Cursul 16

V.a. continue

V.a. simple. v.a. care au un nr finit de valori.

V.a. continue.

(Ω, K, P) ; $f : \Omega \to \mathbb{R}$, are un nr. infinit de valori.

$F : \mathbb{R} \to \mathbb{R}$, $F(x) = P(\omega \in \Omega \mid f(\omega) \le x)$, funcție de repartiție.

f are o lege de repartiție, deci de rep. dacă $\exists\, p : \mathbb{R} \to \mathbb{R}$,

$$F(x) = \int_{-\infty}^{x} p(t)\,dt.$$

Lege de repartiție

lege de prob $p : \mathbb{R} \to \mathbb{R}$

1) $p(x) \ge 0 \quad \forall x \in \mathbb{R}$

2) $p(x)$ ap.t. continuă

3) $p(x)$ integrabilă pe $\mathbb{R}$

$$\int_{-\infty}^{+\infty} p(t)\,dt = 1.$$

Avem în vedere:

(Ω, K, P)

$f : \Omega \to \mathbb{R}$ v.a. cont.

care are F = funcție de repartiție.

are $p(x)$ lege de probabilit.

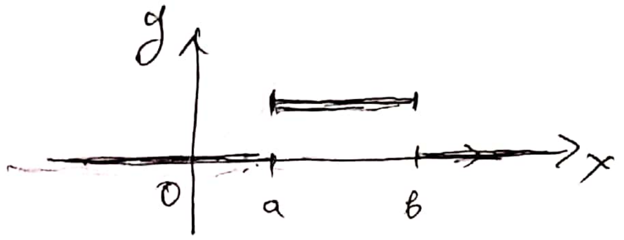
$$\begin{cases} m_f = \int_{-\infty}^{+\infty} x\,p(x)\,dx. \\ \sigma_f^2 = \int_{-\infty}^{+\infty} x^2 p(x)\,dx - m_f^2 \\ \sigma_f = \sqrt{\sigma_f^2}. \end{cases}$$

Legi de prob

16 (2)

1) Lege de prob. uniformă.

$$p:\mathbb{R}\to\mathbb{R},\quad p(x)=\begin{cases}\frac{1}{b-a} & x\in[a,b]\\ 0 & x\notin[a,b].\end{cases}$$



$$\boxed{m_f=\frac{a+b}{2}}$$

$$D_f^2=\frac{(b-a)^2}{12}$$

$$\sigma_f=\frac{b-a}{2\sqrt{3}}$$

$$m_f=\int_{-\infty}^{+\infty}x\,p(x)\,dx=\int_{-\infty}^{a}+\int_a^b\frac{x}{b-a}dx+\int_b^{\infty}=$$

$$=\int_a^b\frac{x}{b-a}dx=\frac{1}{b-a}\frac{x^2}{2}\Big|_a^b=$$

$$=\frac{b^2-a^2}{2(b-a)}=\frac{a+b}{2}$$

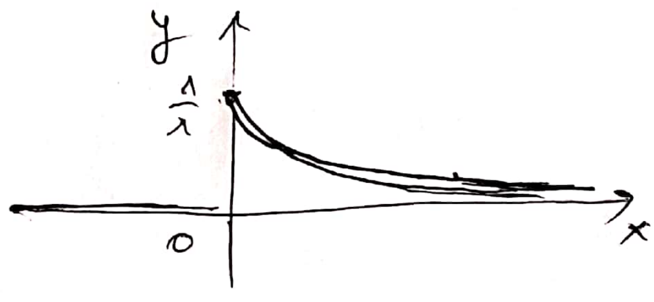
2) Lege de prob exponențială

$$p:\mathbb{R}\to\mathbb{R},\quad p(x)=\begin{cases}\frac{1}{\lambda}\cdot e^{-\frac{x}{\lambda}} & x\in[0,\infty)\\ 0 & x<0.\end{cases}\qquad \lambda>0$$

$$m_f=\lambda.$$

$$D_f^2=\lambda^2$$

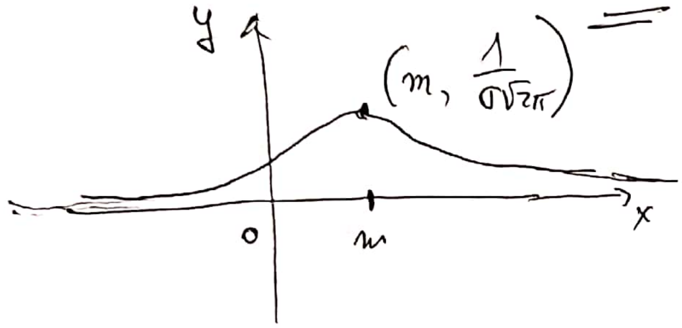
$$\sigma=\lambda.$$



3) **Lege normală** 16 3

$$p : \mathbb{R} \to \mathbb{R} \qquad p(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$\sigma > 0$, $m \in \mathbb{R}$



Clopotul lui Gauss.

f o v.a. care are lege $p(x)$.

$$\begin{cases} m_f = m \\ D_f^2 = \sigma^2 \\ \sigma = \sigma \end{cases}$$

Teoremă

$f(\omega)$ v.a. cu lege de prob $p(x)$.

