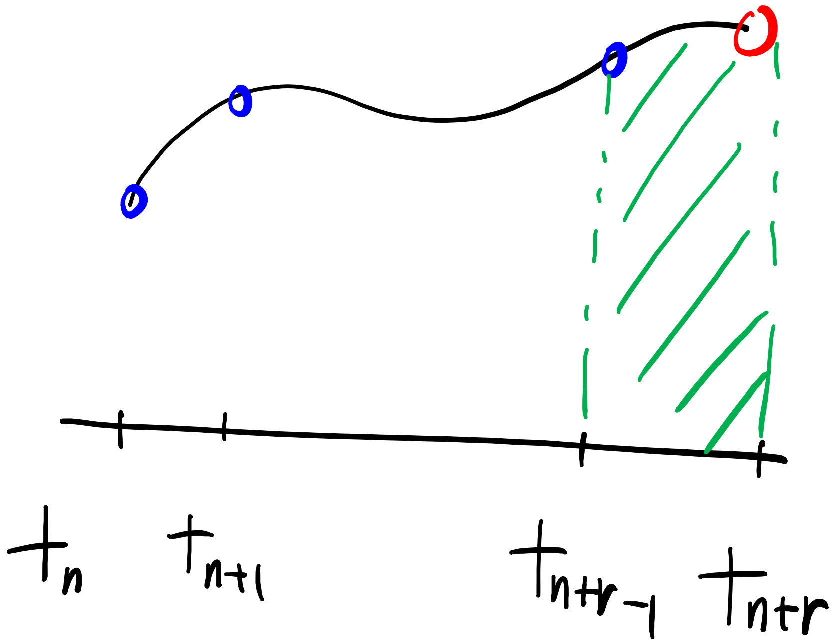
Adams-Bashforth. explicit.

$$U_{n+r} - U_{n+r-1} = h \sum_{k=0}^{r-1} \beta_k f(U_{n+k}, t_{n+k})$$

$\alpha_r = 1$, $\alpha_{r-1} = -1$, $\alpha_j = 0$, $j = 0, \cdots, r-2$.

Determine β_k's $\longrightarrow$ quadrature.

$f(u(s),s)$



extrapolation.

$$u(t_{n+r}) = u(t_{n+r-1}) + \int_{t_{n+r-1}}^{t_{n+r}} f(u(s), s)\, ds.$$

$\downarrow$

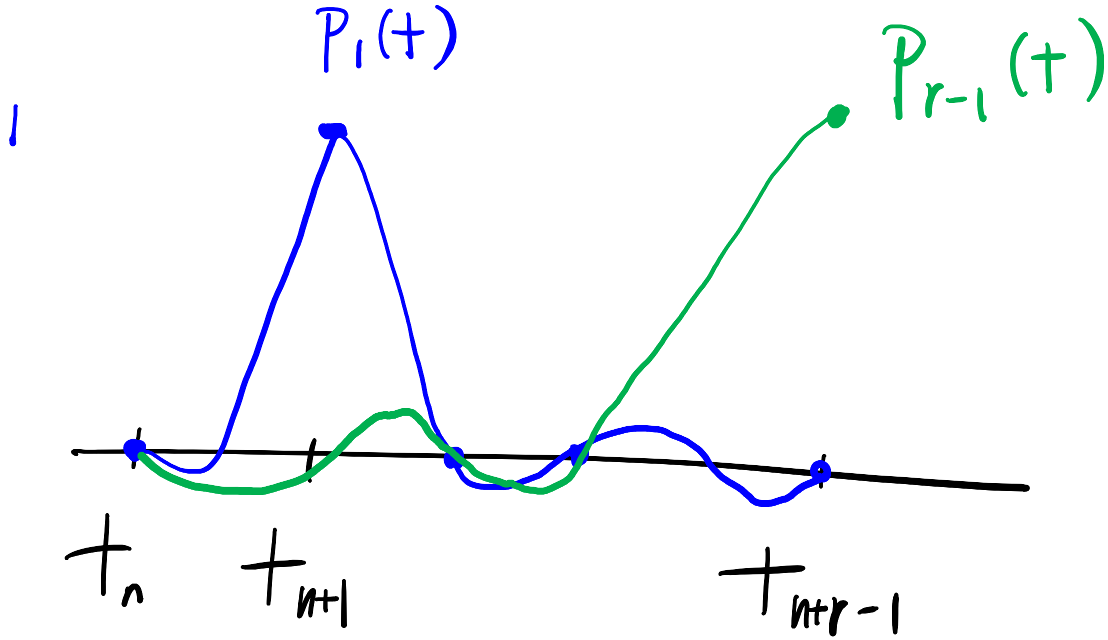
$\tilde{f}(s)$

Lagrange interpolation.

