Math 207, Final Exam, Peterson
December 15, 2016

Name: Key

This exam has seven pages. You may *not* use calculators or similar electronic devices on this exam. Show your work. Let me know if the wording of any part of this exam is unclear.

1. (10 points) Find the standard matrix of the following linear transformation:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ x + 3y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} -y \\ 3y \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

matrix: $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

2. (10 points) Consider the homogeneous system of linear equations given below. For convenience, the general solution to this system is given. It can be shown that the solutions of this system form a subspace W of $\mathbf{R}^4$. (Do *not* show that the solutions form a subspace.) Find a basis for the subspace of solutions. Give the dimension of the subspace.

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & x_3 & - & x_4 & = & 0 \\ x_1 & - & 2x_2 & & & + & x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & x_3 & - & x_4 & = & 0 \end{array}$$

The general solution is $(2r - s, r, 3s, s)$.

$$\begin{bmatrix} 2r-s \\ r \\ 3s \\ s \end{bmatrix} = \begin{bmatrix} 2r \\ r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ 3s \\ s \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$, dimension $= 2$

3. (9 points) The following matrices are singular because of certain column or row properties. Give reason.

(a) $\begin{bmatrix} 3 & 6 & 0 \\ 4 & 0 & 0 \\ 7 & 9 & 0 \end{bmatrix}$ Column 3 is all 0's

(b) $\begin{bmatrix} 2 & 4 & 3 \\ 4 & 8 & 6 \\ 9 & 3 & -7 \end{bmatrix}$ Row 2 = 2 × Row 1.

(c) $\begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 7 \\ -1 & 1 & 8 \end{bmatrix}$ Column 2 = −1 × column 1.

4. (13 points) Determine the inverse of the following 3×3 matrix, if it exists. Do so by using the method of Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 2 \\ -1 & 3 & 7 \end{bmatrix}$$

$$\left[\begin{array}{cccccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & 3 & 7 & 0 & 0 & 1 \end{array}\right] \quad R_3 + R_1$$

$$\left[\begin{array}{cccccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + 2R_2 \\ R_3 + (-1)R_2 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \quad R_2 + (-2)R_3$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Inverse: $\begin{bmatrix} 1 & 2 & 0 \\ -2 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

5. (9 points) Suppose that A and B are 3×3 matrices, and suppose that $|A| = 4$ and $|B| = 6$. Compute the following determinants:

(a) $|AB|$ $= |A| \cdot |B| = 4 \cdot 6 = 24$

(b) $|A^{-1}B|$ $= |A^{-1}| \cdot |B| = \frac{1}{|A|} \cdot |B| = \frac{6}{4} = \frac{3}{2}$

(c) $|2A|$ $= 2^3 |A| = 8 \cdot |A| = 8 \cdot 4 = 32$

6. (15 points) Determine whether the first vector is a linear combination of the other vectors. Show your work.

$$(1,4,7);\ (1,0,2),\ (0,2,1),\ (2,4,3)$$

we want $c_1\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2\begin{bmatrix}0\\2\\1\end{bmatrix} + c_3\begin{bmatrix}2\\4\\3\end{bmatrix} = \begin{bmatrix}1\\4\\7\end{bmatrix}$

$$\begin{aligned} c_1 \quad\quad + 2c_3 &= 1 \\ 2c_2 + 4c_3 &= 4 \\ 2c_1 + c_2 + 3c_3 &= 7 \end{aligned}$$

$\begin{bmatrix}1&0&2&1\\0&2&4&4\\2&1&3&7\end{bmatrix}$ $R_3 + (-2)R_1$

$\begin{bmatrix}1&0&2&1\\0&2&4&4\\0&1&-1&5\end{bmatrix}$ $\frac{1}{2}R_2$

$\begin{bmatrix}1&0&2&1\\0&1&2&2\\0&1&-1&5\end{bmatrix}$ $R_3 + (-1)R_2$

$\begin{bmatrix}1&0&2&1\\0&1&2&2\\0&0&-3&3\end{bmatrix}$ $-\frac{1}{3}R_3$

$\begin{bmatrix}1&0&2&1\\0&1&2&2\\0&0&1&-1\end{bmatrix}$ $R_1 + (-2)R_3$, $R_2 + (-2)R_3$

$\begin{bmatrix}1&0&0&3\\0&1&0&4\\0&0&1&-1\end{bmatrix}$

$c_1 = 3,\ c_2 = 4,\ c_3 = -1.$

Yes, the first vector is a linear combination of the other vectors.

