Math 325K - Lecture 19
Section 7.3 Composition of functions

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Outline

- Definition.
- Properties.

The composition of functions

Definition

Let $f : X \to Y'$ and $g : Y \to Z$ be two functions with the property that the range of f is a subset of the domain Y of g. The **composition** *of f and g is another function $g \circ f : X \to Z$ such that for every $x \in X$, $(g \circ f)(x) = g(f(x))$. The symbol $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x".*

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Remark

To make sure that $g \circ f$ is well-defined, g should be able to map each $f(x)$, and one sufficient condition could be that the range of f is a subset of the domain of g. Note that Y' is not necessarily equal to Y.

Example: formulas of compositions

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For $f : \mathbb{N} \to \mathbb{N}$ *with* $f(n) = n + 1$ *and* $g : \mathbb{N} \to \mathbb{N}$ *with* $g(n) = n^2$*, what would be the formula for* $g \circ f$*? We just need to figure out what is the value of* $(g \circ f)(n)$ *for every* $n \in \mathbb{N}$*.*

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$$(g \circ f)(n) = (n + 1)^2 \quad \forall n \in \mathbb{N}.$$

Example: formulas of compositions

Exercise

Let $f : \mathbb{N} \to \mathbb{N}$ *with* $f(n) = n + 1$ *and* $g : \mathbb{N} \to \mathbb{N}$ *with* $g(n) = n^2$. *Find the formula of* $f \circ g$. *Is it one-to-one? Is it onto?*

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Solution

For every $n \in \mathbb{N}$, $(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$.

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Solution

For every $n \in \mathbb{N}$, $(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$*. To check the one-to-one property, suppose* n_1, n_2 *are positive integers such that* $(f \circ g)(n_1) = (f \circ g)(n_2)$*. Then* $n_1^2 + 1 = n_2^2 + 1$*. So* $0 = n_1^2 - n_2^2 = (n_1 + n_2)(n_1 - n_2)$*. Since* n_1, n_2 *are positive integers,* $n_1 + n_2 > 0$*, hence* $n_1 = n_2$ *and* $f \circ g$ *is one-to-one.*

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Let $f : \mathbb{N} \to \mathbb{N}$ *with* $f(n) = n + 1$ *and* $g : \mathbb{N} \to \mathbb{N}$ *with* $g(n) = n^2$. *Find the formula of* $f \circ g$. *Is it one-to-one? Is it onto?*

Solution

For every $n \in \mathbb{N}$, $(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$. *To check the one-to-one property, suppose* n_1, n_2 *are positive integers such that* $(f \circ g)(n_1) = (f \circ g)(n_2)$. *Then* $n_1^2 + 1 = n_2^2 + 1$. *So* $0 = n_1^2 - n_2^2 = (n_1 + n_2)(n_1 - n_2)$. *Since* n_1, n_2 *are positive integers,* $n_1 + n_2 > 0$, *hence* $n_1 = n_2$ *and* $f \circ g$ *is one-to-one. To check the onto property, note that not many integers are of the form* $n^2 + 1$ *with* $n \in \mathbb{Z}$. *So it is easy to find an example not in the range. Suppose* $n^2 + 1 = 3$, *then* $n^2 = 2, n = \pm\sqrt{2}$, *which is not an integer. So* 3 *is in the co-domain but not in the range of* $f \circ g$ *and* $f \circ g$ *is not onto.*

Identity functions

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Remark

Identity functions have equal domain and co-domain, and they map every element in the domain to itself. So they are one of the simplest functions.

Composition with identity functions

Proposition

Let $f : X \to Y$ be a function and $I_X : X \to X$, $I_Y : Y \to Y$ be the identity functions. Then

$$f \circ I_X = I_Y \circ f = f.$$

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Let $f : X \to Y$ be a function and $I_X : X \to X$, $I_Y : Y \to Y$ be the identity functions. Then

$$f \circ I_X = I_Y \circ f = f.$$

Proof.

For any $x \in X$, we have

$$(f \circ I_X)(x) = f(I_X(x)) = f(x), (I_Y \circ f) = I_Y(f(x)) = f(x).$$

$\square$

Composition of inverse functions

Theorem

If $f : X \to Y$ is a bijection with inverse function f^{-1}, then

$$f \circ f^{-1} = I_Y, f^{-1} \circ f = I_X.$$

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Proof.

