Interpolare polinomială

$f:[a,b] \longrightarrow \mathbb{R}$ continuă

$(x_i)_{i=\overline{1,n+1}}$ - diviziune a intervalului $[a,b]$

$$a = x_1 < x_2 < \dots < x_{n+1} = b.$$

P_n - polinom de interpolare Lagrange, de grad n, a.î. $P_n(x_i) = f(x_i)$, $i=\overline{1,n+1}$

x_i, $i=\overline{1,n+1}$ $\rightarrow$ puncte (noduri) de interpolare

1) Să se afle, conform metodei directe, $P_2(x)$ asociat funcției $f(x) = 2\sqrt{x}+1$, relativ la diviziunea $(0,1,4)$.

$x_1 = 0 \qquad f(x_1) = f(0) = 1 = y_1$

$x_2 = 1 \qquad f(x_2) = f(1) = 3 = y_2$

$x_3 = 4 \qquad f(x_3) = f(4) = 5 = y_3$

$P_2(x) = a_1 + a_2 x + a_3 x^2$, cu condițiile $\begin{cases} P_2(x_1) = y_1 \\ P_2(x_2) = y_2 \\ P_2(x_3) = y_3 \end{cases}$

Deci, $\begin{cases} P_2(0) = 1 \\ P_2(1) = 3 \\ P_2(4) = 5 \end{cases} \Rightarrow \begin{cases} a_1 = 1 \\ a_1 + a_2 + a_3 = 3 & \Rightarrow a_2 + a_3 = 2 \\ a_1 + 4a_2 + 16a_3 = 5 & \Rightarrow 4a_2 + 16a_3 = 4 \Rightarrow \end{cases}$

$$a_2 + 4a_3 = 1$$

$$\begin{array}{r} a_2 + a_3 = 2 \\ a_2 + 4a_3 = 1 \\ \hline -3a_3 = 1 \end{array} \; - \quad \Rightarrow a_3 = -\frac{1}{3}$$

$$a_2 = 2 - a_3 = 2 + \frac{1}{3} = \frac{7}{3}$$

$$P_2(x) = 1 + \frac{7}{3}x - \frac{1}{3}x^2.$$

- Metoda Lagrange

$$P_n(x) = \sum_{k=1}^{n+1} L_{n,k}(x)\cdot y_k, \quad x \in \mathbb{R}$$

unde $P_n(x_i) = y_i$, $i = \overline{1, n+1}$

și $$L_{n,k}(x) = \frac{(x-x_1)\cdot\ldots\cdot(x-x_{k-1})(x-x_{k+1})\cdot\ldots\cdot(x-x_{n+1})}{(x_k-x_1)\cdot\ldots\cdot(x_k-x_{k-1})(x_k-x_{k+1})\cdot\ldots\cdot(x_k-x_{n+1})} = \prod_{\substack{j=1 \\ j\neq k}}^{n+1} \frac{x-x_j}{x_k-x_j}.$$

2) Utilizând metoda Lagrange de determinare a polinomului, determinați polinomul de gradul 2 ce verifică condițiile $f(0)=2$, $f(1)=1$ și $f(4)=4$.

$$\underset{x_1}{f(0)} = \underset{y_1}{2} \qquad \underset{x_2}{f(1)} = \underset{y_2}{1} \qquad \underset{x_3}{f(4)} = \underset{y_3}{4}$$

$$P_2(x) = L_{2,1}(x)\cdot y_1 + L_{2,2}(x)\cdot y_2 + L_{2,3}(x)\cdot y_3$$

$y_1 = 2$

$y_2 = 1$

$y_3 = 4$

$$L_{2,1}(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-1)(x-4)}{(0-1)\cdot(0-4)} = \frac{(x-1)(x-4)}{4} = \frac{x^2-5x+4}{4}$$

$$L_{2,2}(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-4)}{(1-0)(1-4)} = \frac{x(x-4)}{-3} = \frac{x^2-4x}{-3}$$

$$L_{2,3}(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)}{(4-0)(4-1)} = \frac{x(x-1)}{12} = \frac{x^2-x}{12}$$

$$P_2(x) = \frac{x^2-5x+4}{\cancel{4}\,2}\cdot\cancel{2} + \frac{x^2-4x}{-3}\cdot 1 + \frac{x^2-x}{\cancel{12}\,3}\cdot\cancel{4} =$$

$$= x^2\left(\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}}\right) + x\left(-\frac{5}{2} + \frac{4}{3} - \frac{1}{3}\right) + \frac{4}{2} = \frac{x^2}{2} - \frac{3}{2}x + 2$$

• Metoda Newton

$$P_n(x) = c_1 + c_2(x-x_1) + c_3(x-x_1)(x-x_2) + \ldots + c_{n+1}(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)$$

c_i - se det. din $P_n(x_i) = y_i,\ i=\overline{1,n+1}$.

