Extrapolarea Richardson

$$f'(x_i) \simeq \begin{cases} \dfrac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}, & \text{diferențe finite progresive.} \\[2ex] \dfrac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}, & \text{diferențe finite regresive} \\[2ex] \dfrac{f(x_{i+1})-f(x_{i-1})}{x_{i+1}-x_{i-1}}, & \text{diferențe finite centrale.} \end{cases}$$

1) Fie $f:(0,\infty)\to\mathbb{R}$, $f(x)=x+\sqrt{x}$. Aflați valoarea aproximativă a lui f' în punctul $x_0=2$, folosind extrapolarea Richardson (cu diferențe finite progresive) cu $h=1$ și 3 pași.

$$f'(x)=1+\frac{1}{2\sqrt{x}} \Rightarrow f'(2)=1+\frac{1}{2\sqrt{2}} \simeq 1+\frac{1}{2\cdot 1{,}414213}=1+\frac{1}{2{,}828426}=$$

$$=1{,}353553.$$

$$f'(x)\simeq \frac{f(x+h)-f(x)}{h}$$

$$\boxed{\varphi_1(x,h)=\frac{f(x+h)-f(x)}{h}} \quad (*)$$

Notăm $\varphi_1(x,h)=\varphi_1(h)$. La noi $x=x_0=2$.

$\frac{h}{2^n}$	$O(h)$	$O(h^2)$	$O(h^3)$
$\frac{h}{1}=1$	$\varphi_1(1)=1{,}3178$		
$\frac{h}{2}=\frac{1}{2}$	$\varphi_1(\frac{1}{2})=1{,}3338$	$\varphi_2(1)=1{,}3498$	
$\frac{h}{2^2}=\frac{1}{4}$	$\varphi_1(\frac{1}{4})=1{,}3432$	$\varphi_2(\frac{1}{2})=1{,}3526$	$\varphi_3(1)=1{,}35353$.

$$(*)\ \varphi_1(1)=\frac{f(x+1)-f(x)}{1}=\frac{f(2+1)-f(2)}{1}=\frac{f(3)-f(2)}{1}=3+\sqrt{3}-2-\sqrt{2}=$$

$$=1+1{,}7320-1{,}4142=1{,}3178$$

$$(*)\ \varphi_1\left(\tfrac{1}{2}\right)=\frac{f(x+\frac{1}{2})-f(x)}{\frac{1}{2}}=2\left(f(2+\tfrac{1}{2})-f(2)\right)=2\left(f(\tfrac{5}{2})-f(2)\right)=$$

$$= 2\left(\frac{5}{2}+\sqrt{\frac{5}{2}}-2-\sqrt{2}\right) = 2\left(2{,}5+1{,}5811-3{,}4142\right) =$$

$$= 2\cdot 0{,}6669 = 1{,}3338.$$

$$(*)\ \varphi_1\left(\tfrac{1}{4}\right) = \frac{f(x+\frac{1}{4})-f(x)}{\frac{1}{4}} = 4\cdot\left(f(2+\tfrac{1}{4})-f(2)\right) = 4\left(f(\tfrac{9}{4})-f(2)\right) =$$

$$= 4\left(\frac{9}{4}+\sqrt{\frac{9}{4}}-2-\sqrt{2}\right) = 4\left(2{,}25+1{,}5-2-1{,}4142\right) =$$

$$= 4\cdot 0{,}3358 = 1{,}3432$$

$$(**)\ \varphi_2(h) = \varphi_2(1) = \frac{1}{2^{2-1}-1}\left(2^{2-1}\cdot\varphi_1\left(\tfrac{h}{2}\right)-\varphi_1(h)\right) = 2\cdot\varphi_1\left(\tfrac{h}{2}\right)-\varphi_1(h) =$$

$$\boxed{\varphi_n(h) = \frac{1}{2^{n-1}-1}\cdot\left(2^{n-1}\varphi_{n-1}\left(\tfrac{h}{2}\right)-\varphi_{n-1}(h)\right)}\ (**) \rightarrow \text{formula generală}.$$

$$= 2\cdot\varphi_1\left(\tfrac{1}{2}\right)-\varphi_1(1) = 2\cdot 1{,}3338-1{,}3178 = 2{,}6676-1{,}3178 = 1{,}3498.$$

$$(**)\ \varphi_2\left(\tfrac{1}{2}\right) = 2\cdot\varphi_1\left(\tfrac{1}{4}\right)-\varphi_1\left(\tfrac{1}{2}\right) = 2\cdot 1{,}3432-1{,}3338 = 2{,}6864-1{,}3338 =$$

