• Calculul determinantului unei matrici folosind metoda Gauss

1) Folosind Gauss (GPP), calculați determinantul matricei $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ -3 & 4 & 0 \end{pmatrix}$.

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ -3 & 4 & 0 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} -3 & 4 & 0 \\ -1 & 4 & 2 \\ 2 & -1 & 0 \end{pmatrix} \underset{L_3 \leftarrow L_3 + \frac{2}{3}L_1}{\overset{L_2 \leftarrow L_2 - \frac{1}{3}L_1}{\sim}} \begin{pmatrix} -3 & 4 & 0 \\ 0 & \frac{8}{3} & 2 \\ 0 & \frac{5}{3} & 0 \end{pmatrix} \overset{L_3 \leftarrow L_3 - \frac{5}{8}L_2}{\sim}$$

$k=1$: pivot $= \max\{|2|, |-1|, |-3|\} = 3 = a_{31}$

$k=2$: pivot $= \max\{|\frac{8}{3}|, |\frac{5}{3}|\} = \frac{8}{3} = a_{22}$

$$A \sim \begin{pmatrix} -3 & 4 & 0 \\ 0 & \frac{8}{3} & 2 \\ 0 & 0 & -\frac{5}{4} \end{pmatrix}$$

nr. schimbărilor de linii

$$\text{Deci } \det A = (-1)^{1} \cdot \underbrace{(-3) \cdot \frac{8}{3} \cdot \frac{-5}{4}}_{\text{produsul elementelor de pe diagonala principală}} = -10$$

2) Aflați rangul matricei A (cu met. GPP), unde $A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 3 & -1 & -2 & -4 \\ 2 & 3 & -5 & 1 \end{pmatrix}$.

$k=1$, $\max\{|1|, |3|, |2|\} = 3 = a_{21}$

$$A \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 3 & -1 & -2 & -4 \\ 1 & 2 & -3 & 1 \\ 2 & 3 & -5 & 1 \end{pmatrix} \underset{L_3 \leftarrow L_3 - \frac{2}{3}L_1}{\overset{L_2 \leftarrow L_2 - \frac{1}{3}L_1}{\sim}} \begin{pmatrix} 3 & -1 & -2 & -4 \\ 0 & \frac{7}{3} & -\frac{7}{3} & \frac{7}{3} \\ 0 & \frac{11}{3} & -\frac{11}{3} & \frac{11}{3} \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim}$$

$\max\{|\frac{7}{3}|, |\frac{11}{3}|\} = \frac{11}{3} = a_{32}$

$$\sim \begin{pmatrix} 3 & -1 & -2 & -4 \\ 0 & \frac{11}{3} & -\frac{11}{3} & \frac{11}{3} \\ 0 & \frac{7}{3} & -\frac{7}{3} & \frac{7}{3} \end{pmatrix} \overset{L_3 \leftarrow L_3 - \frac{7}{11}L_2}{\sim} \begin{pmatrix} 3 & -1 & -2 & -4 \\ 0 & \frac{11}{3} & -\frac{11}{3} & \frac{11}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rang}(A) = 2$$

(*)

(*) Dacă vreunul din elementele indicate ar fi fost $\neq 0$, atunci rang $A = 3$.
(am fi putut alege un determinant cu diagonala nenulă și sub diagonala principală zerouri, deci $\neq 0$).

3) Fie $A = \begin{pmatrix} 5 & 1 \\ 8 & 2 \end{pmatrix}$. Determinați A^{-1}, folosind metoda GPT.

$$\overline{A} = \left(\begin{array}{cc|cc} 5 & 1 & 1 & 0 \\ 8 & 2 & 0 & 1 \end{array}\right)$$

$A \qquad I_2$

- alegem max. elementelor în modul $\Rightarrow \max\{|5|, |1|, |8|, |2|\} = 8 = a_{21}$
- interschimbăm linii/coloane pentru a muta pivotul 8 pe diagonala principală, primul element
- prin operații algebrice, obținem zerouri sub diagonala principală.

$$\overline{A} \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cc|cc} 8 & 2 & 0 & 1 \\ 5 & 1 & 1 & 0 \end{array}\right) \overset{L_2 \leftarrow L_2 - \frac{5}{8}L_1}{\sim} \left(\begin{array}{cc|cc} 8 & 2 & 0 & 1 \\ 0 & -\frac{1}{4} & 1 & -\frac{5}{8} \end{array}\right)$$