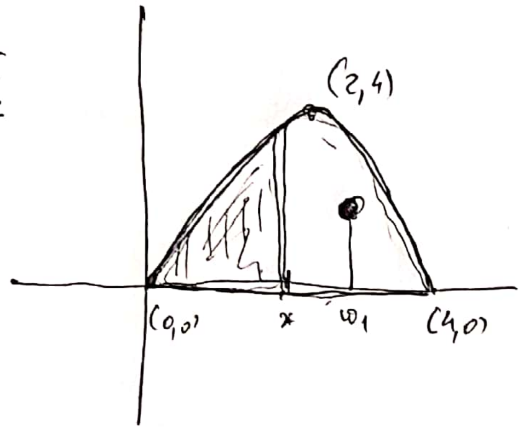
1) $P(\alpha \leq f(\omega) \leq \beta) = \int_\alpha^\beta p(x)\,dx$.

2) Dacă $f(\omega)$ este normal distrib, atunci:

$$P(\alpha \leq f(\omega) \leq \beta) = \int_\alpha^\beta p(x)\,dx = \Phi\left(\frac{\beta - m}{\sigma}\right) - \Phi\left(\frac{\alpha - m}{\sigma}\right).$$

$\Phi(x)$ = funcția erf.

Ex



16
4

$\Omega = \{(x,y) \mid y \le -x^2+4x, \; y \ge 0\}$

$f(\omega) = \omega_1$.

v.a. care monitorizează abscisa.

$0 < x < 4$

• Funcție de repartiție.

$$F(x) = P(\omega \mid f(\omega) \le x) = \frac{\int_0^x (-t^2+4t)\,dt}{\int_0^4 (-t^2+4t)\,dt} = \frac{-\frac{x^3}{3}+2x^2}{\left(-\frac{t^3}{3}+2t^2\right)\Big|_0^4}$$

$$F(x) = \begin{cases} 0 & x<0 \\ 1 & x>4 \end{cases}$$

$$= \frac{-\frac{x^3}{3}+2x^2}{-\frac{64}{3}+32} = \frac{-x^3+6x^2}{-64+96} = \frac{-x^3+6x^2}{32}$$

$$F(x) = \begin{cases} 0 & x<0 \\ \frac{-x^3+6x^2}{32} & 0 \le x \le 4 \\ 1 & x>4 \end{cases}$$

$F(x) = \dfrac{-x^3+6x^2}{32}$ funcție de repartiție.

$$F(x) = \int_0^x p(t)\,dt \qquad \boxed{p(x) = \frac{-3x^2+12x}{32}} \text{ dens. de prob.}$$

$0 \le x \le 4$ în rest 0.

$$F(x) = \int_{-\infty}^{x} p(t)\,dt$$

$$m_f = \int_0^4 x\,p(x)\,dx = \int_0^4 \frac{-3x^3+12x^2}{32}\,dx =$$

$$= \left(-\frac{3}{32}\cdot\frac{x^4}{4} + \frac{12}{32}\cdot\frac{x^3}{3}\right)\Big|_0^4 = \frac{-3\cdot 4^3}{32} + \frac{4\cdot 4^3}{32} = \frac{4^3}{32} = 2.$$

$D_f^2 = \ldots$

$\sigma \ldots$

$$P(3 \le f(\omega) \le 4) = \int_3^4 \frac{-3x^2+12x}{32}\,dx = F(4) - F(3)$$

$= \ldots$

Teorema Cebîşev

$$P(\omega\in\Omega \mid |f(\omega)-m_f|\geq \varepsilon) \leq \frac{D_f^2}{\varepsilon^2}.$$

Teorema celor 3σ

$$P(\omega\in\Omega \mid |f(\omega)-m|< 3\sigma) \approx 1.$$

$P(\omega\in\Omega \mid |f(\omega)-m|\geq 3\sigma)$ este miraculos.
$= 0$

T.L.C. (Teorema fund. a statisticii sub forma lui Liapunov)

f v.a.

