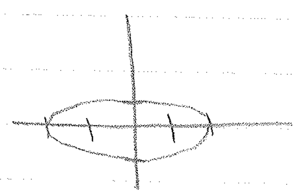
Multivariate Analysis

Def: A function of two independent variables is a correspondence that assigns to each ordered pair (x,y) in the domain a unique partner in the range.
- The domain is a subset of the xyplane, z is the dependent variable, x & y are the independent variables.
- Notation: $z = f(x,y)$
- when $z = f(x,y)$ is given by an equation, the domain will be all the points in the xy plane for which the equation (function) is defined.

Given the equation $z = f(x,y)$ be able to determine its domain.

#1 Ex $\Rightarrow$ $f(x,y) = \sqrt{4-x^2-4y^2}$

$$4-x^2-4y^2 \geq 0$$
$$4 \geq x^2+4y^2$$
$$x^2+4y^2 \leq 4$$
$$\frac{x^2}{4}+\frac{y^2}{1} \leq 1$$



$\boxed{D: -2 \leq x \leq 2}$

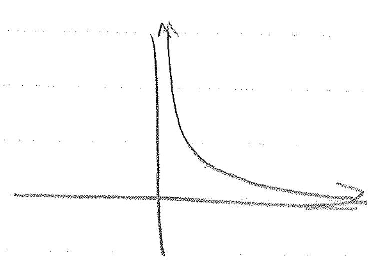
#2 Ex $\Rightarrow$ $f(x,y) = \dfrac{xy}{x-y}$

$$x-y \neq 0$$
$\boxed{D: x \neq y}$

#3 Ex $\Rightarrow$ $f(x,y) = \ln(4-xy)$

$$4-xy > 0$$
$$xy < 4$$
$$x < 4/y$$

$\boxed{D: 0 < x \leq \infty}$



Given the equation for a function of 2 (or more) variables be able to evaluate the function at a given point in its domain.

#4 Ex ⇒ $g(x,y) = \ln(x+y)$ at (e,e)

$$\ln(e+e) = \ln 2e = \ln 2 + \ln e = \boxed{\ln 2 + 1}$$

#5 Ex ⇒ $f(x,y,z) = \sqrt{x+y+z}$ at $(6,6,-3)$

$$\sqrt{6+6-3} = \boxed{3}$$

#6 Ex ⇒ $g(x,y) = \int_x^y 1/t \, dt$ at $(4,1)$

$$\int_4^1 1/t \, dt = \ln|t| \Big]_4^1 = \ln 1 - \ln 4$$
$$0 - \ln 4 = \boxed{-\ln 4}$$

Be able to solve a variety of word problems by evaluation.

#7 Ex ⇒ Doyle's Log Rule $N(d,L) = \left(\frac{d-4}{4}\right)^2 L$ gives the number of board feet N in a log having a diameter of d inches and a length of L feet. Find the number of board feet in a 12 ft. log having a diameter of 22 inches.

$$N(22,12) = \left(\frac{22-4}{4}\right)^2 (12) = \boxed{843 \text{ board feet}}$$

#8 Ex=> The value (in 1994) in dollars of a \$1000 bond purchased in 1994 earning 10% interest annually can be computed using this formula: $V(I,R) = 1000\left(\dfrac{1+0.1(1-R)}{1+I}\right)^{10}$ where V is its value, I is the inflation rate, and R is the tax rate. Find the value of a \$1000 bond if the inflation rate is 3.5% and the tax rate is 28%.

$$V(.035, 0.28) = 1000\left(\frac{1+0.1(1-.28)}{1+.035}\right)^{10} = \boxed{\$1420.84}$$

#9 Ex=> Given that the temperature T in degrees Celsius at any point (x,y) on a circular steel plate of radius 10 feet is given by $T = 600 - 0.75x^2 - 0.75y^2$,

A) What is the domain of this function? $\boxed{\mathbb{R}}$

B) What is the temperature at the center of the plate? (at pt $(0,0)$)

$$T(0,0) = \boxed{600°C}$$

c) Find the temperature at these points:

$T(10,0) = 525°C$

$T(6,8) = 525°C$

$T(5,5) = 562.5°C$

#10 Ex ⇒ The Cobb-Douglas Production Formula (Economics) where x measures the units of labor and y represents the units of capital and f(x,y) represents the number of units produced. Given $f(x,y) = 100x^{0.6}y^{0.4}$

A) Find the production level (units produced) if $x=500$ & $y=1000$ units.

$$f(500,1000) = \boxed{65{,}975.39554}$$

B) Show that if both x & y are doubled, then the output would also be doubled

$$f(1000,2000) = \boxed{131{,}950.7911}$$

c) What would happen if both x & y were tripled?

$$f(1500,3000) = \boxed{197{,}926.1866}$$

#11 Ex ⇒ A manufacturer sells a sophisticated new machine to both foreign and domestic markets. The price of the machines depends on the number of machines made available. If the manufacturer supplies x machines to the domestic market and y machines to the foreign market the machines will sell for $60 - x/5 + y/20$ thousand dollars a piece at home and $50 - x/10 + y/20$ thousand dollars each abroad.

