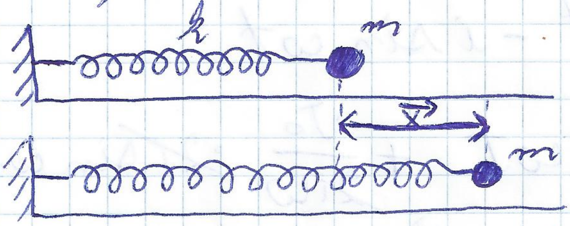
Recapitulare mecanică

I. Subiecte teoretice:

1. Scrieți ecuația de mișcare pentru un corp punctiform de masă m legat de un resort fixat, care are constanta elastică k, în limita de valabilitate a legii Hooke (ecuația oscilatorului liniar armonic). Deduceți elongația, viteza, accelerația corpului, în absența câmpului gravitațional.



$$\left.\begin{aligned} \vec{F}_e &= -k\vec{x} \\ m\vec{a} &= \vec{F}_e \\ \vec{a} &= \ddot{\vec{x}} \end{aligned}\right\} \Rightarrow -k\vec{x} = m\ddot{\vec{x}} \Rightarrow$$

$$\Rightarrow m\ddot{\vec{x}} + k\vec{x} = 0 \Rightarrow \left.\begin{aligned} &\ddot{x} + \frac{k}{m}x = 0 \\ &\omega = \sqrt{\frac{k}{m}} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \ddot{x} + \omega^2 x = 0$$

$$x = c\,e^{kt} \qquad \dot{x} = ck\,e^{kt} \qquad \ddot{x} = ck^2 e^{kt}$$

$$ck^2 e^{kt} + \omega^2 x = 0 \Rightarrow x k^2 + x\omega^2 = 0 \Rightarrow$$

$$\Rightarrow x(k^2 + \omega^2) = 0 \Rightarrow k^2 + \omega^2 = 0 \Rightarrow$$

$$\Rightarrow k^2 = -\omega^2 \Rightarrow k = \pm i\omega$$

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

$$\begin{cases} x(0) = x_0 \\ v(0) = v_0 \end{cases} \qquad x_0 = c_1 + c_2$$

$$\dot{x} = v = i\omega(c_1 e^{i\omega t} - c_2 e^{-i\omega t})$$

$$v_0 = i\omega(c_1 - c_2)$$

$$\begin{cases} x_0 = c_1 + c_2 \\ v_0 = i\omega(c_1 - c_2) \end{cases} \Rightarrow \begin{cases} c_1 = x_0 - c_2 \\ v_0 = i\omega(x_0 - 2c_2) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} c_1 = \frac{v_0 + i\omega x_0}{2i\omega} \\ c_2 = -\frac{v_0 - i\omega x_0}{2i\omega} \end{cases} \Rightarrow \begin{cases} c_1 = \left(\frac{v_0}{i\omega} + x_0\right)/2 \\ c_2 = \left(x_0 - \frac{v_0}{i\omega}\right)/2 \end{cases}$$

$$x(t) = \frac{x_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) + \frac{v_0}{2i\omega}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$\begin{cases} e^{i\omega t} = \cos\omega t + i\sin\omega t \\ e^{-i\omega t} = \cos\omega t - i\sin\omega t \end{cases}$$

elongația $x(t) = \frac{x_0}{\cancel{2}} \cdot \cancel{2}\cos\omega t + \frac{v_0}{\cancel{2i}\omega} \cdot \cancel{2i}\sin\omega t \Rightarrow$

$$\Rightarrow x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

viteza $v(t) = \dot{x}(t) = -x_0\omega\sin\omega t + v_0\cos\omega t$

accelerația $a(t) = \dot{v}(t) = \ddot{x}(t) = -x_0\omega^2\cos\omega t - v_0\omega\sin\omega t =$

$$= -\omega^2\left(x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t\right) = -\omega^2 \cdot x$$

$$x_0 = A\cos\alpha$$

$$v_0 = \pm A\omega\sin\alpha$$

elongația $x(t) = A(\cos\omega t\cos\alpha - \sin\omega t\sin\alpha) = A\cos(\omega t + \alpha)$

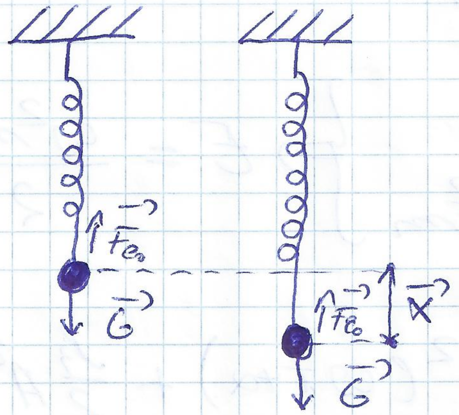
viteza $v(t) = -A\omega\sin\omega t\cos\alpha - A\omega\sin\alpha\cos\omega t =$

$$= -A\omega(\sin\omega t\cos\alpha + \cos\omega t\sin\alpha) =$$

$$= -A\omega\sin(\omega t + \alpha)$$

accelerația $a(t) = -\omega^2 x = -\omega^2 A\cos(\omega t + \alpha)$

2. Scrieți ecuația de mișcare pentru un corp punctiform de masă m suspendat vertical în câmp gravitațional de un resort cu constanta elastică k, în limita de valabilitate a legii Hooke (ecuația oscilatorului liniar armonic). Deduceți elongația mișcării (la soluția ecuației diferențiale omogene adăugați o soluție particulară și apoi găsiți forma explicită a soluției în funcție de condițiile inițiale pentru elongație și viteză).



$$\left.\begin{array}{l}\vec{F}_{e_0} + \vec{G} = 0\\ m\vec{a} = \vec{F}_e + \vec{G}\\ \vec{G} = m\cdot\vec{g}\end{array}\right\} \Rightarrow$$

$$\Rightarrow m\vec{a} = \vec{F}_e + m\cdot\vec{g} \Rightarrow$$

$$\Rightarrow m\cdot\ddot{\vec{x}} = -k\vec{x} + m\cdot\vec{g} \Rightarrow$$

$$\vec{a} = \ddot{\vec{x}} \qquad \vec{F}_{e_0} = -k\cdot\vec{x}$$

$$\Rightarrow \ddot{\vec{x}} + \frac{k}{m}\vec{x} = \vec{g} \quad \Rightarrow \ddot{x} + \omega^2 x = g$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\begin{cases} x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t\\ v(t) = -x_0\omega\sin\omega t + v_0\cos\omega t\end{cases}$$

$$x^2(t) + \frac{v^2(t)}{\omega^2} = \cancel{x^2(0) + \frac{v^2(0)}{\omega^2}} = x_0^2 + \frac{v_0^2}{\omega^2} = A^2 \Rightarrow$$

$$\Rightarrow \begin{cases} x_0 = A\cos\alpha\\ v_0 = \pm\omega A\sin\alpha\end{cases} \Rightarrow x = A\cos\alpha\cos\omega t - A\sin\alpha\sin\omega t$$

$$\Rightarrow x = A\cos(\omega t + \alpha)$$

3. Calculati energiile potentiale si cinetice ale oscilatorului liniar armonic si verificati conservarea energiei mecanice în absenta fortelor disipative

$$E_c = \frac{mv^2}{2} \Rightarrow E_c = \frac{m}{2}\omega^2 A^2 \sin^2(\omega t+\alpha)$$

$$v = -A\omega \sin(\omega t+\alpha)$$

$$E_p = \frac{kx^2}{2} \Rightarrow E_p = \frac{k}{2}A^2\cos^2(\omega t+\alpha)$$

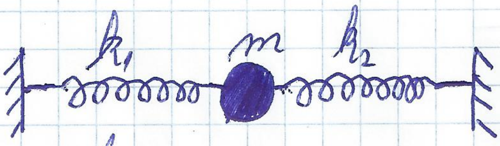
cu $x = A\cos(\omega t+\alpha)$

$$\left.\begin{array}{l} E = E_c + E_p = \dfrac{kA^2}{2} \\ \omega = \sqrt{\dfrac{k}{m}} \Rightarrow k = \omega^2 m \end{array}\right\} \Rightarrow E = \frac{\omega^2 m A}{2}$$

$$\left.\begin{array}{l} E_c + E_p = \dfrac{m}{2}\omega^2 A^2 \sin^2(\omega t+\alpha) + \dfrac{k}{2}A^2\cos^2(\omega t+\alpha) \Rightarrow \\ \omega = \sqrt{\dfrac{k}{m}} \end{array}\right\} \Rightarrow \left.\begin{array}{l} E_c + E_p = \dfrac{k}{2}A^2 \\ k = \omega^2 m \end{array}\right\} \Rightarrow$$

$$\Rightarrow E_c + E_p = \frac{\omega^2 m A}{2} = E \Rightarrow E = E_c + E_p$$

4. Compuneți două oscilații armonice paralele prin metodele: fazorială, trigonometrică, a numerelor complexe.



a) Trigonometric

$$x = A\cos(\omega t+\alpha)$$

$$\begin{cases} x_1 = A_1\cos(\omega t+\alpha_1) \\ x_2 = A_2\cos(\omega t+\alpha_2) \end{cases}$$

$$k_1 = k_2$$

$$x_1+x_2 = A\cos(\omega t+\alpha) = x \qquad A=?\quad \alpha=?$$

$$x_1+x_2 = A_1\cos\omega t\cos\alpha_1 - A_1\sin\omega t\sin\alpha_1 + A_2\cos\omega t\cos\alpha_2 - A_2\sin\omega t\sin\alpha_2 = A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha \Rightarrow$$

$$\Rightarrow \begin{cases} A\cos\alpha = A_1\cos\alpha_1 + A_2\cos\alpha_2 \\ A\sin\alpha = A_1\sin\alpha_1 + A_2\sin\alpha_2 \end{cases} \Rightarrow \begin{cases} A^2\cos^2\alpha = A_1^2\cos^2\alpha_1 + A_2^2\cos^2\alpha_2 + 2A_1A_2\cos\alpha_1\cos\alpha_2 \\ A^2\sin^2\alpha = A_1^2\sin^2\alpha_1 + A_2^2\sin^2\alpha_2 + 2A_1A_2\sin\alpha_1\sin\alpha_2 \end{cases}$$

