Computations with the elliptic Hall algebra

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Abstract

This is a short note explaining how to do computations algorithmically with the elliptic Hall algebra: arbitrary commutators of generators, and the Fock representation.

1 Elliptic Hall algebra

The elliptic Hall algebra $U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ is, in some sense, the quantum affinization of $U_{t_1,t_2}(\widehat{\mathfrak{gl}}_1)$. Its elements are indexed by elements in $\mathbb{Z}^2$.

Definition 1.1. Let $\mathbf{Z} \coloneqq \mathbb{Z}^2$ and $\mathbf{Z}^\times \coloneqq \mathbb{Z}^2 - \{(0,0)\}$.

- For $\mathbf{a} = (a_1, a_2) \in \mathbf{Z}^\times$, define

$$\deg(\mathbf{a}) \coloneqq \gcd(a_1, a_2).$$

  (This is the degree of the corresponding element in the slope subalgebra.)

- For $(\mathbf{a}, \mathbf{b}) \in (\mathbf{Z}^2)^\times$, define

$$\epsilon_{\mathbf{a},\mathbf{b}} \coloneqq \operatorname{sign}(\det(\mathbf{a}, \mathbf{b})) \in \{\pm 1\}.$$

  Let $\Delta_{\mathbf{a},\mathbf{b}}$ be the triangle with vertices $(0,0), \mathbf{a}, \mathbf{a}+\mathbf{b}$.

Also, split $\mathbf{Z}^\times = \mathbf{Z}^+ \oplus \mathbf{Z}^-$, where

$$\mathbf{Z}^+ \coloneqq \{(i,j) \in \mathbf{Z}^\times : i > 0 \text{ or } i = 0,\, j > 0\}, \quad \mathbf{Z}^- \coloneqq -\mathbf{Z}^+.$$

Put $\epsilon_{\mathbf{a}} \coloneqq \pm 1$ for $\mathbf{a} \in \mathbf{Z}^\pm$.

The idea is that for each line of rational slope passing through the origin, there is an associated slope subalgebra generated by the elements indexed by the lattice points with that slope. These slope subalgebras are exactly the quantum Heisenberg algebras $U_{t_1,t_2}(\widehat{\mathfrak{gl}}_1)$, with scaling factor coming from

$$n_d \coloneqq \frac{(1-t_1^d)(1-t_2^d)(1-\hbar^{-d})}{d}$$

where t_1, t_2 are formal variables and $\hbar \coloneqq t_1 t_2$.

Definition 1.2. The **elliptic Hall algebra** $U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ is an associative unital algebra generated over $k(t_1, t_2)$ by elements

$$\{e_{\mathbf{a}} : \mathbf{a} \in \mathbf{Z}^\times\}, \quad \{K_{\mathbf{a}} : \mathbf{a} \in \mathbf{Z}\},$$

with defining relations:

1. elements $K_{\mathbf{a}}$ are central and
$$K_{\mathbf{0}} \coloneqq 1, \quad K_{\mathbf{a}} K_{\mathbf{b}} \coloneqq K_{\mathbf{a}+\mathbf{b}}; \tag{1}$$

2. if $\mathbf{a}, \mathbf{b}$ are collinear, then
$$[e_{\mathbf{a}}, e_{\mathbf{b}}] \coloneqq \delta_{\mathbf{a}+\mathbf{b}} \frac{K_{\mathbf{a}}^{-1} - K_{\mathbf{a}}}{n_{\deg(\mathbf{a})}}; \tag{2}$$

3. if $\mathbf{a}, \mathbf{b}$ are such that $\deg(\mathbf{a}) = 1$ and $\Delta_{\mathbf{a},\mathbf{b}}$ has no interior lattice points, then
$$[e_{\mathbf{a}}, e_{\mathbf{b}}] \coloneqq -\epsilon_{\mathbf{a},\mathbf{b}} K_{\alpha(\mathbf{a},\mathbf{b})} \frac{\Psi_{\mathbf{a}+\mathbf{b}}}{n_1} \tag{3}$$
where
$$\alpha(\mathbf{a}, \mathbf{b}) \coloneqq \frac{1}{2} \left(\epsilon_{\mathbf{a}} \mathbf{a} + \epsilon_{\mathbf{b}} \mathbf{b} - \epsilon_{\mathbf{a}+\mathbf{b}}(\mathbf{a} + \mathbf{b})\right) \cdot \begin{cases} \epsilon_{\mathbf{a}} & \epsilon_{\mathbf{a},\mathbf{b}} = 1 \\ \epsilon_{\mathbf{b}} & \epsilon_{\mathbf{a},\mathbf{b}} = -1 \end{cases}$$
and the elements $\Psi_{\mathbf{a}}$ are defined by
$$\sum_{k=0}^{\infty} \Psi_{k\mathbf{a}} z^k = \exp\left(\sum_{m=1}^{\infty} n_m e_{m\mathbf{a}} z^m\right)$$
for $\mathbf{a} \in \mathbf{Z}$ such that $\deg(\mathbf{a}) = 1$.

