Thm. Gauss-Legendre quadrature w. r points is exact for all polynomials of order $\leq 2r-1$.

Pf: For any $f(x) \in \mathbb{P}_{2r-1}$. $P_r(x) \leftarrow GL$

$$f(x) = g(x) P_r(x) + h(x), \quad g, h \in \mathbb{P}_{r-1}$$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} g(x) P_r(x)\,dx + \int_{-1}^{1} h(x)\,dx$$

apply quad.

$$\sum_{l=1}^{r} w_l \, \cancel{g(x_l) P_r(x_l)} + \sum_{l=1}^{r} w_l \, h(x_l)$$

$\cancel{g(x_l) P_r(x_l)} = 0$ exact (orthogonality & choice of nodes)

$\sum_{l=1}^{r} w_l \, h(x_l)$ exact. (Lag. interpolation)

$\square$.

algebraic accuracy.

More general $f \in C^\infty([-1,1])$

$a=-1, \; b=1$

$$\left| \int_a^b f(x)\,dx - \sum_{l=1}^{r} f(x_l)\,\omega_l \right|$$

$$= \left| \int_a^b \sum_{n=0}^{2r-1} \frac{f^{(n)}(a)}{n!} (x-a)^n\,dx + \sum_{n=2r}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n dx \right.$$

$$\left. - \sum_{l=1}^{r} f(x_l)\,\omega_l \right|$$

Taylor expand as well.

$$\leq C \sup_{a \leq x \leq b} \left| f^{(2r)}(x) \right| (b-a)^{2r+1}$$

G-L quadrature from $[-1,1] \longrightarrow [a, a+h]$

$$\rightarrow \left| \int_a^{a+h} f(x)\,dx - \sum_{l=1}^{r} f(x_l)\, w_l \right| \leq C h^{2r+1} \sup_{a \leq x \leq a+h} \left| f^{(2r)}(x) \right|.$$

G-L w. r points is of order. 2r.

why 3-term recurrence ?

$$\beta_l P_l(x) = (x - \alpha_l) P_{l-1}(x) - \gamma_l P_{l-2}(x)$$

$$+ \sum_{j=0}^{l-3} c_j P_j(x).$$

Project on P_j $(j=0,\dots,l-3)$

$$\int_{-1}^{1} \beta_l \cancel{P_j} P_l \, dx = \int_{-1}^{1} \overset{\in \mathbb{P}_{l-2}}{x P_j} \cancel{P_{l-1}} \, dx - \alpha_l \int_{-1}^{1} \cancel{P_j} P_{l-1} \, dx$$

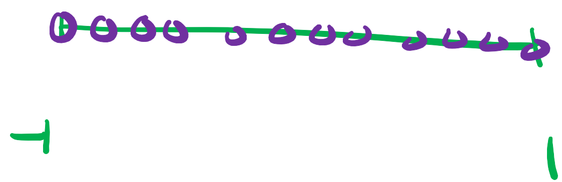
$$- \gamma_l \int_{-1}^{1} P_j \cancel{P_{l-2}} \, dx + \sum_{j'=0}^{l-3} c_{j'} \int_{-1}^{1} P_j P_{j'} \, dx$$

$$\left(\int_{-1}^{1} P_j^2 \, dx \right) c_j = 0 \Rightarrow c_j = 0, \quad j = 0, \dots, l-3.$$

Three-term recurrence.

$$x P_{\ell-1}(x) = \gamma_\ell P_{\ell-2}(x) + \alpha_\ell P_{\ell-1}(x) + \beta_\ell P_\ell(x).$$

"matrix A"



$\{x_i\}_{i=1}^{m}$

$x_1 = -1, \; x_m = 1$

$$A = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{bmatrix}$$

$$P_\ell = \begin{bmatrix} P_\ell(x_1) \\ \vdots \\ P_\ell(x_m) \end{bmatrix}$$

$$V_\ell = [P_0, P_1, \cdots P_{\ell-1}] \in \mathbb{R}^{m \times \ell}$$

$$P_\ell^T P_{\ell'} = 0, \quad \ell \neq \ell'$$

$$AV_\ell = V_{\ell+1} \underbrace{\begin{bmatrix} \alpha_0 & \gamma_1 & & 0 \\ \beta_0 & \alpha_1 & \ddots & \\ & \beta_1 & \ddots & \gamma_{\ell-1} \\ 0 & & \ddots & \alpha_{\ell-1} \\ & & & \beta_{\ell-1} \end{bmatrix}}_{\in \mathbb{R}^{(\ell+1)\times \ell}} \rightarrow \begin{bmatrix} T \\ \hline 0 \quad \beta_{\ell-1} \end{bmatrix}$$

$$AV_\ell = V_\ell T + \beta_{\ell-1} p_\ell \cdot e_\ell^T, \quad e_\ell = \left.\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \underline{1} \end{pmatrix}\right\} \ell$$