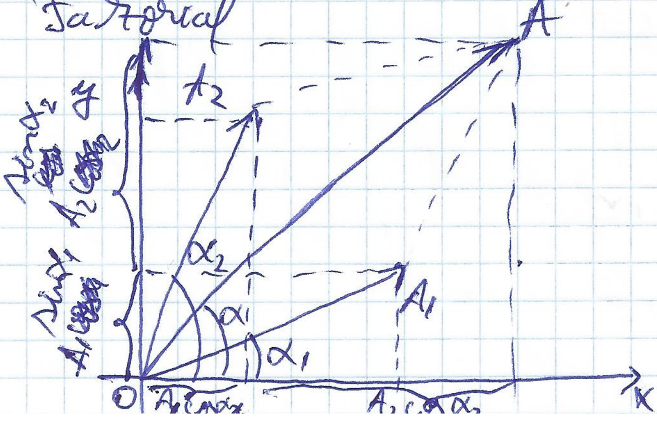
$$\Rightarrow A^2 = A_1^2 + A_2^2 + 2A_1A_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$$

$$\Rightarrow A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_1-\alpha_2)$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_1-\alpha_2)}$$

$$tg\,\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2} \Rightarrow \alpha = arctg\,\frac{A_1\sin\alpha_1 + A_2\cos\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}$$

b) Fazorial



$$\vec{A_1} + \vec{A_2} = \vec{A} \Rightarrow$$

$$\Rightarrow (\vec{A_1}+\vec{A_2})(\vec{A_1}+\vec{A_2}) = \vec{A}\cdot\vec{A}$$

$$\vec{a}\cdot\vec{b} = |a||b|\cos(\widehat{a,b})$$

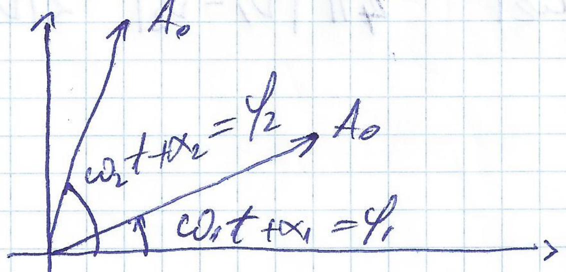
$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_1 - \alpha_2) \Rightarrow$$

$$\Rightarrow A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_1 - \alpha_2)}$$

$$tg\,\alpha = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}$$

5. Prezentați fenomenul de "bătaie" obținut la compunerea oscilațiilor. Câte oscilații cu frecvența semnalului purtător se obțin în intervalul de timp corespunzător unei bătăi?

$$\begin{cases} x_1 = A_0 \cos(\omega_1 t + \alpha_1) \\ x_2 = A_0 \cos(\omega_2 t + \alpha_2) \end{cases}$$

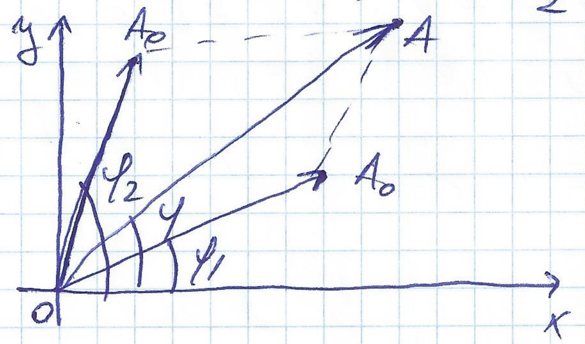


$$|\vec{A_1}| = |\vec{A_2}| = A_0$$

$$\vec{A} \cdot \vec{A} = (\vec{A_1} + \vec{A_2})(\vec{A_1} + \vec{A_2}) \Rightarrow A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_1 - \varphi_2)$$

$$\Rightarrow A^2 = 2A_0^2(1 + \cos(\varphi_1 - \varphi_2)) = 4A_0^2 \cos^2\frac{\varphi_1 - \varphi_2}{2} \Rightarrow$$

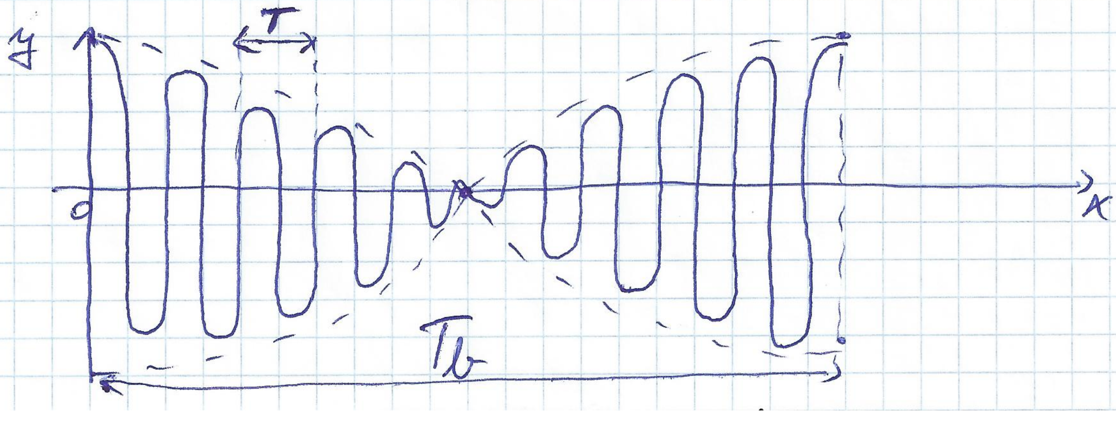
$$\Rightarrow A = 2A_0 \left|\cos\frac{\varphi_1 - \varphi_2}{2}\right| \Rightarrow A = 2A_0 \left|\cos\frac{(\omega_1 - \omega_2)t + \alpha_1 - \alpha_2}{2}\right|$$



$$\varphi_1 = \omega_1 t + \alpha_1$$
$$\varphi_2 = \omega_2 t + \alpha_2$$
$$\varphi = \frac{\varphi_1 + \varphi_2}{2}$$

$$X = A\cos\left(\frac{\omega_1 + \omega_2}{2} t + \frac{\alpha_1 + \alpha_2}{2}\right) \qquad X = X_1 + X_2 \Rightarrow$$

$$\Rightarrow A\cos\left(\frac{\omega_1 + \omega_2}{2} t + \frac{\alpha_1 + \alpha_2}{2}\right) = 2A_0 \cos\left(\frac{\omega_1 - \omega_2}{2} t + \frac{\alpha_1 - \alpha_2}{2}\right)\cos\left(\frac{\omega_1 + \omega_2}{2} t + \frac{\alpha_1 + \alpha_2}{2}\right) = X$$



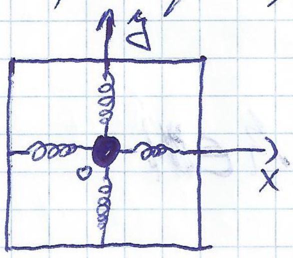
$$\omega = \frac{2\pi}{T} = 2\pi\nu \qquad \omega_p = \frac{\omega_1+\omega_2}{2} \Rightarrow T = \frac{2\pi}{\omega_p} = \frac{4\pi}{\omega_1+\omega_2}$$

$$\omega_m = \frac{|\omega_1-\omega_2|}{2}$$

$$T_b = \frac{2\pi}{2\omega_m} = \frac{2\pi}{|\omega_1-\omega_2|}$$

$$N = \frac{T_b}{T} = \frac{\omega_1+\omega_2}{2|\omega_1-\omega_2|} = \frac{2\pi(\nu_1+\nu_2)}{4\pi|\nu_1-\nu_2|} = \frac{\nu_1+\nu_2}{2|\nu_1-\nu_2|}$$

6. Compuneți două oscilații armonice perpendiculare. Deduceți forma eliptică a traiectoriei în cazul frecvențelor egale. Comentați formele specifice ale elipsei, în funcție de fazele inițiale.



$$\begin{cases} x = A\cos(\omega t+\alpha) \\ y = B\cos(\omega t+\beta) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{x}{A} = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha \quad \big| \cdot \cos\beta \\ \frac{y}{B} = \sin\omega t\cos\beta + \cos\omega t\sin\beta \quad \big| \cos\alpha \end{cases} \Rightarrow$$

$$\Rightarrow \frac{x}{A}\cos\beta - \frac{y}{B}\cos\alpha = -\cos\omega t\sin(\beta-\alpha)$$

$$\begin{cases} \frac{x}{A} = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha \quad \big| \sin\beta \\ \frac{y}{B} = \sin\omega t\cos\beta + \cos\omega t\sin\beta \quad \big| \sin\alpha \end{cases} \Rightarrow$$

$$\Rightarrow \frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha = \sin\omega t \cdot \sin(\beta-\alpha)$$

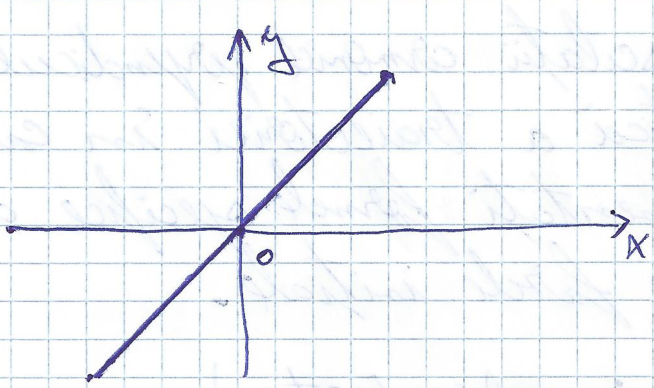
$$\begin{cases} \frac{x}{A}\cos\beta - \frac{y}{B}\cos\alpha = -\cos\omega t\sin(\beta-\alpha) \\ \frac{x}{A}\sin\beta - \frac{y}{B}\sin\alpha = \sin\omega t \cdot \sin(\beta-\alpha) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{x^2}{A^2}\cos^2\beta - \frac{2xy}{AB}\cos\alpha\cos\beta + \frac{y^2}{B^2}\cos^2\alpha = \cos^2\omega t\sin^2(\beta-\alpha) \\ \frac{x^2}{A^2}\sin^2\beta - \frac{2xy}{AB}\sin\alpha\sin\beta + \frac{y^2}{B^2}\sin^2\alpha = \sin^2\omega t\sin^2(\beta-\alpha) \end{cases}$$

$$\Rightarrow \frac{x^2}{A^2} - \frac{2xy}{AB}\cos(\beta-\alpha) + \frac{y^2}{B^2} = \sin^2(\beta-\alpha)$$

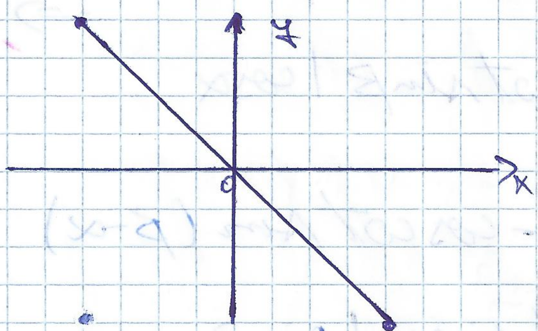
a) $\beta - \alpha = 2k\pi \quad k \in \mathbb{N} \Rightarrow$

$$\Rightarrow \frac{x}{A} - \frac{y}{B} = 0 \Rightarrow y = \frac{B}{A} \cdot x$$