$$= 1{,}3526.$$

$$(**)\ \varphi_3(h) = \frac{1}{2^{3-1}-1}\cdot\left(2^{3-1}\varphi_2\left(\tfrac{h}{2}\right)-\varphi_2(h)\right)$$

$$\varphi_3(1) = \frac{1}{3}\cdot\left(4\cdot\varphi_2\left(\tfrac{h}{2}\right)-\varphi_2(h)\right) = \frac{1}{3}\left(4\cdot\varphi_2\left(\tfrac{1}{2}\right)-\varphi_2(1)\right) =$$

$$= \frac{1}{3}\left(4\cdot 1{,}3526-1{,}3498\right) = \frac{1}{3}\left(5{,}4104-1{,}3498\right) = \frac{1}{3}\cdot 4{,}0606 =$$

$$= 1{,}35353.$$

Integrare numerică

1) Fie $I(f) = \int_1^{1,5} \frac{2x}{x^2-4}\,dx$.

a) Calculați $I(f)$.

b) Calculați formulele de cuadratură ale dreptunghiului, trapezului, Simpson.

c) Calculați erorile reale ale cuadraturilor.

a) $I(f) = \int_1^{1,5} \frac{2x}{x^2-4}\,dx$

Schimbare de variabilă: $t = x^2-4 \Rightarrow dt = 2x\,dx$

$x=1 \to t = 1^2-4 = -3$

$x = 1,5 \to t = 1,5^2-4 = 2,25-4 = -1,75$

$$I(f) = \int_{-3}^{-1,75} \frac{1}{t}\,dt = \ln|t|\,\Big|_{-3}^{-1,75} = \ln 1,75 - \ln 3 = \ln\frac{1,75}{3} = \ln 0,58(3) =$$

$= -0,538997\ldots$

b) $\boxed{I_{dreptunghi} = (b-a)\cdot f\left(\frac{a+b}{2}\right)}$

$$I_{dreptunghi} = (1,5-1)\cdot f\left(\frac{1+1,5}{2}\right) = 0,5\cdot f\left(\frac{2,5}{2}\right) = 0,5\cdot f\left(\frac{5}{4}\right) =$$

$$= 0,5\cdot\frac{2\cdot\frac{5}{4}}{\left(\frac{5}{4}\right)^2-4} = 0,5\cdot\frac{\frac{5}{2}}{\frac{25}{16}-\frac{16}{4}} = 0,5\cdot\frac{\frac{5}{2}}{\frac{25-64}{16}} = \frac{1}{2}\cdot\frac{\frac{5}{2}}{\frac{-39}{16}} =$$

$$= \frac{\frac{5}{4}}{\frac{-39}{16}} = \frac{5}{4}\cdot\frac{-16}{39} = -\frac{20}{39} = -0,512820\ldots$$

$$\boxed{I_{trapez} = \frac{b-a}{2}\left(f(a)+f(b)\right)} = \frac{1,5-1}{2}\cdot\left(f(1)+f(1,5)\right) = 0,25\left(f(1)+f(1,5)\right)$$

$$f(1) = \frac{2\cdot 1}{1^2-4} = \frac{2}{-3} = \frac{-2}{3}$$

$$f(1,5) = \frac{2\cdot 1,5}{1,5^2-4} = \frac{3}{2,25-4} = \frac{3}{-1,75} = \frac{3}{\frac{-175}{100}} = \frac{300}{-175} = -\frac{12}{7}$$

$$I_{trapez} = \frac{1}{4}\cdot\left(-\frac{2}{3}-\frac{12}{7}\right) = \frac{1}{4}\left(\frac{-14-36}{21}\right) = \frac{1}{4}\cdot\frac{-50}{21} = \frac{-25}{42} = -0,595238\ldots$$

$$\boxed{I_{simpson} = \frac{b-a}{2}\left(\frac{1}{3}f(a)+\frac{4}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{3}f(b)\right)} =$$

$$= \frac{1{,}5-1}{2}\cdot\left(\frac{1}{3}f(1)+\frac{4}{3}\cdot f\left(\frac{2{,}5}{2}\right)+\frac{1}{3}f(1{,}5)\right) = 0{,}25\cdot\left(\frac{1}{3}\cdot\frac{-2}{3}+\frac{4}{3}\cdot\frac{-40}{39}+\frac{1}{3}\cdot\frac{-12}{7}\right)$$

