• Interpolare spline liniară

1• Fie $f:[-1,2]\to\mathbb{R}$, $f(x)=x\cdot e^x$. Să se afle funcția spline liniară relativ la diviziunea $(-1,0,2)$.

$I_1=[-1,0)$, $I_2=[0,2]$

$x_1=-1$, $x_2=0$, $x_3=2$, $n+1=3 \Rightarrow \underline{n=2}$

$y_1=f(x_1)=f(-1)=-1\cdot e^{-1}=-\frac{1}{e}$

$y_2=f(x_2)=f(0)=0\cdot e^0=0$

$y_3=f(x_3)=f(2)=2\cdot e^2$

funcția căutată este: $S(x)=\begin{cases} S_1(x), & x\in[-1,0) \\ S_2(x), & x\in[0,2] \end{cases}$, unde

- $S_j(x)=a_j+b_j\cdot(x-x_j)$, $j=\overline{1,2}$
- S - continuă în nodurile x_2 (**)
- $S(x_i)=y_i$, $i=\overline{1,3}$ $\Rightarrow$ (*)

$S(x)=\begin{cases} a_1+b_1(x-x_1), & x\in[-1,0) \\ a_2+b_2(x-x_2), & x\in[0,2] \end{cases} \Rightarrow S(x)=\begin{cases} a_1+b_1\cdot(x+1), & x\in[-1,0) \\ a_2+b_2\cdot x, & x\in[0,2] \end{cases}$

(*) $\begin{cases} S(x_1)=y_1 \\ S(x_2)=y_2 \\ S(x_3)=y_3 \end{cases} \Rightarrow \begin{cases} S(-1)=-\frac{1}{e} \\ S(0)=0 \\ S(2)=2e^2 \end{cases} \Rightarrow \begin{cases} a_1+b_1\cdot(\cancel{-1+1})=-\frac{1}{e} \Rightarrow a_1=-\frac{1}{e} \\ a_2+b_2\cdot 0=0 \Rightarrow a_2=0 \\ a_2+b_2\cdot 2=2e^2 \Rightarrow b_2=e^2 \end{cases}$

(**) $l_s(0)=l_d(0) \Rightarrow a_1+b_1=a_2+b_2\cdot 0 \Rightarrow -\frac{1}{e}+b_1=0 \Rightarrow b_1=\frac{1}{e}$

$\Rightarrow S(x)=\begin{cases} -\frac{1}{e}+\frac{1}{e}(x+1), & x\in[-1,0) \\ 0+e^2\cdot x, & x\in[0,2] \end{cases} = \begin{cases} \frac{1}{e}\cdot x, & x\in[-1,0) \\ e^2\cdot x, & x\in[0,2] \end{cases}$

• Funcția spline pătratică

2. Să se determine funcția spline pătratică pentru funcția $f(x) = x \cdot e^{2x}$ relativ la diviziunea $(-1, 0, 2)$.

$$x_1 = -1 \Rightarrow y_1 = f(x_1) = f(-1) = -e^{-2} = -\frac{1}{e^2}$$
$$x_2 = 0 \Rightarrow y_2 = f(x_2) = f(0) = 0$$
$$x_3 = 2 \Rightarrow y_3 = f(x_3) = f(2) = 2e^4$$

$$I_1 = [-1, 0), \quad I_2 = [0, 2]$$

$$S(x) = \begin{cases} a_1 + b_1(x - x_1) + c_1(x - x_1)^2, & x \in [-1, 0) \\ a_2 + b_2(x - x_2) + c_2(x - x_2)^2, & x \in [0, 2] \end{cases} = \begin{cases} a_1 + b_1 \cdot (x+1) + c_1 \cdot (x+1)^2, & x \in [-1, 0) \\ a_2 + b_2 x + c_2 \cdot x^2, & x \in [0, 2] \end{cases}$$