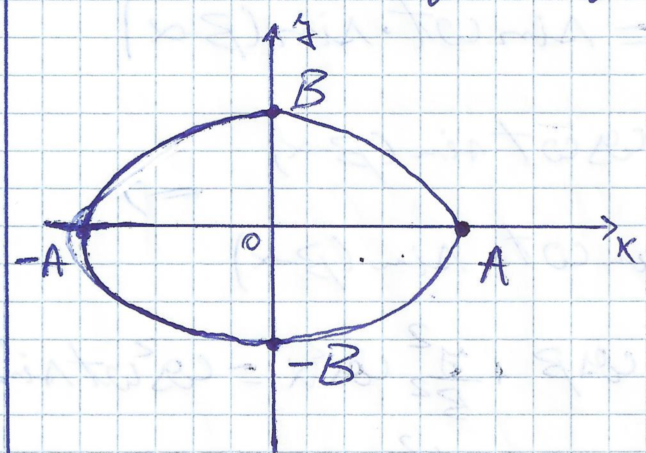


b) $\beta - \alpha = (2k+1)\pi \qquad k \in \mathbb{N}$

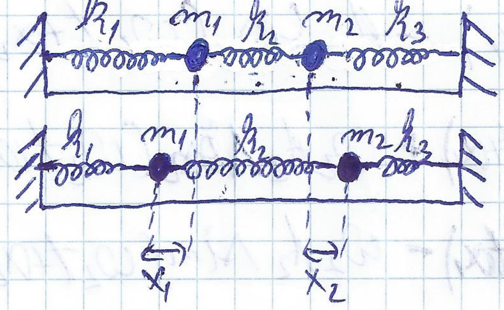
$$\frac{x}{A} + \frac{y}{B} = 0 \Rightarrow y = -\frac{B}{A}x$$



c) $\beta - \alpha = \frac{\pi}{2} \text{ sau } \frac{3\pi}{2} \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$



7. Rezolvați problema oscilațiilor armonice în cazul a două mișcări oscilatorii liniar armonice cuplate (oscilațiile pe perna de aer liniară).



$$\begin{cases} m_1 \cdot \ddot{\vec{x}}_1 = -k_1\vec{x}_1 + k_2\vec{x}_2 - k_2\vec{x}_1 \\ m_2 \cdot \ddot{\vec{x}}_2 = -k_3\vec{x}_2 - k_2\vec{x}_2 + k_2\vec{x}_1 \end{cases}$$

$$\begin{cases} k_2 = k_c \\ m_1 = m_2 = m \\ k_1 = k_3 = k \end{cases} \qquad \begin{cases} \ddot{x}_1 + \frac{k+k_c}{m}x_1 - \frac{k_c}{m}x_2 = 0 \\ \ddot{x}_2 + \frac{k+k_c}{m}x_2 - \frac{k_c}{m}x_1 = 0 \end{cases}$$

$$\begin{cases} \frac{k+k_c}{m} = \omega_0^2 \\ \frac{k_c}{m} = \omega_c^2 \end{cases} \qquad \begin{cases} \ddot{x}_1 + \omega_0^2 x_1 - \omega_c^2 x_2 = 0 \\ \ddot{x}_2 + \omega_0^2 x_2 - \omega_c^2 x_1 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x_1 + x_2 = q_1 \\ x_1 - x_2 = q_2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{q_1+q_2}{2} \\ x_2 = \frac{q_1-q_2}{2} \end{cases} \qquad \begin{cases} \ddot{q}_1 + (\omega_0^2 - \omega_c^2)q_1 = 0 \\ \ddot{q}_2 + (\omega_0^2 + \omega_c^2)q_2 = 0 \end{cases}$$

$$\begin{cases} \omega_1^2 = \omega_0^2 - \omega_c^2 = \frac{k}{m} \\ \omega_2^2 = \omega_0^2 + \omega_c^2 = \frac{k+2k_c}{m} \end{cases}$$

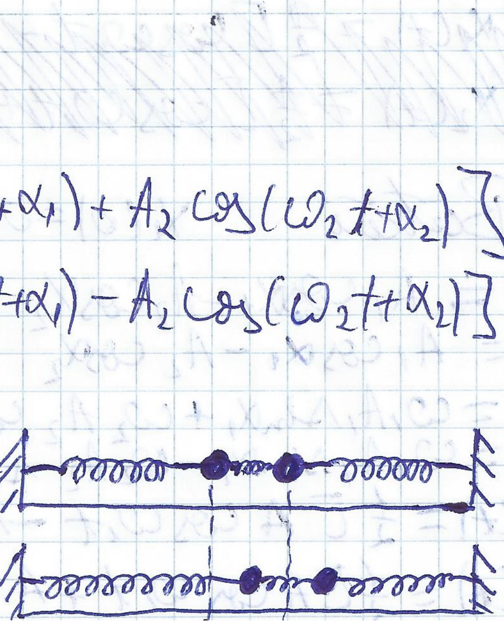
$$\begin{cases} q_1(t) = A_1 \cos(\omega_1 t + \alpha_1) \\ q_2(t) = A_2 \cos(\omega_2 t + \alpha_2) \end{cases}$$

$$\begin{cases} \dot{x}_1 = A_1 \rho e^{\rho t} \\ \ddot{x} = A_1 \rho^2 e^{\rho t} \\ x = A_1 e^{\rho t} \end{cases}$$

$$\begin{cases} x_1(t) = \frac{1}{2}[A_1\cos(\omega_1 t + \alpha_1) + A_2\cos(\omega_2 t + \alpha_2)] \\ x_2(t) = \frac{1}{2}[A_1\cos(\omega_1 t + \alpha_1) - A_2\cos(\omega_2 t + \alpha_2)] \end{cases}$$

a) Oscilații simetrice

$x_1(0) = x_2(0) = A$



$\dot{x}_1(0) = \dot{x}_2(0) = 0$

$x_1(t) = \frac{1}{2}[A_1 \cos(\omega_1 t + \alpha_1) + A_2 \cos(\omega_2 t + \alpha_2)]$

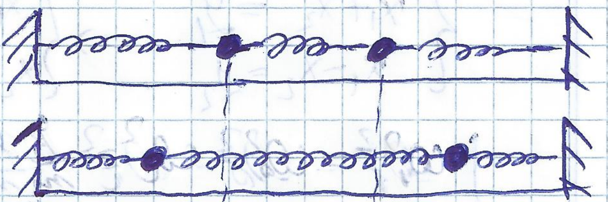
$x_2(t) = \frac{1}{2}[A_1 \cos(\omega_1 t + \alpha_1) - A_2 \cos(\omega_2 t + \alpha_2)]$

$\dot{x}_1(t) = -\frac{1}{2}[\omega_1 A_1 \sin(\omega_1 t + \alpha_1) + \omega_2 A_2 \sin(\omega_2 t + \alpha_2)]$

$\dot{x}_2(t) = -\frac{1}{2}[\omega_1 A_1 \sin(\omega_1 t + \alpha_1) - \omega_2 A_2 \sin(\omega_2 t + \alpha_2)]$

$$\begin{cases} 2A = A_1 \cos\alpha_1 + A_2 \cos\alpha_2 \\ 2A = A_1 \cos\alpha_1 - A_2 \cos\alpha_2 \\ 0 = \omega_1 A_1 \sin\alpha_1 + \omega_2 A_2 \sin\alpha_2 \\ 0 = \omega_1 A_1 \sin\alpha_1 - \omega_2 A_2 \sin\alpha_2 \end{cases} \Rightarrow \begin{cases} A_2 = 0 \\ \alpha_1 = \alpha_2 = 0 \\ A_1 = 2A \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1(t) = A \cos\omega_1 t \\ x_2(t) = A \cos\omega_1 t \end{cases}$$



b) Oscilații antisimetrice

$x_1(0) = A \quad x_2(0) = -A$

$\dot{x}_1(0) = 0 \quad \dot{x}_2(0) = 0$

$$\begin{cases} 2A = A_1 \cos\alpha_1 + A_2 \cos\alpha_2 \\ -2A = A_1 \cos\alpha_1 - A_2 \cos\alpha_2 \\ 0 = \omega_1 A_1 \sin\alpha_1 + \omega_2 A_2 \sin\alpha_2 \\ 0 = \omega_1 A_1 \sin\alpha_1 - \omega_2 A_2 \sin\alpha_2 \end{cases} \Rightarrow \begin{cases} A_1 = 0 \\ A_2 = 2A \\ \alpha_1 = \alpha_2 = 0 \end{cases} \Rightarrow$$

$$\cancel{\Rightarrow \begin{cases} x_1(t) = \frac{A}{2}[\cos\omega_1 t - \cos\omega_2 t] \\ x_2(t) = \frac{A}{2}[\cos\omega_1 t + \cos\omega_2 t] \end{cases}} \Rightarrow \begin{cases} x_1(t) = A\cos(\omega_2 t) \\ x_2(t) = -A\cos(\omega_2 t) \end{cases}$$

c) Bătăi $\quad x_1(0) = 0 \quad x_2(0) = A \quad \dot{x}_1(0) = \dot{x}_2(0) = 0$

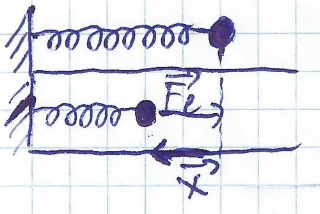
$$\begin{cases} 0 = A_1 \cos\alpha_1 + A_2 \cos\alpha_2 \\ 2A = A_1 \cos\alpha_1 - A_2 \cos\alpha_2 \\ 0 = \omega_1 A_1 \sin\alpha_1 + \omega_2 A_2 \cos\alpha_2 \\ 0 = \omega_1 A_1 \sin\alpha_1 - \omega_2 A_2 \sin\alpha_2 \end{cases} \Rightarrow \begin{cases} A_1 = A \\ A_2 = -A \\ \alpha_1 = \alpha_2 = 0 \end{cases}$$

$$\begin{cases} x_1(t) = \frac{1}{2}[A\cos\omega_1 t - A\cos\omega_2 t] \\ x_2(t) = \frac{1}{2}[A\cos\omega_1 t + A\cos\omega_2 t] \end{cases}$$

