Cursul 12

$\mathcal{E}$ o experiență care se repetă

A un ev. legat de ce exp.

$P(A)$ = șanse de a apare ev. A.

1. Fermat-Pascal.
2. Definiție, metode statistice, L.N.M.
3. Prob. geometrice.
4. Def axiomatică a lui Kolmogorov.

Tripletul Kolmogorov:

$(\Omega, \mathcal{K}, P)$

Ω - spațiul evenimentelor; $\Omega \neq \phi$; Ω = câmpul evenimentelor.

$\mathcal{K}$ = mulțimea evenimentelor:

$$\begin{cases} \Omega \in \mathcal{K} \\ A \in \mathcal{K} \Rightarrow \bar{A} \in \mathcal{K} \\ A, B \in \mathcal{K} \Rightarrow A \cup B \in \mathcal{K} \end{cases}$$

Consecință $\phi \in \mathcal{K}$, $A, B \in \mathcal{K} \Rightarrow A \cap B \in \mathcal{K}$.

Ω se numește câmp borelian de evenimente dacă

$A_1, A_2, \ldots A_n \ldots \in \mathcal{K} \Rightarrow A_1 \cup A_2 \cup \ldots \in \mathcal{K}$.

Emile Borel.

$(\Omega, \mathcal{K}, P)$, Ω, spațiul ev., $\mathcal{K}$ câmp de evenimente.

$P: \mathcal{K} \longrightarrow R$

- $P(A) \geq 0$, $\forall A \in \mathcal{K}$
- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B)$, $A, B \in \mathcal{K}$, $A \cap B = \phi$

Propoziție

(Ω, K, P) câmp. de probabilitate.

1. $P(\phi)=0.$

- $P(\bar{A})=1-P(A).$
- $\begin{cases} \text{Miracol} \\ \text{Ev. aproape sigur.} \end{cases}$ $\begin{cases} \text{Ev. imposibil.} \\ \text{Ev. sigur.} \end{cases}$
- $0 \leq P(A) \leq 1$, $\forall A \in K.$
- $P(A \cup B)=P(A)+P(B)-P(A \cap B)$

Poincaré.

Problema Gauss

$p_1<p_2<p_3<\dots<p_n<\dots$

șirul nr. prime!

$A_1=\{n \in \mathbb{N} \mid p_1 | n\}$; $P(A_1)=\frac{1}{p_1}$; $P(\bar{A}_1)=1-\frac{1}{p_1}.$

$P=\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n \cap \dots$

$P(P)=P(\bar{A}_1) \cdot P(\bar{A}_2) \dots P(\bar{A}_n) \dots = \underbrace{\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \dots \left(1-\frac{1}{p_n}\right)}_{x_n} \dots$

$P(P)=\lim x_n.$

$\frac{1}{x_n}=\frac{1}{\left(1-\frac{1}{p_1}\right)\cdot\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_n}\right)}=\left(1+\frac{1}{p_1}+\frac{1}{p_1^2}\dots\right)\cdot\left(1+\frac{1}{p_2}+\dots\right)\cdot\left(1+\frac{1}{p_n}+\dots\right)$

$\geq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{p_n} \longrightarrow \infty \implies \frac{1}{x_n} \longrightarrow \infty \implies x_n \to 0.$

$\underline{\underline{P(P)=0.}} = \frac{1}{3/17}$

$\mathcal{E}$ exp. aruncări cu 2 zaruri. $p_1 < p_2 < \dots < p_n < \dots$ C (12,3)

A_1 = Ev. ca (n, m) să fie div cu p_1

$P(A_1) = \frac{1}{p_1^2} \Rightarrow P(\bar{A_1}) = 1 - \frac{1}{p_1^2}$.

A_2 = Ev. ca (n, m) să fie div cu p_2.

$P(A_2) = \frac{1}{p_2^2} \Rightarrow P(\bar{A_2}) = 1 - \frac{1}{p_2^2}$.

$A = \bar{A_1} \cap \bar{A_2} \cap \dots$ = ev. ca numerele n, m care apar pe

cele două zaruri sunt prime între ele.

$$P(A) = P(\bar{A_1}) \cdot P(\bar{A_2}) \dots P(\bar{A_n}) \dots = \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \dots \left(1 - \frac{1}{p_n^2}\right) \dots$$

$$\frac{1}{P(A)} = \frac{1}{\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \dots} = \left(1 + \frac{1}{2^2} + \dots\right) \cdot \left(1 + \frac{1}{3^2} + \dots\right) \dots \left(1 + \frac{1}{p_n^2} + \dots\right)$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \Longrightarrow P(A) = \frac{1}{\zeta(2)} = \frac{\pi^2}{6}.$$

Analog pt. trei zaruri.

A ev. ca $(n, m, k) = 1$.

$$P(A) = \frac{1}{\zeta(3)}.$$

Variabile aleatoare simple

(12.

(Ω, K, P) tripletul lui Kolmogorov.

$(A_1, A_2, \dots, A_n)$ sistem complet de evenimente $\begin{cases} P(A_i) \neq \\ A_i \cap A_j = \\ \Omega = A_1 \cup \dots \end{cases}$

$f : \Omega \to \mathbb{R}$

$f(\omega) = \sum_{n=1}^{\infty} \alpha_n \chi_{A_n}(\omega)$ s.n. v. aleat.

Exemple

(1.) Se aruncă un zar.

