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NON-LINEAR TRANSVERSE DYNAMICS FOR STORAGE RINGS WITH APPLICATIONS TO THE LOW-ENERGY ANTIPROTON RING (LEAR) AT CERN

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Abstract

A tensor equation has been used to derive the equations of motion for the curvilinear coordinate system customarily used for particle accelerators. A Hamiltonian formalism, expanded to third order in the canonical variables, has also been developed to describe the transverse motion in an accelerator.

Time-dependent perturbation theory has been applied and computerized using a computer-algebra system. In particular, the perturbations due to magnetic sextupoles have been calculated to second power in the sextupole strength.

The frequency spectra for the horizontal and the vertical betatron motion close to a single resonance have been calculated using time-independent perturbation theory. It has been shown that information about excited resonances and the type of driving field can be derived from the spectra. In particular, it is possible to obtain the amplitude and the phase of a given resonance. The results have been used to study the perturbations in the Low Energy Antiproton Ring (LEAR) at CERN.

Contents

Chapter 1

INTRODUCTION

The study presented here is the result of a stay with the Low Energy Antiproton Ring (LEAR) group at CERN. LEAR was built to provide intense low-energy antiproton beams, by using phase-space cooling, which also makes the deceleration possible. The ultra-slow extraction gives spill times from 15 minutes to 5 hours with 2.0×10^4 to 1.0×10^6 antiprotons per second. The momentum of the extracted beam can be varied from 105 to 1900 MeV/c. LEAR has been mainly operated as a variable-momentum antiproton beam stretcher ring [PLASS80, CHANE87].

The problems that are studied in this document arose from the interest of low-momentum operation, where the beam lifetime was very poor. Perturbations had been observed that were linked to the sextupole configuration on the extracted beam originating from coupled oscillations between the horizontal and the vertical plane. Methods had been developed to measure the perturbations of the beam. The main aim of the presented work has therefore been the analytical study of the non-linear perturbations due to magnetic sextupoles and to relate the theory with possible measurements.

First of all, a brief introduction to accelerator physics is given in order to relate this study to previous work done in the field. In particular, an attempt will be made to point out similarities and differences in the mentioned works. There is no intention of giving a complete survey of all related work. Rather at least one reference has been given, when possible, to previous work that is either similar or closely related to the different parts of this study.

1.1 A short introduction to accelerator physics

The motion of a particle in an accelerator[1] is conveniently described by a reference orbit for a particle with some reference momentum p_0. For a circular accelerator it is customary to choose the periodic orbit which closes upon itself after one turn (closed orbit). The motion for an arbitrary particle is then described by the deviations from this reference orbit.

Synchrotrons, storage rings, and transfer lines are built up by sequences of magnetic elements (referred to as the lattice). In the design of a storage ring one normally starts with a simplified lattice that only contains ideal magnetic dipoles and quadrupoles. The dipoles then define the closed orbit for the reference particle. The magnetic fields can thus be expanded around this orbit, which leads to the equations for particles that deviate from the reference orbit. If one only keeps linear terms, the equations of motion have the following form for the horizontal and the vertical motion respectively:

[1] Accelerator will be used to refer both to accelerators, where particles are confined for a relatively short time, and to storage rings, where particles are stored for a considerably longer time (e.g. days or even weeks).

$$\frac{d^2x}{ds^2} + K_x(s)x = \delta h(s)$$
$$\frac{d^2z}{ds^2} + K_z(s)z = 0 \,, \tag{1}$$

where the momentum deviation δ is defined by

$$\delta = \frac{p - p_0}{p_0} \,. \tag{2}$$

The variables x and z are the deviations from the reference orbit, s is the distance along this, and $K(s)$ and $h(s)$ are functions of s defined by the lattice. Note that $K(s)$ is periodic for a circular accelerator. In this case the corresponding homogeneous equations are known as Hill's equations. Owing to the focusing forces described by $K(s)$, this motion is oscillatory along the reference orbit and is called the betatron motion.

In these equations we have neglected the variation of momentum due to, for example, acceleration or synchrotron radiation. This can be done if the variation of momentum is slow compared to the betatron motion (adiabatic approximation).

The solutions of Eqs. (1) can be written in a pseudoharmonic oscillator form introduced in a classic paper by Courant and Snyder [COURA58]:

$$x = \sqrt{2\, J_x \beta_x(s)} \cos \psi_x(s) \,. \tag{3}$$

The phase function in Eq. (3) is

$$\psi_x(s) = \int_{s_0}^{s} \frac{d\tau}{\beta_x(\tau)} + \phi_x \,. \tag{4}$$

J_x and ϕ_x, which define the amplitude and the phase of the betatron motion, are constants given by the initial conditions of a particle. The amplitude function $\beta_x(s)$ is the solution of the differential equation

$$\frac{d^2\sqrt{\beta_x}}{ds^2} + K(s)\sqrt{\beta_x} - \frac{1}{\beta^{3/2}} = 0 \,. \tag{5}$$

For a circular accelerator, $\beta_x(s)$ is the periodic solution. In this case the number of betatron oscillation wavelengths in one revolution (tune) is

$$Q_x = \frac{1}{2\pi} \int_{s}^{s+C} \frac{d\tau}{\beta_x(\tau)} \,, \tag{6}$$

where C is the circumference of the accelerator. Similar expressions hold for the vertical plane.

For the linear motion [defined by the linear equations (1)] the tune is independent of the amplitude of the betatron motion given by J and of the momentum deviation δ. When higher order-terms in x, z, and δ, are introduced in the equations of motion, this is no longer true. In particular, one normally finds quite a strong dependence of the tune on δ (chromaticity), which can be compensated by the introduction of magnetic sextupoles. In this way, it is possible to remove the linear tune dependence on δ. However, the sextupoles make the equations of motion non-linear and coupled. This can lead to strong amplitude dependence and non-linear momentum dependence if special care is not taken. Higher-order non-linear terms have a similar effect. In general, the tune can then be expanded as

$$Q = Q_0 + \frac{\partial Q}{\partial \delta}\delta + \frac{\partial Q}{\partial J_x}J_x + \frac{\partial Q}{\partial J_z}J_z + \dots . \tag{7}$$

The linear chromaticity is defined by

$$\xi = \frac{1}{Q_0}\frac{\partial Q}{\partial \delta} . \tag{8}$$

The linear equation for the horizontal motion (1) contains an inhomogeneous term $\delta h(s)$. The general solution is then the sum of the homogeneous solution (3) and a particular solution. It is customary do define the particular solution in terms of the linear dispersion function $D_0(s)$ as

$$x_p(s) = \delta D_0(s) , \tag{9}$$

where $D_0(s)$ is a solution of

$$\frac{d^2 D_0}{ds^2} + K_x(s) D_0(s) = h(s) . \tag{10}$$

For a circular accelerator the periodic solution has to be taken.

The appearance of higher-order terms in δ in the equations of motion leads to a momentum dependence of the dispersion function. The non-linear dispersion function can then be expanded as

$$D(s) = D_0(s) + \delta D_1(s) + \dots , \tag{11}$$

where $D_1(s)$ is a function of s given as a solution of the differential equation obtained by replacing $D_0(s)$ with $D(s)$ in Eq. (9), using this in Eq. (1), and collecting terms of equal power in δ. One effect of the non-linear terms is thus to make Q, $D(s)$, $\beta(s)$, and $\psi(s)$ amplitude and momentum dependent[2].

A classification of the influence on the lattice functions of the different linear and non-linear terms is given in Table 1, where 0, 1, and 2 refer to the powers in which the variables x, z, and δ appear in the equations of motion. As an example: a term of the form $x\delta$ contributes to the chromaticity, and the appearance of quadratic terms in x and z in the equations of motion leads to amplitude dependence of Q, $\beta(s)$, and $\psi(s)$. The work described here mainly concerns the terms in the lower left corner of Table 1. Explicit solutions of $D_1(s)$ can be found in Delahaye and Jäger [DELAH86], and a study of coupled synchro-betatron motion can be found in Barber et al. [BARBE87].

[2] Note that $\psi(s)$ and Q are defined by $\beta(s)$ by virtue of Eqs. (4) and (6).

Table 1: Classification of the terms appearing in the equations of motion

x, z \ δ	0	1	2
0	Reference orbit	$D_0(s)$	$D_1(s)$
1	$\beta(s), \psi(s), Q$	ξ	
2	$\frac{\partial \beta}{\partial J}, \frac{\partial \psi}{\partial J}, \frac{\partial Q}{\partial J}$		

1.2 Perturbation theory

To find solutions of the non-linear equations of motion one normally has to apply some sort of perturbation theory. In the author's opinion it is preferable to use a Hamiltonian formulation, particularly if one wants to do higher-order calculations. One possibility is then to use the time-independent perturbation theory [GOLDS80] (or equivalently s-independent, since it is customary to use s as the independent variable for accelerators) This approach originates from celestial mechanics, where it is known as the "Poincaré – von Zeipel" procedure [ZEIPE16]. It has been studied by Moser [MOSER55] who has also made many contributions to the modern treatment of stability problems [MOSER73]. It has been applied to accelerators by Hagedorn [HAGED57] and Schoch [SCHOC57]. Examination of the slowly varying Hamiltonian has led to attempts to parametrize the beam dynamics by the strength of isolated resonances. Work related to the definition and the study of the width of non-linear resonances has been done, for example, by Guignard [GUIGN78].

Since in a real accelerator or storage ring we try to avoid a situation where one or a few resonance terms dominate, the beam dynamics cannot usually be described by the single-resonance theory with a transformed Hamiltonian containing terms which are linear in a perturbation parameter ε (e.g. the multipole strength). Rather, a global approach must be used where the transformed Hamiltonian contains the terms of second power in ε since all the terms linear in ε can be removed by the canonical transformation. Explicit expressions for the Hamiltonian to second power in ε and the generating function to first power in ε are given by Schoch [SCHOC57] and Hagedorn [HAGED57], and more recently by Ando [ANDO84], Courant et al. [COUR84], Ruth [RUTH86] and Parsa [PARSA87]. This method has been extended to third power in ε for the generating function by Willeke [WILLE85B].

The Poincaré – von Zeipel procedure has a disadvantage, which can be stated by using Michelotti's words [MICHE86]:

> "... the Poincaré – von Zeipel procedure suffers from a serious disadvantage. Because the generating function is written in a mixed system of variables, the transformation from the new, "averaged" variables to the old, "exact" variables is only defined implicitly. Practically, then, carrying out the transformation to better than lowest order is accomplished more in principle than in practice."

This problem has been solved by using Lie transforms for which the transformations equations are explicit and can be developed recursively[3] to any power and therefore computerized [HORI66, DEPRI66].

Deprit's recursive algorithm has been applied to accelerators by Michelotti [MICHE85, MICHE86] to study the stability, for example to fourth power in the sextupole strength. Another possibility, which also gives explicit solutions, is to use time-dependent perturbation theory [GOLDS80]. This has been done by Collins [COLLI84], working directly on the differential equations which led to the concept of distortion functions. A similar approach has been chosen by Autin [AUTIN86] and Schonfeld [SCHON86] to find a method to correct the non-linear beam envelope distortions. These cases give solutions to first power in ε. The expressions for the perturbations of the action given by Autin has been integrated analytically for thick magnets [BENGT88A] (see Chapter 7) by using the computer algebra system REDUCE [HEARN85]. Since the time-dependent perturbation theory also is recursive, it has been computerized by using a Hamiltonian formulation and extended to obtain solutions to second power in ε [BENGT87C] (see Chapter 7). This corresponds to a generating function with terms to second power in ε for the time-independent perturbation theory. The results have been used to study the perturbations in LEAR [BENGT88B] (see Chapter 16). They have also been used by Autin to work out an octupolar compensation of the resonance $2Q_x - 2Q_z = 0$ excited by sextupoles in second power of the sextupole strength [AUTIN].

One of the basic ideas in this study, has been to use a recursive formulation for the perturbation theory, in order to be able to computerize the calculations by using a computer algebra system. The use of a computer algebra system also has the big advantage of efficient FORTRAN code generation of the often quite cumbersome expressions that can be used for numerical analysis. Something like 4000 lines of FORTRAN codes were generated for the study described in Chapter 16, and the work presented in a paper with Autin [BENGT88A] involved the solving of over 100 integrals of similar nature giving 6000 lines of code. The aim of the calculations has been to go at least one step further then the first-order calculation in order to obtain some understanding of the importance of the higher-order perturbations.

1.3 Beam measurements

In an accelerator it is possible to obtain information about the beam's motion by using electrostatic pick-up electrodes [ASSEO85, BERNA83]. Since measurements can be done when the tune has been moved close to a single resonance, the resonance theory can be used to describe the observed motion. In particular it is possible to measure the frequency spectra for the betatron motion by using Fourier analysis and signal processing [ASSEO87] (see Chapter 10 and 12). In order to relate the theory to possible measurements, it is preferable to have explicit solutions as a function of s. Not much work had been done in this direction, probably owing to the use of the Poincaré – von Zeipel procedure, with its implicit transformation equations, and because the main concern has been the study of stability, where the explicit s-dependence is not needed. An attempt has been made by Ando [ANDO84] to obtain the frequency spectra for the non-linear betatron motion due to sextupoles by using time-independent perturbation theory. The problem of solving the implicit tranformation did not appear since he neglected the perturbation in the phase variable. His treatment is therefore inconsistent since this perturbation is linear in the sextupole strength. This explains the bad agreement between the analytical results and the numerical simulation.

In the present work, the frequency spectra will be calculated for the case of a single-resonance Hamiltonian (see Chapter 8) by applying time-independent perturbation theory and by using

[3] A function is recursive if it is computable (i.e. can be computed by a Turing machine) [LOFGR72].

successive approximations to the implicit transformation [BENGT87B] (see Chapter 9). The analytical results are also sucessfully compared with numerical integration of the equations of motion (see Chapter 11). It follows from the analytical results that information about which resonances are excited, and the type of driving field can be derived from the frequency spectra. It will also be shown that it is possible to obtain the amplitude and the phase of a given resonance. This can thus be used to analyse tracking data [BENGT87D] (see Chapter 14) and real measurements [BENGT86A, BENGT86B, BENGT87A] (see Chapters 13 and 15).

Another possibility is to reconstruct the transverse phase space from the beam measurements [WILLE85A, MORTO85]. This has been tried at LEAR by Chanel et al. [ASSEO].

1.4 Equations of motion

It is common to use curvilinear coordinates to describe the motion of a particle in an accelerator. The customary approach to obtain differential equations for the linear motion [KOLOM66, STEFF85] and equations of motion to second power in the coordinates [BROWN85B] has been to calculate the derivatives of the unit vectors. An alternative and probably more transparent approach is to use a tensor equation [BENGT87E] (see Chapter 2) as employed in, for example, general relativity. This approach will be used in the present work to derive the general differential equations describing the motion. When they are expanded to second power in the coordinates, we recover the equations given by Brown.

Much has been written using a Hamiltonian formulation but we did not find a Hamiltonian systematically expanded to a particular power in the coordinates leading to Brown's differential equations. Expansion to third power in the coordinates for fixed curvature has been done by Bell [BELL55]. This has been extended to allow for varying curvature [BENGT87E] (see Chapters 3 and 4). The Hamiltonian formulation normally used is non-covariant. However, since it is possible to use a covariant formulation for a charged particle in an external electromagnetic field, such a formulation will be presented (see Appendix A).

Since the perturbative calculations have been computerized using a Hamiltonian formulation and time-dependent perturbation theory, we will also describe this formulation for the dispersion function (see Chapter 6).

The aim for these extensions and applications of the theory was to provide a consistent derivation of the equations used to interpret the measurements made and the strategies proposed to improve the beam stability in LEAR.

Chapter 2

EQUATIONS OF MOTION IN AN EXTERNAL ELECTROMAGNETIC FIELD

To study the dynamics of particle beams, we need the equations of motion for a charged particle in an external electromagnetic field. Since a curvilinear coordinate system is normally used for accelerators, the equations of motion are derived from a tensor equation, valid in any coordinate system. In this approach, which is widely used in general relativistic mechanics, one does not have to work out the derivatives of the unit vectors with respect to s, the distance along the reference curve, as is necessary in the customary approach [BROWN85B, KOLOM66, STEFF85]. The equations of motion are obtained directly from the given metric. We will first derive the exact equations and then expand them to second power in the coordinates to obtain the same equations as Brown, which will be presented below. In Chapter 3 a Hamiltonian formalism will be developed.

2.1 Equations of motion in a curvilinear coordinate system

The equations of motion in a curvilinear system are given by the tensor equation [WEINB72, ANDER67]

$$\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} = \frac{f^\mu}{m_0}, \quad \mu = 0,1,2,3,4\,, \tag{12}$$

where u^μ is the four-velocity, $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, m_0 the rest mass, f^μ the four-force, and τ the proper time. The covariant derivative along the curve $x^\mu(\tau)$ of a contravariant four-vector A^μ is defined by

$$\frac{\mathrm{D}A^\mu}{\mathrm{D}\tau} \equiv \frac{\mathrm{d}A^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}A^\nu\,, \tag{13}$$

where $\Gamma^\mu{}_{\nu\lambda}$ is the affinity given by the metric tensor $g_{\mu\nu}$

$$\Gamma^\mu{}_{\nu\lambda} \equiv \frac{1}{2}g^{\alpha\mu}\left[\frac{\partial g_{\lambda\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\alpha}\right]. \tag{14}$$

The metric tensor $g_{\mu\nu}$ defines the differential distance in a particular coordinate system

$$c^2\,d\tau^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{15}$$

It obeys the following relation:

$$g_{\mu\lambda}\,g^{\lambda\nu} = \delta^\nu{}_\mu\,. \tag{16}$$

The transformation from a contravariant vector A^μ to a covariant vector A_μ is given by

$$A_\mu = g_{\mu\nu} A^\nu ,$$
$$A^\mu = g^{\mu\nu} A_\nu . \tag{17}$$

The four-force f^μ for an electromagnetic field is given by

$$f^\mu = \frac{e}{c} F^\mu{}_\nu \frac{dx^\nu}{d\tau} , \tag{18}$$

where e is the charge of the particle and $F^\mu{}_\nu$ is the electromagnetic field tensor which, in a Cartesian inertial system, takes the form

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} , \quad \tilde{F}^\mu{}_\nu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \tag{19}$$

since the metric tensor is in this case

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{20}$$

The form of the field tensor in a general coordinate system may then be calculated from the transformation rule of a contravariant tensor

$$A'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} A^{\alpha\beta} . \tag{21}$$

By combining Eqs. (12), (13), and (18) we get the equations of motion for a particle in an electromagnetic field:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = \frac{e}{m_0 c} F^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} . \tag{22}$$

In an inertial system this reduces to the well-known Lorentz force law

$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}t} = e(\bar{E} + \bar{v} \times \bar{B}) , \tag{23}$$

where v is the velocity. The time-like part of Eq. (22) assumes the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = e\bar{v} \cdot \bar{E} , \tag{24}$$

since the four-momentum is given by

$$p^{\mu} = m_0 \frac{dx^{\mu}}{d\tau} = m_0 \frac{dt}{d\tau}\frac{dx^{\mu}}{dt} = m_0 \gamma \frac{dx^{\mu}}{dt} = m_0 \gamma (c,\bar{v}) = (\frac{E}{c}, \bar{p}) . \tag{25}$$

It is seen that if the electric field $\bar{E}$ is orthogonal to the velocity or zero then the energy E of the particle is constant.

2.2 Equations of motion for a particle in an accelerator

In accelerator theory it is convenient to use local coordinates for a particle of the form shown in Figure 1 [CAS85][4]. The distance along a reference curve $R(s)$ is denoted by s. The only assumption for this curve is that it should lie in a horizontal plane and have the local curvature

$$h(s) = \frac{1}{\rho(s)} . \tag{26}$$

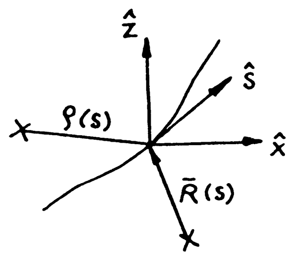

Figure 1: Curvilinear coordinate system used for accelerators

A general vector $\bar{r}$ can then be written as

$$\bar{r}(x,s,z) = \bar{R}(s) + x\hat{x}(s) + z\hat{z}(s) . \tag{27}$$

It follows that

$$d\bar{r}(x,s,z) = dx\,\hat{x} + (1 + hx)ds\,\hat{s} + dz\,\hat{z} , \tag{28}$$

so that

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 = c^2 dt^2 - dx^2 - (1 + hx)^2 ds^2 - dz^2 . \tag{29}$$

---

[4] It would be preferable to use, for example, u, v, and w instead of x, z, and s, since the coordinate system is curvilinear.

From Eqs. (15), (16), and (29) we find for the metric tensor

$$g_{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-(1+hx)^2&0\\0&0&0&-1\end{pmatrix},\quad g^{\mu\nu}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&\dfrac{-1}{(1+hx)^2}&0\\0&0&0&-1\end{pmatrix}. \tag{30}$$

From Eqs. (8) and (10) we may calculate the field tensor to be

$$F^{\mu\nu}=\begin{pmatrix}0&-E_x&\dfrac{-E_s}{1+hx}&-E_z\\ E_x&0&\dfrac{-cB_z}{1+hx}&cB_s\\ \dfrac{E_s}{1+hx}&\dfrac{cB_z}{1+hx}&0&\dfrac{-cB_x}{1+hx}\\ E_z&-cB_s&\dfrac{cB_x}{1+hx}&0\end{pmatrix},$$

$$F^{\mu}{}_{\nu}=\begin{pmatrix}0&E_x&E_s(1+hx)&E_z\\ E_x&0&cB_z(1+hx)&-cB_s\\ \dfrac{E_s}{1+hx}&\dfrac{-cB_z}{1+hx}&0&\dfrac{cB_x}{1+hx}\\ E_z&cB_s&-cB_x(1+hx)&0\end{pmatrix}, \tag{31}$$

where we have used E_s, B_s instead of E_y, B_y.

The only affinities different from zero are from Eq. (14):

$$\Gamma^1_{2\,2}=-h(1+hx)\,,\quad \Gamma^2_{1\,2}=\Gamma^2_{2\,1}=\frac{h}{1+hx}\,,\quad \Gamma^2_{2\,2}=h'\frac{x}{1+hx}\,, \tag{32}$$

where a prime denotes a derivative with respect to s. Equation (22) gives the equations of motion:

$$\begin{aligned}
\frac{dE}{d\tau}&=e\left[\frac{dx}{d\tau}E_x+(1+hx)\frac{ds}{d\tau}E_s+\frac{dz}{d\tau}E_z\right]\\
\frac{d^2x}{d\tau^2}-h(1+hx)\left(\frac{ds}{d\tau}\right)^2&=\frac{e}{m_0}\left[\frac{dt}{d\tau}E_x+(1+hx)\frac{ds}{d\tau}B_z-\frac{dz}{d\tau}B_s\right]\\
\frac{d^2s}{d\tau^2}+\frac{2h}{1+hx}\frac{dx}{d\tau}\frac{ds}{d\tau}+\frac{h'x}{1+hx}\left(\frac{ds}{d\tau}\right)^2&=\frac{e}{m_0}\left[\frac{dt}{d\tau}\frac{E_s}{1+hx}-\frac{dx}{d\tau}\frac{B_z}{1+hx}+\frac{dz}{d\tau}\frac{B_x}{1+hx}\right]\\
\frac{d^2z}{d\tau^2}&=\frac{e}{m_0}\left[\frac{dt}{d\tau}E_z+\frac{dx}{d\tau}B_s-(1+hx)\frac{ds}{d\tau}B_x\right].
\end{aligned} \tag{33}$$

As before, we see from the first equation of (32) that if the electric field is zero then the energy E of the particle is constant. In this case we have

$$E = \gamma m_0 c^2 = m_0 c^2 \frac{\mathrm{d}t}{\mathrm{d}\tau} \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t} = \gamma\frac{d}{dt} = \gamma\frac{ds}{dt}\frac{d}{ds} = \gamma\dot{s}\frac{\mathrm{d}}{\mathrm{d}s} ,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = \gamma^2\dot{s}^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \gamma^2\ddot{s}\frac{\mathrm{d}}{\mathrm{d}s} , \tag{34}$$

so that Eqs. (33) may be simplified to

$$\begin{aligned} x'' + \frac{\ddot{s}}{\dot{s}^2}x' - h(1 + hx) &= \frac{v}{\dot{s}}\frac{e}{p}\left[(1 + hx)B_z - z'B_s\right] \\ \frac{\ddot{s}}{\dot{s}^2}(1 + hx) + 2hx' + h'x &= \frac{v}{\dot{s}}\frac{e}{p}\left[z'B_x - x'B_z\right] \\ z'' + \frac{\ddot{s}}{\dot{s}^2}z' &= \frac{v}{\dot{s}}\frac{e}{p}\left[x'B_s - (1 + hx)B_x\right] , \end{aligned} \tag{35}$$

where a prime denotes the derivative with respect to s and we have used the fact that $p = \gamma m_0 v = mv$, where m is the relativistic mass. The second equation gives

$$\frac{\ddot{s}}{\dot{s}^2} = -\frac{2hx' + h'x}{1 + hx} + \frac{v}{\dot{s}}\frac{e}{p}\left[\frac{z'}{1 + hx}B_x - \frac{x'}{1 + hx}B_z\right] . \tag{36}$$

From Eq. (28) it follows that

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\dot{x}^2 + (1 + hx)^2\dot{s}^2 + \dot{z}^2} = \dot{s}\sqrt{x'^2 + (1 + hx)^2 + z'^2} , \tag{37}$$

where we have used

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} = \dot{s}\frac{\mathrm{d}}{\mathrm{d}s} , \tag{38}$$

so that

$$\frac{v}{\dot{s}} = \sqrt{x'^2 + (1 + hx)^2 + z'^2} . \tag{39}$$

The motion along s can then be described by the two transverse coordinates x and z. Using Eqs. (35), (36), and (39) we get

$$\begin{aligned} x'' - \frac{x'}{1 + hx}(2hx' + h'x) - h(1 + hx) &= \frac{e}{p}\sqrt{x'^2 + (1 + hx)^2 + z'^2} \\ &\quad \times\left\{(1 + hx)\left[1 + \frac{x'^2}{(1 + hx)^2}\right]B_z - z'B_s - \frac{x'z'}{1 + hx}B_x\right\} \\ z'' - \frac{z'}{1 + hx}(2hx' + h'x) &= \frac{e}{p}\sqrt{x'^2 + (1 + hx)^2 + z'^2} \\ &\quad \times\left\{x'B_s - (1 + hx)\left[1 + \frac{z'^2}{(1 + hx)^2}\right]B_x + \frac{x'z'}{1 + hx}B_z\right\} . \end{aligned} \tag{40}$$

The number of degrees of freedom is two since the energy E is conserved and from

$$E = \sqrt{(pc)^2 + (m_0c^2)^2} \tag{41}$$

the total momentum is also conserved. In accelerator literature [BROWN85B, STEFF85, KOLOM66] one normally finds a different equation from (35) for the longitudinal motion. It is derived from Eq. (39) and written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{v}{\dot{s}}\right)^2 = \frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{\dot{s}}\right)^2 = \frac{1}{2}\frac{1}{\dot{s}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{\dot{s}}\right)^2 = -\left(\frac{v}{\dot{s}}\right)^2\frac{\ddot{s}}{\dot{s}^2} = x'x'' + (1 + hx)(hx' + h'x) + z'z'' \tag{42}$$

since v is constant. This equation for s may be verified by solving for x'' and z'' in (35), using this in (42), and comparing the result with the second equation there.

2.3 Expansion of the equations of motion

If we use the expansions

$$\begin{aligned} \frac{1}{1+\alpha} &= 1 - \alpha + \alpha^2 - \alpha^3 + O(4) \\ \sqrt{1+\alpha} &= 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2 + \frac{1}{16}\alpha^3 + O(4) \end{aligned} \tag{43}$$

in Eqs. (40) and only keep terms to second power in the coordinates we find

$$\begin{aligned} &x'' - x'(2hx' + h'x) - h(1 + hx) \\ &\quad = \frac{e}{p}\left[\left(1 + 2hx + h^2x^2 + \frac{3}{2}x'^2 + \frac{1}{2}z'^2\right)B_z - (1 + hx)z'B_s - x'z'B_x\right] + O(3) \\ &z'' - z'(2hx' + h'x) \\ &\quad = \frac{e}{p}\left[(1 + hx)x'B_s - \left(1 + 2hx + h^2x^2 + \frac{1}{2}x'^2 + \frac{3}{2}z'^2\right)B_x + x'z'B_z\right] + O(3) \ . \end{aligned} \tag{44}$$

If we compare with the equations (2.6) derived by Brown and Servranckx, [BROWN85B], we find that they differ by some field terms on the right-hand side, and also on the left-hand side where we have a term $-2hx'^2$ instead of $-hx'^2$ in the horizontal plane and $-2hx'z'$ instead of $-hx'z'$ in the vertical plane. This apparent difference is due to the fact that the fields have not yet been expanded to second power in the coordinates. When this is done, we will see, in the following, that the difference is only in higher order.

We now choose a trajectory for a particle with some momentum p_0 as the reference curve (e.g. the closed orbit for a particle with reference momentum in a circular machine). We then define the momentum deviation δ by

$$\delta \equiv \frac{p - p_0}{p_0} \ , \tag{45}$$

where p is the momentum for an arbitrary particle. The curvature $h(s)$ is then given by the vertical field $B_z(s) = -h(s)p_0/e$. Since the reference curve was assumed to lie in a horizontal plane it follows that B_x is zero at this curve. The difference from Brown is then only in the terms $x'^2\delta/(1 + \delta)$ and

$x'z'\delta/(1 + \delta)$, in the horizontal and vertical planes respectively. Since these terms are of higher (third) power in the variables the equations agree with Brown's expanded equations.

Chapter 3

HAMILTONIAN FORMALISM

There are two ways of carrying out a Hamiltonian formulation for a relativistic particle in an external electromagnetic field [GOLDS80, BARUT80]. Either one works in a specific Lorentz frame (non-covariant formulation) or one attempts to make a fully covariant description. For accelerators the first method has normally been used. A Hamiltonian expanded to third power in the coordinates for fixed curvature has been derived by Bell [BELL55]. It is extended in this chapter in order to allow for varying curvature. This Hamiltonian will therefore lead to the same differential equations of motion as derived in the previous chapter or by Brown and Servranckx [BROWN85B].

A Hamiltonian for the second approach can be found in Appendix A

3.1 Hamiltonian for a specific Lorentz frame

The Hamiltonian for a particle in an external electromagnetic field is given by Barut [BARUT80]:

$$H(\overline{x},\overline{p};t) = e\Phi(\overline{x};t) + c\sqrt{[\overline{p} - q\overline{A}(\overline{x};t)]^2 + m^2_{\ 0}c^2}\ , \tag{46}$$

where $\overline{A}$ is the vector potential, Φ is the scalar potential for the external electromagnetic field,

$$\begin{aligned} \overline{B} &= \nabla\times\overline{A} \\ \overline{E} &= -\frac{\partial\overline{A}}{\partial t} - \nabla\Phi\ , \end{aligned} \tag{47}$$

and $\overline{p}$ the conjugate momenta. Hamilton's equations are

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= \frac{\partial H}{\partial p_i} \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\frac{\partial H}{\partial x_i}\ , \end{aligned} \tag{48}$$

lead to the Lorentz force law (23) [COURA84, BELL85].

3.2 Hamiltonian for an accelerator

The Hamiltonian for the curvilinear system is obtained by a canonical transformation with the generating function [COURA58]:

$$F_3(\bar{r},\bar{p}) = -\bar{p}\cdot\bar{r} = -\bar{p}\cdot[\bar{R}(S) + X\cdot\hat{x}(S) + Z\cdot\hat{z}(S)]\,, \tag{49}$$

where $R(s)$ is the reference curve,

$$\begin{aligned}
&P_x = -\frac{\partial F_3}{\partial X} = \bar{p}\cdot\hat{x}\,, \quad && P_s = -\frac{\partial F_3}{\partial S} = \bar{p}\cdot\hat{s}(1 + hx)\,, \quad && P_z = -\frac{\partial F_3}{\partial Z} = \bar{p}\cdot\hat{z}\,,\\
&x = -\frac{\partial F_3}{\partial p_x} = X\,, && s = -\frac{\partial F_3}{\partial p_s} = S\,, && z = -\frac{\partial F_3}{\partial p_z} = Z\,,\\
&H_1 = H\,.
\end{aligned} \tag{50}$$

The old Hamiltonian may now be transformed and if we again use small letters for the new canonical variables we have

$$H_1(x,s,z,p_x,p_s,p_z;t) = e\Phi + c\sqrt{m_0c^2 + (p_x - eA_x)^2 + \left(\frac{p_s}{1 + hx} - eA_s\right)^2 + (p_z - eA_z)^2}\,, \tag{51}$$

where A_x, A_s, A_z are the components of the vector potential in the curvilinear system[5]. If Φ and A_x, A_s, A_z are time independent, then H_1 is a constant of motion which we may identify as the energy E [e.g. for a coasting beam or in the adiabatic approximation for the longitudinal motion, which can be done when the longitudinal oscillation (synchrotron oscillation) is much slower than the betatron oscillation]. To change from t to s as the independent variable, we take $-p_s$ as a new Hamiltonian [ANDER67, BELL55, COURA84, COURA58, BELL85]. If we take $\Phi = 0$ we get

$$H_1 = E = \sqrt{(pc)^2 + (m_0c^2)^2}\,, \tag{52}$$

where p is the total momentum.

The new Hamiltonian is then

$$H_2(x,z,p_x,p_z;s) = -p_s = -(1 + hx)[eA_s + \sqrt{p^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2}]\,. \tag{53}$$

Hamilton's equations give

---

[5] They are defined by $\bar{A} = A_x\hat{x} + A_s\hat{s} + A_z\hat{z}$, and p_x, p_s, p_z are in fact the covariant components of the vector $\bar{p}$ in the curvilinear system.

$$x' = \frac{dx}{ds} = \frac{\partial H_2}{\partial p_x} = \frac{(1 + hx)(p_x - eA_x)}{\sqrt{\quad}}$$

$$z' = \frac{dz}{ds} = \frac{\partial H_2}{\partial p_z} = \frac{(1 + hx)(p_z - eA_z)}{\sqrt{\quad}}$$

$$p_x' = \frac{dp_x}{ds} = -\frac{\partial H_2}{\partial x} = (1 + hx)e\frac{\partial A_s}{\partial x} + heA_s + h\sqrt{\quad}$$

$$+ (1 + hx)\frac{(p_x - eA_x)e\dfrac{\partial A_x}{\partial x} + (p_z - eA_z)e\dfrac{\partial A_z}{\partial x}}{\sqrt{\quad}} \tag{54}$$

$$p_z' = \frac{dp_z}{ds} = -\frac{\partial H_2}{\partial x} = (1 + hx)e\frac{\partial A_s}{\partial z} + (1 + hx)\frac{(p_x - eA_x)e\dfrac{\partial A_x}{\partial z} + (p_z - eA_z)e\dfrac{\partial A_z}{\partial z}}{\sqrt{\quad}}$$

$$\frac{1}{\dot{s}} = \frac{dt}{ds} = \frac{\partial H_2}{\partial(-H_1)} = -\frac{\partial p}{\partial H_1}\frac{\partial H_2}{\partial p} = -\frac{E}{pc^2}\frac{\partial H_2}{\partial p} = \frac{(1 + hx)E}{c^2\sqrt{\quad}} = \frac{m(1 + hx)}{\sqrt{\quad}}$$

$$\frac{d(-H_1)}{ds} = -\frac{\partial H_2}{\partial t} = 0\,,$$

where

$$\sqrt{\quad} \equiv \sqrt{p^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2}\,. \tag{55}$$

From these equations it follows that

$$\sqrt{x'^2 + (1 + hx)^2 + z'^2} = \frac{p(1 + hx)}{\sqrt{\quad}} = \frac{p}{m\dot{s}} = \frac{mv}{m\dot{s}} = \frac{v}{\dot{s}}\,. \tag{56}$$

We may solve for p_x and p_z in the first two equations of (54) and then take the derivative with respect to s:

$$\begin{aligned} \frac{d}{ds}p_x &= \frac{d}{ds}(\frac{\dot{s}}{v}x' + eA_x) = \frac{\dot{s}}{v}(x'' + \frac{s}{\dot{s}^2}x') + eA'_x \\ \frac{d}{ds}p_z &= \frac{d}{ds}(\frac{\dot{s}}{v}z' + eA_z) = \frac{\dot{s}}{v}(z'' + \frac{s}{\dot{s}^2}z') + eA'_z\,, \end{aligned} \tag{57}$$

where we have used

$$\frac{d}{ds}\dot{s} = \frac{dt}{ds}\frac{d}{dt}\dot{s} = \frac{\ddot{s}}{\dot{s}}\,. \tag{58}$$

Since we assumed that $\partial A_x/\partial t = \partial A_z/\partial t = 0$ we find

$$\begin{aligned} A_x' &= \frac{dA_x}{ds} = x'\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial s} + z'\frac{\partial A_x}{\partial z} \\ A_z' &= \frac{dA_z}{ds} = x'\frac{\partial A_z}{\partial z} + \frac{\partial A_z}{\partial s} + z'\frac{\partial A_z}{\partial z}\,. \end{aligned} \tag{59}$$

Combining Eqs. (54), (56), (58), and (59) gives

$$
\begin{aligned}
x'' + \frac{\ddot{s}}{\dot{s}^2}x' - h(1 + hx) &= \frac{v}{\dot{s}}\frac{e}{p}\left[hA_s + (1 + hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s} - z'\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] \\
z'' + \frac{\ddot{s}}{\dot{s}^2}z' &= \frac{v}{\dot{s}}\frac{e}{p}\left[-\frac{\partial A_z}{\partial s} + (1 + hx)\frac{\partial A_s}{\partial z} + x'\frac{\partial A_x}{\partial z} - x'\frac{\partial A_z}{\partial x}\right],
\end{aligned} \tag{60}
$$

where $\ddot{s}$ is calculated by taking the derivative of the fifth equation in (54)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\dot{s} = \frac{\ddot{s}}{\dot{s}} &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\sqrt{}}{m(1 + hx)} \\
&= -(h'x + hx')\frac{\sqrt{}}{m(1 + hx)^2} - \frac{(p_x - eA_x)(p'_x - eA'_x) + (p_z - eA_z)(p'_z - eA'_z)}{m(1 + hx)\sqrt{}} \\
&= -\dot{s}\frac{2hx' + h'x}{1 + hx} - \frac{e}{m}\frac{1}{(1 + hx)^2}\left\{x'\left[hA_s + (1 + hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right]\right. \\
&\quad \left. - z'\left[\frac{\partial A_z}{\partial x} - (1 + hx)\frac{\partial A_s}{\partial z}\right]\right\}.
\end{aligned} \tag{61}
$$

Since

$$
\overline{B} = \nabla\times\overline{A}\ , \tag{62}
$$

and in an orthogonal coordinate system [WEINB72]

$$
(\nabla\times\overline{A})_i = \frac{h_i}{h_1h_2h_3}\sum_{jk}\varepsilon^{ijk}\frac{\partial}{\partial x^j}h_kA_k\ , \tag{63}
$$

where h_i are the diagonal elements of $g_{\mu\nu}$ in (30), we have

$$
\begin{aligned}
B_x &= \frac{1}{1 + hx}\frac{\partial A_z}{\partial s} - \frac{\partial A_s}{\partial z} \\
B_s &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\
B_z &= \frac{h}{1 + hx}A_s + \frac{\partial A_s}{\partial x} - \frac{1}{1 + hx}\frac{\partial A_x}{\partial s}\ .
\end{aligned} \tag{64}
$$

The equations of motion become

$$
\begin{aligned}
x'' + \frac{\ddot{s}}{\dot{s}^2}x' - h(1 + hx) &= \frac{e}{p}\frac{v}{\dot{s}}[(1 + hx)B_z - z'B_s] \\
\frac{\ddot{s}}{\dot{s}^2}(1 + hx) + 2hx' + h'x &= \frac{e}{p}\frac{v}{\dot{s}}[z'B_x - x'B_z] \\
z'' + \frac{\ddot{s}}{\dot{s}^2}z' &= \frac{e}{p}\frac{v}{\dot{s}}[x'B_s - (1 + hx)B_x]
\end{aligned} \tag{65}
$$

in agreement with Eqs. (40).

