

Lec 3 .

Convergence order

2nd order Taylor method.

Adams - Bashforth method .

$$\|e_{n+1}\| \leq (1+hL)^{n+1} \|e_0\| + \left[(1+hL)^n \|T_0\| + \dots + \|T_n\| \right]$$

$$\leq (1+hL)^{n+1} \|e_0\| + \frac{Mh}{2} \cdot \frac{(1+hL)^{n+1} - 1}{hL}$$

(use $1+x \leq e^x$)

$$(1+hL)^n \leq e^{hLn}, \quad T = Nh$$

$$\Rightarrow \|e_N\| \leq e^{LT} \left(\|e_0\| + \frac{Mh}{2L} \right)$$

→ a priori analysis.

exp. amp. initial error

↓ ↓

LTE $O(h^P)$ order of method

Stability

consistency.

$$\tau \sim O(h^{p+1}) \rightarrow \|e_n\| \sim O(h^p)$$

Stability + consistency \Rightarrow convergence.

check convergence order numerically.

$$u(t) \quad u_h(t_n) \equiv u_n$$

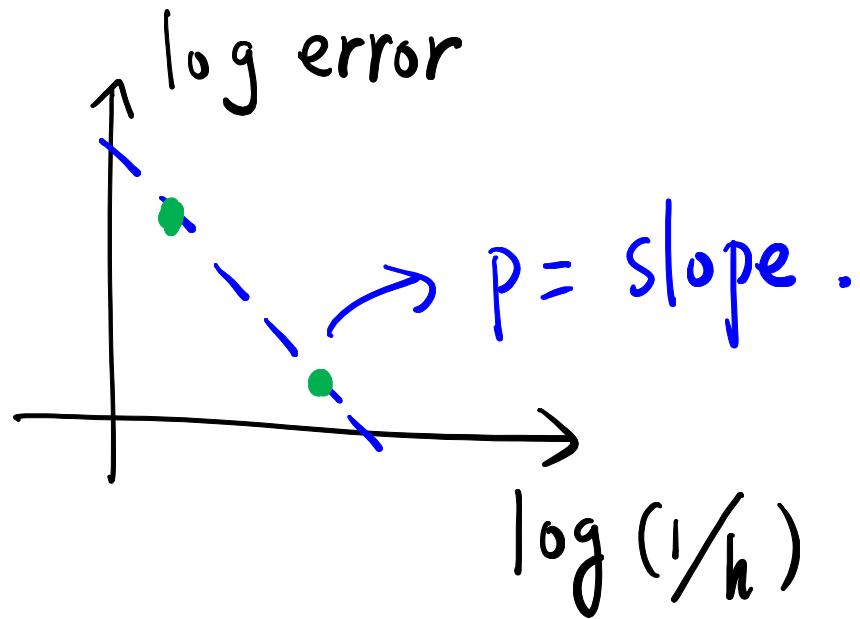
exact sol numer. sol.

$$t \in [0, T] \quad t_n = nh \\ n=0, \dots, N$$

$$u(T) = u_h(T) + \cancel{U(h)} h^P + O(h^{P+1})$$

$$\|u(T) - u_h(T)\| \approx C h^P, \quad c = \|U(T)\|$$

$$\log \|u(\tau) - u_h(\tau)\| \approx \log C - p \log (1/h)$$



$u(\tau)$ not known?

$$u(\tau) \approx u_h(\tau) + v(\tau) h^p$$

$$\approx U_{\frac{h}{2}}(T) + v(T) \frac{h^P}{2^P}$$

$$\approx U_{\frac{h}{4}}(T) + v(T) \frac{h^P}{4^P}$$

$$\| U_h(T) - U_{\frac{h}{2}}(T) \| \approx C h^P \left(1 - \frac{1}{2^P} \right)$$

$$\| U_{\frac{h}{2}}(T) - U_{\frac{h}{4}}(T) \| \approx C h^P \left(\frac{1}{2^P} - \frac{1}{4^P} \right)$$

$$\frac{\|U_h(\tau) - U_{\frac{h}{2}}(\tau)\|}{\|U_{\frac{h}{2}}(\tau) - U_{\frac{h}{4}}(\tau)\|} \approx 2^p$$

↑ everything is computable !