A) Express the revenue R as a function of x and y

$$\left.\begin{aligned} R(x,y)\ \text{domestic} &= 60 - x/5 + y/20 \\ R(x,y)\ \text{foreign} &= 50 - x/10 + y/20 \end{aligned}\right\rangle R(x,y) = 110 - 3x/10 + y/10$$

B) Find the revenue if 500 machines are sold locally & 800 abroad.

$$R(500,800) = 40 \quad \therefore \quad \boxed{\$40{,}000}$$

If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ as $(x,y)\to(x_0,y_0)$ then $f(x,y)\to L$.

note: $f(x,y)$ need not be defined at (x_0,y_0) in order to have a limit there.

⇒ Given a rule for $z=f(x,y)$ be able to evaluate the limit of $f(x,y)$ as (x,y) approaches a fixed point (a,b) in the plane.

#1 Ex ⇒ $\lim\limits_{(x,y)\to(1,1)} \dfrac{x}{\sqrt{x+y}} = \boxed{\dfrac{1}{\sqrt{2}}}$

#2 Ex ⇒ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{(x^2+1)(y^2+1)} = \dfrac{0}{1} = \boxed{0}$

#3 Ex ⇒ $\lim\limits_{(x,y,z)\to(2,0,1)} xe^{yz} = 2e^{0\cdot 1} = \boxed{2}$

#4 Ex ⇒ $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2} = \dfrac{1-\cos 0}{0} = 1-\dfrac{1}{0} = \boxed{-\infty}$

#5 Ex ⇒ $\lim\limits_{(x,y)\to(0,0)} \ln(x^2+y^2) = \ln 0 = \boxed{-\infty}$

⇒ If direct substitution yields an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then try to prove that there is no limit at that point by taking the limit along two different paths and show that these limits are not equal.

#6 Ex $\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{y}{x^2+y^2} = \dfrac{0}{0}$

A) Let $(x,y)\to(0,0)$ along the x-axis $(y=0)$, $\lim \dfrac{0}{x^2} = 0$

B) Let $(x,y)\to(0,0)$ along the y-axis $(x=0)$, $\lim \dfrac{y}{y^2} = \lim \dfrac{1}{y} = \dfrac{1}{0} = \infty$

c) Let $(x,y)\to(0,0)$ along the line $y=x$, $\lim \dfrac{x}{x^2+x^2} = \lim \dfrac{1}{2x} = \dfrac{1}{0} = \infty$

$\therefore$ $\boxed{\text{no limit}}$ $\Rightarrow$ * There is no limit if one of the paths leads to ∞ or if there are two different limits.

#7 Ex $\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{2x-y^2}{2x^2+y} = \dfrac{0}{0}$

A) Along x-axis $\lim \dfrac{2x}{2x^2} = \lim \dfrac{1}{x} = \dfrac{1}{0} = \infty$

B) Along y-axis $\lim \dfrac{-y^2}{y} = \lim -y = 0$

c) Along $y=x$ $\lim \dfrac{2x-x^2}{2x^2+x} = \dfrac{\cancel{x}(2-x)}{\cancel{x}(2x+1)} = 2$

D) Along the parabola $y=x^2$ $\lim \dfrac{2x-x^4}{2x^2+x^2} = \dfrac{\cancel{x}(2-x^3)}{\cancel{x}(2x+x)} = \dfrac{2}{0} = \infty$

#8 Ex ⇒ $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \frac{0}{0}$

A) along $y=mx$ $\lim \frac{3x^2mx}{x^2+(mx)^2} = \frac{x^2(3mx)}{x^2(1+m^2)} = \frac{0}{\#} = 0$

B) along $y=x^2$ $\lim \frac{3x^4}{2x^2} = \frac{3x^2}{2} = 0$

c) along $x=y^2$ $\lim \frac{3y^3}{2y^2} = \frac{3y}{2} = 0$

conclude ∴ the limit is probably at 0.

<u>Partial Derivatives</u>

Def: The process of differentiating a function of several variables with respect to <u>one</u> of these independent variables while keeping the other variables fixed is called <u>partial differentiation</u> and the resulting derivative is called a <u>partial derivative</u>.

If $z=f(x,y)$ then the partial derivative of $f(x,y)$ with respect to x is defined to be $f(x,y)_x = z_x = f_x = \frac{\partial z}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$.

With respect to y $f(x,y)_y = z_y = f_y = \frac{\partial z}{\partial y} = \lim_{\Delta y\to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$

Be able to use this formal definition to find partial derivatives.

#1 Ex => Given $f(x,y) = x^2 + 3y + 10$ find f_x and f_y

$$f_x = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 3y + 10 - (x^2 + 3y + 10)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 \cancel{+3y} \cancel{+10} \cancel{-x^2} \cancel{-3y} \cancel{-10}}{\Delta x}$$

$$\lim_{\Delta x \to 0} 2x + \Delta x = \boxed{2x}$$

$$f_y = \lim_{\Delta y \to 0} \frac{x^2 + 3(y + \Delta y) + 10 - (x^2 + 3y + 10)}{\Delta y}$$

$$\lim_{\Delta y \to 0} \frac{\cancel{x^2} \cancel{+3y} + 3\Delta y \cancel{+10} \cancel{-x^2} \cancel{-3y} \cancel{-10}}{\Delta y}$$

$$\lim_{\Delta y \to 0} 3 = \boxed{3}$$

Ex => Find f_x and f_y for each of the following functions:

