ESERCIZIO 1

$\mathcal{F} = (10, 3, 7, 6)$, troncamento

a) $x = 2021$, calcolare $\tilde{x} := fl(x)$.

$$x = 2021 = +10^4 (0.2021)$$
$$\Rightarrow \tilde{x} = 10^4 (0.202)$$

b) Si determinino tutti i numeri $k \in \mathbb{R} \mid fl(k) = \tilde{x}$.

$$\tilde{x} = 10^4 (0.202)$$
$$\Rightarrow \{\text{def. di troncamento}\}$$
$$k \in [2020, 2030[$$

c) u? Determinare due numeri $z \in \mathcal{F} \mid \frac{|z - x|}{|x|} < u$.

$$u = \beta^{1-t} = 10^{1-3} = 10^{-2}$$
$$\frac{|z - x|}{|x|} < 10^{-2}$$

$$\Rightarrow |z - x| < 10^2 (0.2021)$$
$$|z - x| < 20.21$$

e.g. $z = 10^4 (0.201) \vee z = 10^4 (0.203)$

ESERCIZIO 2

$\mathcal{F} = (10, 3, m, M)$, troncamento

a) Hp. $x_1, x_2 \in \mathbb{R}$, $x_1 \in]0; x_2[$

Th. $fl(x_1) \leq fl(x_2)$

{Definizione di fl}

$fl(x_1) = \tilde{x}_1 = 10^{p_1}\left(\sum_{i=1}^{t} d_{i,1} \beta^{-i}\right)$,

la definizione di x_2 e' analoga;

{Hp: $x_1 \in]0; x_2[$ }

$10^{p_1}\left(\sum_{i=1}^{t} d_{i,1} \beta^{-i}\right) \in \left]0; 10^{p_2}\left(\sum_{i=1}^{t} d_{i,2} \beta^{-i}\right)\right[$

$\Rightarrow$ Th.

Q.E.D.

b) Dimostrare con un esempio che

$fl(x_1) \leq fl(x_2) \not\Rightarrow x_1 \leq x_2$

Si considera $x_1 = 2028$, $x_2 = 2021$

$fl(x_1) = fl(x_2) = 2020 \ \wedge$

$x_1 > x_2$

$\Rightarrow$ Th.

Q.E.D.

ESERCIZIO 3

$F = (10, 3, 3, 3)$, troncamento

a) Trovare $a, b \in F \mid (a \oplus b) \oslash 2 = a$.

$a = \tilde{a} = 10(0.100)$,

$b = \tilde{b} = a$

$\Rightarrow (a \oplus b) = 10(0.100) \oplus 10(0.100)$

$= fl(1+1) = fl(2) = 10(0.200)$

$\Rightarrow (a \oplus b) \oslash 2 = 10(0.200) \oslash 2 =$

$fl(10(0.200)/2) = 10(0.100)$.

b) Trovare $a, b \in F \mid (a \oplus b) \oslash 2 < \min\{a, b\}$

$a = \omega = 10^{-3-1} = 10^{-4}$,

$b = 0$

$\Rightarrow (a \oplus b) = 10^{-4} \oplus 0 = 10^{-4}$

$\Rightarrow (a \oplus b) \oslash 2 = 10^{-4} \oslash 2 =$

$fl(10^{-4}/2) = 0$.

ESERCIZIO 4

$\mathbb{F} = \mathcal{F}(\beta, t, m, M)$

a) ω ? Ω ?

$$\omega = \beta^{-m-1} \quad , \quad \Omega = \beta^{M}(1-\beta^{-t})$$

b) $\frac{1}{\omega} \in \mathbb{F}$? $\frac{1}{\Omega} \in \mathbb{F}$?

$$\frac{1}{\omega} = \frac{1}{\beta^{-m-1}} = \beta^{m+1}$$

$\Rightarrow \frac{1}{\omega} \in \mathbb{F}$ sse $\beta^{m+1} < \beta^{M}(1-\beta^{-t})$

$$\frac{1}{\Omega} = \frac{1}{\beta^{M}(1-\beta^{-t})} = \beta^{-M}(1-\beta^{-t})$$

Si nota che $\beta^{-M}(1-\beta^{-t}) = \beta^{-M}\left(\sum_{k=0}^{\infty}(\beta^{-t})^{k}\right)$ che è periodico