Taylor expansion.

$$u(t_{n+1}) = u(t_n) + \dot{u}(t_n)h + \frac{1}{2} \ddot{u}(t_n)h^2 + \cancel{+ \dots}$$

$$\dot{u}(t_n) = f(u(t_n), t_n) , \quad u(t) \in \mathbb{R}^m$$

$$\ddot{u}_i(t_n) = \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(u(t_n), t_n) \cdot \dot{u}_j(t_n) = f_j(u(t_n), t_n)$$

$$+ \frac{\partial f_i}{\partial t}(u(t_n), t_n)$$

$$(D_u f)_{ij} = \frac{\partial f_i}{\partial u_j} \quad \text{Jacobian matrix}$$

$$\Rightarrow \ddot{u}(t_n) = D_u f(u(t_n), t_n) \cdot f(u(t_n), t_n) \\ + D_t f(u(t_n), t_n)$$

2nd order Taylor scheme.

$$u_{n+1} = u_n + f(u_n, t_n) h + \frac{h^2}{2} [D_u f(u_n, t_n) \cdot \\ \underbrace{f(u_n, t_n) + D_t f(u_n, t_n)}_{\text{}}]$$

almost NEVER used.

- ① Computation of Jacobian.
- ② Higher order method is impossible.

Idea: combine a few evaluations of f
to approximate directional derivative.

$$\begin{aligned} & f(u_n + h f(u_n, t_n), t_{n+1}) \\ \approx & f(u_n, t_n) + h [D_u f(u_n, t_n) \cdot f(u_n, t_n) + D_t f(u_n, t_n)] \\ & + O(h^2) \end{aligned}$$

Define $\bar{u} = u_n + h f(u_n, t_n)$

$$\begin{aligned}
 u_{n+1} &= u_n + h f(u_n, t_n) + \frac{h}{2} [f(\bar{u}, t_{n+1}) - f(u_n, t_n) \\
 &\quad + O(h^2)] \\
 &= u_n + \frac{h}{2} f(u_n, t_n) + \frac{h}{2} f(\bar{u}, t_{n+1}) + O(h^3)
 \end{aligned}$$

Alg. $\bar{u} = u_n + h f(u_n, t_n)$ ← Forward Euler

$$u_{n+1} = u_n + \frac{h}{2} (f(u_n, t_n) + f(\bar{u}, t_{n+1}))$$

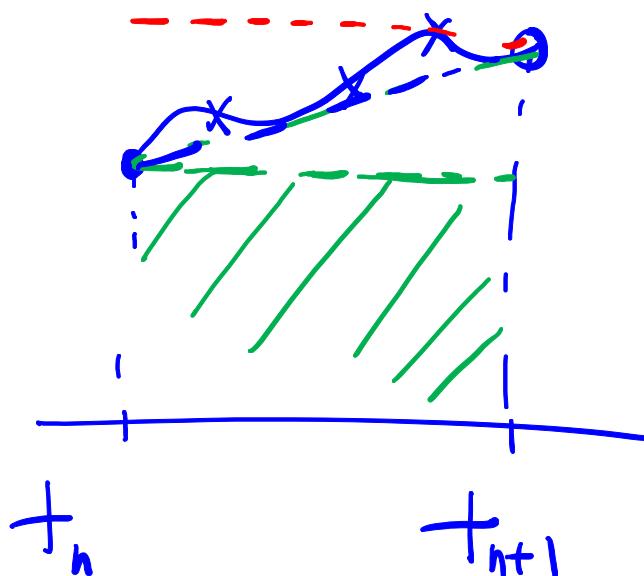
modified Euler's method.

→ one type of Runge-Kutta method.

Integral representation

$$\begin{cases} \dot{u}(t) = f(u(t), t) \\ u(0) = u_0 \end{cases} \Rightarrow u(t) = u_0 + \int_0^t f(u(s), s) ds$$

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f(u(s), s) ds.$$



replace by quadrature.

$$\int_{t_n}^{t_{n+1}} f(u(s), s) ds \approx f(u_n, t_n) \cdot h \quad \begin{matrix} \text{forward} \\ \text{Euler} \end{matrix}$$

$$\approx f(u_{n+1}, t_{n+1}) h \quad \begin{matrix} \text{backward} \\ \text{Euler} \rightarrow \text{implicit} \end{matrix}$$

$$\approx \frac{h}{2} [f(u_n, t_n) + f(u_{n+1}, t_{n+1})]$$

trapezoidal rule
 \rightarrow implicit.

