Lindsey Wingate
Formal Languages and Autonoma, 435
Assignment #3
Due December 7, 2016

Proof:

Assumptions -

Let G=(NT, Σ, R, S) be a Chomsky normal form grammer.

For any sentential form $\alpha \in (\Sigma \cup NT)^*$ derived from a non-terminal $N \in NT$, if the derivation tree for $N \Rightarrow^* \alpha$ has depth k, then $|\alpha| \leq 2^k$.

Fact: Any one derivation from a Chomsky normal form language rule will result in one or two terminals/non-terminals. Any one step cannot produce more than two symbols. A set of iterations may be represented by a binary tree.

Base Case: Let k (the depth of the tree) = 0.

The total number of nodes in the tree is equal to 2^0, or 1.

Proposition: If the derivation tree for $N \Rightarrow^* \alpha$ has depth k, then $|\alpha| \leq 2^k$.

Hypothesis: For any tree with depth k greater than 0, the number of nodes in the tree will be less than or equal to 2^k.

Induction:

Let A represent a binary tree of depth k+1.

Each branch off the original node may be considered its own binary tree, which also conforms to the base case of no more than 2^k leaves.

The amount of leaves that exist in the two branches off the head node must be less than or equal to $2^k + 2^k + 1$ (the head node), or $2^{k+1} + 1$.

Case 1:

In this case, the head node has two children with no grandchildren (3 nodes in the tree total). Therefore, 3 is less than 2^3.

Case 2:

In this case, the head node has two children with one grandchild (4 nodes in the tree total). Therefore, 4 is less than 2^3.

Case 3:

In this case, the head node has two children with two grandchildren (5 nodes in the tree total). Therefore, 5 is less than 2^3.

Case 4:

In this case, the head node has two children with four grandchildren, the maximum number of nodes that may be considered for the third iteration (7 nodes in the tree total). Therefore, 7 is less than 2^3.

Our hypothesis maintains that for any depth k, the total number of nodes in any binary tree is less than or equal to 2^k. $\square$