8. Tratați problema oscilațiilor amortizate pseudo-periodice

$$\vec{R} = -r\dot{\vec{x}}$$

$$m\ddot{\vec{x}} = \vec{R} + \vec{F_e} \Rightarrow m\ddot{\vec{x}} = -r\dot{\vec{x}} - k\vec{x}$$



$$\ddot{\vec{x}} + \frac{r}{m}\dot{\vec{x}} + \frac{k}{m}\vec{x} = 0$$

$$\begin{cases} \frac{r}{m} = 2b \\ \frac{k}{m} = \omega^2 \end{cases}$$

$$\ddot{x} + 2b\dot{x} + \omega^2 x = 0 \qquad x = Ae^{\rho t}$$

$$\rho^2 Ae^{\rho t} + \frac{r}{m}\rho Ae^{\rho t} + \frac{k}{m}Ae^{\rho t} = 0 \Rightarrow$$

$$\Rightarrow \rho^2 x + 2b\rho x + \omega^2 x = 0 \Rightarrow x(\rho^2 + 2b\rho + \omega^2) = 0 \Rightarrow$$

$$\Rightarrow \rho^2 + 2b\rho + \omega^2 = 0$$

$$\Delta = 4b^2 - 4\omega^2 \qquad \rho_{1,2} = \frac{-2b \pm 2\sqrt{b^2-\omega^2}}{2} =$$

$$= -b \pm \sqrt{b^2-\omega^2}$$

$$x = A_1 e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega^2})t} \Rightarrow$$

$$\Rightarrow x = e^{-bt}\left(A_1 e^{t\sqrt{b^2-\omega^2}} + A_2 e^{-t\sqrt{b^2-\omega^2}}\right)$$

$$b < \omega \qquad \omega^2 - b^2 = \omega'^2$$

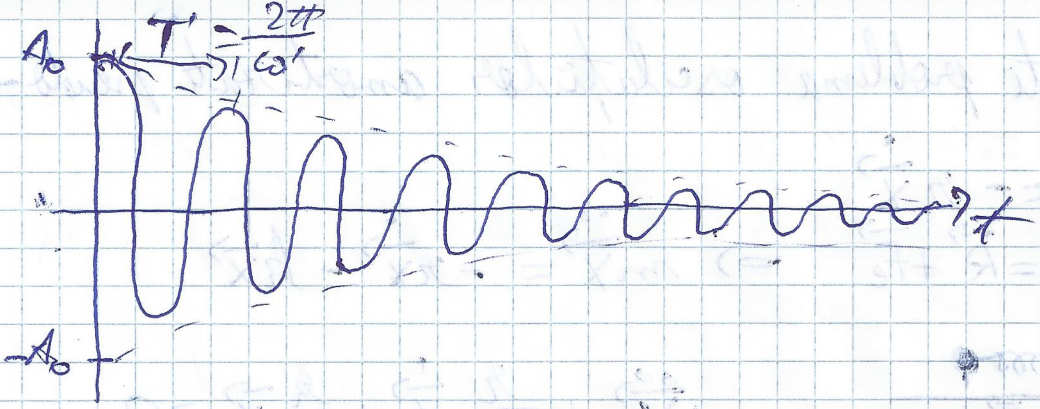
$$\sqrt{b^2-\omega^2} = \sqrt{-\omega'^2} = \pm i\omega'$$

$$x(t) = e^{-bt}\left(A_1 e^{i\omega' t} + A_2 e^{-i\omega' t}\right)$$

$$\begin{cases} A_1 = \frac{1}{2}A_0 e^{i\alpha} \\ A_2 = \frac{1}{2}A_0 e^{-i\alpha} \end{cases}$$

$$x(t) = \frac{A_0}{2}e^{-bt}\left(e^{i(\omega' t+\alpha)} + e^{-i(\omega' t+\alpha)}\right) =$$

$$= A_0 e^{-bt}\cos(\omega' t+\alpha)$$



$$\frac{x(t)}{x(t+T')} = \frac{A_0 e^{-bt}\cos(\omega' t+\alpha)}{A_0 e^{-b(t+T')}\cos[\omega'(t+T')+\alpha]}$$

$$\omega'(t+T') = \omega' t + 2\pi$$

$$\frac{x(t)}{x(t+T')} = e^{+bT'} \Rightarrow \ln\frac{x(t)}{x(t+T')} = bT' = D$$

$$A(\tau) = A_0 e^{-1} = A_0 e^{-b\tau} \Rightarrow -1 = -b\tau \Rightarrow \tau = \frac{1}{b}$$

τ este timp de relaxare

D este decrementul

9. Obțineți elongația, viteza și accelerația în mișcarea oscilatorie forțată în prezența amortizării, în regim staționar. Prezentați fenomenul de rezonanță

$$m\vec{a} = \vec{F_e} + \vec{R} + F_0 \cos \Omega t \Rightarrow$$

$$\Rightarrow m\ddot{x} + r\dot{x} + k x = F_0 \cos \Omega t \Rightarrow$$

$$\Rightarrow \ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos \Omega t \Rightarrow$$

$$\Rightarrow \ddot{x} + 2b\dot{x} + \omega^2 x = \frac{F_0}{m}\cos \Omega t \qquad b = \frac{r}{2m}$$

$$\omega^2 = \frac{k}{m}$$

$$x_p(t) = B\cos(\Omega t + \beta)$$

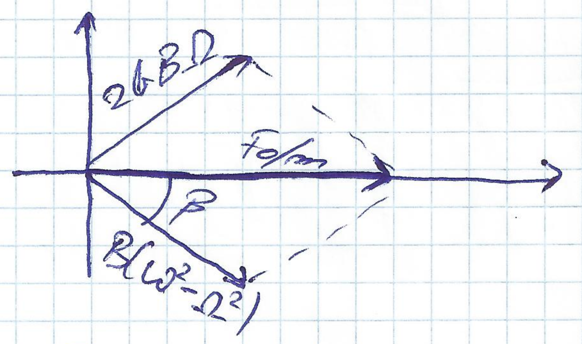
$$\dot{x}_p(t) = -B\Omega \cos(\Omega t + \beta) = B\Omega \cos(\Omega t + \beta + \frac{\pi}{2})$$

$$\ddot{x}_p(t) = -B\Omega^2 \cos(\Omega t + \beta)$$

$$-B\Omega^2 \cos(\Omega t + \beta) + 2bB\Omega \cos(\Omega t + \beta) + \omega^2 B\cos(\Omega t + \beta)$$

$$\Rightarrow B(\omega^2 - \Omega^2)\cos(\Omega t + \beta) + 2bB\Omega \cos(\Omega t + \beta) =$$

$$= \frac{F_0}{m}\cos \Omega t$$



$$\frac{F_0}{m} = B\sqrt{4b^2\Omega^2 + (\omega^2 - \Omega^2)^2} \Rightarrow B = \frac{F_0}{m\sqrt{4b^2\Omega^2 + (\omega^2 - \Omega^2)^2}}$$

$$\operatorname{tg}\beta = \frac{-2b\Omega}{\omega^2 - \Omega^2}$$

$$f(\Omega^2) = 4b^2\Omega^2 + (\omega^2 - \Omega^2)^2 \Rightarrow$$

$$\Rightarrow f(\Omega^2) = \Omega^4 + \Omega^2\, 2(2b^2 - \omega^2) + \omega^4$$

$$\cancel{\Delta = 16b^4 - 16b^2\omega^2}$$

$$\cancel{\Omega^2_{min} = -\frac{\Delta}{4} = 4b^2\omega^2 + 4b^4}$$

$$\Omega^2_{min} = \omega^2 - 2b^2 = \Omega^2_{rezonanță}$$

$$B(\Omega^2_{rezonanță}) = \frac{F_0}{m} \cdot \frac{1}{\sqrt{(\omega^2-\omega^2+2b^2)^2+4b^2(\omega^2-2b^2)}} =$$

$$= \frac{F_0}{m} \cdot \frac{1}{\sqrt{4b^2\omega^2-4b^4}} = \frac{F_0}{m2b\sqrt{\omega^2-b^2}} = B_{max}$$

$$B_{max} = \frac{F_0}{2bm\omega\sqrt{1-\frac{b^2}{\omega^2}}}$$

- Cazul static $\quad B(\Omega=0) = B_{static} = \frac{F_0}{m\omega^2}$

$$\frac{B_{max}}{B_{static}} = \frac{\omega}{2b\sqrt{1-\frac{b^2}{\omega^2}}}$$

10. Calculați puterea activă și puterea reactivă în cazul mișcării oscilatorii forțate în prezența amortizării, în regim staționar.

a) Puterea activă

$$\left.\begin{aligned} P_a &= \frac{dL}{dt} = L'(t) \\ dL &= F\,dx \end{aligned}\right\} \Rightarrow P_a = F\frac{dx}{dt} = F x'(t) = F\dot{x}(t) = $$
$$= F v(t)$$

$$P_a = \overline{Fv} = \frac{1}{T}\int_0^T F v\,dt. \qquad T = \frac{2\pi}{\Omega}$$

$$\frac{1}{T}F_0\int_0^T \cos(\Omega t)(-B)\Omega\cos(\Omega t+\beta)\,dt =$$

$$= -\frac{F_0 B\Omega}{T}\int_0^T \cos\Omega t\cdot\cos(\Omega t+\beta)\,dt =$$

$$= -\frac{F_0 B\Omega}{2T}\int_0^T \cos(2\Omega t+\beta)+\cos\beta\,dt =$$

$$= -\frac{F_0 B\Omega}{2T}T\sin\beta =$$

$$= -\frac{F_0 B\Omega\sin\beta}{2} \Rightarrow P_a = -\frac{F_0 B\Omega\sin\beta}{2}$$

b) Puterea reactivă

$$P_r = \overline{F_r\cdot v} = -r\overline{v^2} = -2bm\overline{v^2} = -\frac{2bm}{T}B^2\Omega^2\int_0^T \sin^2($$
$$\Omega t+\beta)\,dt = -\frac{2bm}{T}B^2\Omega^2\int_0^T \sin^2(\Omega t+\beta)\,dt =$$

$$= -\frac{2bm}{T}B^2\Omega^2\int_0^T \frac{1-\cos(2\Omega+2\beta)}{2}\,dt =$$

$$= -bmB^2\Omega^2 = bmB\Omega\frac{F_0\sin\beta}{2bm} =$$

$$= \frac{B\Omega F_0\sin\beta}{2} \Rightarrow P_r = \frac{F_0 B\Omega\sin\beta}{2}$$

11. Prezentați teoria analizei armonice (Fourier) referitoare la descompunerea funcțiilor periodice în serii trigonometrice.

$$f(t) = f(t+T)$$

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt$$

$$\int_0^T \sin n\omega t \cos k\omega t\,dt = 0$$

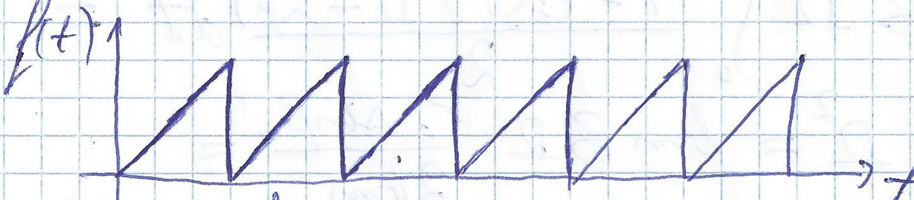
$$\int_0^T \sin n\omega t \sin k\omega t\,dt = \frac{T}{2}\delta_{nk}$$

$$\int_0^T \cos n\omega t \cos k\omega t\,dt = \frac{T}{2}\delta_{nk}$$

$$\delta_{nk} \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

$$\int_0^T \cos(k\omega t) f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{T}{2}\delta_{nk} = \frac{T}{2}a_k \Rightarrow$$

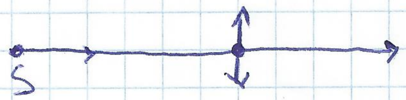
$$\Rightarrow a_n = \frac{2}{T}\int_0^T \cos k\omega t \cdot f(t)\,dt$$



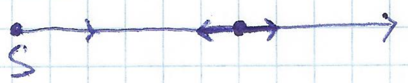
$$\begin{cases} f(t) = \frac{k}{T} \quad t \in [0,T] \\ f(t) = f(t+T) \end{cases}$$

12. Prezentați subiectul: Unda plană. Unda plană monocromată.

O undă se numește undă plană dacă există plane perpendiculare pe direcția de propagare a undei în care punctele oscilează în fază.