3) Să se afle, conform metodei Newton, polinomul P_2 asociat funcției $f(x) = \sqrt{x+1} - 1$, relativ la diviziunea $(-1, 0, 3)$.

$x_1 = -1 \qquad y_1 = f(x_1) = f(-1) = -1$

$x_2 = 0 \qquad y_2 = f(x_2) = f(0) = 0$

$x_3 = 3 \qquad y_3 = f(x_3) = f(3) = 1.$

$$P_2(x) = c_1 + c_2(x-x_1) + c_3(x-x_1)(x-x_2) =$$
$$= c_1 + c_2(x+1) + c_3(x+1)\cdot x$$

$$\begin{cases} P_2(x_1) = y_1 \\ P_2(x_2) = y_2 \\ P_2(x_3) = y_3 \end{cases} \Rightarrow \begin{cases} P_2(-1) = -1 \\ P_2(0) = 0 \\ P_2(3) = 1 \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_1 + c_2 = 0 \Rightarrow c_2 = 1 \\ c_1 + 4c_2 + 12c_3 = 1 \Rightarrow \end{cases}$$

$$\Rightarrow -1 + 4 + 12c_3 = 1 \Rightarrow 12c_3 = -2 \Rightarrow c_3 = \frac{-1}{6}$$

$$P_2(x) = -1 + (x+1) - \frac{1}{6}(x+1)\cdot x$$
$$= -1 + x + 1 - \frac{1}{6}x^2 - \frac{1}{6}x = -\frac{1}{6}x^2 + \frac{5}{6}x.$$

• Metoda Newton cu diferențe divizate.

$$P_n(x) = f[x_1] + f[x_1, x_2]\cdot(x - x_1) + \ldots + f[x_1, x_2, \ldots, x_{n+1}]\cdot(x - x_1)(x - x_2)\cdot\ldots\cdot(x - x_n).$$

unde $f[x_1] = f(x_1)$ → diferența divizată de ordin 0 a lui f în raport cu x_1

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$$ → diferența divizată de ordin 1 a lui f în raport cu x_1 și x_2

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$ → diferența divizată de ordin 2 în raport cu x_1, x_2, x_3

...

$$f[x_1, x_2, \ldots, x_{n+1}] = \frac{f[x_2, x_3, \ldots, x_{n+1}] - f[x_1, x_2, \ldots, x_n]}{x_{n+1} - x_1}$$

→ DD de ordin n a lui f în raport cu $x_1, x_2, \ldots, x_{n+1}$.

Construim un tabel:

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	...	$f[x_1, x_2, \ldots, x_{n+1}]$
x_1	$f[x_1]$				
x_2	$f[x_2]$	$f[x_1, x_2]$			
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$		
⋮	⋮	⋮	$f[x_2, x_3, x_4]$		
			⋮		
x_{n+1}	$f[x_{n+1}]$	$f[x_n, x_{n+1}]$	$f[x_{n-1}, x_n, x_{n+1}]$		$f[x_1, x_2, \ldots, x_{n+1}]$

4) Fie $f : [0,12] \to \mathbb{R}$, cu $f(x) = 2\sqrt{2x+1} - 3x$, $\forall x \in [0,12]$. Utilizând formula de reprezentare Newton cu diferențe divizate, să se determine polinomul de grad 2, relativ la diviziunea $(0, 4, 12)$.

$x_1 = 0$

$x_2 = 4$

$x_3 = 12.$

$$P_2(x) = f[x_1] + f[x_1, x_2]\cdot(x - x_1) + f[x_1, x_2, x_3]\,(x - x_1)(x - x_2)$$
$$= f[0] + f[0,4]\cdot(x-0) + f[0,4,12]\cdot(x-0)(x-4)$$

