Seminar 14 – Medii, dispersii. Probleme recapitulative.

Ex: Fie $f(x) = \frac{2x+1}{k}, x = 0,1,2,3,4$. Se cere:

a) Sa se calculeze k astfel incat f sa fie o densitate de probabilitate si, apoi, sa se calculeze functia de repartitie.

b) Fie X v.a. a carei densitate de probabilitate este f. Sa se calculeze $P(X = 4), P(X \leq 1), P(X > -10)$

b) Sa se calculeze media, dispersia si abaterea medie patratica a variabilei X

a)

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{1}{k} & \frac{3}{k} & \frac{5}{k} & \frac{7}{k} & \frac{9}{k} \end{pmatrix}$$

$$\sum_{i=1}^{4} \frac{2i+1}{k} = 1 = \frac{25}{k} = 1 => k = 25$$

$$F(x) = P(X < x) = \begin{cases} 0, x < 0 \\ \frac{1}{25}, 0 \leq x < 1 \\ \frac{4}{25}, 1 \leq x < 2 \\ \frac{9}{25}, 2 \leq x < 3 \\ \frac{16}{25}, 3 \leq x < 4 \\ 1, 4 \leq x \end{cases}$$

b)

$$P(X = 4) = \frac{9}{25}$$

$$P(X \leq 1) = \frac{4}{25}$$

$$P(X > -10) = 1$$

c)

$$E(X) = \sum_{x=0}^{4} xP(X = x) = \frac{70}{25} = 2{,}8$$

$$D^2(X) = E(X^2) - (E(X))^2 = \sum_{x=0}^{4} x^2 P(X = x) - 2,8^2 = \frac{230}{25} - 2,8^2 = 1{,}36$$

$$D(X) = \sqrt{D^2(X)} = 1{,}16$$

Ex: Fie $f(x) = \left(\frac{k}{7}\right)\left(\frac{1}{2}\right)^x, x = 1,2,3$. Se cere:

a) Sa se calculeze k astfel incat f sa fie o densitate de probabilitate si, apoi, sa se calculeze functia de repartitie.

b) Fie X v.a. a carei densitate de probabilitate este f. Sa se calculeze $P(X \leq 1), P(X > 1), P(2 < X < 6)$

b) Sa se calculeze media, dispersia si abaterea medie patratica a variabilei X

a)

$$X = \begin{pmatrix} 1 & 2 & 3 \\ \frac{k}{14} & \frac{k}{28} & \frac{k}{56} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{4}{7} & \frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

$$\frac{k}{7}\sum_{i=1}^{3}\frac{1}{2^i} = \frac{k}{7} * \frac{7}{8} = \frac{k}{8} = 1 \Rightarrow k = 8$$

b)

$$P(X \leq 1) = \frac{4}{7}$$

$$P(X > 1) = \frac{3}{7}$$

$$P(2 < X < 6) = \frac{1}{7}$$

c)

$$E(X) = \frac{4}{7} + \frac{4}{7} + \frac{3}{7} = \frac{11}{7}$$

$$D^2(X) = \frac{4}{7} + \frac{8}{7} + \frac{9}{7} - \left(\frac{11}{7}\right)^2 = \frac{26}{49}$$

$$D(X) = \sqrt{D^2(X)} = 0{,}728$$

Fie vectorul (X, Y) cu densitatea

$$f(x,y) = \begin{cases} a\sin(x+y), & 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2} \\ 0, & \text{în rest} \end{cases}$$

Să se determine "a", $E(X), E(Y), D^2(X), D^2(Y), \operatorname{cov}(X,Y)$.

$$a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(x+y)\,dydx = 1$$

$$a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(x+y)\,dydx = -a\int_0^{\frac{\pi}{2}}\left((\cos(x+y))|_0^{\frac{\pi}{2}}\right)dx = -a\int_0^{\frac{\pi}{2}}\left(\cos\left(x+\frac{\pi}{2}\right)-\cos x\right)dx$$

$$= -a\int_0^{\frac{\pi}{2}}\left(\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}-\cos x\right)dx = a\left(\int_0^{\frac{\pi}{2}}sin\,x\,dx+\int_0^{\frac{\pi}{2}}\cos x\,dx\right)$$

$$= a\left((-\cos x)|_0^{\frac{\pi}{2}}+(\sin x)|_0^{\frac{\pi}{2}}\right) = a(1+1) = 2a = 1 => a = \frac{1}{2}$$