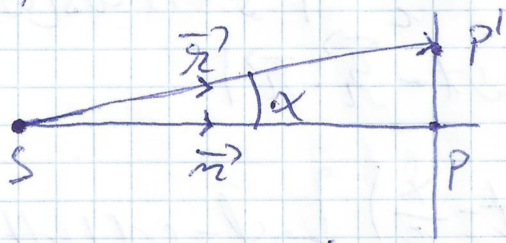
undă transversală



undă longitudinală

$\xi(0,t)$ reprezintă elongatia

$$\xi(x,t) = \xi\left(0, t-\frac{x}{c}\right) = f\left(t-\frac{x}{c}\right)$$



$|n| = 1$

$$\xi_P(x,t) = f\left(t-\frac{x}{c}\right) \qquad \xi_{P'}(\vec{r},t) = f\left(t-\frac{\vec{r}\cdot\vec{n}}{c}\right) = f\left(t-\frac{|\vec{r}|\cos\alpha}{c}\right)$$

$$\xi(0,t) = A\cos\omega t$$

$$\xi(x,t) = A\cos\left[\omega\left(t-\frac{x}{c}\right)\right] = A\cos\left[\frac{2\pi}{T}\left(t-\frac{x}{c}\right)\right] =$$

$$\left.\begin{array}{l} = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi \cdot x}{T\cdot c}\right) \\ \lambda = T\cdot c \end{array}\right\} \Rightarrow \xi(x,t) = A\cos\left(\frac{2\pi t}{T} - \frac{2\pi \cdot x}{\lambda}\right)$$

$$\cancel{\xi(x,t)}\ \xi(x,t+T) = A\cos\left\{\omega\left[(t+T)-\frac{x}{c}\right]\right\} =$$

$$\cancel{= A\cos\left(\omega\left(t-\frac{x}{c}\right)\right)} = A\cos\left(\omega t + 2\pi - \frac{\omega x}{c}\right) =$$

$$= A\cos\left(\omega t - \frac{\omega x}{c}\right) = A\cos\left[\omega\left(t - \frac{x}{c}\right)\right] =$$

$$\xi(x,t) \Rightarrow \xi(x, t+T) = \xi(x,t)$$

$$\left.\begin{array}{l}\xi(x+\Lambda, t) = A\cos\left[\omega\left(t - \frac{x+\Lambda}{c}\right)\right] = A\cos\left[\omega\left(t - \frac{x}{c}\right)\right] \\ \frac{\omega\Lambda}{c} = 2\pi \Rightarrow \Lambda = \frac{2\pi c}{\omega} = T\cdot c = \lambda\end{array}\right\} \Rightarrow$$

$$\Rightarrow \xi(x,t) = \xi(x+\lambda, t)$$

$$\vec{k} = \frac{2\pi}{\lambda}\vec{n}$$

$$\xi(\vec{r},t) = f\left(t - \frac{\vec{r}\cdot\vec{n}}{c}\right)$$

$$\xi(x,t) = A\cos\left[\omega\left(t - \frac{x}{c}\right)\right]$$

$$\xi(\vec{r},t) = A\cos\left[\omega\left(t - \frac{\vec{r}\cdot\vec{n}}{c}\right)\right] = A\cos\left[\omega t - \vec{r}\cdot\frac{2\pi\vec{n}}{\lambda}\right] = A\cos(\omega t - \vec{k}\cdot\vec{r})$$

$$\left.\begin{array}{l}\varphi = \omega t - \vec{k}\cdot\vec{r} = \omega\left(t - \frac{x}{c}\right) \\ x = x + dx \qquad t = t + dt\end{array}\right\} \Rightarrow \varphi = \omega(t+dt) - \frac{\omega(x+dx)}{c}$$

$$\omega\left(t - \frac{x}{c}\right) = \omega\left(t + dt - \frac{x+dx}{c}\right) \Rightarrow \omega dt - \frac{\omega dx}{c} = 0 \Rightarrow$$

$$\Rightarrow dt - \frac{dx}{c} = 0 \Rightarrow c = \frac{dx}{dt} = v_f$$

$$\xi = A\cos(\omega t - \vec{k}\cdot\vec{r}) \Rightarrow = R_e\left[A e^{i(\omega t - \vec{k}\cdot\vec{r})}\right]$$

13. Prezentați subiectul: Ecuația undei plane

$$\begin{cases} x = \frac{d^2 f}{dx^2} - \frac{1}{c^2} \cdot \frac{d^2 f}{dt^2} = 0 \\ \Delta f - \frac{1}{c^2} \cdot \frac{d^2 f}{dt^2} = 0 \end{cases}$$

$$\frac{df}{dt} = \frac{df}{du} \cdot \frac{du}{dt} = \frac{df}{du} \qquad f\left(t - \frac{x}{c}\right) = f(u)$$

$$\frac{d^2 f}{dt^2} = \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d}{du}\left(\frac{df}{du}\right)\frac{du}{dt} = \frac{d^2 f}{du^2}$$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{df}{du}\left(-\frac{1}{c}\right)$$

$$\frac{d^2 f}{dx^2} = \frac{d}{du}\left[\frac{df}{du}\left(-\frac{1}{c}\right)\right]\frac{du}{dx} = \frac{1}{c^2}\frac{d^2 f}{du^2}$$

$$\frac{d^2 f}{dx^2} = \frac{1}{c^2}\frac{d^2 f}{du^2} = 0 \qquad f\left(t - \frac{\vec{r} \cdot \vec{n}}{c}\right) = f(u)$$

$$\left.\begin{aligned} \vec{r} &= x\vec{i} + y\vec{j} + z\vec{k} \\ \vec{n} &= n_x\vec{i} + n_y\vec{j} + n_z\vec{k} \end{aligned}\right\} \Rightarrow \vec{r} \cdot \vec{n} = x n_x + y n_y + z n_z$$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{df}{du}\left(-\frac{n_x}{c}\right)$$

$$\frac{d^2 f}{dx^2} = \frac{d}{du}\left[\frac{df}{du}\left(-\frac{n_x}{c}\right)\right]\frac{du}{dx} = \frac{n_x^2}{c^2}\frac{d^2 f}{du^2}$$

$$\begin{cases} \frac{d^2 f}{dx^2} = \frac{n_x^2}{c^2}\frac{d^2 f}{du^2} \\ \frac{d^2 f}{dy^2} = \frac{n_y^2}{c^2}\frac{d^2 f}{du^2} \\ \frac{d^2 f}{dz^2} = \frac{n_z^2}{c^2} \cdot \frac{d^2 f}{du^2} \end{cases}$$

$$\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dz^2} = \frac{n_x^2 + n_y^2 + n_z^2}{c^2} \cdot \frac{d^2 f}{du^2} = \frac{1}{c^2} \cdot \frac{d^2 f}{du^2}$$

$$\Delta f = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dz^2} = \frac{1}{c^2} \cdot \frac{d^2 f}{du^2}$$

$$\frac{d^2 f}{du^2} = \frac{d^2 f}{dt^2} \Rightarrow \Delta f = \frac{d^2 f}{dt^2}$$

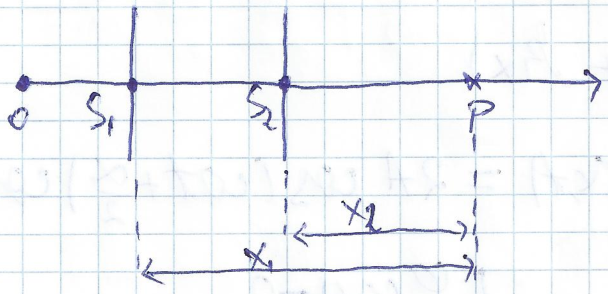
$$\Delta f - \frac{1}{c^2} \cdot \frac{d^2 f}{dt^2} = 0$$

14. Prezentați subiectul: Interferența undelor

a) Interferența undelor plane.

Condiții: 1) undele trebuie să aibă aceeași frecvență.

2) undele trebuie să fie coerente

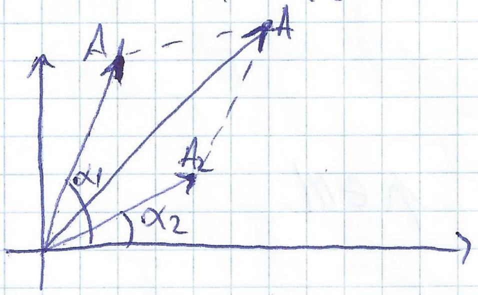


$$\xi_1(x,t) = A_1 \cos(\omega t - k x_1)$$

$$\xi_2(x,t) = A_2 \cos(\omega t - k x_2)$$

$$\xi(x,t) = \xi_1 + \xi_2$$

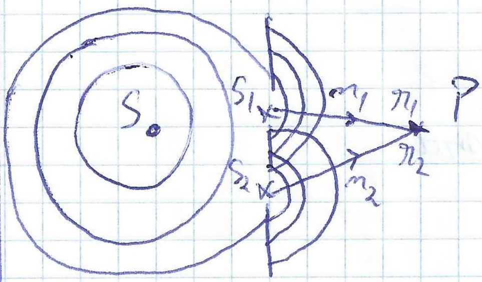
$$\vec{k} = \frac{2\pi}{\lambda} \vec{n}$$



$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos k(x_2 - x_1)} =$$

$$= \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos \frac{2\pi}{\lambda} \Delta x}$$

$$A \begin{cases} \to \text{maxim} \Leftrightarrow \frac{2\pi}{\lambda} \Delta x = 2\pi n \Rightarrow \Delta x = 2n\frac{\lambda}{2} \\ \to \text{minim} \Leftrightarrow k(x_2 - x_1) = 2(n+1)\pi \Rightarrow \Delta x = (2n+1)\frac{\lambda}{2} \end{cases}$$



$$\xi_1(\vec{r}_1, t) = \frac{A}{r_1} \cos(\omega t - k r_1)$$

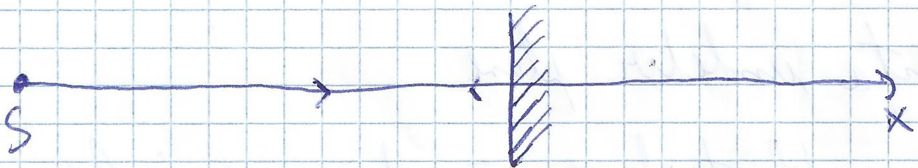
$$A_1(r) = \frac{A}{r_1}$$

$$\xi_2(\vec{r}_2, t) = \frac{A}{r_2} \cos(\omega t - k r_2)$$

$$A_2(r) = \frac{A}{r_2}$$

$$k \Delta r = \begin{cases} \text{max}: \Delta r = 2n\frac{\lambda}{2} \\ \text{minim}: \Delta r = (2n+1)\frac{\lambda}{2} \end{cases}$$