Just note that $f(f^{-1}(y)) = y$ for all $y \in Y$ and $f^{-1}(f(x)) = x$ for all $x \in X$. $\square$

Composition of one-to-one functions

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be one-to-one functions. Then $g \circ f : X \to Z$ is also one-to-one.

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Proof.

By definition, it suffices to show that for any two elements $x_1, x_2 \in X$, if $(g \circ f)(x_1) = (g \circ f)(x_2)$, then $x_1 = x_2$. Since $g(f(x_1)) = g(f(x_2))$ and g is one-to-one, we deduce that $f(x_1) = f(x_2)$. Now since f is one-to-one, we deduce that $x_1 = x_2$. $\square$

Composition of onto functions

Theorem

Let $f : X \to Y$ and $g : Y \to Z$ be onto functions. Then $g \circ f : X \to Z$ is also onto.

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Let $f : X \to Y$ and $g : Y \to Z$ be onto functions. Then $g \circ f : X \to Z$ is also onto.

Proof.

By definition, it suffices to show that for every element $z \in Z$, there exists an element $x \in X$ such that $(g \circ f)(x) = z$, which is $g(f(x)) = z$. Since g is onto, there exists an element $y \in Y$ such that $g(y) = z$. Next since f is onto, there exists an element $x \in X$ such that $f(x) = y$. Then $g(f(x)) = g(y) = z$. $\square$

Similar statements

Exercise

Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ and f are both one-to-one, what about g?

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Solution

g is not necessarily one-to-one, because Y may contain many elements that are not in the range of f, so they won't destroy the one-to-one property of $g \circ f$ while they could do for g.

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Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ and f are both one-to-one, what about g?

Solution

g is not necessarily one-to-one, because Y may contain many elements that are not in the range of f, so they won't destroy the one-to-one property of $g \circ f$ while they could do for g.
A explicit counterexample: $f : \mathbb{N} \to \mathbb{Z}$ with $f(n) = n$; $g : \mathbb{Z} \to \mathbb{Z}$ with $g(n) = n^2$. Then $(g \circ f)(n) = n^2$ for all $n \in \mathbb{N}$ so it's one-to-one, while g is not.

Exercise: computing inverse functions

Exercise

Let $H : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ *be the function with* $H(x) = \frac{x+1}{x-1}$ *for all* $x \in \mathbb{R} - \{1\}$. *Find its inverse function* H^{-1}, *and verify that the composition* $H \circ H^{-1}$ *is indeed an identity function.*

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Solution

For each y *in* $\mathbb{R} - \{1\}$, *we would like to find the unique* $x \in \mathbb{R} - \{1\}$ *such that* $H(x) = y$. *Then* $\frac{x+1}{x-1} = y$. *So* $x + 1 = (x - 1)y = xy - y$. *Then* $y + 1 = xy - x$. *Note that* $y \neq 1$, *so* $x = \frac{y+1}{y-1}$. *So* $H^{-1}(y) = \frac{y+1}{y-1}$, *and it turns out that* $H^{-1} = H$.

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Let $H : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ be the function with $H(x) = \frac{x+1}{x-1}$ for all $x \in \mathbb{R} - \{1\}$. Find its inverse function H^{-1}, and verify that the composition $H \circ H^{-1}$ is indeed an identity function.

Solution

For each y in $\mathbb{R} - \{1\}$, we would like to find the unique $x \in \mathbb{R} - \{1\}$ such that $H(x) = y$. Then $\frac{x+1}{x-1} = y$. So $x + 1 = (x-1)y = xy - y$. Then $y + 1 = xy - x$. Note that $y \neq 1$, so $x = \frac{y+1}{y-1}$. So $H^{-1}(y) = \frac{y+1}{y-1}$, and it turns out that $H^{-1} = H$.For the second question, it suffices to show that $H(H(x)) = x$ for all $x \neq 1$. In fact

$$H(H(x)) = H\left(\frac{x+1}{x-1}\right) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{2x}{2} = x.$$

HW# 10 of this section

Exercise 5, 8(b), 17, 22.