#2 A) $f(x,y) = x^2 + 3y^2 + 7$ $\qquad f_x = 2x \qquad f_y = 6y$

#3 B) $f(x,y) = x^2 - 3xy + y^2$ $\qquad f_x = 2x - 3y \qquad f_y = -3x + 2y$

#4 c) $z = xe^{x/y}$

$$\frac{\partial z}{\partial x} = xe^{x/y}(1/y) + e^{x/y}(1) = \frac{xe^{x/y}}{y} + e^{x/y}$$

$$\frac{\partial z}{\partial y} = xe^{x/y}(-x/y^2) + e^{x/y}(0) = \frac{-x^2 e^{x/y}}{y^2}$$

#5 D) $z = \ln\sqrt{xy} = \frac{1}{2}\ln x + \frac{1}{2}\ln y$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\left(\frac{1}{x}\right) = \frac{1}{2x} \qquad \frac{\partial z}{\partial y} = \frac{1}{2}\left(\frac{1}{y}\right) = \frac{1}{2y}$$

#10 E) $f(x,y) = \sin(3x)\cos(3y)$

$$f_x = \sin(3x)(0) + \cos(3y)\cos(3x)(3) = 3\cos 3x \cos 3y$$

$$f_y = \sin 3x(-\sin(3y)(3)) + \cos(3y)(0) = -3\sin 3x \sin 3y$$

Ex⇒ Evaluate these partial derivatives at the given point:

#7 A) $f(x,y) = \sin^{-1}(xy)$ Evaluate f_x at $(1,0)$

$$f_x = \frac{1}{\sqrt{1+(xy)^2}}\, y \Rightarrow \frac{y}{\sqrt{1-(xy)^2}} \ @(1,0) = \frac{0}{\sqrt{1+0}} = \boxed{0}$$

#4 B) $f(x,y) = \dfrac{4xy}{\sqrt{x^2+y^2}}$ Evaluate f_x at $(1,0)$

$$f_x = \frac{(x^2+y^2)^{1/2}(4y) - (4xy)(1/2)(x^2+y^2)^{-1/2}(2x)}{x^2+y^2}$$

$$= \frac{4y\sqrt{x^2+y^2} - \dfrac{4x^2y}{\sqrt{x^2+y^2}}}{x^2+y^2} \ @(1,0) = \frac{0-0}{1} = \boxed{0}$$

Ex => Find these partial derivatives implicitly. Assume z is defined "implicitly" as a function of x & y.

#9 A) $x^2 + 5xy + z^3 = 1$ Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$

$$2x + 5y + 3z^2 \frac{dz}{dx} = 0$$

$$3z^2 \frac{dz}{dx} = \boxed{\frac{-2x-5y}{3z^2}}$$

$$0 + 5x + 3z^2 \frac{dz}{dy} = 0$$

$$\cancel{3z^2} \frac{dz}{dy} = \boxed{\frac{-5x}{3z^2}}$$

#10 B) $3x^2 + 4y^2 + 2z^2 = 5$ Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$

$$6x + 0 + 4z \frac{dz}{dx} = 0$$

$$\cancel{4z} \frac{dz}{dx} = \frac{-6x}{4z} = \boxed{\frac{-3x}{2z}}$$

$$0 + 8y + 4z \frac{dz}{dy} = 0$$

$$\cancel{4z} \frac{dz}{dy} = \frac{-8y}{4z} = \boxed{\frac{-2y}{z}}$$

Ex ⇒ Find the following higher-order or mixed partial derivatives.
Given $z = x^4 - 3x^2y^2 + y^4$, find:

#11 A) $\frac{\partial^2 z}{\partial x^2} \Rightarrow 4x^3 - 6y^2x \Rightarrow \boxed{12x^2 - 6y^2}$

#12 B) $\frac{\partial^2 z}{\partial y^2} \Rightarrow -6x^2y + 4y^3 \Rightarrow \boxed{-6x^2 + 12y^2}$

#13 C) $\frac{\partial^2 z}{\partial x \partial y} \Rightarrow 4x^3 - 6y^2x \Rightarrow \boxed{-12xy}$

#14 D) $\frac{\partial^2 z}{\partial y \partial x} \Rightarrow -6x^2y + 4y^3 \Rightarrow \boxed{-12xy}$

#15 Ex ⇒ Given $f(x,y,z) = x^2 - 3xy + 4yz + z^3$
show that $f_{xyz} = f_{yxz} = f_{yzx}$

$f_{xyz} \Rightarrow f_x = 2x - 3y$
$f_{xy} = -3$
$f_{xyz} = 0$

$f_{yxz} \Rightarrow f_y = -3x + 4z$
$f_{yx} = -3$
$f_{yxz} = 0$

$f_{yzx} \Rightarrow f_y = -3x + 4z$
$f_{yz} = 4$
$f_{yzx} = 0$

#16 Ex ⇒ If $z = xe^{x^2y}$, evaluate $\frac{\partial z}{\partial x}$ at $(1, \ln 2)$

$$\frac{\partial z}{\partial x} = xe^{x^2y}(2xy) + e^{x^2y}(1) = 2x^2ye^{x^2y} + e^{x^2y}$$

$$@\ (1, \ln 2) = 2(1)(\ln 2)e^{(1)\ln 2} + e^{(1)\ln 2}$$
$$= 2\ln 2\, e^{\ln 2} + e^{\ln 2}$$
$$= \ln 4\,(2) + 2$$
$$= \boxed{\ln 16 + 2}$$