3.3 Expansion of the Hamiltonian

To apply perturbation theory, one has to divide the Hamiltonian into two parts:

$$H \equiv H_0 + V\,, \tag{66}$$

where H_0 is the part of the Hamiltonian for which the equations of motion can be solved. To do this for the Hamiltonian (53) one normally expands the square root. We expand in powers of deviations from a reference orbit x, p_x, z, p_z, and δ, defined by a particle with reference momentum p_0. The Hamiltonian (53) may now be written as

$$H_3(x,z,p_x,p_z;s) = -\,p(1 + hx)\left[\frac{e}{p}A_s + \sqrt{1 - \frac{(p_x - eA_x)^2}{p^2} - \frac{(p_x - eA_z)^2}{p^2}}\,\right]. \tag{67}$$

Scaling of the Hamiltonian with p gives

$$H_4 = \frac{H_3}{p}\,, \quad P_x = \frac{p_x}{p}\,, \quad P_z = \frac{p_z}{p}\,. \tag{68}$$

For simplicity of notation we use small letters for the new canonical variables. We get

$$H_4(x,z,p_x,p_z;s) = -(1 + hx)\left[\frac{e}{p}A_s + \sqrt{1 + \left(p_x - \frac{e}{p}A_x\right)^2 - \left(p_z - \frac{e}{p}A_z\right)^2}\,\right]. \tag{69}$$

The square root may be expanded as

$$\sqrt{1+\alpha} = 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2 + \frac{1}{16}\alpha^3 + O(4)\,. \tag{70}$$

The Hamiltonian to third power is then

$$\begin{aligned} H_4(x,z,p_x,p_z;s) = -\,(1 + hx)\Bigg[&\frac{e}{p_0(1+\delta)}A_s + 1 - \frac{1}{2}\left(p_x - \frac{e}{p_0(1+\delta)}A_x\right)^2 \\ &- \frac{1}{2}\left(p_z - \frac{e}{p_0(1+\delta)}A_z\right)^2\Bigg] + O(4)\,. \end{aligned} \tag{71}$$

Hamilton's equations now give

$$
\begin{aligned}
x' &= \frac{\partial H_4}{\partial p_x} = (1 + hx)\left(p_x - \frac{e}{p_0(1+\delta)}A_x\right) \\
z' &= \frac{\partial H_4}{\partial p_z} = (1 + hx)\left(p_z - \frac{e}{p_0(1+\delta)}A_z\right) \\
p_x' &= -\frac{\partial H_4}{\partial x} = h\left[\frac{e}{p_0(1+\delta)}A_s + 1 - \frac{1}{2}\left(p_x - \frac{e}{p_0(1+\delta)}A_x\right)^2 - \left(p_z - \frac{e}{p_0(1+\delta)}A_z\right)^2\right] \\
&\quad + (1+hx)\frac{e}{p_0(1+\delta)}\left[\frac{\partial A_s}{\partial x} + \left(p_x - \frac{e}{p_0(1+\delta)}A_x\right)\frac{\partial A_x}{\partial x}\right. \\
&\quad \left. + \left(p_z - \frac{e}{p_0(1+\delta)}A_z\right)\frac{\partial A_z}{\partial x}\right] \\
p'_z &= -\frac{\partial H_4}{\partial z} = (1+hx)\frac{e}{p_0(1+\delta)}\left[\frac{\partial A_s}{\partial z} + \left(p_x - \frac{e}{p_0(1+\delta)}A_x\right)\frac{\partial A_x}{\partial z}\right. \\
&\quad \left. + \left(p_z - \frac{e}{p_0(1+\delta)}A_z\right)\frac{\partial A_z}{\partial z}\right] \\
\frac{1}{\dot{s}} &= \frac{\partial(pH_4)}{\partial(-H_1)} - \frac{\partial p}{\partial H_1}\frac{\partial pH_4}{\partial p} = -\frac{E}{pc^2}\frac{\partial(pH_4)}{\partial p} = \\
&= \frac{1}{v}(1+hx)\left[1 + \frac{1}{2}\left(p_x - \frac{e}{p_0(1+\delta)}A_x\right)^2 + \frac{1}{2}\left(p_z - \frac{e}{p_0(1+\delta)}A_z\right)^2\right] \\
\frac{\mathrm{d}(-H_1)}{\mathrm{d}s} &= -\frac{\partial(pH_4)}{\partial t} = 0\,.
\end{aligned}
\tag{72}
$$

In the last two equations the original conjugate momenta given by Eqs. (68) have to be inserted before taking the derivatives after which they are resubstituted. Since the Hamiltonian is expanded to third power, the equations of motion are correct to second power.

Solving for p_x and p_z in Eqs. (72), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}p_x &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{x'}{1+hx} + \frac{e}{p_0(1+\delta)}A_x\right] = \frac{1}{1+hx}\left[x'' - \frac{(h'x + hx')x'}{1+hx} + \frac{1+hx}{1+\delta}\frac{e}{p_0}A_x'\right] \\
\frac{d}{ds}p_z &= \frac{d}{ds}\left[\frac{z'}{1+hx} + \frac{e}{p_0(1+\delta)}A_z\right] = \frac{1}{1+hx}\left[z'' - \frac{(h'x + hx')z'}{1+hx} + \frac{1+hx}{1+\delta}\frac{e}{p_0}A_z'\right].
\end{aligned}
\tag{73}
$$

Putting this equal to p_x' and p_z' in (72), by using Eqs. (59) and only keeping terms to second power in the canonical variables, we find

$$
\begin{aligned}
x'' - x'(h'x + hx') &= h(1+hx)\left[1 - \frac{1}{2}x'^2 - \frac{1}{2}z'^2\right] + (1 - \delta + \delta^2)(1+hx)\frac{e}{p_0} \\
&\quad \times\left[hA_s + (1+hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s} + z'\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \\
z'' - z'(h'x + hx') &= (1 - \delta + \delta^2)\frac{e}{p_0}\left[(1+hx)\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} + x'\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right],
\end{aligned}
\tag{74}
$$

where we have used Eqs. (43). By using Eqs. (64) we finally obtain

$$x'' - h'xx' - \frac{h}{2}(x'^2 - z'^2) - h(1 - hx)$$

$$= (1 - \delta + \delta^2)\frac{e}{p_0}\Big[(1 + 2hx + h^2x^2)B_z - (1 + hx)z'B_s\Big] + O(3) \tag{75}$$

$$z'' - z'(h'x + hx') = (1 - \delta + \delta^2)\frac{e}{p_0}\Big[(1 + hx)x'B_s - (1 + 2hx + h^2x^2)B_x\Big] + O(3)\ .$$

This differs slightly from Eqs. (44) but, by again taking into account that h is given by $B_z = -hp/q$, and that the reference curve was assumed to lie in a horizontal plane so that B_x is zero, it is seen that the difference is only in terms of higher order.

Chapter 4

THE EXPLICIT HAMILTONIAN

In order to obtain the explicit Hamiltonian, we have to find expansions for the vector potential. These are derived in this chapter, from Steffen's field expansions valid in the case of mid-plane symmetry [STEFF85]. By combining the expansions obtained for the vector potential with the Hamiltonian derived in Section 3.1 we get an explicit Hamiltonian to third power in the canonical variables with a curvature that may vary with s.

4.1 Expansions of the magnetic field

We assume that $B_x = B_s = 0$ in the symmetry plane z = 0. The fields can then be expanded in the local curvilinear system as [STEFF85]

$$\begin{aligned}
\frac{e}{p_0}B_x(s) &= k(s)z + m(s)xz + O(3) \\
\frac{e}{p_0}B_s(s) &= -\,h'(s)z + O(2) \\
\frac{e}{p_0}B_z(s) &= -\,h(s) + k(s)x + \frac{1}{2}m(s)x^2 - \frac{1}{2}[m(s) + h(s)k(s) - h''(s)]z^2 + O(3)\,,
\end{aligned} \tag{76}$$

where

$$\begin{aligned}
h &= -\frac{e}{p_0}B_z\,, && \text{dipole} \\
k &= \frac{e}{p_0}\frac{\partial B_z}{\partial x} = \frac{e}{p_0}\frac{\partial B_x}{\partial z}\,, && \text{quadrupole} \\
m &= \frac{e}{p_0}\frac{\partial^2 B_z}{\partial x^2}\,, && \text{sextupole}
\end{aligned} \tag{77}$$

Note that we have changed the sign of $h(s)$ compared to Steffen [STEFF85] and that the field derivatives are defined with respect to the curvilinear system.

4.2 Expansions of the vector potential

The field expansions in Eqs. (76) can be derived from the following vector potential:

$$
\begin{aligned}
\frac{e}{p_0}A_x(s) &= -\frac{1}{2}h'(s)[1 + h(s)x]z^2 + O(3) \\
\frac{e}{p_0}A_s(s) &= [1 + h(s)x]\Big\{ - h(s)x + \frac{1}{2}[k(s) + 3h^2(s)]x^2 - \frac{1}{2}k(s)z^2 + \\
&\qquad \times \frac{1}{6}[m(s) - 4h(s)k(s) - 12h^3(s)]x^3 - \frac{1}{2}[m(s) - h(s)k(s)]xz^2\Big\} + O(4) \\
\frac{e}{p_0}A_z(s) &= 0 + O(3)
\end{aligned}
\tag{78}
$$

in some arbitrary gauge, which simplifies to

$$
\begin{aligned}
\frac{e}{p_0}A_x(s) &= -\frac{1}{2}h'(s)z^2 + O(3) \\
\frac{e}{p_0}A_s(s) &= -h(s)x + \frac{1}{2}[k(s) + h^2(s)]x^2 - \frac{1}{2}k(s)z^2 \\
&\qquad + \frac{1}{6}[m(s) - h(s)k(s) - 3h^3(s)]x^3 - \frac{1}{2}m(s)xz^2 + O(4) \\
\frac{e}{p_0}A_z(s) &= 0 + O(3)
\end{aligned}
\tag{79}
$$

and can be verified by substitution in Eqs. (64).

By using the Hamiltonian (71) expanded to third power in the canonical variables and the expansions (79) of the vector potential, we obtain a Hamiltonian of the form

$$
\begin{aligned}
H_4(x,z,p_x,p_z;s) = &- \delta(1-\delta)hx + \frac{1}{2}(1-\delta)(h^2-k)x^2 + \frac{1}{2}(1-\delta)kz^2 - \frac{1}{6}(m+2hk)x^3 \\
&+ \frac{1}{2}(m+hk)xz^2 + \frac{1}{2}p_x^2 + \frac{1}{2}h'z^2p_x + \frac{1}{2}p_z^2 + \frac{1}{2}hxp_x^2 + \frac{1}{2}hxp_z^2 + O(4)\,.
\end{aligned}
\tag{80}
$$

The part describing the linear motion is

$$
H_0 \equiv -\delta hx + \frac{1}{2}(h^2-k)x^2 + \frac{1}{2}kz^2 + \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 \tag{81}
$$

so that

$$
H_4 \equiv H_0 + V_4\,, \tag{82}
$$

where

$$
\begin{aligned}
V_4 \equiv \delta^2hx &- \frac{1}{2}\delta(h^2-k)x^2 - \frac{1}{2}\delta kz^2 - \frac{1}{6}(m+2hk)x^3 + \frac{1}{2}(m+hk)xz^2 \\
&+ \frac{1}{2}h'z^2p_x + \frac{1}{2}hxp_x^2 + \frac{1}{2}hxp_z^2 + O(4)\,.
\end{aligned}
\tag{83}
$$

Hamilton's equations give

$$
\begin{aligned}
x' &= \frac{\partial H_4}{\partial p_x} = (1 + hx)p_x + \frac{1}{2}h'z^2 \\
z' &= \frac{\partial H_4}{\partial p_z} = (1 + hx)p_z \\
p_x' &= -\frac{\partial H_4}{\partial x} = \delta(1 - \delta)h - (1 - \delta)(h^2 - k)x + \frac{1}{2}(m + 2hk)x^2 \\
&\qquad - \frac{1}{2}(m + hk)z^2 - \frac{1}{2}hp_x^2 - \frac{1}{2}hp_z^2 \\
p_z' &= -\frac{\partial H_4}{\partial z} = -(1 - \delta)kz - (m + hk)xz - h'zp_x \,.
\end{aligned}
\tag{84}
$$

To eliminate p_x and p_z first find

$$
\begin{aligned}
p_x &= \frac{x' - \frac{1}{2}h'z^2}{1 + hx} \\
p_z &= \frac{z'}{1 + hx} \,.
\end{aligned}
\tag{85}
$$

Taking the derivative with respect to s gives

$$
\begin{aligned}
p_x' &= \frac{x'' - \frac{1}{2}h''z^2 - h'zz'}{1 + hx} - \frac{(h'x + hx')(x' - \frac{1}{2}h'z^2)}{(1 + hx)^2} \\
p_z' &= \frac{z''}{1 + hx} - \frac{(h'x + hx')z'}{(1 + hx)^2} \,.
\end{aligned}
\tag{86}
$$

If we put this equal to p_x', p_z' in Eqs. (84) and only keep terms to second power in the canonical variables we find

$$
\begin{aligned}
x'' - h'xx' - \frac{1}{2}hx'^2 - h'zz' + \frac{1}{2}hz'^2 & \\
&= \delta h - \delta^2 h - (h^2 - k)x + \delta(2h^2 - k)x + \frac{1}{2}(m + 4hk - 2h^3)x^2 \\
&\quad - \frac{1}{2}(m + hk - h'')z^2 + O(3) \\
z'' - h'xz' - hx'z' + h'x'z &= -kz + \delta kz - (m + 2hk)xz + O(3)
\end{aligned}
\tag{87}
$$

Chapter 5

THE LINEAR EQUATIONS OF MOTION

Before going further into the study of the non-linear effects, it is useful to discuss the linear equations of motion and their solutions. The linear dispersion function is introduced to obtain a homogeneous differential equation for the horizontal motion. We also introduce the action-angle variables. This chapter follows closely preceding works [COURA58, STEFF85].

5.1 The linear dispersion function

The Hamiltonian for the linear equtions of motion H_0 is given by Eq. (81). Hamilton's equations are then

$$\begin{aligned} x' &= \frac{\partial H_0}{\partial p_x} = p_x \,, \quad p_x{}' = -\frac{\partial H_0}{\partial x} = \delta h - (h^2 - k)x \,, \\ z' &= \frac{\partial H_0}{\partial p_z} = p_z \,, \quad p_z{}' = -\frac{\partial H_0}{\partial z} = -kx \,. \end{aligned} \tag{88}$$

Elimination of p_x, p_z leads to the linear equations of motion

$$\begin{aligned} &x'' + [h^2(s) - k(s)]x = \delta h \\ &z'' + k(s)z = 0 \,. \end{aligned} \tag{89}$$

Since the differential equation for the horizontal motion is inhomogeneous, the general solution is a linear combination of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation.

It is customary to introduce a linear dispersion function for the particular solution in the form [STEFF85]:

$$\begin{aligned} x(s) &= \delta D_0(s) \\ z(s) &= 0 \,. \end{aligned} \tag{90}$$

If this is substituted in Eqs. (89) we find

$$D_0'' + [h^2(s) - k(s)]D_0 = h(s) \,. \tag{91}$$

The linear dispersion function is a solution of this differential equation. For a circular accelerator it is the periodic solution (closed orbit). This orbit is put in the centre of phase space by a canonical transformation generated by [COURA84, RUTH86]:

$$F_2(x,P_x;s) = [x - x_{ref}(s)][P_x + p_{ref}(s)] \,. \tag{92}$$

$p_{ref}(s)$ is obtained from Eqs. (88) and (90) as

$$p_{ref}(s) = x_{ref}' = \delta D_0'(s) \,. \tag{93}$$

The generating function is then

$$F_2(x,P_x;s) = [x - \delta D_0(s)][P_x + \delta D_0'(s)] \tag{94}$$

and the canonical transformation is

$$\begin{aligned} p_x &= \frac{\partial F_2}{\partial x} = P_x + \delta D_0' \\ X &= \frac{\partial F_2}{\partial P_x} = x - \delta D_0 \\ H_{0\beta} &= H_0 + \frac{\partial F_2}{\partial s} \end{aligned} \tag{95}$$

or

$$\begin{aligned} x &= X + \delta D_0 \\ p_x &= P_x + \delta D_0' \\ H_{0\beta} &= H_0 + \frac{\partial F_2}{\partial s} \,. \end{aligned} \tag{96}$$

We find for the new Hamiltonian

$$H_{0\beta} = \frac{1}{2}(h^2 - k)x^2 + \frac{1}{2}kz^2 + \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 \,, \tag{97}$$

where we have used small letters for the new canonical variables. Hamilton's equations are then

$$\begin{aligned} x' &= \frac{\partial H_{0\beta}}{\partial p_x} = p_x \,, \quad p_x' = -\frac{\partial H_{0\beta}}{\partial x} = -(h^2 - k)x \,, \\ z' &= \frac{\partial H_{0\beta}}{\partial p_z} = p_z \,, \quad p_z' = -\frac{\partial H_{0\beta}}{\partial z} = kz \,, \end{aligned} \tag{98}$$

which give

$$\begin{aligned} &x'' + [h^2(s) - k(s)]x = 0 \\ &z'' + k(s)z = 0 \,. \end{aligned} \tag{99}$$

5.2 The betatron motion

The equations of motion (99) are of the general form

$$x'' + K(s)x = 0 \tag{100}$$

and describe an oscillatory motion (betatron motion[6]) along a reference curve with a variable restoring force $K(s)$. $K(s)$ is periodic for a circular accelerator:

$$K(s + C) = K(s)\,, \tag{101}$$

where C is the circumference of the accelerator. Equation (100) can then be recognized as Hill's equation[7]. The general solution of Eq. (100) can be written as [STEFF85, COURA58]

$$\begin{aligned} x(s) &= C(s)x_0 + S(s)x_0' \\ x'(s) &= C'(s)x_0 + S'(s)x_0'\,, \end{aligned} \tag{102}$$

where $C(s)$ and $S(s)$ are two independent solutions of the homogeneous equation with the initial conditions

$$\begin{pmatrix} C_0 & S_0 \\ C_0' & S_0' \end{pmatrix} \cdot = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{103}$$

Equations (102) can be written in matrix form:

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s|s_0)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}. \tag{104}$$

M is refered to as the transfer matrix.

A phase-amplitude form of the solution can be obtained by the ansatz[8] [STEFF85, COURA58]

$$x(s) = \mathrm{Re}\left(\sqrt{\beta_x(s)}\,e^{i\psi_x(s)}\right). \tag{105}$$

$\beta(s)$ is referred to as the beta function and $\psi(s)$ as the betatron phase. Substitution in Eq. (100) leads to

$$(\sqrt{\beta_x})'' + K(s)\sqrt{\beta_x} - \sqrt{\beta}\psi_x'^2 = 0 \tag{106}$$

---

[6] From the motion in a betatron first analysed by Kerst and Serber [KERST41] although the equations of motion in that case were of a harmonic oscillator type.

[7] From the astronomer George William Hill (1838 – 1914), who studied it in relation to the "main problem of the lunar theory", i.e. the problem of finding the moon's motion under the gravitational attraction of the earth and the sun, all bodies treated as point masses [BROUV61].

[8] This may be compared with the WKB approximation in quantum mechanics for a harmonic oscillator with a time-dependent restoring force.

and

$$\frac{\psi_x''}{\psi_x'} = -\frac{\beta_x'}{\beta_x} . \tag{107}$$

This equation can be integrated twice, which gives

$$\psi_x(s) = \int_{s_0}^{s} \frac{d\tau}{\beta_x(\tau)} + \phi_x \tag{108}$$

since the first integration constant can be chosen to be 1. ϕ_x is a constant given by the initial conditions. The difference in phase between two locations in the lattice is called the phase advance and is denoted by μ. It follows that

$$(\sqrt{\beta_x})'' + K(s)\sqrt{\beta_x} - \frac{1}{\beta_x^{3/2}} = 0 . \tag{109}$$

One normally uses another differential equation for β which can be obtained by taking the derivative with respect to s in Eq. (109):

$$\beta_x''' + 4K(s)\beta_x' + 2K'(s)\beta_x = 0 . \tag{110}$$

In the case of a piece-wise constant K we obtain by integrating once

$$\beta_x'' + 4K\beta_x = \text{const.} \tag{111}$$

For a circular accelerator $\beta_x(s)$ is the periodic solution of this equation. In this case the number of betatron oscillation wavelengths in one revolution (the tune) is

$$Q_x \equiv \frac{1}{2\pi} \int_{s}^{s+C} \frac{d\tau}{\beta_x(\tau)} . \tag{112}$$

The general solution of Eq. (99) is

$$x(s) = \sqrt{2J_x\beta_x(s)} \cos\psi_x(s) , \tag{113}$$

where J_x is a constant given by the initial conditions. This solution describes an ellipse in the phase space [STEFF85, COURA58]. The area of the ellipse is

$$E = \pi 2J . \tag{114}$$

The ellipse can be converted into a circle by the linear transformation

$$\tilde{x} = \frac{x}{\sqrt{\beta_x(s)}}$$
$$\tilde{x}' = \frac{1}{\sqrt{\beta_x(s)}}(\alpha_x(s)x + \beta_x(s)x')\,. \tag{115}$$

These coordinates are called the normalized coordinates and define the normalized phase space.

Since J is a constant of motion, different for each particle with given initial conditions, it can be used to characterize a beam of particles. The emittance ε of a beam is the area that includes the phase-space trajectories for some percentage (e.g. 95%) of the particles.

5.3 Action-angle variables for the linear equations of motion

The transformation to action – angle variables can be done by a canonical transformation with the generating function [COURA84, RUTH86]:

$$F_1(x,z,\psi_x,\psi_z;s) = -\frac{x^2}{2\beta_x(s)}[\tan\psi_x + \alpha_x(s)] - \frac{z^2}{2\beta_z(s)}[\tan\psi_z + \alpha_z(s)]\,, \tag{116}$$

where

$$\alpha_x(s) \equiv -\frac{1}{2}\beta_x'(s)\,, \tag{117}$$

so that

$$p_x = \frac{\partial F_1}{\partial x} = -\frac{x}{\beta_x(s)}[\tan\psi_x + \alpha_x(s)]$$
$$J_x = -\frac{\partial F_1}{\partial \psi_x} = \frac{x^2}{2\beta_x(s)\cos^2\psi_x(s)} \tag{118}$$
$$K_{0\beta} = H_{0\beta} + \frac{\partial F_1}{\partial s}$$

or

$$x = \sqrt{2J_x\beta_x(s)}\cos\psi_x(s)$$
$$p_x = -\sqrt{\frac{2J_x}{\beta_x(s)}}[\sin\psi_x(s) + \alpha_x(s)\cos\psi_x(s)]\,. \tag{119}$$

The new Hamiltonian is

$$K_{0\beta}(J_x,J_z,\psi_x,\psi_z) = \frac{J_x}{\beta_x(s)} + \frac{J_z}{\beta_z(s)}\,. \tag{120}$$

Hamilton's equations are

$$J_x' = -\frac{\partial K_{0\beta}}{\partial \psi_x} = 0\,, \quad \psi_x' = \frac{\partial K_0}{\partial J_x} = \frac{1}{\beta_x(s)}\,,$$
$$J_z' = -\frac{\partial K_{0\beta}}{\partial \psi_z} = 0\,, \quad \psi_z' = \frac{\partial K_0}{\partial J_z} = \frac{1}{\beta_z(s)}\,. \tag{121}$$

If these equations are integrated:

$$J_x = J_{0x}\,, \quad \psi_x(s) = \int_{s_0}^{s} \frac{d\tau}{\beta_x(\tau)} + \phi_{0x}\,,$$
$$J_z = J_{0z}\,, \quad \psi_z(s) = \int_{s_0}^{s} \frac{d\tau}{\beta_z(\tau)} + \phi_{0z}\,. \tag{122}$$

where J_{0x}, J_{0z}, ϕ_{0x}, and ϕ_{0z} are constants given by the initial conditions.

Chapter 6

THE NON-LINEAR EQUATIONS OF MOTION

In this chapter we will, by generalizing to the non-linear case, present the non-linear dispersion function corresponding to the equations of motion expanded to second power in the coordinates. Explicit solutions can be found elsewhere [DELAH86]. For higher-order calculations it may be interesting to use a Hamiltonian formulation. This is presented for the mentioned dispersion function. We also obtain the differential equations of motion for the non-linear betatron motion.

6.1 The non-linear dispersion function

The constant terms in the equation for the non-linear horizontal motion can be removed by seeking a solution of Eqs. (87) of the form

$$\begin{aligned} x(s) &= \delta D(s) \\ z(s) &= 0 \, , \end{aligned} \tag{123}$$

where $D(s)$ is the non-linear dispersion function. When this is substituted in Eqs. (87) we find

$$D'' - \delta h' D D' - \frac{1}{2}\delta h D'^2 = (1 - \delta)h - [h^2 - k - \delta(2h^2 - k)]D + \frac{1}{2}\delta(m + 4hk - 2h^3)D^2 \tag{124}$$

so the non-linear dispersion function is a solution of this differential equation. For a circular accelerator it should be the periodic solution (closed orbit). Note that $D(s)$ will depend on δ. This equation can be solved by Taylor expanding $D(s)$ with respect to δ [DELAH86]:

$$D(s) = D(s)\Big|_{\delta=0} + \delta \frac{\partial D(s)}{\partial \delta}\Big|_{\delta=0} + O(\delta^2) \equiv D_0(s) + \delta D_1(s) + O(\delta^2) \, . \tag{125}$$

If this expansion is substituted in Eq. (124) and equal powers of δ collected, we obtain

$$\begin{aligned} &D_0'' + (h^2 - k)D_0 = h \\ &D_1'' + (h^2 - k)D_1 = -h + (2h^2 - k + h'D_0')D_0 + \frac{1}{2}(m + 4hk - 2h^3)D_0^2 + \frac{1}{2}hD_0'^2 \, . \end{aligned} \tag{126}$$

These equations can be integrated analytically [DELAH86]. It is clear that D_0 is the linear dispersion function.

6.2 A Hamiltonian for the non-linear dispersion function

Alternatively, a Hamiltonian for this differential equation can be found by putting $z = 0, p_z = 0$ in Eq. (80) and making a scaling with δ so that

$$D = \frac{x}{\delta}\,, \quad P_D = \frac{p_x}{\delta}\,, \quad H_D = \frac{H_4}{\delta^2}\,, \tag{127}$$

which give

$$H_D(D,P_D;s) = -(1-\delta)hD + \frac{1}{2}(1-\delta)(h^2-k)D^2 - \frac{1}{6}\delta(m+2hk)D^3$$
$$+ \frac{1}{2}(1+\delta hD)P_D^2 + O(5)\,. \tag{128}$$

Hamilton's equations lead then to Eq. (124). Splitting the Hamiltonian as before:

$$H_D \equiv H_{0D} + V_D\,, \tag{129}$$

where

$$H_{0D}(D,P_D;s) = -hD + \frac{1}{2}(h^2-k)D + \frac{1}{2}P_D^2\,, \tag{130}$$

which gives the differential equation (91) for the linear dispersion function D_0 and

$$V_D = \delta hD - \frac{1}{2}\delta(h^2-k)D^2 - \frac{1}{6}\delta(m+2hk)D^3 + \frac{1}{2}\delta hDP_D^2 + O(5)\,. \tag{131}$$

D can now be found by applying some sort of Hamiltonian perturbation theory, based on the solution of H_{0D} which is the linear dispersion function D_0. D_1 is then obtained by first order perturbation theory.

6.3 The non-linear betatron motion

This orbit, given by the dispersion function, is put in the centre of phase space by a canonical transformation generated by Eq. (92). $p_{ref}(s)$ is obtained from Eqs. (84) and (123):

$$p_{ref}(s) = \frac{x'_{ref}}{1+hx_{ref}} = \frac{\delta D'}{1+\delta hD} = \delta D'(1-\delta hD) + O(3)\,. \tag{132}$$

The generating function can then be written

$$F_2(x,P_x) = (x-\delta D)(P_x + \delta D' - \delta^2 hDD')\,, \tag{133}$$

so that

$$
\begin{aligned}
p_x &= \frac{\partial F_2}{\partial x} = P_x + \delta D' - \delta^2 hDD' \\
X &= \frac{\partial F_2}{\partial P_x} = x - \delta D \\
H_{4\beta} &= H_4 + \frac{\partial F_2}{\partial s}
\end{aligned}
\tag{134}
$$

or

$$
\begin{aligned}
x &= X + \delta D \\
p_x &= P_x + \delta D' - \delta^2 hDD' \\
H_{4\beta} &= H_4 + \frac{\partial F_2}{\partial s} \; .
\end{aligned}
\tag{135}
$$

The transformed Hamiltonian is then

$$
\begin{aligned}
H_{4\beta} = {} & \frac{1}{2}\{h^2 - k - \delta[(m + 2hk)D + h^2 - k]\}x^2 \\
& + \frac{1}{2}\{k + \delta[(m + hk)D - k + h'D']\}z^2 - \frac{1}{6}(m + 2hk)x^3 + \frac{1}{2}(m + hk)xz^2 \\
& + \frac{1}{2}(1 + \delta hD)p_x^2 + \frac{1}{2}(1 + \delta hD)p_z^2 + \delta hD'xp_x + \frac{1}{2}h'z^2p_x + \frac{1}{2}hxp_x^2 + \frac{1}{2}hxp_z^2 + O(4) \; ,
\end{aligned}
\tag{136}
$$

where small letters have been used for the new canonical variables. Splitting the Hamiltonian gives

$$
H_{4\beta} = H_{0\beta} + V_{4\beta} \tag{137}
$$

and

$$
\begin{aligned}
V_{4\beta} = {} & -\frac{1}{2}\delta[(m + 2hk)D + h^2 - k]x^2 + \frac{1}{2}\delta[(m + hk)D - k + h'D']z^2 \\
& - \frac{1}{6}(m + 2hk)x^3 + \frac{1}{2}(m + hk)xz^2 + \frac{1}{2}\delta hDp_x^2 \\
& + \frac{1}{2}\delta hDp_z^2 + \delta hD'xp_x + \frac{1}{2}h'z^2p_x + \frac{1}{2}hxp_x^2 + \frac{1}{2}hxp_z^2 + O(4) \; .
\end{aligned}
\tag{138}
$$

Hamilton's equations are

$$
\begin{aligned}
x' &= \frac{\partial H_{4\beta}}{\partial p_x} = (1 + hx + \delta hD)p_x + \delta hD'x + \frac{1}{2}h'z^2 \\
z' &= \frac{\partial H_{4\beta}}{\partial p_z} = (1 + hx + \delta hD)p_z \\
p'_x &= -\frac{\partial H_{4\beta}}{\partial x} = -\{h^2 - k - \delta[(m + 2hk)D + (h^2 - k)]\}x + \frac{1}{2}(m + 2hk)x^2 \\
&\qquad - \frac{1}{2}(m + hk)z^2 - \delta hD'p_x - \frac{1}{2}hp_x^2 - \frac{1}{2}hp_z^2 \\
p'_z &= -\frac{\partial H_{4\beta}}{\partial z} = -\{k + \delta[(m + hk)D - k + h'D']\}z - (m + hk)xz - h'zp_x \,.
\end{aligned}
\tag{139}
$$

Elimination of p_x and p_z leads to

$$
\begin{aligned}
& x'' - h'xx' - \frac{1}{2}hx'^2 - h'zz' + \frac{1}{2}hz'^2 - \delta(hD' + h'D)x' \\
&\qquad = -\{h^2 - k - \delta[(m + 4hk - 2h^3)D + h'D' + 2h^2 - k]\}x \\
&\qquad\quad + \frac{1}{2}(m + 4hk - 2h^3)x^2 - \frac{1}{2}(m + hk - h'')z^2 + O(3) \\
& z'' - h'xz' - hx'z' + h'x'z - \delta(hD' + h'D)z' \\
&\qquad = -\{k + \delta[(m + 2hk)D + h'D' - k]\}z - (m + 2hk)xz + O(3) \,.
\end{aligned}
\tag{140}
$$

Chapter 7

TIME-DEPENDENT PERTURBATION THEORY

Assuming that the real accelerator consists of an ideal machine (with solvable equations of motion) perturbed by small "field errors", the perturbed equations of motion are derived. The formulation used for the perturbed equations of motion closely follows Guignard [GUIGN78], but instead of using two complex amplitudes we are using the action and the phase of the angle variable. However, instead of Fourier-expanding the Hamiltonian and only keeping a dominating term for which the system is integrable[9] (resonance approach), we apply time-dependent perturbation theory [GOLDS80] to the full Hamiltonian. Since the perturbative calculations are recursive, they can be automated by using a computer algebra system. We apply the formalism to the thick sextupole case and make two iterations to obtain the perturbations to second power in the sextupole strength. Note that time-dependent refers to the fact that, in this formulation, what were constants of motion for the linear motion become functions of time (or equivalently s) owing to the perturbation.

7.1 The perturbed equations of motion

The Hamiltonian is divided into two parts [GOLDS80]:

$$H(p,q;t) = H_0(p,q;t) + V(p,q;t)\ , \tag{141}$$

where $H_0(p,q;t)$ is the Hamiltonian of a solvable, unperturbed problem, $V(p,q;t)$ some perturbation, and (p,q) a set of $2n$ canonical variables. Let the solution of the unperturbed system be

$$\begin{cases} q_i \equiv q_i(c,t) \\ p_i \equiv p_i(c,t) \end{cases} \tag{142}$$

where c stands for $2n$ independent constants. Equations (142) can (at least in principle) be inverted:

$$c_k \equiv c_k(q_i,p_i;t)\ , \tag{143}$$

The perturbed motion may now be expressed in terms of $c_k(q_i,p_i;t)$, which are then functions of time owing to the perturbation.

The Poisson bracket for two functions f and g of the canonical variables (p,q) is defined as

---

[9] For an integrable system with m degrees of freedom one can find m independent constants of motion. The motion of the system lies on a smooth manifold of dimension m in the $2m$-dimensional phase space. If this manifold is compact and connected, then it is an invariant torus and the motion on it is quasi-periodic [PERCI86].

$$[f(p,q;t),\, g(p,q;t)] \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right) . \tag{144}$$

The time derivative of a function f of the canonical variables can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = \frac{\partial f}{\partial t} + [f,H] . \tag{145}$$

The time dependence for the functions c_k is then

$$\dot{c}_k = \frac{\partial c_k}{\partial t} + [c_k, H_0 + V] , \qquad k = 1,2,\ldots,2n . \tag{146}$$

For an unperturbed trajectory the following holds:

$$0 = \frac{\partial c_k}{\partial t} + [c_k, H_0] , \qquad k = 1,2,\ldots,2n \tag{147}$$

so from Eqs. (146) and (147) we obtain

$$\dot{c}_k = [c_k, V] , \qquad k = 1,2,\ldots,2n . \tag{148}$$

By writing V as a function of the c_k this may be written

$$\dot{c}_k = \sum_{j=1}^{2n} [c_k, c_j] \frac{\partial V}{\partial c_j} , \qquad j,\, k = 1,2,\ldots,2n . \tag{149}$$

These equations are still exact. They are however normally as hard to solve as the original equations. The perturbative treatment is done by approximating the right-hand sides, after having evaluated the derivatives. In the first iteration the solutions of H_0 are used in the right-hand sides. The solutions of these equations may then be used for a second iteration and so on.

7.2 The perturbed equations of motion for an accelerator

We choose for H_0 the Hamiltonian for the linear betatron motion

$$H_{0\beta}(x, p_x, z, p_z; s) = \frac{1}{2}(h^2 - k)x^2 + \frac{1}{2}kz^2 + \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 . \tag{150}$$

Note that we now use s as the independent variable instead of t. The solutions are of the type

$$\begin{cases} x = \sqrt{2J_x \beta_x(s)} \cos \psi_x(s) \\ p_x = -\sqrt{\dfrac{2J_x}{\beta_x(s)}} [\sin \psi_x(s) + \alpha_x(s) \cos \psi_x(s)] \end{cases} \tag{151}$$

where

$$\psi_x(s) = \int_{s_0}^{s} \frac{d\tau}{\beta_x(\tau)} + \phi_x \tag{152}$$

and J_x and ϕ_x are constants given by the initial conditions. The same holds for the vertical plane. Equations (151) may be inverted for J_x and ϕ_x so that

$$\begin{cases} \phi_x(p_x,x;s) = -\arctan\left[\beta_x(s)\dfrac{p_x}{x} + \alpha_x(s)\right] - \displaystyle\int_{s_0}^{s} \frac{d\tau}{\beta_x(\tau)} \\ J_x(p_x,x;s) = \gamma_x(s)x^2 + \dfrac{1}{2}\beta_x(s)p_x^2 + \alpha_x(s)xp_x = \dfrac{1}{2\beta_x(s)}\{x^2 + [\beta_x(s)p_x + \alpha_x(s)x]^2\} \end{cases} \tag{153}$$

where

$$\gamma_x(s) \equiv \frac{1 + \alpha_x^2(s)}{\beta_x(s)} . \tag{154}$$

Note that $2J_x$ is then the well-known Courant–Snyder invariant [COURA58] since $p_x = x'$ for the linear motion. J_x and ϕ_x are functions of s for a perturbed trajectory. The Poisson brackets are evaluated to

$$[J_x,\phi_x] = -1 , \quad [J_x,J_x] = 0 , \quad [\phi_x,\phi_x] = 0 , \quad [\phi_x,J_x] = 1 . \tag{155}$$

From Eq. (149) we then obtain for the perturbed motion:

$$\begin{aligned} J_x' &= -\frac{\partial V}{\partial \phi_x} = -\frac{\partial V}{\partial \psi_x} , \quad \phi_x' = \frac{\partial V}{\partial J_x} , \\ J_z' &= -\frac{\partial V}{\partial \phi_z} = -\frac{\partial V}{\partial \psi_z} , \quad \phi_z' = \frac{\partial V}{\partial J_z} , \end{aligned} \tag{156}$$

where we have used the fact that

$$\frac{\partial}{\partial \phi} = \frac{\partial \psi}{\partial \phi}\frac{\partial}{\partial \psi} = \frac{\partial}{\partial \psi} . \tag{157}$$

Perturbation theory is then applied by using in the first iteration the solutions (122) to the linear equations of motion in the right-hand sides, after having evaluated the derivatives.

7.3 Automation of the perturbative calculations

Automation of perturbative calculations goes back to the beginning of 1970 when Deprit repeated Delaunay's calculations for the "main problem of the lunar theory", carrying it to at least one higher power in small parameters. Delaunay had already, in 1867 [DELAU67], worked out an analytical solution to the "main problem" by his procedure, which might be described as an early version of the "Poincaré – von Zeipel procedure" (i.e. time-independent perturbation theory for celestial mechanics, known as Moser-like transformations to accelerator physicists). After 20 years of effort without any assistance, having done all the work by hand, he published his results in two volumes, with around 400 pages each, containing algebraic formulae for the perturbed motion of oscullating elements in powers of small quantities up to 7th power, and in some cases to 9th power. The calculations covered more than 500 canonical transformations, involving in all over 10 000 terms. By comparing his work with Deprit's results it was found that in all the calculations Delaunay had made only one mistake (amounting to writing $147-90+9=46$) at the 9th power, all errors resulting from its propagation through other terms giving errors in seven coefficients.

In the Poincaré – von Zeipel method one has to invert the canonical transformation (since the generating function is written in a mixed system of variables) to be able to find the motion as a function of time. This is often difficult to carry out analytically in a systematic fashion. In the 1960s Hori [HORI66] and Deprit [DEPRI66] used Lie transforms for which the tranformation equations were explicit and could be developed recursively to any order. Deprit implemented his algorithm in a computer algebra system, MACSYMA, and reproduced Delaunay's calculations in 1970 [DEPRI70].

Michelotti [MICHE86] has applied Deprit's recursive algorithm to accelerator theory, for example to fourth power in the sextupole strength to study the stability limit.

In our, case we have implemented the calculations for the time-dependent perturbation theory in the computer algebra system REDUCE [HEARN85]. This can be done since the perturbative calculations for a given perturbation are explicit and recursive but involve a large number of algebraic manipulations. The computer algebra system REDUCE can be used to evaluate the partial derivatives to obtain the perturbed equations of motion, to perform the necessary variable substitutions, and to do the final integrations within a lattice element. The result from REDUCE may subsequently be written as a FORTRAN source language file, comprising routines that calculate numerical values for the perturbations for a given lattice.

We have developed a REDUCE program where the input is the solutions to the unperturbed Hamiltonian and the perturbation term. The current program allows one or two iterations (extensions to more iterations can be done owing to the recursive nature of the calculations) and integration within a lattice element or thin element approximation. The output from the program consists of analytical formulae for the perturbation of the action, the phase, and the tune shifts. It also gives expressions for the perturbation per turn that can be used for tracking. The first iteration gives the perturbations linear in the perturbation parameter, and the second iteration gives quadratic terms. The routine for integration within a lattice element is described elsewhere [BENGT88A].

