Lec 27.

$A \succ 0$. $M \succ 0$. $M \approx A$

$$M^{-1} A x = M^{-1} b.$$

Preconditioned SD/CG.

M- inner product $\quad (x,y)_M = x^T M y$

$$(x, M^{-1}Ay)_M = x^T M M^{-1} A y = x^T A M^{-1} M y$$

$$= (M^{-1}Ax, y)_M. \quad \forall x, y.$$

$$\varphi(x) = \frac{1}{2}\left(x, M^{-1}Ax\right)_M - (M^{-1}b, x)_M$$

$$= \frac{1}{2} x^T M M^{-1} A x - b^T M^{-1} M x$$

$$= \frac{1}{2} x^T A x - b^T x.$$

Preconditioned SD.

① $x_{k+1} = x_k + \alpha_k p_k$, $\alpha_k = \dfrac{p_k^T r_k}{p_k^T A p_k} = \dfrac{(p_k, z_k)_M}{(p_k, M^{-1} A p_k)_M}$

② $r_{k+1} = b - A x_{k+1}$

③ $z_{k+1} = M^{-1} r_{k+1}$

④ $p_{k+1} = z_{k+1}$

Preconditioned CG.

①-②-③

④ $p_{k+1} = z_{k+1} + \beta_k p_k$, $\beta_k = -\dfrac{p_k^T A z_{k+1}}{p_k^T A p_k}$

exer. compare formula w. Wiki.

Preconditioned GMRES.

$$M^{-1} A x = M^{-1} b$$

Left preconditioning.

$$A M^{-1} u = b \quad , \quad x = M^{-1} u.$$

Right preconditioning.

$$M_1^{-1} A M_2^{-1} u = M_1^{-1} b. \quad \kappa(M_1^{-1} A M_2^{-1}) \sim O(1).$$

split preconditioning.

$$A \approx LU, \quad M_1 = L, \quad M_2 = U.$$

- Diagonal / block-diagonal.

$$A \approx \begin{pmatrix} A_{11} & & * \\ & \ddots & \\ * & & A_{nn} \end{pmatrix} \quad M = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{pmatrix}$$

often performs well.

A diagonally dominate.

- $A = A_0 + A_1$ ex. $A = \underbrace{-\Delta}_{A_0} + \underbrace{V}_{A_1}$

A_0 ill conditioned. A_0^{-1} is easy.

$I + A_0^{-1} A_1$ well conditioned.

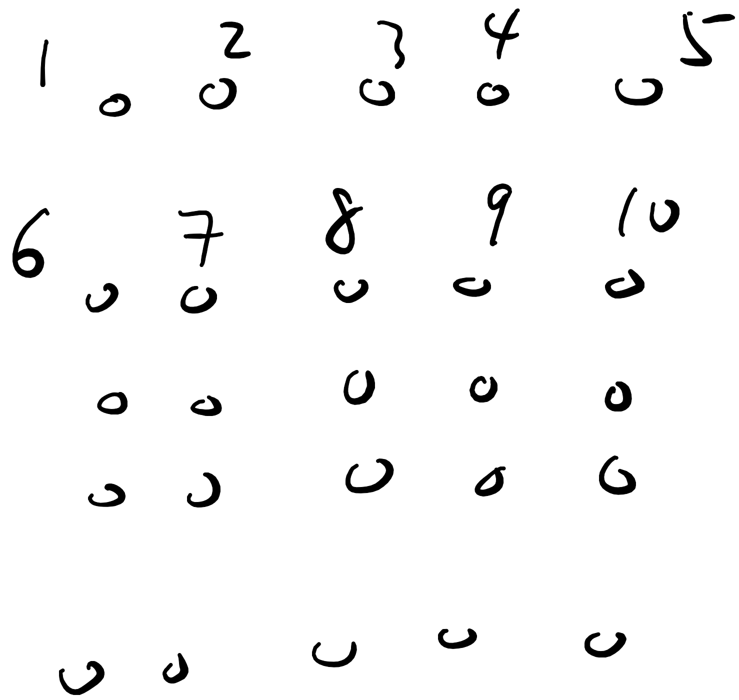
$$\underbrace{(-\Delta - z)}_{A_0} + (V - z)$$

sparse direct solver.