$$= \frac{1}{4}\left(\frac{-2}{9}-\frac{160}{117}-\frac{12}{21}\right) = \frac{1}{4}\cdot\left(\frac{-182-1120-468}{819}\right) = \frac{1}{4}\cdot\frac{-1770}{819}$$

$$f\left(\frac{2{,}5}{2}\right) = f\left(\frac{5}{4}\right) = \frac{2\cdot\frac{5}{4}}{\left(\frac{5}{4}\right)^2-4} = \frac{\frac{5}{2}}{\frac{25}{16}-\frac{16}{4}} = \frac{\frac{5}{2}}{\frac{25-64}{16}} = \frac{5}{2}\cdot\frac{16}{-39} = \frac{-40}{39}$$

$$I_{simpson} = \frac{-1770}{4\cdot 819} = \frac{-295}{546} \approx -0{,}540293\ldots$$

c) Erorile reale ale cuadraturilor:

$$|I(f) - I_{dreptunghi}| \simeq |-0{,}538997 + 0{,}512820| = 0{,}026177.$$

$$|I(f) - I_{trapez}| \simeq |-0{,}538997 + 0{,}595238| = 0{,}056241$$

$$|I(f) - I_{simpson}| \simeq |-0{,}538997 + 0{,}540293| = 0{,}001296$$

d) Erorile teoretice ale cuadraturilor de la b):

$$\varepsilon_{dreptunghi} = \frac{f''(\xi)}{3}(b-a)^3 = \frac{f''(\xi)}{3}\cdot 0{,}125,\quad \xi\in(1, 1.5)$$

$$\varepsilon_{trapez} = \frac{f''(\xi)}{12}\cdot(b-a)^3 = \frac{f''(\xi)}{12}\cdot 0{,}125,\quad \xi\in(1, 1.5)$$

$$\varepsilon_{simpson} = \frac{f^{(4)}(\xi)}{90}\cdot\left(\frac{b-a}{2}\right)^5 = \frac{f^{(4)}(\xi)}{90}\cdot\left(\frac{1}{4}\right)^5 = \frac{f^{(4)}(\xi)}{90}\cdot\frac{1}{1024},$$

$\xi\in(1, 1.5)$

2) Aproximați integrala $I = \int_0^2 \frac{1}{1+x^2}\,dx = \operatorname{arctg} x \Big|_0^2 = \operatorname{arctg} 2 \simeq 1.107149$, folosind formula de cuadratură sumată a trapezului, I^n_{trapez}, $n=\overline{2,3}$.

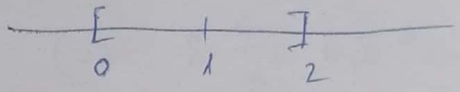
$$\boxed{I^n_{trapez} = \frac{h}{2}\Big(\underset{\overset{\shortparallel}{x_1}}{f(a)} + 2\cdot\sum_{k=2}^{n} f(x_k) + \underset{\overset{\shortparallel}{x_{n+1}}}{f(b)}\Big)}, \quad \boxed{\begin{array}{l} h=\frac{b-a}{n} \\ x_{k+1}=a+k\cdot h,\ k=\overline{0,n} \end{array}}$$

$\underline{n=2}$: $h = \frac{2-0}{2} = 1$

$x_1 = 0+0\cdot h = 0$

$x_2 = a+1\cdot h = 0+1\cdot 1 = 1$

$x_3 = 0+2\cdot h = 0+2\cdot 1 = 2$

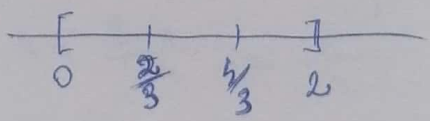


$I^2_{trapez} = \frac{1}{2}\left(f(0)+2\cdot f(1)+f(2)\right) = \frac{1}{2}\cdot\left(1+\cancel{2}\cdot\frac{1}{\cancel{2}}+\frac{1}{5}\right) = \frac{1}{2}\left(2+\frac{1}{5}\right) =$

$= \frac{1}{2}\cdot\frac{11}{5} = \frac{11}{10} = 1,1.$

$\underline{n=3}$: $h = \frac{2-0}{3} = \frac{2}{3}$, $x_{k+1} = 0+k\cdot\frac{2}{3}$, $k=\overline{0,3}$

$x_1 = 0$, $x_2 = \frac{2}{3}$, $x_3 = 2\cdot\frac{2}{3} = \frac{4}{3}$, $x_4 = \cancel{3}\cdot\frac{2}{\cancel{3}} = 2$.