$$E(X) = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}x\sin(x+y)\,dydx = -\frac{1}{2}\int_0^{\frac{\pi}{2}}x\left((\cos(x+y))|_0^{\frac{\pi}{2}}\right)dx$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}}x\left(\cos\left(x+\frac{\pi}{2}\right)-\cos x\right)dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}}x\left(\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}-\cos x\right)dx$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}xsin\,x\,dx+\int_0^{\frac{\pi}{2}}x\cos x\,dx\right) = \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}x\,(-\cos x)'dx+\int_0^{\frac{\pi}{2}}x\,(sin\,x)'dx\right)$$

$$= \frac{1}{2}\left(-(x\cos x)|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dx+(x\sin x)|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x\,dx\right)$$

$$= \frac{1}{2}\left((\sin x)|_0^{\frac{\pi}{2}}+\frac{\pi}{2}+(\cos x)|_0^{\frac{\pi}{2}}\right) = \frac{1}{2}\left(1+\frac{\pi}{2}-1\right) = \frac{\pi}{4} = E(Y)$$

$$D^2(X)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}x^2\sin(x+y)\,dydx-\big(E(X)\big)^2=-\frac{1}{2}\int_0^{\frac{\pi}{2}}x^2\left(\left(\cos(x+y)\right)\Big|_0^{\frac{\pi}{2}}\right)dx-\frac{\pi^2}{16}$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}x^2\left(\cos\left(x+\frac{\pi}{2}\right)-\cos x\right)dx-\frac{\pi^2}{16}$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}x^2\left(\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}-\cos x\right)dx-\frac{\pi^2}{16}$$

$$=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}x^2 sin\,x\,dx+\int_0^{\frac{\pi}{2}}x^2\cos x\,dx\right)-\frac{\pi^2}{16}$$

$$=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}x^2\,(-\cos x)'dx+\int_0^{\frac{\pi}{2}}x^2\,(sin\,x)'dx\right)-\frac{\pi^2}{16}$$

$$=\frac{1}{2}\left(-(x^2\cos x)\Big|_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}x\cos x\,dx+(x^2\sin x)\Big|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\sin x\,dx\right)-\frac{\pi^2}{16}=$$

$$=\frac{1}{2}\left(2\int_0^{\frac{\pi}{2}}x(\sin x)'\,dx+\frac{\pi^2}{4}+2\int_0^{\frac{\pi}{2}}x(cos\,x)'\,dx\right)-\frac{\pi^2}{16}$$

$$=\frac{1}{2}\left(2\left((x\sin x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x\,dx\right)+\frac{\pi^2}{4}+2\left((x\cos x)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\cos x\,dx\right)\right)-\frac{\pi^2}{16}=$$

$$=\frac{1}{2}\left(2\left(\frac{\pi}{2}+(\cos x)\Big|_0^{\frac{\pi}{2}}\right)+\frac{\pi^2}{4}+2\left(-(\sin x)\Big|_0^{\frac{\pi}{2}}\right)\right)-\frac{\pi^2}{16}$$

$$=\left(\left(\frac{\pi}{2}-1\right)+\frac{\pi^2}{8}+(-1)\right)-\frac{\pi^2}{16}=\frac{\pi}{2}+\frac{\pi}{16}^2-2$$

$$cov(X,Y)=E(XY)-E(X)E(Y)$$

$$E(XY)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}xy\sin(x+y)\,dxdy=\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}xy(\sin x\cos y+\cos x\sin y)\,dxdy$$

$$=\frac{1}{2}\left(\left(\int_0^{\frac{\pi}{2}}x\sin x\,dx\right)\left(\int_0^{\frac{\pi}{2}}y\cos y\,dy\right)+\left(\int_0^{\frac{\pi}{2}}x\cos x\,dx\right)\left(\int_0^{\frac{\pi}{2}}y\sin y\,dy\right)\right)$$

$$=\frac{1}{2}2\left(\frac{\pi}{2}-1\right)=\frac{\pi}{2}-1$$

$$cov(X,Y)=\frac{\pi}{2}-1-\frac{\pi^2}{4}$$

Fie vectorul aleator (X, Y) cu densitatea

$$f(x,y) = \begin{cases} ae^{-x-2y}, & x \geq 0, y \geq 0 \\ 0, & \text{în rest} \end{cases}$$

Să se determine "a", funcţia de repartiţie a vectorului (X, Y) şi funcţiile de repartiţie ale variabilelor $X+Y, \frac{X}{Y}, X^2, \sqrt{X}$.

