Lec 16.

Hamiltonian system.

$$H : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$$

$\uparrow$ momentum p $\qquad$ $\uparrow$ space. q

$$\begin{cases} \dot{p} = -\dfrac{\partial H(p,q)}{\partial q} \\[2ex] \dot{q} = \dfrac{\partial H(p,q)}{\partial p} \end{cases}$$

$$H(p,q) = T(p) + V(q) \qquad \text{separable}$$

Ex. N-particles interacting in a potential field $V(q_1, \cdots, q_N)$

$$H(p,q) = \sum_{i=1}^{N} \frac{1}{2m_i} p_i^2 + V(q_1, \cdots, q_N)$$

$$\begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}}_{J^{-1}} \underbrace{\begin{pmatrix} \frac{\partial H(p(t), q(t))}{\partial p} \\ \frac{\partial H(p(t), q(t))}{\partial q} \end{pmatrix}}_{\nabla_u H(u(t))}$$

$$\varphi_t(p_0, q_0) = (p(t), q(t)) := u(t)$$

$$\boxed{\frac{du(t)}{dt} = J^{-1} \nabla_u H(u(t))}$$

$$J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix} \qquad \boxed{J^{-1} = -J = J^T}$$

Area preserving $\Rightarrow$ Consider

Jacobian

$$\Phi_t(p_0, q_0) = \begin{bmatrix} \frac{\partial p(t)}{\partial p_0} & \frac{\partial q(t)}{\partial p_0} \\ \frac{\partial p(t)}{\partial q_0} & \frac{\partial q(t)}{\partial q_0} \end{bmatrix}$$

$$\Phi_t(u_0) = \frac{\partial u(t)}{\partial u_0}$$

Want to show: $\left|\Phi_t(u_0)\right| = 1$.

Thm (Poincaré) Hamiltonian dynamics

$$\Phi_t^T J \Phi_t = J$$

symplectic

$u(t)=\varphi_t(u_0)$

Symplectic $\Rightarrow$ area preserving

$$\det\left(\Phi_t^T J \Phi_t\right) = \det(\Phi_t)^2 \cancel{\det(J)} = \cancel{\det(J)}$$

$$\Rightarrow \det(\Phi_t) = \pm 1. \quad \text{for all } t.$$

$$\det \Phi_t = \det \Phi_0 = 1 \quad (\text{continuity})$$

Pf: $\Phi_0 = I \qquad \Phi_0^T J \Phi_0 = J. \quad t=0$ ✓

$$\frac{d}{dt}\Phi_t = \frac{d}{dt}\nabla_{u_0}\varphi_t(u_0) = \nabla_{u_0}\frac{d}{dt}\varphi_t(u_0)$$

$$= \nabla_{u_0}\left(J^{-1}\nabla_u H(\varphi_t(u_0))\right)$$

$$= J^{-1}\underbrace{\nabla_u^2 H(u(t))}_{\text{Hessian. symmetric}}\cdot\Phi_t$$

$$\frac{d}{dt}\left(\Phi_t^T J \Phi_t\right)$$

$$= \Phi_t^T \nabla_u^2 H \left(-J^{-1}\right) J \Phi_t$$

$$+ \Phi_t^T J J^{-1} \nabla_u^2 H \Phi_t = 0 \qquad \square.$$

Symplectic scheme.

Discrete flow map

$$\varphi_n : u_0 \mapsto u_n(u_0)$$

Jacobian $\quad \Phi_n := \dfrac{\partial \varphi_n(u_0)}{\partial u_0}$

$$\Phi_n^T J \Phi_n = J$$

Lemma: $f: \mathbb{R}^n \to \mathbb{R}^n$. $f \in C^1(\mathbb{R}^n)$

$Df(u)$ is symmetric i.e.

$$\frac{\partial f_i}{\partial u_j} = \frac{\partial f_j}{\partial u_i}$$

Then $\exists$ $H: \mathbb{R}^n \to \mathbb{R}$.

s.t. $f(u) = D_u H(u)$.

Pf:
$$H(u) = \int_0^1 u \cdot f(tu)\, dt$$

$$\frac{\partial H(u)}{\partial u_j} = \int_0^1 f_j(tu)\, dt$$

$$+ \int_0^1 \sum_{i=1}^{N} u_i \frac{\partial f_i(tu)}{\partial u_j} \cdot t\, dt$$

$$= \int_0^1 f_j(tu) + \sum_{i=1}^{N} u_i \cdot \frac{\partial f_j(tu)}{\partial u_i} t\, dt$$

$$= \int_0^1 \frac{d}{dt}\left(t\ f_j(tu)\right) dt$$

$$= f_j(u) \qquad \square.$$

<u>Thm</u>. (Inverse of Poincaré)

$f: \mathbb{R}^{2d} \to \mathbb{R}^{2d}$. $f \in C^1(\mathbb{R}^{2d})$

flow map of $\dot{u} = f(u)$

is symplectic. Then $\exists H$

$$f = J^{-1} D_u H.$$