#17 Ex ⇒ Given $h(x,y) = x^2 - y^2$ find the slope of the surface in the x direction and also in the y direction at the point $(-2,1,3)$ on this surface.

$$h_x = 2x \ @ \ (-2,1,3) = \boxed{-4}$$

$$h_y = -2y \ @ \ (-2,1,3) = \boxed{-2}$$

#18 Ex ⇒ Given $z = \sqrt{49 - x^2 - y^2}$ find the slope of this surface in the y direction at the point $(2,3,6)$.

$$\frac{dz}{dy} = \tfrac{1}{2}(49 - x^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{49 - x^2 - y^2}} \ @ \ (2,3,6) = \frac{-3}{\underbrace{\sqrt{49 - 4 - 9}}_{36}} = \boxed{\frac{-1}{2}}$$

#19 Ex ⇒ Given $z = \tfrac{1}{2}\sin(2x - y)$ find the slope of this surface in the x direction at the point $(\pi/2, \pi/3, \sqrt{3}/4)$ on this surface.

$$\frac{dz}{dx} = \tfrac{1}{2}\cos(2x-y)(2) = \cos(2x-y) \ @ \ (\pi/2, \pi/3, \sqrt{3}/4) = \cos\left(2(\pi/2) - \pi/3\right)$$
$$= \cos(2\pi/3) = \boxed{-1/2}$$

#20 Ex ⇒ Given that the temperature at a point (x,y) on a metal plate in the xy plane is given by $T(x,y) = x^3 + 2xy^2 + y$ degrees.

A) Find the temperature at the pt $(2,1)$ ⇒ $2^3 + (2)(2)(1) + 1 = \boxed{13°}$

B) Find the rate at which the temperature changes as you move "up" (parallel to the y-axis) from this point.

$T_y = 4xy + 1 \ @ \ (2,1) = 4(2)(1) + 1 = \boxed{9°}$

c) Find the rate-of-change in the temperature as you move to the right from this point. $T_x = 3x^2 + 2y^2 \ @ \ (2,1) = 3(4) + 2(1) = \boxed{14°}$

#21 Ex ⇒ The surface of a certain lake is represented by a region in the xy plane such that the depth under the point corresponding to (x,y) is given by the equation $f(x,y) = 300 - 2x^2 - 3y^2$. If a water skier is in the water at the point (x,y), find the rate-of-change in the depth in the direction of the x-axis.

$$\boxed{f_x = -4x}$$

#22 Ex ⇒ The kinetic energy of a body with a mass m and velocity v is given by $K = \frac{1}{2}mv^2$. Show that $\frac{dK}{dm}\frac{d^2k}{dv^2} = K$

$$\frac{dK}{dm} = \frac{1}{2}v^2 \qquad \frac{d^2k}{dv^2} \Rightarrow K_v = mv, \quad K_{vv} = m \qquad \therefore \left(\tfrac{1}{2}v^2\right)(m) = K \quad \underline{\text{true}}$$

#23 Ex ⇒ Suppose the electric potential V at a point (x,y,z) is given by $V = \frac{100}{x^2+y^2+z^2}$ when V is in volts and x, y, z are in inches.
Find the instantaneous rate-of-change of V wrt distance at the point (2,-1,1) in the direction of:

A) the x-axis $V_x = -100(x^2+y^2+z^2)^{-2}(2x) = \frac{-200x}{(x^2+y^2+z^2)^2}$ @ (2,-1,1) $= \frac{-400}{\underbrace{(4+1+1)^2}_{36}} = \boxed{\frac{-100}{9} \text{ in}}$

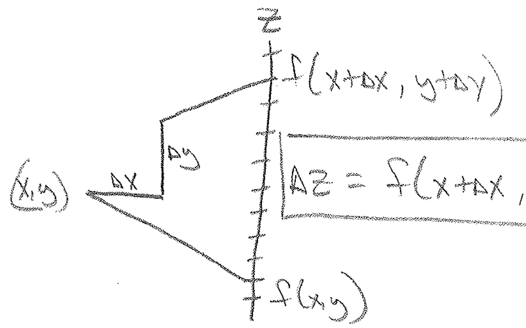
B) the y-axis $V_y = \frac{-200y}{(x^2+y^2+z^2)^2}$ @ (2,-1,1) $= \frac{200}{36} = \boxed{\frac{50}{9} \text{ in}}$

C) the z-axis $V_z = \frac{-200z}{(x^2+y^2+z^2)^2}$ @ (2,-1,1) $= \frac{-200}{36} = \boxed{\frac{-50}{9} \text{ in}}$

Differentials

Consider $z=f(x,y)$ with the point (x,y) being in its domain. Suppose we change the x coordinate by the amount $\overline{\Delta x}$ and the y coordinate by the amount $\overline{\Delta y}$ then these changes in x and y will bring about a change in the dependent variable, too $(\overline{\Delta z})$.

