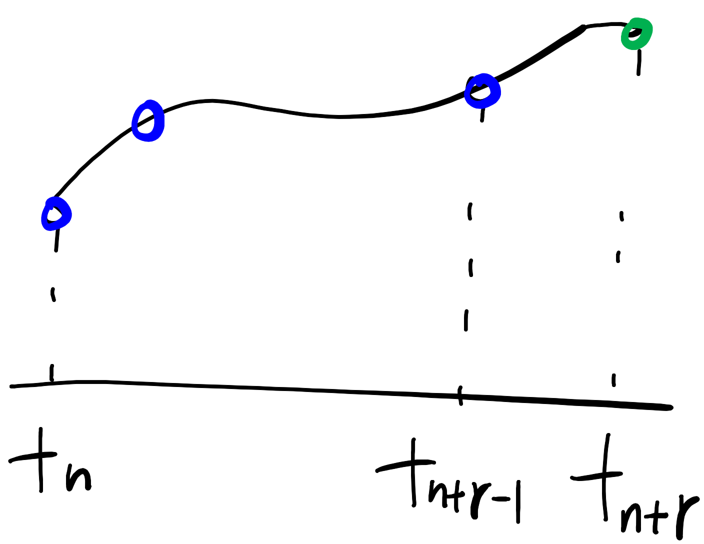
Lec 5.

Consistency of LMM.

Adaptive time stepping
(a posteriori error control).

Zero stability.

(exer) Trapezoidal rule is convergent of order 2.



o AB $\mathbb{P}_{r-1}$

o + o AM. $\mathbb{P}_r$

LTE

$$\tau_n = \sum_{j=0}^{r} \alpha_j u(t_{n+j}) - h \sum_{k=0}^{r} \beta_k f(u(t_{n+k}), t_{n+k})$$

$$= \sum_{j=0}^{r} \alpha_j u(t_{n+j}) - h \sum_{k=0}^{r} \beta_k u'(t_{n+k})$$

$$u(t_{n+j}) = u(t_n) + (jh) u'(t_n) + \cdots + \frac{1}{p!} (jh)^p u^{(p)}(t_n)$$

$$+ O(h^{p+1})$$

$$u'(t_{n+j}) = u'(t_n) + (jh) u^{(2)}(t_n) + \cdots + \frac{1}{p!} (jh)^p u^{(p+1)}(t_n)$$

$$+ O(h^{p+1})$$

$$\tau_n = \sum_{k=0}^{p} c_k h^k u^{(k)}(t_n) + O(h^{p+1})$$

$$C_0 = \sum_{j=0}^{r} \alpha_j$$

$$C_1 = \sum_{j=0}^{r} (\alpha_j \cdot j - \beta_j)$$

$$\vdots$$

$$C_k = \sum_{j=0}^{r} \left(\alpha_j \cdot \frac{j^k}{k!} - \beta_j \cdot \frac{j^{k-1}}{(k-1)!} \right)$$

$$= \frac{1}{k!} \left(\sum_{j=0}^{r} j^k \alpha_j - \beta_j \, j^{k-1} \cdot k \right)$$

algebraic eq. for α's, β's

<u>Thm</u>. LTE is $O(h^{p+1}) \iff C_0 = \cdots = C_p = 0$.

(exer) Check accuracy of AB1, AB2, AM1.

<u>Thm</u>. LTE of AB_n is $O(h^{n+1})$

AM_n is $O(h^{n+2})$.

Sketch. Choose special eqs.

$$\begin{cases} \dot{u}(t) = m t^{m-1}, & m \geq 1. \Rightarrow u(t) = t^m \\ u(0) = 0 \end{cases}$$

$\Rightarrow \tau_\ell = 0$ up to some order.

$\Rightarrow$ Lag. interpolation is exact

$m \le n$. (AB$_n$)

$m \le n+1$. (AM$_n$)

$\Rightarrow$ $\tau_\ell = 0$ gives $c_\ell = 0$.

<u>Ref</u>. [Hai] 3.2.

More compact way.

$$\begin{cases} u'(t) = u(t) \\ u(0) = 1 \end{cases} \Rightarrow u(t) = e^{t}.$$

$$\text{LTE} \quad \tau_n = \sum_{j=0}^{r} \alpha_j e^{(n+j)h} - h \sum_{k=0}^{r} \beta_k e^{(n+k)h}$$

$$= e^{nh} \left(\sum_{j=0}^{r} \alpha_j (e^h)^j - \log(e^h) \sum_{k=0}^{r} \beta_k (e^h)^k \right)$$

Define $z = e^h > 1$

$$\tau_n = O(h^{p+1}) = O((\log z)^{p+1}) = O(|z-1|^{p+1})$$

Define characteristic polynomials of LMM.

$$\rho(z) = \sum_{j=0}^{r} \alpha_j z^j, \quad \sigma(z) = \sum_{k=0}^{r} \beta_k z^k$$

<u>Thm</u>. LTE is $O(h^{p+1})$

$$\iff C_0 = \cdots = C_p = 0$$

$$\iff \rho(z) - \log z \, \sigma(z) = O(|z-1|^{p+1})$$

Ex. AB 1.

