# Driven oscillators and the resonance phenomenon

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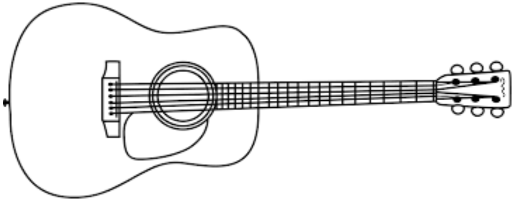
Arts & Métiers Institute of Technology, January 2022

Collapse of the Tacoma bridge (1940).

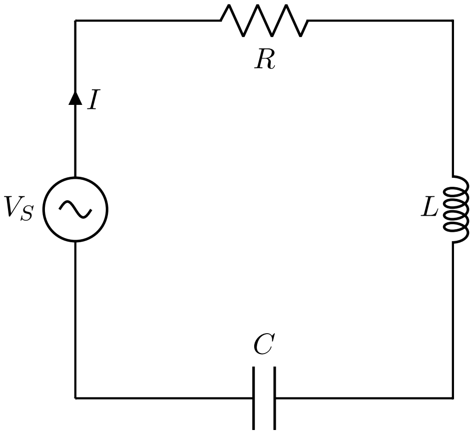



# Forced Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = \gamma\cos(\omega t)$$

## Forced Harmonic oscillator

$$\ddot{x} + 2\xi\dot{x} + x = \gamma\cos(\omega t)$$


$R$
$I$
$V_S$
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$C$

## Forced linear oscillator (no damping)

$$\ddot{x} + x = \gamma \cos(\omega t)$$

# Forced linear oscillator (no damping)

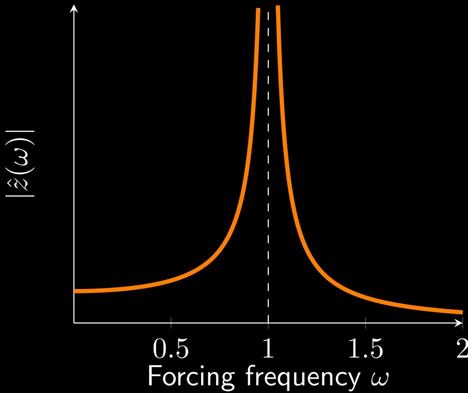
$$\ddot{z} + z = \gamma e^{i\omega t}$$

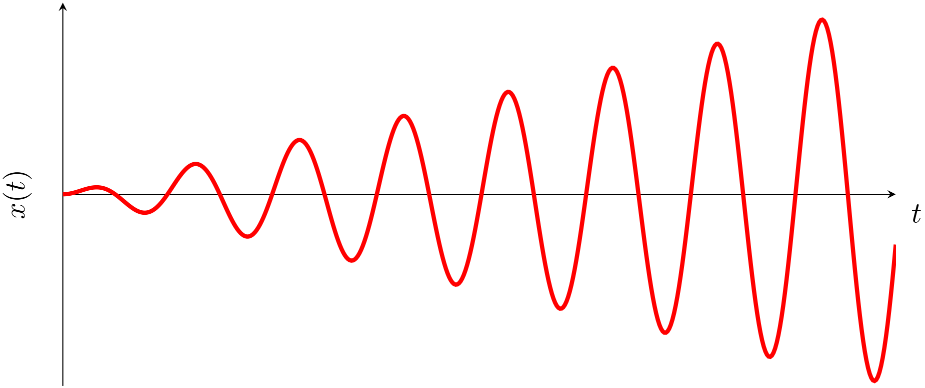
# Forced linear oscillator (no damping)

$$\left(1-\omega^2\right)\hat{z}e^{i\varphi}=\gamma$$

# Forced linear oscillator (no damping)

$$\hat{z} = \frac{\gamma}{1-\omega^2} e^{-i\varphi}$$





$x(t)$
$t$

## Forced linear oscillator (linear damping)

$$\ddot{x} + 2\xi\dot{x} + x = \gamma\cos(\omega t)$$

# Forced linear oscillator (linear damping)

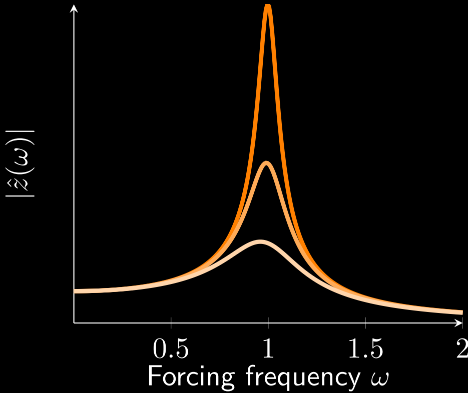
$$\ddot{z} + 2\xi\dot{z} + z = \gamma e^{i\omega t}$$

