

ESERCIZIO 1

$F = (10, 3, 7, 6)$, troncamento

a) $x = 2021$, calcolare $\tilde{x} := \text{fl}(x)$.

$$x = 2021 = +10^4 (0.2021)$$

$$\Rightarrow \tilde{x} = 10^4 (0.202)$$

b) Si determinino tutti i numeri $k \in \mathbb{R} \mid \text{fl}(k) = \tilde{x}$.

$$\tilde{x} = 10^4 (0.202)$$

$$\Rightarrow \left\{ \begin{array}{l} \text{def. di troncamento} \\ k \in [2020, 2030[\end{array} \right.$$

c) u ? Determinare che numeri $z \in F \mid \frac{|z - u|}{|u|} < u$.

$$u = \beta^{1-t} = 10^{1-3} = 10^{-2}$$

$$\frac{|z - u|}{|u|} < 10^{-2}$$

$$\Rightarrow |z - u| < 10^{-2} (0.2021)$$

$$|z - u| < 20.21$$

$$\text{e.g. } z = 10^4 (0.2017) \vee z = 10^4 (0.203)$$

ESERCIZIO 2

$F = (10, 3, m, M)$, fronzimento

a) Hp: $x_1, x_2 \in \mathbb{R}$, $x_1 \in]0; x_2[$
 Th. $fl(x_1) \leq fl(x_2)$

{Definizione di fl }

$$fl(x_1) = \tilde{x}_1 = 10^{P_1} \left(\sum_{i=1}^+ d_i \beta^{-i} \right),$$

la definizione di x_2 è analoga;

{Hp: $x_1 \in]0; x_2[$ }

$$10^{P_1} \left(\sum_{i=1}^+ d_i \beta^{-i} \right) \in]0; 10^{P_2} \left(\sum_{i=1}^+ d_i \beta^{-i} \right)[$$

\Rightarrow Th.

Q.E.D.

b) Dimostrare con un esempio che

$$fl(x_1) \leq fl(x_2) \nRightarrow x_1 \leq x_2$$

Si considera $x_1 = 2028$, $x_2 = 2021$

$$fl(x_1) = fl(x_2) = 2020 \wedge$$

$$x_1 > x_2$$

\Rightarrow Th.

Q.E.D.

ESERCIZIO 3

$F = (10, 3, 3, 3)$, troncamento

a) Trovare $a, b \in F \mid (a \oplus b) \textcircled{1} 2 = a$.

$$a = \tilde{a} = 10(0.100),$$

$$b = \tilde{b} = a$$

$$\Rightarrow (a \oplus b) = 10(0.100) \oplus 10(0.100)$$

$$= \text{fl}(1+1) = \text{fl}(2) = 10(0.200)$$

$$\Rightarrow (a \oplus b) \textcircled{1} 2 = 10(0.200) \textcircled{1} 2 =$$

$$\text{fl}(10(0.200)/_2) = 10(0.100).$$

b) Trovare $a, b \in F \mid (a \oplus b) \textcircled{0} < \min\{a, b\}$

$$a = w = 10^{-3-1} = 10^{-4},$$

$$b = 0$$

$$\Rightarrow (a \oplus b) = 10^{-4} \oplus 0 = 10^{-4}$$

$$\Rightarrow (a \oplus b) \textcircled{0} 2 = 10^{-4} \textcircled{0} 2 =$$

$$\text{fl}(10^{-4}/_2) = 0.$$

Esercizio 4

$$F = F(\beta, t, m, M)$$

a) $w ? \Omega ?$

$$w = \beta^{-m-1}, \quad \Omega = \beta^m(1 - \beta^{-t})$$

b) $\frac{1}{w} \in F ? \quad \frac{1}{\Omega} \in F ?$

$$\frac{1}{w} = \frac{1}{\beta^{-m-1}} = \beta^{m+1}$$

$$\Rightarrow \frac{1}{w} \in F \text{ sse } \beta^{m+1} < \beta^m(1 - \beta^{-t})$$

$$\frac{1}{\Omega} = \frac{1}{\beta^m(1 - \beta^{-t})} = \beta^{-M}(1 - \beta^{-t})$$