$\begin{cases} \Omega = \{1, 2, 3, 4, 5, 6\} \\ K = \text{mult. ev.} \\ P(A) = \dfrac{\text{Nr. caz. fav}}{\text{Nr. total de cazuri}} \end{cases}$

v. a. care monitorizează zarul apărut.

$A_1 = \{1\}, \dots, A_6 = \{6\}$

$f(\omega) = \sum_{n=1}^{6} n \chi_{A_n}$

$X_f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$ matricea de rep.

Exp. Se aruncă două zaruri.

(2) $\Omega = \{(n, m) \dots\}$

$f((\omega_1, \omega_2)) = |\omega_1 - \omega_2|$

v. a. care monitorizează $|\omega_1 - \omega_2|$.

Funcție de repartiție 125

f v.a simplă

$F_f : \mathbb{R} \to \mathbb{R}$

X_f = mulț. de rep.

$F_f(x) = P(\omega \in \Omega \mid f(\omega) \le x)$.

$$X_f = \begin{pmatrix} \alpha_1 & \dots & \alpha_n \\ p_1 & \dots & p_n \end{pmatrix}$$

Media unei v.a.

$$M_f = \alpha_1 \cdot p_1 + \alpha_2 p_2 \dots + \alpha_n p_n.$$

Dispersie

$$D_f^2 = \alpha_1^2 p_1 + \dots + \alpha_n^2 p_n - (\alpha_1 p_1 + \dots + \alpha_n p_n)^2.$$

Abaterea medie pătratică

$$\sigma = \sqrt{D_f^2}.$$

Exemple

1) V.a. care monitorizează nr. zeroi.

2) v.a. care monitorizează $|\omega_1 - \omega_2|$.

3) 3a, 2b, 1c Se face exp. B. fără revenire de 3 ori

v.a. care monitoriz. nr. de bile a extrase.

12.6

4) 3a, 2b, 1c. Se face exp. B cu rev. de 3 ori

v.a. care numără nr. de bile a extr.

5) 3a, 2b, 1c 2a, 3b, 1c 1a, 2b, 3c

Se face câte o extragere din fiecare urnă.

v.a. care număr. nr. de bile a extrase.

(12,7)

Ex. 1 Să se det. v.a. cea mai importantă asoc. aruncării cu un zar și apoi calculați $X_f, F_f, M_f, D_f^2, \sigma_f$.

$f(\omega) = \alpha_1 \chi_{A_1}(\omega) + \alpha_2 \chi_{A_2}(\omega) + \dots + \alpha_6 \chi_{A_6}(\omega)$.

$X_f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$ = media de rep.

$F_f(x): \mathbb{R} \to \mathbb{R}$

$F_f(x) = P(\omega \in \Omega \mid f(\omega) \le x)$.

$F_f(\omega) = \begin{cases} 0 & x = -2 \quad \underline{x < 1} \\ \frac{1}{6} & 1 \le x < 2 \\ \frac{2}{6} & 2 \le x < 3 \\ \frac{3}{6} & 3 \le x < 4 \\ \frac{4}{6} & 4 \le x < 5 \\ \frac{5}{6} & 5 \le x < 6 \\ 1 & x \ge 6 \end{cases}$ funcția de rep.

$M_f = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = \frac{1+2+3+4+5+6}{6}$

$= 3,5$

$D_f^2 = d_1^2 p_1 + \dots + d_n^2 p_n - M_f^2 =$

12,8

$$= \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} - 3,5^2 = \frac{91}{6} - 3,5^2 = \cancel{\ldots}\ \underline{2,91}$$

$$1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{\cancel{6}\cdot 7\cdot 13}{\cancel{6}}$$

$\sigma_f = \sqrt{2,91} = 1,7$ - ab. medie pat.

Ex 2

Se aruncă cu 2 zaruri. ne interesează $|\omega_1 - \omega_2|$

V.A. care numit. ~~acele ex.~~

$$X_f = \begin{array}{cccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \end{array}$$
$$X_f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{6}{36} & \frac{10}{36} & \frac{8}{36} & \frac{6}{36} & \frac{4}{36} & \frac{2}{36} \end{pmatrix}$$

$A_1 = \{(\omega_1,\omega_2) \mid |\omega_1-\omega_2| = 0\} = \{(1,1), (2,2), \ldots, (6,6)\}$.

$A_2 = \{(\omega_1,\omega_2) \mid |\omega_1-\omega_2| = 1\} = \{(2,1),(1,2),(3,2),(2,3),(4,3),(3,4)$
$(5,4),(4,5),(6,5),(5,6)\}$

$A_3 = \{(3,1),(1,3),(4,2),(2,4),(5,1),(3,5),(6,4),(4,6)\}$

$A_4 = \{(4,1),(1,4),(5,2),(2,5),(6,3),(3,6)\}$

$A_5 = \{(5,1),(1,5),(6,2),(2,6)\}$

$A_6 = \{(6,1),(1,6)\}$

$$M_f = 0\cdot\frac{6}{36} + 1\cdot\frac{10}{36} + 2\cdot\frac{8}{36} + 3\cdot\frac{6}{36} + \frac{4}{36} + 5\cdot\frac{2}{3} =$$

$$= 1,94$$

$$D_f^2 = 0^2\cdot\frac{6}{36} + 1^2\cdot\frac{10}{36} + 2^2\cdot\frac{8}{36} + 3^2\cdot\frac{6}{36} + 4^2\cdot\frac{4}{36} + 5^2\cdot\frac{2}{36}$$

$$- 1,94^2 = 2,06$$

$$\sigma = \sqrt{D_f^2} = 1,43.$$

Ex. 3 (3a, 2b, 1c) Se face exp. B. fara rep. de 3 ori

f v.a. care reprezintă nr. de bile a extrase.