Example 1.3. To compute $[e_{(1,1)}, e_{(-1,1)}]$ we use (3) to get

$$[e_{(1,1)}, e_{(-1,1)}] = K_{(1,-1)} \frac{\Psi_{(0,2)}}{n_1} = -\frac{n_2}{n_1} K_{(1,-1)} e_{(0,2)} - \frac{n_1}{2} K_{(1,-1)} e_{(0,1)}^2.$$

2 Ordering and convex paths

Within each slope subalgebra $U_{t_1,t_2}(\widehat{\mathfrak{gl}}_1)$ there is a canonical *ordering* of monomials, by degree. We can extend this to an ordering of monomials in the whole algebra $U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ as follows.

Firstly, the defining relation (1) for $K_{\mathbf{a}}$'s means that each monomial can be written in the form $K_{\mathbf{c}} \prod_i e_{\mathbf{a}_i}$. For the remainder of this section, we disregard the K term. Indices $\mathbf{a} \in \mathbf{Z}^\times$ of the remaining e terms have associated **slopes**

$$\mu(\mathbf{a}) \in [-\pi/2, 3\pi/2),$$

namely the angle formed by $\mathbf{a}$ and the horizontal axis.

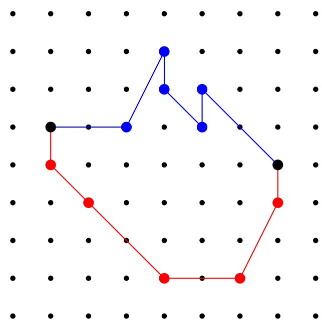
Definition 2.1. Given a sequence $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ of elements in $\mathbf{Z}^\times$, its associated **path** is the broken line in $\mathbf{Z}$ connecting

$$\mathbf{0},\ \mathbf{a}_1,\ \mathbf{a}_1 + \mathbf{a}_2,\ \ldots,\ \mathbf{a}_1 + \cdots + \mathbf{a}_n.$$

It is **convex** if slopes $\mu(\mathbf{a}_i)$ are non-decreasing, i.e.

$$-\frac{\pi}{2} \leq \mu(\mathbf{a}_1) \leq \mu(\mathbf{a}_2) \leq \cdots \leq \mu(\mathbf{a}_n) < \frac{3\pi}{2}.$$

A path p can always be made convex by sorting the $\mathbf{a}_i$ by slope. The resulting convex path is not unique, but the area bounded by it and p is well-defined; call this the **discrepancy** of p.



Clearly paths index monomials $\prod_i e_{\mathbf{a}_i}$, and sorting elements in a monomial require introducing additional monomials involving commutators $[e_{\mathbf{a}_i}, e_{\mathbf{a}_{i+1}}]$ of adjacent (non-convex) elements. For example, if $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ gives a convex path, then

$$e_{\mathbf{a}} e_{\mathbf{c}} e_{\mathbf{b}} = e_{\mathbf{a}} e_{\mathbf{b}} e_{\mathbf{c}} + e_{\mathbf{a}}[e_{\mathbf{c}}, e_{\mathbf{b}}].$$

The following section, which describes how to compute commutators of generators $e_{\mathbf{a}}$, can be viewed both as an algorithm and as a demonstration that the resulting monomials (on the rhs) all have smaller discrepancy than the original monomial (on the lhs). This effectively yields a proof of the following.

Theorem 2.2. *Monomials in e indexed by convex paths span the algebra $U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ as a module over $k(t_1, t_2, K_{\mathbf{a}})$.*

In fact, in addition it is true that such monomials are linearly independent and therefore form a *basis*.

