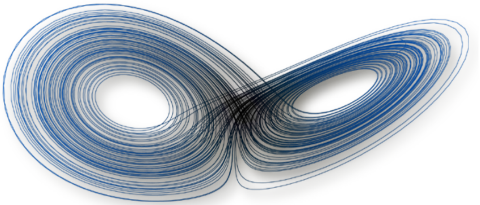

Strongly nonlinear oscillators and relaxation oscillations

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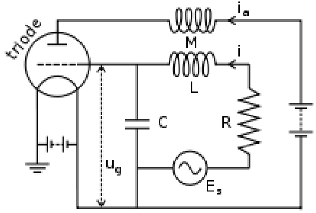
Strongly nonlinear oscillators

Example : van der Pol oscillator

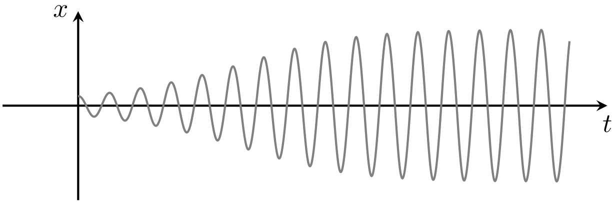
$$\boxed{\textbf{van der Pol osc. :} \quad \ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0}$$



For small values of μ, the dynamics are those of a weakly nonlinear oscillator. But what if μ is much larger than unity? Can we say anything meaningful about them from a theoretical point of view?

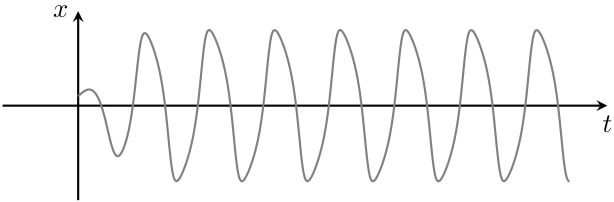
Strongly nonlinear oscillators

Relaxation oscillations



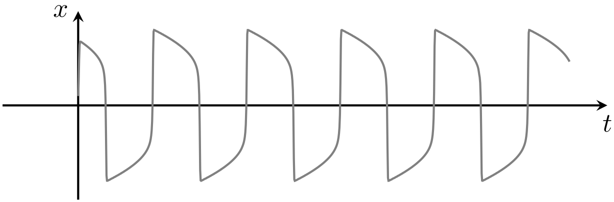
Strongly nonlinear oscillators

Relaxation oscillations



Strongly nonlinear oscillators

Relaxation oscillations



Strongly nonlinear oscillators

Why does it matters?

Strongly nonlinear oscillators

How to study these relaxation oscillations?

$$\ddot{x} + \mu \left(x^2 - 1\right) \dot{x} + x = 0$$

Strongly nonlinear oscillators

How to study these relaxation oscillations?

$$\frac{d}{dt}\underbrace{\left(\dot{x}+\mu\left(\frac{x^3}{3}-x\right)\right)}_{w}+x=0$$

Strongly nonlinear oscillators

How to study these relaxation oscillations?

$$\dot{w} = -x$$
$$\dot{x} = w + \mu\left(x - \frac{x^3}{3}\right)$$

Strongly nonlinear oscillators

How to study these relaxation oscillations?

$$\frac{\dot{w}}{\mu} = -\frac{x}{\mu}$$

$$\frac{\dot{x}}{\mu} = \frac{w}{\mu} + x - \frac{x^3}{3}$$

Strongly nonlinear oscillators

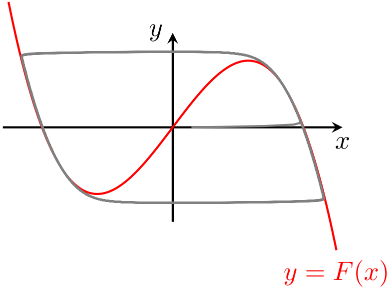
How to study these relaxation oscillations?

$$\dot{y} = -\epsilon x$$

$$\epsilon \dot{x} = y + x - \frac{x^3}{3}$$

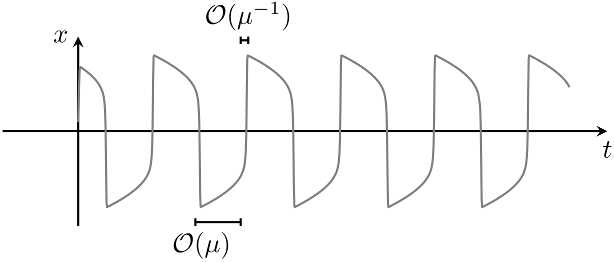
Strongly nonlinear oscillators

Boundary layers in time



Strongly nonlinear oscillators

Boundary layers in time



Strongly nonlinear oscillators

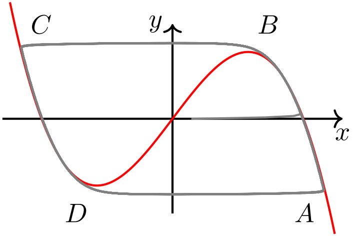
The slow time-scale

On the slow branches, we have $y \simeq F(x)$, hence

$$\frac{dy}{dt} \simeq F'(x)\frac{dx}{dt} = \left(x^2 - 1\right)\frac{dx}{dt}.$$

Using the equations of the system, we can find

$$dt \simeq -\frac{\mu\left(x^2 - 1\right)}{x}dx$$



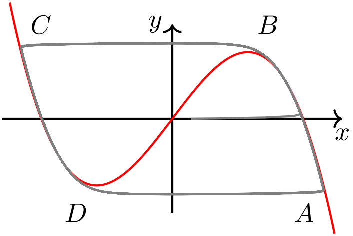
Strongly nonlinear oscillators

The slow time-scale

From symmetry arguments, we can then show that the period of oscillation is

$$T \simeq -2\mu \int_2^1 \frac{x^2 - 1}{x} dx = \mu\left(3 - 2\ln 2\right).$$

which is $\mathcal{O}(\mu)$ as expected.



Boundary layer in time

A linear example

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) = 0$$

Boundary layer in time

A linear example

$$\frac{L}{g\tau^2}\ddot{\theta} + \frac{b}{mgL\tau}\dot{\theta} + \sin(\theta) = 0$$

Boundary layer in time

A linear example

$$\frac{m^2 L^3 g}{b^2}\ddot{\theta} + \dot{\theta} + \sin(\theta) = 0$$

Boundary layer in time

A linear example

$$\epsilon\ddot{\theta} + \dot{\theta} + \sin(\theta) = 0$$

Boundary layer in time

A linear example

$$\epsilon\ddot{\theta} + \dot{\theta} + \theta = 0$$

$$\text{with} \quad \theta_0 = 0 \quad \text{and} \quad \dot{\theta}_0 = 1.$$

Boundary layer in time

A linear example

Boundary layer in time

A linear example

Let us consider the outer layer defined by $t \gg 1$. Using regular perturbation theory, i.e. expanding the solution as

$$x(t,\epsilon) = x_0(t) + \epsilon x_1(t) + \cdots$$

leads to

$$\mathcal{O}(1): \quad \dot{x}_0 + x_0 = 0$$
$$\mathcal{O}(\epsilon): \quad \dot{x}_1 + x_1 = -\ddot{x}_0$$

Note that we do not consider initial conditions since they would need to be applied inside the initial layer.

Boundary layer in time

A linear example

$$\boxed{\textbf{Outer solution :} \quad x_{\mathsf{out}}(t) = \epsilon A e^{-t} + \cdots}$$