Lanczos procedure.

embedded RK. Adaptive time stepping.

estimate τ.

$$u(t_{n+1}) = u(t_n) + h\sum_{i=1}^{r} b_i k_i + \tau + O(h^{p+2})$$

$$u(t_{n+1}) = u(t_n) + h\sum_{i=1}^{r} \tilde{b}_i \tilde{k}_i + O(h^{p+2})$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & \tilde{b}^T \end{array}$$

$k_i = \tilde{k}_i$

$$\tau \approx h\sum_{i=1}^{r} (\tilde{b}_i - b_i) k_i$$

ode45. 4-th & 5-th order, 7-stage scheme

for better (absolute) stability properties.

Absolute stability.

recall zero stability.

$$u' = 0 \qquad \text{special RHS.}$$

abs. stability

$$u' = \lambda u \quad , \quad \lambda \in \mathbb{C}$$

$\mathrm{Re}\,\lambda < 0$. $\quad u(t) = e^{\lambda t} u(0) \quad |u(t)|$ decays exponentially

numerical scheme to produce exp. decay.

$\mathrm{Re}\,\lambda = 0$, $\lambda = i\omega$, $\omega \in \mathbb{R}$.

often require special treatment. $\omega >> 1$.

$\mathrm{Re}\,\lambda > 0$. should just use small time step

to resolve. exp. growth.

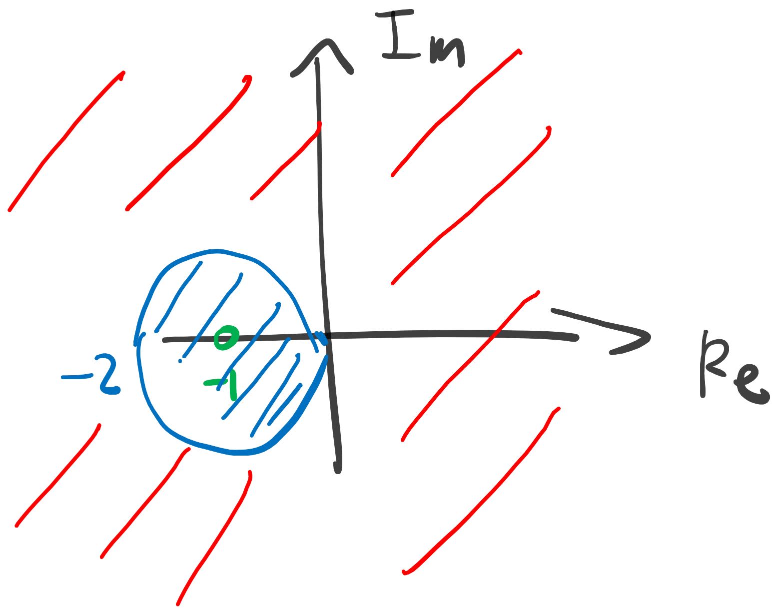
Forward Euler

$$u_{n+1} = u_n + \lambda h\, u_n = (1+\lambda h)\, u_n$$

$$z = \lambda h$$

$$u_n = (1+z)^n u_0$$

$$|1+z| \leq 1$$



region of absolute stability (RAS)

