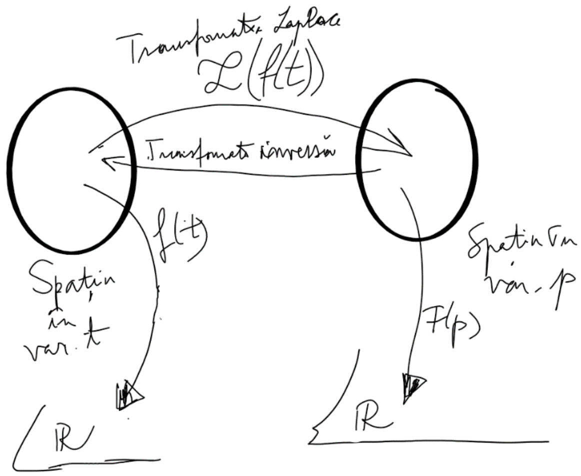
Seminar 5 – Aplicatii ale transformatei Laplace

Definiţia 2. **Transformata Laplace** (sau **funcţia imagine**) a unei funcţii original f este funcţia complexă $F\colon \{p \in \mathbb{C} \mid \operatorname{Re} p > p_0\} \to \mathbb{C}$:

$$F(p) = \int_0^{\infty} f(t)\mathrm{e}^{-pt}dt \tag{2}$$

Se poate demonstra că funcţia imagine F este olomorfă (analitică) în semiplanul $\operatorname{Re} p > p_0$ şi vom nota $F = \mathcal{L}[f]$.



Tabelul 9.1

Nr.	$f(t)$	$F(p)$
1	h(t)	$\frac{1}{p}$
2	t^n	$\frac{n!}{p^{n+1}}$
3	$\mathrm{e}^{\omega t}$	$\frac{1}{p-\omega}$
4	$\sin \omega t,\ \omega > 0$	$\frac{\omega}{p^2+\omega^2}$
5	$\cos \omega t,\ \omega > 0$	$\frac{p}{p^2+\omega^2}$
6	$sh\, \omega t,\ \omega > 0$	$\frac{\omega}{p^2-\omega^2}$
7	$ch\, \omega t,\ \omega > 0$	$\frac{p}{p^2-\omega^2}$
8	$J_n(t)$, (funcţie Bessel)	$\frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$
9	$\frac{\sin t}{t}$	$\operatorname{arcctg} p$
10	$\frac{1}{2}(\sin t - t\cos t)$	$\frac{1}{(p^2+1)^2}$

Funcţiile de la nr. 2-10 care apar în tabelul 9.1. sunt subînţelese a fi înmulţite cu $u(t)$, pentru că , în caz contrar, nu ar fi funcţii original; astfel, de exemplu, prin t^n se înţelege $t^n u(t)$. Această convenţie va fi utilizată şi în continuare.

Proprietatile transformarii Laplace

Liniaritate: $\mathcal{L}[af(t)+bg(t)](p) = a\mathcal{L}[f(t)](p) + b\mathcal{L}[g(t)](p)$

Teorema asemanarii: $\mathcal{L}[f(at)](p) = \frac{1}{a}F\left(\frac{p}{a}\right)$

Teorema intarzierii: $\mathcal{L}[f(t-\tau)](p) = e^{-p\tau}F(p)$

Teorema deplasarii: $\mathcal{L}\left[e^{-\lambda t}f(t)\right](p) = \mathcal{L}[f(t)](p+\lambda) = F(p+\lambda)$

Teorema derivarii imaginii: $\mathcal{L}[t^n f(t)](p) = (-1)^n \frac{d^nF}{dp^n}$

Teorema derivarii originalului: $\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right](p) = p^nF(p) - \left(p^{n-1}f(0+0) + p^{n-2}f'(0+0) + \cdots + f^{(n-1)}(0+0)\right)$

Teorema integrarii originalului: $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right](p) = \frac{F(p)}{p}$

Teorema integrarii imaginii: $\mathcal{L}\left[\frac{f(t)}{t}\right](p) = \int_p^\infty F(v)dv$

Teorema de convolutie: Daca $h(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ atunci $H = F \cdot G$