# Forced linear oscillator (linear damping)

$$\left(2\xi i\omega + 1 - \omega^2\right) \hat{z} e^{i\varphi} = \gamma$$

# Forced linear oscillator (linear damping)

$$\hat{z} = \frac{\gamma}{2\xi i\omega + 1 - \omega^2} e^{-i\varphi}$$



## Conservative nonlinear oscillator

Forced Liénard equation

$$\ddot{x} + g(x) = \gamma \cos(\omega t)$$

## Conservative nonlinear oscillator

Forced Duffing oscillator

$$\ddot{x} + x + \epsilon x^3 = \gamma \cos(\omega t)$$

## Conservative nonlinear oscillator

$$\frac{d^2x}{d\tau^2} + \frac{1}{\omega^2}x + \frac{\epsilon}{\omega^2}x^3 = \frac{\gamma}{\omega^2}\cos(\tau)$$

## Conservative nonlinear oscillator

Retaining only terms of order $\mathcal{O}(\epsilon)$ leads to

$$\frac{d^2x}{d\tau^2} + x = \epsilon\left(\Gamma\cos(\tau) + 2\omega_1 x - x^3\right)$$

where the parameters are defined as $\gamma = \epsilon\Gamma$.

## Conservative nonlinear oscillator

Introducing the power series expansion $x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau)$ yields

$$\begin{aligned} \mathcal{O}(1): &\quad \ddot{x}_0 + x_0 = 0 \\ \mathcal{O}(\epsilon): &\quad \ddot{x}_1 + x_1 = 2\omega_1 x_0 - x_0^3 + \Gamma \cos(\tau) \end{aligned}$$

which we can now solve.

## Conservative nonlinear oscillator

At leading order, the solution is given by $x_0(\tau) = A\cos(\tau)$. At the next order, we have

$$\ddot{x}_1 + x_1 = \underbrace{\left(2\omega_1 A - \frac{3}{4}A^3 + \Gamma\right)}_{=0}\cos(\tau) - \frac{A^3}{4}\cos(3\tau)$$

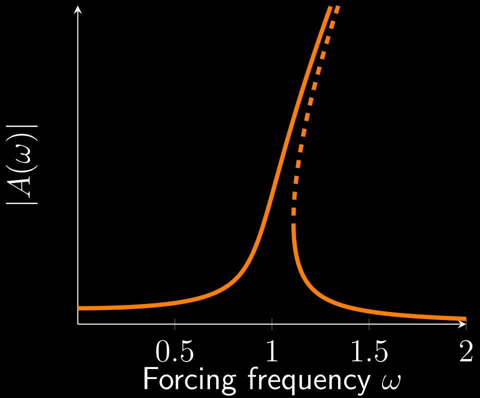
which leads to $\omega_1 = \dfrac{3}{8}A^2 - \dfrac{\Gamma}{2A}$ to avoid secular growth.

## Conservative nonlinear oscillator

In the original variables, this leads to

$$\frac{3}{4}\epsilon A^3 + \left(1 - \omega^2\right) A - \gamma = 0$$

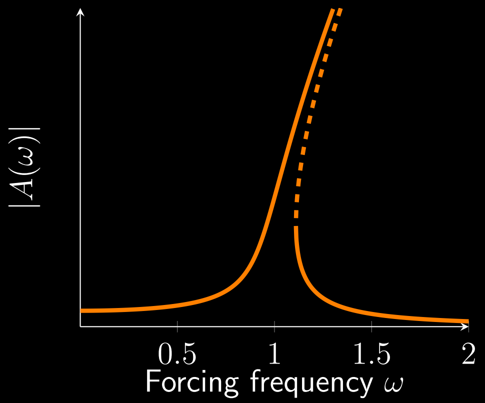
describing the **nonlinear response function** of the system.



