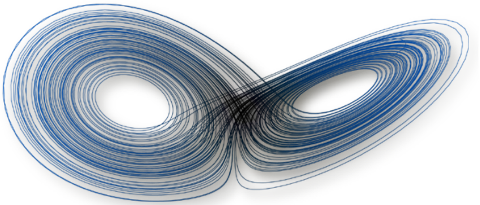

Weakly nonlinear oscillators

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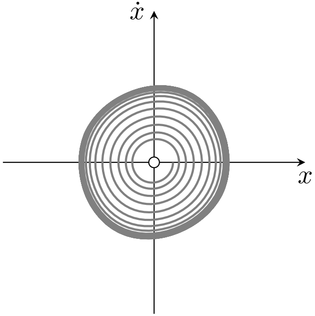
Weakly nonlinear oscillators

Multiple time scales

$$\boxed{\ddot{x} + x + \epsilon h(x, \dot{x}) = 0}$$

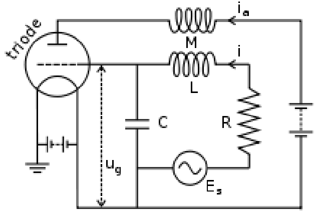
Nonlinear oscillators are often characterized by dynamics at different time scales, e.g. the phase tends to change at a faster rate than the oscillation's amplitude.



Weakly nonlinear oscillators

Examples : the van der Pol oscillator

van der Pol osc. : $$\ddot{x} + x + \epsilon \left(x^2 - 1\right) \dot{x} = 0$$
$$x(0) = 1, \quad \dot{x}(0) = 0$$



It is a canonical example of nonlinear oscillators proposed in 1927 by the Dutch electrical engineer Balthasar van der Pol.

Weakly nonlinear oscillators

Example : van der Pol oscillator

Fast time scale : $\tau = t$ for the evolution of the phase.

Slow time scale : $T = \epsilon t$ for the evolution of the oscillation's amplitude.

Power series expansion : $x(t,\epsilon) = x_0(\tau,T) + \epsilon x_1(\tau,T) + \mathcal{O}(\epsilon^2)$

$$\begin{aligned}
\mathcal{O}(1) : &\quad \frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0 \\
\mathcal{O}(\epsilon) : &\quad \frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \tau \partial T} - \left(x_0^2 - 1\right)\frac{\partial x_0}{\partial \tau}
\end{aligned}$$

Weakly nonlinear oscillators

Example : van der Pol oscillator

$$\boxed{\mathcal{O}(1): \quad \frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0}$$

As usual, the dynamics at order zero are captured by a simple harmonic oscillator. The general solution can be written as

$$x_0(\tau, T) = r(T) \cos\left(\tau + \varphi(T)\right)$$

where $r(T)$ and $\varphi(T)$ are the **slowly varying amplitude and phase** of x_0.

Weakly nonlinear oscillators

Example : van der Pol oscillator

$$\boxed{\mathcal{O}(\epsilon) : \quad \frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\left(\dot{r}\sin(\tau+\varphi) + r\dot{\varphi}\cos(\tau+\varphi)\right) - r\sin(\tau+\varphi)\left(r^2\cos^2(\tau+\varphi) - 1\right)}$$

⚠ The right-hand side contains **resonant** terms which would leads to unphysical **secular growth**.

Trig. identity : $\sin(\tau+\varphi)\cos^2(\tau+\varphi) = \dfrac{1}{4}\left(\sin(\tau+\varphi) + \sin\left(3(\tau+\varphi)\right)\right)$

Weakly nonlinear oscillators

Example : van der Pol oscillator

$$\boxed{\mathcal{O}(\epsilon): \quad \frac{\partial^2 x_1}{\partial \tau^2} + x_1 = \left(-2\dot{r} + r - \frac{r^3}{4}\right)\sin(\tau+\varphi) - 2r\dot{\varphi}\cos(\tau+\varphi) - \frac{r^3}{4}\sin\left(3(\tau+\varphi)\right)}$$

The slowly varying amplitude and phase $r(T)$ and $\varphi(T)$ need to satisfy

$$\frac{dr}{dT} = \frac{1}{8} r\left(4 - r^2\right), \quad \text{and} \quad \frac{d\varphi}{dT} = 0$$

to avoid **secular growth**.

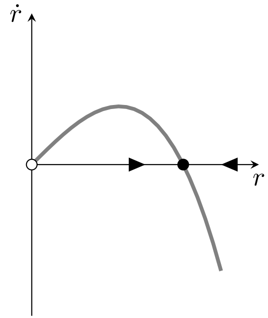
Weakly nonlinear oscillators

Example : van der Pol oscillator

The slowly-varying amplitude obeys

$$\frac{dr}{dt} = \frac{\epsilon}{8} r \left(4 - r^2\right) .$$

It has two fixed points : $r^* = 0$ is linearly unstable while $r^* = 2$ is linearly stable. Hence, as $t \to \infty$, $r(t) \to 2$.



Weakly nonlinear oscillators

Example : van der Pol oscillator

Two-timing : $x(t) = r(t)\cos(t + \varphi_0)$

with $\lim_{t\to\infty} r(t) = 2$ for $\epsilon > 0$