MAXIM TIBERIU
GRUPA 251
NR 12 $\Rightarrow i=2$

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EXAMEN

1. $f:(0,\infty)\times(0,\infty)\to(0,\infty);\ f(x)=kxe^{-(3x+4y)};\ k>0$ - densitate de probabilitate a V.A. (X,Y)

a) f - densitate de repartiție $\Leftrightarrow \begin{cases} f(x,y)\geq 0, \forall x,y\in(0,\infty) \\ \int_0^\infty\int_0^\infty f(x,y)\,dx\,dy=1 \end{cases}$

$$\int_0^\infty\int_0^\infty f(x,y)\,dy\,dx=1 \Leftrightarrow k\int_0^\infty \underbrace{\int_0^\infty xe^{-3x-4y}\,dy}_{I}\,dx=1 \quad (1)$$

$$I=\int_0^\infty xe^{-3x-4y}\,dy = x\int_0^\infty e^{-3x-4y}\,dy$$

S.V. $u=-3x-4y$
$du=-4dy$
$y=0 \Rightarrow u=-3x$
$y\to\infty \Rightarrow u\to-\infty$

$$\Rightarrow I=-\frac{x}{4}\int_{-3x}^{-\infty} e^u\,du =$$

$$= -\frac{e^u x}{4}\Bigg|_{u=-3x}^{-\infty} = \frac{1}{4}e^{-3x}\cdot x \quad (2)$$

$$(1),(2) \Rightarrow K\int_0^{\infty} \frac{xe^{-3x}}{4}dx = 1 \Leftrightarrow \frac{K}{4}\int_0^{\infty} xe^{-3x}dx = 1$$

$f = x$
$g' = e^{-3x}$

$\Rightarrow$

$$\Rightarrow \frac{K}{4}\left(-\frac{xe^{-3x}}{3}\Big|_0^{\infty} - \int_0^{\infty} -\frac{e^{-3x}}{3}dx\right) = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{K}{4}\left(-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right)\Big|_0^{\infty} = 1 \Leftrightarrow \frac{K}{4}\left(-\frac{(3x+1)e^{-3x}}{\cancel{36}\,9}\right)\Big|_0^{\infty} = 1$$

$$\Leftrightarrow \frac{K}{4}\cdot\frac{1}{\cancel{36}\,9} = 1 \Rightarrow \frac{K}{\cancel{144}\,36} = 1 \Rightarrow \cancel{K = 144}\ K = 36$$

b) $f(x,y) = \cancel{144}\,36\, xe^{-3x-4y}$

$$f_x = \int_0^{\infty} f(x,y)dy = \int_0^{\infty} \cancel{144}\,36\, xe^{-3x-4y}dy = \cancel{144}\,36\, x\int_0^{\infty} e^{-3x-4y}$$

Analog cu s.v. de la subpunctul a) $\Rightarrow$
$u = -3x-4y$

$$\Rightarrow \cancel{144x\int}\ \cancel{144}\,36\, x \cdot -\frac{1}{4}\int_{-3x}^{-\infty} e^{u}du = (-\cancel{36}\,9\, e^{u}x)\Big|_{-3x}^{-\infty} \Rightarrow$$

$$f_X(x) = \cancel{36}\,9\, e^{-3x}\cdot x\ ,\ \forall x \in (0,\infty)$$

$$f_Y = \int_0^\infty f(x,y)\,dx = \int_0^\infty \overset{36}{\cancel{144}}\, x e^{-3x} e^{-4y}\,dx = \overset{36}{\cancel{144}}\, e^{-4y}\int_0^\infty x e^{-x}\,dx =$$

$$= \overset{36}{\cancel{144}}\, e^{-4y}\underbrace{\left(-\frac{xe^{-3x}}{3}\Big|_0^\infty - \int_0^\infty -\frac{e^{-3x}}{3}\,dx\right)}_{\text{Analog cu calculele de la a)}} =$$

$$= \overset{36}{\cancel{144}}\, e^{-4y}\left(-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right)\Big|_0^\infty = \left(-48xe^{-3x-4y} - 16e^{-3x-4y}\right)\Big|_0^\infty = \cancel{(-16x}\left[-\overset{4}{\cancel{16}}(3x+1)e^{-3x-4y}\right]\Big|_0^\infty \Rightarrow$$

$$\Rightarrow f_Y(y) = \overset{4}{\cancel{16}}\, e^{-4y},\ \forall\, y \in (0,\infty)$$

d) $\cancel{M(X) = \int_0^\infty 36 e^{-3x}\,dx =}$

$$M(X) = \int_0^\infty x\cdot f_X(x)\,dx = \int_0^\infty \overset{9}{\cancel{36}}\, x\cdot e^{-3x}\cdot x\,dx$$