## Conservative nonlinear oscillator

Nonlinearity prevents the unbounded growth of the oscillations by generating higher-order harmonics.

As the amplitude grows, the resonant forcing frequency changes.



## Dissipative nonlinear oscillator

Forced Duffing oscillator with damping

$$\ddot{x} + \delta\dot{x} + x + \epsilon x^3 = \gamma\cos(\omega t)$$

## Dissipative nonlinear oscillator

Let's illustrate another technique to derive the nonlinear response function, namely **Harmonic Balance**. Assume the solution is of the form $x(t) = A\cos(\omega t) + B\sin(\omega t)$ and inject into the equations.

## Dissipative nonlinear oscillator

$$\left(-\omega^2 A + \omega\delta B + A + \frac{3}{4}\epsilon A^3 + \frac{3}{4}\epsilon A B^2 - \gamma\right)\cos(\omega t)$$
$$+\left(-\omega^2 B - \omega\delta A + \frac{3}{4}\epsilon B^3 + B + \frac{3}{4}\epsilon A^2 B\right)\sin(\omega t)$$
$$+\left(\frac{1}{4}\epsilon A^3 - \frac{3}{4}\epsilon A B^3\right)\cos(3\omega t)$$
$$+\left(\frac{3}{4}\epsilon A^2 B - \frac{1}{4}\epsilon B^3\right)\sin(3\omega t) = 0$$

## Dissipative nonlinear oscillator

Neglecting superharmonics at $3\omega$ leads to the balance equations

$$\left(1-\omega^2\right)A+\omega\delta B+\frac{3}{4}\epsilon A^3+\frac{3}{4}\epsilon AB^2=\gamma$$
$$\left(1-\omega^2\right)B-\omega\delta A+\frac{3}{4}\epsilon B^3+\frac{3}{4}\epsilon A^2B=0.$$

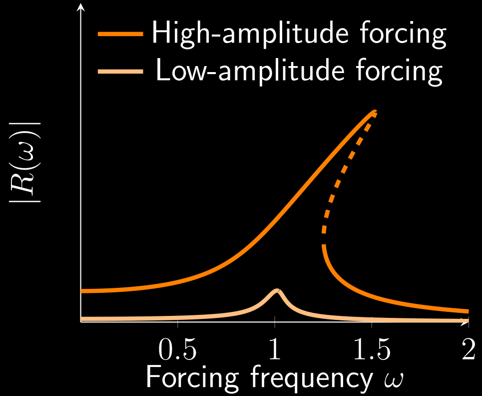
If $B=0$ and $\delta=0$, we recover the response function derived for the undamped case.

## Dissipative nonlinear oscillator

These conditions can be combined into

$$\left[\left(\frac{3}{4}\epsilon R^2 + 1 - \omega^2\right)^2 + (\delta\omega)^2\right] R^2 = \gamma^2$$

with $R = \sqrt{A^2 + B^2}$ the amplitude of the oscillation.

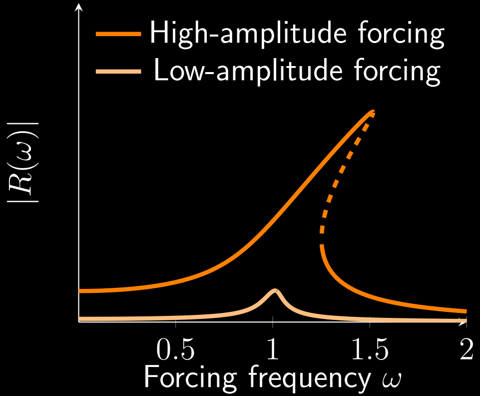


## Dissipative nonlinear oscillator

It can be simplified to

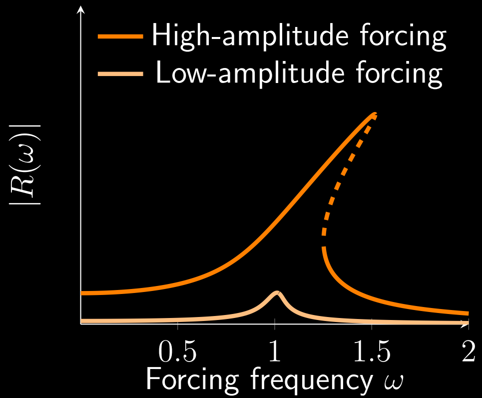
$$\left[\left(\frac{3}{4}R^2 - 2\omega_1\right)^2 + \Delta^2\right] R^2 = \Gamma^2$$

with $1 - \omega^2 = -2\epsilon\omega_1$, $\Delta = \delta/\epsilon$ and $\Gamma = \gamma/\epsilon$.