→ $S(x_j) = f(x_j), \; j = \overline{1,3}$

$$\begin{cases} S(x_1) = y_1 \\ S(x_2) = y_2 \\ S(x_3) = y_3 \end{cases} \Rightarrow \begin{cases} S(-1) = -\frac{1}{e^2} \\ S(0) = 0 \\ S(2) = 2e^4 \end{cases} \Rightarrow \begin{cases} a_1 + b_1 \cdot (-1+1) + c_1 \cdot (-1+1)^2 = -\frac{1}{e^2} \\ a_2 + b_2 \cdot 0 + c_2 \cdot 0^2 = 0 \\ a_2 + b_2 \cdot 2 + c_2 \cdot 2^2 = 2e^4 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a_1 = -\frac{1}{e^2} \\ a_2 = 0 \\ 2b_2 + 4c_2 = 2e^4 \quad (3) \end{cases}$$

→ S - continuă în $\underset{\text{nod interior}}{x_2} \Rightarrow S_1(x_2) = S_2(x_2) \Rightarrow S_1(0) = S_2(0) \Rightarrow a_1 + b_1 + c_1 = a_2 \Rightarrow$

$$\Rightarrow -\frac{1}{e^2} + b_1 + c_1 = 0 \Rightarrow b_1 + c_1 = \frac{1}{e^2} \quad (1)$$

→ $$S'(x) = \begin{cases} b_1 + 2c_1 \cdot (x+1), & x \in [-1, 0) \\ b_2 + 2c_2 x, & x \in [0, 2] \end{cases}$$

S' → continuă în $\underset{\text{nod interior}}{x_2} \Rightarrow S_1'(x_2) = S_2'(x_2) \Rightarrow S_1'(0) = S_2'(0) \Rightarrow$

$$\Rightarrow b_1 + 2c_1 = b_2 \quad (2)$$

→ una din $S'(x_1) = f'(x_1)$ este satisfăcută
$S'(x_3) = f'(x_3)$

considerăm satisfăcută relația $s'(x_1) = f'(x_1) \Rightarrow s'(-1) = f'(-1)$

$$f'(x) = (xe^{2x})' = e^{2x} + x \cdot e^{2x} \cdot 2.$$

$$f'(-1) = e^{-2} + (-1) \cdot e^{-2} \cdot 2 = e^{-2} - 2e^{-2} = -e^{-2}$$

$$s'(-1) = b_1 + 2c_1(\cancel{-1} + \cancel{1}) = b_1 = -e^{-2}$$

din (1) $\Rightarrow c_1 = \frac{1}{e^2} - b_1 = \frac{1}{e^2} + e^{-2} = \frac{1}{e^2} + \frac{1}{e^2} = \frac{2}{e^2}$

din (2) $\Rightarrow -e^{-2} + 2 \cdot \frac{2}{e^2} = b_2 \Rightarrow b_2 = -\frac{1}{e^2} + \frac{4}{e^2} = \frac{3}{e^2}$

din (3) $\Rightarrow 2 \cdot \frac{3}{e^2} + 4c_2 = 2e^4 \Rightarrow 4c_2 = 2e^4 - \frac{6}{e^2} \Rightarrow$

$$\Rightarrow c_2 = \frac{e^4}{2} - \frac{3}{2e^2}$$

$$s(x) = \begin{cases} -\frac{1}{e^2} - \frac{1}{e^2}(x+1) + \frac{2}{e^2}(x+1)^2, & x \in [-1, 0) \\ \frac{3}{e^2}x + \left(\frac{e^4}{2} - \frac{3}{2e^2}\right)x^2, & x \in [0, 2] \end{cases}$$

3. Determinați funcția spline cubică asociată funcției $f(x) = xe^x$ relativ la diviziunea $(-1, 0, 1)$.

$x_1 = -1,\ x_2 = 0,\ x_3 = 1.$, $n = 2$.

$y_1 = f(x_1) = f(-1) = -\frac{1}{e}$ $\qquad I_1 = [-1, 0),\ I_2 = [0, 1].$

$y_2 = f(x_2) = f(0) = 0$

$y_3 = f(x_3) = f(1) = e$

$$S(x) = \begin{cases} a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3, & x \in [-1, 0) \\ a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3, & x \in [0, 1]. \end{cases}$$

$$\Rightarrow S(x) = \begin{cases} a_1 + b_1(x+1) + c_1(x+1)^2 + d_1(x+1)^3, & x \in [-1, 0) \\ a_2 + b_2 x + c_2 x^2 + d_2 x^3, & x \in [0, 1]. \end{cases}$$

$\boxed{a_j = f(x_j),\ j = \overline{1, n}} \Rightarrow a_j = f(x_j),\ j = \overline{1, 2} \Rightarrow a_1 = -\frac{1}{e}$

