

# Conservative, reversible and dissipative dynamical systems

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# Two-dimensional systems

Two major classes of dynamical systems

## Conservative

$$\dot{x} = y$$

$$\dot{y} = -\sin(x)$$

## Dissipative

$$\dot{x} = y$$

$$\dot{y} = -y - x - x^3$$

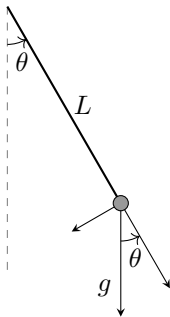
# Two-dimensional systems

## Conservative systems

The equations of motion of many physical systems can be derived from Newton's principles. They are of the form

$$m\ddot{x} = F(x)$$

where  $F(x)$  is independent of both  $\dot{x}$  and  $t$ , i.e. the system is autonomous and frictionless.



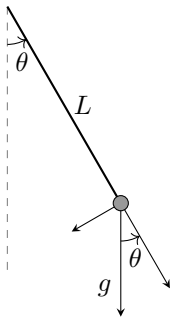
# Two-dimensional systems

## Conservative systems

Under this assumption, we can show that  $F(x)$  derives from a potential, i.e.

$$F(x) = -\frac{dV}{dx}$$

such as the gravitational potential energy  $V(x) = mgx$  or the elastic potential energy of a spring  $V(x) = \frac{1}{2}kx^2$ .



# Two-dimensional systems

## Conservative systems

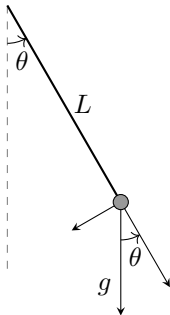
From there, the equations of motion can be rewritten as

$$m\ddot{x} + \frac{dV}{dx} = 0.$$

One can then show that

$$\frac{d}{dt} \left( \frac{1}{2} m \dot{x}^2 + V(x) \right) = 0.$$

The quantity  $\frac{1}{2} m \dot{x}^2 + V(x)$  is thus **conserved** over time.



# Two-dimensional systems

## Conservative systems

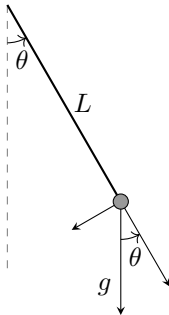
For the pendulum, the equations of motions are

$$\underbrace{mL\ddot{\theta}}_{\text{Acceleration}} + \underbrace{mg \sin(\theta)}_{\text{-Weight}} = 0$$

Multiplying by the velocity  $L\dot{\theta}$  yields

$$\frac{d}{dt} \left( \frac{1}{2} m (L\dot{\theta})^2 - mgL \cos(\theta) \right) = 0.$$

We thus recover that the **total energy** of the system is conserved (in the absence of friction).

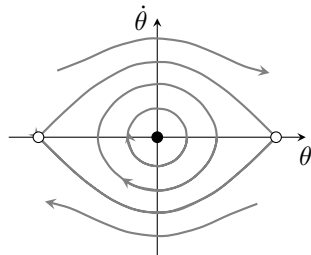


# Two-dimensional systems

## Conservative systems

For a conservative system, the energy remains constant along a given trajectory.

Fixed points cannot be attractive and the trajectories are closed orbits.



# Conservative systems

Hamiltonian formalism

Conservative systems are crucially important in physics and gave rise to **Hamiltonian mechanics** (in opposition to Lagrangian mechanics).

In a suitable reference frame  $(\mathbf{p}, \mathbf{q})$  where  $\mathbf{p}$  and  $\mathbf{q}$  are the **generalized coordinates**, the equations of motion take a special form known as the **Hamilton equations**.

## Hamilton equations

$$\dot{\mathbf{p}} = - \frac{d\mathcal{H}(\mathbf{p}, \mathbf{q})}{d\mathbf{q}}$$

$$\dot{\mathbf{q}} = + \frac{d\mathcal{H}(\mathbf{p}, \mathbf{q})}{d\mathbf{p}}$$



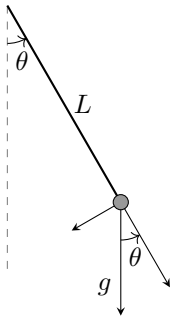
# Two-dimensional systems

## Dissipative systems

Let us reconsider the pendulum example but now taking into damping.  
The equations of motion now read

$$\ddot{\theta} + 2k\dot{\theta} + \sin(\theta) = 0$$

where the term  $2k\dot{\theta}$  models friction at the joint or viscous drag due to the surrounding air.



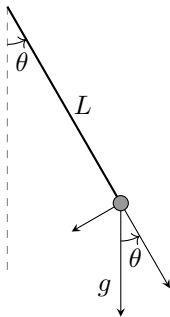
# Two-dimensional systems

## Dissipative systems

Multiplying by  $\dot{\theta}$  to obtain the energy equation yields

$$\frac{d}{dt} \left( \frac{1}{2} \dot{\theta}^2 - \cos(\theta) \right) = -k \dot{\theta}^2 < 0.$$

The total energy is no longer conserved. It decreases over time. We'll say it gets **dissipated** due to friction.

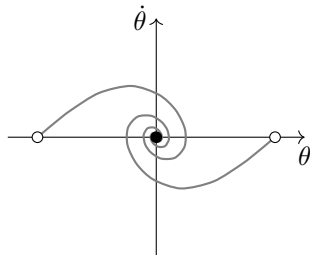


# Two-dimensional systems

## Dissipative systems – Attractors

As  $t \rightarrow \infty$ , the dynamics of a dissipative system settle on an **attractor**.

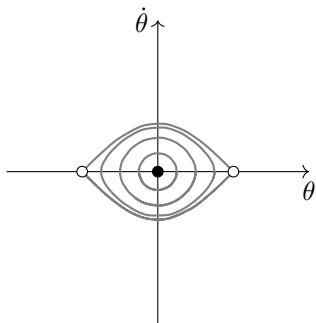
For a two-dimensional system, these attractors can only be **fixed points** or **limit cycles**.



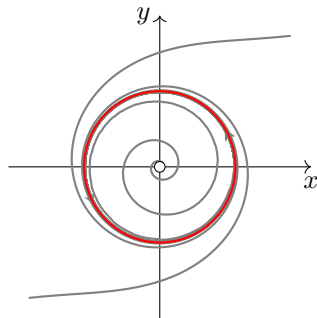
# Two-dimensional systems

Periodic orbits in conservative systems vs. limit cycles in dissipative systems

## Conservative system



## Dissipative system



# Two-dimensional systems

Dissipative systems – Volume contraction in phase space

Thinking of the dynamics  $\mathbf{f}(\mathbf{x})$  as a vector field, dissipativity can be understood as volume contraction in phase space.

Using the divergence theorem, it can easily be shown that a volume  $V(t)$  of initial conditions evolves as

$$\dot{V} = \int_V \nabla \cdot \mathbf{f}(\mathbf{x}) dV.$$

For a conservative system, we thus have  $\nabla \cdot \mathbf{f}(\mathbf{x}) = 0 \ \forall \mathbf{x}$ . For a dissipative system, we have that on average  $\nabla \cdot \mathbf{f}(\mathbf{x}) < 0$ .