Det. M_f, D_f^2, σ_f pt. ac. v.a.

(12.10)

$$
\begin{array}{cccc}
 & A_1 & A_2 & A_3 \\
X_f = & \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}
\end{array}
$$

$$P(A_1) = \frac{C_3^3 + C_3^2 \cdot C_1^1}{C_6^3} = \frac{1+3}{\frac{6!}{3!\cdot 3!}} = \frac{4.}{} = \frac{1}{5}$$

$$P(A_2) = \frac{C_3^2 \cdot C_2^1 + C_3^1 \cdot C_2^1 C_1^1}{C_6^3} = \frac{6+6}{\frac{6!}{3!\cdot 3!}} = \frac{12 \cdot 6 \cdot 6.}{6!} =$$

$$P(A_3) = \frac{C_3^1 \cdot C_2^2 + C_2^2 \cdot C_1^1}{C_6^3} = \frac{4}{}$$

$$M_f = 1$$

$$D_f^2 = 0 + \frac{3}{5} + \frac{4}{5} - 1^2 = \frac{2}{5}$$

$$\sigma = \sqrt{\frac{2}{5}}$$

Scanned with CamScanner

12.11

3a, 2b, 1c   2a, 3b, 1c   1a, 2b, 3c

<u>Ex. lui Poisson</u>

V.a. care monitorizează nr. de bile a extrase la urne ex. Poisson.

$$X_f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{39}{6^3} & \frac{1}{2} & \frac{2}{9} & \frac{1}{6^2} \end{pmatrix}$$

$f(x)$

$$\varphi(x,y,z) = (p_1 x + q_1 y + r_1 z)\cdot(p_2 x + q_2 y + r_2 z)\cdot(p_3 x + q_3 y + r_3 z)$$

$$= \left(\frac{3}{6}x + \frac{2}{6}y + \frac{1}{6}z\right)\cdot\left(\frac{2}{6}x + \frac{3}{6}y + \frac{1}{6}z\right)\cdot\left(\frac{1}{6}x + \frac{2}{6}y + \frac{3}{6}z\right) =$$

$$= \frac{1}{6^3}(3x+2y+z)(2x+3y+z)\cdot(x+2y+3z)$$

Suma coef. termenilor de tipul $x^0 \cdot y^i \cdot z^j$

$$\frac{1}{6^3}(2\cdot3\cdot2 + 2\cdot3\cdot3 + 1\cdot3\cdot2 + 1\cdot1\cdot3) = \frac{12+18+6+3}{6^3} = \frac{39}{6^3}$$

Suma coef. termenilor de tipul $x^1 y^i \cdot z^j$

$$\frac{1}{6^3}\cancel{(\ldots)}\ 3(6+9+3)+2(4+6+2+3)$$

$6 \cdot 2 + 6 \cdot 3 + 33 + 3 \cdot 1 + 3 \cdot 3 + 31$ C (13/12)

Se consideră v.a. care modelează nr. divizorilor nr. n, $n \in \{10, \dots, 20\}$.

Să se det. X_f, M_f, D^2_f, σ_f.

$X_f = ($ 4 3

10 → {1, 2, 5, 10} — 4

~~11~~ — {1, 11} — 2

12 → {1, 2, 3, 4, 6, 12} — 6

~~13~~ — {1, 13} — 2

14 → {1, 2, 7, 14} — 4

15 → {1, 3, 5, 15} — 4

16 → {1, 2, 4, 8, 16} — 5

~~17~~ — {1, 17} — 2

18 — {1, 2, 3, 6, 9, 18} — 6

~~19~~ — {1, 19} — 2

20 = {1, 2, 4, 5, 10, 20} — 6

Cursul 13

(Ω, K, P) - tripletul lui Kolmogorov. (1930)

Ω câmpul de probabilitate.

K mulțimea evenimentelor.

P funcție de probabilitate.

$P(A)$

Exemplu Aruncarea cu un zar. (Experiența).

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$K = \mathcal{P}(\Omega) = \{\phi, \Omega, \{2, 3\} \quad \}$

$P(A) = \frac{\text{caz. fav.}}{\text{caz. total}}$ Fermat - Pascal.

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$

$\frac{1}{2} \cdot \frac{1}{2} =$

Fie $\mathcal{E}$ o experiență și (Ω, K, P)

$A, B \in K$ se numesc independente

dacă $P(A) = P(X)$ unde

X este evenimentul de a apare A dacă s-a produs deja B.

$$P(A) = \frac{P(A \cap B)}{P(B)} \iff \boxed{P(A \cap B) = P(A) \cdot P(B)}.$$

Ex.

$A = \{1, 2, 3\}$; $B = \{2, 3, 4\}$. sunt indep ?

$P(A \cap B) = P(A) \cdot P(B)$?

$\frac{2}{6} = \frac{3}{6} \cdot \frac{3}{6}$ NU.

$P(A) = \frac{3}{6}$. $P(X) = \frac{2}{3}$ NU.

Variabile aleatoare

$(A_1, A_2, \dots, A_n)$ un sistem complet de ev.