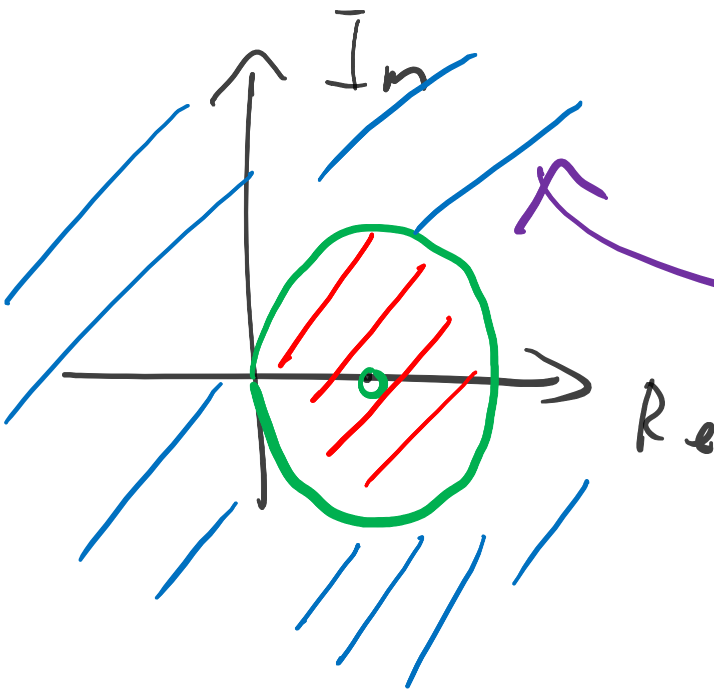
$$RAS := \{ z \mid |1+z| \leq 1 \}.$$

Backward Euler.

$$u_{n+1} = u_n + z\, u_{n+1}$$

$$u_{n+1} = \frac{1}{1-z} u_n$$

$$RAS = \{ z \mid |(1-z)^{-1}| \leq 1 \} \supseteq \{ z \mid \mathrm{Re}\,\lambda < 0 \}$$



A-stable

discrepancy between stability & accuracy (consistency).

General LMM.

$$\sum_{j=0}^{r} \alpha_j u_{n+j} = h \sum_{k=0}^{r} \beta_k f_{n+k}. \qquad f_{n+k} = \lambda u_{n+k}.$$

$$\Longrightarrow \quad \sum_{j=0}^{r} (\alpha_j - z\beta_j) u_{n+j} = 0.$$

which value(s) of z $\Rightarrow$ sols of difference eq.

are stable.

root condition (for a fixed z)

$$\tilde{\rho}(\omega; z) = \rho(\omega) - z\,\sigma(\omega)$$

$$= \sum_{j=0}^{r} (\alpha_j - z\beta_j)\,\omega^j \in \mathbb{P}_r \quad \text{w.r.t. } \omega.$$

1) all roots of $\tilde{\rho}(\omega; z)$ have modulus ≤ 1.

2) if a root $|\omega| = 1$, then ω must be simple.

$RAS = \{ z \mid \tilde{\rho}(\cdot\,; z) \text{ satisfies root condition} \}.$

"brute force" computing RAS.

mesh on $\mathbb{C}$. check root cond. on each point.

Ex. F - Euler.

$$\tilde{\rho}(\omega; z) = \omega - 1 - z \cdot \underline{1} = 0.$$

root $\omega = 1 + z$. $\Rightarrow |1+z| \le 1$

Ex. Trapezoidal.

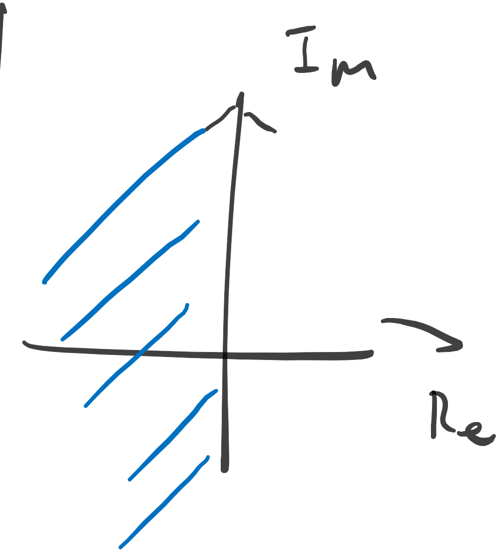
$$\tilde{\rho}(\omega; z) = (\omega - 1) - z\left(\tfrac{1}{2}\omega + \tfrac{1}{2}\right) = 0.$$

$$\Rightarrow \omega = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$$

$$\left|1 + \tfrac{1}{2}z\right| \le \left|1 - \tfrac{1}{2}z\right|$$

$$\Rightarrow 1 + \tfrac{1}{4}|z|^2 + \mathrm{Re}\, z \le 1 + \tfrac{1}{4}|z|^2 - \mathrm{Re}\, z$$

$$\Rightarrow \mathrm{Re}\, z \le 0.$$



A-stable

Ex. Leap frog.

$$u_{n+2} - u_n = 2h\, f_{n+1}$$

zero stable. 2nd order accurate.