Obținem 2 sisteme:

$$\begin{cases} 8y_1 + 2y_2 = 0 \\ -\frac{1}{4}y_2 = 1 \Rightarrow y_2 = -4 \Rightarrow y_1 = 1. \end{cases} \Rightarrow x^{(1)} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

(am folosit coeficienții din stânga liniei, și termenii liberi $\rightarrow$ prima coloană din dreapta liniei)

$$\begin{cases} 8y_1 + 2y_2 = 1 \\ -\frac{1}{4}y_2 = -\frac{5}{8} \Rightarrow y_2 = \frac{5}{2} \Rightarrow y_1 = -\frac{1}{2} \end{cases} \Rightarrow x^{(2)} = \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$$

(am folosit coeficienții din stânga liniei, și termenii liberi $\rightarrow$ a doua coloană din dreapta liniei)

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -4 & \frac{5}{2} \end{pmatrix}$$ (este formată din coloanele $x^{(1)}$ şi $x^{(2)}$).

- se poate face proba: $A = \begin{pmatrix} 5 & 1 \\ 8 & 2 \end{pmatrix}$. $A^{-1} = ?$

- $\det A = 5\cdot 2 - 8\cdot 1 = 10 - 8 = 2 \neq 0$
- $A^T = \begin{pmatrix} 5 & 8 \\ 1 & 2 \end{pmatrix}$; $A^* = \begin{pmatrix} 2 & -1 \\ -8 & 5 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det A} \cdot A^* = \frac{1}{2} \cdot \begin{pmatrix} 2 & -1 \\ -8 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -4 & \frac{5}{2} \end{pmatrix}. \checkmark$$

<u>Obs.</u> Dacă schimbăm şi coloanele între ele pentru a duce pivotul pe diagonala principală, schimbăm şi necunoscutele (adică y_1, y_2).

4) Folosind metoda GPP, det A^{-1}, unde $A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix}$.

$$\overline{A} = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|ccc} -2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 + \frac{1}{2}L_1 \\ \sim \\ L_3 \leftarrow L_3 + L_1 \end{array}$$

$\max\{|1|, |-2|, |2|\} = 2 = a_{21}$

$$\sim \left(\begin{array}{ccc|ccc} -2 & 1 & 2 & 0 & 1 & 0 \\ 0 & \frac{5}{2} & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - \frac{2}{5}L_2}{\sim} \left(\begin{array}{ccc|ccc} -2 & 1 & 2 & 0 & 1 & 0 \\ 0 & \frac{5}{2} & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & -\frac{2}{5} & \frac{4}{5} & 1 \end{array}\right)$$

$\max\{|\frac{5}{2}|, |1|\} = \frac{5}{2} = a_{22}$

- avem 3 sisteme de rezolvat

$$\begin{cases} -2y_1 + y_2 + 2y_3 = 0 \\ \frac{5}{2} y_2 = 1 \\ 2y_3 = -\frac{2}{5} \end{cases}$$

$2y_3 = -\frac{2}{5} \Rightarrow y_3 = \frac{-2}{10} = \frac{-1}{5}$

$\Rightarrow y_2 = \frac{2}{5}$

$\Rightarrow -2y_1 + \frac{2}{5} + \frac{-2}{5} = 0$

$\Rightarrow y_1 = 0$.

$\Rightarrow x^{(1)} = \begin{pmatrix} 0 \\ \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}$

$$\begin{cases} -2y_1 + y_2 + 2y_3 = 1 \\ \frac{5}{2} y_2 = \frac{1}{2} \Rightarrow y_2 = \frac{1}{5} \\ 2y_3 = \frac{4}{5} \Rightarrow y_3 = \frac{2}{5} \end{cases} \Rightarrow -2y_1 + \frac{1}{5} + \frac{4}{5} = 1 \Rightarrow y_1 = 0$$

$x^{(2)} = \begin{pmatrix} 0 \\ \frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$

$$\begin{cases} -2y_1 + y_2 + 2y_3 = 0 \\ \frac{5}{2} y_2 = 0 \quad y_2 = 0 \\ 2y_3 = 1 \Rightarrow y_3 = \frac{1}{2} \end{cases} \Rightarrow -2y_1 + 0 + 1 = 0 \Rightarrow y_1 = \frac{1}{2}$$

$x^{(3)} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$

Deci $A^{-1} = \begin{array}{c} \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & \frac{1}{2} \end{pmatrix} \\ \begin{array}{ccc} x^{(1)} & x^{(2)} & x^{(3)} \end{array} \end{array}$.