Ex: Sa se calculeze transformata Laplace a functiei $f(t) = \begin{cases} 4, & 0 < t < 3 \\ 2, & t > 3 \end{cases}$

$$\begin{aligned}\mathcal{L}[f(t)](p) &= \int_0^\infty f(t)e^{-pt}\,dt = \int_0^3 4e^{-pt}\,dt + \int_3^\infty 2\,e^{-pt}dt = 4\left(\frac{e^{-pt}}{-p}\right)\Bigg|_0^3 + 2\left(\frac{e^{-pt}}{-p}\right)\Bigg|_3^\infty \\ &= 4\frac{e^{-3p}-1}{-p} + 2\frac{0-e^{-3p}}{-p} = \frac{2}{-p}\frac{2e^{-3p}-2-e^{-3p}}{1} = \frac{4-2e^{-3p}}{p}\end{aligned}$$

Ex: Folositi proprietatea de liniaritate pentru a deduce transformata Laplace a functiei $f(t) = 5e^{-7t} + t + 2e^{2t}$

$$\mathcal{L}[f(t)](p) = 5\mathcal{L}[e^{-7t}](p) + \mathcal{L}[t](p) + 2\mathcal{L}[e^{2t}](p) = 5\frac{1}{p+7} + \frac{1}{p^2} + 2\frac{1}{p-2}$$

Ex: Calculati transformata Laplace a functiei $f(t) = e^{at}(t^2+bt+c)$.

$$\begin{aligned}\mathcal{L}[f(t)](p) &= \mathcal{L}[e^{at}t^2](p) + b\mathcal{L}[e^{at}t](p) + c\mathcal{L}[e^{at}](p) \\ &= (-1)^2\frac{d^2}{dp^2}\mathcal{L}[e^{at}](p) - b\frac{d}{dp}\mathcal{L}[e^{at}](p) + c\frac{1}{p-a} \\ &= \frac{d^2}{dp^2}\left(\frac{1}{p-a}\right) - b\frac{d}{dp}\left(\frac{1}{p-a}\right) + \frac{c}{p-a} = -\frac{d}{dp}\left(\frac{1}{(p-a)^2}\right) + \frac{b}{(p-a)^2} + \frac{c}{p-a} \\ &= \frac{2}{(p-a)^3} + \frac{b}{(p-a)^2} + \frac{c}{p-a}\end{aligned}$$

Ex: Sa se gaseasca functia original ale carei transformata Laplace este functia $F(p) = \frac{p+3}{p^3+4p^2}$

Vom desparti in fractii simple

$$F(p)=\frac{p+3}{p^2(p+4)}=\frac{ap+b}{p^2}+\frac{c}{p+4}=\frac{ap^2+4ap+bp+4b+cp^2}{p^2(p+4)}$$

$$\begin{cases} a+c=0 \\ 4a+b=1 \\ 4b=3 \end{cases} => \begin{cases} b=\dfrac{3}{4} \\ a=\dfrac{1}{16} \\ c=-\dfrac{1}{16} \end{cases}$$

$$F(p)=\frac{1}{16}\frac{p+12}{p^2}-\frac{1}{16}\frac{1}{p+4}=\frac{1}{16}\frac{1}{p}+\frac{3}{4}\frac{1}{p^2}-\frac{1}{16}\frac{1}{p+4}$$

$$f(t)=\frac{1}{16}+\frac{3}{4}t-\frac{1}{16}e^{-4t}$$

Ex: Sa se calculeze transformata Laplace a functiei $h(t)=\int_0^t \tau^2\cos 2(t-\tau)\,d\tau$

Vom aplica teorema convolutiei

$$\mathcal{L}[h(t)](p)=H(p)=F(p)\cdot G(p)=\mathcal{L}[t^2](p)\cdot\mathcal{L}[\cos 2t](p)=\frac{2}{p^3}\frac{p}{p^2+4}=\frac{2}{p^2(p^2+4)}$$

Ex: Sa se calculeze transformata Laplace a functiei $f(t)=t\,\mathrm{sh}\,3t$

$$\mathcal{L}[f(t)](p)=\mathcal{L}[t\,sh\,3t](p)=-\frac{d}{dp}\mathcal{L}[sh\,3t](p)=-\frac{d}{dp}\left(\frac{3}{p^2-9}\right)=\frac{6p}{(p^2-9)^2}$$

Ex: Sa se calculeze transformata Laplace a functiei $f(t)=e^{t-1}\sin(t-1)$

$$\mathcal{L}[t\,sh\,3t](p)=-\frac{d}{dp}\mathcal{L}[sh\,3t](p)=-\frac{d}{dp}\left(\frac{3}{p^2-9}\right)=3\frac{2p}{(p^2-9)^2}=\frac{6p}{(p^2-9)^2}$$