If $z=f(x,y)$



$$\boxed{\Delta z = f(x+\Delta x,\ y+\Delta y) - f(x,y)}$$

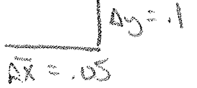
Ex ⇒ Given $z = 3x^2 - xy$

\#1 A) find $\overline{\Delta z}$ ⇒ $3(x+\Delta x)^2 - (x+\Delta x)(y+\Delta y) - (3x^2 - xy)$

$$= 3x^2 + 6x\overline{\Delta x} + 3\overline{\Delta x}^2 - xy - x\overline{\Delta y} - \overline{\Delta x}\,y - \overline{\Delta x}\,\overline{\Delta y} - 3x^2 + xy$$

$$= 6x\overline{\Delta x} + 3\overline{\Delta x}^2 - x\overline{\Delta y} - y\overline{\Delta x} - \overline{\Delta x}\,\overline{\Delta y}$$

\#2 B) Evaluate $\overline{\Delta z}$ if $\overline{\Delta x} = 0.05$ and $\overline{\Delta y} = 0.1$ are the incremental changes from the point $(2,3)$ in the domain of this function.



$$z(2,3) = 3(4) - (2)(3) = 6$$
$$z(2.05, 3.1) = 6.2525$$
$$\overline{\Delta z} = 6.2525 - 6 = 0.2525$$

$$\Delta z = 6(2)(.05) + 3(.05)^2 - 2(.1) - (3)(.05) - (.05)(.1) = \boxed{0.2525}$$

Differentials are used to approximate the change in the dependent variable (Δz) brought about by changes in the independent variables (Δx and Δy).

Def: $dx = \Delta x$, the differential of x is the same as the increment in x
$dy = \Delta y$, the differential of y is the same as the increment in y
but dz (the total differential of the dependent variable) is defined to be $\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$

$\Rightarrow$ In $z = f(x,y)$

$$\left.\begin{array}{l} dx = \Delta x \\ dy = \Delta y \end{array}\right\} \text{independent variables}$$

$$\underset{\text{approximate change}}{dz} = \underset{\text{actual change}}{\Delta z}$$

Given $z = f(x,y)$ be able to find the total differential of z using the formula $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

#1) Ex $\Rightarrow$ find the total differential for $z = \frac{x^2}{y}$.

$$\frac{\partial z}{\partial x} = \frac{2x}{y} \qquad \frac{\partial z}{\partial y} = \frac{-x^2}{y^2}$$

$$\boxed{dz = \frac{2x}{y}dx - \frac{x^2}{y^2}dy}$$

Given $z = f(x,y)$ and Δx and Δy from a given point (x_0, y_0) be able to find Δz (the actual change) and dz (the approximate change).

#2 Ex ⇒ Given $z = x^2 + 3xy - y^2$. If x changes from 2 to 2.05 and y changes from 3 to 2.96 find Δx, Δy, Δz, and dz.

$\Delta x = dx = 0.05$

$\Delta y = dy = -0.04$

$\Delta z = f(2.05, 2.96) - f(2,3) = 13.6449 - 13 = \underline{0.6449 \text{ (exact)}}$

$dz = (2x+3y)dx + (3x-2y)dy = (4+9)(.05) + (6-6)(-.04) = \underline{0.65 \text{ (approx.)}}$

#3 Ex ⇒ Given $f(x,y) = x \sin y$ find Δz and dz from the point $(1,2)$ if $\Delta x = dx = 0.5$ and $\Delta y = dy = 0.1$.

$\Delta z = f(1.5, 2.1) - f(1,2) = (1.5)(\sin 2.1) - (1)(\sin 2) = 0.3855 \text{ (exact)}$

$dz = \sin y \, dx + x \cos y \, dy = (\sin 2)(.5) + (1)(\cos 2)(.1) = \underline{0.413 \text{ (approx)}}$

#4 Ex ⇒ Suppose a can 12 cm tall and having a radius of 3 cm is reduced in size by reducing its height by 0.2 cm and reducing its radius by 0.3 cm. Estimate the change in the volume of this can.

$V = \pi r^2 h \qquad pt\,(r,h) = (3,12) \qquad \Delta r = -0.3 \qquad \Delta h = -0.2$

$dz = 2\pi r h \, dr + \pi r^2 dh = (2\pi)(3)(12)(-.3) + (\pi)(9)(-0.2) = \boxed{-73.51 \text{ cm}^3}$

-23.4π

Using differentials in <u>Error Analysis</u>

Suppose z is a quantity which is to be computed using measured values for x and y. If the measurements are "off" then there will be a propogated error in calculating z.

note: dx is the maximum error in measuring x $= \Delta x$

dy is the maximum error in measuring y $= \Delta y$

Δz is the maximum possible error in calculating z.

dz will be used to approximate this maximum propogated error due to the error in measuring x & y

note: $\frac{dx}{x}$ and $\frac{dy}{y}$ are the relative (or percentage) error in measuring x and y respectively and $\frac{dz}{z}$ is the relative error in calculating z.