$A_i \cap A_j = \Phi$

$\Omega = A_1 \cup \dots \cup A_n$

$P(A_i) \neq 0.$

funcția lui Dirichlet

$f : \Omega \to \mathbb{R}$ $\quad f(\omega) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(\omega)$

$$X_f = \begin{array}{c} \begin{matrix} A_1 & A_2 & & A_n \end{matrix} \\ \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ p_1 & p_2 & & p_n \end{pmatrix} \end{array} = \text{= tabelul de repartiție}$$

$p_i = P(A_i).$

Variabile aleatoare independente.

$$f : X_f \begin{array}{c} \begin{matrix} A_1 & \dots & A_n \end{matrix} \\ \begin{pmatrix} \alpha_1 & \dots & \alpha_n \\ p_1 & \dots & p_n \end{pmatrix} \end{array} \qquad g : X_g \begin{array}{c} \begin{matrix} B_1 & \dots & B_m \end{matrix} \\ \begin{pmatrix} \alpha_1' & \dots & \alpha_m' \\ p_1' & \dots & p_m' \end{pmatrix} \end{array}$$

s.n. indep. $\Longleftrightarrow$ A_i, B_j sunt independente.

$P(A_i \cap B_j) = P(A_i) \cdot P(B_j)$

Adunarea v.a.

f, g două v.a. pe (Ω, K, P)

$$X_f = \begin{array}{c} \begin{matrix} A_1 & A_2 & A_3 \end{matrix} \\ \begin{pmatrix} -1 & 0 & 1 \\ 0,2 & 0,4 & 0,4 \end{pmatrix} \end{array} \qquad X_g = \begin{array}{c} \begin{matrix} B_1 & B_2 \end{matrix} \\ \begin{pmatrix} -1 & 1 \\ 0,6 & 0,4 \end{pmatrix} \end{array}$$

$$X_{f+g} = \begin{pmatrix} -2 & 0 & -1 & 1 & 0 & 2 \\ P(A_1 \cap B_1) & P(A_1 \cap B_2) & P(A_2 \cap B_1) & P(A_2 \cap B_2) & P(A_3 \cap B_1) & P(A_3 \cap B_2) \end{pmatrix}$$

$$= \begin{array}{cccccc} & P(A_1)\cdot P(B_2) & & & & \\ \end{array}$$

$$= \begin{pmatrix} -2 & 0 & -1 & 1 & 0 & 2 \\ 0{,}12 & 0{,}08 & 0{,}24 & 0{,}16 & 0{,}24 & 0{,}16 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0{,}12 & 0{,}24 & 0{,}32 & 0{,}16 & 0{,}16 \end{pmatrix}$$

$$\begin{array}{r} 12 \\ 24 \\ 32 \\ 16 \\ 16 \\ \hline 100 \end{array}$$

<u>Variabile aleatoare bernoulliene</u> cu număr finit de ...

cu eveniment . cu ... $P(A)=p$ şi $P(\bar{A})=q$, $p+q=1$.

$$X_f = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}.$$

Se face un şir de experienţe.

f_1 v.a. care numără apariţia ev. A la exp. 1

$$X_{f_1} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

f_2 v.a. ________________ 2

$$X_{f_2} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

f_3 v.a. $\quad X_{f_3} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$ - - -

$\bar{g}_1 = \bar{f}_1$

$\bar{g}_2 = \bar{f}_1 + \bar{f}_2$

$\bar{g}_3 = \bar{f}_1 + \bar{f}_2 + \bar{f}_3$

. . . ——

sunt v.a. care monitorizează apariția ev. A când se fac mai multe exp.

S.n. frecvențele absolute cumulate.

$$\bar{g}_1 = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

$$\bar{g}_2 = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ q^2 & 2pq & p^2 \end{pmatrix}$$

$$\bar{g}_3 = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ q^3 & 3q^2p & 3qp^2 & p^3 \end{pmatrix}$$

⋮

$\bar{v}_n = \dfrac{\bar{g}_n}{n}$ = se numesc v.a. ale frecvenței relative.

$$\bar{v}_1 = \begin{pmatrix} \frac{0}{1} & \frac{1}{1} \\ q & p \end{pmatrix}$$

$$\bar{v}_2 = \begin{pmatrix} \frac{0}{1} & \frac{1}{2} & \frac{2}{2} \\ q^2 & 2pq & p^2 \end{pmatrix}$$

$$\bar{v}_3 = \begin{pmatrix} \frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} \\ q^3 & 3q^2p & 3qp^2 & p^3 \end{pmatrix} \ldots$$

<u>Deci avem</u>

f_n v.a. care monitorizează ev. A <u>la pasul n</u>

g_n v.a. care monitorizează frecvența absolută cumulată la pasul 1,2,...,n.

ν_n v.a care monitorizează frecvența relativă cumulată la pasul 1,2,...,n.

<u>Def</u> (Ω, K, P)

f_n, f v.a.

<u>Se spune</u> $f_n \xrightarrow{P} f \iff$

$\forall \varepsilon > 0 \quad \lim\limits_{n \to \infty} P\left(\omega \in \Omega \,\middle|\, |f_n - f|(\omega) \geq \varepsilon\right) = 0.$

L.N.M

1) <u>Forma empirică</u>

Fie (Ω, K, P), $A \in K$ a.î. $P(A) = p$.

Se face un șir de experiențe.

Notăm g_n frecvența absolută cumulată a ev. A.

ν_n frecvența relativă ale apariției ev. A.

