Math 2603 - Lecture 25
Section 13.2 Coloring Graphs

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Coloring of graphs

Motivating example: final exam schedule

Remark

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Example

Suppose the curriculum of 4 students are shown in the table below:

Name	*Courses*
Alice	*Calculus, Discrete Math*
Bob	*Algebra, Calculus, Number Theory*
Carol	*Discrete Math, Number Theory, Topology*
Dave	*Algebra, Topology*

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Name	Courses
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How many different time slots it takes to arrange the final exams of the courses?

Reformulate the problem

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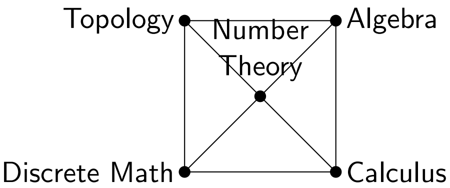

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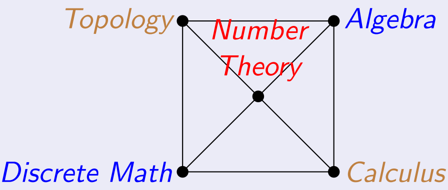
Since some student has 3 courses, we need at least 3 time slots. 3 is enough, as illustrated by the colors.



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There are other situations where we want distinct colors for neighbors - maps. If we treat different regions in a map as vertices, and two vertices are connected if and only if the regions are adjacent on the map, then we want to color the regions such that any two adjacent vertices have different colors.

Remark

Then we have the idea of coloring of graphs, and we systematically introduce it.

Definition

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A **coloring** *of a graph is an assignment of colors to the vertices so that adjacent vertices have different colors. An n-***coloring** *is a coloring with n colors. The* **chromatic number** *of a graph $\mathcal{G}$, denoted $\chi(\mathcal{G})$, is the minimal value of n for which an n-coloring of $\mathcal{G}$ exists.*

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In the final exam case, our graph admits a 3 *coloring, and the chromatic number of that graph is* 3*.*

Remark

Keep in mind that we always color the vertices, not the edges (which is a different story to tell).

Example: complete graphs

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Example

What is $\chi(\mathcal{K}_n)$ for positive integer $n \geq 3$?

Solution

On one hand, n colors are enough - just color all vertices with distinct colors; one the other hand, if we have less than n colors, by Pigeonhole Principle, there exist two vertices with the same color, a contradiction! So $\chi(\mathcal{K}_n) = n$.

Example: bipartite graphs

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Solution

First, 2 *colors are enough, just assign different colors to the different parts of the vertices. Then the endpoints of any edge have distinct colors. Since there is at least one edge,* 1 *color may not work. So the chromatic number is* 2.

Example: cycles

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Apparently we need any least 2 colors. If there are exactly 2 colors, then any pair of adjacent vertices must have distinct colors, so the coloring is "alternating". This works when n is even, but not when n is odd, and in that case we need one more color. Hence

$$\chi(\mathcal{C}_n) = \begin{cases} 2, & \text{if } 2 \mid n; \\ 3, & \text{if } 2 \nmid n; \end{cases}.$$

An upper bound of $\chi(\mathcal{G})$

Theorem

Let $\Delta(\mathcal{G})$ *be the maximum of the degrees of the vertices of a graph* $\mathcal{G}$*. Then* $\chi(\mathcal{G}) \leq 1 + \Delta(\mathcal{G})$.

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Theorem

Let $\Delta(\mathcal{G})$ be the maximum of the degrees of the vertices of a graph $\mathcal{G}$. Then $\chi(\mathcal{G}) \leq 1 + \Delta(\mathcal{G})$.

Remark

The intuition is: for any vertex, it is adjacent to at most $\Delta(\mathcal{G})$ other vertices. In the worst case, its neighbors use up $\Delta(\mathcal{G})$ colors. Hence if there are more than this number of colors, we can always find an unused color for the chosen vertex.

The Proof

Proof.

We apply induction on the number of vertices in $\mathcal{G}$. The basis step is trivial. Suppose the statement is true when $\mathcal{G}$ has k vertices ($k \in \mathbb{N}$) and suppose now $\mathcal{G}$ has $(k+1)$ vertices.

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The Four Color Theorem

The problem

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Remark

It may take at least 4 colors: there is a subgraph K_4 of Luxembourg, Belgium, France, and Germany.

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The graphs corresponding to maps are all planar. So we have the following conjecture.

Conjecture

If $\mathcal{G}$ is a planar graph, then $\chi(\mathcal{G}) \leq 4$.

A weaker result

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If $\mathcal{G}$ is a planar graph, then there exists a vertex of $\mathcal{G}$ whose degree is at most 5.

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Lemma

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Proof of lemma.

Otherwise, each degree is at least 6, by Euler's formula, $2E = \sum_v \deg(v) \geq 6V$, and $E \geq 3V$, a contradiction to the inequality $E \leq 3V - 6$. $\square$

Sketch of proof

Proof.

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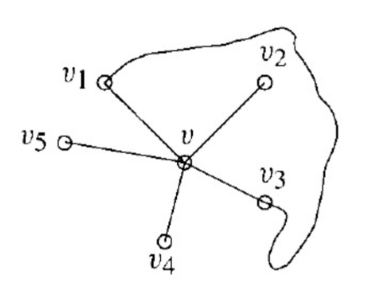
Proof.



Finally we want to adjust the colors of the v_i to spare a color for v. If there is no path from v_1 and v_3 only containing colors 1 and 3, we can do it.

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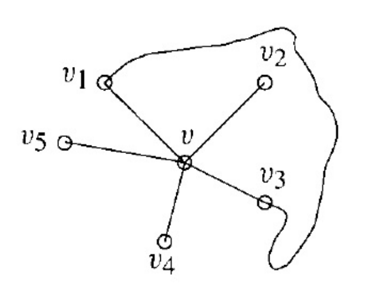

Finally we want to adjust the colors of the v_i to spare a color for v. If there is no path from v_1 and v_3 only containing colors 1 and 3, we can do it. If there is such a path, then it separates v_2 and v_4, and thus no such a path connecting v_2 and v_4, and we can still do the adjustment.

□

The theorem

Remark

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Theorem (Four color theorem)

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Theorem (Four color theorem)

If $\mathcal{G}$ is a planar graph, then $\chi(\mathcal{G}) \leq 4$.

Remark

Later scholars made progress to reduce the number of cases, but it's still open to find a manual proof of the theorem.

Chromatic polynomials

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Remark

When $k < \chi(\mathcal{G})$, $P(\mathcal{G}, k) = 0$.

Example: complete graphs

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$P(\mathcal{K}_n, k) = P(k, n) = k(k-1)(k-2)\cdots(k-n+1)$.

Proof.

Every coloring of $\mathcal{K}_n$ involves an order sequence of n distinct colors. So the number is the number of n-permutations among k elements. $\square$

The nice property

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Remark

$P(\mathcal{G}, x)$ could be recursively computed from the ones of graphs homeomorphic to it.

Homework Assignment # 14 - today

Section 13.2 Exercise 2, 4(d)(f), 7, 12, 17.