#1 Ex => A box is measured to the following dimensions: $x = 50$ cm, $y = 20$ cm, $z = 15$ cm. If the possible error in measuring each dimension is 0.1 cm then:

A) estimate the propagated error in calculating the volume of this box. $V = xyz$

$$dV = yz\,dx + xz\,dy + xy\,dz$$
$$= (20)(15)(.1) + (50)(15)(.1) + (50)(20)(.1) = \boxed{205 \text{ cm}^3}$$

B) What would be the relative error in measuring x?

$$\frac{dx}{x} = \frac{.1}{50} = 0.002 = \boxed{0.2\%}$$

C) What would be the relative error in calculating the volume?

$$\frac{dV}{V} = \frac{205}{(50)(20)(15)} = 0.013\overline{6} = \boxed{1.3\overline{6}\%}$$

#2 Ex ⇒ The radius r and the height h of a right circular cylinder are measured with possible errors of 4% and 2% respectively. Approximate the maximum possible percentage error in computing the volume of this cylinder.

$$V = \pi r^2 h \qquad \frac{dr}{r} = 0.04 \qquad \frac{dh}{h} = 0.02 \qquad \text{want } \frac{dV}{V}$$

$$\frac{dV}{V} = \frac{2\pi r h\, dr + \pi r^2 dh}{\pi r^2 h} = \frac{2dr}{r} + \frac{dh}{h} = 2(.04) + 0.02 = .1$$

$$\therefore \frac{dv}{v} = \boxed{10\%}$$

Section II - Multivariate Analysis

Chain Rules for functions of several variables

Case I - The dependent variable w is a function of two intermediate variables x and y which are each a function of a single independent variable t, thus $w=f(x,y)$ and $x=g(t)$ and $y=h(t)$. Goal is to find dw/dt.

$$\begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

Thus $\boxed{\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}}$

#1 Ex ⇒ If $z = x^2y + 3xy^4$ where $x = e^t$ and $y = \sin t$

A) Draw the diagram and write the chain rule for finding dz/dt.

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

B) Find dz/dt and evaluate it when $t=0$

$$dz/dt = (2xy + 3y^4)e^t + (x^2 + 12xy^3)\cos t$$

when $t=0$ $\quad x = e^0 = 1 \quad y = \sin 0 = 0$

$\therefore \; dz/dt = (0+0)(1) + (1+0)(1) = \boxed{1}$

*2 Ex ⇒ Suppose $S = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where $x_1 = 4\cos t$, $x_2 = 2\sin 2t$, $y_1 = 2\sin t$, $y_2 = 3\cos 2t$.

A) Draw the diagram showing dependent variable S, the intermediate variables, and the independent variable.

$$\begin{tikzcd}[column sep=small] & & S \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\ x_1 \arrow[d] & x_2 \arrow[d] & y_1 \arrow[d] & y_2 \arrow[d] \\ t & t & t & t \end{tikzcd}$$

B) Write the chain rule for finding ds/dt then evaluate ds/dt when $t = \pi$.

$$\frac{ds}{dt} = \frac{ds}{dx_1}\frac{dx_1}{dt} + \frac{ds}{dx_2}\frac{dx_2}{dt} + \frac{ds}{dy_1}\frac{dy_1}{dt} + \frac{ds}{dy_2}\frac{dy_2}{dt}$$

$$= \tfrac{1}{2}\left[(x_2-x_1)^2 + (y_2-y_1)^2\right]^{-1/2}(2)(x_2-x_1)(-1)\, dx_1/dt + \tfrac{1}{2}\left[(x_2-x_1)^2 + (y_2-y_1)^2\right]^{-1/2}(2)(x_2-x_1)(1)\, dx_2/dt$$

$$+ \tfrac{1}{2}\left[(x_2-x_1)^2 + (y_2-y_1)^2\right]^{-1/2}(2)(y_2-y_1)(-1)\, dy_1/dt + \tfrac{1}{2}\left[(x_2-x_1)^2 + (y_2-y_1)^2\right]^{-1/2}(2)(y_2-y_1)(1)\, dy_2/dt$$

$x_1 = 4\cos\pi = -4$ $\quad x_2 = 2\sin 2\pi = 0$ $\quad y_1 = 2\sin\pi = 0$ $\quad y_2 = 3\cos 2\pi = 3$

$$= \tfrac{1}{2}(16+9)^{-1/2}(2)(4)(-1)(-4\sin t) + \tfrac{1}{2}(25)^{-1/2}(2)(4)(1)(2\cos 2t(2)) + \tfrac{1}{2}(25)^{-1/2}(2)(3)(-1)(2\cos t) + \tfrac{1}{2}(25)^{-1/2}(2)(3)(1)(-3\sin 2t(2))$$

$$= \frac{-8}{10}(0) + \frac{8}{10}(4) - \frac{6}{10}(-2) + \frac{6}{10}(0)$$

$$\frac{32}{10} + \frac{12}{10} = \frac{44}{10} = \boxed{\frac{22}{5}}$$

<u>Case II</u> - The dependent variable w is a function of two (or more) intermediate variables (such as x & y, etc) which are themselves functions of <u>two</u> independent variables (s & t). Thus $w = f(x,y)$ & $x = h(s,t)$ & $y = g(s,t)$. Goal is to find dw/ds or dw/dt for this case.

