Seminar 1

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$\downarrow$

$= P_B(A)$

$$P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1) \cdot P_{A_1}(A_2) \cdot P_{A_1 \cap A_2} P(A_3) \cdot \ldots \cdot P_{A_1 \cap A_2 \cap \ldots \cap A_{m-1}}(A_m)$$

$\rightarrow$ Intersecția a n evenimente condiționate succesiv

Formula probabilității totale:

$$P(B) = \sum_{i=1}^{m} P(A_i) \cdot P_{A_i}(B)$$

$\downarrow$

eveniment oarecare

Formula lui Bayes:

$$P_B(A_i) = \frac{P_{A_i}(B) \cdot P(A_i)}{\sum_{j=1}^{m} P(A_j) \cdot P_{A_j}(B)} = \frac{P(B \cap A_i)}{P(B)}$$

Inegalitatea lui Boole:

$$P(A \cap B) \geq P(A) + P(B) - 1$$

Variabile aleatoare

f. de distribuție (repartiție)

V.A : • discretă $\rightarrow X(E)$ (mulțimea valorilor variabilei aleatoare) cel mult numărabilă

• continuă $\rightarrow X(E)$ mulțime infinită de nr. reale

1. Variabile aleatoare discrete simple

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$ $\rightarrow$ tablou de repartiție

x_i = valorile variabilei aleatoare
p_i = probabilitățile

$F: \mathbb{R} \rightarrow [0,1]$

$$F_X(x) = \begin{cases} 0, & x < x_1 \\ p_1, & x_1 \leq x \leq x_2 \\ p_1 + p_2, & x_2 \leq x < x_3 \\ \vdots \\ 1, & x \geq n \end{cases}$$

funcția de repartiție

ex.

$$X : \begin{pmatrix} 0 & 1 & 2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix} \quad Y : \begin{pmatrix} -1 & 1 \\ 0.5 & 0.5 \end{pmatrix}$$

$$3X = \begin{pmatrix} 0 & 3 & 6 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 0 & 1 & 8 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

$$X+Y = \begin{pmatrix} 0 & 1 & 2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0-1 & 0+1 & 1-1 & 1+1 & 2-1 & 2+1 \\ 0.5\cdot 0.3 & 0.15 & 0.25 & 0.25 & 0.1 & 0.1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 2 & 1 & 3 \\ 0.15 & 0.15 & 0.25 & 0.25 & 0.1 & 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 & 2 & 3 \\ 0.15 & 0.25 & 0.25 & 0.25 & 0.1 \end{pmatrix}$$

$$X \cdot Y = \begin{pmatrix} 0 & 1 & 2 \\ 0.3 & 0.5 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 1 & -2 & 2 \\ 0.15 & 0.15 & 0.25 & 0.25 & 0.1 & 0.1 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0.1 & 0.25 & 0.3 & 0.25 & 0.1 \end{pmatrix}$$

Media variabilei aleatoare:

- discret: $m = E[X] = x_1 \cdot p_1 + x_2 \cdot p_2 + \ldots + x_n \cdot p_n = \sum_{i=1}^{n} x_i p_i$

Pt. $n \to \infty$, seria trebuie să fie convergentă

• continuă: $m = E[x] = \int_{-\infty}^{+\infty} x \cdot f(x)\,dx$

Pt. $x \in [a, b]$, $m = \int_{a}^{b} x \cdot f(x)\,dx$

Dispersia lui X

X variabilă aleatoare

$E(X) = \mu$

Dispersia lui X: $Var(X) = E((X-\mu)^2)$

Deviația standard σ a lui X e $\sigma = \sqrt{Var(X)}$

$Var(X) = E((X-\mu)^2) = \sum_{i=1}^{n} p(x_i)(x_i - \mu)^2$

ex.

$E(X)$, $Var(X)$, σ

$X \sim \begin{pmatrix} 1 & 3 & 5 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$E(X) = 1 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 5 \cdot \frac{1}{2} = 1 + \frac{5}{2} = \frac{7}{2}$

$Var(X) = E((X - \frac{7}{2})^2)$

x	1	3	5
$p(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$(x-\frac{7}{2})^2$	$(1-\frac{7}{2})^2$	$(3-\frac{7}{2})^2$	$(5-\frac{7}{2})^2$

$$= \begin{array}{ccc} 1 & 3 & 5 \\ \hline \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \hline \frac{25}{4} & \frac{1}{4} & \frac{9}{4} \end{array}$$

$$Var(X) = \frac{1}{4} \cdot \frac{25}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{9}{4} = \frac{11}{4}$$

$$\sigma = \sqrt{Var(X)} = \sqrt{\frac{11}{4}} = \sqrt{11 \cdot \frac{1}{4}} = \frac{1}{2}\sqrt{11}$$

$$P(a < X \leq b) = F(b) - F(a)$$

<u>v.a. continue</u>:

$$P(a < x \leq b) = P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b)$$
$$= \int_a^b f(x)\, dx$$