$P_k(t)$ polynomials. $0 \le k \le r-1$

$$P_k(t_{n+\ell}) = \delta_{k\ell} = \begin{cases} 1, & k=\ell \\ 0, & k \ne \ell. \end{cases}$$

Kronecker δ-symbol



$$\tilde{f}(t) = \sum_{k=0}^{r-1} f_{n+k} \, P_k(t)$$

$$\Rightarrow \tilde{f}(t_{n+\ell}) = \sum_{k=0}^{r-1} f_{n+k} \underbrace{P_k(t_{n+\ell})}_{\delta_{k\ell}} = f_{n+\ell}$$

interpolary.

Uniquely determine $P_k(t) \in \mathbb{P}_{r-1}$

$$P_k(t) = \prod_{\substack{j=0 \\ j \neq k}}^{r-1} \left(\frac{t - t_{n+j}}{t_{n+k} - t_{n+j}} \right)$$

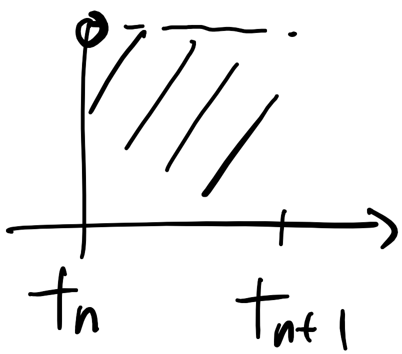
$$u_{n+r} - u_{n+r-1} = \int_{t_{n+r-1}}^{t_{n+r}} \sum_{k=0}^{r-1} f_{n+k} P_k(s)\, ds$$

$$= h \sum_{k=0}^{r-1} \beta_k f_{n+k}.$$

$$\beta_k = \frac{1}{h} \int_{t_{n+r-1}}^{t_{n+r}} P_k(s)\, ds$$

evaluate

Ex. $r=1$



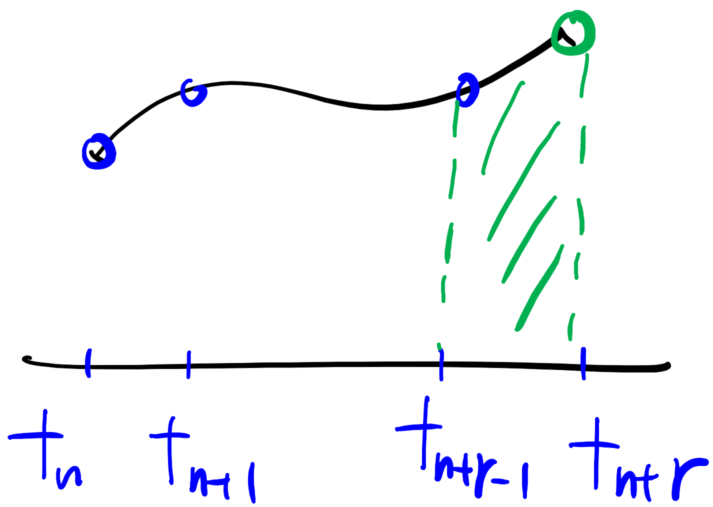
Forward Euler.

$r=2$.

$$u_{n+2} - u_{n+1} = h\left(-\frac{1}{2} f_n + \frac{3}{2} f_{n+1}\right)$$

AB_n

Adams - Moulton (AM_n)



$$P_k(t) \in \mathbb{P}_r$$

$$P_k(t) = \prod_{\substack{j=0 \\ j \neq k}}^{r} \left(\frac{t - t_{n+j}}{t_{n+k} - t_{n+j}} \right)$$

$r = 1$. $\quad P_0(t) = \dfrac{t - t_{n+1}}{t_n - t_{n+1}}$, $\quad P_1(t) = \dfrac{t - t_n}{t_{n+1} - t_n}$

$\beta_0 = \beta_1 = \frac{1}{2}$

$$u_{n+1} = u_n + \frac{h}{2}\left(f_n + f_{n+1} \right)$$

How to start LMM ?