Factorizare LU

1) Să se rezolve prin metoda LU cu GFP sistemul $Ax=b$, unde

$$A=\begin{pmatrix}4 & 0 & 1\\ 2 & 1 & 1\\ 1 & 3 & 1\end{pmatrix},\quad b=\begin{pmatrix}11\\ 8\\ 8\end{pmatrix}.$$

$w=(1,2,3) \rightarrow$ vectorul cu pozițiile inițiale ale liniilor matricei A.

$k=1$: $A=\begin{pmatrix}\boxed{4} & 0 & 1\\ 2 & 1 & 1\\ 1 & 3 & 1\end{pmatrix}$

multiplicatorii $m_{21}=\dfrac{a_{21}^{(1)}}{a_{11}^{(1)}}=\dfrac{2}{4}=\dfrac{1}{2}$, $m_{31}=\dfrac{a_{31}^{(1)}}{a_{11}^{(1)}}=\dfrac{1}{4}$.

$L_2 \leftarrow L_2-\frac{1}{2}L_1$

$L_3 \leftarrow L_3-\frac{1}{4}L_1$

$$\sim \begin{pmatrix}\boxed{4} & 0 & 1\\ 0 & 1 & \frac{1}{2}\\ 0 & 3 & \frac{3}{4}\end{pmatrix}$$

$k=2$: $A \sim \begin{pmatrix}\boxed{4} & 0 & 1\\ 0 & \boxed{1} & \frac{1}{2}\\ 0 & 3 & \frac{3}{4}\end{pmatrix}$

multiplicatorul $m_{32}=\dfrac{a_{32}^{(2)}}{a_{22}^{(2)}}=\dfrac{3}{1}=3$

$L_3 \leftarrow L_3-3L_2$

$$\sim \begin{pmatrix}\boxed{4} & 0 & 1\\ 0 & \boxed{1} & \frac{1}{2}\\ 0 & 0 & \boxed{-\frac{3}{4}}\end{pmatrix}$$

$$L=\begin{pmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ \frac{1}{4} & 3 & 1\end{pmatrix},\quad U=\begin{pmatrix}4 & 0 & 1\\ 0 & 1 & \frac{1}{2}\\ 0 & 0 & -\frac{3}{4}\end{pmatrix}$$

$$\left(L=\begin{pmatrix}1 & 0 & 0\\ m_{21} & 1 & 0\\ m_{31} & m_{32} & 1\end{pmatrix}\right)$$

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$b_1' = b_{w_1}$, $b_2' = b_{w_2}$, $b_3' = b_{w_3}$ $\Rightarrow$ $b' = b = \begin{pmatrix} 11 \\ 8 \\ 8 \end{pmatrix}$

$$Ax = b \Rightarrow L\underbrace{Ux}_{y} = b' \qquad \begin{array}{l} \bullet\ Ly = b' \\ \bullet\ Ux = y \end{array}$$

$$Ly = b \rightarrow \begin{cases} y_1 = 11 \\ \frac{1}{2}y_1 + y_2 = 8 & \Rightarrow \frac{11}{2} + y_2 = 8 \Rightarrow y_2 = 8 - \frac{11}{2} = \frac{5}{2} \\ \frac{1}{4}y_1 + 3y_2 + y_3 = 8 & \Rightarrow \frac{11}{4} + 3 \cdot \frac{5}{2} + y_3 = 8 \Rightarrow \end{cases}$$

$$\Rightarrow \frac{11}{4} + \frac{15}{2} + y_3 = 8 \Rightarrow \frac{11}{4} + \frac{30}{4} + y_3 = 8 \Rightarrow \frac{41}{4} + y_3 = 8 \Rightarrow y_3 = 8 - \frac{41}{4} = \frac{-9}{4}$$

$$Ux = y \Rightarrow \begin{cases} 4x_1 + x_3 = 11 & \Rightarrow 4x_1 + 3 = 11 \Rightarrow x_1 = 2 \\ x_2 + \frac{1}{2}x_3 = \frac{5}{2} & \Rightarrow x_2 + \frac{3}{2} = \frac{5}{2} \Rightarrow x_2 = 1 \\ -\frac{3}{4}x_3 = -\frac{9}{4} & \Rightarrow x_3 = 3 \end{cases}$$

