Ex. Forward Euler. $\alpha_1 = 1, \alpha_0 = -1, \beta_1 = 0, \beta_0 = 1$

$$C_0 = \sum_{j=0}^{1} \alpha_j = 1 - 1 = 0.$$

$$C_1 = \sum_{j=0}^{1} (j \alpha_j - \beta_j) = 1 - 1 = 0.$$

$$C_2 = \frac{1}{2!} \sum_{j=0}^{1} (j^2 \alpha_j - 2 j \beta_j) = \frac{1}{2} \neq 0.$$

$\Rightarrow$ LTE is $O(h^2)$

<u>Thm</u>. LTE of r step AB is $O(h^{r+1})$

AM is $O(h^{r+2})$

Sketch

<u>Choose</u> special eqs. $\tau_n = 0$.

$\Rightarrow$ algebraic conditions of $\{\alpha_j\}, \{\beta_k\}$

$\Rightarrow$ eqs. needed as shown above.

$\Rightarrow$ Lag. interp. is exact for certain polynomials.

Pf: $\begin{cases} u'(t) = 0 \\ u(0) = 1 \end{cases} \Rightarrow u(t) \equiv 1.$

any AB is exact for this.

$$u(t_{n+r}) - u(t_{n+r-1}) = \int_{t_{n+r-1}}^{t_{n+r}} f(u(s), s)\, ds$$

$$= 0 \quad \text{(using numerical scheme)}$$

$$\tau_n = \sum_{j=0}^{r} \alpha_j \longrightarrow C_0 \text{ condition.}$$

Second.

$$\begin{cases} u'(t) = 1 \\ u(0) = 0 \end{cases} \Rightarrow \quad u(t) = t.$$

any AB is exact.

$$0 = \tau_n = \sum_{j=0}^{r} \alpha_j \underbrace{\left[(n+j)h\right]} - h\sum_{k=0}^{r} \beta_k \, 1$$

$$= nh\left(\underbrace{\sum_{j=0}^{r} \alpha_j}_{\substack{\parallel \\ 0}}\right) + h\left(\underbrace{\sum_{j=0}^{r} (\alpha_j j - \beta_j)}_{C_1 \text{ cond.}}\right)$$

In general. $(m \geq 1)$

$$\begin{cases} \dot{u}(t) = m t^{m-1} \\ u(0) = 0. \end{cases} \Rightarrow u(t) = t^m$$

$1 \leq m \leq r$ Lag interp is exact.

$$0 = \tau_n = \sum_{j=0}^{r} \alpha_j \underbrace{[(n+j)h]}^{m} - h \sum_{k=0}^{r} \beta_k \, m[(n+j)h]^{m-1}$$

$$= h^m n^m \left(\sum_{j=0}^{r} \alpha_j\right) + h^m \sum_{l=1}^{m} \left\{ n^{m-l} \binom{m}{l} \left[\sum_{j=0}^{r} \left(\alpha_j j^l - l \beta_j j^{l-1}\right) \right] \right\}$$

$\Rightarrow C_0 = \cdots = C_r = 0.$ $\square$

Another perspective of order condition.

Choose yet another special eq.

$$\begin{cases} u'(t) = u(t) \\ u(0) = 1 \end{cases} \Rightarrow u(t) = e^{t}$$

LTE $\tau_n = \sum_{j=0}^{r} \alpha_j e^{(n+j)h} - h \sum_{k=0}^{r} \beta_k e^{(n+k)h}$

$= e^{nh} \left(\sum_{j=0}^{r} \alpha_j (e^h)^j - \log(e^h) \sum_{k=0}^{r} \beta_k (e^h)^k \right)$

$z = e^h$, $z > 1$. $O(h^{p+1}) \sim O(|z-1|^{p+1})$

$$= e^{nh}\left(\sum_{j=0}^{r} \alpha_j z^j - \log z \sum_{k=0}^{r} \beta_k z^k\right)$$

2 polynomials of z

$$\rho(z) = \sum_{j=0}^{r} \alpha_j z^j, \qquad \sigma(z) = \sum_{k=0}^{r} \beta_k z^k.$$

<u>Thm</u>. Consistency condition

$$\rho(z) - \log z\, \sigma(z) = O\left(|z-1|^{p+1}\right)$$

$$\Longleftrightarrow \quad c_0 = \cdots = c_p = 0.$$

Ex. Euler.