$a_2 = 0$

$\boxed{b_1 = f'(x_1)}$

$f'(x) = (xe^x)' = e^x + xe^x = e^x(x+1) \Rightarrow f'(x_1) = f'(-1) = 0 \Big\} \Rightarrow b_1 = 0.$

$\boxed{b_{n+1} = f'(x_{n+1})} \Rightarrow b_3 = f'(x_3) = f'(1) = 2e$

$\boxed{b_{j-1} + 4b_j + b_{j+1} = \frac{3}{h}(f(x_{j+1}) - f(x_{j-1})),\ j = \overline{2,n}} \Rightarrow j = 2,\ h = x_{j+1} - x_j$

$\Rightarrow \underbrace{b_1}_{0} + 4b_2 + \underbrace{b_3}_{2e} = \frac{3}{1}\cdot(\underbrace{f(x_3)}_{e} - \underbrace{f(x_1)}_{-\frac{1}{e}})$ (div. echidistanta)

$\Rightarrow 4b_2 + 2e = 3\left(e + \frac{1}{e}\right) \Rightarrow 4b_2 = e + 3\cdot\frac{1}{e} \Rightarrow b_2 = \frac{e+3e^{-1}}{4}$

$\boxed{d_j = -\frac{2}{h^3}(f(x_{j+1}) - f(x_j)) + \frac{1}{h^2}(b_{j+1} + b_j),\ j = \overline{1,n}}$ $\rightarrow 2$

$\Rightarrow d_1 = -2\cdot(f(x_2) - f(x_1)) + \frac{1}{1}(b_2 + b_1) =$

$= -2\left(0 - \left(-\frac{1}{e}\right)\right) + \frac{e+3e^{-1}}{4} + 0 = -2e^{-1} + \frac{e+3e^{-1}}{4} = \frac{-5e^{-1}+e}{4}$

$d_2 = -\frac{2}{1}(f(x_3) - f(x_2)) + \frac{1}{1}(b_3 + b_2) =$

$= \cancel{-2\cdot(e-0)} + \frac{e+3e^{-1}}{4} + \cancel{2e} = \frac{e+3e^{-1}}{4}$

$\boxed{c_j = \frac{3}{h^2}(f(x_{j+1}) - f(x_j)) - \frac{b_{j+1}+2b_j}{h},\ j = \overline{1,n}}$ $\rightarrow 2$

$c_1 = \frac{3}{1}(\underbrace{f(x_2)}_{0} - \underbrace{f(x_1)}_{-e^{-1}}) - \frac{b_2+2b_1}{1} =$

$= 3(0 + e^{-1}) - \left(\frac{e+3e^{-1}}{4} + 2\cdot 0\right) =$

$= 3e^{-1} - \frac{e+3e^{-1}}{4} = \frac{9e^{-1}-e}{4}$

$c_2 = \frac{3}{1}(\underbrace{f(x_3)}_{e} - \underbrace{f(x_2)}_{0}) - \frac{b_3+2b_2}{1} =$

$= 3e - \left(2e + 2\cdot\frac{e+3e^{-1}}{4}\right) = 3e - 2e - \frac{e+3e^{-1}}{2} = e - \frac{e+3e^{-1}}{2}$

$= \frac{e-3e^{-1}}{2}$

$$S(x)=\begin{cases} -\frac{1}{e}+0(x+1)+\frac{9e^{-1}-e}{4}(x+1)^2+\frac{-5e^{-1}+e}{4}(x+1)^3, & x\in[-1,0) \\ 0+\frac{e+3e^{-1}}{4}x+\frac{e-3e^{-1}}{2}x^2+\frac{e+3e^{-1}}{4}x^3, & x\in[0,1] \end{cases}$$

Verificare:

$\Rightarrow S(x_j)=f(x_j),\ j=\overline{1,n+1} \Rightarrow \begin{cases} S(-1)=-\frac{1}{e} \checkmark \\ S(0)=0 \checkmark \\ S(1)=\frac{e+3e^{-1}}{4}+\frac{e-3e^{-1}}{2}+\frac{e+3e^{-1}}{4}= \end{cases}$

$$=\frac{\cancel{2}(e+3e^{-1})}{\cancel{4}_2}+\frac{e-3e^{-1}}{2}=\frac{e}{2}+\frac{e}{2}=e \checkmark$$