15. Prezentați subiectul: Unde staționare

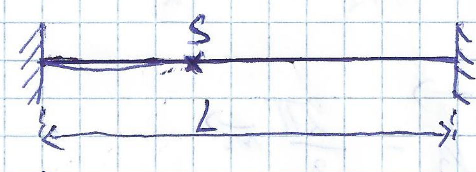


$$\xi_{+}(x,t) = A\cos(\omega t - kx)$$

$$\xi_{-}(x,t) = A\cos(\omega t + kx)$$

$$\xi_{e}(x,t) = \xi_{+}(x,t) + \xi_{-}(x,t) = 2A\cos\left(\omega t + \frac{\alpha}{2}\right)\cos\left(kx + \frac{\alpha}{2}\right)$$

a)



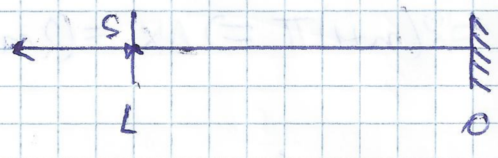
$$\begin{cases} \xi(0,t) = 0 \\ \xi(L,t) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{\alpha}{2} = (2m+1)\frac{\pi}{2} \\ k\cdot L + \frac{\alpha}{2} = (2m+1)\frac{\pi}{2} \end{cases}$$

$$k\cdot L = 2p\frac{\pi}{2} = p\pi \qquad p \in N$$

$$\frac{2\pi}{\lambda}\cdot L = p\pi \Rightarrow L = p\frac{\lambda}{2}$$

b)



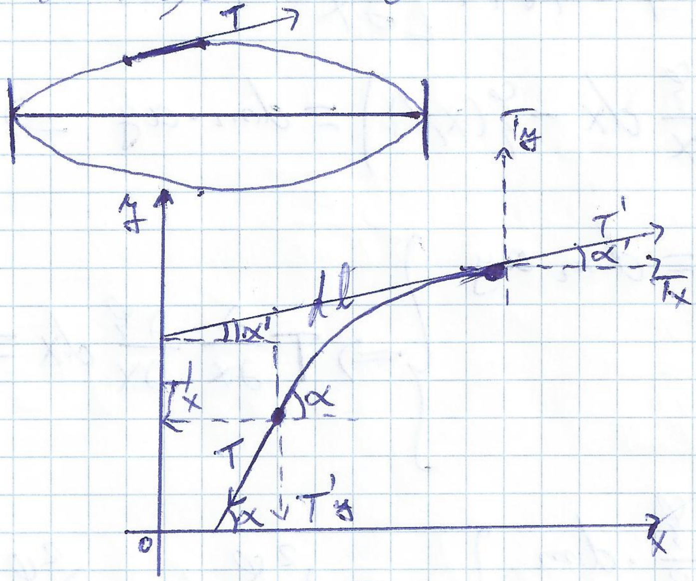
$$\xi(0,t) = 0 \qquad \xi(L,t) = max$$

$$\begin{cases} \frac{\alpha}{2} = (2m+1)\frac{\pi}{2} \\ k\cdot L + \frac{\alpha}{2} = 2m\frac{\pi}{2} = m\pi \end{cases}$$

$$kL = (2p+1)\frac{\pi}{2} \Rightarrow \frac{2\pi}{\lambda}L = (2p+1)\frac{\pi}{2} \Rightarrow$$

$$\Rightarrow L = \frac{(2p+1)\lambda}{4}$$

16. Prezentați subiectul: Coarda vibrantă



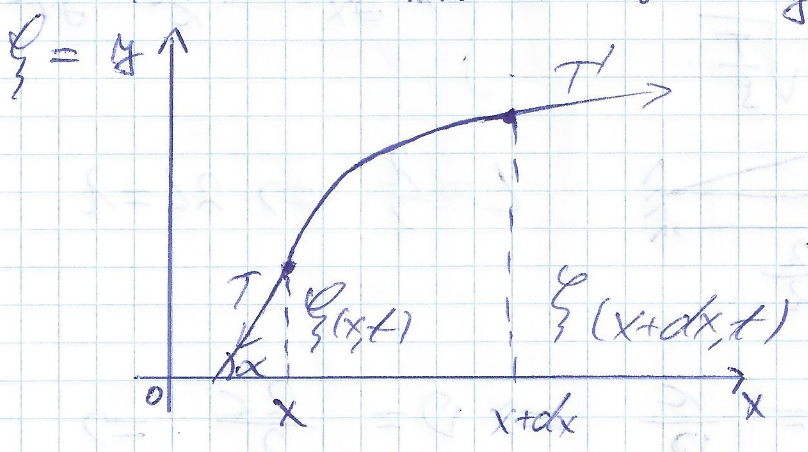
$m\vec{a} = \vec{F}$

$Ox: T'_x - T_x = dm \cdot a_x \Rightarrow T'\cos\alpha' - T\cos\alpha = 0 \Rightarrow$

$\Rightarrow T' = T \qquad\qquad \alpha \simeq 0$

$Oy: T'_y - T_y = dm \cdot a_y \Rightarrow T'\sin\alpha' - T\sin\alpha = dm \cdot a_y$; $T' = T$

$\Rightarrow T\sin\alpha' - T\sin\alpha = dm \cdot a_y$



$tg\alpha = \frac{\partial \xi(x,t)}{\partial x} \qquad tg\alpha \simeq \sin\alpha$

$T(tg\alpha' - tg\alpha) = dm \cdot a_y \Rightarrow$

$\Rightarrow T\left(\frac{\partial \xi(x+dx,t)}{\partial x} - \frac{\partial \xi(x,t)}{\partial x}\right) = dm \cdot a_y \Rightarrow$

$\Rightarrow T\frac{\partial}{\partial x}\left(\xi(x+dx,t) - \xi(x,t)\right) = dm \cdot a_y$

$$\xi(x+dx,t) = \xi(x,t) + \frac{\partial \xi}{\partial x} dx$$

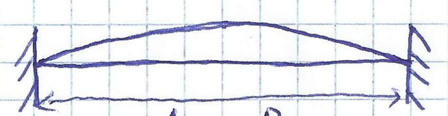
$$T\frac{\partial}{\partial x}\left(\xi(x,t) + \frac{\partial \xi}{\partial x} dx - \xi(x,t)\right) = dm \cdot a_y \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow T\frac{\partial}{\partial x}\frac{\partial \xi}{\partial x} dx = dm\, a_y \\ &a_y = \frac{\partial^2 \xi}{\partial t^2} \end{aligned}\right\} \Rightarrow T\frac{\partial}{\partial x}\frac{\partial \xi}{\partial x} dx = \frac{\partial^2 \xi}{\partial t^2} \cdot dm \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow T\frac{\partial^2 \xi}{\partial x^2} dx = \frac{\partial^2 \xi}{\partial t^2} \cdot dm \\ &dm = \rho\, dx \end{aligned}\right\} \Rightarrow T\frac{\partial^2 \xi}{\partial x^2} dx = \frac{\partial^2 \xi}{\partial t^2} \rho\, dx \Rightarrow$$

$$\Rightarrow T\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} \rho \Rightarrow \frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 \xi}{\partial t^2} \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial^2 \xi}{\partial x^2} - \frac{1}{\frac{T}{\rho}}\frac{\partial^2 \xi}{\partial t^2} = 0 \\ &c = \sqrt{\frac{T}{\rho}} \end{aligned}\right\} \Rightarrow \frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$



$L = \frac{\lambda}{2}$

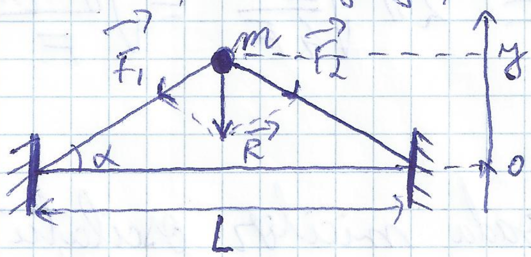
$$L = \frac{\lambda}{2} \Rightarrow 2L = \lambda$$

$$\frac{\lambda}{2} = c \cdot T = \frac{c}{\nu} \Rightarrow \nu = \frac{2c}{\lambda} \Rightarrow$$

$$\Rightarrow \nu = \frac{2\cdot\sqrt{\frac{T}{\rho}}}{2L} \Rightarrow \nu = \frac{1}{L}\sqrt{\frac{T}{\rho}}$$

II Probleme:

1. În mijlocul unei corzi elastice orizontale de lungime L, întinsă cu forța constantă F este suspendat un corp de masă m. Să se afle perioada micilor sale oscilații. Neglijați câmpul gravitațional.



$\vec{R} = \vec{F_1} + \vec{F_2} \Rightarrow \vec{R}\cdot\vec{R} = (\vec{F_1}+\vec{F_2})(\vec{F_1}+\vec{F_2}) \Rightarrow R^2 = F_1^2 + F_2^2 + 2F_1F_2 \cdot \cos(\widehat{F_1,F_2})$

$|\vec{F_1}| = |\vec{F_2}| = F$

$\Rightarrow R^2 = 2F^2 + 2F^2\cos(\widehat{F_1,F_2}) \Rightarrow$

$\Rightarrow R^2 = 2F^2(1+\cos(\widehat{F_1,F_2})) \Rightarrow R^2 = 2F^2 \cdot 2\cos^2\frac{\widehat{F_1,F_2}}{2} \Rightarrow$

$\Rightarrow R = 2F\cos\frac{\widehat{F_1,F_2}}{2} = 2F\sin\alpha \Rightarrow$

$\Rightarrow R = 2F\sin\alpha$

$1+\cos a = 2\cos^2\frac{a}{2}$

$\cos\frac{\widehat{F_1,F_2}}{2} = \sin\alpha$

$\vec{R} = m\cdot\vec{a}$

$\vec{a} = \ddot{\vec{y}}$

$y = A\cos(\omega t+\alpha) \Rightarrow \dot{y} = -A\omega\sin(\omega t+\alpha) \Rightarrow \ddot{y} = -A\omega^2\cos(\omega t+\alpha)$

$\Rightarrow R = -m\ddot{y}$

$tg\,\alpha = \frac{y}{L/2} = \frac{2y}{L}$ $\qquad$ $tg\,\alpha \simeq \sin\alpha \simeq \alpha$ pentru unghiuri mici

$\left.\begin{array}{l} R = 2F\sin\alpha = \frac{4Fy}{L} \\ R = -m\ddot{y} \end{array}\right\} \Rightarrow \frac{4F}{L}y = -m\ddot{y} \Rightarrow$

$$\Rightarrow m\ddot{y} + \frac{4F}{L}y = 0 \Rightarrow \ddot{y} + \frac{4F}{Lm}y = 0$$

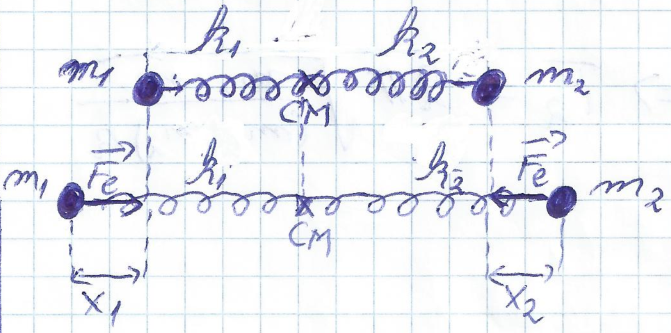
$$\ddot{y} + \omega^2 y = 0$$

$$\Rightarrow \omega^2 = \frac{4F}{Lm} \Rightarrow \omega = \sqrt{\frac{4F}{Lm}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{4F}{Lm}}} = 2\pi\sqrt{\frac{Lm}{4F}} = \sqrt{\frac{\pi^2 Lm}{F}}$$

$$T = \sqrt{\frac{\pi^2 \cdot L \cdot m}{F}}$$ perioada micilor oscilații

2. De capetele unui resort cu constanta elastică k sunt prinse două bile de masă $m_{1,2}$. Neglijând forța gravitațională să se afle perioada de oscilație a resortului, inițial întins și apoi lăsat liber.