$\nu_n = \frac{g_n}{n}$

Atunci în „imensa maj. a cazurilor"

$$\nu_n \longrightarrow p.$$

Pt. $n \gg 30$. $\nu_n \approx p$. **Bernoulli**.

2. **Forma matematică a L.N.M.**

$(\Omega, \mathcal{K}, P)$, $A \in \mathcal{K}$ a.î. $P(A) = p$.

Se face un șir de experiențe.

$\overline{f_n}$, $\overline{g_n}$ v.a. frecv. absolută cumulată:

$\overline{\nu_n}$ v.a. frecv. relativă

a lui ap. ev. A.

Atunci: $\overline{\nu_n} \xrightarrow{P} \binom{p}{\#}$.

Asta este L.N.M sub formă

empirică și sub formă matematică.

Exemplu Să se det. v. a. care asociază numărul divizorilor lui n, $1 \le n \le 10$

1 — 1 $\{1\}$
2 — 2 $\{1,2\}$
3 — 2 $\{1,3\}$
4 — 3 $\{1,2,4\}$
5 — 2 $\{1,5\}$
6 — 4 $\{1,2,3,6\}$
7 — 2 $\{1,7\}$
8 — 4 $\{1,2,4,8\}$
9 — 3 $\{1,3,9\}$
10 — 4 $\{1,2,5,10\}$

$$X_f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{10} & \frac{4}{10} & \frac{2}{10} & \frac{3}{10} \end{pmatrix}$$

M_f $\quad$ M_f

D_f $\quad$ D_f

σ_f $\quad$ σ

Problema lui Gauss

Se alege un nr. natural.

Care este șansa, prob. ca numărul ales să fie prim? A = nr. ales e prim $P(A) = 0$.

- Prob. pe numere naturale.
- A_5 = nr. ca să se div cu 5.

$$P(A_5) = \frac{1}{5}.$$

- P ? pune între ele.

<u>Prob. pe mult. nr. $\mathbb{N}$</u>

§6

$$P(n \in A) = \frac{1}{5}$$

<u>A mult. nr. div cu 5.</u>

$$\lim_{n \to \infty} \frac{\mathrm{Card}\{1,2 \ldots n\} \cap A}{n} = \frac{1}{5}.$$

$p \in \mathbb{N}$. A_p. mult. lui p.

$$\boxed{P(A_p) = \frac{1}{p}.}$$

$$\boxed{P(\overline{A_p}) = 1 - \frac{1}{p}.}$$

P = <u>mult. nr. prime.</u> ; <u>$p_1 < p_2 < p_3 < \ldots$</u>

$$P = \overline{A}_{p_1} \cap \overline{A}_{p_2} \cap \overline{A}_{p_3} \ldots$$

p, q nr. prime A_p și A_q sunt ev. indep.

$\overline{A}_p$ și $\overline{A}_q$ — " —

$$P(A_p \cap A_q) = P(A_p) \cdot P(A_q)$$

$$P(P) = P\left(\overline{A}_{p_1} \cap \overline{A}_{p_2} \cap \overline{A}_3 \dots\right)$$

$$= P(\overline{A}_{p_1}) \cdot P(\overline{A}_{p_2}) \cdot P(\overline{A}_{p_3}) \dots$$

$$= \left(1-\frac{1}{p_1}\right) \cdot \left(1-\frac{1}{p_2}\right) \cdot \left(1-\frac{1}{p_3}\right) \dots \dots = ?$$

$$P(P) = \lim_{n \to \infty} \left(1-\frac{1}{p_1}\right) \dots \left(1-\frac{1}{p_n}\right) = 0$$

Ex

$$\lim_{n \to \infty} \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)$$

$$\boxed{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) \dots \left(1-\frac{1}{p_n}\right) = 0}$$

Ex

$$\underline{P(P) = 0}$$

Cebâșev cel mai mare Rus

Kolmogorov sovietic

$\boxed{\forall n \in \mathbb{N} \text{ între } n \text{ și } 2n \text{ se află totdeauna un prim.}}$

$\zeta(2)$

$$P(P) = 0 = \frac{1}{\zeta(1)}$$

$$x_n = \boxed{[2^n ; 2^{n+1}]}$$

$$\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

$x + 1000 \dots$

x — — —

$$\boxed{\left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{5}\right)\cdots\left(1-\tfrac{1}{p_n}\right) \longrightarrow 0}$$

$\frac{1}{2}$ is $\frac{1}{p_1}$, $\frac{1}{3}$ is $\frac{1}{p_2}$, $\frac{1}{5}$ is $\frac{1}{p_3}$, $\frac{1}{p_n}$ is $\frac{1}{p_n}$

f şi g v. indep

$$X_f = \begin{pmatrix} -1 & 0 & 1 \\ 0,4 & 0,2 & 0,4 \end{pmatrix}$$

$$X_g = \begin{pmatrix} -1 & 0 & 2 \\ 0,4 & 0,4 & 0,2 \end{pmatrix}$$

X_{f+g} şi $X_{f \cdot g}$

$$X_{f+g} = \begin{array}{c} \begin{matrix} A_1 & A_2 & A_3 \end{matrix} \\ \begin{pmatrix} -1 & 0 & 1 \\ 0,4 & 0,2 & 0,4 \end{pmatrix} \end{array} + \begin{array}{c} \begin{matrix} B_1 & B_2 & B_3 \end{matrix} \\ \begin{pmatrix} -1 & 0 & 2 \\ 0,4 & 0,4 & 0,2 \end{pmatrix} \end{array} =$$

$$= \begin{pmatrix} -2 & \boxed{-1} & 1 & \boxed{-1} & 0 & 2 & 0 & 1 & 3 \\ 0,16 & 0,16 & 0,08 & 0,08 & 0,08 & 0,04 & 0,16 & 0,16 & 0,08 \end{pmatrix}$$