2) Să se rezolve prin metoda LU cu GPP sistemul $Ax = b$, unde

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 11 \\ 8 \\ 8 \end{pmatrix}.$$

$w = (1, 2, 3) \rightarrow$ vectorul cu pozițiile inițiale ale liniilor matricei A.

$k = 1$. $\rightarrow \max\{|4|, |2|, |1|\} = 4 = a_{11}$

$$A \sim \begin{pmatrix} \boxed{4} & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

multiplicatorii : $m_{21} = \dfrac{a_{21}^{(1)}}{a_{11}^{(1)}} = \dfrac{1}{2}$, $m_{31} = \dfrac{a_{31}^{(1)}}{a_{11}^{(1)}} = \dfrac{1}{4}$.

$$A \underset{L_3 \leftarrow L_3 - \frac{1}{4}L_1}{\overset{L_2 \leftarrow L_2 - \frac{1}{2}L_1}{\sim}} \begin{pmatrix} \boxed{4} & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 3 & \frac{3}{4} \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{pmatrix}$$

$k = 2$. $\rightarrow \max\{|1|, |3|\} = 3 = a_{32} \Rightarrow L_2 \longleftrightarrow L_3$

$p = 3$ (linia pe care se află pivotul) $\quad w_2 \longleftrightarrow w_3$

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$$A \sim \begin{pmatrix} \boxed{4} & 0 & 1 \\ 0 & \boxed{3} & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$w = (1, 3, 2)$$

deoarece $k > 1$, se interschimbă și subliniii în matricea L, i.e.

$$l_{p,1:k-1} \leftrightarrow l_{k,1:k-1}, \text{ deci } l_{3,1:1} \leftrightarrow l_{2,1:1}$$

$$l_{3,1} \leftrightarrow l_{2,1}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

multiplicatorul $m_{32} = \dfrac{a_{32}^{(2)}}{a_{22}^{(2)}} = \dfrac{1}{3}$

$$A \overset{L_3 \leftarrow L_3 - \frac{1}{3}L_2}{\sim} \begin{pmatrix} \boxed{4} & 0 & 1 \\ 0 & \boxed{3} & \frac{3}{4} \\ 0 & 0 & \boxed{\frac{1}{4}} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad w = (1, 3, 2) \Rightarrow b' = \begin{pmatrix} b_1 \\ b_3 \\ b_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ 8 \end{pmatrix}$$

$$Ly = b' \Rightarrow \begin{cases} y_1 = 11 \\ \frac{1}{4}y_1 + y_2 = 8 \Rightarrow \frac{11}{4} + y_2 = 8 \Rightarrow y_2 = 8 - \frac{11}{4} = \frac{21}{4} \\ \frac{1}{2}y_1 + \frac{1}{3}y_2 + y_3 = 8 \end{cases} \Rightarrow \frac{11}{2} + \frac{1}{3} \cdot \frac{21}{4} + y_3 = 8 \Rightarrow \frac{11}{2} + \frac{7}{4} + y_3 = 8$$

$$\Rightarrow \frac{29}{4} + y_3 = 8 \Rightarrow y_3 = \frac{3}{4}$$

$$Ux = y \Rightarrow \begin{cases} 4x_1 + x_3 = 11 \\ 3x_2 + \frac{3}{4}x_3 = \frac{21}{4} \Rightarrow 3x_2 + \frac{3}{4} \cdot 3 = \frac{21}{4} \Rightarrow 3x_2 = \frac{21}{4} - \frac{9}{4} = \frac{12}{4} = 3 \\ \frac{1}{4}x_3 = \frac{3}{4} \Rightarrow x_3 = 3 \end{cases}$$

$$x_2 = 1$$

$$4x_1 + 3 = 11 \Rightarrow x_1 = 2$$