Si nota che $\beta^{-M}(1 - \beta^{-t}) = \beta^{-M} \left(\sum_{k=0}^{\infty} (\beta^{-t})^k \right)$ che e' periodico
 $\Rightarrow \frac{1}{\Omega} \notin F$

c) Esaminare il caso $\beta = 2, t = 8, m = M = 6$

$$\Rightarrow \omega = 2^{-7}, \Omega = 2^6(1 - 2^{-8})$$

$$\Rightarrow \frac{1}{\omega} \notin \mathbb{F}, \frac{1}{\Omega} \notin \mathbb{F}.$$

ESERCIZIO 5

Determinare il massimo $N \in \mathbb{Z}$ ($\forall x \in \mathbb{Z}, n \in [-N; N]$).

$$x = f(l(x)) \wedge x \in F(2, t, m, M)).$$

Quanto vale N in aritmetica IEEE in doppia precisione?

$$x = 2^P \cdot f \text{ con } f \in]0; 1[$$

$$\Leftrightarrow x = 2^P (0.d_1d_2d_3 \dots d_t)_2 = 2^P \left(\sum_{i=1}^{t+1} d_i \cdot 2^{-i} \right)$$

Per essere intero: $P+t \geq 0 \Leftrightarrow P \geq -t$

Poiché si cerca il massimo: $P = t$

$$\Rightarrow N = 2^{t+1} (1 - 2^{-t}) = 2^{t+1} - 2$$

Nel caso IEEE754 (binary 64) :

$$N = 2^{t+1} (1 - 2^{-t}) =$$

$$= 2^{12} (1 - 2^{-11}) = 4094$$

ESERCIZIO 6

$$f = (x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$$

a) Studiare il condizionamento di $f(x)$.

$$\varepsilon_{in} = \frac{f(\tilde{x}) - f(x)}{f(x)} \quad \text{con } x = (x_1, \dots, x_n)$$

$$\doteq \sum_{i=1}^n c_{x_i} \varepsilon_{x_i}, \quad \text{con } c_{x_i} = \frac{x_i}{f(x)} \frac{\partial}{\partial x_i} f$$

$$\Rightarrow |\varepsilon_{in}| = \frac{1}{|f(x)|} \cdot \left| \sum_{i=1}^n 2x_i \varepsilon_{x_i} \right| \leq \frac{1}{|f(x)|} \sum_{i=1}^n 2x_i^2 |\varepsilon_{x_i}|$$

$$\leq \frac{1}{|f(x)|} \sum_{i=1}^n 2x_i^2 u =$$

$$= 2u \cdot \frac{\sum_{i=1}^n x_i^2}{f(x)} = 2u.$$

Il problema e' ben condizionato.

b) Si fornisca un algoritmo per il calcolo
di $f(x_1)$ e se ne studi la stabilità.

