Lec 12.

Theoretical challenge

$$\|e_N\| \le e^{C_1 T}\|e_0\| + e^{C_2 T}\cdot \frac{\tau}{h}$$

$\dot{u} = -\lambda u$, $\lambda \in \mathbb{R}$, $\lambda >> 0$.

$\dot{u} = -\lambda (u - \varphi) + \dot{\varphi}$

$$u_{n+1} = u_n - \lambda h u_{n+1}$$

$$u(t_{n+1}) = u(t_n) - \lambda h u(t_{n+1}) + \tau$$

$$e_{n+1} = e_n - \lambda h e_{n+1} + \tau$$

Before:

$$\|e_{n+1}\| \le \|e_n\| + |\lambda h| \|e_{n+1}\| + \|\tau\|$$

Now (method 1)

$$e_{n+1} \cdot e_{n+1} = e_n \cdot e_{n+1} - \lambda h\, e_{n+1} \cdot e_{n+1} + e_{n+1} \cdot \tau$$

$$\|e_{n+1}\|^2 = e_n \cdot e_{n+1} \underbrace{- \lambda h \|e_{n+1}\|^2}_{\le 0} + e_{n+1} \cdot \tau$$

$$\le \|e_n\| \cdot \underline{\|e_{n+1}\|} + \underline{\|e_{n+1}\|} \cdot \|\tau\|$$

$$\Rightarrow \|e_{n+1}\| \le \|e_n\| + \|\tau\|$$

$$\Rightarrow \|e_N\| \le \|e_0\| + N \max \|\tau\| \rightarrow T \left(\frac{\max \|\tau\|}{h} \right)$$

Independent of Lip Const!

method 2.

$$(1+\lambda h)\, u_{n+1} = u_n$$

$$u_{n+1} = \frac{1}{1+\lambda h} u_n$$

$$u(t_{n+1}) = \frac{1}{1+\lambda h} u(t_n) + \tau.$$

$$e_{n+1} = \frac{1}{1+\lambda h} e_n + \tau$$

$$\|e_{n+1}\| \le \left|\frac{1}{1+\lambda h}\right| \|e_n\| + \|\tau\|$$

$$\|e_N\| \le \underbrace{\left|\frac{1}{1+\lambda h}\right|^N \|e_0\|}_{\text{goes to } 0 \text{ exp.}} + \frac{1}{1-\left|\frac{1}{1+\lambda h}\right|} \underbrace{\max\|\tau\|}_{\text{improved order by } h}$$

$z = \lambda h$, $z \gg 0$.

goes to 0

exp.

improved

order.

by h

(Slightly) more general.

RK applied to P-R.

$$\vec{k} = f(u_n \vec{e} + hA\vec{k})$$

$$= -L(u_n \vec{e} + hA\vec{k} - \varphi \vec{e}) + \dot{\varphi} \vec{e}$$

$$\vec{k} = (I + hLA)^{-1} [-L u_n + L\varphi + \dot{\varphi}] \vec{e}$$

$$u_{n+1} = u_n + h \vec{b}^T \vec{k}$$

$$= R(z)\, u_n + h\, \vec{b}^T (I + hLA)^{-1} [L\varphi + \dot{\varphi}]\, \vec{e}$$

$$u(t_{n+1}) = R(z)\, u(t_n) + \qquad + \tau$$

$$\Rightarrow e_{n+1} = R(z)\, e_n + \tau$$

$$\xRightarrow{\text{Back-Euler}} \|e_N\| \le |R(z)|^N \|e_0\| + \frac{1}{1-|R(z)|} \max \|\tau\|$$

Require $|R(z)| < 1$ to make sense

Care about $\lim_{z \to \infty} |R(z)| = 0$

L-stable

A-stable

Ex. Backward Euler.

$$R(z) = \frac{1}{1+z} \longrightarrow \text{L-stable}$$

Ex. Trapezoidal.

$$R(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}$$

$$\lim_{z\to\infty} R(z) = -1. \quad \text{NOT L-stable}$$

oscillatory behavior.

Contractive (a.k.a. dissipative) ODE

$$\begin{cases} \dot{u} = f(u,t) \\ u(0) = u_0 \end{cases} \qquad \begin{cases} \dot{v} = f(v,t) \\ v(0) = v_0 \end{cases}$$

$$E(t) = \| u(t) - v(t) \|^2$$

Def ODE is contractive if

$$\frac{dE(t)}{dt} \le 0 \quad \text{for any } u, v.$$

$$\frac{dE(t)}{dt} = 2\,\mathrm{Re}\left(u(t) - v(t),\ \dot{u}(t) - \dot{v}(t) \right)$$

$$= 2\,\mathrm{Re}\left(u(t) - v(t),\ f(u(t),t) - f(v(t),t) \right)$$

$$\le 0.$$

<u>Def</u> If f satisfies

$$\mathrm{Re}\big(f(u,t) - f(v,t),\ u-v \big) \leq 0$$

$\forall\ u, v, t$. Then f is called contractive.

