Factorizare Cholesky

1) Fie $A = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 5 & 5 \\ 6 & 5 & 11 \end{pmatrix}$.

a) Verificați că A este simetrică, pozitiv definită.

b) Determinați factorizarea Cholesky.

c) Rezolvați sistemul $Ax = \begin{pmatrix} 14 \\ 9 \\ 23 \end{pmatrix}$.

a) • $A^T = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 5 & 5 \\ 6 & 5 & 11 \end{pmatrix}$

$A = A^T \Rightarrow A$ - simetrică

• $A = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 5 & 5 \\ 6 & 5 & 11 \end{pmatrix}$

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 20 - 4 = 16 > 0$

$\Delta_3 = \det A = 220 + 60 + 60 - 180 - 44 - 100 = 16 > 0$

$\Rightarrow A$ - pozitiv definită

b) $A = L \cdot L^T$, unde L este inferior triunghiulară

$$\begin{pmatrix} 4 & 2 & 6 \\ 2 & 5 & 5 \\ 6 & 5 & 11 \end{pmatrix} = \begin{pmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ l_4 & l_5 & l_6 \end{pmatrix} \cdot \begin{pmatrix} l_1 & l_2 & l_4 \\ 0 & l_3 & l_5 \\ 0 & 0 & l_6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & 2 & 6 \\ 2 & 5 & 5 \\ 6 & 5 & 11 \end{pmatrix} = \begin{pmatrix} l_1^2 & l_1 l_2 & l_1 l_4 \\ l_1 l_2 & l_2^2 + l_3^2 & l_2 l_4 + l_3 l_5 \\ l_1 l_4 & l_2 l_4 + l_3 l_5 & l_4^2 + l_5^2 + l_6^2 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} l_1^2 = 4 & \Rightarrow l_1 = 2 \\ l_1 l_2 = 2 & \Rightarrow l_2 = 1 \\ l_1 l_4 = 6 & \Rightarrow l_4 = 3 \\ l_2^2 + l_3^2 = 5 & \Rightarrow l_3 = 2 \\ l_2 l_4 + l_3 l_5 = 5 & \Rightarrow 1\cdot 3 + 2l_5 = 5 \Rightarrow l_5 = 1 \\ l_4^2 + l_5^2 + l_6^2 = 11 & \Rightarrow 9+1+l_6^2 = 11 \Rightarrow l_6 = 1 \end{cases}$$

$$\Rightarrow \quad L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

c) $Ax = b \Rightarrow \underbrace{LL^T x}_{y} = b$ $\quad\quad$ • $Ly = b$

• $L^T x = y$

• $Ly = b \Rightarrow$

$$\begin{aligned} 2y_1 &= 14 && \Rightarrow y_1 = 7 \\ y_1 + 2y_2 &= 9 && \Rightarrow y_2 = 1 \\ 3y_1 + y_2 + y_3 &= 23 && \Rightarrow 21+1+y_3 = 23 \Rightarrow y_3 = 1 \end{aligned}$$

$$y = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

• $L^T x = y \Rightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{cases} 2x_1 + x_2 + 3x_3 = 7 & \Rightarrow 2x_1 + 0 + 3 = 7 \Rightarrow x_1 = 2 \\ 2x_2 + x_3 = 1 & \Rightarrow 2x_2 = 0 \Rightarrow x_2 = 0 \\ x_3 = 1 \end{cases}$$

$$x = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

2) Fie sistemul liniar: $\begin{cases} x_1-2x_2+2x_3=4 \\ -2x_1+5x_2-3x_3=-7 \\ 2x_1-3x_2+6x_3=10 \end{cases}$

a) Să se scrie sistemul sub forma $Ax=b$, determinându-se matricea A şi vectorul b.

b) Să se arate că matricea A corespunzătoare sistemului liniar este simetrică şi pozitiv definită.

c) Să se determine factorizarea Cholesky a matricei A.

d) Folosind factorizarea Cholesky a matricei A, să se determine soluţia sistemului liniar.

a) $A=\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 6 \end{pmatrix}$, $b=\begin{pmatrix} 4 \\ -7 \\ 10 \end{pmatrix}$

b) A simetrică: $A=A^T$

$$A^T=\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 6 \end{pmatrix}=A \checkmark$$