$\Rightarrow \frac{1}{\Omega} \notin \mathbb{F}$

c) Esaminare il caso $\beta = 2$, $t = 8$, $m = M = 6$

$$\Rightarrow \omega = 2^{-7}, \quad \Omega = 2^{6}(1-2^{-8})$$

$$\Rightarrow \frac{1}{\omega} \notin F, \quad \frac{1}{\Omega} \notin F.$$

ESERCIZIO 5

Determinare il massimo $N \in \mathbb{Z} \,|\, (\forall x \in \mathbb{Z},\ x \in [-N;N].$
$x = fl(x) \wedge x \in F(2,t,m,M))$.
Quanto vale N in aritmetica IEEE in doppia precisione?

$$x = 2^{p} \cdot f \quad \text{con } f \in]0;1[$$

$$\Leftrightarrow x = 2^{p}(0.d_1 d_2 d_3 \dots d_t)_2 = 2^{p}\left(\sum_{i=1}^{t} d_i\, 2^{-i}\right)$$

Per essere intero: $p - t \geq 0 \Leftrightarrow p \geq t$

Poiché si cerca il massimo: $p = t$

$$\Rightarrow N = 2^{t+1}(1-2^{-t}) = 2^{t+1} - 2$$

Nel caso IEEE754 (binary 64 :

$$N = 2^{t+1}(1-2^{-t}) =$$

$$= 2^{12}(1-2^{-11}) = 4094$$

ESERCIZIO 6

$$f = (x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2$$

a) Studiare il condizionamento di $f(x)$.

$$\varepsilon_{in} = \frac{f(\tilde{x}) - f(x)}{f(x)} \quad \text{con } x = (x_1, \dots, x_n)$$

$$\doteq \sum_{i=1}^{n} c_{x_i} \varepsilon_{x_i}, \quad \text{con } c_{x_i} = \frac{x_i}{f(x)} \frac{\partial f}{\partial x_i}$$

$$\Rightarrow |\varepsilon_{in}| = \frac{1}{|f(x)|} \cdot \left| \sum_{i=1}^{n} 2x_i^2 \varepsilon_{x_i} \right| \leq \frac{1}{f(x)} \sum_{i=1}^{n} 2x_i^2 |\varepsilon_{x_i}|$$

$$\leq \frac{1}{f(x)} \sum_{i=1}^{n} 2x_i^2 u =$$

$$= 2u \cdot \frac{\sum_{i=1}^{n} x_i^2}{f(x)} = 2u.$$

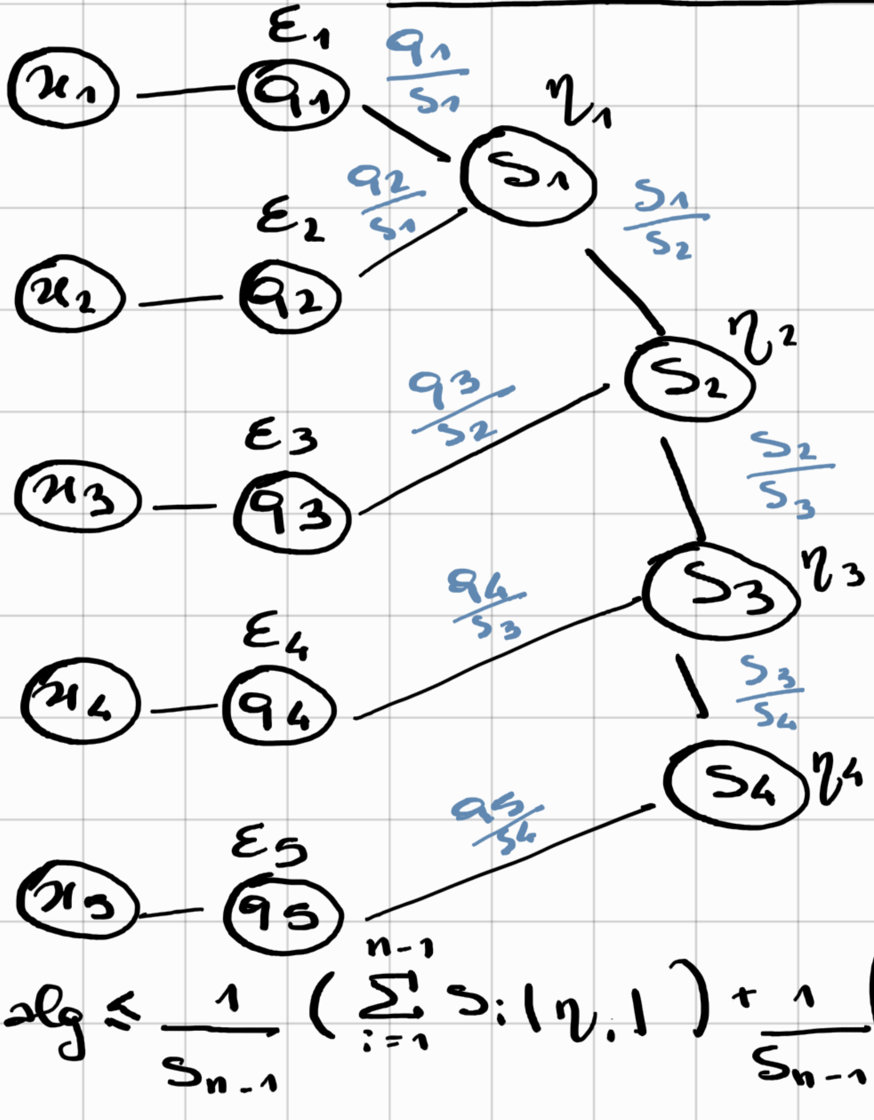
Il problema e' ben condizionato.