$$\tilde{\rho}(\omega; z) = \omega^2 - 1 - z\, 2 \cdot \omega = 0$$

$$\omega = \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1}$$

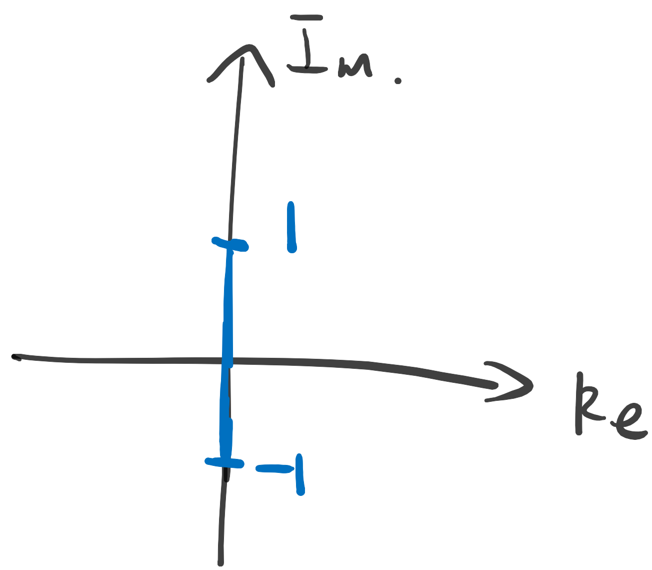
$|\omega| \leq 1$. $\omega_1 \omega_2 = -1 \Rightarrow |\omega_1| \cdot |\omega_2| = 1$

$$\Rightarrow |\omega_1| = |\omega_2| = 1 .$$

$$\omega = e^{i\theta}, \quad \theta \in [0, 2\pi).$$

$$e^{2i\theta} - 1 = 2z e^{i\theta}$$

$$\Rightarrow z = \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) = i \sin\theta$$



only works

1) eg. purely oscillatory

2) small time step.

General trend.

For LMM. RAS general shrinks

w.r.t increase of order.

$\Rightarrow$ high order LMM. more difficult to use

Boundary locus method for drawing RAS.

RAS is a "proper" region.

∂RAS is a line

characterize $|\omega| = 1$. $\omega = e^{i\theta}$, $\theta \in [0, 2\pi)$.

$$\hat{\rho}(\omega; z) = \rho(e^{i\theta}) - z\,\sigma(e^{i\theta}) = 0.$$

$$\Rightarrow \quad z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

pick a point z inside both interior & exterior to see whether region is stable.

RAS for RK methods

$$k_i = f\left(u_n + h\sum_{j=1}^{r} a_{ij} k_j\right) = \lambda\left(u_n + h\sum_j a_{ij}k_j\right)$$

$$u_{n+1} = u_n + h\sum_i b_i k_i$$

$$\vec{k} = \lambda u_n \vec{e} + \lambda h A \vec{k}$$

$$\Rightarrow \vec{k} = (I - zA)^{-1}\vec{e} \cdot \lambda u_n$$

$$u_{n+1} = u_n + \vec{b}^T (I - zA)^{-1}\vec{e} \cdot z\, u_n$$

$$= R(z)\, u_n. \qquad R(z) = 1 + z\,\vec{b}^T (I - zA)^{-1}\vec{e}$$

Def. RAS for RK method

$$\{ z \in \mathbb{C} \mid |R(z)| \le 1 \}.$$

Ex. GL 1.

$$\begin{array}{c|c} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$k = f(u_n + \tfrac{1}{2} h k)$$

$$u_{n+1} = u_n + h k$$

$\tilde{u}_{n+\frac{1}{2}}$

u_n $\qquad$ u_{n+1}

implicit midpoint rule.

$$R(z) = 1 + z \cdot \left(1 - \tfrac{1}{2} z\right)^{-1} = \frac{1 + \frac{1}{2} z}{1 - \frac{1}{2} z}$$

$|R(z)| \le 1 \Rightarrow \mathrm{Re}\, z \le 0.$ A-stable

Ex.

$$\begin{array}{|cc}
0 & \\
\frac{1}{2} & 0 \\
\hline
0 & 1
\end{array}$$

$$R(z) = 1 + z \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ -\frac{1}{2} z & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} z & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \left(1 + \frac{1}{2} z\right) = 1 + z + \frac{1}{2} z^2$$

Boundary locus.

$$R(z) = 1 + z + \frac{1}{2} z^2 = e^{i\theta}, \quad \theta \in [0, 2\pi).$$

$$z = -1 \pm \sqrt{1 - 2(1 - e^{i\theta})}$$