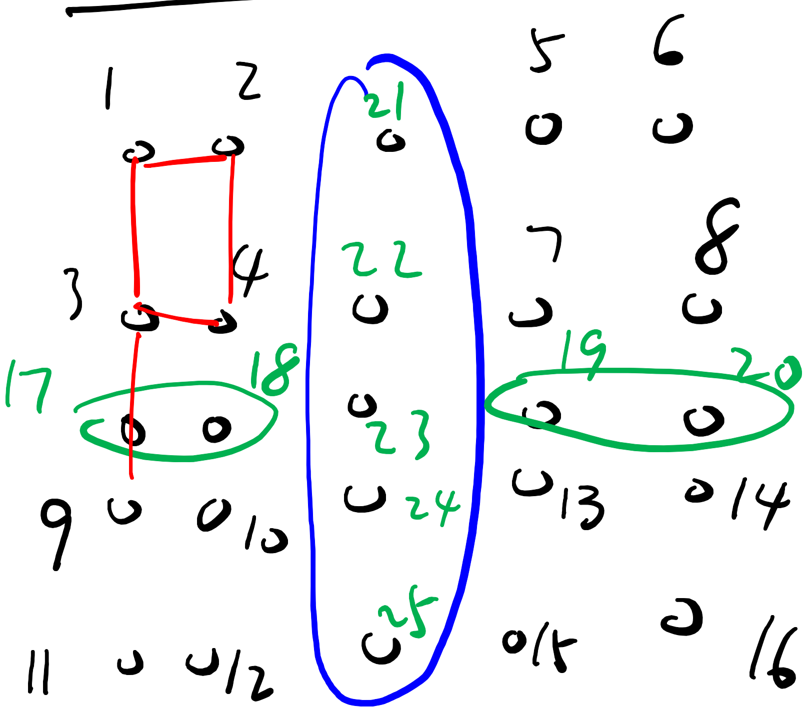
$Ax=b$. A sparse.

Gaussian elimination for sparse matrices.

$\boxed{\text{reordering}}$



natural ordering.



nested disection.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$A_{11} \in \mathbb{R}$, $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$

$A_{11} \neq 0$.

$$A = \begin{pmatrix} 1 & 0 \\ A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & I \end{pmatrix}$$

$$S = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

Schur complement

$$A = L_1 \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} U_1$$

$$S = \tilde{L}_2 \begin{pmatrix} 1 & 0 \\ 0 & \tilde{S} \end{pmatrix} \tilde{U}_2$$

$$L_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{L}_2 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{U}_2 \end{pmatrix}$$

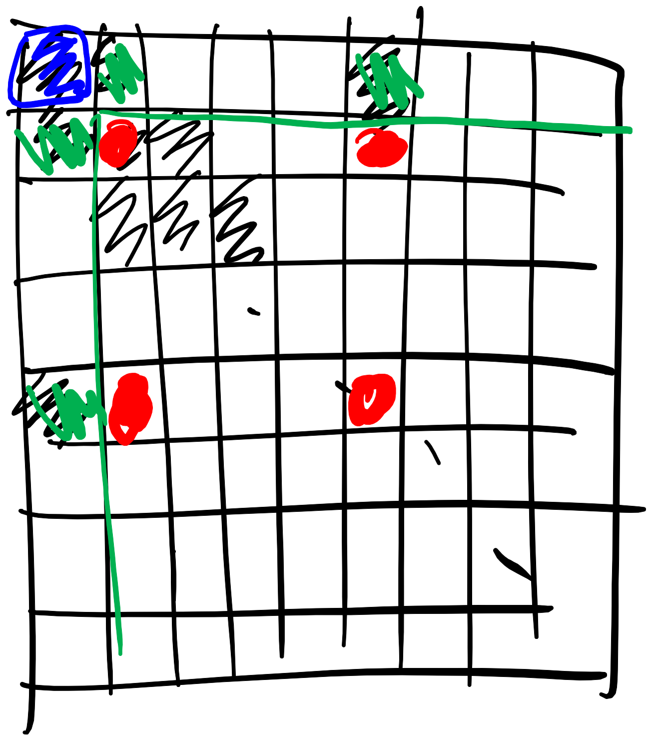
$$A = L_1 L_2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & \tilde{S} \end{pmatrix} U_2 U_1$$

$$\cdots$$

$$= \underbrace{L_1 L_2 \cdots L_n}_{L} \underbrace{U_n \cdots U_1}_{U}$$

$$L_1 L_2 = \begin{pmatrix} 1 & 0 \\ l_1 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{L}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ l_1 & \tilde{L}_2 \end{pmatrix}$$



● fill-in in Schur complement.

Alg. (Sparse LU, conceptual).

for $k = 1, \cdots, n$

$$L_{kk} = 1$$

$$L_{ik} = A_{ik} A_{kk}^{-1} \quad , \ i > k. \quad A_{ik} \neq 0$$

$$U_{kj} = A_{kj} \quad , \ j > k \ , \ A_{kj} \neq 0$$

$$A_{ij} \leftarrow A_{ij} - L_{ik} U_{kj} \ , \quad i, j > k.$$

SuperLU / SuperLU_DIST.