Pentru ca functia data sa fie o densitate de repartitie trebuie sa avem

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = 1$$

$$a\int_0^\infty \int_0^\infty e^{-x-2y}\,dxdy = a\left(\int_0^\infty e^{-2y}\,dy\right)\left(\int_0^\infty e^{-x}\,dx\right) = a\left(\left(-\frac{1}{2}e^{-2y}\right)\Big|_0^\infty\right)((-e^{-x})|_0^\infty) = a\frac{1}{2}1 = \frac{a}{2}$$
$$= 1$$

$$a = 2$$

Fie $Z = X + Y \Rightarrow X = Z - Y$

$$F_Z(z) = F_{X+Y}(z) = 2\int_0^z \int_0^{z-y} e^{-x-2y}\,dxdy = 2\int_0^z e^{-2y}\left(\int_0^{z-y} e^{-x}\,dx\right)dy$$
$$= 2\int_0^z e^{-2y}\left(-(e^{-x})|_0^{z-y}\right)dy = 2\int_0^z e^{-2y}\,(1-e^{y-z})dy$$
$$= 2\left(\int_0^z e^{-2y}\,dy - \int_0^z e^{-y-z}\,dy\right) = 2\left(\left(-\frac{1}{2}e^{-2y}\right)\Big|_0^z - e^{-z}(-(e^{-y})|_0^z)\right)$$
$$= 2\left(-\frac{1}{2}(e^{-z}-1) + e^{-z}(e^{-z}-1)\right) = (e^{-z}-1)(-1+2e^{-z})$$
$$= -e^{-z} + 2e^{-2z} + 1 - 2e^{-z} = 1 - 3e^{-z} + 2e^{-2z}$$

Fie $T = \frac{X}{Y} \Rightarrow X = TY$.

$$x, y \geq 0, x = ty, t \geq 0$$

$$F_T(t) = F_{\frac{X}{Y}}(t) = 2\int_0^\infty e^{-2y}\int_0^{ty} e^{-x}\,dxdy = 2\int_0^\infty e^{-2y}\left((-e^{-x})|_0^{ty}\right)dy = 2\int_0^\infty e^{-2y}\,(1-e^{-ty})dy$$
$$= 2\left(\int_0^\infty e^{-2y}\,dy - \int_0^\infty e^{-y(2+t)}dy\right) = 2\left(\left(-\frac{e^{-2y}}{2}\right)\Big|_0^\infty - \left(-\frac{e^{-y(2+t)}}{2+t}\right)\Big|_0^\infty\right)$$
$$= 2\left(\frac{1}{2} + \frac{-1}{2+t}\right) = 2\frac{2+t-2}{2(2+t)} = \frac{t}{t+2}$$

Fie $M = X^2 \Rightarrow X = \sqrt{M}$

$$x, y \geq 0, x \in (0, \sqrt{t})$$

$$F_M(t) = 2\int_0^\infty \int_0^{\sqrt{t}} e^{-x-2y}\,dxdy = 2\left(\int_0^\infty e^{-2y}\,dy\right)\left(\int_0^{\sqrt{t}} e^{-x}\,dx\right) = 2\frac{1}{2}\left((-e^{-x})|_0^{\sqrt{t}}\right) = 1 - e^{-\sqrt{t}}$$

Fie $N = \sqrt{X} => X = N^2$

$$N \geq 0, x, y \geq 0$$

$$F_N(t) = 2\int_0^{\infty} e^{-2y} dy \int_0^{t^2} e^{-x}\, dx = (-e^{-x})|_0^{t^2} = 1 - e^{-t^2}$$

Dacă X, Y sunt v. a. independente, arătaţi că v. a. $U = \max(X, Y)$, $V = \min(X, Y)$ au funcţiile de repartiţie $F_U(t) = F_X(t) \cdot F_Y(t), F_V(t) = = 1 - [(1 - F_X(t))(1 - F_Y(t))]$.

$$F_U(t) = P(U < t) = P(\max(X, Y) < t) = P(X < t, Y < t) = P(X < t)P(Y < t) = F_X(t)F_Y(t)$$

$$F_V(t) = P(V < t) = P(\min(X, Y) < t) = 1 - P(\min(X, Y) \geq t) = 1 - P(X \geq t, Y \geq t)$$
$$= 1 - P(X \geq t)P(Y \geq t) = 1 - \big(1 - P(X < t)\big)\big(1 - P(Y < t)\big)$$
$$= 1 - \big(1 - F_X(t)\big)\big(1 - F_Y(t)\big)$$

Fie X şi Y variabile aleatoare pentru care $E(X) = -2, E(Y) = = 4, D^2(X) = 4, D^2(Y) = 9$, iar coeficientul de corelaţie $\rho(X, Y) = -0,5$. Să se calculeze valoarea medie a variabilei $Z = 3X^2 - 2XY + +Y^2 - 3$.