The unperturbed solution depends on the type of linear element in which the perturbation is located. The linear solution is given by the transfer matrix, which we list for some common cases [STEFF85].

Straight section

$$M_{x,z} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} . \qquad (158)$$

Sector magnet

$$M_x = \begin{pmatrix} \cos\phi & \rho\sin\phi \\ -\dfrac{1}{\rho}\sin\phi & \cos\phi \end{pmatrix}, \quad M_z = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \phi \equiv \frac{s}{\rho}, \tag{159}$$

where ρ is the radius of curvature.

Edge focusing (hard edge approximation)

$$M_{x,z} = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{\rho}\tan\varepsilon & 1 \end{pmatrix} \tag{160}$$

where ε is the entrance or exit angle.

Quadrupole

$$\begin{aligned} M_{x,z} &= \begin{pmatrix} \cos\phi & \dfrac{s}{\phi}\sin\phi \\ -\dfrac{\phi}{s}\sin\phi & \cos\phi \end{pmatrix}, \quad K > 0 \\ M_{x,z} &= \begin{pmatrix} \cosh\phi & \dfrac{s}{\phi}\sinh\phi \\ \dfrac{\phi}{s}\sinh\phi & \cosh\phi \end{pmatrix}, \quad K < 0 \end{aligned} \tag{161}$$

where

$$\phi \equiv s\sqrt{|K|}\ . \tag{162}$$

7.4 Perturbations from thick sextupoles

Perturbations in the action have been calculated and FORTRAN coded for thick elements to first power in the multipole strength for the following cases: dipole, quadrupole, sextupole, and octupole [BENGT88A].

As an example of what has been calculated with a computer we take the case of thick sextupoles. If we only consider pure sextupoles we have from the Hamiltonian (83)

$$\begin{aligned} V(s) = -\frac{1}{6}m(s)(x^3 - 3xz^2) = &-\frac{\sqrt{2}}{3}m(s)[J_x^{3/2}(s)\beta_x^{3/2}(s)\cos^3\psi_x(s) \\ &- 3\sqrt{J_x(s)}J_z(s)\sqrt{\beta_x(s)}\beta_z(s)\cos\psi_x(s)\cos^2\psi_z(s)]\ , \end{aligned} \tag{163}$$

where we have used (151). The perturbed equations of motion are given by Eqs. (156):

$$
\begin{aligned}
J_x' &= -\frac{\partial V}{\partial \psi_x} \\
&= -\sqrt{2}m(s)\left[J_x^{3/2}(s)\beta_x^{3/2}(s)\sin\psi_x(s)\cos^2\psi_x(s) - \sqrt{J_x(s)}J_z(s)\sqrt{\beta_x(s)}\beta_z(s)\sin\psi_x(s)\cos^2\psi_z(s)\right] \\
J_z' &= -\frac{\partial V}{\partial \psi_z} = 2\sqrt{2}m(s)\sqrt{J_x(s)}J_z(s)\sqrt{\beta_x(s)}\beta_z(s)\cos\psi_x(s)\sin\psi_z(s)\cos\psi_z(s) \\
\phi_x' &= \frac{\partial V}{\partial J_x} = -\frac{\sqrt{2}}{2}m(s)\left[\sqrt{J_x(s)}\beta_x^{3/2}(s)\cos^3\psi_x(s) - \frac{J_z(s)}{\sqrt{J_x(s)}}\sqrt{\beta_x(s)}\beta_z(s)\cos\psi_x(s)\cos^2\psi_z(s)\right] \\
\phi_z' &= \frac{\partial V}{\partial J_z} = \sqrt{2}m(s)\sqrt{J_x(s)}\sqrt{\beta_x(s)}\beta_z(s)\cos\psi_x(s)\cos^2\psi_z(s)\ .
\end{aligned}
\tag{164}
$$

In the first iteration the right-hand sides are approximated by the solutions to the linear equations (151):

$$
\begin{aligned}
J(s) &= J_0 \\
\psi(s) &= \psi_0(s) \equiv \int_{s_0}^{s} \frac{d\tau}{\beta(\tau)} + \phi_0\ ,
\end{aligned}
\tag{165}
$$

where J_0 and ϕ_0 are constants given by the initial conditions. The equations are integrated as follows:

$$
\begin{aligned}
\Delta_1 J_x(s) &= -\int_{s_0}^{s}\sqrt{2}m(\sigma)\left[J_{0x}^{3/2}\beta_x^{3/2}(\sigma)\sin\psi_{0x}(\sigma)\cos^2\psi_{0x}(\sigma)\right. \\
&\qquad \left. - \left[\sqrt{J_{0x}}J_{0z}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\sin\psi_{0x}(\sigma)\cos^2\psi_{0z}(\sigma)\right]\right] d\sigma \\
\Delta_1 J_z(s) &= \int_{s_0}^{s} 2\sqrt{2}m(\sigma)\sqrt{J_{0x}}J_{0z}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\cos\psi_{0x}(\sigma)\sin\psi_{0z}(\sigma)\cos\psi_{0z}(\sigma)\, d\sigma \\
\Delta_1 \phi_x(s) &= -\int_{s_0}^{s}\frac{\sqrt{2}}{2}m(\sigma)\left[\sqrt{J_{0x}}\beta_x^{3/2}(\sigma)\cos^3\psi_{0x}(\sigma) - \frac{J_{0z}}{\sqrt{J_{0x}}}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\cos\psi_{0x}(\sigma)\cos^2\psi_{0z}(\sigma)\right] d\sigma \\
\Delta_1 \phi_z(s) &= \int_{s_0}^{s}\sqrt{2}m(\sigma)\sqrt{J_{0x}}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\cos\psi_{0x}(\sigma)\cos^2\psi_{0z}(\sigma)\, d\sigma\ .
\end{aligned}
\tag{166}
$$

The perturbation at s_0 can then be written as a sum over the number of turns N:

$$\Delta_1 J_x(s_0 + NC) = -\sum_{n=0}^{N}\int_{s_0}^{s_0+C} \sqrt{2}m(\sigma)\Big\{J_{0x}^{3/2}\beta_x^{3/2}(\sigma)\sin[\psi_{0x}(\sigma) + n2\pi Q_x]\cos^2[\psi_{0x}(\sigma) + n2\pi Q_x]$$

$$- \sqrt{J_{0x}}J_{0z}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\sin[\psi_{0x}(\sigma) + n2\pi Q_x]\cos^2[\psi_{0z}(\sigma) + n2\pi Q_z]\Big\}\,d\sigma$$

$$\Delta_1 J_z(s_0 + NC) = \sum_{n=0}^{N}\int_{s_0}^{s_0+C} 2\sqrt{2}m(\sigma)\sqrt{J_{0x}}J_{0z}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\cos[\psi_{0x}(\sigma) + n2\pi Q_x]$$

$$\times \sin[\psi_{0z}(\sigma) + n2\pi Q_z]\cos[\psi_{0z}(\sigma) + n2\pi Q_z]\,d\sigma \tag{167}$$

$$\Delta_1 \phi_x(s_0 + NC) = -\sum_{n=0}^{N}\int_{s_0}^{s_0+C} \frac{\sqrt{2}}{2}m(\sigma)\Big\{\sqrt{J_{0x}}\beta_x^{3/2}(\sigma)\cos^3[\psi_{0x}(\sigma) + n2\pi Q_x]$$

$$- \frac{J_{0z}}{\sqrt{J_{0x}}}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\cos[\psi_{0x}(\sigma) + n2\pi Q_x]\cos^2[\psi_{0z}(\sigma) + n2\pi Q_z]\Big\}\,d\sigma$$

$$\Delta_1 \phi_z(s_0 + NC) = \sum_{n=0}^{N}\int_{s_0}^{s_0+C} \sqrt{2}m(s)\sqrt{J_{0x}}\sqrt{\beta_x(\sigma)}\beta_z(\sigma)\cos[\psi_{0x}(\sigma) + n2\pi Q_x]\cos^2[\psi_{0z}(\sigma) + n2\pi Q_z]\,d\sigma$$

since

$$\psi(s + nC) = \psi(s) + n2\pi Q\,. \tag{168}$$

If we assume that $m(s)$ is constant within an element, the integrals can be divided into a sum over elements

$$
\begin{aligned}
\Delta_1 J_x(s_0 + NC) = & - \sum_{n=0}^{N} \sum_{k=1}^{K} \int_0^{l_k} \sqrt{2} m(\sigma) \Big\{ J_{0x}^{3/2} \beta_x^{3/2}(\sigma) \sin [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& \times \cos^2 [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& - \sqrt{J_{0x}} J_{0z} \sqrt{\beta_x(\sigma)} \beta_z(\sigma) \sin [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& \times \cos^2 [\psi_{0z}(\sigma) + \psi_{0z}(s_k) + n2\pi Q_z] \Big\} \mathrm{d}\sigma \\
\Delta_1 J_z(s_0 + NC) = & \sum_{n=0}^{N} \sum_{k=1}^{K} \int_0^{l_k} 2\sqrt{2} m(\sigma) \sqrt{J_{0x}} J_{0z} \sqrt{\beta_x(\sigma)} \beta_z(\sigma) \cos [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& \times \sin [\psi_{0z}(\sigma) + \psi_{0z}(s_k) + n2\pi Q_z] \cos [\psi_{0z}(\sigma) + \psi_{0z}(s_k) + n2\pi Q_z] \mathrm{d}\sigma \\
\Delta_1 \phi_x(s_0 + NC) = & - \sum_{n=0}^{N} \sum_{k=1}^{K} \int_0^{l_k} \frac{\sqrt{2}}{2} m(\sigma) \Big\{ \sqrt{J_{0x}} \beta_x^{3/2}(\sigma) \cos^3 [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& - \frac{J_{0z}}{\sqrt{J_{0x}}} \sqrt{\beta_x(\sigma)} \beta_z(\sigma) \cos [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& \times \cos^2 [\psi_{0z}(\sigma) + \psi_{0z}(s_k) + n2\pi Q_z] \Big\} \mathrm{d}\sigma \\
\Delta_1 \phi_z(s_0 + NC) = & \sum_{n=0}^{N} \sum_{k=1}^{K} \int_0^{l_k} \sqrt{2} m(\sigma) \sqrt{J_{0x}} \sqrt{\beta_x(\sigma)} \beta_z(\sigma) \cos [\psi_{0x}(\sigma) + \psi_{0x}(s_k) + n2\pi Q_x] \\
& \times \cos^2 [\psi_{0z}(\sigma) + \psi_{0z}(s_k) + n2\pi Q_z] \mathrm{d}\sigma \, ,
\end{aligned} \tag{169}
$$

where l is the length of an element, an index k refers to the values at the entrance of element k, and K is the number of elements.

The s-dependence for $\beta(s)$ and $\psi(s)$ within an element is given by the transfer matrix (104)

$$
\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M_x(s|s_0) \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \tag{170}
$$

and similarly for the vertical plane. We have [AUTIN83]

$$
\begin{aligned}
\sqrt{\beta(\sigma)} \sin \psi(\sigma) &= \frac{a_{12}(\sigma)}{\sqrt{\beta_1}} \\
\sqrt{\beta(\sigma)} \cos \psi(\sigma) &= \sqrt{\beta_1} [a_{11}(\sigma) - \frac{\alpha_1}{\beta_1} a_{12}(\sigma)] \\
\beta(\sigma) &= \beta_1 a_{11}^2(\sigma) - 2\alpha_1 a_{11}(\sigma) a_{12}(\sigma) + \gamma_1 a_{12}^2(\sigma) \, ,
\end{aligned} \tag{171}
$$

where σ is the distance inside the element and β_1, α_1, and γ_1 are the values at the entrance of the element. This gives

$$
\begin{aligned}
\sqrt{\beta(\sigma)}\sin[\psi(\sigma)+u] &= \sqrt{\beta_1}\left\{\frac{a_{12}(\sigma)}{\beta_1}\cos u + \left[a_{11}(\sigma) - \frac{\alpha_1}{\beta_1}a_{12}(\sigma)\right]\sin u\right\} \\
\sqrt{\beta(\sigma)}\cos[\psi(\sigma)+u] &= \sqrt{\beta_1}\left\{\left[a_{11}(\sigma) - \frac{\alpha_1}{\beta_1}a_{12}(\sigma)\right]\cos u - \frac{a_{12}(\sigma)}{\beta_1}\sin u\right\} \\
\beta(\sigma) &= \beta_1 a_{11}^2(\sigma) - 2\alpha_1 a_{11}(\sigma)a_{12}(\sigma) + \gamma_1 a_{12}^2(\sigma)\,.
\end{aligned}
\tag{172}
$$

If the elements are placed in straight sections we have, from Eqs. (158),

$$
M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \tag{173}
$$

It is then straightforward, but cumbersome, to perform the integrations since the integrand is transformed into a polynomial in σ. The result can be found in Appendix B

The sum over the number of turns N can be evaluated by using

$$
\sum_{n=0}^{N} e^{i(u+n2v)} = e^{iu}\sum_{n=0}^{N} e^{in2v} = e^{iu}\frac{1 - e^{i(N+1)2v}}{1 - e^{i2v}}\,, \tag{174}
$$

so that

$$
\begin{aligned}
\sum_{n=0}^{N}\sin(u + n2v) &= \frac{\cos(u-v) - \cos[u + (2N+1)v]}{2\sin v} \\
\sum_{n=0}^{N}\cos(u + n2v) &= \frac{-\sin(u-v) + \sin[u + (2N+1)v]}{2\sin v}\,.
\end{aligned}
\tag{175}
$$

Since the first term is constant and the other oscillating with the number of turns N, we have for the average perturbation:

$$
\begin{aligned}
\sum_{n=0}^{N}\sin(u + n2v) &= \frac{\cos(u-v)}{2\sin v} \\
\sum_{n=0}^{N}\cos(u + n2v) &= -\frac{\sin(u-v)}{2\sin v}\,.
\end{aligned}
\tag{176}
$$

The average perturbations to first power in the sextupole strength can be found in Appendix C The perturbations of the action contain the following resonances

$$
\begin{aligned}
&Q_x = p_1\,, & &3Q_x = p_3\,, \\
&Q_x - 2Q_z = p_3\,, & &Q_x + 2Q_z = p_4\,,
\end{aligned}
\tag{177}
$$

where

$$
p_1, \ldots, p_4 = \text{integer}\,. \tag{178}
$$

The thin-element approximation is obtained by taking the limit $l \to 0$, but keeping the "integrated strength" (ml) constant. The average perturbations (330) to (333) then simplify to

$$
\begin{aligned}
<\Delta_1 J_x(s_0)> &= \sum_{k=1}^{K} \frac{\sqrt{2}}{8}(ml)_k \sqrt{J_{0x}\beta_{kx}} \Big\{ - (J_{0x}\beta_{kx} - 2J_{0z}\beta_{kz}) \frac{\cos[\psi_{0x}(s_k) - \pi Q_x]}{\sin \pi Q_x} \\
&\quad - J_{0x}\beta_{kx} \frac{\cos[3\psi_{0x}(s_k) - 3\pi Q_x]}{\sin 3\pi Q_x} \\
&\quad + J_{0z}\beta_{kz} \frac{\cos[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin \pi(Q_x + 2Q_z)} \\
&\quad + J_{0z}\beta_{kz} \frac{\cos[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin \pi(Q_x - 2Q_z)} \Big\} \\
<\Delta_1 J_z(s_0)> &= \sum_{k=1}^{K} \frac{\sqrt{2}}{4}(ml)_k \sqrt{J_{0x}\beta_{kx}} J_{0z}\beta_{kz} \Big\{ \frac{\cos[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin \pi(Q_x + 2Q_z)} \\
&\quad - \frac{\cos[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin \pi(Q_x - 2Q_z)} \Big\} \\
<\Delta_1 \phi_x(s_0)> &= \sum_{k=1}^{K} \frac{\sqrt{2}}{16}(ml)_k \sqrt{\frac{\beta_{kx}}{J_{0x}}} \Big\{ (3J_{0x}\beta_{kx} - 2J_{0z}\beta_{kz}) \frac{\sin[\psi_{0x}(s_k) - \pi Q_x]}{\sin \pi Q_x} \\
&\quad + J_{0x}\beta_{kx} \frac{\sin[3\psi_{0x}(s_k) - 3\pi Q_x]}{\sin 3\pi Q_x} \\
&\quad - J_{0z}\beta_{kz} \frac{\sin[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin \pi(Q_x + 2Q_z)} \\
&\quad - J_{0z}\beta_{kz} \frac{\sin[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin \pi(Q_x - 2Q_z)} \Big\} \\
<\Delta_1 \phi_z(s_0)> &= \sum_{k=1}^{K} \frac{\sqrt{2}}{8}(ml)_k \sqrt{J_{0x}\beta_{kx}}\, \beta_{kz} \Big\{ - \frac{\sin[\psi_{0x}(s_k) - \pi Q_x]}{\sin \pi Q_x} \\
&\quad - J_{0z}\beta_{kz} \frac{\sin[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin \pi(Q_x + 2Q_z)} \\
&\quad - J_{0z}\beta_{kz} \frac{\sin[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin \pi(Q_x - 2Q_z)} \Big\} .
\end{aligned}
\tag{179}
$$

The perturbations in the coordinates x and z are given by Eqs. (151) so that

$$
\begin{aligned}
x(s) &= \sqrt{2J_x(s)\beta_x(s)} \cos \psi_x(s) \\
&= \sqrt{2\left[J_{0x} + <\Delta_1 J_x(s)> + \widetilde{\Delta_1 J_x}(s)\right]\beta_x(s)} \cos[\psi_{0x}(s) + <\Delta_1\phi_x(s)> + \widetilde{\Delta_1\phi_x}(s)] \\
&= \sqrt{2J_{1x}(s)\beta_x(s)} \left[\left(1 + \frac{\widetilde{\Delta_1 J_x}(s)}{2J_{0x}}\right) \cos\psi_{1x}(s) - \widetilde{\Delta_1\phi_x}(s) \sin\psi_{1x}(s) \right] + O[(ml)^2] ,
\end{aligned}
\tag{180}
$$

where we have developed the cosine and expanded. A tilde denotes the N-dependent part of the perturbation and

$$
\begin{aligned}
J_{1x}(s) &\equiv J_{0x} + \ <\Delta_1 J_x(s)> \\
\psi_{1x}(s) &\equiv \psi_{0x}(s) + \ <\Delta_1 \psi_x(s)> \ .
\end{aligned}
\tag{181}
$$

Note that $<\Delta_1 J_x(s)>$ is constant for a given position s. The result is

$$
\begin{aligned}
x(s_0) = \sqrt{2J_{1x}(s_0)\beta_x(s_0)}\Bigg\{&\cos\,[\psi_{1x}(s_0) + N2\pi Q_x] \\
&+ \sum_{k=1}^{K} \frac{\sqrt{2}}{16}(ml)_k \sqrt{\frac{\beta_{kx}}{J_{0x}}}\Bigg[2(\beta_{kx}J_{0x} - \beta_{kz}J_{0z})\frac{\cos\,[\psi_{1x}(s_0) - [\psi_{0x}(s_k) + \pi Q_x]]}{\sin\pi Q_x} \\
&- \beta_{kx}J_{0x}\frac{\cos\,[\psi_{1x}(s_0) + \psi_{0x}(s_k) + \pi Q_x + N4\pi Q_x]}{\sin\pi Q_x} \\
&+ \beta_{kx}J_{0x}\frac{\cos\,[\psi_{1x}(s_0) - 3[\psi_{0x}(s_k) + \pi Q_x] - N4\pi Q_x]}{\sin 3\pi Q_x} \\
&- \beta_{kz}J_{0z}\frac{\cos\,[\psi_{1x}(s_0) - [\psi_{0x}(s_k) + \pi Q_x] - 2[\psi_{0z}(s_k) + \pi Q_z] - N4\pi Q_z]}{\sin\pi(Q_x + 2Q_z)} \\
&- \beta_{kz}J_{0z}\frac{\cos\,[\psi_{1x}(s_0) - [\psi_{0x}(s_k) + \pi Q_x] + 2[\psi_{0z}(s_k) + \pi Q_z] + N4\pi Q_z]}{\sin\pi(Q_x - 2Q_z)}\Bigg] + O[(ml)^2]\Bigg\}
\end{aligned}
\tag{182}
$$

$$
\begin{aligned}
z(s_0) = \sqrt{2J_{1z}(s_0)\beta_z(s_0)}\Bigg\{&\cos\,[\psi_{1z}(s) + N2\pi Q_z] \\
&+ \sum_{k=1}^{K} \frac{\sqrt{2}}{8}(ml)_k \sqrt{J_{0x}\beta_{kx}}\,\beta_{kz}\Bigg[-\frac{\cos\,[\psi_{1z}(s_0) - [\psi_{0x}(s_k) + \pi Q_x] - N2\pi(Q_x - Q_z)]}{\sin\pi Q_x} \\
&+ \frac{\cos\,[\psi_{1z}(s_0) + \psi_{0x}(s_k) + \pi Q_x + N2\pi(Q_x + Q_z)]}{\sin\pi Q_x} \\
&- \frac{\cos\,[\psi_{1z}(s_0) - [\psi_{0x}(s_k) + \pi Q_x] - 2[\psi_{0z}(s_k) + \pi Q_z] - N2\pi(Q_x + Q_z)]}{\sin\pi(Q_x + 2Q_z)} \\
&- \frac{\cos\,[\psi_{1z}(s_0) + \psi_{0x}(s_k) + \pi Q_x - 2[\psi_{0z}(s_k) + \pi Q_z] + N2\pi(Q_x - Q_z)]}{\sin\pi(Q_x - 2Q_z)}\Bigg] + O[(ml)^2]\Bigg\} \ .
\end{aligned}
$$

The frequency associated with each term is given by the N-dependence. In the horizontal plane we find the frequencies

$$
Q_x\,, \quad 2Q_x\,, \quad 2Q_z \tag{183}
$$

and in the vertical plane

$$
Q_z\,, \quad Q_x - Q_z\,, \quad Q_x + Q_z\,. \tag{184}
$$

We may now make a second iteration by approximating the right-hand sides of Eqs. (164) with the solutions from the first iteration. Note that the solutions containing the N-dependence should be used. In order to simplify the calculations it would be convenient to apply the thin-element approximation directly. However, this requires some care.

In the first iteration we could have applied the thin-element approximation directly in Eqs. (164) by replacing

$$m(s) \quad \text{by} \quad (ml)_k \delta(s - s_k) \tag{185}$$

so that the integrals become trivial and Eqs. (179) can bc obtained directly. The solutions are of the form

$$\begin{aligned} \Delta_1 J(s_0 + NC) &= \sum_{n=0}^{N} \sum_{k=1}^{K} f_1(n,s_k) \\ \Delta_1 \phi(s_0 + NC) &= \sum_{n=0}^{N} \sum_{k=1}^{K} g_1(n,s_k) \,, \end{aligned} \tag{186}$$

where $f_1(n,s_k)$ and $g_1(n,s_k)$ are some oscillating functions of N. The second iteration gives us a solution of the form

$$\Delta_2 J(s_0 + NC) = \sum_{m=0}^{N} \sum_{j=1}^{K} f_2[\Delta_1 J(s_0 + mC), \Delta_1 \phi(s_0 + mC), s_j] \,, \tag{187}$$

The perturbation from the first iteration just before the position of an element is given by Eqs. (186) with the sum over k and N replaced by

$$N = \begin{cases} n \,, & s_k < s_j \\ n - 1 \,, & s_k \geq s_j \,. \end{cases} \tag{188}$$

However, we also obtain a contribution to the first iteration from the motion inside the element j. Neglecting it leads to a secular term in the perturbation of the action implying that the motion is unstable, which would be unphysical. If we had used a thick-element integration instead of the thin-element approximation we would obtain something like

$$\Delta_1 J(s - s_j) = a_1 s + O(s^2) \,, \quad s \leq l_j \tag{189}$$

for the motion inside element j. When this is used in the second iteration it is multiplied by a polynomial in s including a constant term b_0. This integrated gives

$$\Delta_2 J(s - s_j) = \frac{1}{2} a_1 b_0 s^2 + O(s^3) \,, \quad s \leq l_j \,. \tag{190}$$

After the passage of element j we have

$$\Delta_2 J(s - s_j) = \frac{1}{2} a_1 b_0 l_j^2 + O(l_j^3) \,, \quad s \geq l_j \,. \tag{191}$$

A correct thin-lens approximation is then obtained by adding a term

$$
\begin{aligned}
&\frac{1}{2} f_1(N-1, s_j) \\
&\frac{1}{2} g_1(N-1, s_j)
\end{aligned}
\tag{192}
$$

to Eqs. (186) The second iteration can now be performed in a similar way to the first one.

In the perturbations of the phase we find terms of the form

$$
\Delta_2 \phi(s_0) = \sum_{n=0}^{N} \sum_{j=1}^{K} a(s_j) \,, \tag{193}
$$

where a is not oscillating with N. The sum over N goes to infinity. However, this term can be interpreted as a tune shift

$$
\Delta Q = \sum_{j=1}^{K} \frac{a(s_j)}{2\pi} \tag{194}
$$

and can therefore be removed so that the sum over the remaining oscillating terms can be evaluated. The tune shifts are

$$\Delta_2 Q_x = \sum_{j=1}^{K}\sum_{k=1}^{K}\frac{1}{64\pi}(ml)_j(ml)_k\sqrt{\beta_{jx}\beta_{kx}}$$

$$\left\{ -\left[3\beta_{jx}\beta_{kx}J_{0x} - (\beta_{kx}\beta_{jz} + 3\beta_{jx}\beta_{kz})J_{0z}\right]\frac{\cos\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) + (2N_0 - 1)\pi Q_x\right]}{\sin \pi Q_x}\right.$$

$$- \beta_{jx}\beta_{kx}J_{0x}\frac{\cos 3\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) + (2N_0 - 1)\pi Q_x\right]}{\sin 3\pi Q_x}$$

$$- 2\beta_{jz}\beta_{kz}J_{0z}\frac{\cos\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) + 2[\psi_{0z}(s_j) - \psi_{0z}(s_k)] + (2N_0 - 1)\pi(Q_x + 2Q_z)\right]}{\sin \pi(Q_x + 2Q_z)}$$

$$\left. + 2\beta_{jz}\beta_{kz}J_{0z}\frac{\cos\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) - 2[\psi_{0z}(s_j) - \psi_{0z}(s_k)] + (2N_0 - 1)\pi(Q_x - 2Q_z)\right]}{\sin \pi(Q_x - 2Q_z)}\right\}$$

$$\Delta_2 Q_z = \sum_{j=1}^{K}\sum_{k=1}^{K}\frac{1}{64\pi}(ml)_j(ml)_k\sqrt{\beta_{jx}\beta_{kx}} \tag{195}$$

$$\left\{ 4(\beta_{kx}\beta_{jz}J_{0x} - \beta_{jz}\beta_{kz}J_{0z})\frac{\cos\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) + (2N_0 - 1)\pi Q_x\right]}{\sin \pi Q_x}\right.$$

$$- \beta_{jz}\beta_{kz}(2J_{0x} + J_{0z})$$

$$\times \frac{\cos\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) + 2[\psi_{0z}(s_j) - \psi_{0z}(s_k)] + (2N_0 - 1)\pi(Q_x + 2Q_z)\right]}{\sin \pi(Q_x + 2Q_z)}$$

$$+ \beta_{jz}\beta_{kz}(2J_{0x} - J_{0z})$$

$$\left. \times \frac{\cos\left[\psi_{0x}(s_j) - \psi_{0x}(s_k) - 2[\psi_{0z}(s_j) - \psi_{0z}(s_k)] + (2N_0 - 1)\pi(Q_x - 2Q_z)\right]}{\sin \pi(Q_x - 2Q_z)}\right\},$$

where

$$N_0 = \begin{cases} 0, & s_k < s_j \\ 1, & s_k \geq s_j . \end{cases} \tag{196}$$

By taking into account the sum over j and k, it is found that the tune shifts can be written as

$$\begin{pmatrix}\Delta_2 Q_x \\ \Delta_2 Q_z\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix} J_{0x} \\ J_{0z}\end{pmatrix}. \tag{197}$$

Since the perturbations of the action and the phase will not be studied in detail, they are not presented here. We only mention the fact that one finds the following resonances in the perturbation of the action:

$$\begin{matrix} 2Q_x = p_1, & 2Q_x - 2Q_z = p_3, & 4Q_x = p_5, \\ 2Q_z = p_2, & 2Q_x + 2Q_z = p_4, & 4Q_z = p_6 . \end{matrix} \quad p_1,\dots,p_6 = \text{integer} \tag{198}$$

Chapter 8

SINGLE-RESONANCE THEORY

We present a short summary of the resonance theory that can be found in the literature [SCHOC57, HAGED57, GUIGN78, COURA84, ANDO84] since the single-resonance Hamiltonian will be used in the next chapter.

We assume the perturbation to be of the form

$$V(s) = V_{m_x,m_z}(s) x^{m_x} z^{m_z} , \tag{199}$$

where m_x and m_z are positive integers. Instead of using the canonical transformation with the generating function (116) we will use a transformation where the new phase increases linearly with s for the linear motion. The canonical transformation is generated by [ANDO84]:

$$F_1(x,z,\phi_{1x},\phi_{1z};s) = -\frac{x^2}{2\beta_x(s)}\{\tan[\phi_{1x} + W_x(s)] + \alpha_x(s)\} - \frac{z^2}{2\beta_z(s)}\{\tan[\phi_{1z} + W_z(s)] + \alpha_z(s)\} , \tag{200}$$

where

$$W_x(s) = \int_{s_0}^{s} \frac{d\tau}{\beta_x}(\tau) - Q_x \frac{s - s_0}{R} \tag{201}$$

and R is the average radius of the accelerator ($C/2\pi$). It follows that

$$\begin{aligned} p_x &= \frac{\partial F_1}{\partial x} = -\frac{x}{\beta_x(x)}\{\tan[\phi_{1x} + W_x(s)] + \alpha_x(s)\} \\ J_{1x} &= -\frac{\partial F_1}{\partial \phi_{1x}} = \frac{x^2}{2\beta_x(s)\cos^2[\phi_{1x} + W_x(s)]} \\ K_{0\beta} &= H_{0\beta} + \frac{\partial F_1}{\partial s} \end{aligned} \tag{202}$$

or

$$\begin{aligned} x &= \sqrt{2J_{1x}\beta_x(s)}\cos[\phi_{1x} + W_x(s)] \\ p_x &= -\sqrt{\frac{2J_{1x}}{\beta_x(s)}}\{\sin[\phi_{1x} + W_x(s)] + \alpha_x(s)\cos[\phi_{1x} + W_x(s)]\} . \end{aligned} \tag{203}$$

Similar expressions hold for z and p_z. The new Hamiltonian for the linear motion is

$$K_{0\beta}(J_{1x},J_{1z},\phi_{1x},\phi_{1z};s) = \frac{1}{R}(Q_x J_{1x} + Q_z J_{1z}) \,. \tag{204}$$

Hamilton's equations are

$$\begin{aligned} J_{1x}{}' &= -\frac{\partial K_{0\beta}}{\partial \phi_{1x}} = 0 \,, \quad \phi_{1x}{}' = \frac{\partial K_{0\beta}}{\partial J_{1x}} = \frac{Q_x}{R} \,, \\ J_{1z}{}' &= -\frac{\partial K_{0\beta}}{\partial \phi_{1z}} = 0 \,, \quad \phi_{1z}{}' = \frac{\partial K_{0\beta}}{\partial J_{1z}} = \frac{Q_z}{R} \,, \end{aligned} \tag{205}$$

with the solutions

$$\begin{aligned} J_{1x} &= \text{const.} \,, \quad \phi_{1x} = Q_x \frac{s - s_0}{R} + \phi_{1x}(s_0) \,, \\ J_{1z} &= \text{const.} \,, \quad \phi_{1z} = Q_z \frac{s - s_0}{R} + \phi_{1z}(s_0) \,. \end{aligned} \tag{206}$$

Equation (203) can be written in the form

$$x = \sqrt{\frac{J_{1x}\beta_x(s)}{2}} \left\{ e^{i[\phi_{1x} + W_x(s)]} + e^{-i[\phi_{1x} + W_x(s)]} \right\} \tag{207}$$

and similarly for z so that

$$V(s) = \frac{1}{R} J_{1x}^{m_x/2} J_{1z}^{m_z/2} \sum_{j=0}^{m_x} \sum_{l=0}^{m_z} h_{jklm} e^{i[n_x\phi_{1x}(s) + n_z\phi_{1z}(s)]} \,, \tag{208}$$

where

$$h_{jklm} = \frac{R}{2^{N/2}} V_{m_x,m_z} \binom{m_x}{j} \binom{m_z}{l} \beta_x^{m_x/2}(s) \beta_z^{m_z/2}(s) e^{i[n_x W_x(s) + n_z W_z(s)]} \tag{209}$$

and

$$\begin{aligned} m_x &= j + k \,, \qquad m_z = l + m \,, \\ n_x &= j - k \,, \qquad n_z = l - m \,, \\ N &= j + k + l + m \,. \end{aligned} \tag{210}$$

Since h_{jklm} is periodic for a circular accelerator, it can be expanded into a Fourier series so that

$$V(s) = \frac{1}{R} J_{1x}^{m_x/2} J_{1z}^{m_z/2} \sum_{j=0}^{m_x} \sum_{l=0}^{m_z} \sum_{p=-\infty}^{\infty} h_{jklmp} e^{i[n_x\phi_{1x} + n_z\phi_{1z} - p(s-s_0)/R]} \,, \tag{211}$$

where the Fourier component h_{jklmp} is calculated from

$$h_{jklmp} = \frac{1}{2^{N/2}}\binom{m_x}{j}\binom{m_z}{l}\frac{1}{2\pi}\int_{s_0}^{s_0+2\pi R} \beta_x^{m_x/2}(s)\beta_z^{m_z/2}(s)V_{m_x,m_z}\,e^{i[n_xW_x(s)+n_zW_z(s)+p(s-s_0)/R]}\,ds\,. \quad (212)$$

Note that this definition differs from that used by Guignard [GUIGN78] by a factor of $R^{(N-2)/2}$.

The perturbation will contain a slowly varying term if

$$n_xQ_x + n_zQ_z = p\,. \quad (213)$$

These terms are called resonance terms. N is the order and p is the harmonic of the perturbation driving the resonance. If n_x and n_z have the same sign the resonance is called a sum resonance, otherwise it is called a difference resonance. If the working point is close to a single resonance then the perturbation will be dominated by this term so that the other terms in (211) can be neglected:

$$V(\phi_{1x},\phi_{1z},J_{1x},J_{1z};s) = \frac{2}{R}|\kappa|J_{1x}^{m_x/2}J_{1z}^{m_z/2}\cos\psi_1\,, \quad (214)$$

where

$$\psi_1 = n_x\phi_{1x} + n_z\phi_{1z} + \phi_\kappa - p\frac{s-s_0}{R} \quad (215)$$

and the amplitude $|\kappa|$ and phase ϕ_κ of the resonance are defined by the Fourier component for this resonance h_{jklmp} as

$$\kappa = |\kappa|e^{i\phi_\kappa} = h_{jklmp} \quad (216)$$

since

$$h_{kjml\,-p} = h^*_{jklmp}\,. \quad (217)$$

Note that a similar treatment can be applied to terms containing p_x or p_z by using the expressions in Eqs. (203).

The Hamiltonian for the perturbed motion is

$$K_{1\beta} = K_{0\beta} + V \quad (218)$$

so that the perturbed equations of motion are

$$
\begin{aligned}
J_{1x}' &= -\frac{\partial K_{1\beta}}{\partial \phi_{1x}} = \frac{2n_x}{R}|\kappa| J_{1x}^{m_x/2} J_{1z}^{m_z/2} \sin\psi_1 \\
J_{1z}' &= -\frac{\partial K_{1\beta}}{\partial \phi_{1z}} = \frac{2n_z}{R}|\kappa| J_{1x}^{m_x/2} J_{1z}^{m_z/2} \sin\psi_1 \\
\phi_{1x}' &= \frac{\partial K_{1\beta}}{\partial J_{1x}} = \frac{Q_x}{R} + \frac{m_x}{R}|\kappa| J_{1x}^{(m_x-2)/2} J_{1z}^{m_z/2} \cos\psi_1 \\
\phi_{1z}' &= \frac{\partial K_{1\beta}}{\partial J_{1z}} = \frac{Q_z}{R} + \frac{m_z}{R}|\kappa| J_{1x}^{m_x/2} J_{1z}^{(m_z-2)/2} \cos\psi_1 \,.
\end{aligned}
\tag{219}
$$

By applying a canonical transformation generated by [COURA84, RUTH86]

$$
F_2 = (\phi_{1x},\phi_{1z},J_{2x},J_{2z};s) = (n_x\phi_{1x} + n_z\phi_{1z} - p\frac{s-s_0}{R})\,J_{2x} + \phi_{1z}J_{2z}\,,
\tag{220}
$$

for which

$$
\begin{aligned}
\phi_{2x} &= \frac{\partial F_2}{\partial J_{2x}} = n_x\phi_{1x} + n_z\phi_{1z} - p\frac{s-s_0}{R}\,, &\quad J_{1x} &= \frac{\partial F_2}{\partial \phi_{1x}} = n_x J_{2x}\,, \\
\phi_{2z} &= \frac{\partial F_2}{\partial J_{2z}} = \phi_{1z}\,, & J_{1z} &= \frac{\partial F_2}{\partial \phi_{1z}} = n_z J_{2x} + J_{2z}\,, \\
K_{2\beta} &= K_{1\beta} + \frac{\partial F_2}{\partial s}\,,
\end{aligned}
\tag{221}
$$

we find the Hamiltonian

$$
K_{2\beta} = \frac{1}{R}(n_xQ_x + n_zQ_z - p)J_{2x} + \frac{1}{R}Q_zJ_{2z} + \frac{2}{R}|\kappa|(n_xJ_{2x})^{m_x/2}(n_zJ_{2x} + J_{2z})^{m_z/2}\cos(\phi_{2x} + \phi_\kappa)\,.
\tag{222}
$$

The new Hamiltonian is independent of s and ϕ_{2z}. It follows that J_{2z} and $K_{2\beta}$ are invariants or in the old variables

$$
\begin{aligned}
&\frac{J_{1x}}{n_x} - \frac{J_{1z}}{n_z} = \text{const.} \\
&Q_xJ_{1x} + Q_zJ_{1z} - \frac{p}{n_x}J_{1x} + 2|\kappa| J_{1x}^{m_x/2} J_{1z}^{m_z/2}\cos\psi_1 = \text{const.}
\end{aligned}
\tag{223}
$$

This implies that the motion is integrable in this case. From the form of the first invariant it is clear that the motion is bounded for a difference resonance.

Chapter 9

TIME-INDEPENDENT PERTURBATION THEORY

In order to relate the theory with measurements, it is preferable to have the explicit solutions as a function of s. Not much work has been done in this direction, probably owing to the use of Poincaré – von Zeipel procedure which has implicit transformation equations. Another reason may be that the main concern has been the study of stability, for which the explicit s-dependence is not needed. An attempt in this direction to obtain the frequency spectra for the non-linear betatron motion due to sextupoles has been done by Ando [ANDO84]. The problem of solving the implicit equations did not appear because he neglected the perturbation on the phase variable. This is inconsistent since the perturbation in the phase variable is linear in the sextupole strength, which explains the bad agreement with the numerical simulation. We will apply the Poincaré – von Zeipel's procedure to the single-resonance Hamiltonian from the previous chapter. Explicit solutions are obtained by using successive approximations to the transformation equations. This leads to the frequency spectra for the betatron motion close to a single resonance. Time-independent (or equivalently s-independent) refers to the fact that, in this case, we try to find the new constants of motion instead of the variation of the old ones due to the perturbation.