3 Computing commutators

The procedure for computing $[e_{\mathbf{a}}, e_{\mathbf{b}}]$ for general $\mathbf{a}, \mathbf{b}$ depends on whether $\Delta_{\mathbf{a},\mathbf{b}}$ has interior points. The idea is to rewrite the desired commutator in terms of other commutators $[e_{\mathbf{a}'}, e_{\mathbf{b}'}]$ such that $\Delta_{\mathbf{a}',\mathbf{b}'}$ has smaller area (and therefore the associated monomials have smaller discrepancy). Eventually we hit the following base case, where the defining relation (3) is applicable.

Lemma 3.1. *If $\Delta_{\mathbf{a},\mathbf{b}}$ has area $1/2$, then $\deg \mathbf{a} = \deg \mathbf{b} = 1$ and it contains no interior lattice points.*

Proof. Pick's formula for a triangle in $\mathbb{Z}^2$ says

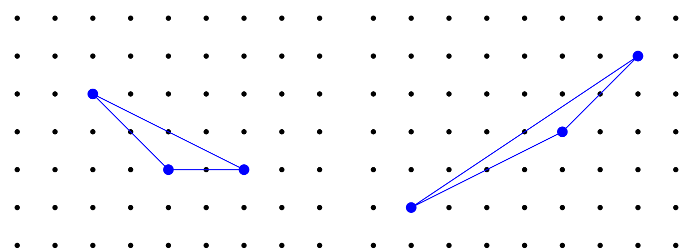
$$\text{area} = \#(\text{interior points}) + \frac{1}{2}\#(\text{boundary points}) - 1. \tag{4}$$

There are at least three boundary points, coming from the three vertices of the triangle. Hence if the lhs is $1/2$, there can be no extra interior or boundary points. $\square$

It is important to recognize that we only need to compute a commutator $[e_{\mathbf{a}}, e_{\mathbf{b}}]$ when $\mu(e_{\mathbf{a}}) > \mu(e_{\mathbf{b}})$.

3.1 No interior lattice points

Suppose $\Delta_{\mathbf{a},\mathbf{b}}$ has no interior points. It is not necessarily the case that (3) is immediately applicable; perhaps neither $\mathbf{a}$ nor $\mathbf{b}$ have degree 1.



To understand such cases, it is best to use an $\mathrm{SL}(2,\mathbb{Z})$ action on the algebra, induced by the natural $\mathrm{SL}(2,\mathbb{Z})$ action on indices $\mathbf{a} \in \mathbf{Z}$.

Lemma 3.2. *The $\mathrm{SL}(2,\mathbb{Z})$ action preserves degrees, areas of triangles, and whether triangles contain interior lattice points.*

Proof. Note that the determinant of a matrix $\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ contains factors $\deg(\mathbf{p}_1)$ and $\deg(\mathbf{p}_2)$, both of which therefore are 1. Given $\mathbf{a}$, by a change of basis we may assume $\mathbf{a} = (\deg \mathbf{a}, 0)$, and therefore

$$\deg(\mathbf{Pa}) = \deg(\mathbf{a}) \deg(\mathbf{p}_1) = \deg(\mathbf{a}).$$

It is clear that $\mathrm{SL}(2,\mathbb{Z})$ preserves areas. Finally, Pick's formula (4) writes the number of interior points in $\Delta_{\mathbf{a},\mathbf{b}}$ terms of area and number of boundary points

$$\#(\text{boundary points}) = \deg(\mathbf{a}) + \deg(\mathbf{b}) + \deg(\mathbf{a}+\mathbf{b}),$$

both of which we just showed are preserved by $\mathrm{SL}(2,\mathbb{Z})$. $\square$

However, note that the $\mathrm{SL}(2,\mathbb{Z})$ action *does not* preserve $\epsilon_{\mathbf{a}}$ and therefore does not preserve the quantity $\alpha(\mathbf{a},\mathbf{b})$ in (3). So we cannot always use it to compute commutators. It preserves commutators only in the case when $\mathbf{a},\mathbf{b} \in \mathbf{Z}^+$, so that $\alpha(\mathbf{a},\mathbf{b}) = 0$, or we set all $K_{\mathbf{a}} = 1$.

Proposition 3.3. *If $\Delta_{\mathbf{a},\mathbf{b}}$ has no interior lattice points, then either:*

1. *one or both of $\mathbf{a}, \mathbf{b}$ is degree 1;*
2. $\deg(\mathbf{a}) = \deg(\mathbf{b}) = 2$.