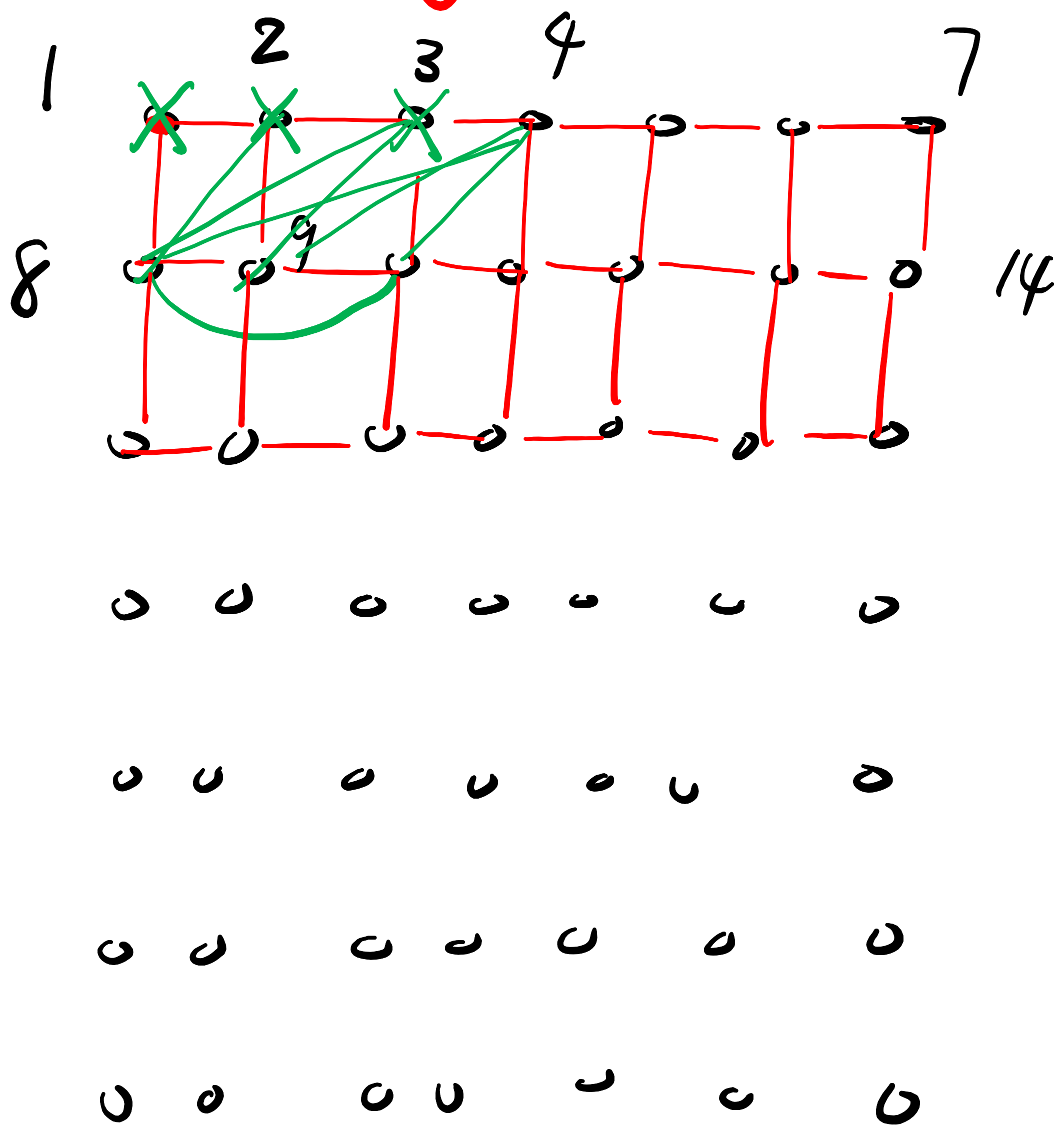
Sparse Cholesky.

$$A > 0. \qquad A = LL^T$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^{\frac{1}{2}} & \\ A_{21}A_{11}^{-\frac{1}{2}} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} A_{11}^{\frac{1}{2}} & A_{11}^{-\frac{1}{2}}A_{12} \\ 0 & I \end{pmatrix}$$

...

Natural ordering.



after eliminating 7,

A[8:14, 8:14]

dense.