Ex. AB 2. $\quad u_{n+2} = u_{n+1} + h\left(-\frac{1}{2} f_n + \frac{3}{2} f_{n+1}\right)$

$u_1 = ?$

Idea: Use AB1

$$u_1 = u_0 + h f_0$$

Expect

$$\|e_N\| \le C_1 \|e_0\| + C_2 \|e_1\| + \frac{C_3}{h} \tau_{AB2}$$

no error

LTE of AB1 $O(h^2)$

$O(h^3)$

Further suggests to use p-th order LMM. All initial values should be obtained by methods w. accuracy at least order $p-1$.

Ex. Linear Eq.

$$\begin{cases} \dot{u}(t) = A(t)\, u(t) \\ u(0) = u_0 \end{cases} . \quad u(t) \in \mathbb{R}^d . \; A(t) \in \mathbb{R}^{d \times d}$$

Trapezoidal.

$$u_{n+1} = u_n + \frac{h}{2}\left(A(t_n)\, u_n + A(t_{n+1})\, u_{n+1} \right)$$

$$\Rightarrow U_{n+1} = \left[I - \frac{h}{2} A(t_{n+1})\right]^{-1} \left[I + \frac{h}{2} A(t_n)\right] U_n$$

Generally.

$$U_{n+1} = U_n + \frac{h}{2}\left[f(U_n, t_n) + f(U_{n+1}, t_{n+1}) \right]$$

Fixed point problem. w.r.t. U_{n+1}

$$x = T(x). \qquad x \in \mathbb{R}^d. \; T: \mathbb{R}^d \to \mathbb{R}^d$$

Simplest idea: Fixed point iteration.

<u>Alg</u>. $k=0$. $x^{(0)}$. $r^{(0)} = x^{(0)} - T(x^{(0)})$

residual

while $\left(\|r^{(k)}\| > \tau \right)$

$$x^{(k+1)} = T\left(x^{(k)}\right)$$

$k \leftarrow k+1$

First assume fixed point x^* exists.

T. LC is α

$$x^{(k+1)} = T\left(x^{(k)}\right)$$

$$x^* = T(x^*)$$

$$\|x^{(k+1)} - x^*\| = \|T\left(x^{(k)}\right) - T(x^*)\|$$

$$\leq \alpha \|x^{(k)} - x^*\|$$

$$\Rightarrow \|e^{(k+1)}\| \leq \alpha \|e^{(k)}\| \leq \cdots \leq \alpha^{k+1} \|e^{(0)}\|$$

$\alpha > 1$: cannot prove convergence

often means divergence

$\alpha < 1$: guaranteed convergence.

<u>Def</u> (Contraction map) $T: \mathbb{R}^d \to \mathbb{R}^d$

$$\|T(u) - T(v)\| \leq \alpha \|u - v\|. \quad \alpha < 1.$$

<u>Thm</u>. T contraction map. Fixed pt problem

$x = T(x)$. sol. exists. unique. &

fixed pt iteration converges globally.

Trapezoidal.

$$T(x) = u_n + \frac{h}{2}\left[f(u_n, t_n) + f(x, t_{n+1})\right]$$

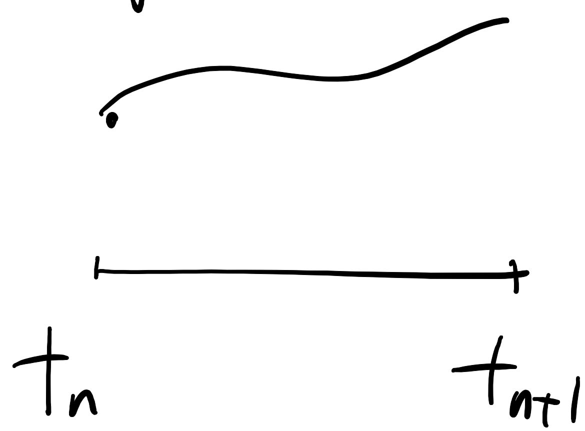
$$\|T(x) - T(y)\| = \frac{h}{2}\|f(x, t_{n+1}) - f(y, t_{n+1})\|$$

$$\leq \frac{hL}{2}\|x-y\|$$

$\Rightarrow h < \frac{2}{L}$ guaranteed convergence.

Implicit method + Fixed pt iteration = bad idea.

why implicit at all?



reach t_{n+1} w. alternative routes than dynamics.

① Acceleration method

② preconditioning.