$$\rho(z) = z-1$$

$$\sigma(z) = 1.$$

$$\rho(z) - \log z \, \sigma(z)$$

$$= z-1 - \log(1+(z-1))$$

$$= \frac{1}{2}(z-1)^2 + O((z-1)^3)$$

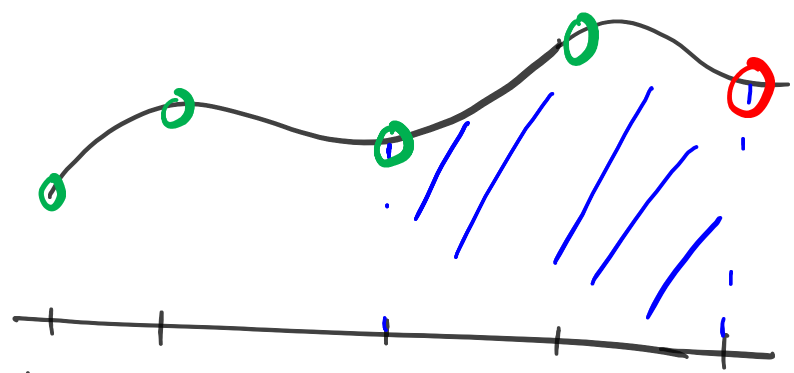
LTE: $O(h^2)$

common another change of var.

$$\zeta = z-1$$

Other types of LMM.

$$u_{n+r} - u_{n+r-2} = \int_{t_{n+r-2}}^{t_{n+r}} f(u(s), s)\, ds$$



o Nyström

o + o Milne.

Adaptive time stepping

Monitor the LTE τ

"Milne device"

Alg. if $(\tau > \delta h)$

$h \leftarrow \frac{h}{2}$ $\qquad\qquad$ $\tilde{\delta} < \delta$

if $(\tau < \tilde{\delta} h)$

$h \rightarrow 2h$

Estimate τ

$$\sum_{j=0}^{r} \alpha_j U_{n+j} = h \sum_{k=0}^{r} \beta_k f_{n+k}$$

$$\sum_{j=0}^{\tilde{r}} \tilde{\alpha}_j U_{n+j} = h \sum_{k=0}^{\tilde{r}} \tilde{\beta}_r f_{n+k}$$ ← easy to evaluate (usually explicit)

LTE $\quad \tau = u(t_{n+r}) - U_{n+r} = \underline{c}\, h^{p+1} u^{(p+1)}(t_n) + O(h^{p+2})$

$$\tilde{\tau} = u(t_{n+r}) - \tilde{U}_{n+r} = \underline{\tilde{c}}\, h^{p+1} u^{(p+1)}(t_n) + O(h^{p+2})$$

$$\underbrace{\tilde{u}_{n+r} - u_{n+r}}_{\text{Computable}} = (\underbrace{c - \tilde{c}}_{\text{Computable}})\, h^{p+1} u^{(p+1)}(t_n) + O(h^{p+2})$$

$$\|\tau\| \sim |c|\, h^{p+1} \|u^{(p+1)}(t_n)\| \sim \left|\frac{c}{c-\tilde{c}}\right| \|\tilde{u}_{n+r} - u_{n+r}\|$$

Ex. Trapezoidal. monitored by AB2

$$u_{n+1} = u_n + \frac{h}{2}(f_n + f_{n+1}) \qquad \tau = -\frac{1}{12} h^3 u^{(3)}(t_n) + O(h^4)$$

$$\tilde{u}_{n+1} = u_n + h\left(\frac{3}{2} f_n - f_{n-1}\right) \qquad \tilde{\tau} = \frac{5}{12} h^3 u^{(3)}(t_n) + O(h^4)$$

$$\|\tau\| = \left| \frac{-\frac{1}{12}}{-\frac{1}{12} - \frac{5}{12}} \right| \|u_{n+1} - \tilde{u}_{n+1}\| = \frac{1}{6} \|u_{n+1} - \tilde{u}_{n+1}\|$$

Zero stability

independent of LTE.

$$\|e_N\| \le \mathrm{Amp}\left(\|e_0\| + \cdots \|e_{r-1}\|\right) + \widetilde{\mathrm{Amp}} \cdot \mathrm{LTE}$$

focus on

Boil down to

$$\begin{cases} \dot{u}(t) = 0 \\ u(0) = 0 \end{cases}$$ <- why called zero stability

$u_0 = e_0$, $u_1 = e_1$, $\cdots$, $u_{r-1} = e_{r-1}$

$e_0, \cdots, e_{r-1}$ arbitrarily small

$$\|u_N\| \overset{?}{\lesssim} C(T)\left(\|e_0\| + \cdots + \|e_{r-1}\|\right).$$

$T = Nh$

Ex. $\begin{cases} u_{n+2} - 3u_{n+1} + 2u_n = 0 \\ u_0 = e_0, \quad u_1 = e_1 \end{cases}$

Difference eq.

$$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2.$$

general sol.

$$u_n = c_1 + c_2 2^n = (2e_0 - e_1) + (e_1 - e_0) 2^n.$$

$$\begin{cases} u_0 = e_0 = c_1 + c_2 \\ u_1 = e_1 = c_1 + 2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 2e_0 - e_1 \\ c_2 = e_1 - e_0 \end{cases}$$

Fact on difference eq.