The Hamiltonian is divided into two parts:

$$K_{1\beta}(\phi_{1x},\phi_{1z},J_{1x},J_{1z};s) = K_{0\beta}(J_{1x},J_{1z};s) + V(\phi_{1x},\phi_{1z},J_{1x},J_{1z};s)\ , \tag{224}$$

where V is some perturbation to the unperturbed system. The idea is to find a canonical transformation where the new action variable is a constant of motion. This is normally possible only to some power in the perturbation parameter assumed to be small, so that the new action variable can depend on the angle and s in higher order. It is possible to find a canonical transformation to a new phase space where the new action variables are constants of motion. The generating function for the canonical transformation is [GUIGN78, COURA84, RUTH86, ANDO84]:

$$F_2(\phi_{1x},\phi_{1z},J_{2x},J_{2z};s) = \phi_{1x}J_{2x} + \phi_{1z}J_{2z} + G(\phi_{1x},\phi_{1z},J_{2x},J_{2z};s)\ , \tag{225}$$

where

$$G(\phi_{1x},\phi_{1z},J_{2x},J_{2z};s) = -2\frac{|\kappa|}{e}J_{2x}^{m_x/2}J_{2z}^{m_z/2}\sin\psi_1\ . \tag{226}$$

The distance from the resonance e is given by

$$e = n_xQ_x + n_zQ_z - p \tag{227}$$

and ψ_1 is defined by Eq. (215). The canonical transformation is then

$$
\begin{aligned}
\phi_{2x} &= \frac{\partial F_2}{\partial J_{2x}} = \phi_{1x} - m_x\frac{|\kappa|}{e}J_{2x}^{(m_x-2)/2}J_{2z}^{m_z/2}\sin\psi_1 \\
\phi_{2z} &= \frac{\partial F_2}{\partial J_{2z}} = \phi_{1z} - m_z\frac{|\kappa|}{e}J_{2x}^{m_x/2}J_{2z}^{(m_z-2)/2}\sin\psi_1 \\
J_{1x} &= \frac{\partial F_2}{\partial \phi_{1x}} = J_{2x} - 2n_x\frac{|\kappa|}{e}J_{2x}^{m_x/2}J_{2z}^{m_z/2}\cos\psi_1 \\
J_{1z} &= \frac{\partial F_2}{\partial \psi_{1z}} = J_{2z} - 2n_z\frac{|\kappa|}{e}J_{2x}^{m_x/2}J_{2z}^{m_z/2}\cos\psi_1 \,.
\end{aligned}
\tag{228}
$$

The new Hamiltonian is given by [COURA84, RUTH86]:

$$
\begin{aligned}
K_{2\beta}(\phi_{2x},\phi_{2z},J_{2x},J_{2z};s) &= K_{0\beta}(J_{2x},J_{2z};s) + \frac{\partial V}{\partial J_{2x}}\frac{\partial G}{\partial \phi_{1x}} + \frac{\partial V}{\partial J_{2z}}\frac{\partial G}{\partial \phi_{1z}} \\
&\quad + \frac{1}{2}\frac{\partial^2 K_{0\beta}}{\partial J_{2x}^2}\left(\frac{\partial G}{\partial \phi_{1x}}\right)^2 + \frac{1}{2}\frac{\partial^2 K_{0\beta}}{\partial J_{2z}^2}\left(\frac{\partial G}{\partial \phi_{1z}}\right)^2 \\
&= \frac{1}{R}(Q_xJ_{2x} + Q_zJ_{2z}) - \frac{|\kappa|^2}{Re}J_{2x}^{m_x-1}J_{2z}^{m_z-1}\,[n_xm_xJ_{2z} + n_zm_zJ_{2x}][1+\cos 2\psi_1]\,,
\end{aligned}
\tag{229}
$$

where Eq. (214) has been used.

The oscillating term $\cos 2\psi_1$ has zero average and may therefore be neglected as we are only interested in the long-term behaviour. The Hamiltonian is then only a function of the new action variables. Note that the perturbation in the new Hamiltonian is of second power in $|\kappa|$ so that the new action variable is now a constant of motion to second power.

Neglecting the oscillating part gives us the Hamiltonian

$$
K_{1\beta}(\phi_{2x},\phi_{2z},J_{2x},J_{2z};s) = \frac{1}{R}(Q_xJ_{2x} + Q_zJ_{2z}) - \frac{|\kappa|^2}{Re}J_{2x}^{m_x-1}J_{2z}^{m_z-1}\,[n_xm_xJ_{2z} + n_zm_zJ_{2x}]\,. \tag{230}
$$

Hamilton's equations are

$$
\begin{aligned}
J_{2x}' &= -\frac{\partial K_{2\beta}}{\partial \phi_{2x}} = 0 \\
J_{1z}' &= -\frac{\partial K_{2\beta}}{\partial \phi_{2z}} = 0 \\
\phi_{2x}' &= \frac{\partial K_{2\beta}}{\partial J_{2x}} = \frac{1}{R}Q_x - \frac{|\kappa|^2}{Re}J_{2x}^{m_x-2}J_{2z}^{m_z-2}\left[n_xm_x(m_x-1)J_{2z}^2 + n_zm_xm_zJ_{2x}J_{2z}\right] \\
\phi_{2z}' &= \frac{\partial K_{2\beta}}{\partial J_{2z}} = \frac{1}{R}Q_z - \frac{|\kappa|^2}{Re}J_{2x}^{m_x-2}J_{2z}^{m_z-2}\left[n_xm_xm_zJ_{2x}J_{2z} + n_zm_z(m_z-1)J_{2x}^2\right].
\end{aligned}
\tag{231}
$$

We find that the new action variables are constants of motion as expected. The equations for the phase variables are easily integrated giving

$$
\begin{aligned}
\phi_{2x}(s) &= (Q_x + \Delta Q_x)\frac{s - s_0}{R} + \phi_{2x}(s_0) \\
\phi_{2z}(s) &= (Q_z + \Delta Q_z)\frac{s - s_0}{R} + \phi_{2z}(s_0) \, ,
\end{aligned}
\tag{232}
$$

where the tune shifts ΔQ are

$$
\begin{aligned}
\Delta Q_x &= -\frac{|\kappa|^2}{e} J_{2x}^{m_x-2} J_{2z}^{m_z-2}\left[n_x m_x (m_x - 1) J_{2z}^2 + n_z m_x m_z J_{2x} J_{2z}\right] \\
\Delta Q_z &= -\frac{|\kappa|^2}{e} J_{2x}^{m_x-2} J_{2z}^{m_z-2}\left[n_x m_x m_z J_{2x} J_{1z} + n_z m_z (m_z - 1) J_{2x}^2\right] .
\end{aligned}
\tag{233}
$$

The new phases are related to the old ones by Eqs. (228) so that

$$
\begin{aligned}
\phi_{2x}(s_0) &= \phi_{1x}(s_0) - m_x \frac{|\kappa|}{e} J_{2x}^{(m_x-2)/2} J_{2z}^{m_z/2} \sin\psi_1(s_0) \\
\phi_{2z}(s_0) &= \phi_{1z}(s_0) - m_z \frac{|\kappa|}{e} J_{2x}^{m_x/2} J_{2z}^{(m_z-2)/2} \sin\psi_1(s_0) \, ,
\end{aligned}
\tag{234}
$$

where

$$
\psi_1(s_0) = n_x \phi_{1x}(s_0) + n_z \phi_{1z}(s_0) + \phi_\kappa \, . \tag{235}
$$

To find the motion in the original phase space, Eqs. (228) have to be solved for ϕ_{1x} and ϕ_{1z} . Since this cannot be done analytically it is done by successive approximations:

$$
\begin{aligned}
\phi_{1x}^{(1)} &= \phi_{2x} \, , \quad & \phi_{1x}^{(2)} &= \phi_{2x} + m_x \Delta_{1x} \sin\psi_2 \, , \\
\phi_{1z}^{(1)} &= \phi_{2z} \, , \quad & \phi_{1z}^{(2)} &= \phi_{2z} + m_z \Delta_{1z} \sin\psi_2 \, ,
\end{aligned}
\tag{236}
$$

where

$$
\begin{aligned}
\Delta_x &= \frac{|\kappa|}{e} J_{2x}^{(m_x-2)/2} J_{2z}^{m_z/2} \\
\Delta_z &= \frac{|\kappa|}{e} J_{2x}^{m_x/2} J_{2z}^{(m_z-2)/2}
\end{aligned}
\tag{237}
$$

and

$$
\psi_2 = n_x \phi_{2x} + n_z \phi_{2z} + \phi_\kappa - p\frac{s - s_0}{R} \, . \tag{238}
$$

The relations between the new action variables and the old ones can be obtained in a similar way

$$
\begin{aligned}
J_{2x}^{(1)} &= J_{1x} \, , \quad & J_{2x}^{(2)} &= J_{1x} + 2n_x \frac{|\kappa|}{e} J_{1x}^{m_x/2} J_{1z}^{m_z/2} \cos\psi_1 \, , \\
J_{2z}^{(1)} &= J_{1z} \, , \quad & J_{2z}^{(2)} &= J_{1z} + 2n_z \frac{|\kappa|}{e} J_{1x}^{m_x/2} J_{1z}^{m_z/2} \cos\psi_1 \, .
\end{aligned}
\tag{239}
$$

From Eqs. (215) and (238) it follows that

$$\psi_1 = \psi_2 + [n_x m_x \Delta_x + n_z m_z \Delta_z] \sin\psi_2 \,. \tag{240}$$

The motion in the coordinates x and z is given by Eq. (203) so that

$$\begin{aligned} x &= \sqrt{2J_{1x}\beta_x(s)}\,\{\cos[\psi_{1x}(s) + W_x(s)] - n_x\Delta_x \cos\psi_1(s) \cos[\psi_{1x}(s) + W_x(s)]\} + O(|\kappa|^2) \\ z &= \sqrt{2J_{1z}\beta_z(s)}\,\{\cos[\psi_{1z}(s) + W_z(s)] - n_z\Delta_z \cos\psi_1(s) \cos[\psi_{1x}(s) + W_z(s)]\} + O(|\kappa|^2) \,, \end{aligned} \tag{241}$$

where we have used Eqs. (228) and expanded the square root. From Eq. (236) it follows that

$$\cos\psi_{1x}(s) = \cos\psi_{2x}(s) - m_x\Delta_x \sin\psi_2(s) \sin\phi_{2x}(s) + O(|\kappa|^2) \,, \tag{242}$$

so that finally

$$\begin{aligned} x(s) &= \sqrt{2J_{2x}\beta_x(s)}\,\{\cos[\phi_{2x}(s) + W_x(s)] + k\Delta_x \cos[\psi_2(s) + \phi_{2x}(s) + W_x(s)] \\ &\quad - j\Delta_x \cos[\psi_2(s) - \phi_{2x}(s) - W_x(s)]\} + O(|\kappa|^2) \\ z(s) &= \sqrt{2J_{2z}\beta_z(s)}\,\{\cos[\phi_{2z}(s) + W_z(s)] + m\Delta_z \cos[\psi_2(s) + \phi_{2z}(s) + W_z(s)] \\ &\quad - l\Delta_z \cos[\psi_2(s) - \phi_{2z}(s) - W_z(s)]\} + O(|\kappa|^2) \,. \end{aligned} \tag{243}$$

The frequencies and corresponding amplitudes and phases are listed in Table 2 and Table 3.

Table 2: Frequencies for the horizontal betatron motion

Frequency	**Amplitude**	**Phase**
$Q_x + \Delta Q_x$	$\sqrt{2J_{2x}\beta_x(s_0)}$	$\phi_{2x}(s_0)$
$(n_x - 1)(Q_x + \Delta Q_x) + n_z(Q_z + \Delta Q_z)$	$\sqrt{2J_{2x}\beta_x(s_0)}\, j\Delta_x$	$(n_x - 1)\phi_{2x}(s_0) + n_z\phi_{2z}(s_0) + \phi_\kappa$
$(n_x + 1)(Q_x + \Delta Q_x) + n_z(Q_z + \Delta Q_z)$	$\sqrt{2J_{2x}\beta_x(s_0)}\, k\Delta_x$	$(n_x + 1)\phi_{2x}(s_0) + n_z\phi_{2z}(s_0) + \phi_\kappa$

Note that

$$\Delta = \begin{cases} > 0 \,, & e > 0 \\ < 0 \,, & e < 0 \end{cases} \tag{244}$$

so that π should be added to the phase if the corresponding term in Eqs. (243) is negative. Note also that the sign of the phase should be changed if the numerical value of the frequency is negative.

Table 3: Frequencies for the vertical betatron motion

Frequency	**Amplitude**	**Phase**
$Q_z + \Delta Q_z$	$\sqrt{2 J_{2z} \beta_z(s_0)}$	$\phi_{2z}(s_0)$
$n_x(Q_x + \Delta Q_x) + (n_z - 1)(Q_z + \Delta Q_z)$	$\sqrt{2 J_{2z} \beta_z(s_0)}\, l \Delta_z$	$n_x \phi_{2x}(s_0) + (n_z - 1)\, \phi_{2z}(s_0) + \phi_\kappa$
$n_x(Q_x + \Delta Q_x) + (n_z + 1)(Q_z + \Delta Q_z)$	$\sqrt{2 J_{2z} \beta_z(s_0)}\, m \Delta_z$	$n_x \phi_{1x}(s_0) + (n_z + 1)\, \phi_{2z}(s_0) + \phi_\kappa$

Chapter 10

THE DISCRETE FOURIER TRANSFORM

The betatron frequency spectra calculated in the previous chapter can be obtained from simulation of the motion or from beam measurements by using Fourier analysis in discrete time, known as the discrete Fourier transform (for example [KUNT80]). Since it will be used extensively in the later chapters we give a short introduction.

A function $f(t)$ that is periodic with period T in some continuous variable t can under quite general conditions be expanded as a Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i 2n\pi t/T} , \tag{245}$$

where the Fourier coefficients c_n are complex constants given by

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i n 2\pi t/T} \, \mathrm{d}t . \tag{246}$$

For an aperiodic function $f(t)$ this is generalized to

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, \mathrm{d}\omega , \tag{247}$$

where the Fourier transform $F(\omega)$ is given by

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, \mathrm{d}t . \tag{248}$$

If the function $f(t)$ is only defined for some sequence of equidistant points $k\Delta t$ where $k = 0,1,\ldots,N-1'$, one can define

$$x_k = f(k\Delta t) = \sum_{n=0}^{N-1} X_n e^{i 2\pi k\Delta m/N} , \quad k = 0,1,\ldots,N-1 , \tag{249}$$

where the complex constants X_n are given by the Discrete Fourier Transform (DFT) as

$$X_n = \frac{1}{N} \sum_{k=0}^{N-1} x_k e^{-i 2\pi k\Delta m/N} , \quad n = 0,1,\ldots,N-1 . \tag{250}$$

From the form of Eqs. (249) and (250) it is clear that both x_k and X_k are periodic:

$$\begin{aligned} x_k &= x_{k+jN} , \\ X_n &= X_{n+jN} , \end{aligned} \qquad j = \text{integer} \tag{251}$$

if we extend the domain for which they are defined. It can be shown that X_n are a set of samples from the Fourier transform (248) of the continuous function $f(t)$. Furthermore, these samples unambigiously specify the samples x_k.

The DFT can be used for frequency analysis by a digital computer of a signal continuous in time. The N samples x_K are stored in the computer and the DFT is calculated. A straightforward calculation, as given by Eq. (249), is however quite time consuming since the calculation time is proportional to N^2. In the case when N is a power of two one can use an algorithm for the calculation which is known as the Fast Fourier Transform (FFT) [COOLE65], for which the calculation time is proportional to $N \cdot {}^2\log N$.

Owing to the fact that we only take samples of the signal there is a limitation, in the height of the frequencies that one can resolve. It follows from Eqs. (250) and (251) that the Fourier transform of the sampled signal at ω_n is the sum of the values of the Fourier transform of the continuous signal at the frequencies

$$\omega_n + j\omega_s , \quad j = \text{any integer} , \tag{252}$$

where

$$\begin{aligned} \omega_n &= 2\pi \frac{n}{N} \\ \omega_s &= \frac{2\pi}{\Delta t} . \end{aligned} \tag{253}$$

For the sampled signal one thus cannot distinguish the contributions from these frequencies. This effect is known as aliasing. This implies that the highest frequency that can be resolved is ω_s.

The truncation of the signal to N samples leads to the Gibbs phenomenon[10]. This can be improved by a technique known as windowing. The window is a suitable finite sequence

$$p(k\Delta t) = \begin{cases} \neq 0 , & 0 \leq k \leq N-1 \\ = 0 , & \text{otherwise} . \end{cases} \tag{254}$$

The Fourier transform (248) is then

$$P(\omega) = \frac{1}{2\pi} \sum_{K = -(N-1)/2}^{(N-1)/2} p(k\Delta t)\, e^{-ik2\pi\omega} . \tag{255}$$

The Fourier transform of the sequence

[10] The Fourier series approximation of a discontinuity leads to ripples before and after the discontinuity [CARSL30].

$$g(k\Delta t) = p(k\Delta t) f(t) \tag{256}$$

is given by

$$G(\omega) = P(\omega) * F(\omega) = \int_{-\infty}^{\infty} P(\tau) F(\omega - \tau)\, \mathrm{d}\tau \ . \tag{257}$$

The ripple can then be reduced by a suitable choice of $p(k\Delta t)$. In our case we have used a sine-window

$$p(k\Delta t) = \begin{cases} \sin \dfrac{k\pi}{N} \ , & 0 \leq k \leq N - 1 \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{258}$$

The Fourier transform (248) can be written in phase-amplitude form as

$$F(\omega) = A(\omega) e^{i\phi(\omega)} = |F(\omega)| e^{i \arg F(\omega)} \ . \tag{259}$$

The amplitude function $A(\omega)$ gives the frequency spectrum. Since the DFT only contains discrete samples of the amplitude function, interpolation methods are necessary to obtain accurate values, with a limited number of samples, for the frequencies and the corresponding amplitudes and phases. Eventual damping of the signal will then, however, introduce an error in the interpolation.

An elegant method to solve this problem has been developed by Asseo [ASSEO87]. We will describe the key idea briefly.

We start with a sequence of samples of the form

$$f(k\Delta t) = A \cos(\omega k \Delta t + \phi) \ , \qquad k = 0,1,\dots,N-1 \ . \tag{260}$$

We now introduce an amplitude modulation by calculating

$$\begin{aligned} g(k\Delta t) &= f(k\Delta t) \cos(\Delta\omega k \Delta t) = A \cos(\omega k \Delta t + \phi) \cos(\Delta\omega k\Delta t) \\ &= \frac{1}{2} A \{ \cos[(\omega + \Delta\omega) k\Delta t + \phi] + \cos[(\omega - \Delta\omega) k \Delta t + \phi] \} \end{aligned} \tag{261}$$

The result is two new frequencies, $\omega \pm \Delta\omega$, with half the amplitude of the original one but with the same phase! This is applied in an iterative way, where one first locates a peak in the frequency spectra, calculates the approximate frequency by interpolation, and subsequently amplitude modulates the samples so that in a new DFT one peak is moved to the middle of two samples of the Fourier transform. This reduces the effect of the damping in the interpolation for the frequency. By making a few iterations it is thus possible to get an accurate value for the frequency. Having the frequency it is then possible to make a modulation that puts one peak close to a sample of the Fourier transform, which reduces the effect of the damping in the estimation of the phase. By using this technique we obtain the frequency of a peak and the corresponding amplitude and phase to high accuracy even if the signal is damped and contains noise.

Chapter 11

NUMERICAL SIMULATION OF THE MOTION FOR A SINGLE RESONANCE

In order to verify the analytical results given in Chapter 9, one can simulate the perturbed motion for a single resonance by numerical integration of the perturbed equations of motion (219). This will be done by using the Runge – Kutta algorithm [DAHLQ74]. Since it may be interesting to study this perturbed motion when the initial action is zero in one of the two planes, we have used Guignard's formalism [GUIGN78] with two complex amplitudes instead of the action-angle variables. This is because the phase is not defined when the action is zero. The simulated cases are shown in Table 4.

Table 4: Simulated resonances

j	k	l	m	p	Resonance	Type
1	0	1	0	5	$Q_x + Q_z = 5$	linear coupling resonance
3	0	0	0	7	$3Q_x = 7$	sextupolar resonance
1	0	2	0	8	$Q_x - 2Q_z = -3$	sextupolar difference resonance
1	0	0	2	−3	$Q_x + 2Q_z = 8$	sextupolar sum resonance

The simulation has been done with

$$\begin{aligned} |h_{1010\,5}| &= 3.5\times 10^{-3} && \text{measured} \\ |h_{3000\,7}| &= 0.14 && \text{used for extraction} \\ |h_{1002\,-3}| &= 0.41 && \text{calculated} \\ |h_{1020\,8}| &= 3.54 && \text{calculated} \end{aligned} \tag{262}$$

Calculated and measured values were obtained by using the methods described elsewhere [GUIGN78, BENGT86B][11].

A tune diagram is shown in Appendix D, where the normal working point at $Q_x = 2.305$, $Q_z = 2.725$ and the working point for extraction at $Q_x = 2.325$, $Q_z = 2.725$ have been marked. The width[12] of the resonances has also been plotted. Note that in the second case the

[11] The measurements were done by M. Chanel

simulation has been done at the working point for extraction.

The simulations give an idea of the perturbations due to the different resonances. The DFT method described in Chapter 10 has been applied to get the frequency spectra of the non-linear betatron motion and the corresponding amplitudes and phases. The results are shown in Figure 2 to Figure 5. In order to compare with the perturbation theory in Chapter 9 we have calculated J_{2x} and J_{2z} from Eqs. (239) by iteration using the given values for J_{1x}, J_{1z}, $\phi_{1x}(s_0)$, and $\phi_{1z}(s_0)$. These values have then been used in the first two equations of (228) to get $\phi_{2x}(s_0)$ and $\phi_{2z}(s_0)$. The calculated values are shown in the second column and the values obtained from the simulation are shown in the third column in each figure. The values related to the horizontal plane have been put above the values related to the vertical plane. Note that the vertical tune $Q_z = 2.725$ appears at 0.275 with negative phase owing to aliasing. We conclude that the agreement is quite good.

[12] The width of a sum resonance is defined as the region around a resonance in the tune diagram where the amplitude can grow to infinity, whereas for a difference it is the region where the extreme values of the amplitude may be reached [GUIGN78].

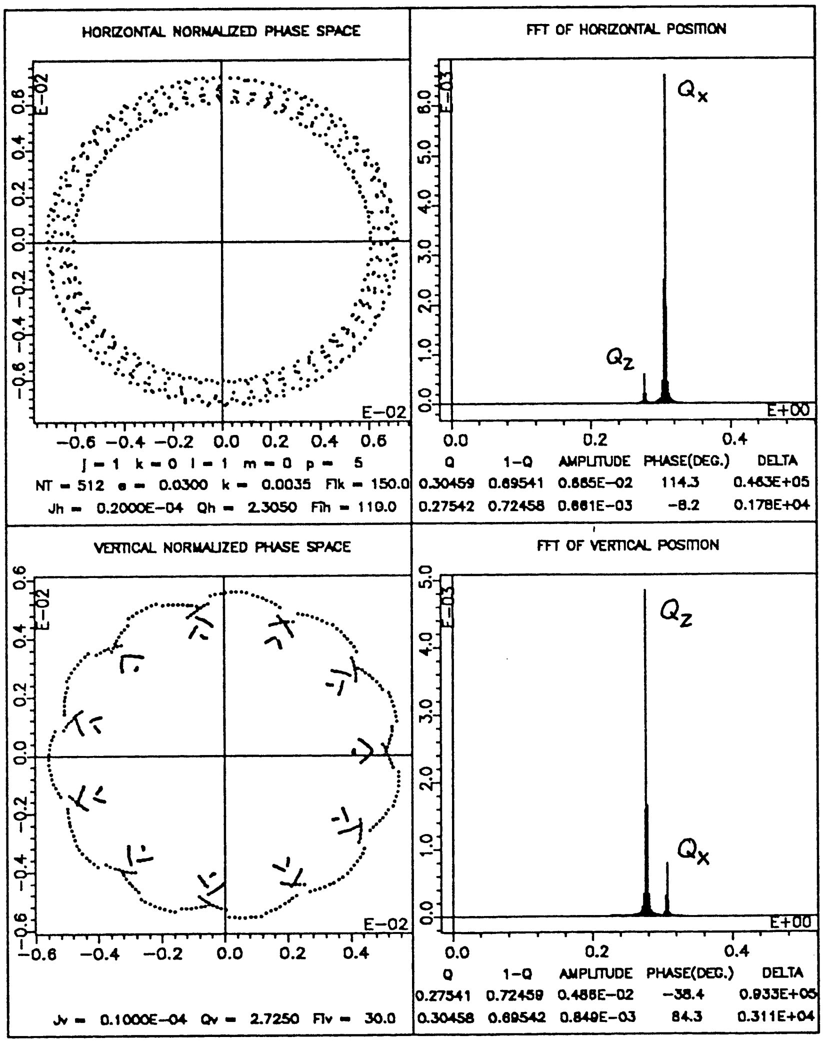

Figure 2: Simulation of linear coupling

$\sqrt{2\beta_x(s_0)\,J_{2x}}$	6.52×10^{-3}	6.65×10^{-3}
$\phi_{2x}(s_0)$	114.6°	114.3°
ΔQ_x	-4.1×10^{-4}	-4.1×10^{-4}
$\sqrt{2\beta_x(s_0)\,J_{2z}}\,\Delta_x$	5.53×10^{-4}	6.61×10^{-4}
$\phi_{2z}(s_0)+\phi_\kappa$	8.6°	8.2°
$\sqrt{2\beta_z(s_0)\,J_{2z}}$	4.74×10^{-3}	4.86×10^{-3}
$\phi_{2z}(s_0)$	38.6°	38.4°
ΔQ_z	-4.1×10^{-4}	-4.1×10^{-4}
$\sqrt{2\beta_z(s_0)\,J_{2x}}\,\Delta_z$	7.60×10^{-4}	8.49×10^{-4}
$\phi_{2x}(s_0)+\phi_\kappa$	84.6°	84.3°

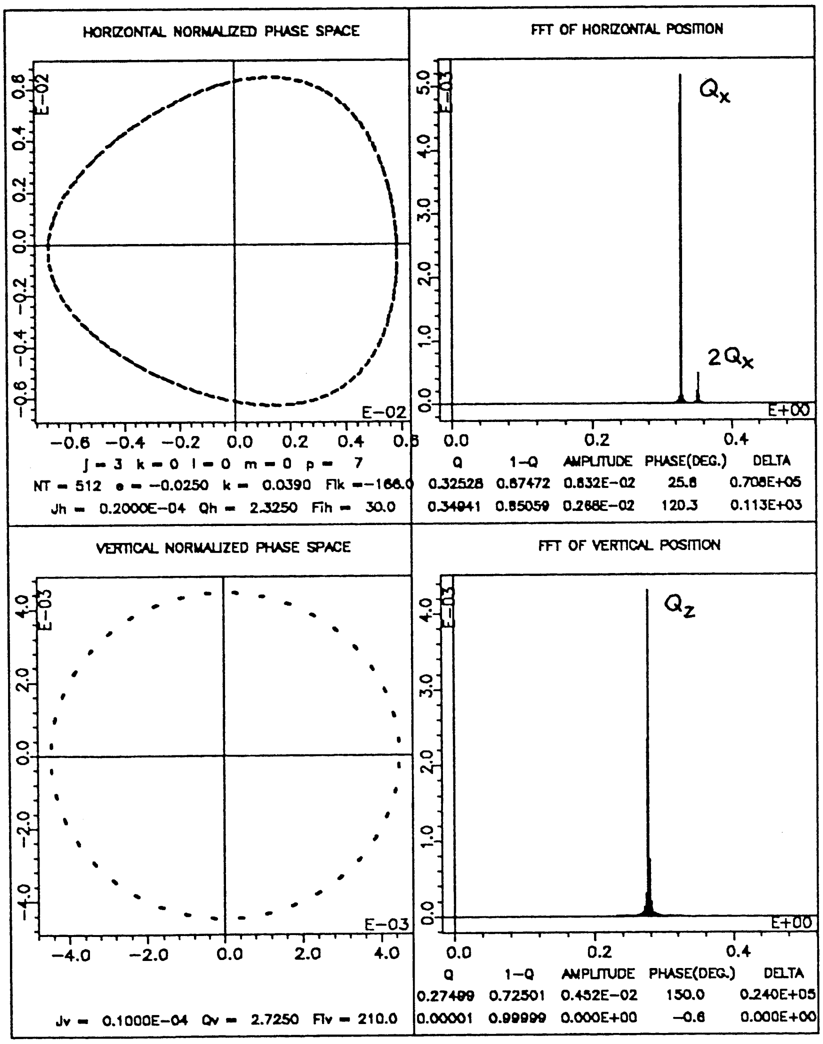

Figure 3: Simulation of a sextupolar resonance

$\sqrt{2\beta_x(s_0)\,J_{2x}}$	6.22×10^{-3}	6.32×10^{-3}
$\phi_{2x}(s_0)$	26.0°	25.6°
ΔQ_x	2.6×10^{-4}	2.8×10^{-4}
$\sqrt{2\beta_x(s_0)\,J_{2x}}3\Delta_x$	4.52×10^{-3}	2.68×10^{-3}
$2\phi_{2x}(s_0) + \phi_\kappa$	−114.1°	−120.3°

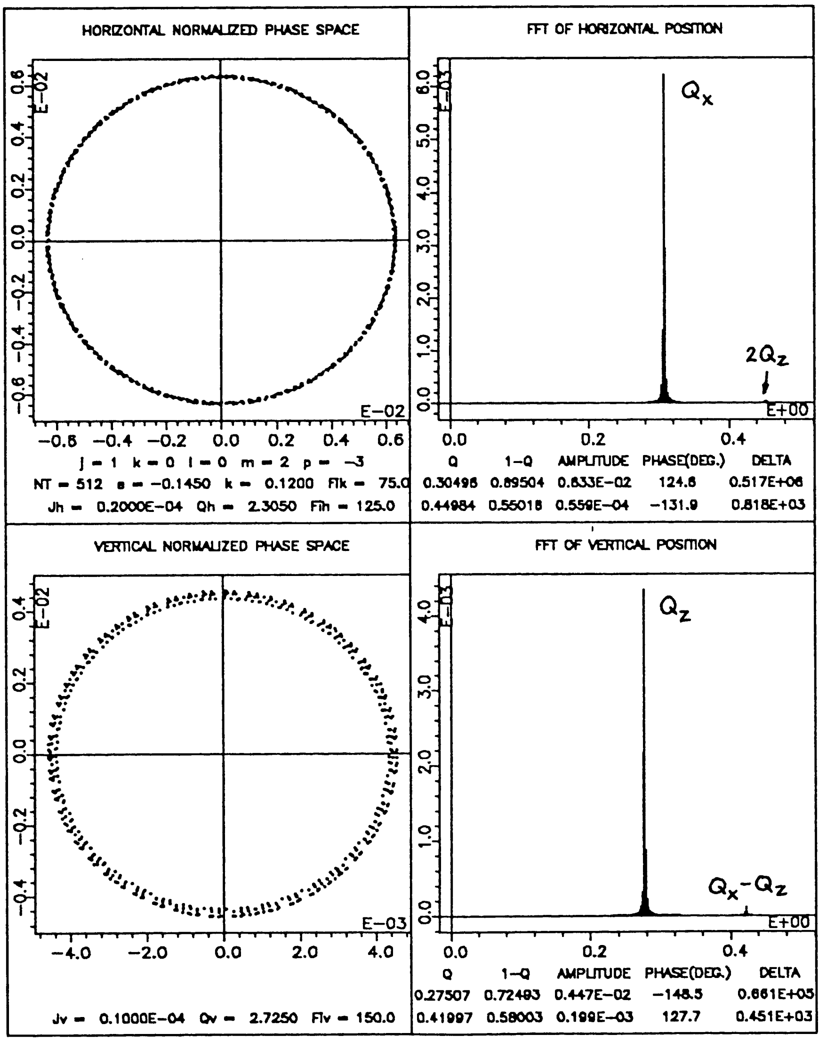

Figure 4: Simulation of a sextupolar difference resonance

$\sqrt{2\beta_x(s_0)\,J_{2x}}$	6.33×10^{-3}	6.33×10^{-3}
$\phi_{2x}(s_0)$	124.6°	124.6°
ΔQ_x	-5×10^{-5}	-4×10^{-5}
$\sqrt{2\beta_x(s_0)\,J_{2x}}\,\Delta_x$	4.10×10^{-5}	5.59×10^{-5}
$2\phi_{2z}(s_0) + \phi_\kappa$	− 138.0°	− 131.9°
$\sqrt{2\beta_z(s_0)\,J_{2z}}$	4.45×10^{-3}	4.47×10^{-3}
$\phi_{2z}(s_0)$	148.5°	148.5°
ΔQ_z	-7×10^{-5}	-7×10^{-5}
$\sqrt{2\beta_z(s_0)\,J_{2z}}\,2\,\Delta_z$	1.17×10^{-4}	1.99×10^{-4}
$\phi_{2x}(s_0) - \phi_{2z}(s_0) + \phi_\kappa$	128.9°	127.7°

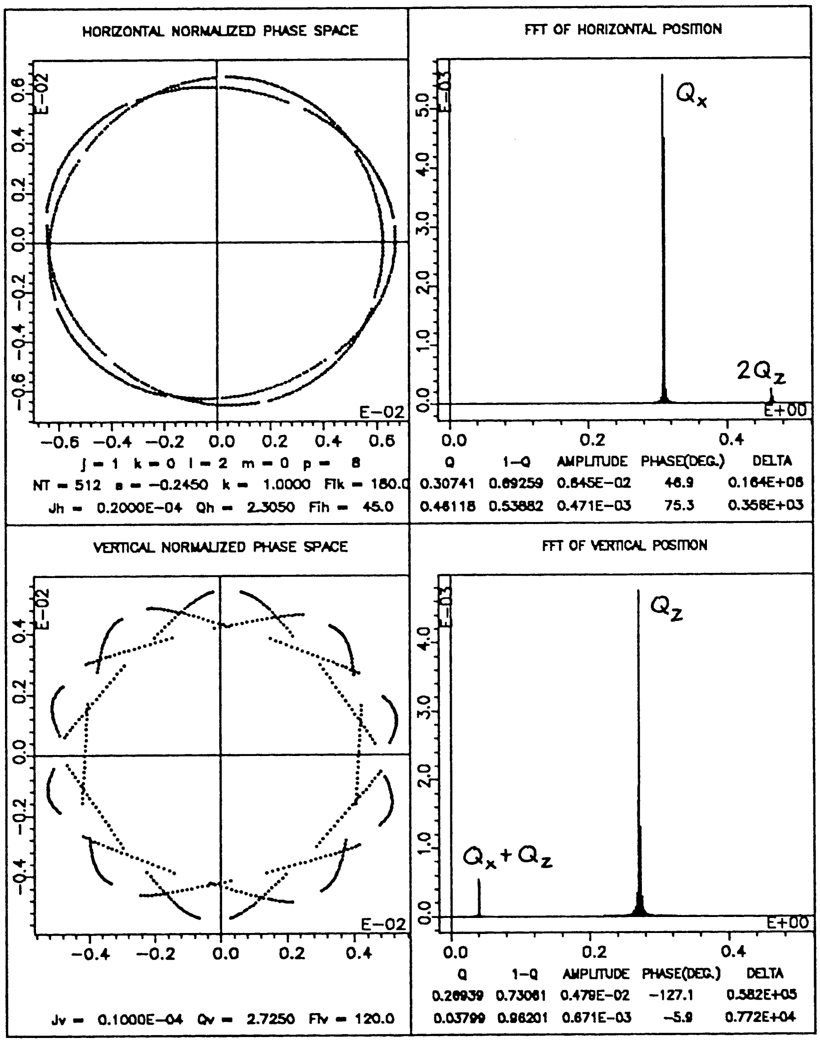

Figure 5: Simulation of a sextupolar sum resonance

$\sqrt{2\beta_x(s_0)\,J_{2x}}$	6.38×10^{-3}	6.45×10^{-3}
$\phi_{2x}(s_0)$	46.9°	46.9°
ΔQ_x	2.19×10^{-3}	2.41×10^{-3}
$\sqrt{2\beta_x(s_0)\,J_{2x}}\,\Delta_x$	2.19×10^{-4}	4.71×10^{-4}
$2\phi_{2z}(s_0)+\phi_\kappa$	74.4°	75.3°
$\sqrt{2\beta_z(s_0)\,J_{2z}}$	4.63×10^{-3}	4.79×10^{-3}
$\psi_{2z}(s_0)$	127.2°	127.1°
ΔQ_z	5.25×10^{-3}	5.61×10^{-3}
$\sqrt{2\beta_z(s_0)\,J_{2z}}\,2\,\Delta_z$	6.03×10^{-4}	6.71×10^{-4}
$\phi_{2x}(s_0)+\phi_{2z}(s_0)+\phi_\kappa$	−5.9°	−5.9°

Chapter 12

MEASUREMENTS OF THE PHASE ADVANCE IN LEAR

It is possible to measure the centre of the charge in a bunched beam by using electrostatic pick-ups [BERNA83] to get the average position of the bunch at one location in the accelerator for each turn. Such a system has been developed for LEAR (Papy-Q system) [ASSEO85]. It has been widely used to monitor the tune of the betatron motion. In this case one excites strong coherent betatron oscillations by kicking a well-cooled[13] beam with, for example, the injection kicker. The signal from a pick-up is then Fourier-analysed by using the DFT technique. This gives the betatron frequency. However, since one also obtains the corresponding phase, one can measure the phase advance between two pick-ups by storing the data from two pick-ups in the same plane. It is possible to make a comparison with the calculated lattice functions from programs such as COMFORT [WOODL83][14] used in the design of an accelerator.

Since the injection kicker only kicks in the horizontal plane the measurements are restricted to this plane. A typical result is shown in Figure 6 and the measured phase advances are presented in Table 5.

Table 5: Phase advance

Pick-ups	**Measured phase advance** (degrees)	**Calculated by COMFORT** (degrees)
UEH13 − UEH14	15.4	16.0
UEH14 − UEH23	192.1	191.2
UEH21 − UEH22	120.7	118.3
UEH22 − UEH23	34.1	36.3
UEH23 − UEH24	15.9	16.0

Each measured value is from a single measurement. The calculated values have been obtained from COMFORT data for the working point $Q_{0x} = 2.305$, $Q_{0z} = 2.725$ with adjustment for the measured tune $Q_x \approx 2.302$ by the linear interpolation

---

13 By a cool beam here is meant a beam where the particles' transverse momenta are small compared to their longitudinal momenta and the spreads in the longitudinal momenta are small (compare with a laser beam).

14 The lattice calculations were done by R. Giannini (unpublished).

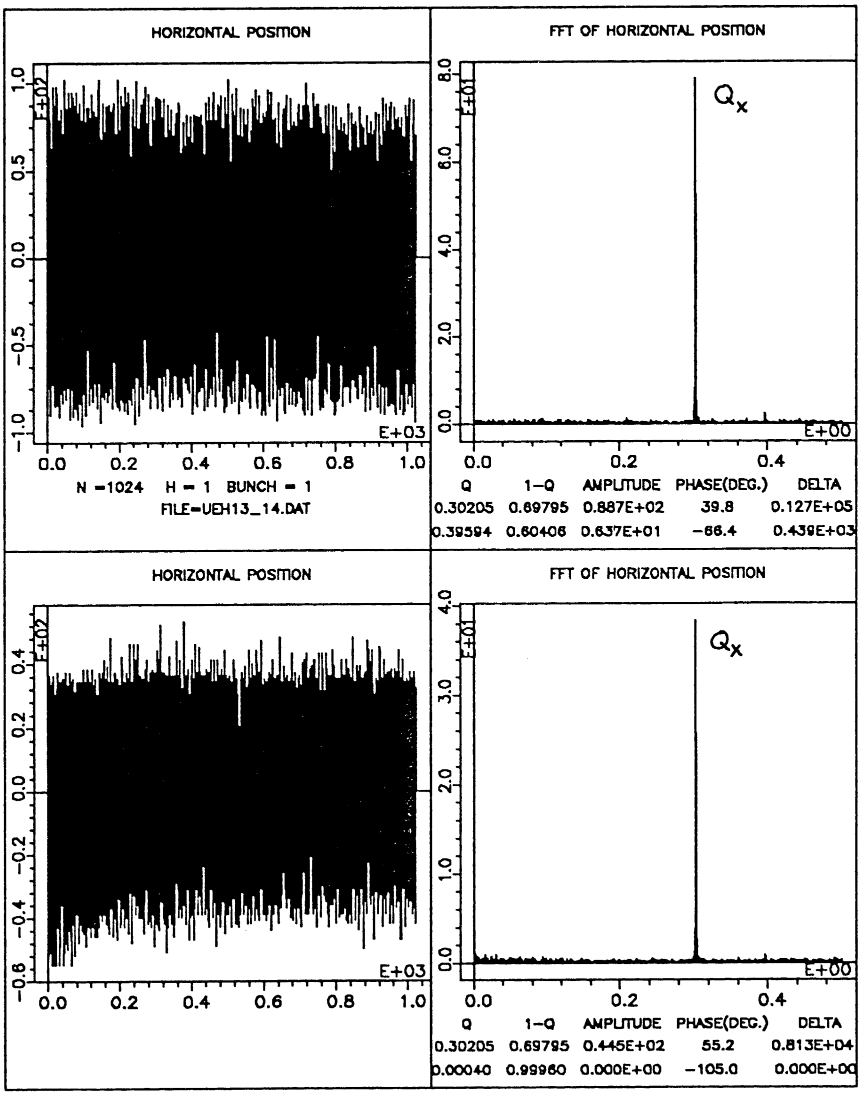

Figure 6: Measurement of the phase advance

$$\Delta\mu_c = \Delta\mu \left[1 + \frac{Q_x - Q_{0x}}{Q_{0x}} \right] . \tag{263}$$

We conclude that the agreement is quite good.