$I^3_{trapez} = \frac{1}{2}\cdot\frac{2}{3}\cdot\left(f(0)+2\cdot f\left(\frac{2}{3}\right)+2\cdot f\left(\frac{4}{3}\right)+f(2)\right) =$

$= \frac{1}{3}\cdot\left(1+2\cdot\frac{1}{1+\frac{4}{9}}+2\cdot\frac{1}{1+\frac{16}{9}}+\frac{1}{5}\right) =$

$= \frac{1}{3}\left(1+2\cdot\frac{9}{13}+2\cdot\frac{9}{25}+\frac{1}{5}\right) = \frac{1}{3}\cdot\left(\overset{325}{1}+\overset{25)}{\frac{18}{13}}+\overset{13)}{\frac{18}{25}}+\overset{65)}{\frac{1}{5}}\right) =$

$= \frac{1}{3}\cdot\frac{325+450+234+65}{325} = \frac{1}{3}\cdot\frac{1074}{325} = 1,101538$

3) De câte subintervale egale ale intervalului $[0,2]$ este nevoie pentru a aproxima integrala exactă $I(f)=\int_0^2 (x^3-1)\,dx$ cu o eroare absolută de cel mult 10^{-6}, folosind formula de cuadratură sumată a trapezului I^n_{trapez}?

$$\text{Eroarea} = I(f) - I^n_{trapez} = \frac{b-a}{12}\cdot f''(\xi)\cdot h^2,\ \xi\in(a,b)$$

la noi: $a=0,\ b=2,\ h=\frac{b-a}{n}=\frac{2}{n}$

$f(x)=x^3-1$

$f''(x)=6x$

$$|E_n(f)|\le 10^{-6} \iff \left|\frac{2}{12}\cdot 6\cdot\xi\cdot\left(\frac{2}{n}\right)^2\right|\le 10^{-6},\ \xi\in(0,2) \iff$$

$$\iff \xi\cdot\frac{4}{n^2}\le 10^{-6},\ \xi\in(0,2) \iff 2\cdot\frac{4}{n^2}\le 10^{-6} \iff 8\le n^2\cdot 10^{-6}$$

$n^2\ge 8\cdot 10^6$, deci $n\ge 2\sqrt{2}\cdot 10^3 \cong 2,828427\cdot 10^3 = 2828,427$

luăm $n=2829$.

4) De câte subintervale egale ale intervalului $[0,1]$ este nevoie pentru a aproxima integrala exactă $I(f)=\int_0^1 \frac{1}{1+x}\,dx$ cu o eroare absolută de cel mult 10^{-10}, folosind formula de cuadratură sumată a dreptunghiului, $I^n_{dreptunghi}$?

$$\text{Eroarea} = I(f) - I^n_{dreptunghi} = \frac{b-a}{6}\cdot f''(\xi)\cdot h^2,\ \xi\in(a,b)$$

la noi: $a=0,\ b=1,\ h=\frac{b-a}{2n}=\frac{1}{2n}$

$f(x)=\frac{1}{1+x}$

$f'(x)=\frac{-1}{(1+x)^2},\quad f''(x)=\frac{2(x+1)}{(x+1)^4}=\frac{2}{(x+1)^3}$

$$|E_n(f)|\le 10^{-10} \iff \left|\frac{1}{6}\cdot\frac{2}{(\xi+1)^3}\cdot\left(\frac{1}{2n}\right)^2\right|\le 10^{-10},\ \xi\in(0,1)$$

$$\iff \frac{1}{3}\cdot\frac{1}{(\xi+1)^3}\cdot\frac{1}{4n^2}\le 10^{-10},\ \xi\in(0,1)$$

$$\frac{10^{10}}{12(\xi+1)^3}\le n^2,\ \xi\in(0,1) \Rightarrow n^2\ge\frac{10^{10}}{12\cdot 1^3}=\frac{10^{10}}{12}\Rightarrow$$

$$\Rightarrow n\ge\sqrt{\frac{10^{10}}{12}}=\frac{10^5}{2\sqrt{3}}=$$

$\simeq 28{,}867{,}5$ $\quad$ luăm $n = 28868$.