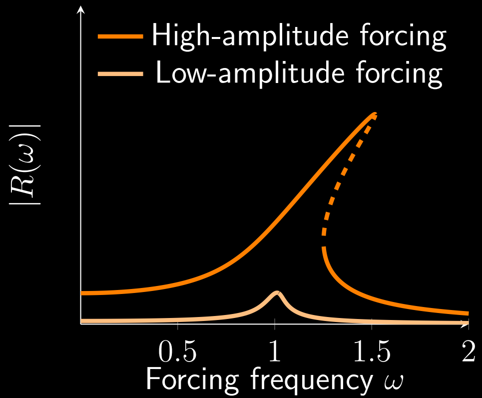
## Dissipative nonlinear oscillator

As $\Gamma$ increases, $R(\omega)$ switches from being single-valued to multi-valued because of a **saddle-node bifurcation**.



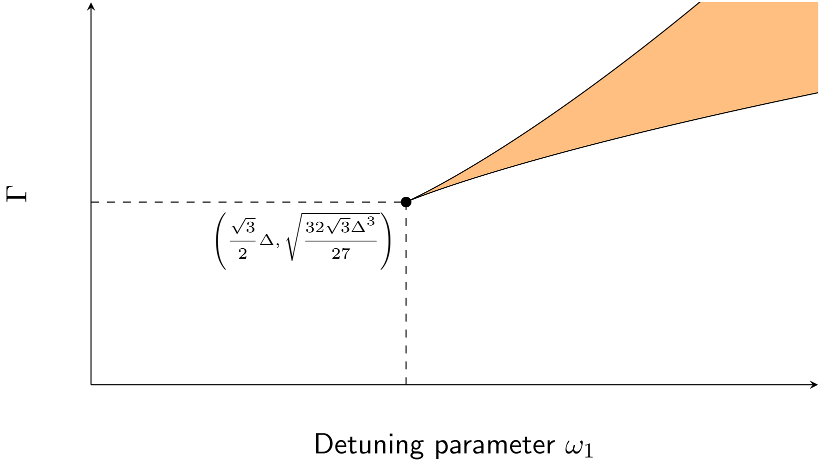
## Dissipative nonlinear oscillator

What are the critical values of $\Gamma$ and $\omega_1$ at which this bifurcation happen ?