Chapter 13

MEASUREMENT AND COMPENSATION OF $Q_x + Q_z = 5$ IN LEAR

In this Chapter we will show that it is possible to obtain information about the fields the beam feels by studying the frequency spectra of the betatron motion. The frequency spectra are obtained by kicking a well-cooled bunched beam. The motion is sampled at each turn by using electrostatic pick-ups [BERNA83] and the Papy-Q system [ASSEO85]. The samples are Fourier-analysed by using the DFT technique [ASSEO87] shortly described in Chapter 10 In particular it is possible to derive the type of driving field and to measure the amplitude and the phase of a given resonance.

13.1 The resonance $Q_x + Q_z = 5$

The resonance $Q_x + Q_z = 5$ can be excited by both skew quadrupolar and skew octupolar fields. One can also have a contribution from the resonance $2Q_x + 2Q_z = 10$ driven by normal octupolar fields. These cases are listed in Table 6.

Table 6: Possible excitations of $Q_x + Q_z = 5$

j	k	l	m	p	Type
1	0	1	0	5	skew quadrupole
2	1	1	0	5	skew octupole
1	0	2	1	5	skew octupole
2	0	2	0	10	octupole

To correctly compensate this resonance it is therefore necessary to distinguish between the different driving sources. From Chapter 9 it is clear that the frequency spectra for the betatron motion are different for the cases listed. This is verified by simulation using the method described in Chapter 11 The results are shown in Figure 7 to Figure 10. We conclude that it is possible to obtain information about the type of driving field by studying the frequency spectra.

The frequency spectra are obtained by using the same method as in Chapter 12 However, to be sure that the different contributions to the resonance are excited the beam should be kicked in both planes. This can be done by mis-steering the injection. The result is shown in Figure 11.

The spectra show that the main driving term is from skew quadrupoles. Note that if skew octupoles were present we would expect contributions from both cases listed in Table 6.

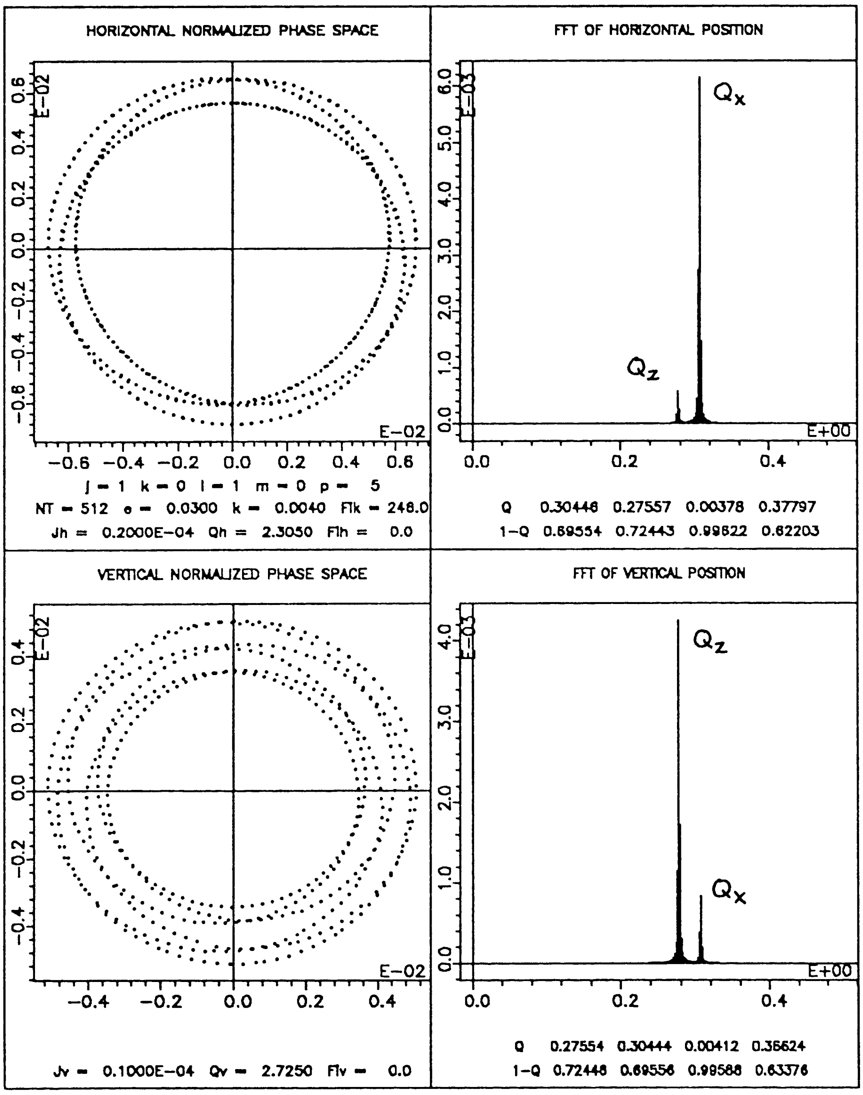

Figure 7: Simulation of skew quadrupoles

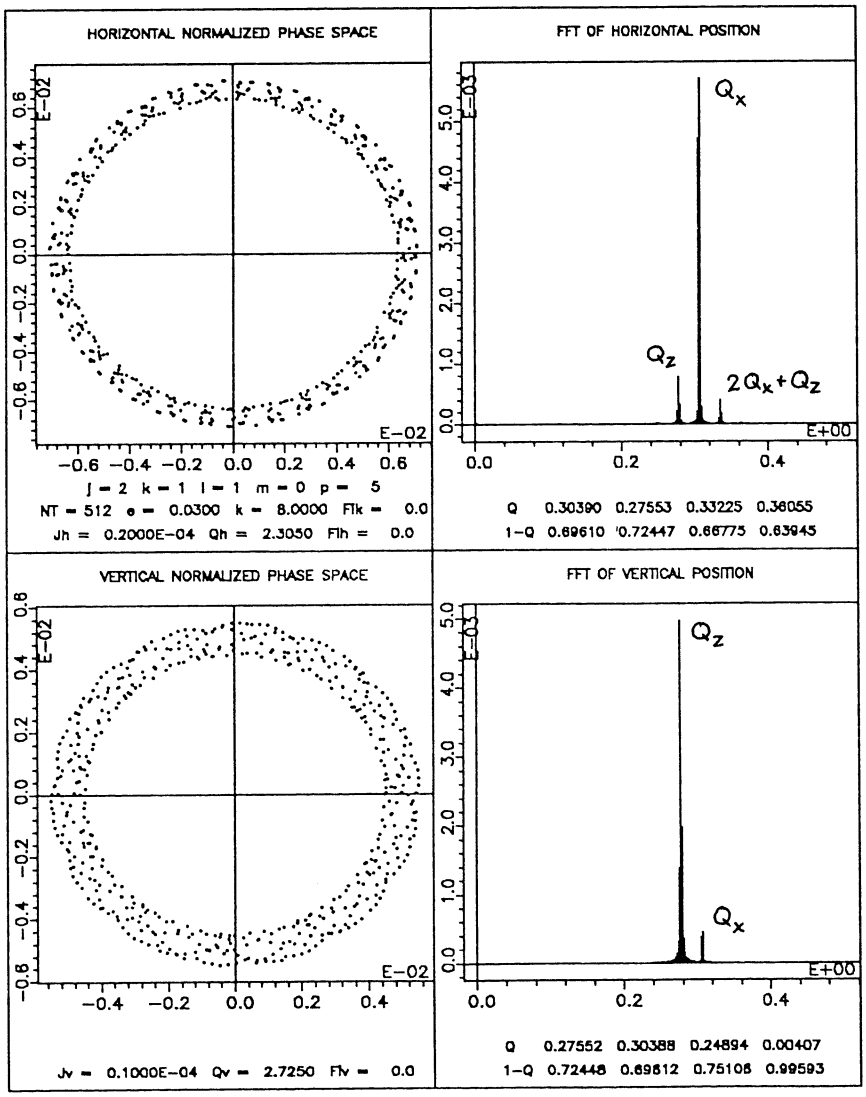

Figure 8: Simulation of skew octupoles

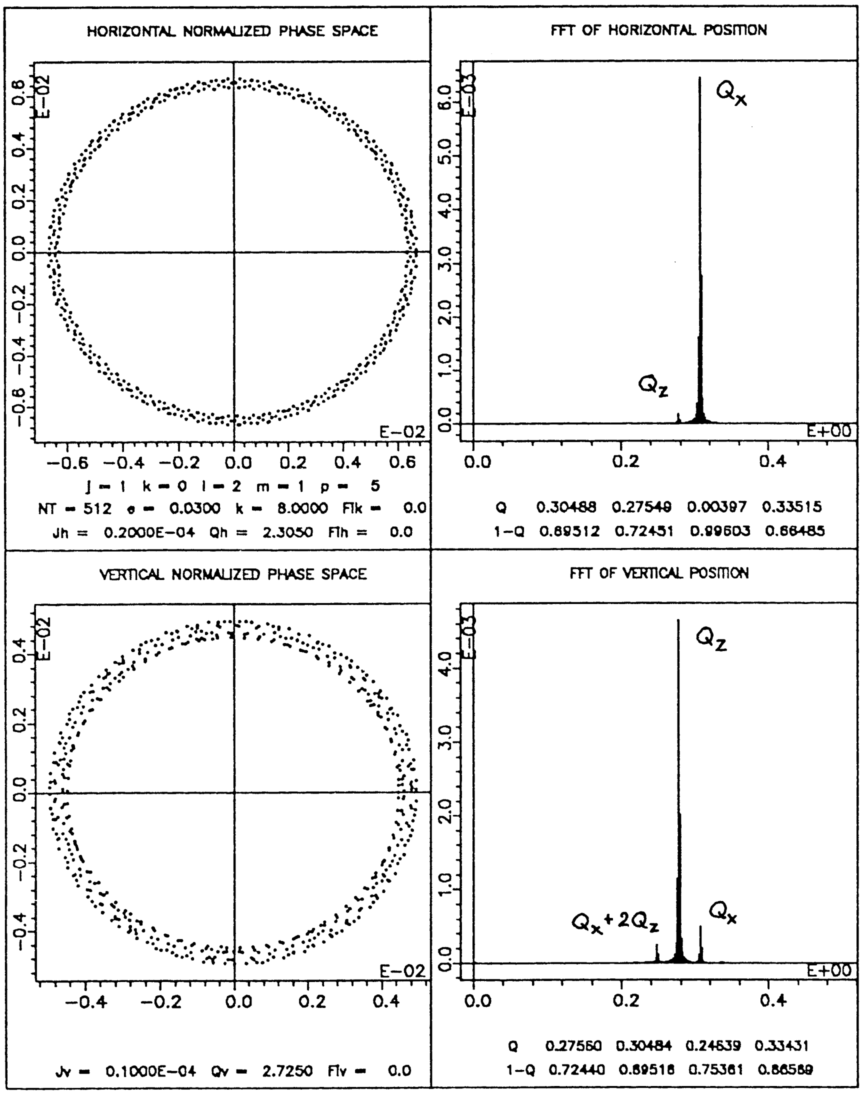

Figure 9: Simulation of skew octupoles

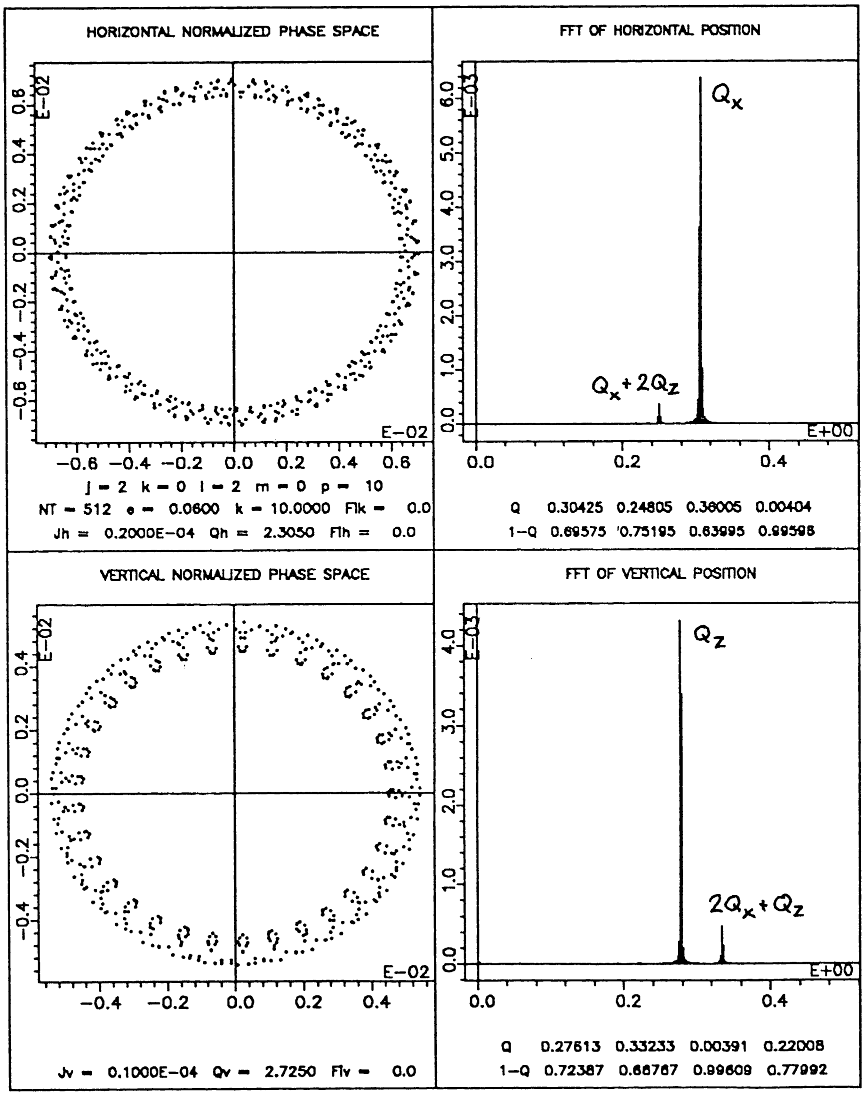

Figure 10: Simulation of normal octupoles

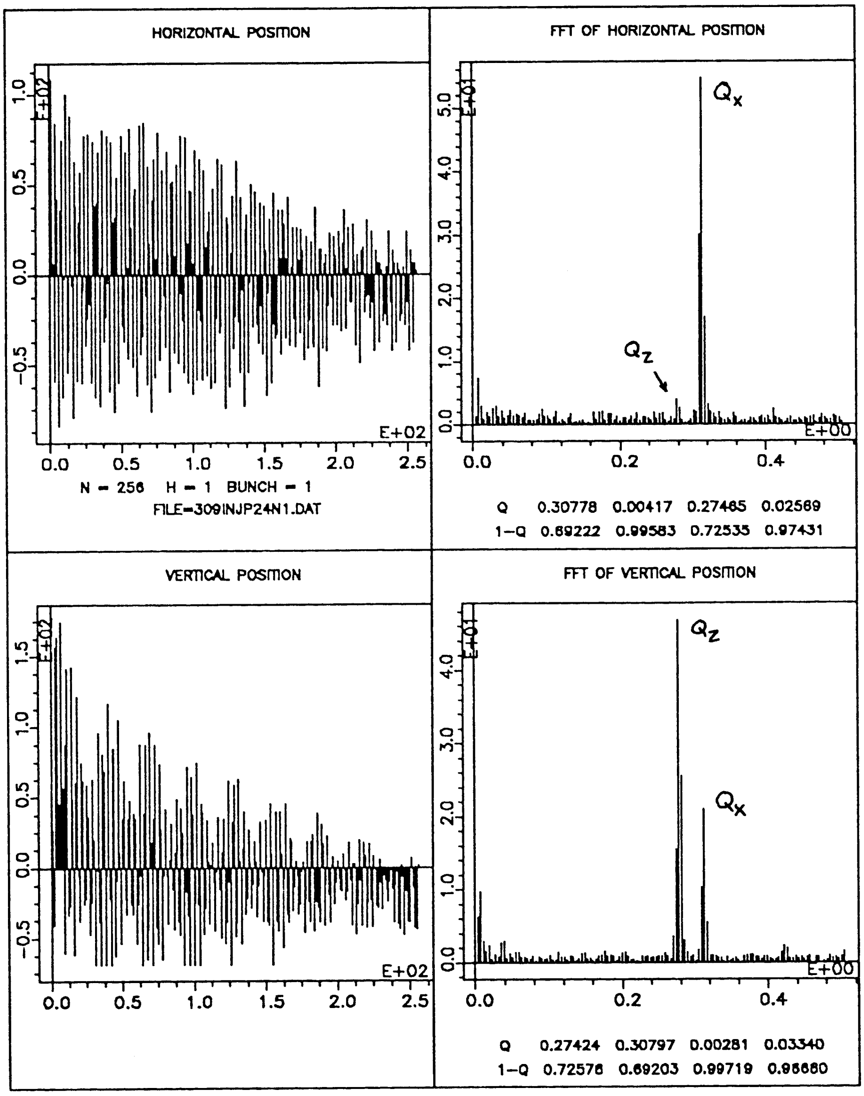

Figure 11: Measurement done at injection, e = 0.034

13.2 Measurements of the amplitude and the phase of $Q_x + Q_z = 5$

The frequency spectra for skew quadrupoles are obtained from the theory in Chapter 9 The result is presented in Table 7 and Table 8. From the tables it follows that

$$|\kappa| = \frac{ea_z^{(1)}}{a_x^{(0)}}\sqrt{\frac{\beta_x(s_0)}{\beta_z(s_0)}} = \frac{ea_x^{(1)}}{a_z^{(0)}}\sqrt{\frac{\beta_z(s_0)}{\beta_x(s_0)}}\ , \tag{264}$$

$$\phi_k = \phi_z^{(1)} - \phi_x^{(0)} = \phi_x^{(1)} - \phi_z^{(0)}\ ,$$

where e can be approximated by the measured tunes $Q_x^{(0)}$, $Q_z^{(0)}$, and $\beta_x(s_0)$, $\beta_z(s_0)$ are known from the lattice calculations for the linear motion. It follows that the amplitude and the phase of the resonance can be obtained from the measured frequency spectra.

Table 7: Frequency spectra for the horizontal plane

Frequency	Amplitude	Phase
$Q_x^{(0)} = Q_x + \Delta Q_x$	$a_x^{(0)} = \sqrt{2J_{2x}\beta_x(s_0)}$	$\phi_x^{(0)} = \phi_{2x}(s_0)$
$Q_x^{(1)} = Q_z + \Delta Q_z$	$a_x^{(1)} = \sqrt{2J_{2z}\beta_x(s_0)}\frac{\lvert\kappa\rvert}{e}$	$\phi_x^{(1)} = \phi_{2z}(s_0) + \phi_\kappa$

Table 8: Frequency spectra for the vertical plane

Frequency	Amplitude	Phase
$Q_z^{(0)} = Q_z + \Delta Q_z$	$a_z^{(0)} = \sqrt{2J_{2z}\beta_z(s_0)}$	$\phi_z^{(0)} = \phi_{2z}(s_0)$
$Q_z^{(1)} = Q_x + \Delta Q_x$	$a_z^{(1)} = \sqrt{2J_{2x}\beta_z(s_0)}\frac{\lvert\kappa\rvert}{e}$	$\phi_z^{(1)} = \phi_{2x}(s_0) + \phi_\kappa$

In the case of skew quadrupoles it is sufficient to kick in the horizontal plane to excite the resonance. The measurements are done by giving a well-cooled bunched beam a kick of about 0.50 mrad (10/5 kV on kicker KFH42). The betatron oscillations are sampled with a horizontal and a vertical pick-up using the Papy-Q system, and the frequency spectra are obtained by the DFT technique. An example is shown in Figure 12. Note that the amplifier gain may vary between the horizontal and vertical planes and also between different pictures in the following.

Owing to the fact that the difference in total amplification for the two pick-ups is not known, we can only obtain a relative value for $|\kappa|$. The result from a single measurement is

$$|\kappa| \sim 3.2\times 10^{-2}\,, \quad (\phi_\kappa)_{\mathrm{UEH24}} = 42° \tag{265}$$

at the pick-ups UEH24, UEV24. The phase of the resonance with respect to $s = 0$ is then

$$(\phi_\kappa)_0 = (\phi_\kappa)_{\mathrm{UEH24}} + (\mu_x)_{\mathrm{UEH24}} + (\mu_z)_{\mathrm{UEH24}} = 69°\ . \tag{266}$$

The resonance can be compensated by tilting one of the normal quadrupoles close to the position given by ϕ_κ. The quadrupole QDN31 was used for which

$$(\phi_\kappa)_0 = 66°\ . \tag{267}$$

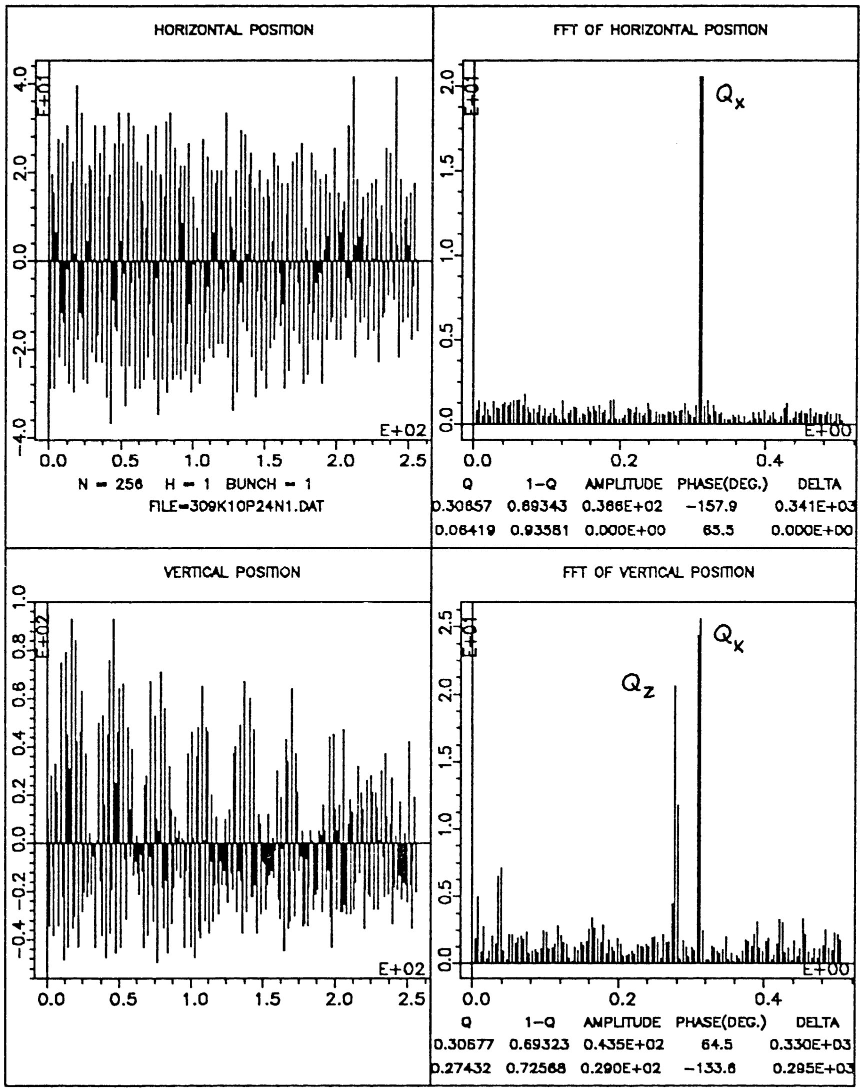

Figure 12: Measurement done before compensation, e = 0.032

Since we did not know the absolute value for $|\kappa|$ the tilt was tried out in three steps. The final tilt was 3.2 mrad. Measurements done after the compensation clearly indicate that the coupling has been reduced, which can be seen in Figure 13.

The measurements can be improved by moving closer to the resonance. We then find

$$|\kappa| \sim (2.2 \pm 0.014) \times 10^{-3}, \quad (\phi_\kappa)_{\mathrm{UEH24}} = 58 \pm 5.4^\circ \tag{268}$$

as the average from two measurements. An example is shown in Figure 14.

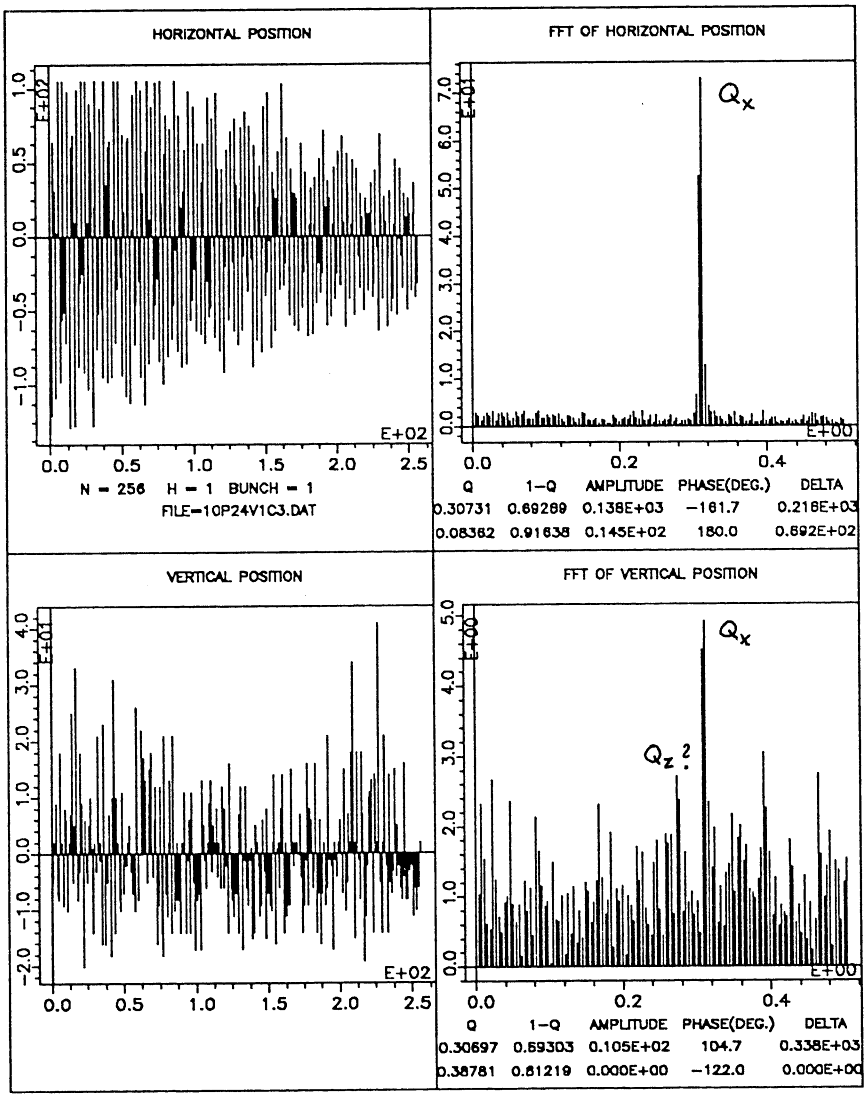

Figure 13: Measurement done after compensation, e = 0.033

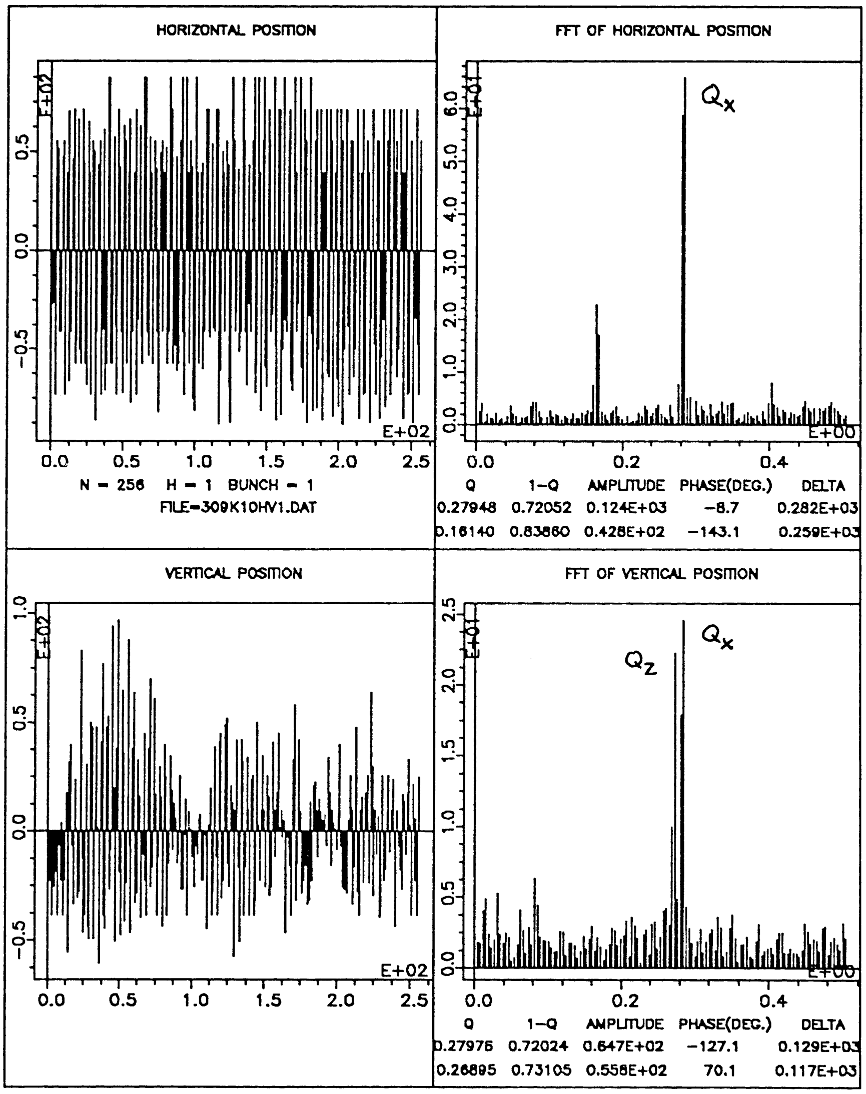

Figure 14: Measurement done after compensation, e = 0.011

13.3 Source of the skew quadrupolar fields

The mechanical alignment errors have been measured [GRAND88] and are presented in Table 9. A plus sign indicates a tilt towards the centre of the accelerator. k is -1.26 for horizontally focusing quadrupoles (QFN) and 1.40 for horizontally defocusing (QDN). From the vector potential for a quadrupole it follows that

$$\frac{e}{p_0} A_s^q = \frac{1}{2} k (x_t^2 - z_t^2) = \frac{1}{2} k [(x \cos\phi + z \sin\phi)^2 - (z \cos\phi - x \sin\phi)^2] \tag{269}$$

so that the skew quadrupolar component is

Table 9: Mechanical alignment errors for the quadrupoles in LEAR

Element	Tilt (mrad)	Element	Tilt (mrad)
QDN11	0.38	QDN31	−3.2
QFN11	0.18	QFN31	0.31
QFN12	−0.40	QFN32	−0.29
QDN12	0.12	QDN32	0.28
QDN21	−0.25	QDN41	−0.53
QFN21	0.02	QDF41	0.06
QFN22	−0.49	QDF42	−0.45
QDN22	−0.10	QDN42	0.07

$$\frac{e}{p_0} A_s^{sq} = (k \sin 2\phi) xz \approx (2k\phi) xz \ , \tag{270}$$

where we have used the fact that ϕ is small. The contribution to the resonance is calculated from Eqs. (71) and (209). Since the lengths of the quadrupoles are small compared to the variation of the beta function and the phase advance one can use the thin-element approximation for the evaluation of the integral in Eq. (209). The contribution from QDN31 is

$$(h_{1010\,5})_{comp} = 4.0 \times 10^{-3} e^{i\,(66^\circ + 180^\circ)} \tag{271}$$

and from the rest of the quadrupoles it is

$$(h_{1010\,5})_{sq} = 7.4 \times 10^{-4} e^{i\,192^\circ} \tag{272}$$

for the working point $Q_x = 2.3$, $Q_z = 2.73$.

Simulations of these contributions have been done by using FORTRAN code generated by the method described in Chapter 7 for thin skew quadrupoles. The result is shown in Figure 15 and Figure 16. Figure 15 may be compared with Figure 7 since the simulation for the latter case was done with the values in Eq. (271). Note that, in the former case, we also have a small contribution from $Q_x - Q_z = p$. It is clear that the perturbations from the skew quadrupoles in Figure 16 are much smaller than what we find for the compensation in Figure 15. There must be another source which gives the main contribution to the resonance.

The closed-orbit distortions in the vertical plane for LEAR are not corrected owing to the lack of vertical correction dipoles. The measured vertical orbit is shown in Figure 17, where we have marked the positions of the sextupoles. Note that the measurement of the closed orbit only gives samples at the pick-up locations. The values between the pick-ups are extrapolated by using spline fitting[15]. Owing to the fact at that the closed orbit does not pass through the centre of the sextupoles, we have a skew quadrupole component given by

[15] The measurements and calculations for the closed orbit have been done by M. Chanel and D. Manglunki.

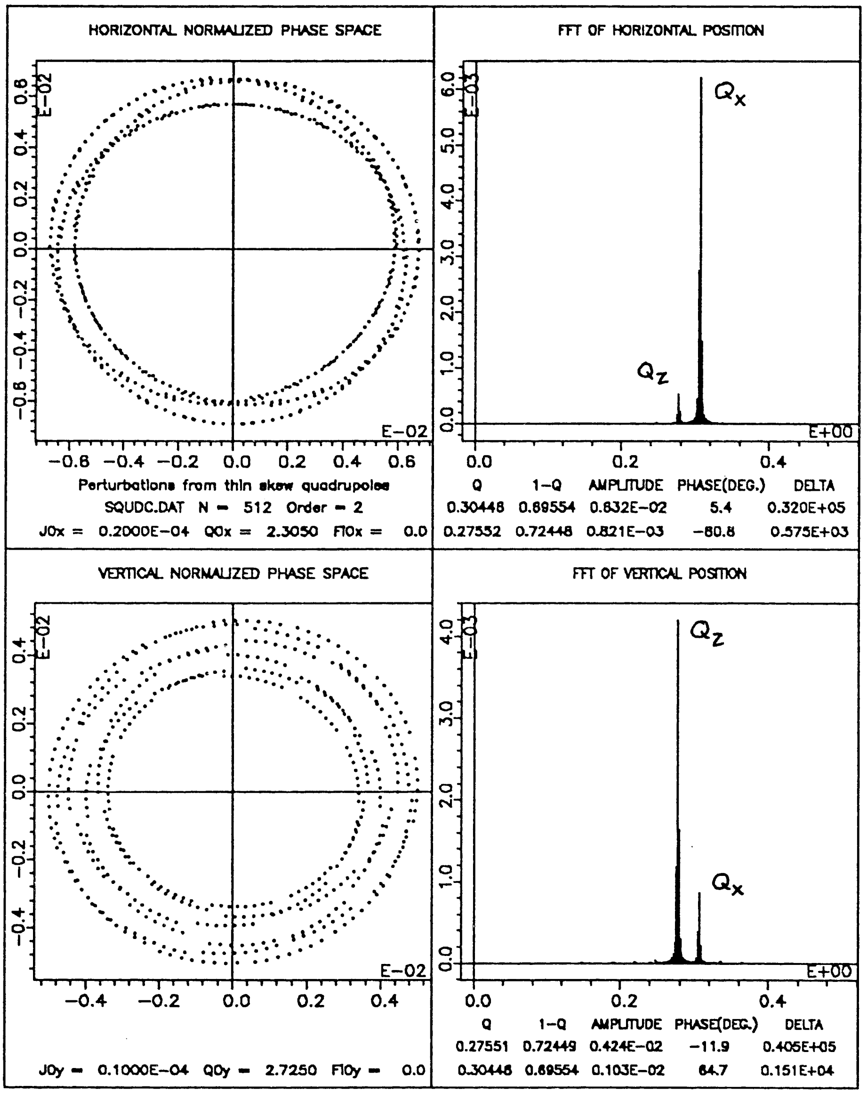

Figure 15: Contribution from QDN31

$$\frac{e}{p_0} A_s^s = \frac{1}{2} m x z^2 = \frac{1}{2} m(x_0 + X)(z_0 + Z)^2 \tag{273}$$

so that

$$\frac{e}{p_0} A_x^{\mathrm{sq}} = (m z_0) X Z \tag{274}$$

giving a contribution

$$(h_{1010\,s})_{\mathrm{co}} = 3.4 \times 10^{-3} e^{i\,108^\circ} . \tag{275}$$

A simulation is shown in Figure 18.

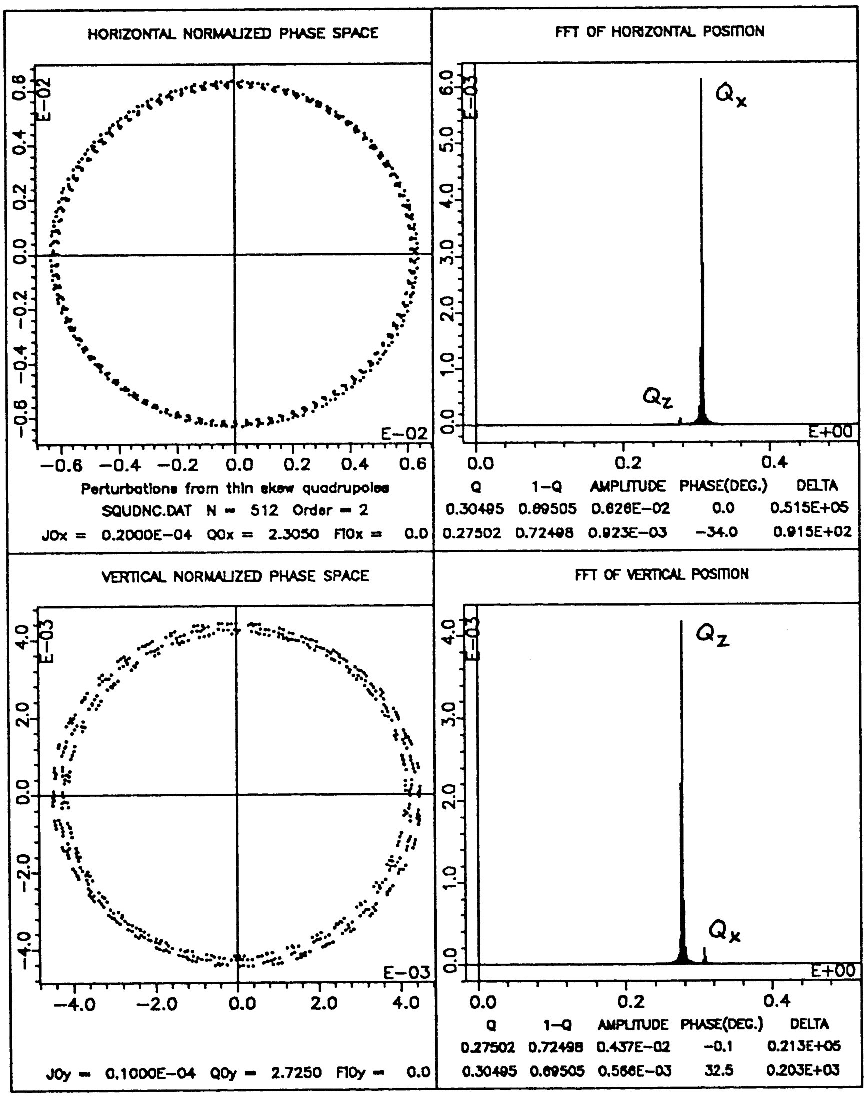

Figure 16: Contribution from mechanical alignment errors

It is clear that the closed-orbit distortions in the sextupoles give a contribution to $Q_x + Q_z = 5$ of the same order as that observed. The phase of this contribution, given by Eq. (275) does not agree so well with the measured one given by Eq. (266) probably owing to the limited knowledge of the closed orbit. This may be explained by the fact that the phase of the resonance is more sensitive to the details of the closed orbit than the amplitude.