Proof. Let $a \coloneqq \deg(\mathbf{a})$ and $b \coloneqq \deg(\mathbf{b})$. Use the $\mathrm{SL}(2,\mathbb{Z})$ action to move

$$\mathbf{a} \mapsto (a,0), \quad \mathbf{b} \mapsto (0,b).$$

By the lemma, the triangle $(0,0),(a,0),(0,b)$ also has no interior points. This is only possible in the specified cases. ☐

Computing the commutator in case 1 of the proposition is handled by the defining relation (3). In case 2, we introduce intermediate points

$$\mathbf{c}^{\pm} \coloneqq \frac{1}{2}(\mathbf{a} \pm \mathbf{b}).$$

Using the $\mathrm{SL}(2,\mathbb{Z})$ action, we can map

$$\mathbf{a} \mapsto (0,2), \quad \mathbf{b} \mapsto (2,0), \quad \mathbf{c}^{\pm} \mapsto (\pm 1, 1). \tag{5}$$

The lemma implies that $\Delta_{\mathbf{c}^+,\mathbf{c}^-}$ contains no interior points, and (3) gives

$$[e_{\mathbf{c}^+}, e_{\mathbf{c}^-}] = x e_{\mathbf{a}} + y$$

for some elements $x, y \in U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$. In particular, x is invertible and y involves only $e_{\mathbf{a}/2}$. Using the Jacobi identity we can rewrite

$$\begin{aligned}
[e_{\mathbf{a}}, e_{\mathbf{b}}] &= x^{-1}\left([[e_{\mathbf{c}^+}, e_{\mathbf{c}^-}], e_{\mathbf{b}}] - [y, e_{\mathbf{b}}]\right) \\
&= x^{-1}\left(-[[e_{\mathbf{b}}, e_{\mathbf{c}^+}], e_{\mathbf{c}^-}] - [[e_{\mathbf{c}^-}, e_{\mathbf{b}}], e_{\mathbf{c}^+}] - [y, e_{\mathbf{b}}]\right).
\end{aligned}$$

Using the model situation given in (5), one can verify that each of these commutators involve triangles with strictly smaller area. (In this expression, there is exactly one commutator which involves a triangle containing interior points; this is handled by the next section.)

3.2 Interior lattice points

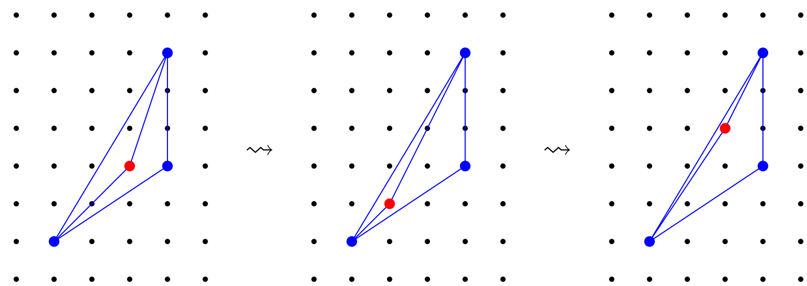
Suppose $\Delta_{\mathbf{a},\mathbf{b}}$ has an interior point. Then to compute $[e_{\mathbf{a}}, e_{\mathbf{b}}]$, we will find a preferred interior point $\mathbf{c}$ and rewrite the desired commutator in terms of $[e_{\mathbf{a}}, e_{\mathbf{c}}]$ and $[e_{\mathbf{b}}, e_{\mathbf{c}}]$. Since $\mathbf{c}$ is an interior point, the corresponding triangles have smaller area, as desired.

Definition 3.4. An interior point $\mathbf{c} \in \Delta_{\mathbf{a},\mathbf{b}}$ is **primitive** if:

1. $\deg(\mathbf{c}) = \deg(\mathbf{a} - \mathbf{c}) = 1$;
2. the triangle $\mathbf{0}, \mathbf{a}, \mathbf{c}$ has no interior points.

Lemma 3.5. *If $\Delta_{\mathbf{a},\mathbf{b}}$ has an interior point, it has a primitive interior point.*

Proof. It is clearly easy to find points satisfying condition 2. Then if $\deg(\mathbf{c}) \neq 1$, replace $\mathbf{c}$ with the point closest to $\mathbf{0}$ on the line $\mathbf{0c}$. Then if $\deg(\mathbf{a}-\mathbf{c}) \neq 1$, similarly replace $\mathbf{c}$ with the point closest to $\mathbf{a}$ on the line $\mathbf{ac}$.