Ex: Sa se rezolve ecuatia

$$x(t)=\cos t+\int_0^t (t-\tau)e^{t-\tau}x(\tau)d\tau$$

Notam $\mathcal{L}[x(t)](p)=X(p)$ si aplicam transformata Laplace in ambii membri

$$X(p)=\mathcal{L}[\cos t](p)+\mathcal{L}\left[\int_0^t x(\tau)(t-\tau)e^{t-\tau}d\tau\right](p)$$

$$X(p)=\frac{p}{p^2+1}+X(p)\cdot\mathcal{L}[te^t](p)=\frac{p}{p^2+1}+X(p)\left(-\frac{d}{dp}\mathcal{L}[e^t](p)\right)=\frac{p}{p^2+1}+X(p)\left(-\frac{d}{dp}\frac{1}{p-1}\right)$$
$$=\frac{p}{p^2+1}+X(p)\frac{1}{(p-1)^2}$$

$$X(p)\left(1-\frac{1}{(p-1)^2}\right)=\frac{p}{p^2+1}$$

$$X(p)\frac{p(p-2)}{(p-1)^2}=\frac{p}{p^2+1}$$

$$X(p)=\frac{(p-1)^2}{(p-2)(p^2+1)}=\frac{p^2-2p+1}{(p-2)(p^2+1)}=\frac{a}{p-2}+\frac{bp+c}{p^2+1}=\frac{ap^2+a+bp^2+cp-2bp-2c}{(p-2)(p^2+1)}$$

$$\begin{cases} a+b=1 \\ c-2b=-2 \\ a-2c=1 \end{cases} => \begin{cases} a=\frac{1}{5} \\ b=\frac{4}{5} \\ c=-\frac{2}{5} \end{cases}$$

$$X(p)=\frac{1}{5}\frac{1}{p-2}+\frac{4}{5}\frac{p}{p^2+1}-\frac{2}{5}\frac{1}{p^2+1}$$

$$x(t)=\frac{1}{5}e^{2t}+\frac{4}{5}\cos t-\frac{2}{5}\sin t$$

Ex: Sa se rezolve problema Cauchy $x''+x=\cos t\,, x(0)=1, x'(0)=2$

Aplicam transformata Laplace si notam $\mathcal{L}[x(t)](p)=X(p)$

$$\mathcal{L}[x''(t)](p)+X(p)=\mathcal{L}[\cos t](p)$$

$$\mathcal{L}[x''(t)](p)=p^2X(p)-\left(px(0)+x'^{(0)}\right)=p^2X(p)-(p+2)$$

$$p^2X(p)-(p+2)+X(p)=\frac{p}{p^2+1}$$

$$X(p)(p^2+1)=\frac{p}{p^2+1}+p+2$$

$$X(p)=\frac{p}{(p^2+1)^2}+\frac{p+2}{p^2+1}=\frac{p}{(p^2+1)^2}+\frac{p}{p^2+1}+\frac{2}{p^2+1}$$

Sa calculam

$$\mathcal{L}[t\sin t](p)=-\frac{d}{dp}\mathcal{L}[\sin t](p)=-\frac{d}{dp}\left(\frac{1}{p^2+1}\right)=-\frac{-2p}{(p^2+1)^2}=\frac{2p}{(p^2+1)^2}$$

$$x(t)=\frac{1}{2}t\sin t+\cos t+2\sin t$$

Ex: Sa se rezolve ecuatia diferentiala cu conditia initiala (problema Cauchy)

$$\begin{cases} y'-3y=4e^{5t} \\ y(0)=6 \end{cases}$$

$$Y(p)=\mathcal{L}[y(t)](p)$$

$$\mathcal{L}[y'(t)](p)-3\mathcal{L}[y(t)](p)=4\mathcal{L}[e^{5t}](p)$$

$$pY(p) - 6 - 3Y(p) = 4\frac{1}{p-5}$$

$$Y(p)(p-3) = \frac{4}{p-5} + 6$$

$$Y(p) = 4\frac{1}{(p-5)(p-3)} + \frac{6}{p-3} = 4\left(\frac{a}{p-5} + \frac{b}{p-3}\right) + \frac{6}{p-3} = 4\left(\frac{ap-3a+bp-5b}{(p-5)(p-3)}\right) + \frac{6}{p-3}$$

$$\begin{cases} a+b=0 \\ -3a-5b=1 \end{cases} \Rightarrow \begin{cases} a=-b \\ -2a=1 \end{cases} \Rightarrow \begin{cases} a=-\dfrac{1}{2} \\ b=\dfrac{1}{2} \end{cases}$$