A pozitiv definită: $\Delta_1=1>0$

$$\Delta_2=\begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix}=5-4=1>0$$

$$\Delta_3=\det A=30+\cancel{12}+\cancel{12}-20-9-\cancel{24}=1>0$$

$\Rightarrow A$ - pozitiv definită.

c) $A=L\cdot L^T$, unde L este inferior triunghiulară

$$\begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 6 \end{pmatrix}=\begin{pmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ l_4 & l_5 & l_6 \end{pmatrix}\cdot\begin{pmatrix} l_1 & l_2 & l_4 \\ 0 & l_3 & l_5 \\ 0 & 0 & l_6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 6 \end{pmatrix} = \begin{pmatrix} l_1^2 & l_1 l_2 & l_1 l_4 \\ l_1 l_2 & l_2^2 + l_3^2 & l_2 l_4 + l_3 l_5 \\ l_1 l_4 & l_2 l_4 + l_3 l_5 & l_4^2 + l_5^2 + l_6^2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} l_1^2 = 1 & \Rightarrow l_1 = 1 \\ l_1 l_2 = -2 & \Rightarrow l_2 = -2 \\ l_1 l_4 = 2 & \Rightarrow l_4 = 2 \\ l_2^2 + l_3^2 = 5 & \Rightarrow (-2)^2 + l_3^2 = 5 \Rightarrow l_3 = 1 \\ l_2 l_4 + l_3 l_5 = -3 & \Rightarrow (-2) \cdot 2 + l_5 = -3 \Rightarrow l_5 = 1 \\ l_4^2 + l_5^2 + l_6^2 = 6 & \Rightarrow 2^2 + 1^2 + l_6^2 = 6 \Rightarrow l_6^2 = 1 \Rightarrow l_6 = 1. \end{cases}$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

d) $Ax = b \Rightarrow L\underbrace{L^T x}_{y} = b \Rightarrow$ • $Ly = b$
• $L^T x = y$

• $Ly = b \Rightarrow \begin{cases} y_1 = 4 \\ -2y_1 + y_2 = -7 & \Rightarrow -8 + y_2 = -7 \Rightarrow y_2 = 1 \\ 2y_1 + y_2 + y_3 = 10 & \Rightarrow 8 + 1 + y_3 = 10 \Rightarrow y_3 = 1. \end{cases}$

• $L^T x = y \Rightarrow \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} x_1 - 2x_2 + 2x_3 = 4 \\ x_2 + x_3 = 1 & \Rightarrow x_2 = 0 \\ x_3 = 1 \end{cases}$

$$\Rightarrow x_1 - 0 + 2 = 4 \Rightarrow x_1 = 2.$$

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• Factorizare QR. Metoda Givens

$A = QR$, Q - matrice ortogonală ($Q^TQ = QQ^T = I_n$)

R - matrice superior triunghiulară

Definim $R^{(ij)} = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & c & \cdots & s & & \\ & & \vdots & & \vdots & & \\ & & -s & \cdots & c & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$ (la intersecția liniilor i,j cu coloanele i,j se află elementele c, s; 1 - pe diag. principală; 0 - în rest)

matricea de rotație Givens.

Se aplică succesiv rotații Givens matricei A, până când A se transformă într-o matrice superior triunghiulară. Matricea obținută este matricea R. Q^T se obține prin înmulțirea succesivă a matricelor de rotație Givens.

Vrem elem. de pe poziția $j,i = 0 \Rightarrow s = \dfrac{a_{ji}}{\sqrt{a_{ii}^2 + a_{ji}^2}}$, $c = \dfrac{a_{ii}}{\sqrt{a_{ii}^2 + a_{ji}^2}}$.