# The cusp catastrophe

Forced nonlinear oscillator



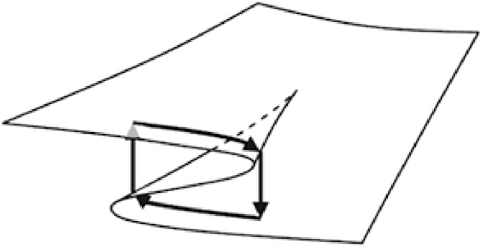
## The cusp catastrophe

Forced nonlinear oscillator

The response function is of the form

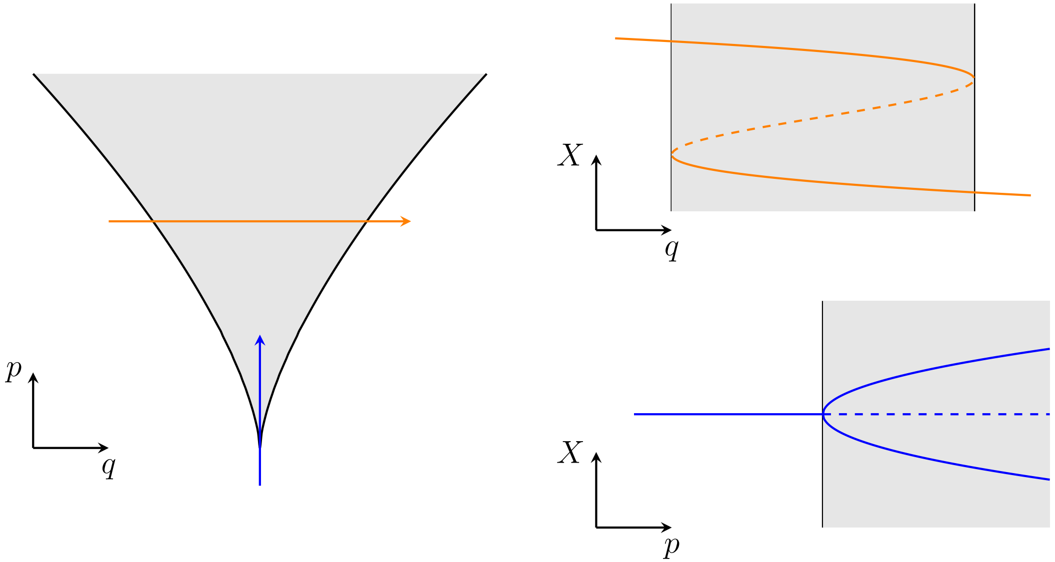
$$X^3 + pX + q = 0.$$

This is the canonical form of the **cusp catastrophe**.



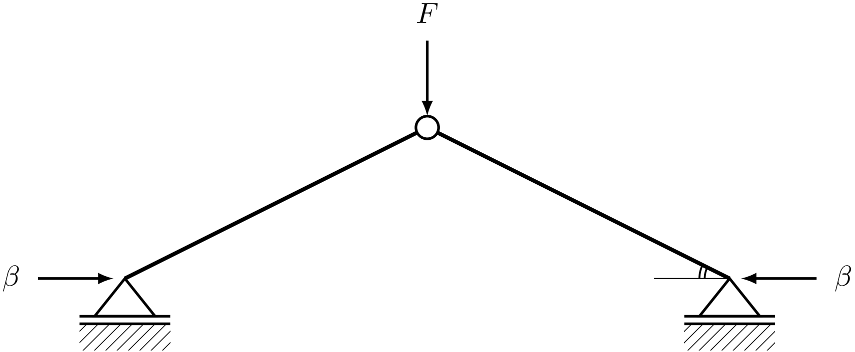
## The cusp catastrophe

Forced nonlinear oscillator



## The cusp catastrophe

Buckling beam

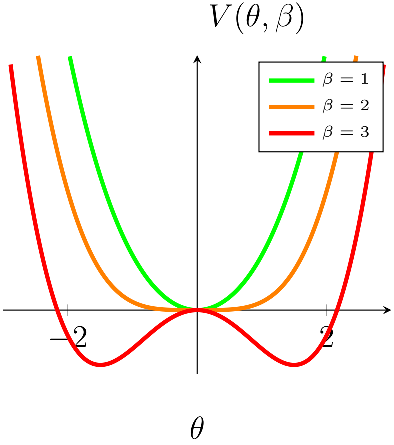


## The cusp catastrophe

Buckling beam

Total energy in the system

$$V = 2\mu\theta^2 + F\sin(\theta) - 2\beta\,(1 - \cos(\theta))$$



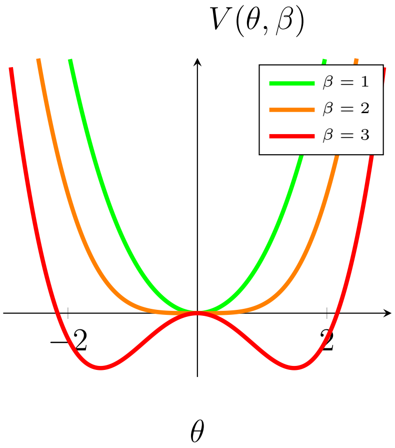

Potential for $F = 0$

## The cusp catastrophe

Buckling beam

Taylor expansion (and $\beta = 2\mu + b$)