<u>Ex</u>. P-R.

$$f(u,t) = -L\,(u-\varphi) + \dot{\varphi}$$

$$\left(-L\,(u-\cancel{\varphi})+\cancel{\dot{\varphi}}+L(v-\cancel{\varphi})-\cancel{\dot{\varphi}},\ u-v\right)$$

$$=-L\,\|u-v\|^2\leq 0\,.$$

Structure-preserving

<u>Def</u> A RK scheme is contractive (a.k.a. B-stable) if for any numerical solution $\{u_n\}$, $\{v_n\}$

$$\|u_{n+1} - v_{n+1}\| \leq \|u_n - v_n\|$$

Why B-stable ?

$$u_{n+1} = u_n + h\vec{b}^T\vec{k}(u_n)$$

$$u(t_{n+1}) = u(t_n) + h\vec{b}^T\vec{k}(u(t_n)) + \tau$$

If B-stable

$$\|u_{n+1} - u(t_{n+1})\| \le \|u_n - u(t_n)\| + \|\tau\|$$

$$\Rightarrow \|e_N\| \le \|e_0\| + \underbrace{N \max \|\tau\|}_{T \frac{\max\|\tau\|}{h}}$$

<u>Thm</u>. B stable $\Rightarrow$ A-stable

Pf: $\dot{u} = \lambda u$

Contractive $\Longleftrightarrow$

$$\mathrm{Re}\left(\lambda(u-v),\, u-v\right)$$

$$= (\mathrm{Re}\,\lambda)\,\|u-v\|^2 \le 0 \Longleftrightarrow \mathrm{Re}\,\lambda \le 0$$

$\{U_n\} \leftarrow$ RK scheme

$\{V_n\}$, $\quad V_0 = 0 \Longrightarrow U_n \equiv 0.$

$$\overset{\text{B-stable}}{\Longrightarrow} \quad \|U_n - \underset{\overset{\shortparallel}{0}}{V_n}\| \le \|U_{n-1} - \underset{\overset{\shortparallel}{0}}{U_{n-1}}\| \le \cdots \le \|U_0 - \underset{\overset{\shortparallel}{0}}{V_0}\|$$

$\Rightarrow \|u_n\| \le \|u_0\| \quad \forall n$

NOT blowing up $\Rightarrow \lambda h = z \in$ RAS

B-stable $\not\Rightarrow$ L-stable.

Counter example: GL-1. $\begin{array}{c|c} & \frac{1}{2} \\ \hline & 1 \end{array}$

① GL-1 B stable (next HW)

② $R(z) = \dfrac{1+\frac{z}{2}}{1-\frac{z}{2}}$. $\lim\limits_{z\to\infty} |R(z)| = 1$.

<u>Ex.</u> Backward Euler is B-stable

$$\begin{cases} u_{n+1} = u_n + h f(u_{n+1}) \\ v_{n+1} = v_n + h f(v_{n+1}) \end{cases}$$

$$\|u_{n+1} - v_{n+1}\|^2 = \mathrm{Re}\left(u_{n+1} - v_{n+1},\ u_n + h f(u_{n+1}) - v_n - h f(v_{n+1}) \right)$$

$$= \mathrm{Re}\left(u_{n+1} - v_{n+1},\ u_n - v_n \right)$$

$$+ \underbrace{h\, \mathrm{Re}\left(u_{n+1} - v_{n+1},\ f(u_{n+1}) - f(v_{n+1}) \right)}_{\leq 0}$$

$$\leq \| u_{n+1} - U_{n+1} \| \cdot \| u_n - U_n \|$$

$$\Rightarrow \| u_{n+1} - U_{n+1} \| \leq \| u_n - U_n \| \qquad \square .$$

Hamiltonian system.

Newtonian dynamics

position $q(t)$

velocity $v(t)$

or

momentum $p(t) = m v(t)$

$$m \dot{v} = f(q)$$

or

$$m \ddot{q} = f(q)$$

Assume f is conservative

$$f(q) = -\nabla_q V(q)$$

$$\begin{cases} \dot{p} = -\nabla_q V(q) \leftarrow \text{Newton's law} \\ \dot{q} = \dfrac{p}{m} \end{cases}$$

Hamiltonian (total energy)

$$H(p,q) = \text{“} \frac{1}{2} m v^2 + V(q) \text{”}$$

$$= \frac{p^2}{2m} + V(q)$$

H is conserved along trajectory.

$$\frac{d}{dt} H\big(p(t), q(t)\big) = \frac{1}{m} p \cdot \dot{p} + D_q V(q) \cdot \dot{q}$$

$$= \frac{1}{m} p \cdot \left(-\nabla_q V(q)\right) + \nabla_q V(q) \cdot \frac{p}{m} = 0$$

Hamiltonian $\quad H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$

$$\underset{\text{momentum}}{\underset{\uparrow}{\mathbb{R}^d}} \times \underset{\text{position}}{\underset{\uparrow}{\mathbb{R}^d}}$$

$$\begin{cases} \dot{p}(t) = -\dfrac{\partial H}{\partial q}\big(p(t), q(t)\big) \\[2ex] \dot{q}(t) = \dfrac{\partial H}{\partial p}\big(p(t), q(t)\big) \end{cases}$$

Hamiltonian dynamics.