$T = ?$

$\vec{p} = (m_1 + m_2)\vec{v_c} = 0 \Rightarrow \vec{v_c} = 0$

$\vec{v_c}$ este viteza centrului de masă (CM)

$$\left.\begin{array}{l} F_e = kx \\ F_e = k_1 x_1 \\ F_e = k_2 x_2 \end{array}\right\} \Rightarrow \left.\begin{cases} x = \dfrac{F_e}{k} \\ x_1 = \dfrac{F_e}{k_1} \\ x_2 = \dfrac{F_e}{k_2} \end{cases} \begin{array}{l} \\ \\ \\ x = x_1 + x_2 \end{array}\right\} \Rightarrow \frac{F_e}{k} = \frac{F_e}{k_1} + \frac{F_e}{k_2} \Rightarrow$$

$$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\begin{cases} m_1 \cdot l_1 = m_2 \cdot l_2 \\ m_1(l_1 + x_1) = m_2(l_2 + x_2) \end{cases} \Rightarrow m_1 x_1 = m_2 x_2 \Rightarrow$$

$$\Rightarrow m_1 \frac{F_e}{k_1} = m_2 \frac{F_e}{k_2} \Rightarrow \frac{m_1}{k_1} = \frac{m_2}{k_2}$$

$$\begin{cases} \dfrac{m_1}{k_1} = \dfrac{m_2}{k_2} \\ \dfrac{1}{k} = \dfrac{1}{k_1} + \dfrac{1}{k_2} \end{cases} \Rightarrow \begin{cases} m_2 k_1 = m_1 k_2 \\ \dfrac{1}{k} = \dfrac{k_1 + k_2}{k_1 k_2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} k_1 = \dfrac{m_1 k_2}{m_2} \\ k = \dfrac{k_1 k_2}{k_1 + k_2} \end{cases} \Rightarrow \begin{cases} k_1 = \dfrac{m_1 k_2}{m_2} \\ k = \dfrac{m_1 k_2}{m_1 + m_2} \end{cases} \Rightarrow$$

$$\Longrightarrow \begin{cases} k_1 = \dfrac{(m_1+m_2)k}{m_2} \\ k_2 = \dfrac{(m_1+m_2)k}{m_1} \end{cases}$$

$$T_1 = 2\pi\sqrt{\frac{m_1}{k_1}} \Longrightarrow T_1 = 2\pi\sqrt{\frac{m_1 \cdot m_2}{(m_1+m_2)k}}$$

$$T_2 = 2\pi\sqrt{\frac{m_2}{k_2}} \Longrightarrow T_2 = 2\pi\sqrt{\frac{m_1 \cdot m_2}{(m_1+m_2)k}}$$

3. O particulă deplasată din poziția de echilibru cu A_0 este lăsată liberă. Ce distanță parcurge particula până la oprirea sa completă? Se cunoaște decrementul logaritmic D.

$$\begin{cases} x = Ae^{-bt}\cos(\omega' t+\alpha) \\ v = \dot{x} = -Ab\,e^{-bt}\cos(\omega' t+\alpha) - Ae^{-bt}\omega'\sin(\omega' t+\alpha) \end{cases}$$

Condiții inițiale $\begin{cases} x(0) = A_0 = A\cos\alpha \\ v(0) = 0 = -Ab\cos\alpha - A\omega'\sin\alpha \end{cases} \Rightarrow$

$\Rightarrow -Ab\cos\alpha = A\omega'\sin\alpha \Rightarrow -b\cos\alpha = \omega'\sin\alpha \Rightarrow$

$\Rightarrow \dfrac{\sin\alpha}{\cos\alpha} = -\dfrac{b}{\omega'} \Rightarrow tg\alpha = -\dfrac{b}{\omega'}$

$v(0) = 0 \Rightarrow tg\alpha = -\dfrac{b}{\omega'}$

$v(t) = 0 \Rightarrow -Ab\,e^{-bt}\cos(\omega' t+\alpha) - Ae^{-bt}\omega'\sin(\omega' t+\alpha) = 0 \Rightarrow$

$\Rightarrow \dfrac{\sin(\omega' t+\alpha)}{\cos(\omega' t+\alpha)} = -\dfrac{b}{\omega'} \Rightarrow tg(\omega' t+\alpha) = -\dfrac{b}{\omega'}$

$tg\alpha = tg(\omega' t+\alpha) \Rightarrow \omega' t_n = n\pi \quad n\in\mathbb{N}$

$t=0 \longrightarrow x(0) = A_0$

$\vdots$

$t = t_n \longrightarrow x(t_n) = Ae^{-bt_n}\cos(\omega' t_n+\alpha) =$

$= Ae^{-\frac{bn\pi}{\omega'}}\cos(n\pi+\alpha) \qquad (-1)^n = \cos(n\pi+\alpha)$

$t_n = \dfrac{n\pi}{\omega'}$

$D = bT' = \dfrac{b\,2\pi}{\omega'} \Rightarrow \dfrac{b}{\omega'} = \dfrac{D}{2\pi}$

Pozițiile când corpul se oprește și se întoarce:

$x(0) = A_0 \qquad x(t_1) = -A_0e^{-\frac{D}{2}} \qquad x(t_2) = +A_0e^{-D} \ldots$

$$\ldots \quad x(t_n) = (-1)^n A_0 e^{-\frac{nD}{2}}$$

Distanța parcursă:

$$d = x(0) + 2|x(t_1)| + 2|x(t_2)| + \ldots + 2|x(t_n)| =$$

$$= A_0 + 2A_0 e^{-\frac{D}{2}} + 2A_0 e^{-D} + \ldots + 2A_0 e^{-\frac{nD}{2}} =$$

$$= A_0 + 2A_0 \left(e^{-\frac{D}{2}} + e^{-D} + \ldots + e^{-\frac{nD}{2}}\right) =$$

$$= A_0 + 2A_0 e^{-\frac{D}{2}} \frac{e^{-\frac{nD}{2}} - 1}{e^{-\frac{D}{2}} - 1} \Rightarrow$$

$$\Rightarrow d = A_0 + 2A_0 \frac{e^{-\frac{(n+1)D}{2}} - e^{-\frac{D}{2}}}{e^{-\frac{D}{2}} - 1}$$

? d mai poate fi și scris ca //nu e de la mine

$$d = x(0) + 2|x(t_1)| + 2|x(t_2)| + \ldots + 2|x(t_n)| =$$

$$= A_0 + 2A_0 e^{-\frac{D}{2}} + 2A_0 e^{-D} + \ldots + 2A_0 e^{-\frac{nD}{2}} =$$

$$= \sum_{n=0}^{D} 2\left|(-1)^n A_0 e^{-\frac{nD}{2}}\right|$$

4. Să se afle amplitudinea inițială A_0 și faza inițială α a oscilațiilor amortizate, știind constantele m, k, r și condițiile ~~...~~ inițiale: poziția x_0 și viteza inițială v_0 la $t=0$.

$$\begin{cases} x = A_0 e^{-bt}\cos(\omega' t+\alpha) \\ \omega' = \sqrt{\omega_0^2 - b^2} \\ v = \dot{x} = -A_0 b e^{-bt}\cos(\omega' t+\alpha) - A_0\omega' e^{-bt}\sin(\omega' t+\alpha) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x(0) = x_0 = A_0\cos\alpha \\ v(0) = v_0 = -A_0 b\cos\alpha - A_0\omega'\sin\alpha \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_0 = A_0\cos\alpha \\ v_0 = -A_0(b\cos\alpha + \omega'\sin\alpha) \end{cases} \Rightarrow \begin{cases} \cos\alpha = \frac{x_0}{A_0} \\ v_0 = -x_0 b - A_0\omega'\sqrt{1-\frac{x_0^2}{A_0^2}} \end{cases}$$

$$\sin\alpha = \sqrt{1-\frac{x_0^2}{A_0^2}}$$

$$\Rightarrow \begin{cases} \cos\alpha = \frac{x_0}{A_0} \\ v_0 = -x_0 b - \sqrt{\omega_0^2 A_0^2 - A_0^2 b^2 - \omega_0^2 x_0^2 + b^2 x_0^2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos\alpha = \frac{x_0}{A_0} \\ v_0 + x_0 b = -\sqrt{\omega_0^2 A_0^2 - A_0^2 b^2 - \omega_0^2 x_0^2 + b^2 x_0^2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos\alpha = \frac{x_0}{A_0} \\ v_0^2 + 2v_0 x_0 b + \cancel{x_0^2 b^2} = \omega_0^2 A_0^2 - A_0^2 b^2 - \omega_0^2 x_0^2 + \cancel{b^2 x_0^2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos\alpha = \frac{x_0}{A_0} \\ v_0^2 + 2v_0 x_0 b + \omega_0^2 x_0^2 = A_0^2(\omega_0^2 - b^2) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \cos\alpha = \sqrt{\frac{\omega'^2 x_0^2}{v_0^2 + 2v_0 x_0 b + \omega_0^2 x_0^2}} \\ A_0 = \sqrt{\frac{v_0^2 + 2v_0 x_0 b + \omega_0^2 x_0^2}{\omega'^2}} \end{cases} \Rightarrow \begin{cases} \alpha = \arccos\sqrt{\frac{\omega'^2 x_0^2}{v_0^2 + 2v_0 x_0 b + \omega_0^2 x_0^2}} \\ A_0 = \sqrt{\frac{v_0^2 + 2v_0 x_0 b + \omega_0^2 x_0^2}{\omega'^2}} \end{cases}$$

5. O coardă întinsă cu forța $F_1 = 160\,N$ generează bătaia de frecvența $\nu_b = 20\,Hz$ când se află în apropierea unui diapazon. Întinsă cu $F_2 = 250\,N$, ea vibrează la unison cu diapazonul. Să se afle frecvența diapazonului.

$$\left.\begin{aligned} T_b &= \frac{2\pi}{\omega_b} \\ \omega_b &= |\omega_2 - \omega_1| \end{aligned}\right\} \Rightarrow$$

~~$T_b = |\nu_2 - \nu_1| = \nu_b$~~ Fie ν_2 frecvența diapazonului.

$$\Rightarrow \left.\begin{aligned} T_b &= \frac{2\pi}{|\omega_2 - \omega_1|} \\ \omega &= 2\pi\nu \end{aligned}\right\} \Rightarrow \left.\begin{aligned} T_b &= \frac{1}{|\nu_2 - \nu_1|} \\ T_b &= \frac{1}{\nu_b} \end{aligned}\right\} \Rightarrow T_b = \frac{1}{|\nu_2 - \nu_1|} = \frac{1}{\nu_b}$$