The resulting choice of $\mathbf{c}$ is a primitive interior point. (Equivalently, pick $\mathbf{c}$ to be one of the interior points with minimal distance to the line $\mathbf{0a}$.) $\square$

After choosing a primitive interior point $\mathbf{c} \in \Delta_{\mathbf{a},\mathbf{b}}$, we can follow the procedure of the previous section. Since $\Delta_{\mathbf{c},\mathbf{a}-\mathbf{c}}$ has no interior points, (3) gives

$$[e_{\mathbf{c}}, e_{\mathbf{a}-\mathbf{c}}] = x e_{\mathbf{a}} + y$$

for some elements $x, y \in U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$. In particular, x is invertible and y involves only $e_{\mathbf{a}/k}$ for k dividing $\deg \mathbf{a}$. Using the Jacobi identity we can rewrite

$$\begin{aligned}[e_{\mathbf{a}}, e_{\mathbf{b}}] &= x^{-1}\left([[e_{\mathbf{c}}, e_{\mathbf{a}-\mathbf{c}}], e_{\mathbf{b}}] - [y, e_{\mathbf{b}}]\right) \\ &= x^{-1}\left(-[[e_{\mathbf{b}}, e_{\mathbf{c}}], e_{\mathbf{a}-\mathbf{c}}] - [[e_{\mathbf{a}-\mathbf{c}}, e_{\mathbf{b}}], e_{\mathbf{c}}] - [y, e_{\mathbf{b}}]\right).\end{aligned}$$

Since $\deg(\mathbf{c}) = 1$, for the purposes of arguing that the discrepancy has decreased we may as well pretend all terms in the first nested commutator $[e_{\mathbf{b}}, e_{\mathbf{c}}]$ are actually $e_{\mathbf{b}+\mathbf{c}}$. Similarly, replace the second nested commutator with $e_{\mathbf{a}-\mathbf{c}+\mathbf{b}}$. It is not hard then to see that the discrepancy has decreased in all three of the rhs terms.

4 Fock representation

There is a geometric realization of the elliptic Hall algebra via 1-step correspondences in the K-theory of the Hilbert scheme

$$\mathrm{Hilb} \coloneqq \bigoplus_{n \geq 0} \mathrm{Hilb}_n, \quad \mathrm{Hilb}_n \coloneqq \mathrm{Hilb}(\mathbb{C}^2, n),$$

due to Schiffmann and Vasserot in [2] (and also Feigin and Tsymbaliuk independently, but via the shuffle algebra). Recall that if $(\mathbb{C}^\times)^2$ acts on $\mathbb{C}^2$ with weights t_1, t_2, then

$$K_T(\mathrm{Hilb}) \cong \mathbb{Q}[p_1, p_2, \ldots]^{S_\infty} \otimes \mathbb{Q}[t_1^\pm, t_2^\pm]$$

is a ring of symmetric functions. Skyscraper sheaves $\mathcal{O}_\lambda$ at T-fixed points $\lambda \in K_T(\mathrm{Hilb})$ correspond to Macdonald polynomials $\tilde{H}_\lambda$ with Haiman's normalization. The coefficients of $\tilde{H}_\lambda$ in the Schur basis are *modified* q,t-Kostka numbers.

Definition 4.1 ([2])**.** Nakajima correspondences in $H_T(\mathrm{Hilb})$ are constructed by convolution with nested Hilbert schemes

$$Z_{n,n+k} \coloneqq \{(I,J) : J \subset I\} \subset \mathrm{Hilb}_n \times \mathrm{Hilb}_{n+k}\,.$$

In K-theory, use the *tautological sheaf* $\boldsymbol{\tau}_{n,n+m} \in K_T(Z_{n,n+k})$, whose fibers are

$$\boldsymbol{\tau}_{n,n+m}\big|_{(I,J)} \coloneqq I/J.$$

Because $Z_{n,n+k}$ for $k > 1$ are badly singular, in K-theory we work only with 1-step correspondences $\boldsymbol{\tau}_{n,n+1}$. Convolution with $\tau_{n,n+k}$ defines operators

$$\begin{aligned} \boldsymbol{f}_{-1,\ell} &\coloneqq \prod_n \boldsymbol{\tau}^{\ell}_{n,n+1}, \quad \boldsymbol{f}_{1,\ell} \coloneqq \prod_n \boldsymbol{\tau}^{\ell}_{n+1,n} \\ \boldsymbol{f}_{0,k} &\coloneqq \prod_n \Psi_k \boldsymbol{\tau}_{n,n}, \quad \boldsymbol{f}_{0,-k} \coloneqq \prod_n \Psi_k \boldsymbol{\tau}^*_{n,n}, \quad k \geq 1. \end{aligned}$$

Here $\boldsymbol{\tau}_{n+1,n}$ denotes convolution in the reverse direction, and $\boldsymbol{\tau}^*_{n,n}$ is the dual bundle. The Ψ_k are Adams operations; we can write $\Psi_{-k}(\boldsymbol{\tau}_{n,n}) = \Psi_k(\boldsymbol{\tau}^*_{n,n})$.