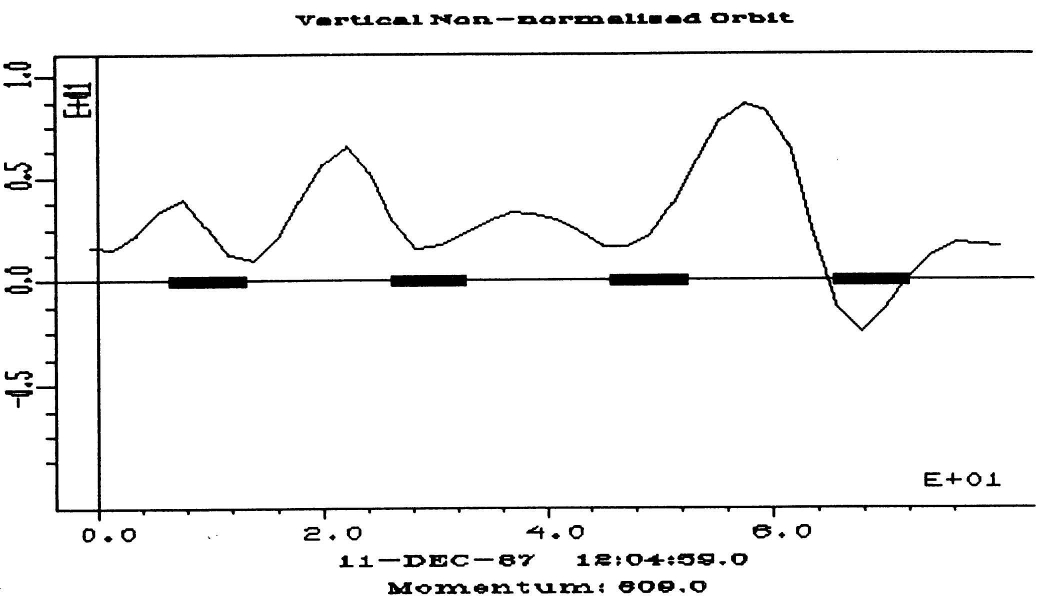

Figure 17: Vertical closed orbit distortions

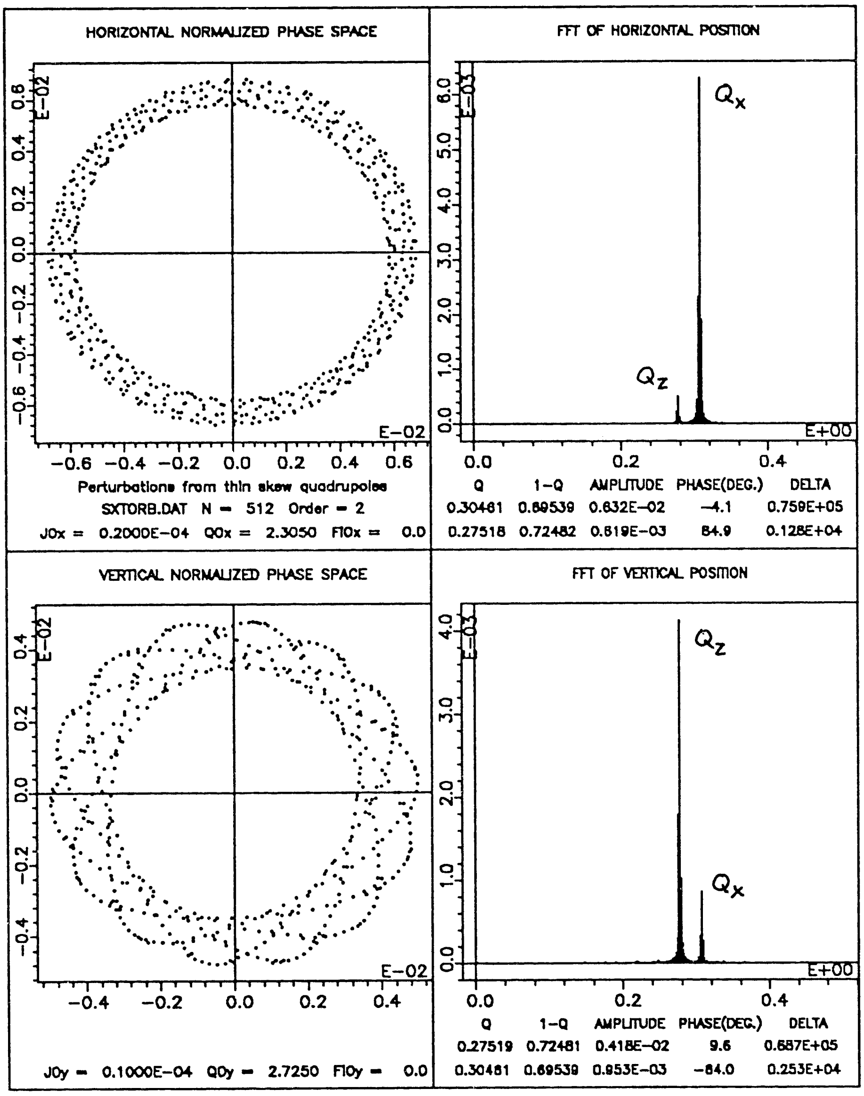

Figure 18: Contribution from the vertical closed orbit in the sextupoles

Chapter 14

STUDY OF THE ULTRASLOW EXTRACTION IN LEAR BY TRACKING

The DFT technique mentioned in Chapter 10, and applied in Chapters 11, 12, and 13, is also very useful for the analysis of tracking data. In this chapter it will be used to analyse tracking done at the extraction working point. By using the theory from Chapter 9 we obtain information about which resonances are excited.

In particular, we find that the systematic sextupolar resonance $Q_x + 2Q_z = 8$ is strongly excited[16]. This leads to coupled motion for the horizontal and the vertical plane which affects the efficiency of the extraction. It is also expected to affect the stability of the motion. The resonance has been compensated by using two additional sextupoles[17]. This led to an increase of the acceptance at extraction by a factor of 4 in emittance.

14.1 Tracking at the extraction working point

Owing to the long spill times (typically one hour) in LEAR, stochastic extraction is used [HARDT81]. This method reduces the modulation of the spill rate owing to ripples on the power supplies.

The extraction is done by exciting the resonance $3Q_x=7$. However, the working point is kept around 2.325 for the stack. The particles are moved to the resonance by tuning the horizontal chromaticity and accelerating the particles longitudinally with noise so that they make a random walk towards the resonance. By a special tuning of the horizontal chromaticity together with that of the amplitude and phase of the resonance, it is possible to obtain an alignment of the outgoing horizontal separatrices for different horizontal amplitudes [HARDT81].

The working point for extraction is

$$Q_x = 2.325\,, \qquad Q_z = 2.725\,. \tag{276}$$

The chromaticities are tuned to

$$\xi_x = 0.53\,, \qquad \xi_z = 0 \tag{277}$$

---

[16] A resonance is called systematic if its harmonic is a multiple of the periodicity of the lattice.

[17] The compensation was worked out by M. Chanel. Compensation of the systematic resonance $Q_x + 2Q_z = 8$ was foreseen during the construction of LEAR [JAGER83, PLASS80]. The limited life time at low momentum and some limitations on the extraction initiated these new studies.

and the extraction resonance $3Q_x = 7$ is excited with

$$A_7 = 6.0, \quad \phi_7 = 40° \tag{278}$$

in normalized units [HARDT81, GIANN81]. These are related to the Fourier component $h_{3000\,7}$ by

$$\begin{aligned} |h_{3000\,7}| &= \frac{\sqrt{\beta_n}}{\sqrt{2}\,24\pi} \\ \phi_\kappa &= 3\phi_7 + 180° , \end{aligned} \tag{279}$$

where $\beta_n = 6$, so that

$$|h_{3000\,7}| = 0.14, \quad \phi_\kappa = -60° . \tag{280}$$

The particles are also given a horizontal bump to make them approach the electromagnetic septum used for the extraction.

We have done tracking with DIMAT [BROWN85A], during these conditions[18]. One particle has been tracked 1024 turns for different horizontal and vertical amplitudes. We have then applied the DFT technique described in Chapter 10 to the horizontal and vertical positions to obtain the frequency spectra and tune shift with amplitude.

The action J has been varied by

$$J_x : \ 0.5 \rightarrow 15 \text{ mm·mrad}, \qquad J_z : \ 0 \rightarrow 15 \text{ mm·mrad} . \tag{281}$$

The results are shown in Figure 19 to Figure 22.

In the frequency spectra we find large amplitudes for the frequencies

$$2Q_x , \quad 2Q_z \tag{282}$$

in the horizontal plane, and

$$Q_x + Q_z , \quad Q_x - Q_z , \quad 3Q_z \tag{283}$$

in the vertical plane. The corresponding resonances are found from Chapter 9 to be

$$\begin{aligned} 3Q_x &= 7 , & Q_x - 2Q_z &= -3 , \\ Q_x + 2Q_z &= 8 , & 4Q_z &= 11 . \end{aligned} \tag{284}$$

By applying the DFT technique we can obtain the tune shifts as functions of amplitude [compare with Eq. (197)]. A least squares fit of the data gives

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_z \end{pmatrix} = \begin{pmatrix} 28.1 & 543 \\ 709 & 194 \end{pmatrix} \begin{pmatrix} J_x \\ J_z \end{pmatrix} \tag{285}$$

---

[18] The LEAR lattice for DIMAT was created by A. Mosnier (Scientific Associate in the LEAR group from CEA Saclay during 1983 – 1984), who also adopted DIMAT for the simulation of the ultraslow extraction.

for J given in mm·mrad.

We conclude from the tracking that the systematic resonance $Q_x + 2Q_z = 8$ is strongly excited, as expected from the arrangement of the sextupoles for the chromaticity correction. This leads to coupled motion in the transverse plane. The coupling can be reduced by compensation of the resonance with two extra sextupoles.

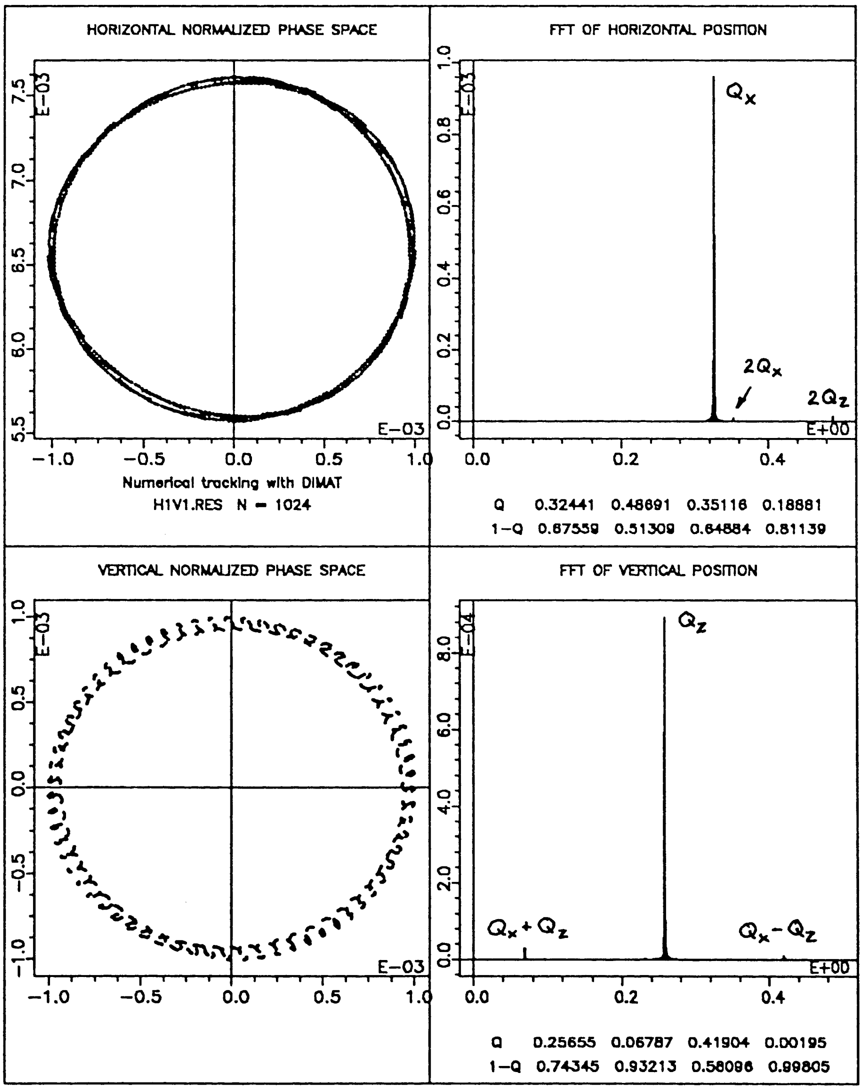

Figure 19: Tracking with DIMAT, $J_x = 0.5$, $J_z = 0.5$

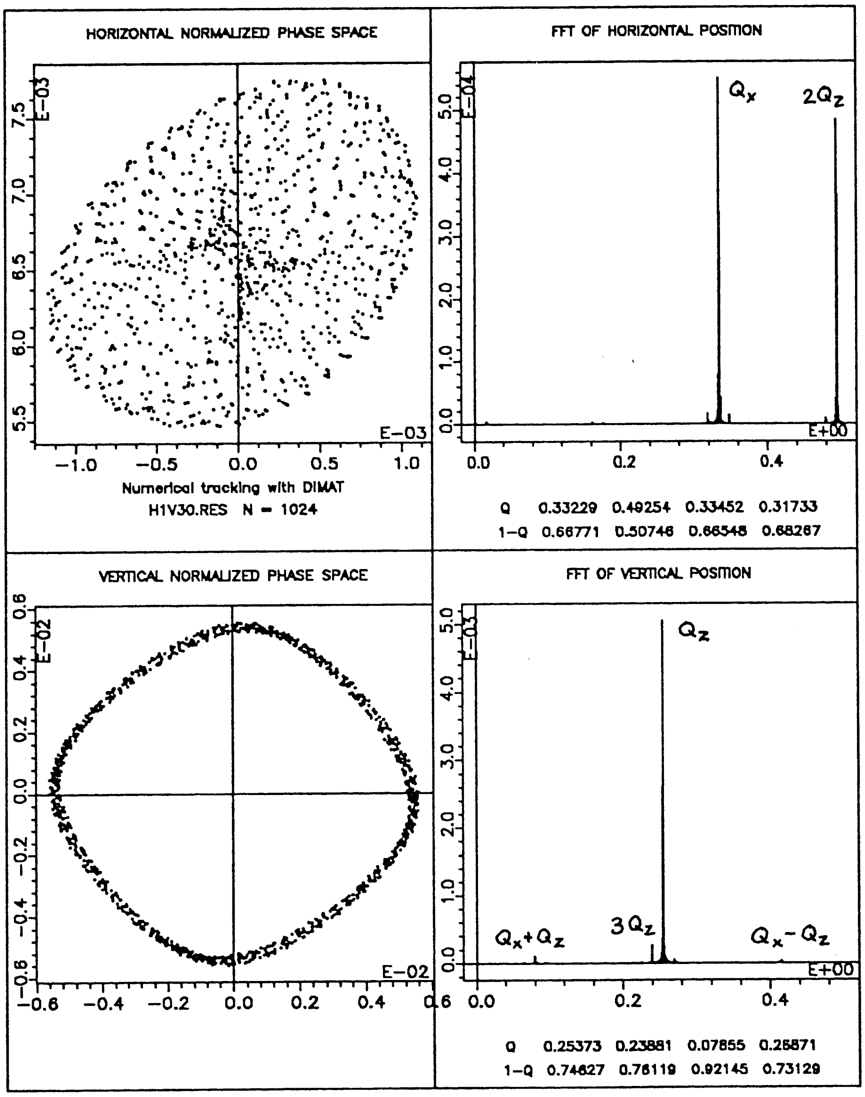

Figure 20: Tracking with DIMAT, $J_x = 0.5$, $J_z = 15$

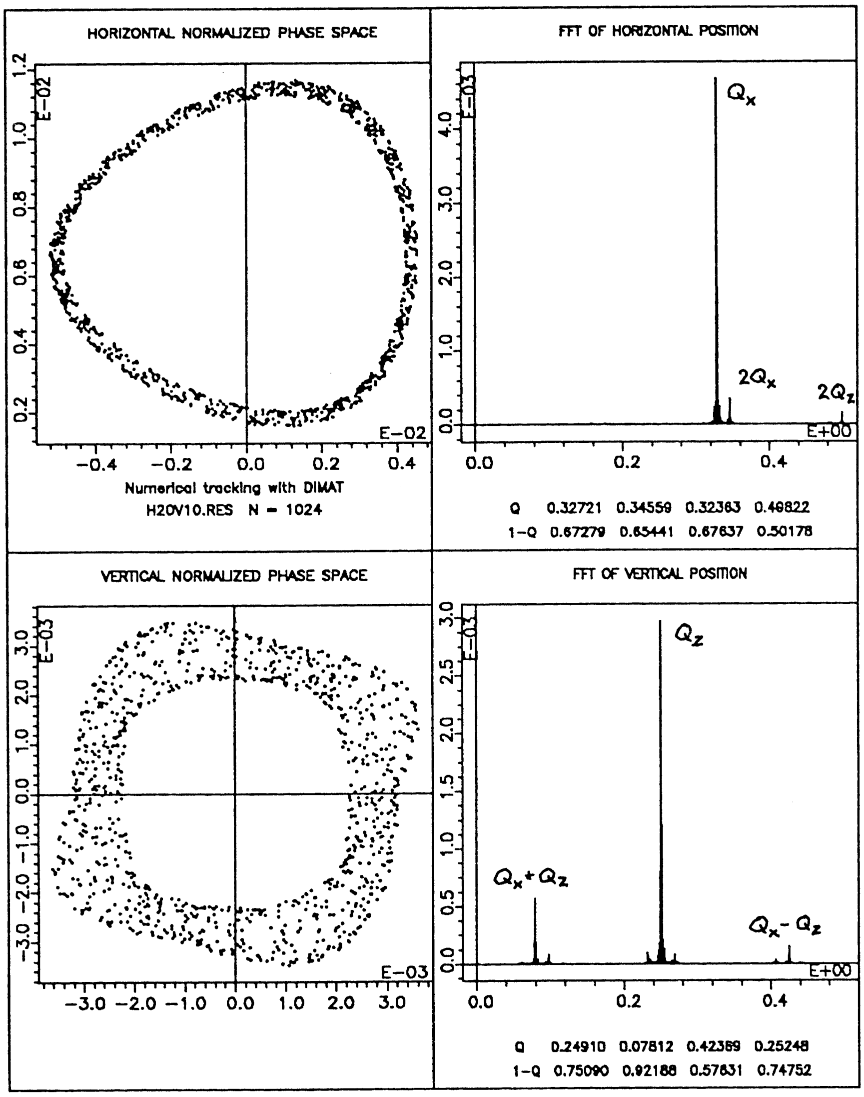

Figure 21: Tracking with DIMAT, $J_x = 10$, $J_z = 5$

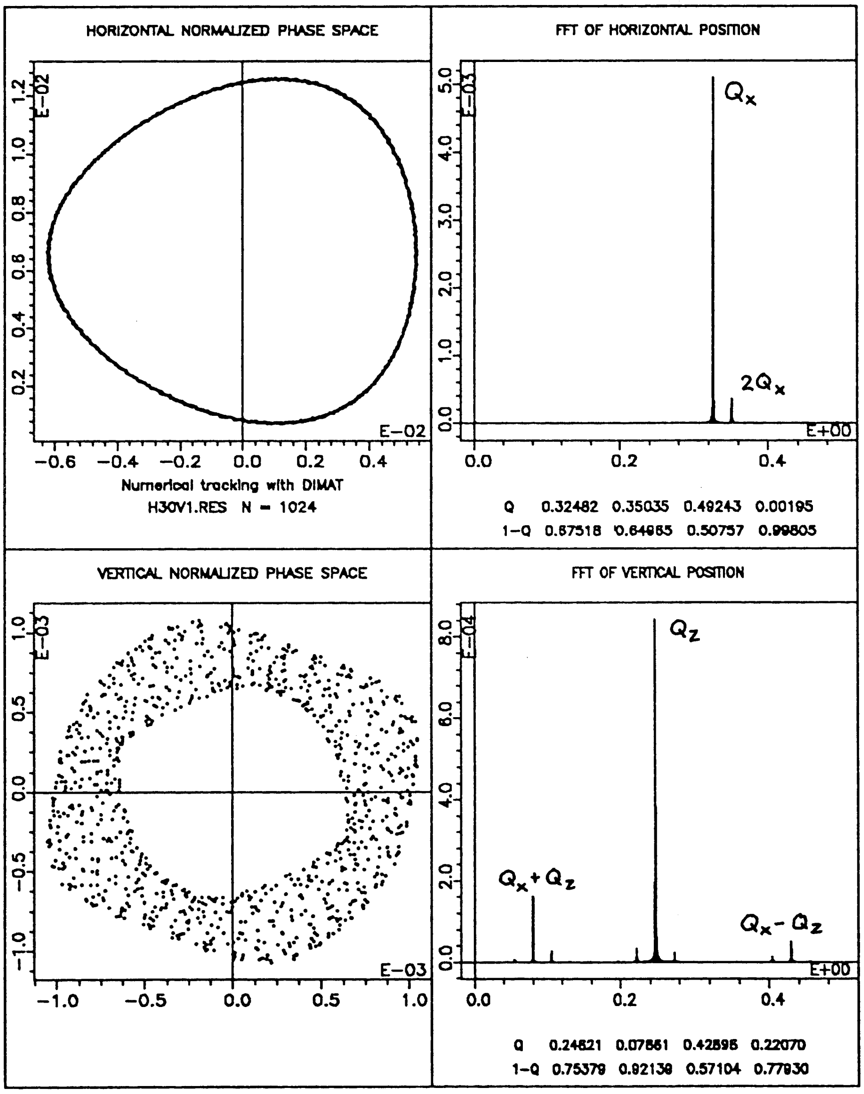

Figure 22: Tracking with DIMAT, $J_x = 15$, $J_z = 0.5$

14.2 Simulation of the ultraslow extraction

The tracking is in this case done under the same conditions as in Section 14.1 However, we now add an increment $\Delta p/p_0$ to the particles, longitudinal momenta for each turn. This simulates the longitudinal acceleration of the particle caused by the noise, so that it moves towards the extraction resonance $3Q_x = 7$. It is then following the separatrices of the extraction resonance and is assumed to be extracted when its horizontal position is larger than the position of the electromagnetic septum ($x_{ext} = 35$ mm).

We have simulated the extraction for 35 particles where J has been varied in equidistant steps as

$$J_x : 0.5 \rightarrow 5 \text{ mm·mrad in 5 steps}, \quad J_z : 0 \rightarrow 5 \text{ mm·mrad in 7 steps}. \tag{286}$$

The result is shown in Figure 23. We observe the motion at the entrance of the electromagnetic septum. In the upper part of the figure without compensation of the systematic resonance $Q_x + 2Q_z = 8$, we find that the outgoing separatrices do not overlap for different vertical amplitudes, owing to the coupling between the horizontal and vertical planes. In the lower part we find the case when the resonance has been compensated by adding two sextupoles. The coupling is reduced so that we recover the alignment of the outgoing separatrices.

We conclude that the coupling between the horizontal and the vertical planes destroys the alignment of the outgoing separatrices for different vertical amplitudes. This leads to an increase of the emittance of the extracted beam and a greater apparent thickness of the electromagnetic septum. The coupling is successfully removed by the compensation. In the vertical plane we find that the maximum amplitude is reduced by a factor of 2 corresponding to a factor of 4 in emittance. This reduces the losses during the extraction process since particles with large amplitudes will hit the vacuum chamber. The large vertical amplitudes appear in the uncompensated case at the last turns before extraction, owing to the large horizontal amplitudes and the coupling. The compensation made it possible to move the electromagnetic septum an additional 7 mm from the centre of the vacuum chamber with, a corresponding reduction of the voltage by around 30%.

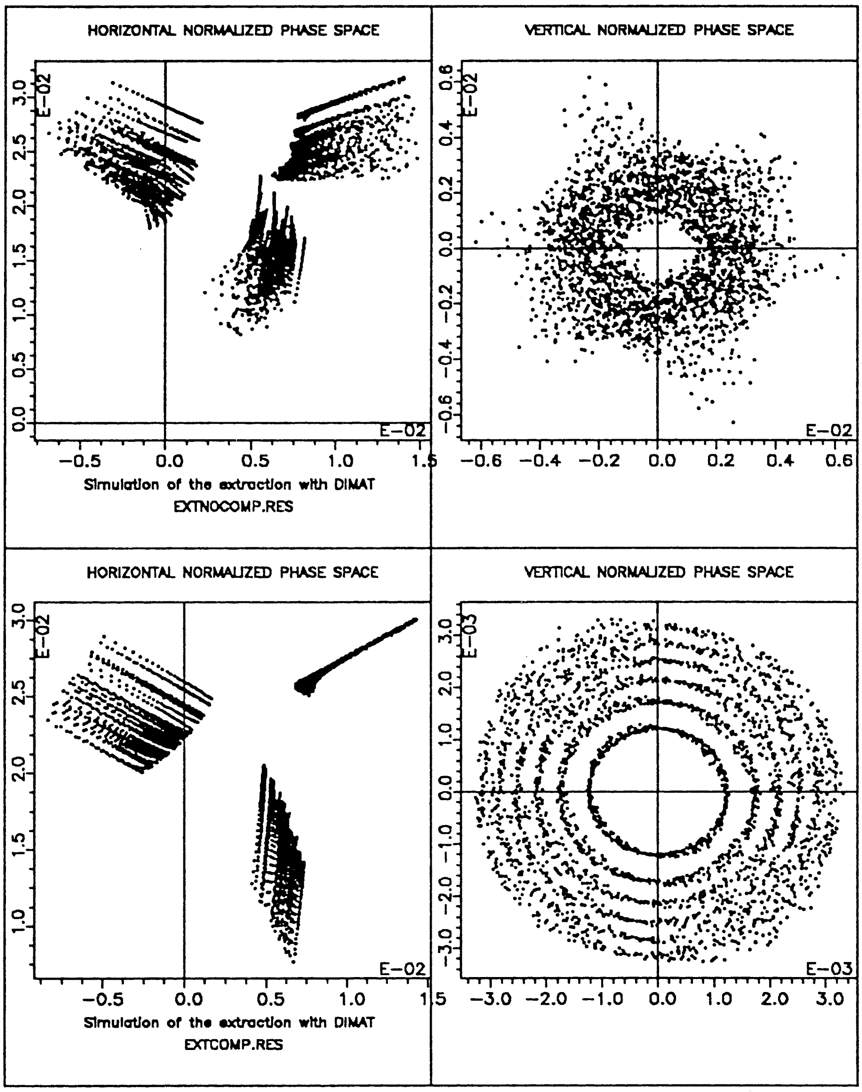

Figure 23: Simulation of the extraction by DIMAT

Chapter 15

MEASUREMENT OF 3Q$_X$ = 7 IN LEAR

In this chapter we will apply the method used in Chapter 13 to measure the phase of the extraction resonance $3Q_x = 7$ for a known excitation.

We have excited the extraction resonance $3Q_x = 7$ with

$$A_7 = 4.5, \quad \phi_7 = 40° \tag{287}$$

in normalized units or from Eqs. (279)

$$h_{3000\,7} = 0.10, \quad \phi_\kappa = -60° \tag{288}$$

for the working point $Q_x = 2.33$, $Q_z = 2.725$. The frequency spectra are obtained from Chapter 9 and given in Table 10.

Table 10: Frequency spectra for the horizontal plane

Frequency	Amplitude	Phase
$Q_x^{(0)} = Q_x + \Delta Q_x$	$a_x^{(0)} = \sqrt{2J_{2x}\beta_x(s_0)}$	$\phi_x^{(0)} = \phi_{2x}(s_0)$
$Q_x^{(1)} = 2(Q_x + \Delta Q_x)$	$a_x^{(1)} = \sqrt{2\beta_x(s_0)}\,J_{2x}3\frac{\lvert\kappa\rvert}{e}$	$\phi_x^{(1)} = 2\phi_{2x}(s_0) + \phi_\kappa$

We have the following relations

$$\begin{aligned} \lvert\kappa\rvert &= \frac{e\sqrt{2\beta_x(s_0)}\,a_x^{(1)}}{(a_x^{(0)})^2} \quad . \\ \phi_k &= \phi_x^{(1)} - 2\phi_x^{(0)}. \end{aligned} \tag{289}$$

An example of the measurements is shown in Figure 24.

Since we do not know the total gain for the pick-up amplifiers we can only get the phase which is

$$(\phi_\kappa)_{\mathrm{UEH23}} = 110° \pm 3.0° \tag{290}$$

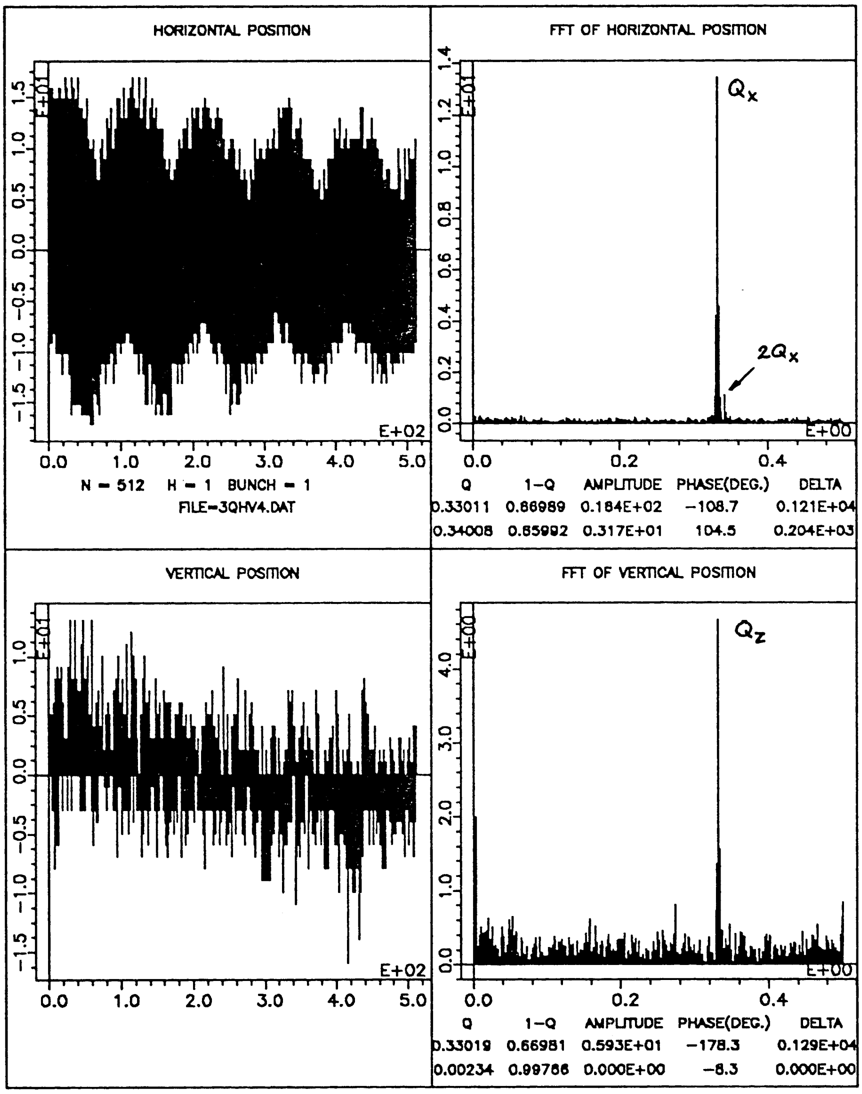

Figure 24: Measurement of $3Q_x = 7$

at pick-up UEH23. The value has been obtained from the average of three measurements. The phase at $s = 0$ then becomes

$$(\phi_x)_0 = (\phi_x)_{\mathrm{UEH23}} + 3(\mu_x)_{\mathrm{UEH23}} = -\ 52^\circ\ , \tag{291}$$

which can be compared with the excitation given in (288).

Chapter 16

ANALYTICAL STUDY OF THE SEXTUPOLAR PERTURBATIONS IN LEAR

The time-dependent perturbation theory, developed in Chapter 9, complements the resonance theory when there is no dominating resonance term in the Hamiltonian. In particular, it can be applied to study the remaining sextupolar perturbations when a correction scheme has been worked out. In this chapter we will study the perturbations due to the sextupolar arrangement in LEAR[19].

16.1 Perturbations from sextupoles

From Chapter 7 it follows that the perturbations to second power in the sextupole strength can be written as:

$$J(s) = J_0 + <\Delta_1 J(s)> + \widetilde{\Delta_1 J}(s) + <\Delta_2 J(s)> + \widetilde{\Delta_2 J}(s) \ , \tag{292}$$

where $< >$ denotes the average value and a tilde the oscillating part. For the long-term behaviour we are mainly interested in the average perturbations. The average perturbations of the action can then be plotted as a function of the horizontal and the vertical tune. To compress both the horizontal and the vertical planes in one plot we define the function:

$$\Delta J \equiv \min\{\max[\,|<\Delta_1 J_x(s)> + <\Delta_2 J_x(s)>|,\ |<\Delta_1 J_z(s)> + <\Delta_2 J_z(s)>|\,],\ 1\} \ , \tag{293}$$

where we have limited the value to 1 since the perturbation tends to infinity on the resonance. Note that we may interpret a perturbation in $(\Delta_1 J + \Delta_2 J)/J_0$ of the order of -1 as an indication of instability because the argument of the square root in Eq. (180) then becomes negative. The perturbation theory is however only expected to be valid for small perturbations. We will use contour plots to represent the average perturbations as a function of Q_x and Q_z. The range of the function 0 to 1 will be divided into 10 steps. Tune diagrams found in Appendix D can be used for guidance.

The average perturbations depend on a particle's initial phase and may be zero for particular cases. It is therefore preferable to complement the study by using the s-dependent part of the perturbations for tracking. The tracking is analysed by the DFT technique presented in Chapter 10 In addition, we calculate the amplitude-dependent tune shifts and present them in the form of (197). The thin-element approximation will be used in all the calculations. The average perturbations are calculated for J_{0x} = 20 mm·mrad and J_{0z} = 10 mm·mrad.

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[19] The lattice functions for LEAR were provided by M. Chanel

16.2 Perturbations for the normal working point

For the normal working point there is no deliberate excitation of the extraction resonance. The sextupoles are used to tune the chromaticity to zero. The working point is $Q_x = 2.305$, $Q_z = 2.725$. A contour plot [BRUN88] of the average perturbations without compensation of the systematic resonance $Q_x + 2Q_z = 8$ is presented in Figure 25 together with the the amplitude-dependent tune shifts. The plot shows a strong excitation of $Q_x + 2Q_z = 8$ as expected from the sextupole arrangement. This is confirmed by the tracking; see Figure 26. When the compensation is applied the resonance almost disappears; see Figure 27. Since the compensation only has twofold symmetry, we find a stronger excitation of $Q_x = 2$, which however is far from the working point. We also note a small excitation of $2Q_x + 2Q_z = 10$. The tune shifts are strongly reduced. From the tracking shown in Figure 28 we find that there only remains a small contribution from $Q_x - 2Q_z = -3$ excited by the compensation.

16.3 Perturbations at extraction

The working point at extraction is $Q_x = 2.325$, $Q_z = 2.725$. The extraction resonance is excited and the horizontal chromaticity is no longer tuned to zero (see Chapter 14). From Figure 29 we conclude that the excitation of the extraction resonance $3Q_x = 7$ leads to an excitation of $Q_x - 2Q_z = -3$. We also find a small excitation of $4Q_z = 11$. This agrees with the tracking shown in Figure 30. The frequency $3Q_z$ in the vertical plane related to the resonance $4Q_z = 11$ cannot be resolved since it overlaps with the vertical betatron frequency. The shape of the vertical phase space however shows its appearance. Note that the tracking was done for the working point $Q_x = 2.3242$, $Q_z = 2.7428$ with the same amplitudes as used for Figure 21. It can therefore be compared with the numerical tracking done by DIMAT.

After compensation there remains only a small perturbation from $Q_x - 2Q_z = -3$, which can be seen in Figure 31 and Figure 32 As before the amplitude-dependent tune shifts are strongly reduced by the compensation.