5) Aproximați integrala $I = \int_{-1}^{1} -x^2 - x + 2 \, dx$, folosind formula de cuadratură sumată Simpson, $I^{n}_{Simpson}$, $n = 2$.

$$I^{n}_{Simpson} = \frac{h}{3}\left(f(x_1) + 4 \cdot \sum_{k=1}^{n} f(x_{2k}) + 2 \sum_{k=1}^{n-1} f(x_{2k+1}) + f(x_{2n+1})\right)$$

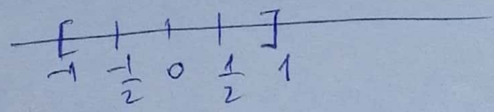
$$h = \frac{b-a}{2n}, \quad x_k = a + (k-1) \cdot h, \quad k = \overline{1, 2n+1}$$

$n = 2$, $h = \frac{1-(-1)}{2 \cdot 2} = \frac{2}{4} = \frac{1}{2}$.

$x_k = -1 + (k-1) \cdot \frac{1}{2}$, $k = \overline{1, 2 \cdot 2+1} = \overline{1,5}$

$x_1 = -1$

$x_2 = -1 + \frac{1}{2} = -\frac{1}{2}$



$x_3 = -1 + 1 = 0$

$x_4 = -1 + \frac{3}{2} = \frac{1}{2}$

$x_5 = -1 + 4 \cdot \frac{1}{2} = 1$.

$$I^{2}_{Simpson} = \frac{1}{3} \cdot \frac{1}{2} \cdot \left(f(-1) + 4 \cdot \sum_{k=1}^{2} f(x_{2k}) + 2 \cdot \sum_{k=1}^{1} f(x_{2k+1}) + f(x_5)\right)$$

$$= \frac{1}{6}\left(f(-1) + 4 \cdot f(x_2) + 4 \cdot f(x_4) + 2 \cdot f(x_3) + f(x_5)\right) =$$

$$= \frac{1}{6}\left(f(-1) + 4 \cdot f\left(-\frac{1}{2}\right) + 4 \cdot f\left(\frac{1}{2}\right) + 2 \cdot f(0) + f(1)\right)$$

$f(0) = 2$, $f(1) = -1 - 1 + 2 = 0$, $f(-1) = -1 + 1 + 2 = 2$

$f\left(-\frac{1}{2}\right) = -\frac{1}{4} + {}^{2)}\frac{1}{2} + 2 = \frac{1}{4} + {}^{4)}2 = \frac{9}{4}$

$f\left(\frac{1}{2}\right) = -\frac{1}{4} - {}^{2)}\frac{1}{2} + 2 = -\frac{3}{4} + {}^{4)}2 = \frac{5}{4}$

$$I^{2}_{Simpson} = \frac{1}{6}\left(2 + \cancel{4} \cdot \frac{9}{\cancel{4}} + \cancel{4} \cdot \frac{5}{\cancel{4}} + 2 \cdot 2 + 0\right) = \frac{1}{6} \cdot 20 = 3{,}(3).$$

analitic, $I = \left(-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right)\Big|_{-1}^{1} =$

$= -\frac{1}{3} - \frac{1}{2} + 2 - \left(-\frac{-1}{3} - \frac{1}{2} - 2\right) =$

$= -\frac{1}{3} - \cancel{\frac{1}{2}} + 2 - \frac{1}{3} + \cancel{\frac{1}{2}} + 2 = -\frac{2}{3} + 4 = \frac{10}{3} = 3,(3).$

6) Aproximați integrala $I = \int_{-1}^{1} -x^2 - x + 2\, dx$, folosind formula de cuadratură sumată a dreptunghiului, $n=2$.

$$\boxed{I^n_{dreptunghi} = 2h \cdot \sum_{k=1}^{n} f(x_{2k}), \quad h = \frac{b-a}{2n}, \quad x_k = a + (k-1)h, \quad k = \overline{1, 2n+1}}$$

$n = 2$, $h = \frac{1-(-1)}{2 \cdot 2} = \frac{2}{4} = \frac{1}{2}$

$x_k = -1 + (k-1) \cdot \frac{1}{2}$, $k = \overline{1,5}$

$x_1 = -1$, $x_2 = -\frac{1}{2}$, $x_3 = 0$, $x_4 = \frac{1}{2}$, $x_5 = 1$.

$I^2_{dreptunghi} = 2 \cdot \frac{1}{2} \cdot \sum_{k=1}^{2} f(x_{2k}) = f(x_2) + f(x_4) =$

$= f(-\frac{1}{2}) + f(\frac{1}{2}) = \frac{9}{4} + \frac{5}{4} = \frac{14}{4} = 3,5.$