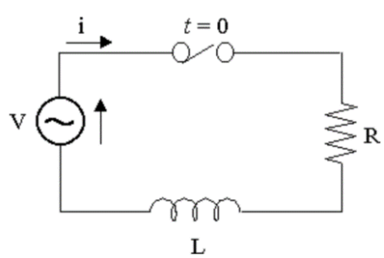
$$Y(p) = \frac{2}{p-3} - \frac{2}{p-5} + \frac{6}{p-3} = \frac{8}{p-3} - \frac{2}{p-5}$$

$$y(t) = 8e^{3t} - 2e^{5t}$$

Ex: Sa se rezolve problema Cauchy

$$\begin{cases} x' - x + 2y = 0 \\ x'' + 2y' = 2t - \cos 2t \\ x(0) = 0 \\ x'(0) = 2 \\ y(0) = -1 \end{cases}$$

Ex: Sa se rezolve $i(t)$ din circuitul electric cu $V(t) = 10\sin 5t\ V, R = 4W, L = 2H\ , i(0) = 0$



Ecuatia este $Ri + L\frac{di}{dt} = V$

Aplicam transformata Laplace in ambii membri, notam $\mathcal{L}[i(t)](p) = I(p)$ si gasim

$$RI(p) + L\mathcal{L}[i'(t)](p) = \mathcal{L}[V(t)](p)$$

$$4I(p) + 2\big(pI(p) - i(0)\big) = 10\frac{5}{p^2+25}$$

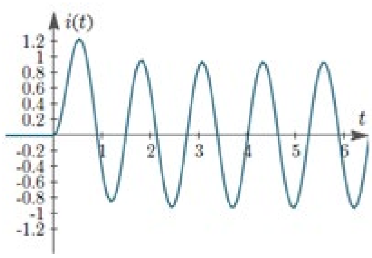
$$2I(p)(2+p) = 25\frac{1}{p^2+25}$$

$$I(p)=25\frac{1}{(p^2+25)(p+2)}=25\left(\frac{ap+b}{p^2+25}+\frac{c}{p+2}\right)=25\left(\frac{ap^2+2ap+bp+2b+cp^2+25c}{(p^2+25)(p+2)}\right)$$

$$\begin{cases} a+c=0 \\ 2a+b=0 \\ 2b+25c=1 \end{cases} => \begin{cases} c=-a \\ b=-2a \\ -4a-25a=1 \end{cases} => \begin{cases} a=-\dfrac{1}{29} \\ b=\dfrac{2}{29} \\ c=\dfrac{1}{29} \end{cases}$$

$$I(p)=\frac{25}{29}\left(\frac{-p+2}{p^2+25}+\frac{1}{p+2}\right)=\frac{25}{29}\left(-\frac{p}{p^2+25}+\frac{2}{p^2+25}+\frac{1}{p+2}\right)$$

$$i(t)=\frac{25}{29}\left(-\cos 5t+\frac{2}{5}\sin 5t+e^{-2t}\right)$$



Ex: Sa se calculeze integrala

$$I=\int_0^\infty \frac{\sin x}{x}\,dx$$

Notam $I(t)=\int_0^\infty \frac{\sin tx}{x}\,dx$

$$\mathcal{L}[I(t)](p)=\int_0^\infty e^{-pt}\int_0^\infty \frac{\sin tx}{x}\,dxdt=\int_0^\infty \frac{1}{x}\left(\int_0^\infty e^{-pt}\sin tx\,dt\right)dx=\int_0^\infty \frac{1}{x}\mathcal{L}[\sin tx](p)\,dx$$
$$=\int_0^\infty \frac{1}{x}\frac{x}{p^2+x^2}\,dx=\int_0^\infty \frac{1}{p^2+x^2}\,dx=\frac{1}{p}\left(arctan\frac{x}{p}\right)\Big|_0^\infty=\frac{\pi}{2p}$$

Atunci

$$I(t)=\frac{\pi}{2}$$

$$I=I(1)=\frac{\pi}{2}=I(2)=I(2021)$$