Our notation will be *reversed*: the direction of convolution is

$$\boldsymbol{\tau}_{n,m} \colon K_T(\mathrm{Hilb}_n) \to K_T(\mathrm{Hilb}_m).$$

This is so that $e_{(a,b)}$ is a *lowering* operator for $a > 0$. The net effect is that our $\boldsymbol{\tau}_{n,m}$ is what Schiffmann and Vasserot call $\boldsymbol{\tau}_{m,n}$, and similarly for the $\boldsymbol{f}$.

Theorem 4.2 ([2])**.** *If in* $U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ *we specialize*

$$K_{(1,0)} = \hbar^{1/2}, \quad K_{(0,1)} = 1, \tag{6}$$

then it is isomorphic via

$$\begin{aligned} e_{(1,\ell)} &\mapsto \boldsymbol{f}_{1,1-\ell} \\ e_{(-1,\ell)} &\mapsto -\hbar^{1/2} \boldsymbol{f}_{-1,-\ell} \\ e_{(0,k)} &\mapsto \mathrm{sign}(k) \left(-\boldsymbol{f}_{0,k} + \frac{1}{(1-t_1^k)(1-t_2^k)} \right) \end{aligned}$$

to the algebra generated by all $\boldsymbol{f}$*.*

For computational purposes, it is better to write the formulas for this action in terms of the *vertical* generators $e_{(0,m)}$ and the *horizontal* generators $e_{(m,0)}$.

- (Vertical) The character of $\boldsymbol{\tau}_{n,n}$ at a fixed point I_λ is just

$$\sum_{\square \in \lambda} t_1^{x(\square)} t_2^{y(\square)} = \sum_{i=1}^{\infty} \frac{1 - t_1^{\lambda_i}}{1 - t_1} t_2^{i-1}.$$

The Adams operation Ψ^k is simply $t_1 \mapsto t_1^k$ and $t_2 \mapsto t_2^k$. It immediately follows that

$$e_{(0,k)}(\tilde{H}_\lambda) = \operatorname{sign}(k) \left(\frac{1}{1-t_1^k} \sum_{i=1}^{\infty} t_1^{k\lambda_i} t_2^{k(i-1)} \right) \tilde{H}_\lambda. \tag{7}$$

So the $e_{(0,k)}$ are some sort of Macdonald operators.

- (Horizontal) From the commutation relation (2) in $U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$,

  $$[e_{(m,0)}, e_{(-m,0)}] = \frac{-m\hbar^{m/2}}{(1-t_1^m)(1-t_2^m)}.$$

  In [2], Schiffmann and Vasserot check the normalization

  $$e_{(1,0)} = \frac{\partial}{\partial p_1},$$

  so it follows that for $m > 0$,

  $$\begin{aligned} e_{(-m,0)} &= \frac{-\hbar^{m/2}}{(1-t_1^m)(1-t_2^m)} p_m \\ e_{(m,0)} &= m \frac{\partial}{\partial p_m}. \end{aligned} \tag{8}$$

The formulas (6), (7), (8) are essentially the ones given by Okounkov and Smirnov in [1]. They suffice to algorithmically determine actions of all other generators, by a recursive process analogous to that of section 3.2. To determine the action of $e_{(a,b)}$ which is neither horizontal nor vertical, consider the triangle $\{\mathbf{0}, (a,0), (a,b)\}$.

1. If it has a primitive interior point, use it to recurse as in section 3.2.
2. If it has no primitive interior point, recurse using the point $(a/|a|, 0)$. (One would like to use the point $(a,0)$ directly, but the defining relation (3) requires at least one vector of degree 1.)

References

[1] A. Okounkov and A. Smirnov. Quantum difference equation for Nakajima varieties. 2016.

[2] O. Schiffmann and E. Vasserot. The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$. *Duke Math. J.*, 162(2):279–366, 2013.