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = 0. \qquad \rho(z) = \sum_{j=0}^{r} \alpha_j z^j$$

$$\rho(z) = \alpha_r (z - z_1)^{\mu_1} \cdots (z - z_m)^{\mu_m}.$$

$z_1, \dots, z_m \in \mathbb{C}$. $\sum_{l=1}^{m} \mu_l = r$, $\mu_l \in \mathbb{N}_+$

general sol. $u_j = \sum_{k=1}^{m} \sum_{l=0}^{\mu_k - 1} c_{kl}\, j^l z_k^j$

Def An r-step LMM is zero stable

if $\rho(z)$ satisfies the root condition.

1) $|z_k| \le 1$

2) if $|z_k| = 1$, z_k is simple (i.e. $\mu_k = 1$).

Ex. Adams method. $U_{n+r} - U_{n+r-1} = \cdots$

$$\rho(z) = z^r - z^{r-1} = z^{r-1}(z-1)$$

1 simple root, 0 root w. multiplicity $r-1$ ✓

Ex. $u_{n+2} - 2u_{n+1} + u_n = \frac{h}{2}(f_{n+2} - f_n)$

(exer) $\rho(z) - \log z \, \sigma(z) = -\frac{1}{12}(z-1)^4 + O(|z-1|^5)$

$\rho(z) = (z-1)^2$ $\times$

1 double root.

$\rho(z) - \log(z)\,\sigma(z) = O(|z-1|^{p+1})$

$z \to 1$

$\Rightarrow \rho(1) = 0 \Rightarrow 1$ is always a root.

Thm LMM

Consistency + Stability $\Rightarrow$ Convergence.

(LTE) (zero)

Sketch: $\sum_{j=0}^{r} \alpha_j u_{n+j} = h \sum_{k=0}^{r} \beta_k f_{n+k}$. $\alpha_r = 1$

vectorization.

$$\tilde{u}_n = \begin{bmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+r-1} \end{bmatrix} \in \mathbb{R}^{dr}, \quad \tilde{u}_{n+1} = \begin{bmatrix} u_{n+1} \\ \vdots \\ u_{n+r} \end{bmatrix}$$

identity $\in \mathbb{R}^{d \times d}$

$$\tilde{f}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=0}^{r} \beta_k f_{n+k} \end{bmatrix}. \quad A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots 0 & I \\ -\alpha_0 & & \cdots & & -\alpha_{r-1} \end{bmatrix} \in \mathbb{R}^{dr \times dr}$$

$$\tilde{u}_{n+1} = A\tilde{u}_n + h\tilde{\hat{f}}_n \qquad \tilde{\tau}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tau_n \end{bmatrix}$$

$$\hat{u}(t_{n+1}) = A\hat{u}(t_n) + h\hat{f}(t_n) + \tilde{\tau}_n$$

$$\tilde{e}_n = \hat{u}(t_n) - \tilde{u}_n \in \mathbb{R}^{dr}$$

$$\underbrace{\|e_n\|_2}_{\text{in } \mathbb{R}^d} \le \underbrace{\|\tilde{e}_n\|_2}_{\text{in } \mathbb{R}^{dr}} \leftarrow \text{prove bounded.}$$

$$\tilde{e}_{n+1} = A\tilde{e}_n + h\left(\tilde{\hat{f}}(t_n) - \tilde{\hat{f}}_n\right) + \tilde{\tau}_n$$

$$= A^{n+1}\tilde{e}_0 + \sum_{j=0}^{n} A^{n-j}\left[h\left(\tilde{f}(t_j) - \tilde{f}_j\right) + \tilde{\tau}_j\right]$$

$$\|\tilde{e}_{n+1}\|_2 \le \|A^{n+1}\|_2 \|\tilde{e}_0\|_2 + \sum_{j=0}^{n} \|A^{n-j}\|_2 \left[h\|\tilde{f}(t_j) - \tilde{f}_j\|_2 + \|\tilde{\tau}_j\|_2\right]$$

① $\|A^{n+1}\|_2$ must be bounded. $\Longleftrightarrow$ zero stability.

Jordan decomposition of A.

$$A = VJV^{-1}$$

$$\|A^n\| = \|VJV^{-1}VJV^{-1}\cdots VJV^{-1}\|$$

$$= \|VJ^nV^{-1}\| \leq \|V\|\|J^n\|\|V^{-1}\|$$

② relate $\|\tilde{f}(t_n) - \tilde{f}_n\|_2$ to $\|e_n\| \to \|\tilde{e}_n\|$

③ bound $\|\tilde{\tau}_n\| \leftarrow$ LTE

④ $\|\tilde{e}_{n+1}\| \leq \sum_{j=0}^{n} \left(* \|\tilde{e}_j\| + * \|\tilde{\tau}_j\| \right)$

discrete Gronwall's inequality.

$$\Rightarrow \quad \|\tilde{e}_n\| \leq \star \, \|\tilde{e}_0\| + \star \max \|\tilde{\tau}_j\| .$$