$\cancel{M(X) = 36\int_0^\infty x e^{-3x}\,dx}$

$\cancel{f = x}$

$g' = e^{-3x}$

$$\Rightarrow M(X) = \cancel{36}\left[-\frac{xe^{-3x}}{3}\Big|_0^\infty - \underbrace{\int_0^\infty -\frac{e^{-3x}}{3}\,dx}_{\frac{e^{-3x}}{9}}\right]$$

$$\cancel{M(X) = 36}\left(\frac{-xe^{-3x}}{3} - \frac{e^{-3x}}{9}\right)\Big|_0^\infty = \left(-12xe^{-3x} - 4e^{-3x}\right)\Big|_0^\infty =$$

$$M(X)=\int_0^\infty \overset{9}{\cancel{36}}x^2e^{-3x}\,dx=\overset{9}{\cancel{36}}\int_0^\infty x^2e^{-3x}\,dx \quad \Rightarrow$$

$$f=\cancel{2x}\,x^2$$
$$g'=e^{-3x}$$

$$\Rightarrow M(X)=\overset{9}{\cancel{36}}\left[\frac{-x^2e^{-3x}}{3}\Big|_0^\infty-\int_0^\infty-\frac{2xe^{-3x}}{3}\,dx\right\}=$$

$$=\overset{9}{\cancel{36}}\left[\frac{-x^2e^{-3x}}{3}\Big|_0^\infty+\frac{2}{3}\underbrace{\int_0^\infty xe^{-3x}\,dx}_{\text{calculată anterior}}\right]=$$

$$=\overset{9}{\cancel{36}}\left[\frac{-x^2e^{-3x}}{3}+\frac{2xe^{-3x}}{9}+\frac{2e^{-3x}}{27}\right]\Big|_0^\infty=$$

$$=\left(-\underset{3}{\cancel{12}}x^2e^{-3x}-\underset{1}{\cancel{8}}xe^{-3x}-\frac{8e^{-3x}}{3}\right)\Big|_0^\infty=\frac{2}{3}$$

$$Var(X)=M(X^2)-(M(X))^2$$

$$M(X^2)=\int_0^\infty x^2f_X(x)\,dx=\int_0^\infty \overset{9}{\cancel{36}}x^3e^{-3x}\,dx=\underset{9}{\cancel{36}}\int_0^\infty x^3e^{-3x}\,dx \quad \Rightarrow$$

$$f=x^3$$
$$g'=e^{-3x}$$

$$\Rightarrow M(X^2)=\overset{9}{\cancel{36}}\left[-\frac{x^3e^{-3x}}{3}\Big|_0^\infty-\int_0^\infty -x^2\cancel{e}e^{-3x}\,dx\right\}$$

$$M(X^2)=\overset{9}{\cancel{36}}\left[-\frac{x^3e^{-3x}}{3}\Big|_0^\infty+\int_0^\infty x^2e^{-3x}\,dx\right)=$$

$\rightarrow$ calculată anterior

$$M(x^2) = \frac{36}{9}\left\{-\frac{x^3e^{-3x}}{3} - \frac{x^2e^{-3x}}{3} - \frac{2xe^{-3x}}{9} - \frac{2e^{-3x}}{27}\right\}\Big|_0^{\infty}$$

$$M(x^2) = \left[-\frac{12}{3}x^3e^{-3x} - \frac{12}{3}x^2e^{-3x} - \frac{8}{1}xe^{-3x} - \frac{8e^{-3x}}{3}\right]\Big|_0^{\infty}$$

$$M(x^2) = \frac{8}{3}$$

$$Var(X) = M(x^2) - (M(x))^2 = \frac{8}{3} - \left(\frac{8}{3}\right)^2 = \frac{8}{3} - \frac{64}{9}$$

$$\cancel{Var(X) = \frac{24}{9} - \frac{64}{9} = \frac{-40}{9}} \quad Var(X) = \frac{6-4}{9} = \frac{2}{9}$$

2.