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_1, x_2, \dots, x_{n+1}]$
0	$f[0]=2$		
4	$f[4]=-6$	$f[0,4] = -2$	
12	$f[12] = -26$	$f[4,12] = -\frac{5}{2}$	$f[x_1, x_2, x_3] = -\frac{1}{24}$

$$f[0] = f(0) = 2$$
$$f[4] = f(4) = 2\cdot\sqrt{2\cdot 4+1} - 3\cdot 4 = 2\cdot 3 - 12 = -6.$$
$$f[12] = f(12) = 2\sqrt{2\cdot 12+1} - 3\cdot 12 = 2\cdot 5 - 36 = -26$$

$$f[0,4] = \frac{f(4) - f(0)}{4-0} = \frac{-6-2}{4} = -2$$
$$f[4,12] = \frac{f(12) - f(4)}{12-4} = \frac{-26+6}{8} = \frac{-20}{8} = \frac{-5}{2}$$
$$f[0,4,12] = \frac{f[4,12] - f[0,4]}{12-0} = \frac{-\frac{5}{2} - (-2)}{12} = \frac{-\frac{5}{2}+2}{12} = \frac{-\frac{1}{2}}{12} = \frac{-1}{24}$$

$$P_2(x) = 2 + (-2)\cdot x + \left(\frac{-1}{24}\right)x(x-4)$$
$$= 2 - 2x - \frac{1}{24}x^2 + \frac{4}{24}x$$
$$= -\frac{1}{24}x^2 + \frac{1}{6}x - 2x + 2$$
$$= -\frac{1}{24}x^2 - \frac{11}{6}x + 2$$

5) Să se afle polinomul de interpolare Lagrange $P_2(x)$ al funcției $f(x)=\sin x$ relativ la diviziunea $(-\frac{\pi}{2}, 0, \frac{\pi}{2})$ conform metodelor directă și Lagrange.

met. directă:

$x_1 = -\frac{\pi}{2}$ $\quad y_1 = f(x_1) = f(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1$

$x_2 = 0$ $\quad y_2 = f(x_2) = f(0) = \sin 0 = 0$

$x_3 = \frac{\pi}{2}$ $\quad y_3 = f(x_3) = f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1.$

$P_2(x) = a_1 + a_2 x + a_3 x^2.$

$$\begin{cases} P_2(x_1) = y_1 \\ P_2(x_2) = y_2 \\ P_2(x_3) = y_3 \end{cases} \Rightarrow \begin{cases} a_1 - \frac{\pi}{2}a_2 + \frac{\pi^2}{4}a_3 = -1 \\ a_1 = 0 \\ a_1 + \frac{\pi}{2}a_2 + \frac{\pi^2}{4}a_3 = 1 \end{cases} \Rightarrow \begin{cases} -\frac{\pi}{2}a_2 + \frac{\pi^2}{4}a_3 = -1 \\ \frac{\pi}{2}a_2 + \frac{\pi^2}{4}a_3 = 1 \end{cases}$$

$$\frac{2\pi^2}{4}a_3 = 0 \Rightarrow a_3 = 0$$

$(1) \rightarrow 0 - \frac{\pi}{2}a_2 + 0 = -1 \Rightarrow a_2 = -1 \cdot \frac{-2}{\pi} = \frac{2}{\pi}$

$P_2(x) = 0 + \frac{2}{\pi}x + 0 \cdot x^2 = \frac{2}{\pi}x.$

met. Lagrange:

$$P_2(x) = L_{2,1}(x) \cdot y_1 + L_{2,2}(x) \cdot y_2 + L_{2,3}(x) \cdot y_3$$

$y_1 = -1,\ y_2 = 0,\ y_3 = 1.$ (vezi mai sus)

$$L_{2,1}(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-\frac{\pi}{2})}{(-\frac{\pi}{2}-0)(-\frac{\pi}{2}-\frac{\pi}{2})} = \frac{x(x-\frac{\pi}{2})}{-\frac{\pi}{2}\cdot(-\pi)} = \frac{x(x-\frac{\pi}{2})}{\frac{\pi^2}{2}} = \frac{2}{\pi^2}x(x-\frac{\pi}{2})$$

$$L_{2,2}(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x+\frac{\pi}{2})(x-\frac{\pi}{2})}{(0+\frac{\pi}{2})(0-\frac{\pi}{2})} = \frac{x^2-\frac{\pi^2}{4}}{-\frac{\pi^2}{4}} = -\frac{4}{\pi^2}\left(x^2-\frac{\pi^2}{4}\right)$$

$$L_{2,3}(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x+\frac{\pi}{2})\cdot(x-0)}{(+\frac{\pi}{2}+\frac{\pi}{2})(\frac{\pi}{2}-0)} = \frac{x(x+\frac{\pi}{2})}{\pi\cdot\frac{\pi}{2}} =$$