b) Si fornisca un algoritmo per il calcolo di $f(x)$ e se ne studi la stabilita'.

```
s = 0;
q = 0;
for i=1:N
    q = x(i) * x(i)
    s = s+q
end
```

$$\varepsilon_{alg} := \frac{1}{s_{n-1}}\left(\sum_{i=1}^{n-1} s_i \eta_i\right) + \frac{1}{s_{n-1}}\left(\sum_{i=1}^{n} q_i \varepsilon_i\right) \quad (*)$$

Disegnino per capire come funzioni



$(*)$ $$\varepsilon_{alg} \le \frac{1}{s_{n-1}}\left(\sum_{i=1}^{n-1} s_i |\eta_i|\right) + \frac{1}{s_{n-1}}\left(\sum_{i=1}^{n} q_i |\varepsilon_i|\right)$$

$\le nu$.

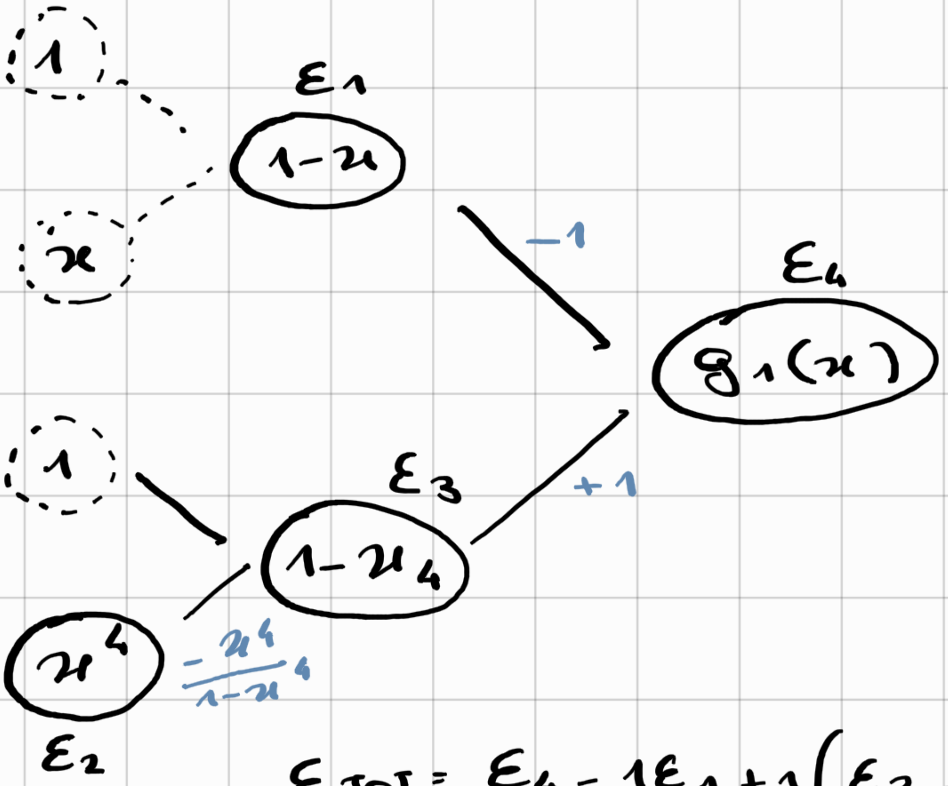
ESERCIZIO 7

$$\frac{1-x^4}{1-x} = 1+x+x^2+x^3$$

Quale dei due algoritmi è più stabile?