$$S = 0;$$

$$q = 0;$$

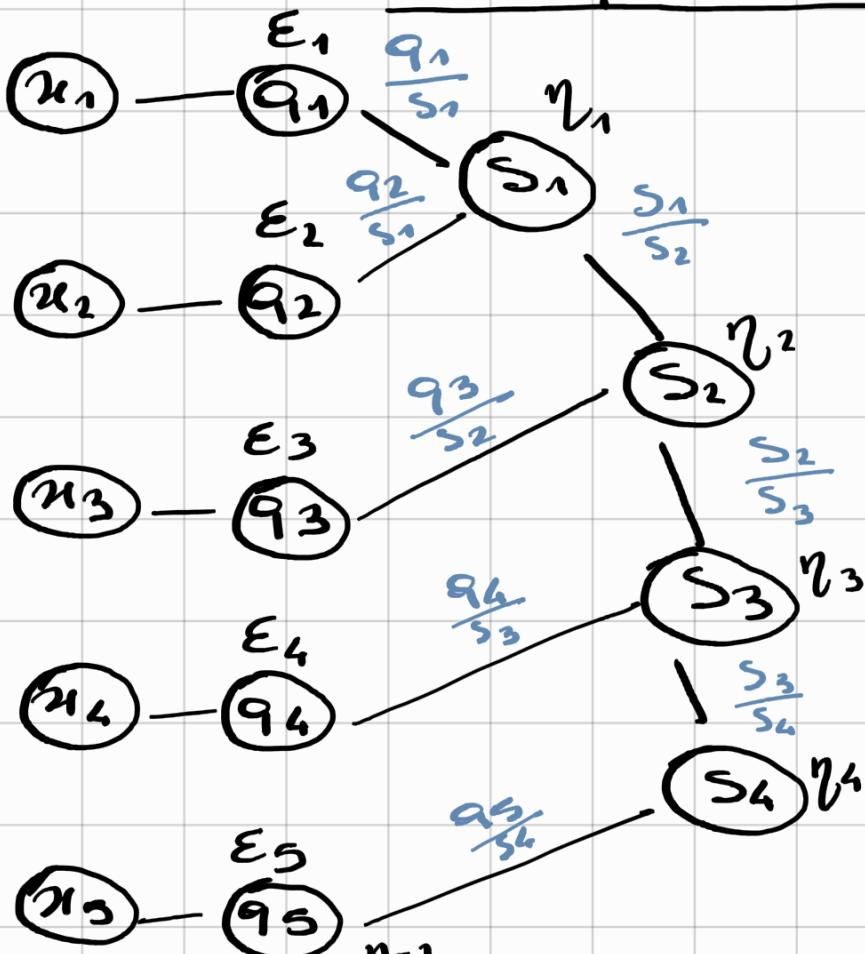
for $i = 1 : N$

$$q = x(i) * x(i)$$

$$S = S + q$$

end

Disegnino per capire
come funzioni



$$\textcircled{4} \quad E_{\text{alg}} \leq \frac{1}{S_{n-1}} \left(\sum_{i=1}^{n-1} S_i (\eta_i) \right) + \frac{1}{S_{n-1}} \left(\sum_{i=1}^n q_i \epsilon_i \right) \leq n u.$$

ESERCIZIO 7

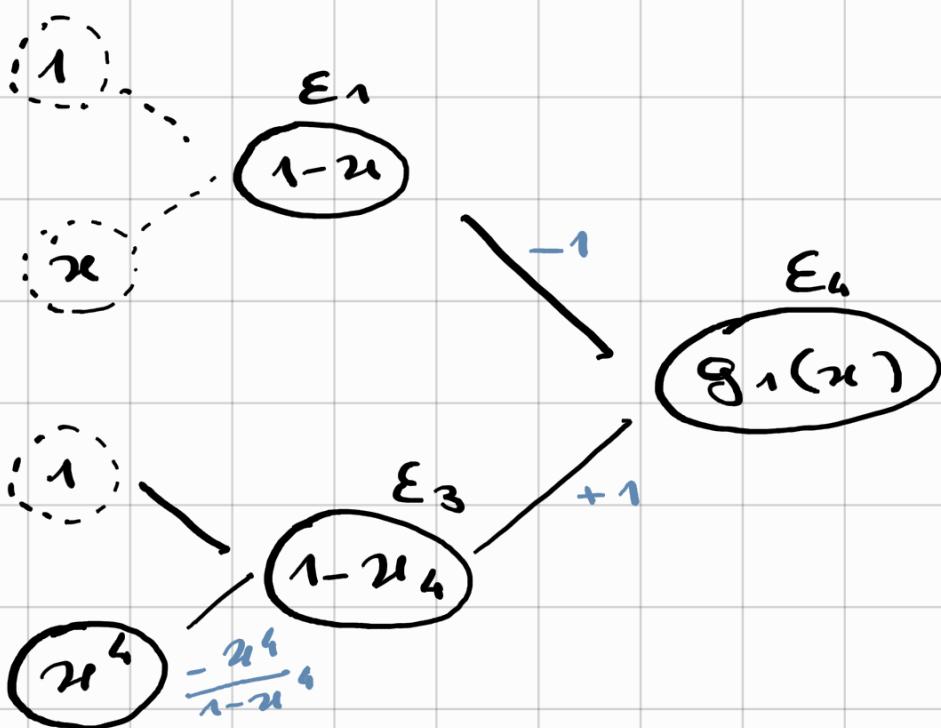
$$\frac{1-u^4}{1-u} = 1+u+u^2+u^3$$

Quale dei due algoritmi è più stabile?