$f_1, \dots, f_n$ — independente, din legea lui f

$$P(\alpha\leq f\leq\beta) = \phi\left(\frac{\beta-m}{\sigma}\right)-\phi\left(\frac{\alpha-m}{\sigma}\right)$$

$$P(\alpha \leq f_1+\dots+f_n \leq \beta) \approx \phi\left(\frac{\beta-nm}{\sigma\sqrt{n}}\right)-\phi\left(\frac{\alpha-nm}{\sigma\sqrt{n}}\right).$$

pt. n „mare"

Miracolul Gauss-Laplace

$f_1, \dots, f_n$ v.a. — indep., din lege lui f

m, σ

$f_1+f_2+\dots+f_n$ pt. n mare se comportă ca o v.a. normală cu

$$M = n\cdot m$$
$$\Sigma^2 = n\sigma^2$$
$$\Sigma = \sigma\sqrt{n}.$$

Scanned with CamScanner

<u>Problemă</u> Câștigul zilnic al unui jucător la ruletă este repartizat uniform în intervalul $[-45, 55]$. Care este prob. ca el să câștige 1000 euro în 100 zile. Dar în 150 zile?

$$p(x) = \begin{cases} \frac{1}{100} & ; -45 \le x \le 55 \\ 0 & \text{în rest.} \end{cases}$$

$m = 5$

$\sigma = \frac{100}{\sqrt{12}}$

$f_1, f_2, \ldots, f_{100}$ v. a. care monitoriz. câștigul zilnic

$$P(f_1 + \ldots + f_{100} \geq 1000) \overset{TLG.}{=} \phi(\infty) - \phi\left(\frac{1000 - 100 \cdot m}{\sigma\sqrt{100}}\right) =$$

$$= 1 - \phi\left(\frac{1000 - 100 \cdot 5}{\frac{1000}{\sqrt{12}}}\right) = 1 - \phi\left(\frac{500}{\frac{1000}{2\sqrt{3}}}\right) = 1 - \phi(\sqrt{3})$$

$$= 1 - \phi(1{,}73) = 1 - 0{,}95 = 0{,}05 = 5\%$$

$$P(f_1 + \ldots + f_{150} \geq 1000) = \phi(\infty) - \phi\left(\frac{1000 - 150 \cdot m}{\sigma\sqrt{150}}\right)$$

$$= 1 - \phi\left(\frac{1000 - 150 \cdot 5}{\frac{100}{2\sqrt{3}} \cdot \sqrt{150}}\right) = 1 - \phi\left(\frac{250 \cdot \sqrt{3}}{50 \cdot \sqrt{150}}\right) = 1 - \phi\left(\frac{5\sqrt{3}}{\sqrt{150}}\right)$$

$$= 1 - \phi(0{,}70) = 1 - 0{,}75 = 0{,}25 = 25\%.$$

1) De ce băncile nu dau faliment în cond. de stabilitate.

V.a. care monitorizează un cont bancar.

$$X_f = \begin{pmatrix} -100 & 0 & 200 & -50 \\ 0,3 & 0,2 & 0,3 & 0,2 \end{pmatrix}.$$

a) Determinați m, D^2, σ.

b) Banca are 500 clienți.

Determinați $P(f_1+\dots+f_{500} \le 0)$; $P(f_1+\dots+f_{500} \geq 100)$.

$m = -30+0+60-10 = 20$

$D^2 = 10.000 \cdot 0,3 + 40000 \cdot 0,3 + 2500 \cdot 0,2 =$

$= 3000 + 12000 + 500 = 15.500 - 400 = 15.100$

$\sigma = \sqrt{15100} = 123$

$P(f_1+\dots+f_{500} \le 0) = \Phi\left(\frac{0-500 \cdot 20}{123 \cdot \sqrt{500}}\right) - \Phi(-\infty)$

$= \Phi\left(\frac{-10000}{123 \cdot \sqrt{500}}\right) = \Phi(-3,63) = 0.$

$f(\omega)$ = vârsta decesului unui individ dintr-o 17 8
cantitate.

$f(\omega) = 90$ $\quad$ $f(\omega)$ v.a continuă.

$f(\omega) = 40$

$f(\omega)$ v.a. normal distribuită

$m = 70$

$\sigma = 3$

$$P\left(\omega\in\Omega \mid 65 \le f(\omega) \le 80\right) =$$

$$= \int_{65}^{80} p(x)\,dx = \Phi\left(\frac{\beta - m}{\sigma}\right) - \Phi\left(\frac{\alpha - m}{\sigma}\right) =$$

$$= \Phi\left(\frac{80-70}{3}\right) - \Phi\left(\frac{65-70}{3}\right) = \qquad \Phi(x) + \Phi(-x) = 1$$

$$= \Phi\left(\frac{10}{3}\right) - \Phi\left(-\frac{5}{3}\right) = \Phi(3{,}33) - \Phi(-1{,}66) =$$

$$= \Phi(3{,}33) - 1 + \Phi(1{,}66) = 1 - 1 + \Phi(1{,}66) =$$

$$= \Phi(1{,}66) = 0{,}95$$