X \ Y	0	3	4	p_i
-1	0.2	0.15	0.1	0.45
0	0	0	0.15	0.15
1	0.1	0.2	0.1	0.4
q_j	0.3	0.35	0.35	1

a) $X: \begin{pmatrix} -1 & 0 & 1 \\ 0.45 & 0.15 & 0.4 \end{pmatrix}$

$Y: \begin{pmatrix} 0 & 3 & 4 \\ 0.3 & 0.35 & 0.35 \end{pmatrix}$

b) $E(X) = -1\cdot 0{,}45 + 0\cdot 0{,}15 + 1\cdot 0{,}4 = -0{,}45 + 0{,}4 = -0{,}05$

$Var(X) = E(X^2) - (E(X))^2$

$\cancel{E}\{X^2\} = \begin{pmatrix} 0 & 1 \\ 0.15 & 0.85 \end{pmatrix} \Rightarrow E(X^2) = 0{,}85$

$Var(X) = 0.85 - (-0{,}05)^2 = 0{,}85 - 0{,}0025$

$Var(X) = 0{,}8475$

$E(Y) = 0\cdot 0{,}3 + 3\cdot 0{,}35 + 4\cdot 0{,}35$

$E(Y) = 0 + 1{,}05 + 1{,}4 = 2{,}45$

$Var(Y) = E(Y^2) - (E(Y))^2$

$\cancel{E}\ Y^2 = \begin{pmatrix} 0 & 9 & 16 \\ 0{,}3 & 0{,}35 & 0{,}35 \end{pmatrix}$

$E(Y^2) = 0\cdot 0{,}3 + 9\cdot 0{,}35 + 16\cdot 0{,}35 = 25\cdot 0{,}35 = 8{,}75$

$Var(Y) = 8{,}75 - (2{,}45)^2 = 2{,}74$

c) $\text{cov}(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

$$XY = \begin{pmatrix} -1\cdot 0 & -1\cdot 3 & -1\cdot 4 & 0\cdot 0 & 0\cdot 3 & 0\cdot 4 & 1\cdot 0 & 1\cdot 3 & 1\cdot 4 \\ 0,2 & 0,15 & 0,1 & 0 & 0 & 0,15 & 0,1 & 0,2 & 0,1 \end{pmatrix}$$

$$XY = \begin{pmatrix} -4 & -3 & 0 & 3 & 4 \\ 0,1 & 0,15 & 0,45 & 0,2 & 0,1 \end{pmatrix}$$

$$E(XY) = -4 \cdot 0,1 + (-3) \cdot 0,15 + 0 \cdot 0,45 + 3 \cdot 0,2 + 4 \cdot 0,1$$

$$E(XY) = \cancel{-0,4} - 0,45 + 0 + 0,6 \cancel{+0,4} = \cancel{0,25}\ 0,15$$

$$\text{cov}(X,Y) = 0,15 - (-0,05 \cdot 2,45) = 0,15 + 0.12\cancel{3}$$

$$\text{cov}(X,Y) = 0,27$$

1. b) verificare independență

X,Y - independente $\Rightarrow f_X(x)\, f_Y(y) = f(x,y)$

$$\left.\begin{matrix} f_X(x) = 36x e^{-3x} \\ f_Y(x) = 16 e^{-4y} \end{matrix}\right| \Rightarrow f_X(x) \cdot f_Y(y) = \overset{9}{\cancel{36}} x e^{-3x} \cdot \overset{4}{\cancel{16}} e^{-4y} =$$

$$= \cancel{9}\, 36x e^{-3x-4y} = f(x,y) \Rightarrow$$

$\Rightarrow X,Y$ - independente

1. c) $F_X(2) - F_Y(1) = ?$ $F_X : \mathbb{R} \to [0,1]$ - f. de rep a X

$F_Y : \mathbb{R} \to [0,1] \to$ f. de rep. a Y

$$F_X(2) = \int_{-\infty}^{2} f_X(t)\,dt = \int_0^2 9\cancel{x}te^{-3\cancel{x}t}\,d\cancel{x}t = 9\underbrace{\int_0^2 te^{-3t}\,dt}_{\text{calculată}} \Rightarrow$$

$$\Rightarrow F_X(2) = 9\left(-\frac{te^{-3t}}{3} - \frac{e^{-3t}}{9}\right)\Big|_0^2 = \cancel{9}\left(-3\cancel{x}te^{-3t} - e^{-3t}\right)\Big|_0^2$$

$$= e^{-6}(e^6 - 7)$$

$$F_Y(1) = \int_{-\infty}^{1} f_Y(t)\,dt = \int_0^1 4e^{-4\cancel{x}t}\,dt = 4\underbrace{\int_0^1 e^{-4t}\,dt}_{\text{calculată}} \Rightarrow$$

$$\Rightarrow F_Y(1) = 4\left(-\frac{\cancel{x}te^{-4t}}{4} - \frac{e^{-4t}}{16}\right)\Big|_0^1 = \left(-te^{-4t} - \frac{e^{-4t}}{4}\right)\Big|_0^1$$

$$F_Y(1) = \frac{e^{-4}(e^4 - 5)}{4}$$

$$F_X(2) - F_Y(1) = e^{-6}(e^6 - 7) - \frac{e^{-4}(e^4 - 5)}{4}$$

3. $f:(0,1)\times\left(\frac{1}{2},1\right)\to f(x,\theta)=\frac{\theta}{1-\theta}x^{\frac{2\theta-1}{1-\theta}}$