$$\begin{tikzcd} & & w \arrow[dll] \arrow[drr] & & \\ x \arrow[d] \arrow[dr] & & & & y \arrow[dl] \arrow[d] \\ s & t & & s & t \end{tikzcd}$$

$$\boxed{\text{Thus } \frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}}$$ and

#1 Ex ⇒ Find dw/ds and dw/dt for $w = 2xy$ where $x = s^2 + t^2$ and $y = s/t$ using the chain rule.

$$\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds}$$

$$= 2y(2s) + 2x(1/t) = 2(s/t)(2s) + 2(s^2+t^2)(1/t) = \frac{4s^2}{t} + \frac{2s^2+2t^2}{t} = \boxed{\frac{6s^2+2t^2}{t}}$$

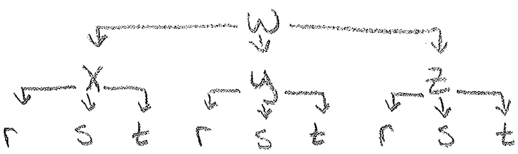
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= 2y(2t) + 2x(-s/t^2) = 2(s/t)(2t) + 2(s^2+t^2)(-s/t^2) = 4s\frac{(t^2)}{(t^2)} - \frac{2s^3}{t^2} - \frac{2st^2}{t^2}$$

$$= \boxed{\frac{2st^2 - 2s^3}{t^2}}$$

#2 Ex $\Rightarrow$ If $w = x^4y + y^2z^3$ where $x = rse^t$ and $y = rs^2e^{-t}$ and $z = r^2s(\sin t)$ then:

A) Sketch a diagram showing the dependent variable w, the intermediate variables, and the independent variables.



B) Write the chain rule for finding dw/ds then evaluate dw/ds when $r=2$, $s=1$, and $t=0$.

$$\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} + \frac{\partial w}{\partial z}\frac{dz}{ds}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

$x = (2)(1)(e^0) = 2 \qquad y = (2)(1)(e^{-0}) = 2 \qquad z = (4)(1)(0) = 0$

$$= 4(8)(2)(2)(1) + (16 + 2(0))(2)(2)(1)(1) + (3)(4)(0)$$

$$= 128 + 64 + 0 = \boxed{192}$$

#3 Ex $\Rightarrow$ Find dw/ds and dw/dt when $s=1$ and $t=2\pi$ for the function given by $w = xy + yz + xz$ where $x = s\cos t$, $y = s\sin t$, $z = t$.

$$\frac{dw}{ds} = (y+z)\cos t + (x+z)\sin t + (y+x)0$$

$$x = 1\cos 2\pi = 1 \quad y = 1\sin 2\pi = 0 \quad z = 2\pi$$

$$= 2\pi(1) + (1+2\pi)(0) + 0 = \boxed{2\pi}$$

$$\frac{dw}{dt} = (y+z)(-s\sin t) + (x+z)s\cos t + (y+x)1$$

$$= 2\pi(0) + (1+2\pi)(1) + (0+1)(1) = 1 + 2\pi + 1 = \boxed{2+2\pi}$$

<u>Directional Derivatives</u>

Theorem: If f is a differentiable function of x & y, then the directional derivative of f in the direction of the unit vector $u = \cos\theta\, i + \sin\theta\, j$ is $D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$.

From (x,y,z) we want to be able to find the rate-of-change in z wrt a unit change in any direction (slope in any direction).

From a point (x,y) in the direction given by u we want to be able to find the value of a directional derivative.

#1 Ex $\Rightarrow$ Find the directional derivative of $f(x,y) = 4 - x^2 - \frac{1}{4}y^2$ at $(1,2)$ in the direction of $u = \cos\frac{\pi}{3}\, i + \sin\frac{\pi}{3}\, j$.

$$D_u f(x,y) = (-2x)\cos\theta + \left(-\tfrac{1}{2}y\right)\sin\theta$$

$$\theta = \pi/3, \quad x = 1, \quad y = 2$$

$$= (-2)(1)\left(\tfrac{1}{2}\right) + \left(-\tfrac{1}{2}\right)(2)\left(\tfrac{\sqrt{3}}{2}\right) = -1 - \tfrac{\sqrt{3}}{2} \approx \boxed{-1.866}$$

Ex ⇒ Find the directional derivative of the following:

#2 A) $f(x,y) = x^2 \sin 2y$ at $(1, \pi/2)$ in the direction of $v = 3i - 4j$

$$u = \frac{\langle 3,-4 \rangle}{\sqrt{9+16}} = \tfrac{3}{5}i - \tfrac{4}{5}j = \cos\theta\, i + \sin\theta\, j$$

$$D_u f(x,y) = (2x\sin 2y)\cos\theta + (2x^2\cos 2y)\sin\theta$$
$$= (2)(1)(\sin\pi)(3/5) + (2)(1)(\cos 2\pi)(-4/5)$$
$$0 + (-2)(-4/5) = \boxed{8/5}$$

#3 B) $f(x,y) = x^3 - 3xy + 4y^2$ at $(1,2)$ in the direction of $u = \cos\frac{\pi}{6} i + \sin\frac{\pi}{6} j$

$$D_u f(x,y) = (3x^2 - 3y)(\cos \pi/6) + (-3x + 8y)(\sin \pi/6)$$
$$= (-3)(\sqrt{3}/2) + (13)(1/2) \approx \boxed{3.9019}$$