$\rightarrow$ S continuă în nodurile $x_{j+1}, j=\overline{1,n-1}$

S - cont. în $x_2=0$: $S_1(0)=S_2(0) \Leftrightarrow$

$$\Leftrightarrow -\frac{1}{e}+0+\frac{9e^{-1}-e}{4}+\frac{-5e^{-1}+e}{4}=0$$

$$\frac{-4e^{-1}+9e^{-1}-\cancel{e}-5e^{-1}+\cancel{e}}{4}=0 \quad (A).$$

$\rightarrow$ S' continuă în nodurile $x_{j+1}, j=\overline{1,n-1}$

$$S'(x)=\begin{cases} \frac{9e^{-1}-e}{2}(x+1)+\frac{3}{4}(-5e^{-1}+e)(x+1)^2, & x\in[-1,0) \\ \frac{e+3e^{-1}}{4}+(e-3e^{-1})x+\frac{3}{4}(e+3e^{-1})x^2, & x\in(0,1] \end{cases}$$

S' cont. în $x_2=0 \Leftrightarrow S_1'(0)=S_2'(0) \Leftrightarrow$

$$\Leftrightarrow \frac{9e^{-1}-e}{2}+\frac{3}{4}(-5e^{-1}+e)=\frac{e+3e^{-1}}{4} \Rightarrow$$

$$\Rightarrow 18e^{-1}-2e-15e^{-1}+3e=e+3e^{-1} \quad (A).$$

$\rightarrow$ S'' continuă în $x_{j+1}, j=\overline{1,n-1}$

$$S''(x)=\begin{cases} \frac{9e^{-1}-e}{2}+\frac{3}{2}\cdot(-5e^{-1}+e)(x+1), & x\in[-1,0) \\ (e-3e^{-1})+\frac{3}{2}(e+3e^{-1})x, & x\in(0,1]. \end{cases}$$

S'' cont. în $x_2=0 \Leftrightarrow S_1''(0)=S_2''(0) \Leftrightarrow$

$$\frac{9e^{-1}-e}{2} + \frac{3}{2}(-5e^{-1}+e) = e-3e^{-1} \iff$$

$$\iff 9e^{-1}-e-15e^{-1}+3e = 2e-6e^{-1} \iff$$

$$\iff -6e^{-1}+2e = 2e-6e^{-1} \text{ (A)}$$

$\Rightarrow \begin{cases} s'(x_1)=f'(x_1) \\ s'(x_{n+1})=f'(x_{n+1}) \end{cases}$ $\quad s'(-1)=f'(-1) \to 0=0$ (A)

$s'(1)=f'(1) \iff$

$$\iff \frac{e+3e^{-1}}{4} + e-3e^{-1} + \frac{3}{4}(e+3e^{-1}) = 2e$$

$$\iff e+\cancel{3e^{-1}}+4e-\cancel{12e^{-1}}+3e+\cancel{9e^{-1}} = 8e \text{ (A)}$$

- **Derivarea numerică**

$f'(x) \simeq \frac{f(x+h)-f(x)}{h}$ → diferențe finite progresive

$f'(x) \simeq \frac{f(x)-f(x-h)}{h}$ → diferențe finite regresive

$f'(x) \simeq \frac{f(x+h)-f(x-h)}{2h}$ → diferențe finite centrale

1° Fie $f:[0,\infty) \to \mathbb{R}$, $f(x)=x\cdot e^x$. Considerăm:

x	1,8	1,9	2,0	2,1	2,2
$f(x)$	10,889365	12,703199	14,778112	17,148937	19,855030

Folosind aproximarea cu diferențe finite centrale, aproximați $f'(2,0)$.

$x = 2,0 = 2$, $h = 0,1$.

$$f'(x) \simeq \frac{f(x+h)-f(x-h)}{2h} \Rightarrow f'(2,0) \simeq \frac{f(2,0+0,1)-f(2,0-0,1)}{2\cdot 0,1} =$$

$$= \frac{f(2,1)-f(1,9)}{0,2} = \frac{17,148937-12,703199}{0,2} \simeq \frac{4,445738}{0,2} \simeq 22,22869.$$