~~$L(\theta,$~~

Fie $X_1, X_2, \ldots, X_n$ - eșantion de selecție de volum n cu valorile $x_1, x_2, \ldots, x_n$.

$$L(\theta; x_1, x_2, \ldots, x_n)=\prod_{i=1}^{n} f_\theta(x_i)=\prod_{i=1}^{n}\frac{\theta}{1-\theta}x_i^{\frac{2\theta-1}{1-\theta}}=$$

$$=\left(\frac{\theta}{1-\theta}\right)^n\prod_{i=1}^{n}x_i^{\frac{2\theta-1}{1-\theta}}$$

logaritmăm $\Rightarrow \ln L(\theta; x_1, x_2, \ldots, x_n)=n\ln\frac{\theta}{1-\theta}+\frac{2\theta-1}{1-\theta}\ln\prod_{i=1}^{n}x_i$

$$\ln L(\theta; x_1, x_2 \ldots x_n)=n\left(\ln\theta+\ln\frac{1}{1-\theta}\right)+\frac{2\theta-1}{1-\theta}\ln\prod_{i=1}^{n}x_i=$$

$$=n\ln\theta+n\ln\frac{1}{1-\theta}+\frac{2\theta-1}{1-\theta}\ln\prod_{i=1}^{n}x_i=$$

$$=n\ln\theta+n(\ln 1-\ln(1-\theta))+\frac{2\theta-1}{1-\theta}\ln\prod_{i=1}^{n}x_i=$$

$$=n\ln\theta+(-n\ln(1-\theta))+\frac{2\theta-1}{1-\theta}\ln\prod_{i=1}^{n}x_i$$

$$=n\ln\theta-n\ln(1-\theta)+\frac{2\theta-1}{1-\theta}\ln\prod_{i=1}^{n}x_i$$

$$\frac{\partial\ln L(\theta; x_1, x_2, \ldots, x_n)}{\partial\theta}=\frac{n}{\theta}+\frac{n}{1-\theta}+\frac{1}{(1-\theta)^2}\ln\prod_{i=1}^{n}x_i$$

$$\frac{\partial \ln L(\theta; x_1, x_2, \dots, x_n)}{\partial \theta} = 0 \Leftrightarrow \frac{n}{\theta} + \frac{n}{(1-\theta)} + \frac{\ln \prod_{i=1}^{n} x_i}{(1-\theta)^2} = 0 \Leftrightarrow$$

$$\underline{\theta(1-\theta)^2}$$

$$\Leftrightarrow \frac{n(1-\theta)^2 + n\theta(1-\theta)^{\cancel{2}} + \theta \ln \prod_{i=1}^{n} x_i}{\theta(1-\theta)^2} = 0$$

$$n(1-\theta)^2 + n\theta(1-\theta) + \theta \ln \prod_{i=1}^{n} x_i = 0 \Leftrightarrow$$

$$\Leftrightarrow n(1 - 2\theta + \cancel{\theta^2} + \theta - \cancel{\theta^2}) + \theta \ln \prod_{i=1}^{n} x_i = 0 \Leftrightarrow$$

$$\Leftrightarrow n - n\theta + \theta \ln \prod_{i=1}^{n} x_i = 0 \Rightarrow \theta\left(\ln \prod_{i=1}^{n} x_i - n\right) = -n \Rightarrow$$

$$\Rightarrow \hat{\theta} = \frac{-n}{\ln \prod_{i=1}^{n} x_i - n}$$

Vf.

$$D = \frac{\partial^2 \ln L(\theta; x_1, x_2, \dots, x_n)}{\partial \theta^2} = \cancel{\frac{(1+\theta)(3 \ln \prod_{i=1}^{n} - n)}{\theta^2 (1-\theta)^3}} \quad \cancel{(1)}$$

$$\theta \in \left(\frac{1}{2}, 1\right) \quad (2)$$

$$(1), (2) \Rightarrow \frac{\partial^2 \ln L(\theta; x_1, x_2, \dots, x_n)}{\partial \theta^2} < 0 \Rightarrow \hat{\theta} \text{ - maxim } \checkmark$$

$$D = -\frac{(2 \ln \prod_{i=1} - 2n)\theta^2 + 3n\theta - n}{\theta^2 (\theta - 1)^3} \quad (1)$$