<u>v.a. discrete</u>:

$$P(a < X < b) = F(b) - F(a) - P(X = b)$$
$$P(a \leq X < b) = F(b) - F(a) - P(X = b) + P(X = a)$$
$$P(a \leq X \leq b) = F(b) - F(a) + P(X = a)$$

<u>Media și momentele unei v.a. continue</u>

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \qquad \text{media}$$

$$Var(X) = \int_{-\infty}^{\infty} (x - \underset{E(X)}{\underset{\parallel}{m}})^2 \cdot f(x)\, dx \qquad \text{dispersia (varianța)}$$

În practică se fol.:

$$Var(X) = E(X^2) - E(X)^2$$

$$\mu_r = \int_{-\infty}^{\infty} (x-\underset{\overset{\shortparallel}{E(x)}}{m})^r \cdot f(x)\, dx$$ moment centrat de ordin r

$$m_r = \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx$$ moment inițial de ordin r

$f: \mathbb{R} \to \mathbb{R}$ densitate de probabilitate dacă:

- $f(x) \geq 0 \ \forall x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$

f. de repartiție:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = P(X \leq x)$$

Probleme cu v.a. continue

1.

X v.a. continuă definită prin densitatea de probabilitate:

$$f(x) = \begin{cases} c \cdot (4x - 2x^2), & 0 < x < 2, \quad c \in \mathbb{R} \\ 0, & \text{în rest} \end{cases}$$

a) $c = ?$

b) $P(X > 1)$

c) $E(X)$, $Var(X)$

d) f. de repartiție a v.a. X

a) f. densitate de prob. $\Leftrightarrow \begin{cases} 1)\ f(x) \geq 0 \\ 2) \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{cases}$

1) $c(4x - 2x^2) \geq 0$

$c \cdot x(4 - 2x) \geq 0$

$2cx(2 - x) \geq 0$

Cum $0 < x < 2$, $2c\underbrace{x}_{da}\underbrace{(2-x)}_{da} \geq 0$

$c > 0$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} c(4x-2x^2)\,dx + \int_{2}^{\infty} 0\,dx = 1$

$$\int_{0}^{2} c(4x-2x^2)\,dx = 1$$

$$\int_{0}^{2} (4cx - 2cx^2)\,dx = \left(\frac{4cx^2}{2} - \frac{2cx^3}{3}\right)\Big|_0^2 =$$

$$= 2cx^2\Big|_0^2 - \frac{2cx^3}{3}\Big|_0^2 = 8c - \frac{16c}{3} =$$

$$= c\left(8 - \frac{16}{3}\right) = c \cdot \frac{8}{3} = 1$$

$$\Rightarrow c = \frac{3}{8} > 0 \quad \text{adv.}$$

b) $P(X>1) = \int_{1}^{\infty} f(x)\,dx = \int_{1}^{2} f(x)\,dx + 0 =$

$$= \int_{1}^{2} \frac{3}{8}(4x-2x^2)\,dx = \ldots = \frac{1}{2}$$

c) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_{-\infty}^{0} x \cdot f(x)\,dx + \int_{0}^{2} x \cdot f(x)\,dx + \int_{2}^{\infty} x f(x)\,dx = \int_{0}^{2} \frac{3}{8}(4x^2-2x^3)\,dx = \ldots = 1$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_0^2 x^2 \cdot f(x)\, dx = \dots$$

calcule

$Var(x) = \dots$ înlocuiesc

d)

$$f(t) = \begin{cases} 0, & t \leq 0 \\ \frac{3}{8}(4t - 2t^2), & t \in (0,2) \\ 0, & t \geq 2 \end{cases}$$

I. $x \leq 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0\, dt = 0$

II. $x \in (0,2) \Rightarrow F(x) = \int_{-\infty}^{0} 0\, dt + \int_0^x \frac{3}{8}(4t - 2t^2)dt = \dots = \frac{3}{4}x^2 - \frac{1}{4}x^3$

III. $x \geq 2 \Rightarrow F(x) = \int_{-\infty}^{0} 0\, dt + \int_0^2 \frac{3}{8}(4t^2 - 2t)dt + \int_2^x 0\, dt = \dots = 1$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{3}{4} \cdot x^2 - \frac{1}{4} \cdot x^3, & x \in (0,2) \\ 1, & x \geq 2 \end{cases}$$

$$P(X>1) = 1 - P(X \leq 1) = 1 - F(1) = \frac{1}{2}$$