Lec 23.

Arnoldi $\{r, Ar, \cdots, A^{k-1} r\}$

$$A V_k = V_{k+1} \tilde{H}_k$$

H_k

$$\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1,k} \\ h_{21} & h_{22} & & \vdots \\ & h_{32} & \ddots & \vdots \\ & & h_{k,k-1} & h_{kk} \end{pmatrix}$$
$$\begin{matrix} 0 & \cdots & 0 & h_{k+1,k} \end{matrix}$$

FOM. $r = b - A(x_0 + V_k y)$

$r \perp \operatorname{span} V_k$

$0 = V_k^T r \Rightarrow H_k y = \beta e_1 \longleftarrow k \times k$

$\beta = \| b - A x_0 \|$

Problems of FOM

1. Residual can be unbounded.

2. keep entire history
   (before restarting)

Generalized minimal residual method (GMRES).

Default for general linear systems.

$$\min \|b - Ax\|_2$$

$$\text{s.t.} \quad x = x_0 + V_k y_k, \quad y_k \in \mathbb{R}^k$$

$$b - A(x_0 + V_k y_k)$$

$$= \beta V_k e_1 - V_{k+1} \tilde{H}_k y_k$$

$$= V_{k+1} \left(\beta \tilde{e}_1 - \tilde{H}_k y_k \right)$$

$$\min_{y_k} \| \beta \tilde{e}_1 - \tilde{H}_k y_k \|_2 \leftarrow \text{least-squares}$$

Least-squares

$$\min \|Ax - b\|_2^2$$

① $A^T A x = A^T b$.

② QR factorization

$$A \in \mathbb{R}^{m \times n}, \quad m > n.$$

$$A = \underset{\substack{\cap \\ \mathbb{R}^{m\times m}}}{Q} R \in \mathbb{R}^{m\times n}$$

$$R = \begin{array}{c} \begin{matrix} n \end{matrix} \\ \begin{pmatrix} * \\ 0 \end{pmatrix} \end{array} \begin{array}{l} n \\ m-n \end{array}$$

$$\| QRx - QQ^T b \|_2$$

$$= \| Rx - Q^T b \|_2$$

$$\left\| \begin{pmatrix} * \\ 0 \end{pmatrix} x - \begin{pmatrix} * \\ * \end{pmatrix} \right\|_2$$

error

Practical implementation of GMRES

① keep track of residual

w.o. reconstructing x.

② Givens - QR.

avoid repeatedly doing QR

for $\tilde{H}_k$.

$$
G(i,j;\theta) =
\begin{array}{c c}
 & \begin{array}{cccccccc} & & & i & & j & & \end{array} \\
\begin{array}{c} \\ \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \\ \end{array} &
\begin{bmatrix}
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & 1 & & & & & \\
 & & & \cos\theta & & \sin\theta & & \\
 & & & & \ddots & & & \\
 & & & -\sin\theta & & \cos\theta & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{bmatrix}
\end{array}
$$

$$
G(i,j,\theta)
\begin{array}{c c}
\begin{pmatrix} * \\ \vdots \\ a \\ \vdots \\ b \\ \vdots \\ * \end{pmatrix} &
\begin{array}{c} \\ \\ \leftarrow i \\ \\ \leftarrow j \\ \\ \\ \end{array}
\end{array}
$$

$$
\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}
$$

$$
\Rightarrow \quad -\sin\theta\, a + \cos\theta\, b = 0
$$

$$\Rightarrow \quad \theta = \arctan\left(\frac{a}{b}\right), \quad b \neq 0$$

Ex.

$$\tilde{H}_2 = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ 0 & h_{32} \end{pmatrix} \xrightarrow{QR} \begin{pmatrix} * & * \\ & * \\ & 0 \end{pmatrix}$$

① eliminate h_{21}, $G(1,2,\theta)$

$$\longrightarrow \begin{pmatrix} * & * \\ 0 & * \\ 0 & h_{32} \end{pmatrix}$$

② eliminate h_{32}. $G(2,3,\theta')$

$$\longrightarrow \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix} \quad R$$

$$\| \tilde{H}_2 y - \beta \tilde{e}_1 \|$$

$$\longrightarrow \| G(2,3,\theta') G(1,2,\theta) \tilde{H}_2 - G(2,3,\theta') G(1,2,\theta) \beta \tilde{e}_1 \|_2$$

$$\longrightarrow \left\| \begin{pmatrix} * & * \\ & * \\ & 0 \end{pmatrix} y - \begin{pmatrix} * \\ * \\ * \end{pmatrix} \right\|_2$$

Ex.

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ 0 & h_{32} & h_{33} \\ 0 & 0 & h_{43} \end{pmatrix} \xrightarrow{G(1,2,\theta)} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & h_{32} & h_{33} \\ 0 & 0 & h_{43} \end{pmatrix}$$

$$\xrightarrow{G(2,3,\theta')} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & h_{43} \end{pmatrix}$$

$$\xrightarrow{G(3,4,\theta'')} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$$

k-th step. work $O(k)$