$$g_1(u) = \frac{1-u^4}{1-u}$$

Hp : { Esistono libreria per }

le potenze



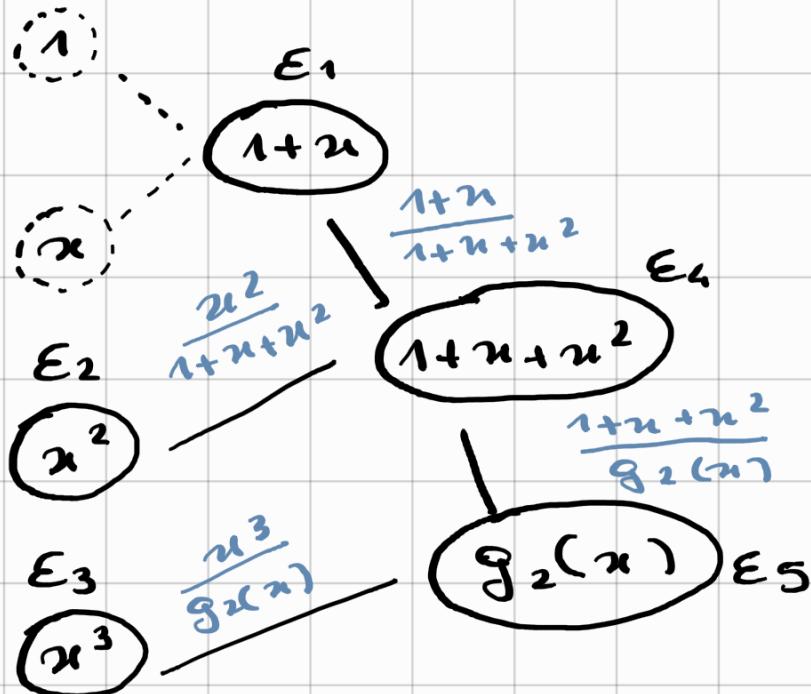
$$E_{\text{TOT}} = E_4 - 1E_1 + 1 \left(E_3 - \frac{u^4}{1-u^4} E_2 \right)$$

$$\Rightarrow |E_{\text{TOT}}| \leq |E_4| + |E_1| + |E_3| + \left| \frac{u^4}{1-u^4} \right| |E_2|$$

$$\leq u \left(3 + \frac{u^4}{|1-u^4|} \right) =$$

$$= 3u + \frac{u u^4}{|1-u^4|}; \text{ risulta instabile per } u \rightarrow \pm 1.$$

$$g_2(x) = 1+x+x^2+x^3$$



$$\begin{aligned} \epsilon_{\text{TOT}} &= \epsilon_5 + \frac{1+x+x^2}{1+x+x^2+x^3} \left(\epsilon_4 + \frac{1+x}{1+x+x^2} \epsilon_1 + \frac{x^2}{1+x+x^2} \epsilon_2 \right) + \\ &\quad + \frac{x^3}{1+x+x^2+x^3} \epsilon_3 \end{aligned}$$

$$\begin{aligned} \Rightarrow |\epsilon_{\text{TOT}}| &\leq |\epsilon_5| + \left| \frac{1+x+x^2}{1+x+x^2+x^3} \right| \left(|\epsilon_4| + \left| \frac{1+x}{1+x+x^2} \right| |\epsilon_1| \right. \\ &\quad \left. + \left| \frac{x^2}{1+x+x^2} \right| |\epsilon_2| \right) + \left| \frac{x^3}{1+x+x^2+x^3} \right| |\epsilon_3| \end{aligned}$$

$$\begin{aligned} &\leq u + \frac{1+x+x^2}{1+x+x^2+x^3} \left(u + \left| \frac{1+x}{1+x+x^2} \right| u + \frac{x^2}{1+x+x^2} u \right) + \\ &\quad + \left| \frac{x^3}{1+x+x^2+x^3} \right| u \end{aligned}$$

Non e' limitata se si annulla almeno uno dei denominatori:

$$1+n+n^2+n^3=0 \quad n^2(1+n)+n(1+n)=(n^2+1)(1+n)$$

$$1+n+n^2=0 \quad \cancel{1+n}$$

\Rightarrow non e' stabile

per $n \rightarrow \pm 1$.