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & 1-\lambda & 9 \\ -1 & 1 & 1-\lambda \end{vmatrix} =$$

7. (18 points) Find the characteristic polynomial, eigenvalues, and corresponding eigenspaces of the following 3×3 matrix:

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 9 \\ -1 & 1 & 1 \end{bmatrix}$$

Note: Feel free to write on the next page.

$$(3-\lambda)\begin{vmatrix} 1-\lambda & 9 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)\left((1-\lambda)(1-\lambda) - 9 \cdot 1\right)$$

$$= (3-\lambda)(1^2 - 2\lambda + \lambda^2 - 9) = (3-\lambda)(\lambda^2 - 2\lambda - 8)$$

$$= (3-\lambda)(\lambda-4)(\lambda+2) \qquad \lambda = 3,\ \lambda = 4,\ \lambda = -2$$

$\lambda = 3$: $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 9 & 0 \\ -1 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \Leftrightarrow R_2$ $\begin{bmatrix} 1 & -2 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 \end{bmatrix}$ $R_3 + R_1$

$\begin{bmatrix} 1 & -2 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{bmatrix}$ $R_1 + (-2)R_3$, $R_2 \Leftrightarrow R_3$ $\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $-1R_2$

$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 = 5x_3 = 5r$

$x_2 = 7x_3 = 7r$

$x_3 = r$

$\vec{x} = \begin{bmatrix} 5r \\ 7r \\ r \end{bmatrix}$

For $\lambda = 3$: $\left\{ r\begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix} \right\}$

$\lambda = 4$: $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -3 & 9 & 0 \\ -1 & 1 & -3 & 0 \end{bmatrix}$ $R_2 + R_1$, $R_3 + (-1)R_1$, Then $(-1)R_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix}$ $-\frac{1}{3}R_2$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix}$

$x_1 = 0$

$x_2 = 3x_3 = 3r$

$x_3 = r$

For $\lambda = 4$: $\left\{ r\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

For $\lambda = -2$:

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 1 & 3 & 9 & 0 \\ -1 & 1 & 3 & 0 \end{bmatrix} \quad \tfrac{1}{5} R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 9 & 0 \\ -1 & 1 & 3 & 0 \end{bmatrix} \quad \begin{matrix} R_2 + (-1) R_1 \\ R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 9 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \quad \tfrac{1}{3} R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \quad R_3 + (-1) R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 + 3x_3 = 0$

$x_2 = -3r, \quad x_3 = r$

For $\lambda = -2$: $\left\{ r \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

8. (16 points) Diagonalize the following matrix (if possible). Give the similarity transformation.

$$\begin{bmatrix} 1 & 6 \\ -1 & 6 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 6 \\ -1 & 6-\lambda \end{vmatrix} =$$

$$(1-\lambda)(6-\lambda) - 6(-1) = 6 - 7\lambda + \lambda^2 + 6 = \lambda^2 - 7\lambda + 12 =$$

$$(\lambda - 4)(\lambda - 3), \quad \lambda = 4, \ \lambda = 3$$

For $\lambda = 4$: $\begin{bmatrix} 1-4 & 6 & 0 \\ -1 & 6-4 & 0 \end{bmatrix}$ $\begin{bmatrix} -3 & 6 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ $-\frac{1}{3}R_1$

$\begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ $x_1 - 2x_2 = 0$ $x_1 = 2r$, $x_2 = r$

$\vec{x} = \begin{bmatrix} 2r \\ r \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For $\lambda = 3$:

$\begin{bmatrix} 1-3 & 6 & 0 \\ -1 & 6-3 & 0 \end{bmatrix}$ $\begin{bmatrix} -2 & 6 & 0 \\ -1 & 3 & 0 \end{bmatrix}$ $-\frac{1}{2}R_1$ $\begin{bmatrix} 1 & -3 & 0 \\ -1 & 3 & 0 \end{bmatrix}$

$x_1 - 3x_2 = 0$, $x_1 = 3x_2$ $x_1 = 3s$, $x_2 = s$

$\vec{x} = \begin{bmatrix} 3s \\ s \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. $\vec{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. $C = [\vec{x}_1 \ \vec{x}_2] = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

$$C^{-1} = \frac{1}{2\cdot 1 - 3\cdot 1}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$C^{-1}AC = D$

$$\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$