$$\rho(z) = z-1 \quad , \quad \sigma(z) = 1$$

$$\rho(z) - \log(1+(z-1))\,\sigma(z)$$

$\xi = z-1$

$$= \xi - \log(1+\xi)$$

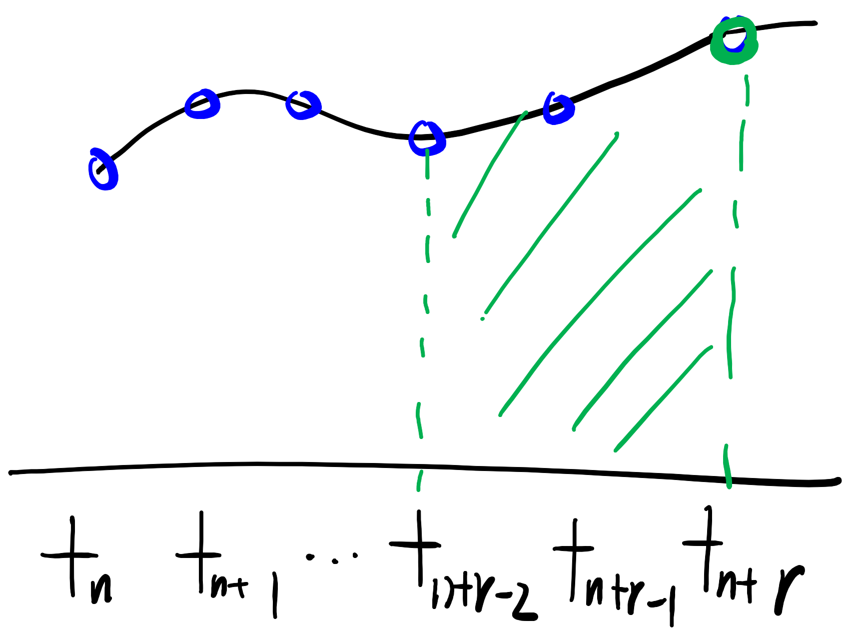
$$= \frac{1}{2}\xi^2 + O(\xi^3)$$

$$\Rightarrow \text{LTE is } O(h^2)$$

Other types of LMM.

$$u(t_{n+r}) - u(t_{n+r-2}) = \int_{t_{n+r-2}}^{t_{n+r}} f(u(s), s)\, ds$$

quad. approx.



o Nyström

o + o Milne

Adaptive time stepping. "Milne device"

Monitor LTE τ.

<u>Alg</u>. if $(\tau > \delta h)$

$$h \leftarrow \frac{h}{2}$$

$\tilde{\delta} < \delta$

if $(\tau < \tilde{\delta} h)$

$$h \to 2h$$

Estimate τ.

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = h \sum_{k=0}^{r} \beta_k f_{n+k}.$$

$$\sum_{j=0}^{r} \tilde{\alpha}_j u_{n+j} = h \sum_{k=0}^{\tilde{r}} \tilde{\beta}_k f_{n+k}$$ ← explicit

easy to evaluate

LTE $$\tau = C h^{p+1} u^{(p+1)}(t_n) + \cancel{O(h^{p+2})} = u(t_{n+r}) - \underset{\sim}{u_{n+r}}$$

$$\tilde{\tau} = \tilde{C} h^{p+1} u^{(p+1)}(t_n) + \cancel{O(h^{p+2})} = u(t_{n+r}) - \underset{\sim}{\tilde{u}_{n+r}}$$

$$\tilde{u}_{n+r} - u_{n+r} = (c - \tilde{c})\, h^{p+1} u^{(p+1)}(t_n) + O(\cancel{h}^{p+2})$$

$$\|\tau\| \approx |c|\, h^{p+1} \, \| u^{(p+1)}(t_n) \|$$

$$\|\tilde{u}_{n+r} - u_{n+r}\| \approx |c - \tilde{c}|\, h^{p+1} \, \| u^{(p+1)}(t_n) \|$$

$$\Rightarrow \|\tau\| \approx \left| \frac{c}{c - \tilde{c}} \right| \cdot \|\tilde{u}_{n+r} - u_{n+r}\| .$$

<u>Corollary</u>. $\tilde{\tau} = O(h^{p+2}) \Rightarrow \tilde{c} = 0$.

$$\Rightarrow \quad \|\tau\| \approx \|\tilde{u}_{n+r} - u_{n+r}\|$$

approx true sol.

Ex. Trapezoidal monitored by AB2.

$$u_{n+1} = u_n + \frac{h}{2}(f_n + f_{n+1}) \quad \rightarrow \tau \approx -\frac{1}{12}h^3 u^{(3)}(t_n)$$

$$\tilde{u}_{n+1} = u_n + h\left(\frac{3}{2}f_n - f_{n-1}\right) \rightarrow \tilde{\tau} \approx \frac{5}{12}h^3 u^{(3)}(t_n)$$

$$\|\tau\| \approx \frac{1}{6}\|u_{n+1} - \tilde{u}_{n+1}\|$$

Zero stability

independent of LTE.

$$\|e_N\| \le C_1 (\|e_0\| + \cdots + \|e_{r-1}\|) + C_2 \frac{LTE}{h}$$

zero stable : $C_1, C_2 < \infty$ as $h \to 0$.