$Ax = b \Rightarrow QRx = b \;\Big|\cdot Q^T$ la stg. $\Rightarrow \underbrace{Q^TQ}_{I_n}Rx = Q^Tb \Rightarrow Rx = \underbrace{Q^Tb}_{=b'} \Rightarrow Rx = b'$

(R - sup. triunghiulară)

1) Fie matricea $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Să se afle factorizarea QR prin metoda Givens. Să se rezolve sistemul $Ax = b$, unde $b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$.

vrem $\begin{cases} a_{21} = 0 \\ a_{31} = 0 \\ a_{32} = 0. \end{cases}$

$a_{21} = 1 \neq 0 \Rightarrow$ aplic $R^{(12)}$ $(i=1, j=2)$

$c = \dfrac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}} = \dfrac{1}{\sqrt{1+1}} = \dfrac{\sqrt{2}}{2}$, $s = \dfrac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}} = \dfrac{1}{\sqrt{1+1}} = \dfrac{\sqrt{2}}{2}$

$$R^{(12)} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \longleftarrow R^{(12)} A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 1 \end{pmatrix}$$

în matricea curentă:

$a_{32} \neq 0 \Rightarrow$ Aplic $R^{(23)}$

$$c = \frac{a_{22}}{\sqrt{a_{22}^2 + a_{32}^2}} = \frac{-\frac{\sqrt{2}}{2}}{\sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + 1^2}} = \frac{-\frac{\sqrt{2}}{2}}{\sqrt{\frac{2}{4} + \frac{4}{4}}} = \frac{-\frac{\sqrt{2}}{2}}{\sqrt{\frac{6}{4}}} =$$

$$= -\frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{6}} = \frac{-\sqrt{2}}{\sqrt{6}} = -\frac{1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

$$s = \frac{a_{32}}{\sqrt{a_{22}^2 + a_{32}^2}} = \frac{1}{\sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + 1^2}} = \frac{1}{\sqrt{\frac{2}{4} + \frac{4}{4}}} = \frac{1}{\sqrt{\frac{6}{4}}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$

$$R^{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$$A \leftarrow R^{(23)} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{3} \\ 0 & 0 & -\frac{2\sqrt{3}}{3} \end{pmatrix} = R$$

$$Q^T = R^{(23)} R^{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ 0 & -\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix}$$

$$b' = Q^T b = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} + \sqrt{2} \\ \frac{\sqrt{6}}{6} - \frac{2\sqrt{6}}{6} + \frac{5\sqrt{6}}{3} \\ \frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} - \frac{5\sqrt{3}}{3} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{2} \\ \frac{9\sqrt{6}}{6} \\ -\frac{6\sqrt{3}}{3} \end{pmatrix} = \begin{pmatrix} \frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{6}}{2} \\ -2\sqrt{3} \end{pmatrix}$$

$$Rx = b' \Rightarrow \begin{cases} \sqrt{2}\, x_1 + \frac{\sqrt{2}}{2} x_2 + \frac{\sqrt{2}}{2} x_3 = \frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{6}}{6} x_2 + \frac{\sqrt{6}}{6} x_3 = \frac{3\sqrt{6}}{2} \\ -\frac{2\sqrt{3}}{3} x_3 = -2\sqrt{3} \Rightarrow x_3 = 3 \end{cases} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{6}}{2} x_2 + \frac{\sqrt{6}}{\cancel{6}_{2}} \cdot \cancel{3} = \frac{3\sqrt{6}}{2} \Rightarrow \frac{\sqrt{6}}{2}(x_2 + 1) = \frac{3\sqrt{6}}{2} \Rightarrow x_2 = \frac{3\cancel{\sqrt{6}}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{\sqrt{6}}} - 1 = 2$$

$$\sqrt{2}\, x_1 + \frac{\sqrt{2}}{2} \cdot 2 + \frac{\sqrt{2}}{2} \cdot 3 = \frac{3\sqrt{2}}{2} \Rightarrow \sqrt{2}\, x_1 + \frac{5\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \Rightarrow$$

$$\Rightarrow x_1 \sqrt{2} = \frac{3\sqrt{2}}{2} - \frac{5\sqrt{2}}{2} = \frac{-2\sqrt{2}}{2} = -\sqrt{2} \Rightarrow x_1 = -1.$$

2) Fie matricea $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 6 \end{pmatrix}$. Să se afle factorizarea QR prin metoda Givens. Să se rezolve sistemul $Ax = b$, unde $b = \begin{pmatrix} 4 \\ -7 \\ 10 \end{pmatrix}$.