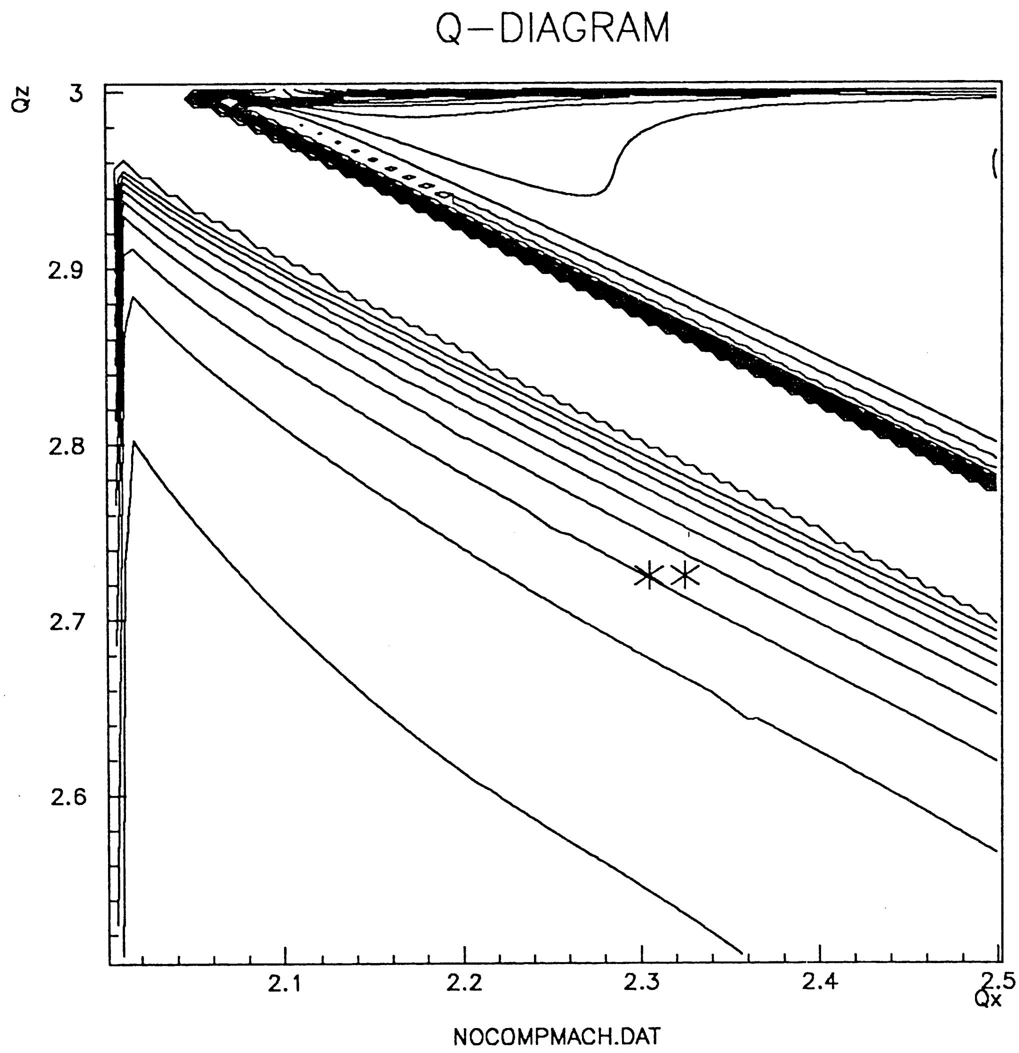

Figure 25: Perturbations in the action without compensation

$$\begin{pmatrix}\Delta Q_x \\ \Delta Q_z\end{pmatrix} = \begin{pmatrix}12.2 & 262 \\ 262 & 93.0\end{pmatrix}\begin{pmatrix}J_{0x} \\ J_{0z}\end{pmatrix}$$

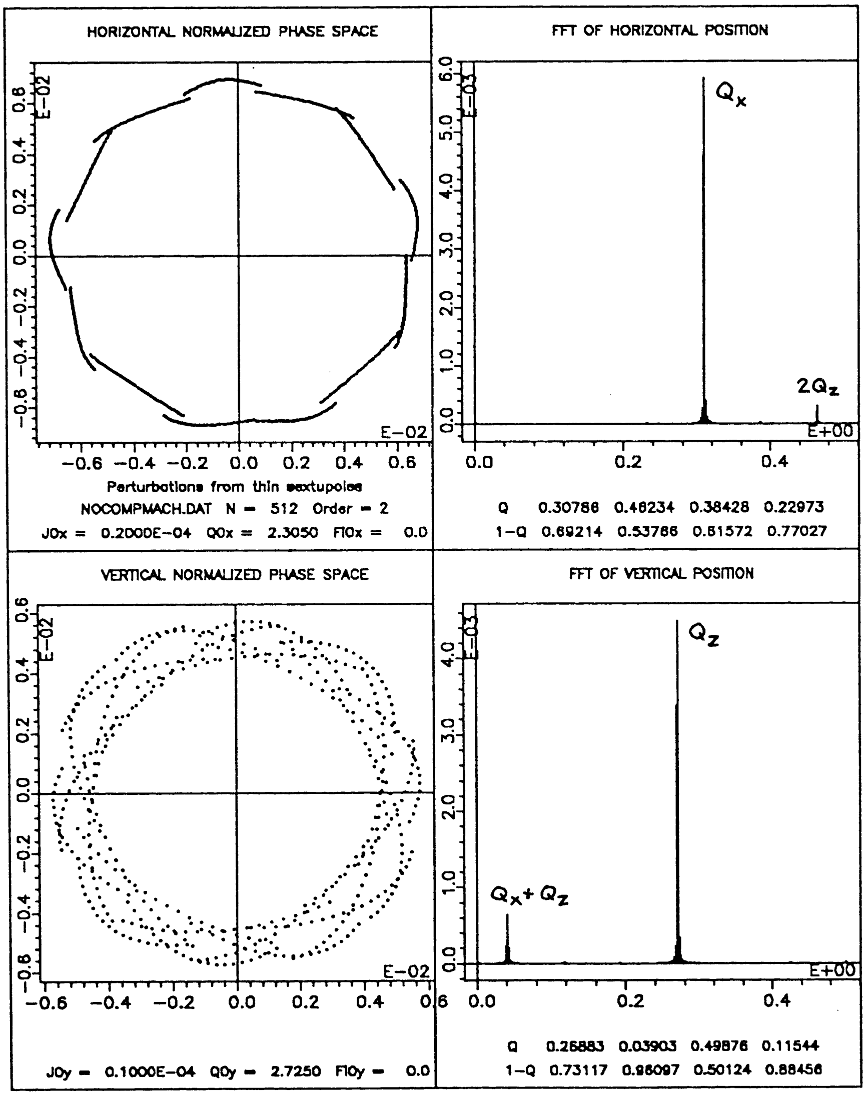

Figure 26: Tracking without compensation

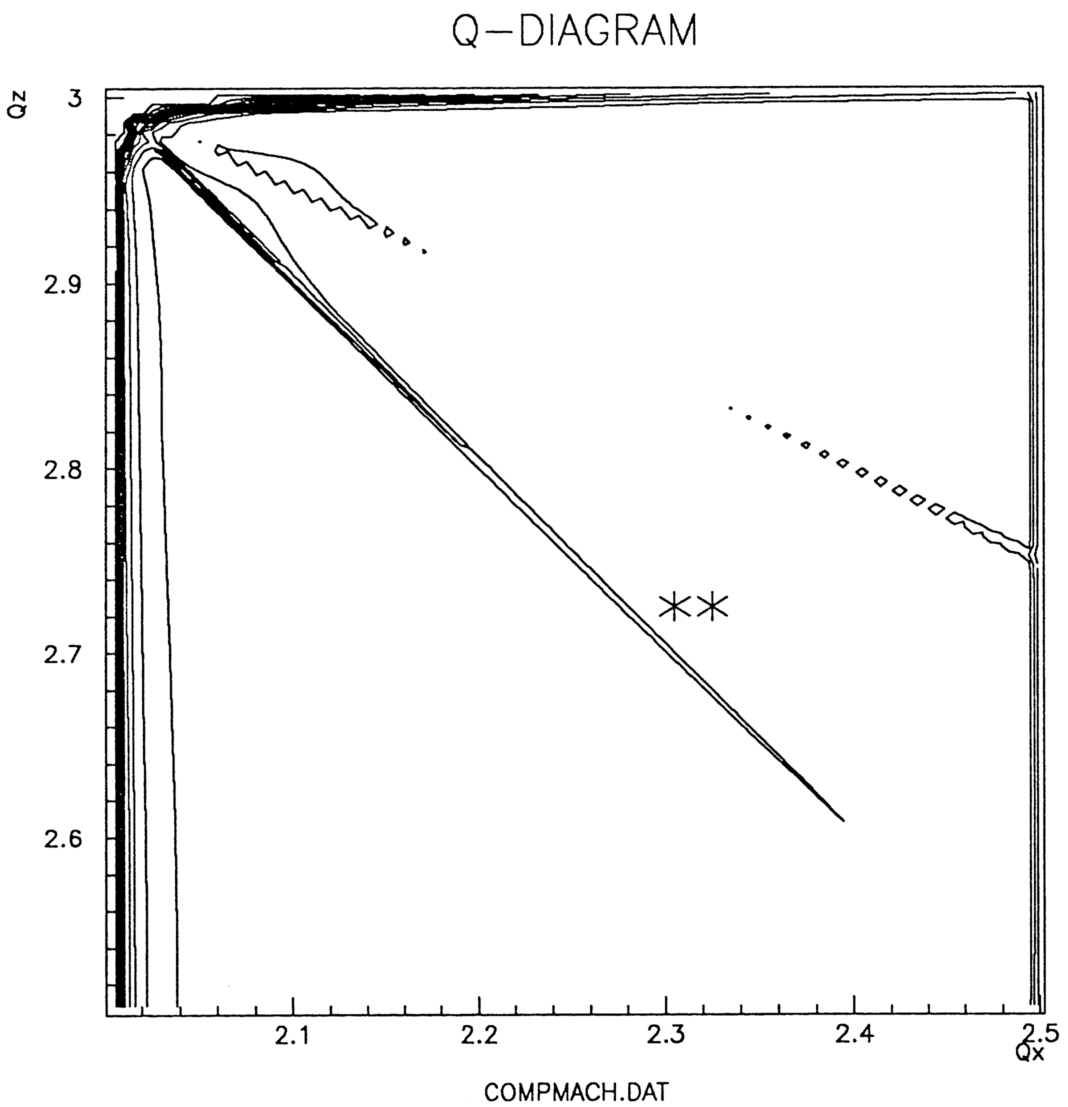

Figure 27: Perturbations in the action with compensation

$$\begin{pmatrix}\Delta Q_x\\ \Delta Q_z\end{pmatrix} = \begin{pmatrix}8.0 & 13.7\\ 13.7 & -28.4\end{pmatrix}\begin{pmatrix}J_{0x}\\ J_{0z}\end{pmatrix}$$

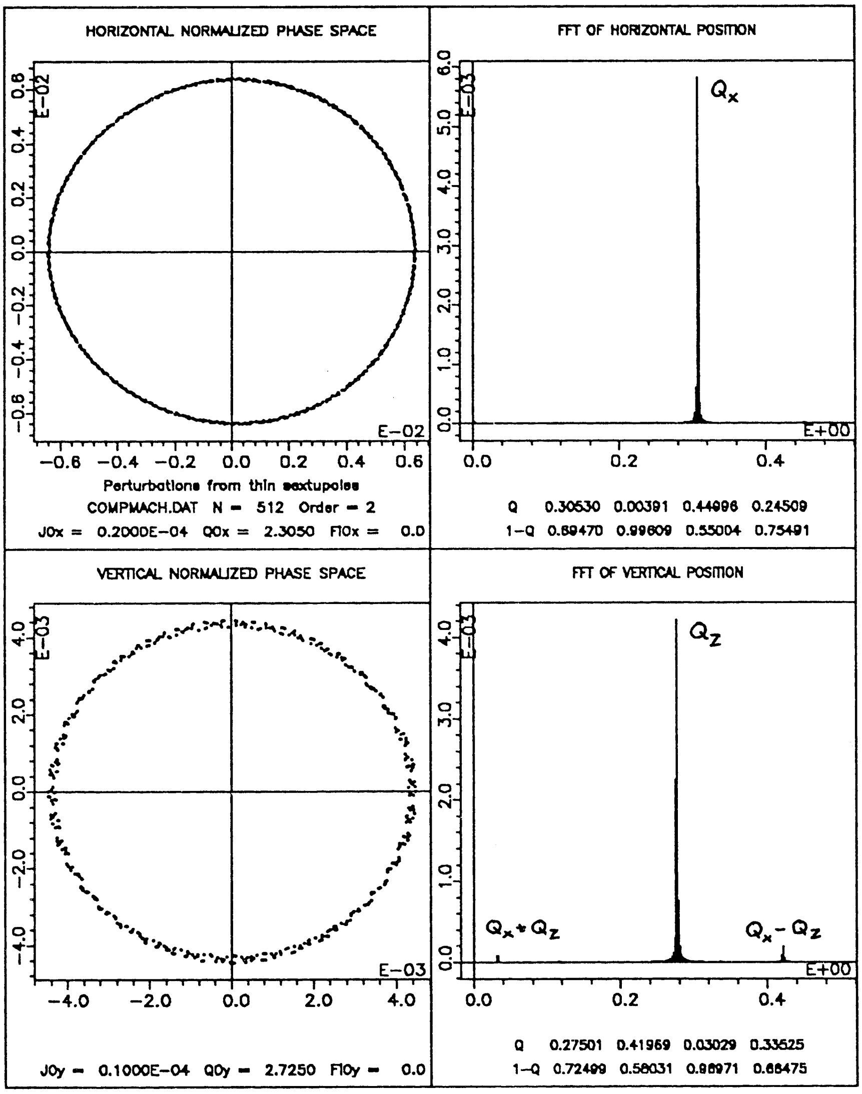

Figure 28: Tracking with compensation

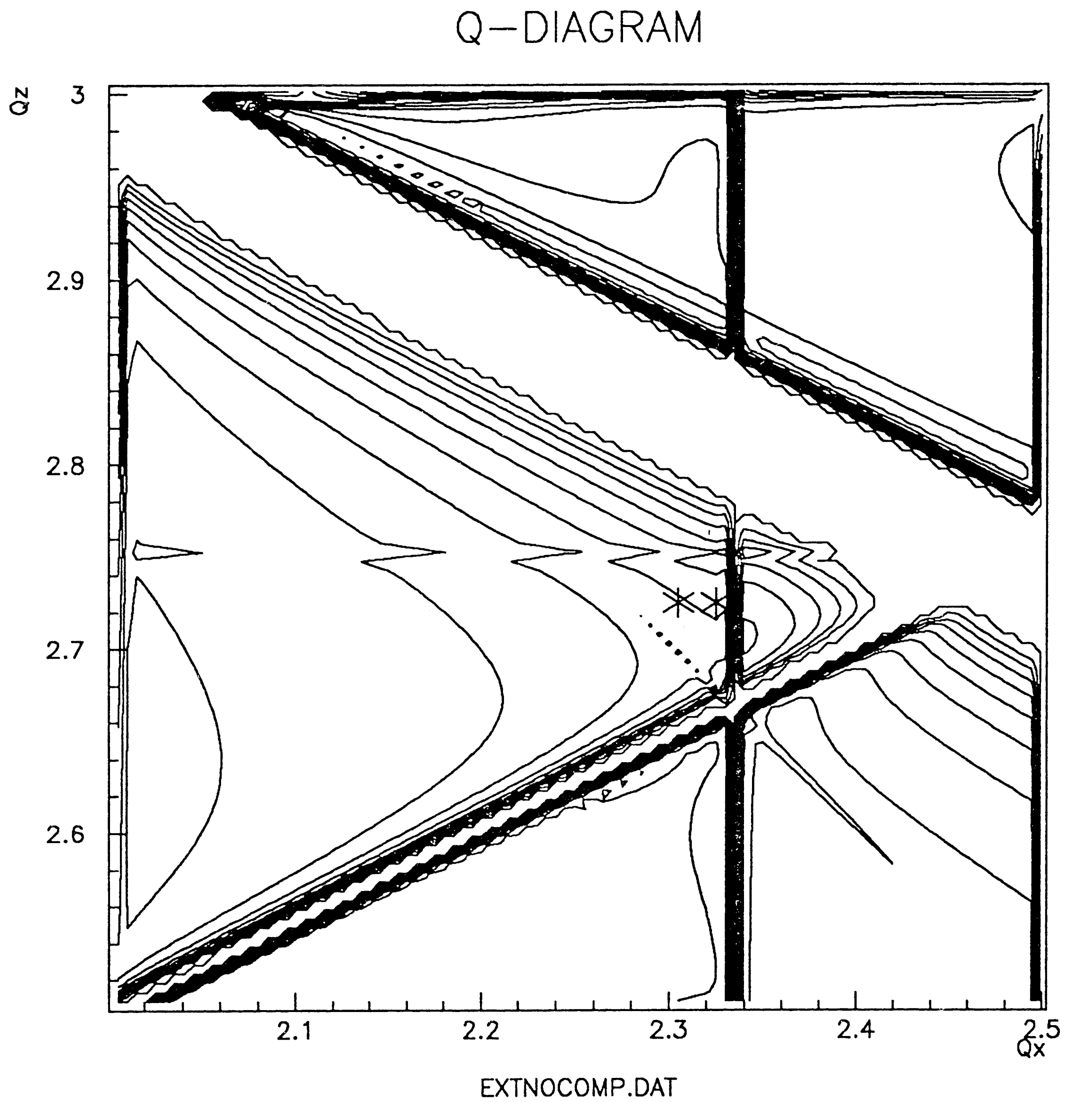

Figure 29: Perturbations in the action at extraction without compensation

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_z \end{pmatrix} = \begin{pmatrix} 31.2 & 310 \\ 310 & 131 \end{pmatrix} \begin{pmatrix} J_{0x} \\ J_{0z} \end{pmatrix}$$

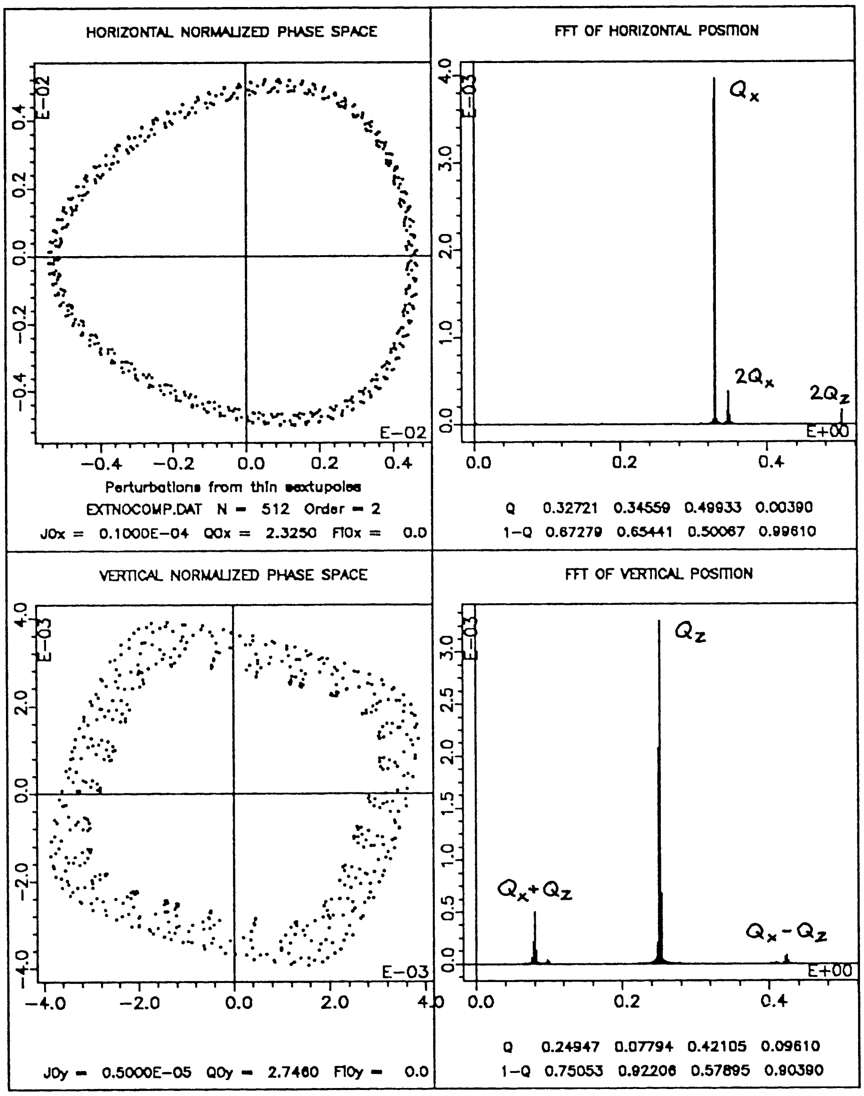

Figure 30: Tracking at extraction without compensation

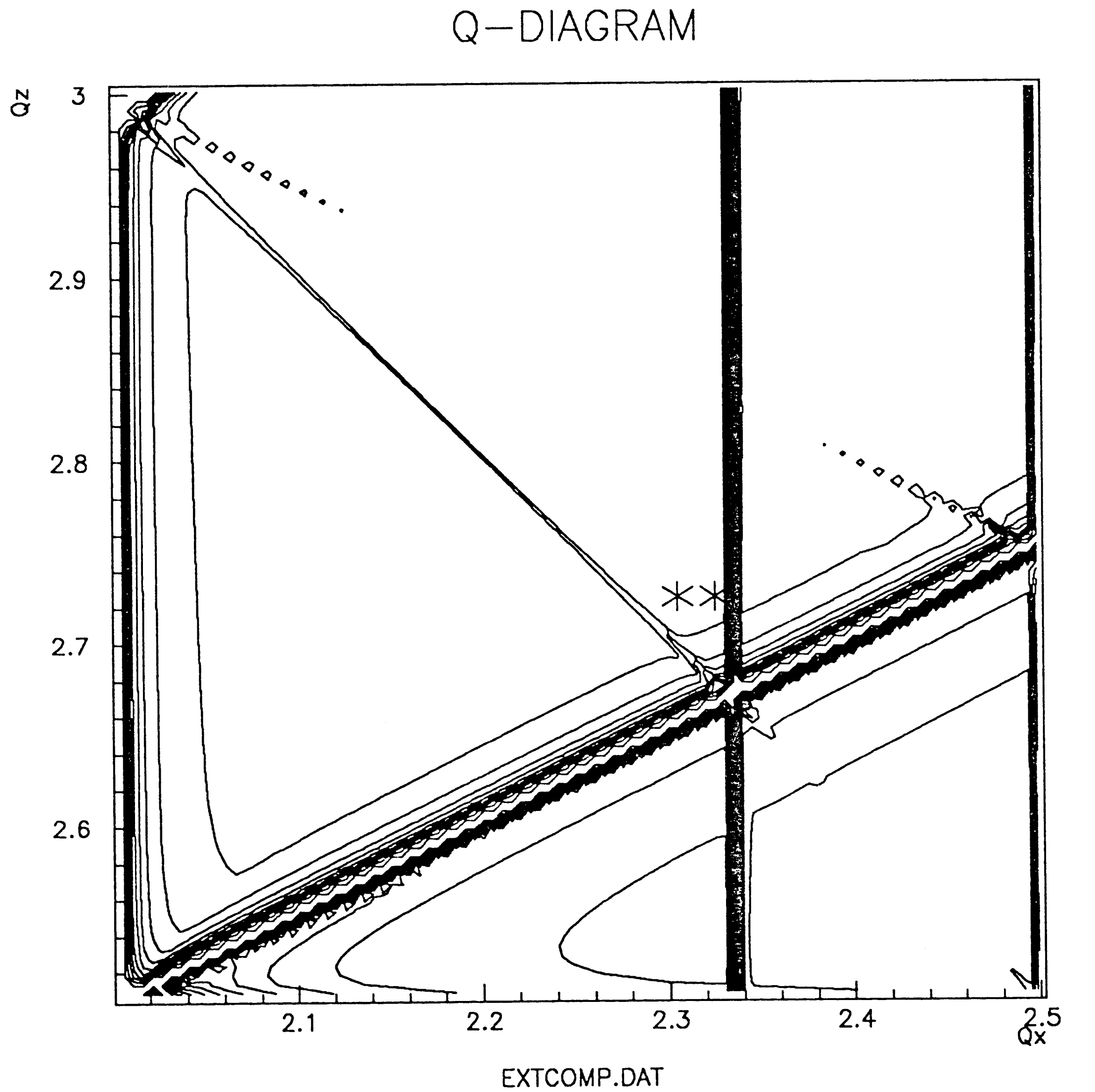

Figure 31: Perturbations in the action with compensation

$$\begin{pmatrix}\Delta Q_x\\ \Delta Q_z\end{pmatrix} = \begin{pmatrix}22.4 & 0.25\\ 0.25 & -17.2\end{pmatrix}\begin{pmatrix}J_{0x}\\ J_{0z}\end{pmatrix}$$

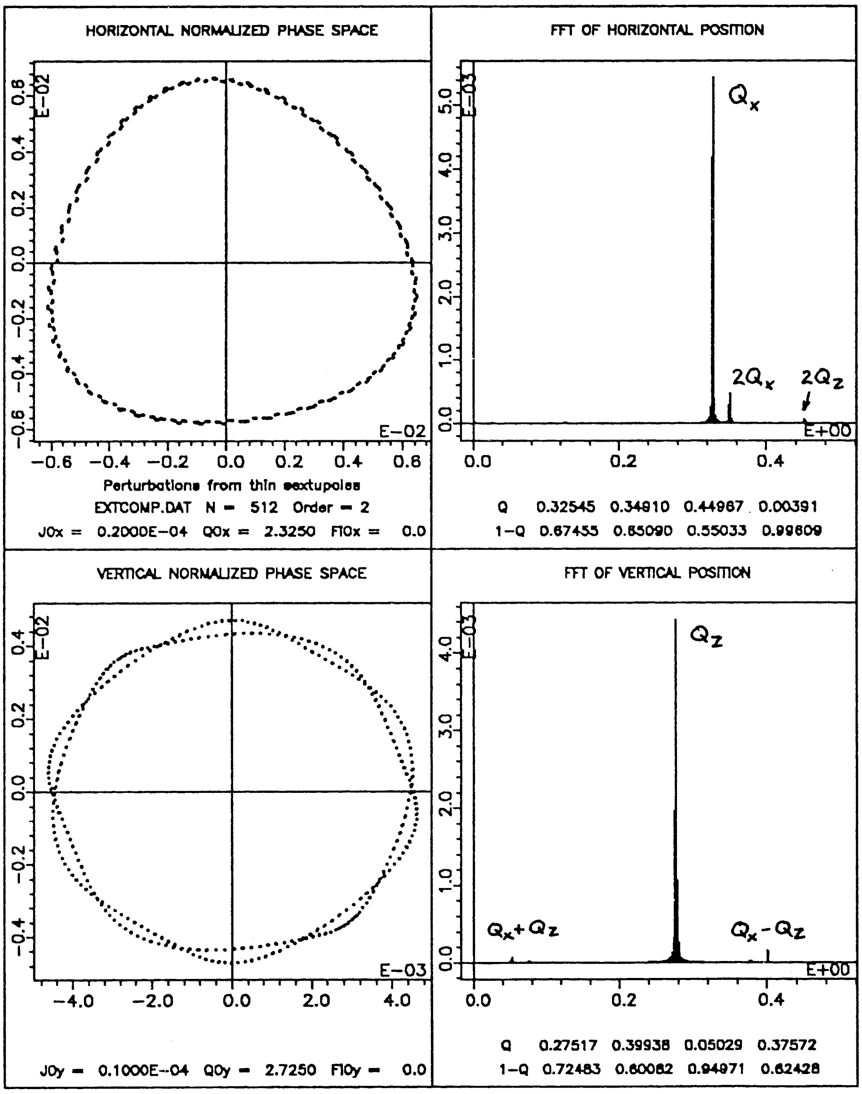

Figure 32: Tracking at extraction with compensation

16.4 Experimental results

The compensation of the systematic resonance $Q_x + 2Q_z = 8$ in LEAR led to increased stability and performance, particularly at low momentum[20]. Without the compensation, the beam lifetime was 13 minutes at a momentum of 100 MeV/c and there was no effect of the stochastic cooling. The compensation increased the lifetime to 35 minutes without stochastic cooling and to 55 minutes with cooling. Measurements of the the amplitude-dependent tune shifts were done by kicking a well-cooled beam with different kicks in the horizontal plane. The result is presented in Figure 33. Before compensation we have for the tune shift:

[20] The results presented in this section were obtained by M. Chanel et al. [BENGT86B, CHANE87].

$$a_{21} = 320 \tag{294}$$

and after compensation:

$$a_{21} < 30 \,. \tag{295}$$

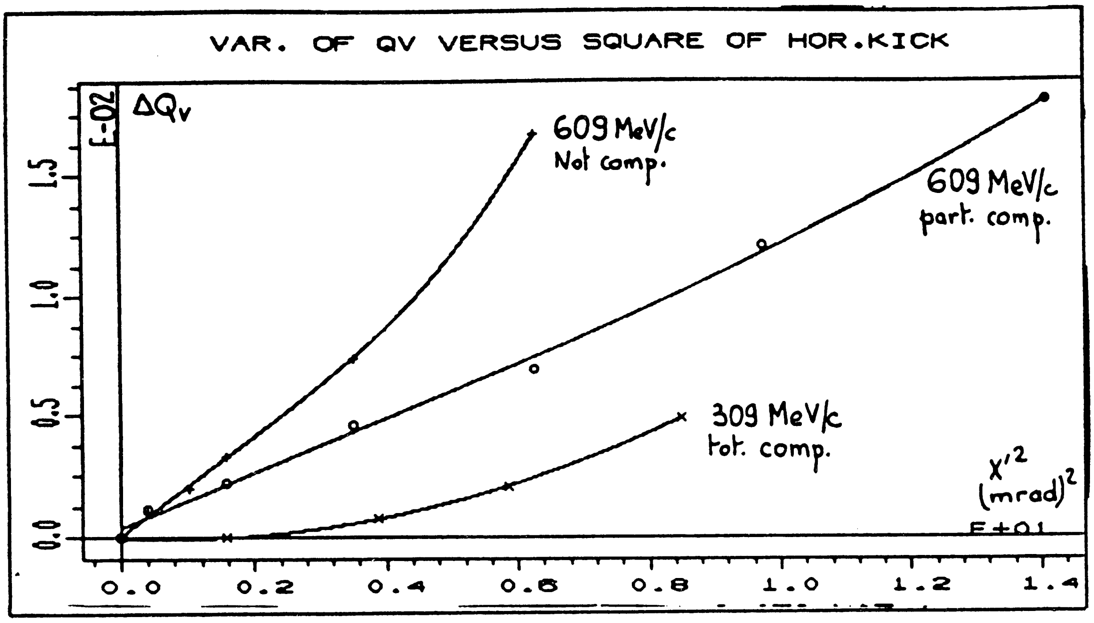

Figure 33: Measured tune shifts

16.5 Perturbations from the closed-orbit distortions

We found a small contribution to $2Q_x + 2Q_z = 10$ from the sextupoles. To compare we will also present the contribution from the closed-orbit distortions in the sextupoles shown in Figure 34. It is clear that this contribution is dominant.

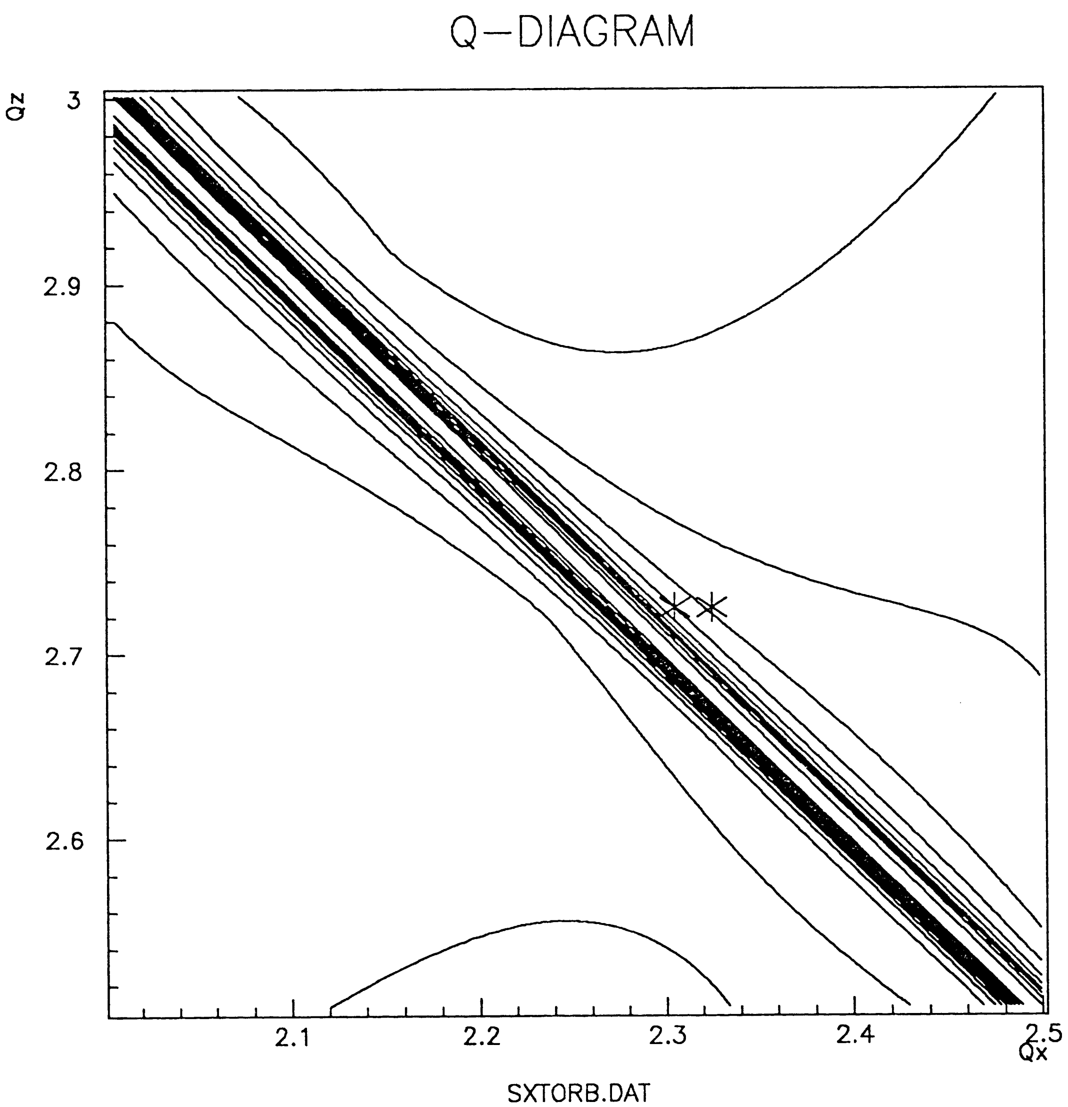

Figure 34: Perturbations from the closed orbit in the sextupoles

16.6 Sextupolar contributions from dipoles

Without going into details I should just like to point out that the terms in the equation for the horizontal motion

$$h'xx' \,, \quad \frac{1}{2}hx'^2 \,, \quad h'zz' \,, \quad \frac{1}{2}hz' \tag{296}$$

and for the vertical motion

$$h'xz' \,, \quad hx'z' \,, \quad h'x'z \tag{297}$$

from Eqs. (87) will give sextupole-like contributions. This can be seen by using Eqs. (151), the approximation

$$x' = p_x \tag{298}$$

valid for the linear motion, and similar expressions for the vertical plane. As an illustration we have done tracking with DIMAT with a working point close to the resonance $Q_x + 2Q_z = 8$ for a lattice only containing dipoles and quadrupoles. The result is shown in Figure 35. Note that DIMAT models the end-fields of the dipoles.

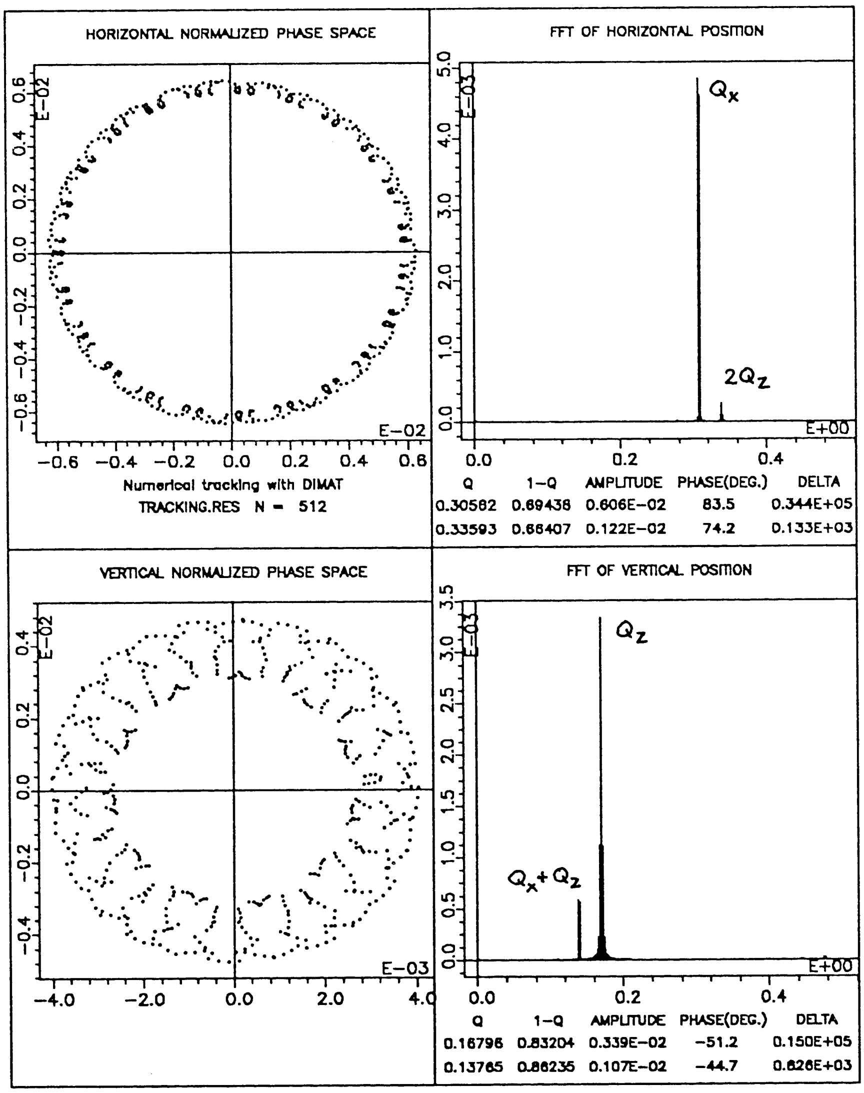

Figure 35: Dipolar contributions to $Q_x + 2Q_z = 8$

From the spectra and the theory in Chapter 9 we find the that the contribution to $Q_x + 2Q_z = 8$ is

$$|h_{1020\,8}| = 1.12\,, \quad \phi_\kappa = 181^\circ\,. \tag{299}$$

If this is added to the calculated contributions from the sextupoles (262) one finds

$$|h_{1020\,8}| = 4.66 \tag{300}$$

since the phase is the same. This can be compared with the measured value [BENGT86B]

$$|h_{1020\,8}| = 5.52\,. \tag{301}$$

The contribution from the dipoles to the tune shifts can be included in the values presented in Figure 29. Since the tune shift is quadratic in $|h_{1020\,8}|$ we find

$$a_{12} = a_{21} = 537\,. \tag{302}$$

This value can be compared with the values obtained from tracking (285). The assymetry between a_{12} and a_{21} in (285) is probably owing to the higher-order terms in the sextupole strength that have been neglected.

Chapter 17

SUMMARY AND CONCLUSIONS

We have used a tensor equation to derive the equations of motion in the curvilinear coordinate system used for accelerators. This formalism is probably more transparent than the normal approach, where it is necessary to calculate the change of the unit vectors. We then present a Hamiltonian expanded to third order in the coordinates where the curvative is allowed to vary with s, the distance along the reference curve.

By using time-dependent perturbation theory (to the Hamiltonian formalism) it was possible to make automatic calculations for a given perturbation by using a computer algebra system. The result is obtained in both the form of analytical formulae for the perturbations and as FORTRAN subroutines that can be used for numerical studies. Calculations for sextupoles and skew quadrupoles were done to second power in the multipole strength and used to study the perturbations in LEAR.

Application of the time-independent perturbation theory to the single resonance Hamiltonian led to analytical expressions for the frequency spectra of the perturbed betatron motion.

These results together with methods for signal processing proved to be useful for the analysis of tracking data and beam measurements. In particular, we have found a method to obtain information from beam measurements that can be used to compensate correctly the perturbations on the beam in an accelerator.

Acknowledgements

The work that has been presented is the result of a stay with the LEAR group at CERN, which is operating and developing the Low-Energy Antiproton Ring (LEAR).

I should like to express my gratitude to P. Lefèvre and D. Möhl for their overall guidance. During the work I have had a lot of stimulating discussions, in particular with the following people: B. Autin, M. Chanel, R. Giannini, W. Hardt and D.Möhl. All beam measurements were done together with M. Chanel and the LEAR team. They made extensive use of the signal-processing algorithms developed by E. Asseo.

I should also like to thank M. Eriksson, my supervisor at the University of Lund. In addition, I should like to express my gratitude for the useful suggestions for improvement of this document that I received from B. Autin, M. Chanel, M. Eriksson, L. Hedin and D. Möhl.

Finally I should like to thank the Scientific Reports Service at CERN and C. Bengtsson for their assistance during the writing and typing of this document.

Appendix A

MANIFESTLY COVARIANT HAMILTONIAN FORMALISM

In the following we will present a manifestly covariant Hamiltonian formalism for a charged particle in an external electromagnetic field.

A.1 The covariant Hamiltonian

In the case of a particle in an external electromagnetic field it is possible to define a covariant Hamiltonian valid in any coordinate system by [BARUT80]:

$$H(x,p) = \frac{1}{2m_0c}(p_\mu - eA_\mu(x))(p^\mu - eA^\mu(x)) = \frac{1}{2m_0c}\left[p_\mu p^\mu - 2ep_\mu A^\mu(x) + e^2A_\mu(x)A^\mu(x)\right], \quad (303)$$

where

$$A^\mu \equiv (\frac{\Phi}{c},\bar{A}), \quad p^\mu \equiv m_0 u^\mu + eA^\mu, \quad u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma(c,\bar{v}). \quad (304)$$

Hamilton's equations are in a Cartesian system:

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \frac{\partial H}{\partial p_\mu} = \frac{1}{m_0c}(p^\mu - eA^\mu) \\ \frac{dp^\mu}{d\tau} &= -\frac{\partial H}{\partial x_\mu} = \frac{e}{m_0c}(p^\nu - eA^\nu)\partial A_\nu . \end{aligned} \quad (305)$$

Since

$$\frac{dA^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\partial_\nu A^\mu \quad (306)$$

and solving (305) for p^μ

$$p^\mu = m_0c\frac{dx^\mu}{d\tau} + eA^\mu , \quad (307)$$

gives in the second equation of (305)

$$\frac{d^2x^\mu}{d\tau^2} = \frac{e}{m_0c}(\partial^\mu A_\nu - \partial_\nu A^\mu)\frac{dx^\nu}{d\tau} = \frac{e}{m_0c}\tilde{F}^\mu{}_\nu\frac{dx^\nu}{d\tau} , \quad (308)$$

where we have used the definition of the electromagnetic field tensor [BARUT80]. This may be compared with Eqs. (22).