$$E(Z) = 3E(X^2) - 2E(XY) + E(Y^2) - 3$$

$$D^2(X) = E(X^2) - \big(E(X)\big)^2 => E(X^2) = D^2(X) + \big(E(X)\big)^2 = 4 + 4 = 8$$

$$E(Y^2) = D^2(Y) + \big(E(Y)\big)^2 = 9 + 16 = 25$$

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{D^2(X)D^2(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{D^2(X)D^2(Y)}} => E(XY) = \rho(X, Y)\sqrt{D^2(X)D^2(Y)} + E(X)E(Y)$$
$$= -0{,}5 * 2 * 3 - 8 = -11$$

$$E(Z) = 3E(X^2) - 2E(XY) + E(Y^2) - 3 = 24 + 22 + 25 - 3 = 68$$

Un aparat este format din 10 subansamble. Probabilitatea ca un subansamblu sa functioneze fara defectare pe o perioada t este egala $p = \frac{1}{20}$, pentru fiecare subansamblu, iar iesirile din functiune ale acestora sunt independente. Sa se afle probabilitatile ca sa se defecteze:

a) cel putin un subansamblu

b) exact unul

c) exact doua

d) cel putin doua

Fie X v.a. egala cu numarul de subansamble care se defecteaza in perioada t, care are repartitie binomiala $n = 10, p = \frac{1}{20}$

a)

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) = 1 - C_{10}^{0} p^{0} (1-p)^{10} = 0{,}401$$

b)

$$P(X = 1) = C_{10}^{1} p^{1} (1-p)^{9} = 0{,}315$$

c)

$$P(X = 2) = C_{10}^{2} p^{2} (1-p)^{8} = 0{,}075$$

d)

$$P(X \geq 2) = P(X \geq 1) - P(X = 1) = 0{,}401 - 0{,}315 = 0{,}085$$

La un examen exista 7 bilete din probabilitati, 5 din statistica si 8 din analiza complexa. Un student extrage, la intamplare, 3 bilete.

a) Care este probabilitatea ca subiectele sa fie din cele 3 capitole?

b) Care este probabilitatea ca studentul sa extraga subiecte in ordinea: analiza complexa, probabilitati, statistica.

a) Vom folosi schema bilei fara intoarcere

$$P = \frac{C_7^1 C_5^1 C_8^1}{C_{20}^3}$$

b)

$$P = \frac{8}{20} * \frac{7}{19} * \frac{5}{18}$$

Fie X numarul vorbitorilor activi dintr-un grup de 8 vorbitori independenti. Presupunem ca un vorbitor este activ cu probabilitate $p = \frac{1}{3}$. Determinati probabilitatea ca numarul de vorbitori activi sa fie mai mare decat 6.

V.a. X are o repartitie binomiala cu $n = 8, p = \frac{1}{3}$

$$P(X>6)=P(X=7)+P(X=8)=C_8^7p^7(1-p)^1+C_8^8p^8(1-p)^0=8p^7(1-p)+p^8=\frac{8*2+1}{3^8}$$
$$=0{,}003$$

Probabilitatea ca o convorbire telefonica sa nu dureze mai mult de t minute este modelata ca o v.a. $X\sim Poisson\left(\lambda=\frac{1}{3}\right)$. Sa se scrie $F_X(x)$. Care este probabilitatea ca o convorbire sa dureze intre 5 si 10 minute? Dar daca repartitia este $Exp\left(\lambda=\frac{1}{3}\right)$?

Daca repartitia este Poisson cu $\lambda=\frac{1}{3}$, atunci

$$F_X(x)=P(X<x)=\sum_{i=0}^{x-1}e^{-\lambda}\lambda^i/i!=e^{-\frac{1}{3}}\left(\sum_{i=0}^{x-1}\frac{1}{3^i i!}\right)$$
$$P(5\le X\le 10)=F(11)-F(5)=2{,}6*10^{-5}$$

Daca repartitia este Exponentiala cu $\lambda=\frac{1}{3}$, atunci

$$F_X(x)=P(X<x)=\int_0^x\lambda e^{-\lambda x}dx$$
$$P(5\le X\le 10)=\int_5^{10}\lambda e^{-\lambda x}\,dx=-\left(e^{-\lambda x}\right)\Big|_5^{10}=e^{-\frac{5}{3}}-e^{-\frac{10}{3}}=0{,}153$$

Numarul de vizualizari al unui site intr-un interval de timp este o v.a. cu repartitie Poisson. El are o medie de $\alpha=2$ vizualizari pe secunda. Care este probabilitatea ca sa nu existe nici o vizualizare intr-un interval de 0,25 secunde? Care este probabilitatea ca sa fie mai mult de 2 vizualizari intr-un interval de o secunda?