$a_{21} \neq 0$ aplic $R^{(12)}$

$$c = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{21}^2}} = \frac{1}{\sqrt{1 + (-2)^2}} = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$s = \frac{a_{21}}{\sqrt{a_{11}^2 + a_{21}^2}} = \frac{-2}{\sqrt{1+(-2)^2}} = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

$$R^{(12)} = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} & 0 \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \leftarrow R^{(12)} A = \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} & 0 \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ -2 & 5 & -3 \\ 2 & -3 & 6 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & -\frac{12\sqrt{5}}{5} & \frac{8\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 2 & -3 & 6 \end{pmatrix}$$

$a_{31} \neq 0$ aplic $R^{(13)}$

$$c = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{31}^2}} = \frac{\sqrt{5}}{\sqrt{5+4}} = \frac{\sqrt{5}}{3}$$

$$s = \frac{a_{31}}{\sqrt{a_{11}^2 + a_{31}^2}} = \frac{2}{\sqrt{5+4}} = \frac{2}{3}$$

$$R^{(13)} = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}$$

$$A \leftarrow R^{(13)}A = \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{-12\sqrt{5}}{5} & \frac{8\sqrt{5}}{5} \\ 0 & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 2 & -3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -6 & \frac{20}{3} \\ 0 & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 0 & \frac{3\sqrt{5}}{5} & \frac{14\sqrt{5}}{15} \end{pmatrix}$$

$a_{32} \neq 0$ aplic $R^{(23)}$

$$c = \frac{a_{22}}{\sqrt{a_{22}^2 + a_{32}^2}} = \frac{\frac{\sqrt{5}}{5}}{\sqrt{\frac{5}{25} + \frac{9 \cdot 5}{25}}} = \frac{\frac{\sqrt{5}}{5}}{\sqrt{\frac{50}{25}}} = \frac{\sqrt{5}}{5} \cdot \frac{5}{\sqrt{50}} = \frac{\sqrt{5}}{\sqrt{50}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

$$s = \frac{a_{32}}{\sqrt{a_{22}^2 + a_{32}^2}} = \frac{\frac{3\sqrt{5}}{5}}{\sqrt{\frac{50}{25}}} = \frac{3\sqrt{5}}{5} \cdot \frac{5}{\sqrt{50}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}.$$

$$R^{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \\ 0 & -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{pmatrix}$$

$$A \leftarrow R^{(23)}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \\ 0 & -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{pmatrix} \begin{pmatrix} 3 & -6 & \frac{20}{3} \\ 0 & \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 0 & \frac{3\sqrt{5}}{5} & \frac{14\sqrt{5}}{15} \end{pmatrix} = \begin{pmatrix} 3 & -6 & \frac{20}{3} \\ 0 & \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{6} \end{pmatrix} = R.$$

$$Q^T = R^{(23)}R^{(13)}R^{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{10}}{10} & \frac{3\sqrt{10}}{10} \\ 0 & -\frac{3\sqrt{10}}{10} & \frac{\sqrt{10}}{10} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{5}}{5} & \frac{-2\sqrt{5}}{5} & 0 \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} \\ -\frac{\sqrt{10}}{15} & \frac{\sqrt{10}}{10} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{10}}{15} & -\frac{3\sqrt{10}}{10} & \frac{\sqrt{2}}{6} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} \end{pmatrix}$$

$$b' = Q^T b = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} \end{pmatrix} \begin{pmatrix} 4 \\ -7 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{38}{3} \\ \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6} \end{pmatrix}$$

$$Rx = b' \Rightarrow \begin{cases} 3x_1 - 6x_2 + \frac{20}{3}x_3 \overset{(1)}{=} \frac{38}{3} \\ \sqrt{2}\,x_2 + \frac{3\sqrt{2}}{2}\,x_3 \overset{(2)}{=} \frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6}x_3 = \frac{\sqrt{2}}{6} \Rightarrow x_3 = 1 \end{cases}$$

$$\overset{(2)}{\Rightarrow} \sqrt{2}\,x_2 + \cancel{\frac{3\sqrt{2}}{2}} \cdot 1 = \cancel{\frac{3\sqrt{2}}{2}} \Rightarrow \sqrt{2}\,x_2 = 0 \Rightarrow x_2 = 0.$$

$$\overset{(1)}{\Rightarrow} 3x_1 - 6 \cdot 0 + \frac{20}{3} \cdot 1 = \frac{38}{3} \Rightarrow 3x_1 = \frac{38}{3} - \frac{20}{3} = \frac{18}{3} = 6. \Rightarrow x_1 = 2.$$