$P(A_1 \cap B_1)$
$= P(A_1) \cdot P(B_1)$

0,40
0,48
0,12

$$= \begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 3 \\ 0,16 & 0,24 & 0,24 & 0,24 & 0,04 & 0,08 \end{pmatrix}$$

$$X_{fg} = \begin{pmatrix} 1 & 0 & -2 & 0 & -1 & 0 & 2 \\ 0,16 & 0,16 & 0,08 & 0,2 & 0,16 & 0,16 & 0,08 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 0,08 & 0,16 & 0,52 & 0,14 & 0,08 \end{pmatrix} ??$$

0,08
0,16
0,52
0,14
0,08

0,98

Seminar 10 S (10.1)

1. (3a, 2b, 1c) Se face exp. B cu rev. de 3 ori. Care este prob. să avem extrase 2a și 1b?

2. (3a, 2b, 1c) Se face exp. B fără rev. de 3 ori. Care este prob. să avem extrase 2a și 1b?

3). (3a, 2b, 1c) (2a, 3b, 1c) (1a, 2b, 3c)

Se face exp. Poisson. Care este prob. să avem extr. 2a și 1b?

1/B

$$B(x,y,z) = (px + qy + rz)^3 =$$

$$= \left(\frac{3}{6}x + \frac{2}{6}y + \frac{1}{6}z\right)^3 = \frac{1}{6^3}(3x+2y+z)^3 =$$

Coef. monom. x^2y: $\frac{1}{6^3} \cdot \frac{3!}{2!\,1!\,0!} \cdot 3^2 2^1 = \frac{1}{6^2} \cdot 9 = \frac{9}{36} = \frac{3}{12} = \frac{1}{4}$

$$P(A) = \frac{C_3^2 C_2^1}{C_6^3} = \frac{6}{\frac{6!}{6\cdot 6}} = \frac{6\cdot 6\cdot 6}{6!} = \frac{36}{120} = 0{,}3$$

$$\operatorname{Rez} f(z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!}\left[f(z)\cdot(z-z_0)^n\right]^{(n-1)}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

§ 10.2

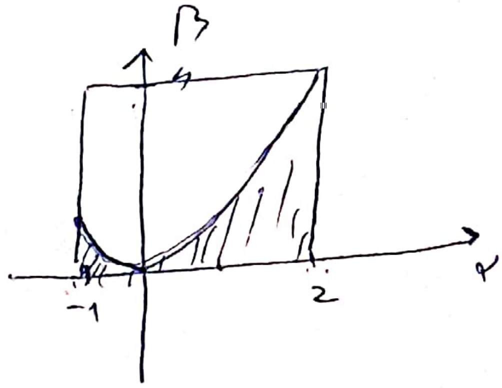
Seminar

- Bernoulli fără rev, cu rev, Poisson.

- $x^2 + 2\alpha x + \beta = 0$

$-1 \le \alpha \le 2$

$0 \le \beta \le 4$



$\Delta = 4\alpha^2 - 4\beta = 0$

$\beta = \alpha^2$

a) A ev. ca răd. să fie reale.

b) B ev. ca răd. să fie egale.

c) C ev. ca răd să fie complexe.

Să se calculeze P(A), P(B), P(C).

- Se aruncă 3 zaruri. Care este șansa, prob. ca cu cele 3 numere să se poată forma un Δ?

- Se alege aleator un triunghi din mulțimea triunghiurilor. Care este prob. ca acest Δ să fie ascuțitunghic? Dar dreptunghic? Dar obtuzunghic?

(6.8)

$$P(x,y,z) = \left(\frac{3}{6}x + \frac{2}{6}y + \frac{1}{6}z\right)\cdot\left(\frac{2}{6}x - \frac{3}{6}y + \frac{1}{6}z\right)\left(\frac{1}{6}x - \frac{2}{6}y + \frac{3}{6}z\right)$$

Coef mul xyz

$$= \frac{1}{6^3}(3x + 2y + z)\cdot(2x + 2y + z)(x + 2y + 3z)$$

$$= \frac{1}{6^3}(3\cdot 2\cdot 2 + 3\cdot 1\cdot 3 + 2\cdot 2\cdot 1) = \frac{25}{6^3} = 0{,}11$$

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\[2mm] \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases}$$

$\underline{z^2 = i} = \underline{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}$

$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$z_2 = \cos\frac{\frac{\pi}{2} + 2\pi}{2} + i\sin\frac{\frac{\pi}{2}+2\pi}{2} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$$P(A) = \frac{\text{Aire hachée}}{\text{Aire totale}} = \frac{3}{12} = \frac{1}{4}$$

$$Ah = \int_{-1}^{2} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = 3$$

(10,4)