$c = \sqrt{\frac{F}{\rho}}$ viteza de propagare a perturbației în aer

$$\left.\begin{aligned} l &= n\frac{\lambda}{2} \\ \lambda &= T \cdot c \end{aligned}\right\} \Rightarrow \left.\begin{aligned} l &= n\frac{T \cdot c}{2} \\ T &= \frac{1}{\nu} \end{aligned}\right\} \Rightarrow l = n\frac{c}{2\nu}$$

l este lungimea corzii

$$n = 1 \Rightarrow \left.\begin{cases} \nu_1 = \dfrac{c_1}{2l} = \sqrt{\dfrac{F_1}{\rho}} \cdot \dfrac{1}{2l} \\ \nu_2 = \dfrac{c_2}{2l} = \sqrt{\dfrac{F_2}{\rho}} \cdot \dfrac{1}{2l} \end{cases}\right\} \Rightarrow \frac{\nu_2}{\nu_1} =$$

$$= \frac{\sqrt{F_2}}{\sqrt{F_1}} = \frac{\sqrt{250}}{\sqrt{160}} = \frac{50}{40} = \frac{5}{4}\ N$$

$$\begin{cases} \dfrac{1}{|\nu_2 - \nu_1|} = \dfrac{1}{\nu_b} = \dfrac{1}{20} \\ \dfrac{\nu_2}{\nu_1} = \dfrac{5}{4} \end{cases} \Rightarrow \begin{cases} \nu_2 = \dfrac{5\nu_1}{4} \\ \dfrac{1}{\left|\dfrac{5\nu_1}{4} - \nu_1\right|} = \dfrac{1}{20} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \nu_2 = \dfrac{5\nu_1}{4} \\ \dfrac{4}{\nu_1} = \dfrac{1}{20} \end{cases} \Rightarrow \begin{cases} \nu_2 = \dfrac{5\nu_1}{4} \\ \nu_1 = 80\,Hz \end{cases} \Rightarrow \begin{cases} \nu_2 = 100\,Hz \\ \nu_1 = 80\,Hz \end{cases}$$

6. Sunetul fundamental emis de o coardă generează bătăi de ~~o~~ frecvența $\nu_b = 20\,Hz$ când se află în apropierea unui diapazon. Dacă se scurtează coarda cu $f = 0{,}01$ din lungimea ei, ea intră în rezonanță cu diapazonul. Să se afle frecvența diapazonului. Fie ν_2 frecvența diapazonului.

$$\left.\begin{matrix} T_b = \dfrac{2\pi}{\omega_b} \\ \omega_b = |\omega_2 - \omega_1| \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} T_b = \dfrac{2\pi}{|\omega_2 - \omega_1|} \\ \omega = 2\pi\nu \end{matrix}\right\} \Rightarrow \left.\begin{matrix} T_b = \dfrac{1}{|\nu_2 - \nu_1|} \\ T_b = \dfrac{1}{\nu_b} \end{matrix}\right\} \Rightarrow \frac{1}{|\nu_2 - \nu_1|} = \frac{1}{\nu_b}$$

$$\left.\begin{matrix} \lambda_1 = 2l = \dfrac{c}{\nu_1} \\ \lambda_2 = 2l(1-f) = \dfrac{c}{\nu_2} \end{matrix}\right\} \Rightarrow \frac{\nu_2}{\nu_1} = \frac{\dfrac{c}{2l(1-f)}}{\dfrac{c}{2l}} = \frac{1}{1-f}$$

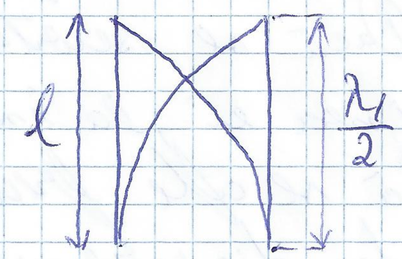
$$\begin{cases} \dfrac{1}{|\nu_2 - \nu_1|} = \dfrac{1}{\nu_b} \\ \dfrac{\nu_2}{\nu_1} = \dfrac{1}{1-f} \end{cases} \Rightarrow \begin{cases} \dfrac{1}{|\nu_2 - \nu_1|} = \dfrac{1}{20} \\ \dfrac{\nu_2}{\nu_1} = \dfrac{1}{0{,}99} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \nu_2 = \dfrac{\nu_1}{0{,}99} \\ \dfrac{1}{\left|\dfrac{\nu_1}{0{,}99} - \nu_1\right|} = \dfrac{1}{20} \end{cases} \Rightarrow \begin{cases} \nu_2 = \dfrac{\nu_1}{0{,}99} \\ \dfrac{0{,}99}{0{,}01\,\nu_1} = \dfrac{1}{20} \end{cases} \Rightarrow$$

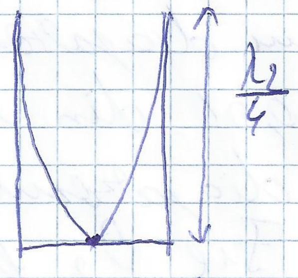
$$\Rightarrow \begin{cases} \nu_2 = \dfrac{\nu_1}{0{,}99} \\ \nu_1 = 1980 \end{cases} \Rightarrow \begin{cases} \nu_2 = 2000\,Hz \\ \nu_1 = 1980\,Hz \end{cases}$$

- Răspuns: frecvența diapazonului este de $2000\,Hz$

7. Un tub sonor închis emite tonul fundamental de frecvență $\nu = 250\,Hz$. Cunoscând viteza sunetului în aer $c = 340\,m/s$, să se afle lungimea tubului și frecvența tonului fundamental emis de acest tub dacă îl deschidem.



tub deschis



tub închis

$$\lambda_1 = \frac{c}{\nu_1} = 2l \Rightarrow \nu_1 = \frac{c}{2l}$$
$$\lambda_2 = \frac{c}{\nu_2} = 4l \Rightarrow \nu_2 = \frac{c}{4l}$$

$$\cancel{\Rightarrow \frac{\nu_2}{\nu_1} = \frac{\frac{c}{2l}}{\frac{c}{4l}} = 2l \Rightarrow \frac{\nu_2}{\nu_1} = \frac{c}{\nu_1} \Rightarrow \nu_2 = 340\,Hz}$$

$$\cancel{2l = \frac{c}{2\nu_2}}$$

$$\nu_2 = 250\,Hz$$

$$4l = \frac{c}{\nu_2} \Rightarrow l = \frac{c}{4\nu_2} = 0{,}34 \Rightarrow \boxed{l = 0{,}34\,m}$$

$$\cancel{\frac{\nu_2}{\nu_1} = 2l \Rightarrow \nu_1 = \frac{\nu_2}{2l} = 367{,}64\,Hz}$$

$$\frac{\nu_2}{\nu_1} = \frac{\frac{c}{4l}}{\frac{c}{2l}} = \frac{1}{2} \Rightarrow \nu_1 = 2\nu_2$$
$$\nu_2 = 250\,Hz$$
$$\Rightarrow \boxed{\nu_1 = 500\,Hz}$$

8. Dezvoltați în serie trigonometrică oscilațiile în "dinți de fierăstrău": $f(t) = hT^{-1} \cdot t$, pentru $t \in [0, T]$, cu $f(t) = f(t + nT)$, $n \in \mathbb{Z}$, $h = ct$

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega t)\, dt$$

$$n = 0 \Rightarrow \begin{cases} a_0 = \frac{1}{T} \int_0^T f(t) \cos 0\, dt = \frac{1}{T} \int_0^T f(t)\, dt \\ b_0 = \frac{1}{T} \int_0^T f(t) \sin 0\, dt = 0 \end{cases}$$

$$f(t) = \frac{h}{T} \cdot t \Rightarrow a_n = \frac{2}{T} \int_0^T \frac{ht}{T} \cos \omega n t\, dt =$$

$$= \frac{2}{T} \int_0^T \frac{ct^2}{T} \cos \omega n t\, dt =$$

$$= \frac{2c}{T^2} \left(t^2 \frac{\sin \omega n t}{\omega n} \Big|_0^T - \int_0^T 2t \frac{\sin \omega n t}{\omega n}\, dt \right) =$$

$$= \frac{2c}{T^2} \left(T^2 \frac{\sin \omega n t}{\omega n} \Big|_0 + 2t \frac{\cos \omega n t}{(\omega n)^2} \Big|_0^T - 2 \int_0^T \frac{\cos \omega n t}{(\omega n)^2} dt \right.$$

$$= \frac{2c}{T^2} \left(T^2 \frac{\sin \omega n T}{\omega n} + 2T \frac{\cos \omega n T}{(\omega n)^2} - 2 \frac{\sin \omega n t}{(\omega n)^2} \right)$$

$$\omega \cdot T = \omega \cdot \frac{2\pi}{\omega} = 2\pi \Rightarrow$$

$\Rightarrow a_n = 0$, dacă $n \neq 0$

$$b_n = \frac{2}{T}\int_0^T \frac{h}{T}\, t \sin n\omega t \, dt =$$

$$= \frac{2}{T}\int_0^T \frac{ct^2}{T} \sin n\omega t \, dt =$$

$$= \frac{2c}{T^2}\int_0^T t^2 \sin n\omega t \, dt =$$

$$= \frac{2c}{T^2}\left[-t^2 \frac{\cos n\omega t}{n\omega}\Big|_0^T + 2\int_0^T t \frac{\cos n\omega t}{n\omega} dt \right] =$$

$$= \frac{2C}{T^2}\left[-T^2 \frac{\cos n\omega T}{n\omega} + 2t \frac{\sin n\omega t}{(n\omega)^2}\Big|_0^T - 2\int_0^T \frac{\sin \omega n t}{(\omega n)^2} dt \right] =$$

$$= \frac{2C}{T^2}\left[-T^2 \frac{\cos n\omega T}{n\omega} + 2T \frac{\sin n\omega T}{(n\omega)^2} - 2\frac{\cos \omega n T}{(\omega n)^3} \right] \Rightarrow$$

$$\omega \cdot T = 2\pi$$

$$\Rightarrow b_n = -\frac{h}{n\pi}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt = \frac{1}{T}\int_0^T \frac{h}{T}\cdot t \, dt =$$

$$= \frac{h}{T^2}\cdot\int_0^T t\, dt = \frac{h}{T^2}\cdot\frac{T^2}{2} = \frac{h}{2} \Rightarrow a_0 = \frac{h}{2}$$

$$f(t) = \frac{h}{2} + \sum_{n=1}^{\infty} -\frac{h}{n\pi} = \frac{h}{2} + \frac{h}{\pi}\sum_{n=1}^{\infty} -\frac{1}{n} \Rightarrow$$

$$\cancel{\Rightarrow f(t) = \frac{h}{2} - h\sum_{n=1}^{\infty}} \Rightarrow$$

$$\Rightarrow f(t) = \frac{h}{2} - \frac{h}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}$$