Fie X v.a. egala cu numarul de vizualizari in 0,25 secunde, care are repartitie Poisson cu $\lambda=2*0{,}25=0{,}5$

$$P(X=0)=\frac{e^{-\lambda}\lambda^0}{0!}=e^{-\lambda}=0{,}606$$

Fie Y v.a. egala cu numarul de vizualizari intr-o secunda, care are repartitie Poisson

$$P(Y\ge 2)=1-P(Y=0)-P(Y=1)=1-e^{-2}-\frac{e^{-2}2^1}{1!}=1-3e^{-2}=1-0{,}406=0{,}593$$

Fie $X\sim U[0,2\pi], Y=\cos X, Z=\sin Y$. Determinati $\rho(Y,Z)$.

$$E(Y)=\int_0^{2\pi}\cos x\,dx=(\sin x)|_0^{2\pi}=0,$$

$$E(Z) = \int_0^{2\pi} \sin x \, dx = (-\cos x)|_0^{2\pi} = 0$$

$$cov(Y,Z) = E\left(\left(Y - E(Y)\right)\left(Z - E(Z)\right)\right) = \int_0^{2\pi} \sin x \cos x \, dx = 0$$

Intre orele 7 si 8 dimineata, un metrou pleaca dintr-o statie la si 3, 5, 8, 10, 13, 15, 18, 20, ... minute dupa ora 7. Sa se determine probabilitatea ca o persoana care soseste in statie sa astepte mai putin de 1 minut pana la plecarea primului tren, daca se presupune ca momentul sosirii persoanei urmeaza o repartiei uniforma in intervalul de timp de la 7 la 8.

Avem $4 * 6 = 24$ plecari de metrou in intervalul de la 7 la 8.

Fie X v.a. egala cu durata de timp de asteptare dintre momentul sosirii in statie si momentul plecarii metroului, exprimata in minute.

$$P(X < 1) = P(X = 0) = P(sa\ soseasca\ in\ statie\ in\ acelasi\ minut\ in\ care\ pleaca\ si\ metroul) = \frac{24}{60} = \frac{2}{5}$$

Probabilitatea ca un condensator sa iasa din functiune intr-un interval de timp t este 0,03. Sa se determine probabilitatea ca, in intrevalul respectiv de timp, din 100 de condensatoare sa iasa din functiune mai putin de 2.

Fie X v.a. reprezentand defectarea unui condensator in intervalul de timp, care are o repartitie binomiala cu $n = 100, p = 0{,}03$

$$P(X < 2) = \sum_{i=0}^{1} C_{100}^{i} p^{i} (1-p)^{100-i} = C_{100}^{0} p^{0} (1-p)^{100} + C_{100}^{1} p^{1} (1-p)^{99} = 0{,}195$$

Populatia Nicosiei este ¾ greaca si ¼ turca, iar 1/5 dintre greci si 1/10 dintre turci vorbesc engleza. Un strain intalneste un locuitor al Nicosiei care vorbeste engleza. Care este probabilitatea ca el sa fie grec?

Procentul grecilor care vorbesc engleza este de $\frac{3}{4} * \frac{1}{5} = \frac{3}{20} = \frac{6}{40}$, iar cel al turcilor care vorbesc engleza este de $\frac{1}{4} * \frac{1}{10} = \frac{1}{40}$. Asadar probabilitatea sa fie grec este $\frac{\frac{6}{40}}{\frac{7}{40}} = \frac{6}{7}$

La un examen se prezinta 100 de participanti care au de raspuns la 10 de intrebari grila, cu cate 5 raspunsuri fiecare. Participantii raspund aleator la intrebari. Care este probabilitatea ca cel putin unul dintre ei sa nimereasca cel putin 5 grile?

Fie X v.a. egala cu evenimentul ca un participant sa raspunda corect la o grila, $p = \frac{1}{5}$

Fie Y v.a. egala numarul de raspunsuri corecte al unui participant, repartitie binomiala $n = 10, p = \frac{1}{5}$

$$P(Y \geq 5) = \sum_{i=5}^{10} C_{10}^{i} \left(\frac{1}{5}\right)^{i} \left(\frac{4}{5}\right)^{10-i} = 0{,}033.$$

Considerand 100 de participanti obtinem o repartitie binomiala cu $n = 100, p = 0{,}033$

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - C_{100}^{0} p^{0} (1-p)^{100-0} = 0{,}965$$