A.2 The covariant Hamiltonian for an accelerator

The metric tensor is in this case given by (30). The normal components of the vector potential A_x, A_s, and A_z in the curvilinear system (see subsection 3.2) are related to the contravariant and covariant components by

$$
\begin{aligned}
A^1 &= A_x\,, & A^2 &= \frac{A_s}{(1+hx)}\,, & A^3 &= A_z\,, & A^0 &= \frac{\Phi}{c}\,,\\
A_1 &= -\,A_x\,, & A_2 &= -\,(1+hx)A_s\,, & A_3 &= A_z\,, & A_0 &= \frac{\Phi}{c}\,.
\end{aligned}
\tag{309}
$$

If we use x, s, z instead of 1, 2, 3 to label the covariant components of the conjugate momenta, we find using (17) and (303) that

$$
H = \frac{1}{2m_0}\left[\left(\frac{E}{c} - e\frac{\Phi}{c}\right)^2 - (p_x + eA_x)^2 - \left(\frac{p_s}{1+xh} + eA_s\right)^2 - (p_z + eA_z)^2\right]. \tag{310}
$$

We assume that there is no electric field, $\Phi = 0$, and that the vector potentials are time independent:

$$
\frac{\partial A_x}{\partial t} = \frac{\partial A_s}{\partial t} = \frac{\partial A_z}{\partial t} = 0\,. \tag{311}
$$

The Hamiltonian is then time independent and simplified to

$$
H = \frac{1}{2m_0}\left[\frac{E^2}{c^2} - (p_x + eA_x)^2 - \left(\frac{p_s}{1+hx} + eA_s\right)^2 - (p_z + eA_z)^2\right]. \tag{312}
$$

Hamilton's equations give

$$
\begin{aligned}
c\frac{\mathrm{d}t}{\mathrm{d}\tau} &= c\frac{\partial H}{\partial E} = \frac{E}{m_0 c} = \gamma c \\
\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}\tau} &= -\frac{1}{c}\frac{\partial H}{\partial t} = 0 \\
\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \frac{\partial H}{\partial p_x} = -\frac{1}{m_0}(p_x + eA_x) \\
\frac{\mathrm{d}p_x}{\mathrm{d}\tau} &= -\frac{\partial H}{\partial x} \\
&= \frac{e}{m_0}\left\{(p_x + eA_x)\frac{\partial A_x}{\partial x} + \left(\frac{p_s}{1+hx} + eA_s\right)\left(\frac{\partial A_s}{\partial x} - \frac{hp_x}{e(1+hx)^2}\right) + (p_z + eA_z\frac{\partial A_z}{\partial x}\right\} \\
\frac{\mathrm{d}s}{\mathrm{d}\tau} &= \frac{\partial H}{\partial p_s} = -\frac{1}{m_0}\frac{1}{1+hx}\left(\frac{p_s}{1+hx} + eA_s\right) \qquad (313) \\
\frac{\mathrm{d}p_s}{\mathrm{d}\tau} &= -\frac{\partial H}{\partial s} \\
&= \frac{e}{m_0}\left\{(p_x + eA_x)\frac{\partial A_x}{\partial s} + \left(\frac{p_s}{1+hx} + eA_s\right)\left(\frac{\partial A_s}{\partial s} - \frac{'hx'}{e(1+hx)^2}\right) + (p_z + eA_z)\frac{\partial A_z}{\partial s}\right\} \\
\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{\partial H}{\partial p_z} = -\frac{1}{m_0}(p_z + eA_z) \\
\frac{\mathrm{d}p_z}{\mathrm{d}\tau} &= -\frac{\partial H}{\partial z} = \frac{e}{m_0}\left\{(p_x + eA_x)\frac{\partial A_x}{\partial z} + \left(\frac{p_s}{1+hx} + eA_s\right)\frac{\partial A_s}{\partial z} + (p_z + eA_z)\frac{\partial A_z}{\partial z}\right\}.
\end{aligned}
$$

From the first equation one has

$$
\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t} = \gamma\frac{\mathrm{d}}{\mathrm{d}t} = \gamma\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} = \gamma\dot{s}\frac{\mathrm{d}}{\mathrm{d}s}\,. \qquad (314)
$$

Using this in Eqs. (313) we obtain

$$
\begin{aligned}
x' &= \frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{1}{p}\frac{v}{\dot{s}}(p_x + eA_x) \\
z' &= \frac{\mathrm{d}z}{\mathrm{d}s} = -\frac{1}{p}\frac{v}{\dot{s}}(p_z + eA_z) \\
\dot{s} &= -\frac{v}{p}\frac{1}{1+hx}\left(\frac{p_s}{1+hx} + eA_s\right) \\
p_x' &= -e\left[x'\frac{\partial A_x}{\partial x} + hA_s + (1+hx)\frac{\partial a_s}{\partial x} + z'\frac{\partial A_z}{\partial x}\right] - p\frac{\dot{s}}{v}(1+hx) \qquad (315) \\
p_z' &= \frac{\mathrm{d}p_z}{\mathrm{d}s} = -e\left[x'\frac{\partial A_x}{\partial z} + (1+hx)\frac{\partial A_s}{\partial z} + z'\frac{\partial A_z}{\partial z}\right] \\
p_s' &= \frac{\mathrm{d}p_s}{\mathrm{d}s} = -e\left[x'\frac{\partial A_x}{\partial s} + (1+hx)\frac{\partial A_s}{\partial s} + h'xA_s + z'\frac{\partial A_z}{\partial s}\right] - p\frac{\dot{s}}{v}(1+hx)h'x\,.
\end{aligned}
$$

Let us first solve for p_x, p_s, and p_z in the first three equations of (315). We then take the derivative with respect to s for p_x, p_s, and p_z and put this equal to the last three equations using Eq. (58) bearing in mind that the vector potential was assumed to be time independent,

$$
\begin{aligned}
A'_x &= x'\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial s} + z'\frac{\partial A_x}{\partial z} \\
A'_s &= x'\frac{\partial A_s}{\partial x} + \frac{\partial A_s}{\partial s} + z'\frac{\partial A_s}{\partial z} \\
A'_z &= x'\frac{\partial A_z}{\partial x} + \frac{\partial A_z}{\partial s} + z'\frac{\partial A_z}{\partial z} \,,
\end{aligned}
\tag{316}
$$

so that we find

$$
\begin{aligned}
&x'' + x'\frac{\ddot{s}}{\dot{s}} - h(1 + hx) = \frac{v}{\dot{s}}\frac{e}{p}\left[hA_s + (1 + hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s} + z'\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \\
&\frac{\ddot{s}}{\dot{s}^2}(1 + hx) + 2hx' + h'x \\
&\qquad = \frac{v}{\dot{s}}\frac{e}{p}\left[x'\left(\frac{1}{1 + hx}\frac{\partial A_x}{\partial s} - \frac{h}{1 + hx}A_s - \frac{\partial A_x}{\partial s}\right) + z'\left(\frac{1}{1 + hx}\frac{\partial A_z}{\partial s} - \frac{\partial A_s}{\partial z}\right)\right] \\
&z'' + z'\frac{\ddot{s}}{\dot{s}^2} = \frac{v}{\dot{s}}\frac{e}{p}\left[x'\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + (1 + hx)\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right].
\end{aligned}
\tag{317}
$$

From (64) it follows that

$$
\begin{aligned}
x'' + x'\frac{\ddot{s}}{\dot{s}} - h(1 + hx) &= \frac{v}{\dot{s}}\frac{e}{p}[(1 + hx)B_z - z'B_s] \\
\frac{\ddot{s}}{\dot{s}^2}(1 + hx) + 2hx' + h'x &= \frac{v}{\dot{s}}\frac{e}{p}[z'B_x - x'B_z] \\
z'' + z'\frac{\ddot{s}}{\dot{s}^2} &= \frac{v}{\dot{s}}\frac{e}{p}[x'B_s - (1 + hx)B_x] \,.
\end{aligned}
\tag{318}
$$

The second equation of (318) gives

$$
\frac{\ddot{s}}{\dot{s}^2} = -\frac{2hx' + h'x}{1 + hx} + \frac{v}{\dot{s}}\frac{e}{p}[z'B_x - x'B_z] \,,
\tag{319}
$$

and from the metric tensor $g_{\mu\nu}$

$$
\mathrm{d}\bar{r} = \mathrm{d}x\hat{x} + (1 + hx)\mathrm{d}s\hat{s} + \mathrm{d}z\hat{z}
\tag{320}
$$

we obtain

$$
v = \frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = \sqrt{\dot{x}^2 + (1 + hx)\dot{s}^2 + \dot{z}^2} = \dot{s}\sqrt{x'^2 + (1 + hx)^2 + z'^2}
\tag{321}
$$

so that

$$
\frac{v}{\dot{s}} = \sqrt{x'^2 + (1 + hx)^2 + z'^2} \,.
\tag{322}
$$

As expected, Eqs. (318) agree with Eqs. (35) obtained from the tensor equation.

The development of canonical transformations, Poisson brackets, and the Hamilton – Jacobi theory can also be applied to the covariant Hamiltonian [BARUT80].

If we introduce an eight-dimensional phase space (p^μ, x^μ), the canonical tranformations can be defined by the equation

$$p_\mu u^\mu - H = p'_\mu u'^\mu - H' + \frac{dF}{d\tau} , \tag{323}$$

where u^μ and u'^μ are four-velocities. In the particular case, for instance, where

$$F \equiv F(x^\mu , x'^\mu , \tau) , \tag{324}$$

we have

$$H' = H + \frac{\partial F}{\partial \tau} , \quad p_\mu = \frac{\partial F}{\partial x^\mu} , \quad p'_\mu = \frac{\partial F}{\partial x'^\mu} . \tag{325}$$

Appendix B

PERTURBATIONS FROM THICK SEXTUPOLES

$$
\begin{aligned}
\Delta_1 J_x(s_0 + NC) = \sum_{n=0}^{N}\sum_{k=1}^{K}\frac{\sqrt{2}}{48} m_k\sqrt{J_{0x}\beta_{kx}}\Bigg\{ & -\Bigg[6l_k^2\left(J_{0x} - \frac{2J_{0z}\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\frac{1}{\beta_{kx}}(J_{0x}\alpha_{kx} - 2J_{0z}\alpha_{kz}) \\
& + 3l_k^4\frac{1}{\beta_{kx}}\left(\frac{J_{0x}}{\beta_{kx}}(1 + \alpha_{kx}^2) - \frac{2J_{0z}}{\beta_{kz}}(1 + \alpha_{kz}^2)\right)\Bigg]\cos[\psi_{0x}(s_k) + n2\pi Q_x] \\
& - \Bigg[12l_k(J_{0x}\beta_{kx} - 2J_{0z}\beta_{kz}) - 6l_k^2\left(3J_{0x}\alpha_{kx} - 4J_{0z}\alpha_{kz} - \frac{2J_{0z}\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) \\
& - 4l_k^3\left(\frac{J_{0x}}{\beta_{kx}}(1 + 3\alpha_{kx}^2) - \frac{4J_{0z}\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{2J_{0z}}{\beta_{kz}}(1 + \alpha_{kz}^2)\right) \\
& + 3l_k^4\frac{\alpha_{kx}}{\beta_{kx}}\left(\frac{J_{0x}}{\beta_{kx}}(1 + \alpha_{kx}^2) - \frac{2J_{0z}}{\beta_{kz}}(2 + \alpha_{kz}^2)\right)\Bigg]\sin[\psi_{0x}(s_k) + n2\pi Q_x] \\
& - 3J_{0x}\left[6l_k^2 - 8l_k^3\frac{\alpha_{kx}}{\beta_{kx}} - l_k^4\frac{1}{\beta_{kx}^2}(1 - 3\alpha_{kx}^2)\right]\cos[3\psi_{0x}(s_k) + n6\pi Q_x] \\
& - 3J_{0x}\left[4l_k\beta_{kx} - 6l_k^2\alpha_{kx} - 4l_k^3\frac{1}{\beta_{kx}}(1 - \alpha_{kx}^2) + l_k^4\frac{\alpha_{kx}}{\beta_{kx}^2}(3 - \alpha_{kx}^2)\right] \\
& \times\sin[3\psi_{0x}(s_k) + n6\pi Q_x] + J_{0z}\Bigg[6l_k^2\left(2 + \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} + \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& - 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1 - 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2)\Bigg]\cos[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) + n2\pi(Q_x + 2Q_z)] \\
& + J_{0z}\Bigg[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) - 4l_k^3\left(\frac{2}{\beta_{kx}} + \frac{1}{\beta_{kz}} - \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx} + 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2)\Bigg]\sin[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) + n2\pi(Q_x + 2Q_z)] \\
& - J_{0z}\Bigg[6l_k^2\left(2 - \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} - \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1 + 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2)\Bigg]\cos[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) + n2\pi(Q_x - 2Q_z)] \\
& + J_{0z}\Bigg[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) + 4l_k^3\left(\frac{2}{\beta_{kx}} - \frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx} - 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2)\Bigg]\sin[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) + n2\pi(Q_x - 2Q_z)]\Bigg\}
\end{aligned}
\tag{326}
$$

$$
\begin{aligned}
\Delta_1 J_z(s_0 + NC) = \sum_{n=0}^{N} \sum_{k=1}^{K} \frac{\sqrt{2}}{24} m_k \sqrt{J_{0x}\beta_{kx}} J_{0z} \Bigg\{ \Bigg[& 6l_k^2\left(2 + \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} + \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& - 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(1 - 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\Bigg] \cos\left[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) + n2\pi(Q_x + 2Q_z)\right] \\
& + \Bigg[12 l_k \beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) - 4l_k^3\left(\frac{2}{\beta_{kx}} + \frac{1}{\beta_{kz}} - \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} + 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\Bigg] \sin\left[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) + n2\pi(Q_x + 2Q_z)\right] \\
& + \Bigg[6l_k^2\left(2 - \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} - \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(1 + 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\Bigg] \cos\left[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) + n2\pi(Q_x - 2Q_z)\right] \\
& - \Bigg[12 l_k \beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) + 4l_k^3\left(\frac{2}{\beta_{kx}} - \frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} - 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\Bigg] \sin\left[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) + n2\pi(Q_x - 2Q_z)\right] \Bigg\} .
\end{aligned}
\tag{327}
$$

$$
\begin{aligned}
\Delta_1\phi_x(s_0+NC) = {} & \sum_{n=0}^{N}\sum_{k=1}^{K}\frac{\sqrt{2}}{96}m_k\sqrt{J_{0x}\beta_{kx}}\Big\{-\big[12l_k(3J_{0x}\beta_{kx}-2J_{0z}\beta_{kz})\\
& -6l_k^2\left(9J_{0x}\alpha_{kx}-4J_{0z}\alpha_{kz}-\frac{2J_{0z}\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right)\\
& -4l_k^3\left(\frac{3J_{0x}}{\beta_{kx}}(1+3\alpha_{kx}^2)-\frac{4J_{0z}\alpha_{kx}\alpha_{kz}}{\beta_{kx}}-\frac{2J_{0z}}{\beta_{kz}}(1+\alpha_{kz}^2)\right)\\
& +3l_k^4\frac{\alpha_{kx}}{\beta_{kx}}\left(\frac{3J_{0x}}{\beta_{kx}}(1+\alpha_{kx}^2)-\frac{2J_{0z}}{\beta_{kz}}(1+\alpha_{kz}^2)\right)\Bigg]\cos[\psi_{0x}(s_k)+n2\pi Q_x]\\
& +\Bigg[6l_k^2\left(3J_{0x}-\frac{2J_{0z}\beta_{kz}}{\beta_{kx}}\right)-8l_k^3\frac{1}{\beta_{kx}}(3J_{0x}\alpha_{kx}-2J_{0z}\alpha_{kz})\\
& +3l_k^4\frac{1}{\beta_{kx}}\left(\frac{3J_{0x}}{\beta_{kx}}(1+\alpha_{kx}^2)-\frac{2J_{0z}}{\beta_{kz}}(1+\alpha_{kz}^2)\right)\Bigg]\sin[\psi_{0x}(s_k)+n2\pi Q_x]\\
& -3J_{0x}\left[4l_k\beta_{kx}-6l_k^2\alpha_{kx}-4l_k^3\frac{1}{\beta_{kx}}(1-\alpha_{kx}^2)+l_k^4\frac{\alpha_{kx}}{\beta_{kx}^2}(3-\alpha_{kx}^2)\right]\\
& \times\cos[3\psi_{0x}(s_k)+n6\pi Q_x]\\
& +3J_{0x}\left[6l_k^2-8l_k^3\frac{\alpha_{kx}}{\beta_{kx}}-l_k^4\frac{1}{\beta_{kx}^2}(1-3\alpha_{kx}^2)\right]\sin[3\psi_{0x}(s_k)+n6\pi Q_x]\\
& +J_{0z}\Bigg[12l_k\beta_{kz}-6l_k^2\left(2\alpha_{kz}+\frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right)-4l_k^3\left(\frac{2}{\beta_{kx}}+\frac{1}{\beta_{kz}}-\frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}}-\frac{\alpha_{kz}^2}{\beta_{kz}}\right)\\
& +3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx}+2\alpha_{kz}-\alpha_{kx}\alpha_{kz}^2)\Bigg]\cos[\psi_{0x}(s_k)+2\psi_{0z}(s_z)+n2\pi(Q_x+2Q_z)]\\
& -J_{0z}\Bigg[6l_k^2\left(2+\frac{\beta_{kz}}{\beta_{kx}}\right)-8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}}+\frac{\alpha_{kz}}{\beta_{kz}}+\frac{\alpha_{kz}}{\beta_{kx}}\right)\\
& -3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1-2\alpha_{kx}\alpha_{kz}-\alpha_{kz}^2)\Bigg]\sin[\psi_{0x}(s_k)+2\psi_{0z}(s_k)+n2\pi(Q_x+2Q_z)]\\
& +J_{0z}\Bigg[12l_k\beta_{kz}-6l_k^2\left(2\alpha_{kz}+\frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right)+4l_k^3\left(\frac{2}{\beta_{kx}}-\frac{1}{\beta_{kz}}+\frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}}+\frac{\alpha_{kz}^2}{\beta_{kz}}\right)\\
& +3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx}-2\alpha_{kz}-\alpha_{kx}\alpha_{kz}^2)\Bigg]\cos[\psi_{0x}(s_k)-2\psi_{0z}(s_k)+n2\pi(Q_x-2Q_z)]\\
& +J_{0z}\Bigg[6l_k^2\left(2-\frac{\beta_{kz}}{\beta_{kx}}\right)-8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}}+\frac{\alpha_{kz}}{\beta_{kz}}-\frac{\alpha_{kz}}{\beta_{kx}}\right)\\
& +3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1+2\alpha_{kx}\alpha_{kz}-\alpha_{kz}^2)\Bigg]\sin[\psi_{0x}(s_k)-2\psi_{0z}(s_k)+n2\pi(Q_x-2Q_z)]\Big\}\,.
\end{aligned}
\tag{328}
$$

$$
\begin{aligned}
\Delta_1\phi_z(s_0+NC) = {} & \sum_{n=0}^{N}\sum_{k=1}^{K}\frac{\sqrt{2}}{48}m_k\sqrt{J_{0x}\beta_{kx}}J_{0z}\Bigg\{\Bigg[12l_k\beta_{kz}-6l_k^2\left(2\alpha_{kz}+\frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right)\\
& +4l_k^3\left(\frac{1}{\beta_{kz}}+\frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}}+\frac{\alpha_{kz}^2}{\beta_{kz}}\right)-3l_k^4\frac{\alpha_{kx}}{\beta_{kx}\beta_{kz}}(1+\alpha_{kz}^2)\Bigg]\\
& \times\cos[\psi_{0x}(s_k)+n2\pi Q_x]\\
& -\left[6l_k^2\frac{\beta_{kz}}{\beta_{kx}}-8l_k^3\frac{\alpha_{kz}}{\beta_{kx}}+3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1+\alpha_{kz}^2)\right]\sin[\psi_{0x}(s_k)+n2\pi Q_x]\\
& +\Bigg[12l_k\beta_{kz}-6l_k^2\left(\frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}+2\alpha_{kz}\right)\\
& -4l_k^3\left(\frac{2}{\beta_{kx}}+\frac{1}{\beta_{kz}}-\frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}}-\frac{\alpha_{kz}^2}{\beta_{kz}}\right)+3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx}+2\alpha_{kz}-\alpha_{kx}\alpha_{kz}^2)\Bigg]\\
& \times\cos[\psi_{0x}(s_k)+2\psi_{0z}(s_k)+n2\pi(Q_x+2Q_z)]\\
& -\Bigg[6l_k^2\left(2+\frac{\beta_{kz}}{\beta_{kx}}\right)-8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}}+\frac{\alpha_{kz}}{\beta_{kz}}+\frac{\alpha_{kz}}{\beta_{kx}}\right)\\
& -3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1-2\alpha_{kx}\alpha_{kz}-\alpha_{kz}^2)\Bigg]\sin[\psi_{0x}(s_k)+2\psi_{0z}(s_k)+n2\pi(Q_x+2Q_z)]\\
& +\Bigg[12l_k\beta_{kz}-6l_k^2\left(2\alpha_{kz}+\frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right)+4l_k^3\left(\frac{2}{\beta_{kx}}-\frac{1}{\beta_{kz}}+\frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}}+\frac{\alpha_{kz}^2}{\beta_{kx}}\right)\\
& +3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx}-2\alpha_{kz}-\alpha_{kx}\alpha_{kz}^2)\Bigg]\cos[\psi_{0x}(s_k)-2\psi_{0z}(s_k)+n2\pi(Q_x-2Q_z)]\\
& +\Bigg[6l_k^2\left(2-\frac{\beta_{kz}}{\beta_{kx}}\right)-8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}}+\frac{\alpha_{kz}}{\beta_{kz}}-\frac{\alpha_{kz}}{\beta_{kx}}\right)\\
& +3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}(1+2\alpha_{kx}\alpha_{kz}-\alpha_{kz}^2)\Bigg]\sin[\psi_{0x}(s_k)-2\psi_{0z}(s_k)+n2\pi(Q_x-2Q_z)]\Bigg\}\,.
\end{aligned}
\tag{329}
$$

Appendix C

AVERAGE PERTURBATIONS FROM THICK SEXTUPOLES

$$
\begin{aligned}
<\Delta_1 J_x(s_0)> = \sum_{k=1}^{K} \frac{\sqrt{2}}{96} m_k \sqrt{J_{0x}\beta_{kx}} \Bigg\{ \Bigg[& 6l_k^2\left(J_{0x} - \frac{2J_{0z}\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3 \frac{1}{\beta_{kx}}(J_{0x}\alpha_{kx} - 2J_{0z}\alpha_{kz}) \\
& + 3l_k^4 \frac{1}{\beta_{kx}}\left(\frac{J_{0x}}{\beta_{kx}}(1 + \alpha_{kx}^2) - \frac{2J_{0z}}{\beta_{kz}}(1 + \alpha_{kz}^2)\right)\Bigg] \frac{\sin[\psi_{0x}(s_k) - \pi Q_x]}{\sin \pi Q_x} \\
& - \Bigg[12l_k(J_{0x}\beta_{kx} - 2J_{0z}\beta_{kz}) - 6l_k^2\left(3J_{0x}\alpha_{kx} - 4J_{0z}\alpha_{kz} - \frac{2J_{0z}\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) \\
& - 4l_k^3\left(\frac{J_{0x}}{\beta_{kx}}(1+3\alpha_{kx}^2) - \frac{4J_{0z}\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{2J_{0z}}{\beta_{kz}}(1 + \alpha_{kz}^2)\right) \\
& + 3l_k^4 \frac{\alpha_{kx}}{\beta_{kx}}\left(\frac{J_{0x}}{\beta_{kx}}(1 + \alpha_{kx}^2) - \frac{2J_{0z}}{\beta_{kz}}(2 + \alpha_{kz}^2)\right)\Bigg] \frac{\cos[\psi_{0x}(s_k) - \pi Q_x]}{\sin \pi Q_x} \\
& + 3J_{0x}\left[6l_k^2 - 8l_k^3 \frac{\alpha_{kx}}{\beta_{kx}} - l_k^4 \frac{1}{\beta_{kx}^2}(1 - 3\alpha_{kx}^2)\right] \frac{\sin[3\psi_{0x}(s_k) - 3\pi Q_x]}{\sin 3\pi Q_x} \\
& - 3J_{0x}\left[4l_k\beta_{kx} - 6l_k^2\alpha_{kx} - 4l_k^3 \frac{1}{\beta_{kx}}(1 - \alpha_{kx}^2) + l_k^4 \frac{\alpha_{kx}}{\beta_{kx}^2}(3 - \alpha_{kx}^2)\right] \\
& \times \frac{\cos[3\psi_{0x}(s_k) - 3\pi Q_x]}{\sin 3\pi Q_x} - J_{0z}\Bigg[6l_k^2\left(2 + \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} + \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& - 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}(1 - 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2)\Bigg] \frac{\sin[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin \pi(Q_x + 2Q_z)} \\
& + J_{0z}\Bigg[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) - 4l_k^3\left(\frac{2}{\beta_{kx}} + \frac{1}{\beta_{kz}} - \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx} + 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2)\Bigg] \frac{\cos[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin \pi(Q_x + 2Q_z)} \\
& + J_{0z}\Bigg[6l_k^2\left(2 - \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} - \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}(1 + 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2)\Bigg] \frac{\sin[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin \pi(Q_x - 2Q_z)} \\
& + J_{0z}\Bigg[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) + 4l_k^3\left(\frac{2}{\beta_{kx}} - \frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}(\alpha_{kx} - 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2)\Bigg] \frac{\cos[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin \pi(Q_x - 2Q_z)} \Bigg\}.
\end{aligned}
\tag{330}
$$

$$
\begin{aligned}
<\Delta_1 J_z(s_0)> = \sum_{k=1}^{K} \frac{\sqrt{2}}{48} m_k \sqrt{J_{0x}\beta_{kx}} J_{0z} \Bigg\{ & - \Bigg[6l_k^2\left(2 + \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} + \frac{\alpha_{kz}}{\beta_{kx}}\right) \\
& - 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(1 - 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\Bigg] \frac{\sin\left[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)\right]}{\sin \pi Q_x + 2Q_z)} \\
& + \Bigg[12 l_k \beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) - 4l_k^3\left(\frac{2}{\beta_{kx}} + \frac{1}{\beta_{kz}} - \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} + 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\Bigg] \frac{\cos\left[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)\right]}{\sin \pi(Q_x + 2Q_z)} \\
& - \Bigg[6l_k^2\left(2 - \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} - \frac{\alpha_{kz}}{\beta_{kx}}\right) + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(1 + 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\Bigg] \\
& \times \frac{\sin\left[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)\right]}{\sin \pi(Q_x - 2Q_z)} \\
& - \Bigg[12 l_k \beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) + 4l_k^3\left(\frac{2}{\beta_{kx}} - \frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kz}}\right) \\
& + 3l_k^4 \frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} - 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\Bigg] \frac{\cos\left[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)\right]}{\sin \pi(Q_x - 2Q_z)} \Bigg\} .
\end{aligned}
\tag{331}
$$

$$
\begin{aligned}
<\Delta_1\phi_x(s_0)> = \sum_{k=1}^{K}\frac{\sqrt{2}}{192}m_k\sqrt{\frac{\beta_{kx}}{J_{0x}}}\Bigg\{\Big[&12l_k(3J_{0x}\beta_{kx} - 2J_{0z}\beta_{kz}) \\
&- 6l_k^2\left(9J_{0x}\alpha_{kx} - 4J_{0z}\alpha_{kz} - \frac{2J_{0z}\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) \\
&- 4l_k^3\left(\frac{3J_{0x}}{\beta_{kx}}\left(1 + 3\alpha_{kx}^2\right) - \frac{4J_{0z}\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{2J_{0z}}{\beta_{kz}}\left(1+\alpha_{kz}^2\right)\right) \\
&+ 3l_k^4\frac{\alpha_{kx}}{\beta_{kx}}\left(\frac{3J_{0x}}{\beta_{kx}}\left(1 + \alpha_{kx}^2\right) - \frac{2J_{0z}}{\beta_{kz}}\left(1 + \alpha_{kz}^2\right)\right)\Big]\frac{\sin\left[\psi_{0x}(s_k) - \pi Q_x\right]}{\sin\pi Q_x} \\
&+ \left[6l_k^2\left(3J_{0x} - \frac{2J_{0z}\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\frac{1}{\beta_{kx}}(3J_{0x}\alpha_{kx} - 2J_{0z}\alpha_{kz})\right. \\
&\left.+ 3l_k^4\frac{1}{\beta_{kx}}\left(\frac{3J_{0x}}{\beta_{kx}}\left(1 + \alpha_{kx}^2\right) - \frac{2J_{0z}}{\beta_{kz}}\left(1 + \alpha_{kz}^2\right)\right)\right]\frac{\cos\left[\psi_{0x}(s_k) - \pi Q_x\right]}{\sin\pi Q_x} \\
&+ 3J_{0x}\left[4l_k\beta_{kx} - 6l_k^2\alpha_{kx} - 4l_k^3\frac{1}{\beta_{kx}}\left(1 - \alpha_{kx}^2\right) + l_k^4\frac{\alpha_{kx}}{\beta_{kx}^2}\left(3 - \alpha_{kx}^2\right)\right] \\
&\times\frac{\sin\left[3\psi_{0x}(s_k) - 3\pi Q_x\right]}{\sin 3\pi Q_x} \\
&+ 3J_{0x}\left[6l_k^2 - 8l_k^3\frac{\alpha_{kx}}{\beta_{kx}} - l_k^4\frac{1}{\beta_{kx}^2}\left(1 - 3\alpha_{kx}^2\right)\right]\frac{\cos\left[3\psi_{0x}(s_k) - 3\pi Q_x\right]}{\sin 3\pi Q_x} \\
&- J_{0z}\left[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) - 4l_k^3\left(\frac{2}{\beta_{kx}} + \frac{1}{\beta_{kz}} - \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{\alpha_{kz}^2}{\beta_{kz}}\right)\right. \\
&\left.+ 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} + 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\right]\frac{\sin\left[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)\right]}{\sin\pi(Q_x + 2Q_z)} \\
&- J_{0z}\left[6l_k^2\left(2 + \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} + \frac{\alpha_{kz}}{\beta_{kx}}\right)\right. \\
&\left.- 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(1 - 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\right]\frac{\cos\left[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)\right]}{\sin\pi(Q_x + 2Q_z)} \\
&- J_{0z}\left[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) + 4l_k^3\left(\frac{2}{\beta_{kx}} - \frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kz}}\right)\right. \\
&\left.+ 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} - 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\right]\frac{\sin\left[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)\right]}{\sin\pi(Q_x - 2Q_z)} \\
&+ J_{0z}\left[6l_k^2\left(2 - \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} - \frac{\alpha_{kz}}{\beta_{kx}}\right)\right. \\
&\left.+ 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(1 + 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\right]\frac{\cos\left[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)\right]}{\sin\pi(Q_x - 2Q_z)}\Bigg\}.
\end{aligned}
\tag{332}
$$

$$
\begin{aligned}
<\Delta_1\phi_z(s_0)> = \sum_{k=1}^{K}\frac{\sqrt{2}}{96}m_k\sqrt{J_{0x}\beta_{kx}}\Bigg\{ & -\Bigg[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) \\
& + 4l_k^3\left(\frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kz}}\right) - 3l_k^4\frac{\alpha_{kx}}{\beta_{kx}\beta_{kz}}\left(1 + \alpha_{kz}^2\right)\Bigg] \\
& \times \frac{\sin\,[\psi_{0x}(s_k) - \pi Q_x)]}{\sin\pi Q_x} \\
& - \left[6l_k^2\frac{\beta_{kz}}{\beta_{kx}} - 8l_k^3\frac{\alpha_{kz}}{\beta_{kx}} + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(1 + \alpha_{kz}^2\right)\right]\frac{\cos\,[\psi_{0x}(s_k) - \pi Q_x]}{\sin\pi Q_x} \\
& - \Bigg[12l_k\beta_{kz} - 6l_k^2\left(\frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}} + 2\alpha_{kz}\right) \\
& - 4l_k^3\left(\frac{2}{\beta_{kx}} + \frac{1}{\beta_{kz}} - \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} - \frac{\alpha_{kz}^2}{\beta_{kz}}\right) + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} + 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\Bigg] \\
& \times \frac{\sin\,[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin\pi(Q_x + 2Q_z)} \\
& - \left[6l_k^2\left(2 + \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} + \frac{\alpha_{kz}}{\beta_{kx}}\right) - 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(1 - 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\right] \\
& \times \frac{\cos\,[\psi_{0x}(s_k) + 2\psi_{0z}(s_k) - \pi(Q_x + 2Q_z)]}{\sin\pi(Q_x + 2Q_z)} \\
& - \Bigg[12l_k\beta_{kz} - 6l_k^2\left(2\alpha_{kz} + \frac{\alpha_{kx}\beta_{kz}}{\beta_{kx}}\right) + 4l_k^3\left(\frac{2}{\beta_{kx}} - \frac{1}{\beta_{kz}} + \frac{2\alpha_{kx}\alpha_{kz}}{\beta_{kx}} + \frac{\alpha_{kz}^2}{\beta_{kx}}\right) \\
& + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(\alpha_{kx} - 2\alpha_{kz} - \alpha_{kx}\alpha_{kz}^2\right)\Bigg]\frac{\sin\,[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin\pi(Q_x - 2Q_z)} \\
& + \left[6l_k^2\left(2 - \frac{\beta_{kz}}{\beta_{kx}}\right) - 8l_k^3\left(\frac{\alpha_{kx}}{\beta_{kx}} + \frac{\alpha_{kz}}{\beta_{kz}} - \frac{\alpha_{kz}}{\beta_{kx}}\right) + 3l_k^4\frac{1}{\beta_{kx}\beta_{kz}}\left(1 + 2\alpha_{kx}\alpha_{kz} - \alpha_{kz}^2\right)\right] \\
& \times \frac{\cos\,[\psi_{0x}(s_k) - 2\psi_{0z}(s_k) - \pi(Q_x - 2Q_z)]}{\sin\pi(Q_x - 2Q_z)}\Bigg\}\,.
\end{aligned}
\tag{333}
$$

Appendix D

TUNE DIAGRAMS

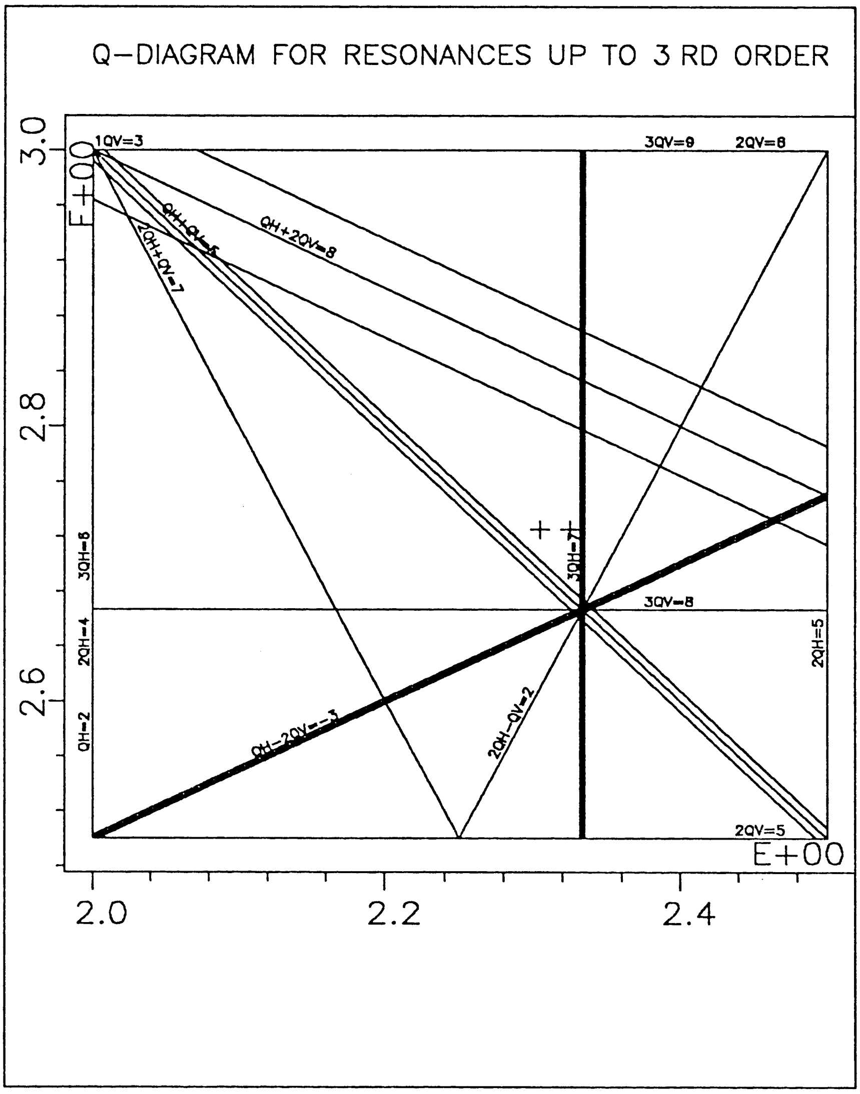

Figure 36: Tune diagram with resonances up to the third order. The bandwith of the resonances has been calculated for $J_{0x} = 20$, $J_{0z} = 10$ mm·mrad and

$$|h_{1010\,5}| = 3.5\times10^{-3}$$
$$|h_{3000\,7}| = 0.14$$
$$|h_{1002\,-3}| = 0.41$$
$$|h_{1020\,8}| = 3.54$$

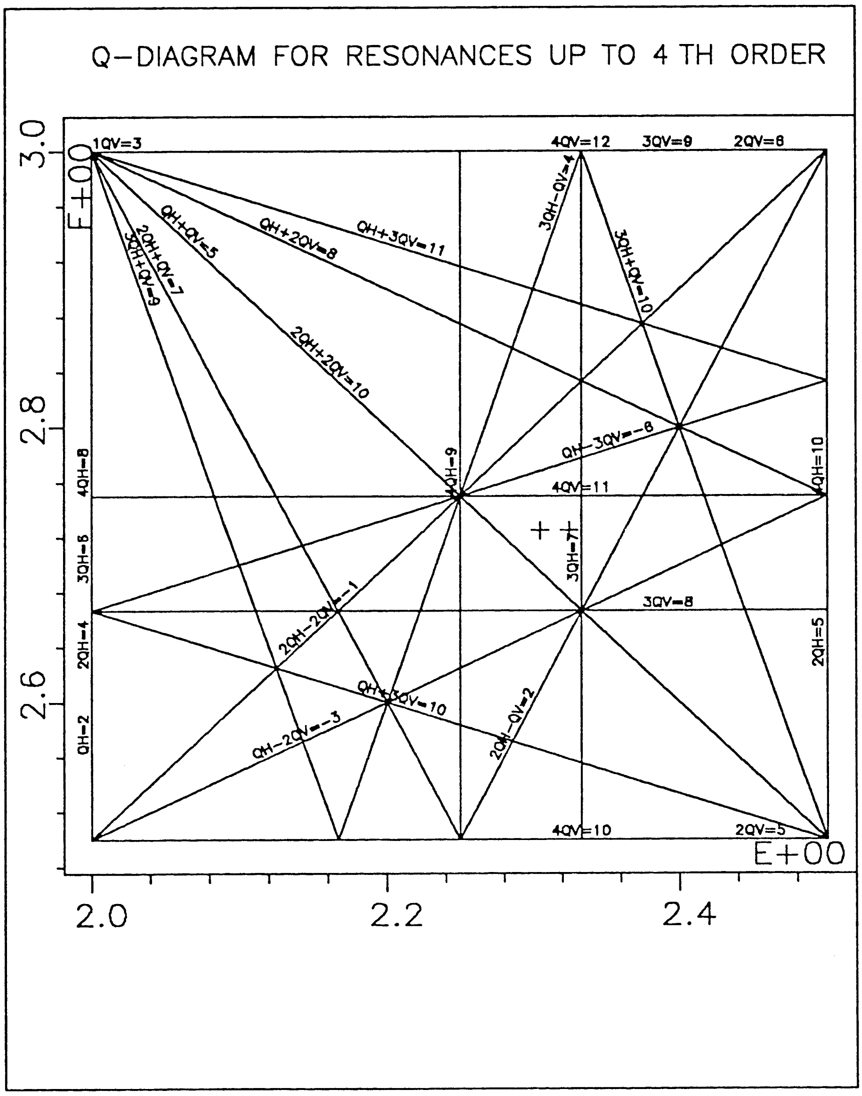

Figure 37: Tune diagram with resonances up to the fourth order

Appendix E

SYMBOLS AND NOTATIONS

R	mean radius of the accelerator
C	circumference of the accelerator
κ	excitation coefficient of the resonance
N	order of a resonance or number of turns
μ_x, μ_z	betatron phase advance
ψ_x, ψ_z	betatron phase function
β_x, β_z	betatron amplitude function
J_x, J_z	action variable
Q_x, Q_z	tune
δ	momentum deviation
B_x, B_z, B_s	magnetic field
A_x, A_z, A_s	vector potential
K_x, K_z	restoring forces
h	curvature
k	quadrupole strength
m	sextupole strength
ϕ	phase of the betatron motion
D_0	linear orbit dispersion function
D	orbit dispersion function
H, H_0	Hamiltonian
V	perturbation added to the Hamiltonian
prime	derivative with respect to s
p_x, p_z, p_s	conjugate momenta
e	charge of a particle or distance from a resonance
l	length of a magnetic multipole
x, z	transverse coordinates in the curvilinear coordinate system
s	distance along the reference curve
ρ	local radius
$g_{\mu\nu}$	metric tensor

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