$$V = \frac{2\mu + b}{12}\theta^4 - \frac{F}{6}\theta^3 - b\theta^2 + m\theta$$



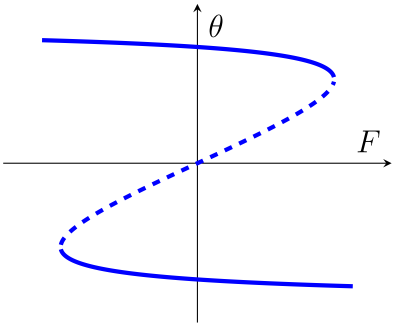

Potential for $F = 0$

## The cusp catastrophe

Buckling beam

Equilibria are stationary points

$$\frac{2\mu + b}{3}\theta^3 - \frac{F}{2}\theta^2 - 2b\theta + F = 0$$



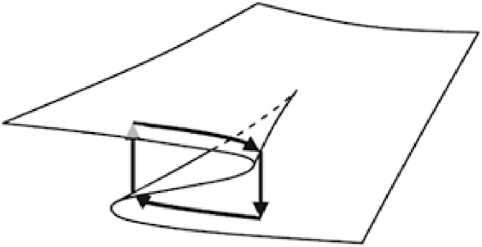

Fixed points for $(\mu, b) = (1, 1.1)$

## The cusp catastrophe

Buckling beam

Canonical form of the cusp catastrophe

$$X^3 + pX + q = 0$$



## The cusp catastrophe

Burger's equation

<u>Convex conservation laws</u>

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f} = 0$$
$$u(\boldsymbol{x}, 0) = \varphi_0(\boldsymbol{x})$$

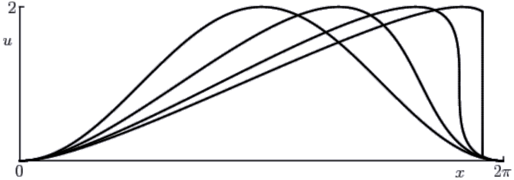
- $\boldsymbol{x}$ is 1-space variable,
- $\boldsymbol{f} = f(u)$, i.e. flux depends only on the state variable,
- $f(u)$ is $C^\infty$ and convex, i.e. $f''(u) > 0$.

# The cusp catastrophe

## Burger's equation

Burger's equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$
$$u(x,0) = \varphi_0(x)$$



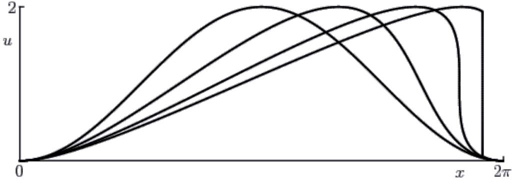
## The cusp catastrophe

Burger's equation

Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \varphi_0(x)$$



The following equation is quite useful in solving this PDE. Let

$$\begin{aligned}\frac{dx}{dt} &= u(x(t), t)\\ x(0) &= x_0.\end{aligned}$$

It is known as the **characteristic equation**.

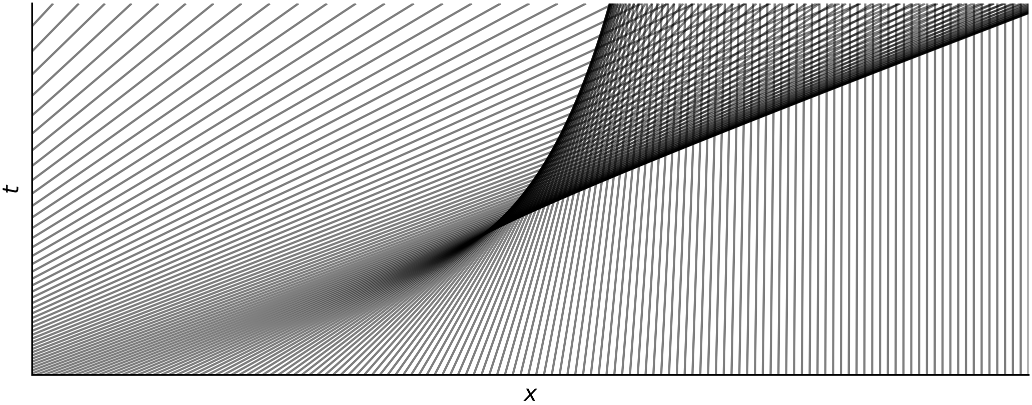
Differentiating with respect to time $t$ yields

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \\ &= 0.\end{aligned}$$

Hence, $(x(t), t)$ is a straight line called a **characteristic**. Furthermore, the slope is constant and given by $u(x(t), t) = u(x_0, 0) = \varphi x_0$.

# The cusp catastrophe

Burger's equation



**Thank you for your attention**

**Any question ?**