1. (5,3,2)	$f_1 = 0$	$\nu_1 = \frac{f_1}{1} = 0$
2. (1,2,3)	$f_2 = 0$	$\nu_2 = \frac{f_2}{2} = 0$
3. (5,5,2)	$f_3 = 1$	$\nu_3 = \frac{1}{3} = 0{,}33$
4. (4,4,3)	$f_4 = 2$	$\nu_4 = \frac{2}{4} = 0{,}50$
5. (3,3,3)	$f_5 = 3$	$\nu_5 = \frac{3}{5} = 0{,}60$
6. (4,2,5)	$f_6 = 4$	$\nu_6 = \frac{4}{6} = 0{,}67$
7. (2,4,3)	$f_7 = 5$	$\nu_7 = \frac{5}{7} = 0{,}71$
8 (6,1,5)	$f_8 = 5$	$\nu_8 = \frac{5}{8} = 0{,}62$
9. (4,1,5)	$f_9 = 5$	$\nu_9 = \frac{5}{9} = 0{,}56$
10 (1,6,1)	$f_{10} = 5$	$\nu_{10} = \frac{5}{10} = 0{,}5$
11 (6,1,5)	$f_{11} = 5$	$\nu_{11} = 0{,}45$
12 (4,1,3)	$f_{12} = 5$	$\nu_{12} = 0{,}42$
13 (3,2,2)	$f_{13} = 6$	$\nu_{13} = 0{,}46$
14 (3,4,2)	$f_{14} = 7$	$\nu_{14} = 0{,}50$
13 (6,1,6)	$f_{15} = 8$	$\nu_{15} = 0{,}53$
14 (4,1,6)	$f_{16} = 8$	$\nu_{16} = 0{,}5$
15 (4,4,6)	$f_{17} = 9$	$\nu_{17} = 0{,}53$
16. (1,2,4)	$f_{18} = 9$	$\nu_{18} = 0{,}5$
17. (3,3,4)	$f_{19} = 10$	$\nu_{19} = 0{,}52$
18. (1,2,4)	$f_{20} = 10$	$\nu_{20} = 0{,}5$
19. (1,3,6)	$f_{21} = 10$	$\nu_{21} = 0{,}47$

A s.n. mirocel

$A \neq \phi$

$P(A) = 0.$

Ev. impos

$A = \phi$

Ev. sigur

$A = \Omega$

$P(A) = 1$

Ev. ap. sig

$A \neq \Omega$

$P(A) = 1$

125

22.	(1,1,2)	$f_{22} = 10$	$\nu_{22} = 0{,}5$
23.	(7,1,5)	$f_{23} = 11$	$\nu_{23} = 0{,}47$
24.	(5,1,7)	$f_{24} = 12$	$\nu_{24} = 0{,}5$.
25.	(1,1,5)	$f_{25} = 12$	$\nu_{25} = 0{,}48$
26	(1,2,3)	$f_{26} = 12$	$\nu_{26} = 0{,}46$
27	(1,3,4)	$f_{27} = 12$	$\nu_{27} = 0{,}44$
28	(1,6,4)	$f_{28} = 12$	$\nu_{28} = 0{,}42$
29	(4,3,6)	$f_{29} = 13$	$\nu_{29} = 0{,}44$
30	(5,5,2)	$f_{30} = 14$	$\nu_{30} = 0{,}46$.

$$\boxed{P(A) \approx 0{,}5}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$e^{z} =$$