$$g_1(x) = \frac{1-x^4}{1-x}$$

Hp: $\left\{ \exists \right.$ funzione di libreria per le potenze $\left. \right\}$



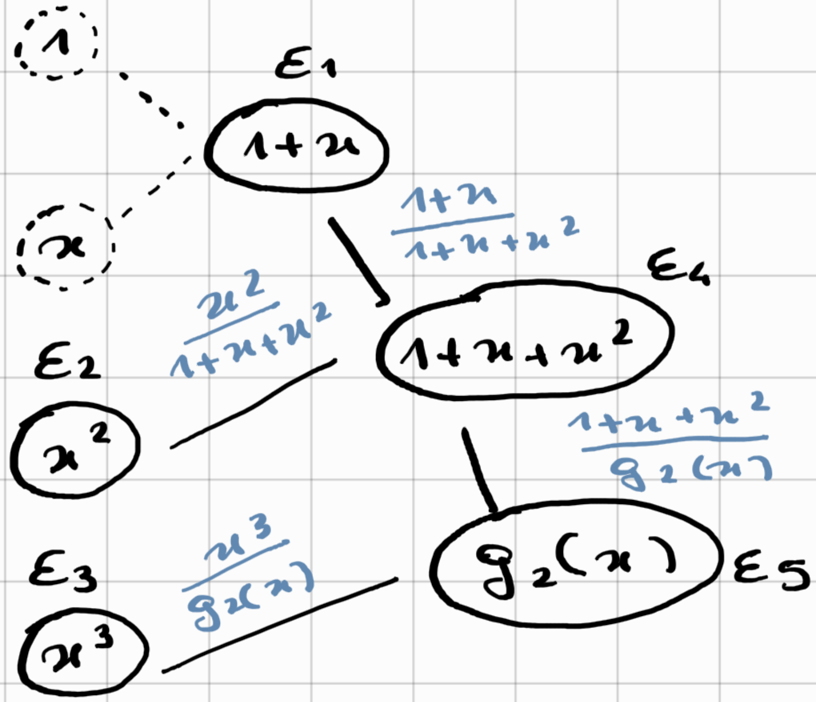
$$\varepsilon_{TOT} = \varepsilon_4 - 1\varepsilon_1 + 1\left(\varepsilon_3 - \frac{x^4}{1-x^4}\varepsilon_2\right)$$

$$\Rightarrow |\varepsilon_{TOT}| \le |\varepsilon_4| + |\varepsilon_1| + |\varepsilon_3| + \left|\frac{x^4}{1-x^4}\right||\varepsilon_2|$$

$$\le u\left(3 + \frac{x^4}{|1-x^4|}\right) =$$

$$= 3u + \frac{u x^4}{|1-x^4|}$$; risulta instabile per $x \to \pm 1$.

$$g_2(x) = 1+x+x^2+x^3$$



$$\varepsilon_{TOT} = \varepsilon_5 + \frac{1+x+x^2}{1+x+x^2+x^3}\left(\varepsilon_4 + \frac{1+x}{1+x+x^2}\varepsilon_1 + \frac{x^2}{1+x+x^2}\varepsilon_2\right) + \frac{x^3}{1+x+x^2+x^3}\varepsilon_3$$

$$\Rightarrow |\varepsilon_{TOT}| \leq |\varepsilon_5| + \left|\frac{1+x+x^2}{1+x+x^2+x^3}\right|\left(\left(|\varepsilon_4| + \left|\frac{1+x}{1+x+x^2}\right||\varepsilon_1| + \frac{x^2}{|1+x+x^2|}|\varepsilon_2|\right) + \left|\frac{x^3}{1+x+x^2+x^3}\right||\varepsilon_3|\right.$$

$$\leq u + \frac{1+x+x^2}{1+x+x^2+x^3}\left(u + \left|\frac{1+x}{1+x+x^2}\right|u + \frac{x^2}{|1+x+x^4|}\mu\right) + \left|\frac{x^3}{1+x+x^2+x^3}\right|u$$

Non è limitata se si annulla almeno uno dei denominatori:

$1+x+x^2+x^3=0 \qquad x^2(1+x)+1(1+x)=(x^2+1)(1+x)$

$1+x+x^2=0 \quad \nexists x$

$\Rightarrow$ non è stabile per $x \to \pm 1$.