#4 c) $f(x,y) = x^3 y^2$ at $(-1,2)$ in the direction of $V = 4i - 3j$

$$u = \frac{\langle 4,-3 \rangle}{\sqrt{16+9}} = \tfrac{4}{5}i - \tfrac{3}{5}j$$

$$D_u f(x,y) = (3x^2y^2)(4/5) + (2x^3y)(-3/5)$$
$$= (12)(4/5) + (-4)(-3/5) = \boxed{12}$$

Gradient of a function

Def: Let $z=f(x,y)$ be a function of x & y such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x,y)$ is the vector $\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$. ($\nabla f$ is read "del f")

#1 Ex ⇒ Find the gradient of $f(x,y) = y\ln x + xy^2$ at the point $(1,2)$.

$$\nabla f(x,y) = (y/x + y^2)i + (\ln x + 2xy)j$$
$$= \boxed{6i + 4j}$$

Alternative form of the Directional Derivative

$$D_u f(x,y) = \nabla f(x,y) \cdot u$$

Ex ⇒ Find the following directional derivatives using the gradient vector.

#2 A) $f(x,y) = 3x^2 - 2y^2$ at $(-3/4, 0)$ in the direction from $P(-3/4, 0)$ to $Q(0,1)$.

$$\vec{PQ} = \langle 3/4, 1\rangle \qquad \vec{u} = \frac{\langle 3/4, 1\rangle}{\sqrt{9/16 + 1}} = \frac{\langle 3/4, 1\rangle}{5/4} = 3/5\, i + 4/5\, j$$

$$\nabla f(x,y) = (6x)i + (-4y)j = -9/2\, i + 0j$$

$$D_u f(x,y) = \langle -9/2, 0\rangle \cdot \langle 3/5, 4/5\rangle = -27/10 + 0 = \boxed{-27/10}$$

#3 B) $f(x,y) = \sin x + e^{xy}$ at $(0,1)$ in the direction of $u = 2i - j$.

$\nabla f(x,y) = (\cos x + e^{xy}(y))i + e^{xy}(x)j$

$= [1 + (1)(1)]i + (1)(0)j = 2i + 0j$

$D_u f(x,y) = \langle 2,0 \rangle \cdot \langle 2,-1 \rangle = \boxed{4}$

#4 Ex ⇒ Given $f(x,y) = x^2 - 4xy$ find:

A) $f_x = \boxed{2x - 4y}$

B) $f_y = \boxed{-4x}$

c) $\nabla f(x,y) = \langle 2x - 4y, -4x \rangle$

D) $D_u f(1,2)$ in the direction of $V = \langle -3, 4 \rangle$

$\nabla f = \langle -6, -4 \rangle$ $u = \langle -3/5, 4/5 \rangle$ $D_u f(x,y) = 18/5 - 16/5 = \boxed{2/5}$

#5 Ex ⇒ Given $f(x,y,z) = x^2 + yz - 2xy - z^2$ find:

A) $\nabla f(x,y,z)$ at pt $(2,1,3)$

$= \langle 2x - 2y, z - 2x, y - 2z \rangle = \boxed{\langle 2, -1, -5 \rangle}$

B) Evaluate $D_u f(2,1,3)$ in the direction of $U = 2i - 2j + k$

$= \langle 2,-1,-5 \rangle \cdot \langle 2,-2,1 \rangle = 4 + 2 - 5 = \boxed{1}$

check ~~C)~~ Evaluate $D_u f(2,1,3)$ in the direction of $\nabla f(2,1,3)$ $(u = \nabla f)$

$= \langle 2,-1,-5 \rangle \cdot \langle 2,1,3 \rangle = 4 - 1 - 15 = \boxed{-12}$

The maximum value of a directional derivative $D_u f(x,y)$ will occur when the direction vector $\vec{u}$ is in the same direction as the gradient vector $\nabla f(x,y)$. The maximum value will be the length of this gradient vector $||\nabla f(x,y)||$.

#1 Ex ⇒ Given $f(x,y) = 2x^2 + 3xy + 4y^2$, find the maximum value of the directional derivative at the point $(1,1)$.

$$\nabla f = \langle 4x+3y,\ 3x+8y \rangle = \langle 7, 11 \rangle \Rightarrow ||\langle 7,11 \rangle|| = \sqrt{49+121} = \boxed{\sqrt{170}}$$

#2 Ex ⇒ Given $f(x,y,z) = x^2 + y^2 - 4z$ find:

A) ∇f at the pt $(2,-1,1)$

$$\nabla f = \langle 2x, 2y, -4 \rangle \Rightarrow \langle 4, -2, -4 \rangle$$

B) the directional derivative of f at the point $(2,-1,1)$ in the direction of $u = 2i - j - 3k$.

$$D_u = \langle 4,-2,-4 \rangle \cdot \langle 2,-1,-3 \rangle = 8 + 2 + 12 = \boxed{22}$$

C) the maximum value of the directional derivative at $(2,-1,1)$.

$$= \sqrt{16+4+16} = \boxed{6}$$