$$= \frac{2}{\pi^2}\, x\left(x+\frac{\pi}{2}\right)$$

$$P_2(x) = \frac{2}{\pi^2}\cdot x(x-\frac{\pi}{2})\cdot(-1) + 0 + \frac{2}{\pi^2}\, x(x+\frac{\pi}{2})\cdot 1$$

$$= \cancel{-\frac{2}{\pi^2}x^2} + \frac{\cancel{2}}{\pi^{\cancel{2}}}\cdot\frac{\cancel{\pi}}{\cancel{2}}x + \cancel{\frac{2}{\pi^2}x^2} + \frac{\cancel{2}}{\pi^{\cancel{2}}}\frac{\cancel{\pi}}{\cancel{2}}x = \frac{1}{\pi}x + \frac{1}{\pi}x = \frac{2}{\pi}\cdot x.$$

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metoda Newton

$$P_2(x) = c_1 + c_2(x-x_1) + c_3(x-x_1)(x-x_2) =$$

$$= c_1 + c_2(x+\frac{\pi}{2}) + c_3(x+\frac{\pi}{2})\cdot(x-0).$$

$$\begin{cases} P_2(x_1) = y_1 \\ P_2(x_2) = y_2 \\ P_2(x_3) = y_3 \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_1 + c_2\cdot\frac{\pi}{2} = 0 \\ c_1 + c_2\cdot(\frac{\pi}{2}+\frac{\pi}{2}) + c_3\cdot(\frac{\pi}{2}+\frac{\pi}{2})\cdot\frac{\pi}{2} = 1 \end{cases} \Rightarrow \frac{\pi}{2}c_2 = 1 \Rightarrow c_2 = \frac{2}{\pi}$$

$$c_1 + \pi c_2 + c_3\cdot\frac{\pi^2}{2} = 1 \Rightarrow -1 + \cancel{\pi}\cdot\frac{2}{\cancel{\pi}} + c_3\cdot\frac{\pi^2}{2} = 1 \Rightarrow c_3 = 0$$

$$P_2(x) = -1 + \frac{2}{\pi}\cdot(x+\frac{\pi}{2}) + 0 = \cancel{-1} + \frac{2}{\pi}\cdot x + \cancel{1} = \frac{2}{\pi}\cdot x.$$

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metoda Newton cu diferențe divizate $\qquad x_1 = -\frac{\pi}{2},\ x_2 = 0,\ x_3 = \frac{\pi}{2}$

$$P_2(x) = f[x_1] + f[x_1,x_2]\cdot(x-x_1) + f[x_1,x_2,x_3]\cdot(x-x_1)(x-x_2)$$

$$= f(-\frac{\pi}{2}) + f[-\frac{\pi}{2},0]\cdot(x+\frac{\pi}{2}) + f[-\frac{\pi}{2},0,\frac{\pi}{2}]\cdot(x+\frac{\pi}{2})\cdot x$$

x_i	$f[x_i]$	$f[x_i;x_{i+1}]$	$f[x_1,x_2,\dots,x_{n+1}]$
$-\frac{\pi}{2}$	$f[-\frac{\pi}{2}] = -1$		
0	$f[0] = 0$	$f[-\frac{\pi}{2},0] = \frac{2}{\pi}$	
$\frac{\pi}{2}$	$f[\frac{\pi}{2}] = 1$	$f[0,\frac{\pi}{2}] = \frac{2}{\pi}$	$f[-\frac{\pi}{2},0,\frac{\pi}{2}] = 0$

$$f[-\frac{\pi}{2};0] = \frac{f(0)-f(-\frac{\pi}{2})}{0-(-\frac{\pi}{2})} = \frac{0+1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$f[0,\tfrac{\pi}{2}] = \frac{f[\frac{\pi}{2}]-f[0]}{\frac{\pi}{2}-0} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$$f[-\tfrac{\pi}{2},0,\tfrac{\pi}{2}] = \frac{f[0,\frac{\pi}{2}]-f[-\frac{\pi}{2},0]}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)} = \frac{\frac{2}{\pi}-\frac{2}{\pi}}{\pi} = 0.$$

$$P_2(x) = -1 + \frac{2}{\pi}\left(x+\frac{\pi}{2}\right) + 0 = \cancel{-1} + \frac{2}{\pi}\cdot x + \cancel{1} = \frac{2}{\pi}\cdot x.$$

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