$$e^{2+\pi i} = e^{2}(\cos\pi + i\sin\pi) = -e^{2}.$$

10.6

1. (96, 37, 47) $f_1 = 0$ $r_1 = 0$

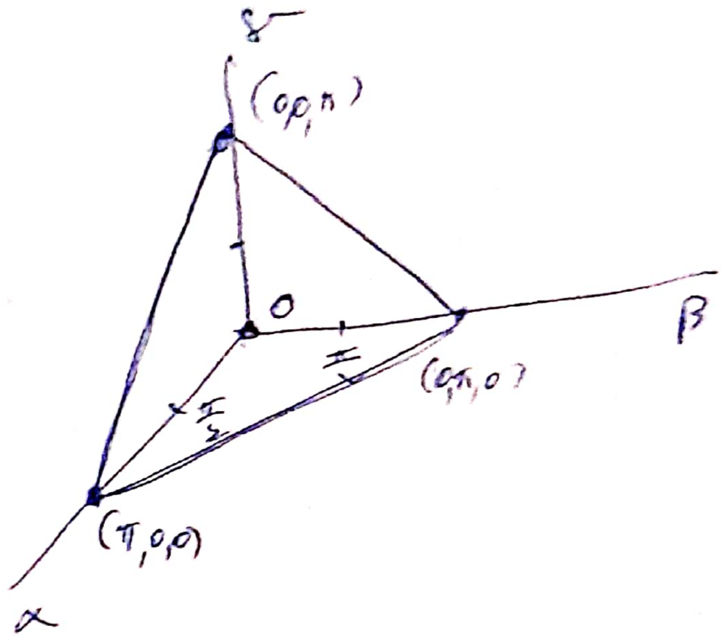
2. (23, 57, 160) $f_2 = 0$ $r_2 = 0$

3. (74, 56, 50) $f_3 = 1$ $r_3 = 0{,}33$

4. (82, 63, 35) $f_4 = 2$ $r_4 = 0{,}5$

5. (26, 68, 86) $f_5 = 3$ $r_5 = 0{,}6$

⋮



$0 \leq \alpha, \beta, \gamma \leq \pi$

$\alpha + \beta + \gamma = \pi$

$0 \leq \alpha \leq \frac{\pi}{2}$

$0 \leq \beta \leq$

$\gamma = \pi - \alpha - \beta$

$\pi \geq \alpha \geq 0$

$\pi \geq \beta \geq 0$

$\pi \geq \pi - \alpha - \beta \geq 0$

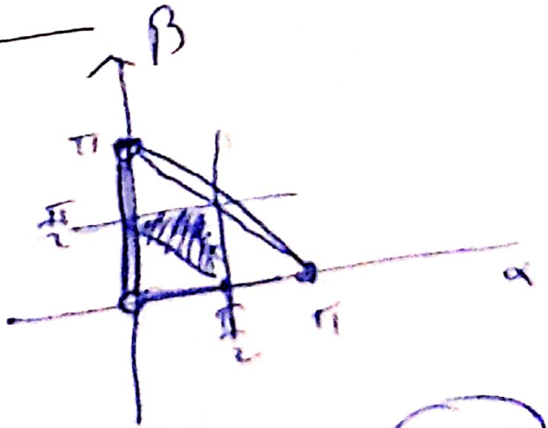
$0 \leq \alpha \leq \frac{\pi}{2}$

$0 \leq \beta \leq \frac{\pi}{2}$

$0 \leq \pi - \alpha - \beta \leq \frac{\pi}{2}$

$\alpha + \beta \leq \pi$

$\alpha + \beta \geq$



$P(A) = \frac{1}{4} = 0{,}25$

$\begin{cases} \alpha = \\ \beta = \end{cases}$ $\quad \Delta = \alpha^2 - \beta \geq 0 \quad f_n \quad \nu_n$

(10.8)

$\nu_n \approx 0,25$

Sol1 Fermat-Pascal

$P(A) = \frac{}{6^3}$

Sol2

1. $(1,4,3)$	$f_1 = 0$	$\nu_1 = \frac{f_1}{1} = 0.$
2. $(4,3,6)$	$f_2 = 1$	$\nu_2 = \frac{f_2}{2} = 0,5$
3. $(1,5,4)$	$f_3 = 1$	$\nu_3 = \frac{1}{3} = 0,33$
4. $(2,1,6)$	$f_4 = 1$	$\nu_4 = \frac{1}{4} = 0,25$
5. $(6,6,3)$	$f_5 = 2$	$\nu_5 = \frac{2}{5} = 0,4.$
6. $(1,6,1)$	$f_6 = 2$	$\nu_6 = \frac{2}{6} = 0,33$
7. $(2,2,3)$	$f_7 = 3$	$\nu_7 = \frac{3}{7} = 0,42$
8. $(5,3,2)$	$f_8 = 3$	$\nu_8 = \frac{3}{8} = 0,37$
9 $(2,1,6)$	$f_9 = 3$	$\nu_9 = \frac{3}{9} = 0,33$
10 $(1,6,4)$	$f_{10} = 3$	$\nu_{10} = \frac{3}{10} = 0,3.$

(10.9)

11 (1,1,6)	$f_{11}=3$	$r_{11}=0,27$
12 (6,3,4)	$f_{12}=4$	$r_{12}=0,33$
13 (2,4,3)	$f_{13}=5$	$r_{13}=0,38$
14 (3,2,4)	$f_{14}=5$	$r_{14}=0,35$
15 (2,5,4)	$f_{15}=6$	$r_{15}=0,4$
16 (6,1,4)	$f_{16}=6$	$r_{16}=0,37$
17 (2,3,4)	$f_{17}=7$	$r_{17}=0,41$
18 (2,2,4)	$f_{18}=7$	$r_{18}=0,38$
(6,6,5)	$f_{19}=8$	$r_{19}=0,42$
(6,4,1)	$f_{20}=9$	$r_{20}=0,45$
(6,5,5)	$f_{21}=10$	$r_{21}=0,47$
(4,3,2)	$f_{22}=11$	$r_{22}=0,5$
(3,3,4)	$f_{23}=12$	$r_{23}=0,52$
(4,1,6)	$f_{24}=12$	$r_{24}=0,5$
(2,6,6)	$f_{25}=13$	$r_{25}=0,52$
(5,1,4)	$f_{26}=13$	$r_{26}=0,5$
(2,4,6)	$f_{27}=13$	$r_{27}=0,48$
(2,3,6)	$f_{28}=13$	$r_{28}=0,46$
(1,2,4)	$f_{29}=13$	$r_{29}=0,44$
(3,2,1)	$f_{30}=14$	$r_{30}=0,46$

$P(A) \simeq 0,48$

$\simeq 0,51$

$$P = \left(\frac{3}{6}x + \frac{2}{6}y + \frac{1}{6}z\right)\left(\frac{2}{6}x + \frac{3}{6}y + \frac{1}{6}z\right)\left(\frac{1}{6}x + \frac{2}{6}y + \frac{3}{6}z\right) S \quad 10\,10$$

Coef $\underline{\underline{x^2y}}$ $\quad P(C) = 11$

$$P(B) = \frac{C_3^2 \cdot C_2^1}{C_6^3} = \frac{3 \cdot 2}{\frac{6!}{3! \cdot 3!}} = 0{,}2$$

$\mathbb{N}$ $\quad A \subset \mathbb{N}$.

$$P(A) = \lim_{n \to \infty} \frac{\text{Card } \{1, 2, \dots, n\} \cap A}{n}$$

prob. pe $\mathbb{N}$.

$P \subset \mathbb{N}$ mult. nr. nat

$P =$ mult. nr. pări

$$P(P) = \lim_{n \to \infty} \frac{\text{Card } \{1, \dots, n\} \cap P}{n} = 0$$

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