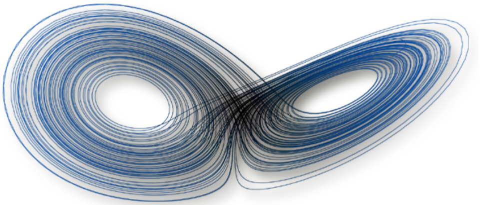


First-order systems

Jean-Christophe Loiseau

jean-christophe.loiseau@ensam.eu
Laboratoire DynFluid
Arts et Métiers, France.

First-order system

Flow on the real number line

First-order systems

$$\dot{x} = f(x)$$

- $x(t)$ a real-valued function of time t,
- $f(x)$ a smooth real-valued function of x and does not explicitely depend on time t.

$$\dot{x} = \sum_{k=0}^{N} a_k x^k, \quad \text{with } a_k \in \mathbb{R} \quad \forall k$$

$$\dot{x} = \sin x$$

$$\dot{x} = \frac{1}{x}$$

$$\vdots$$

First-order systems

A motivating example

Consider the system

$$\dot{x} = \sin x.$$

Its solution is implicitely defined by

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|.$$

Although exact, this formula is not very intuitive. Our goal will instead be to develop a more intuitive geometric way of thinking.

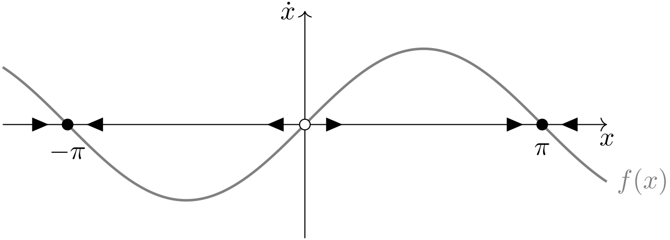
$$\begin{aligned}
\frac{dx}{dt} &= \sin x \\
dt &= \frac{dx}{\sin x} \\
t &= \int \csc x \; dx \\
&= -\ln|\csc x - \cot x| + C \\
&\vdots
\end{aligned}$$

First-oder systems

A motivating example

Points for which $f(x) = 0$ are equilibrium points. Some fixed points are attractive, other repulsive. For our problem, these fixed points are given by

$$x^* = n\pi \quad \forall n \in \mathbb{Z}$$

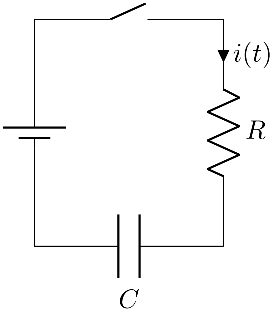


First-order systems

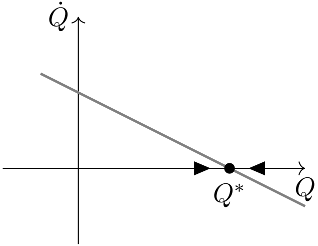
A motivating example

First-order systems

Examples



Equation : $\dot{Q} = \frac{V_0}{RC} - \frac{Q}{RC}$



First-order systems

Linear stability

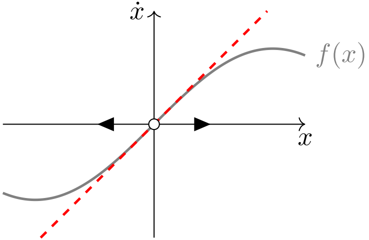
Let x^* be a fixed point and $\eta = x - x^*$ a small perturbation. Then

$$\dot{\eta} = \frac{d}{dt}(x - x^*) = \dot{x}.$$

For η infinitesimally small, we can write

$$\dot{\eta} = f'(x^*)\eta + \mathcal{O}(\eta^2).$$

Neglecting $\mathcal{O}(\eta^2)$, we arrive at the **linearization of the dynamics about the fixed point** x^*.



First-order systems

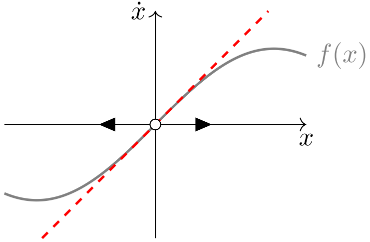
Linear stability

The solution is then given by

$$\eta(t) = \exp\left(f'(x^*)t\right)\eta_0.$$

Fixed points can be classified based on $\operatorname{sign} f'(x^*)$.

- If $f'(x^*) < 0$, $\eta(t)$ decays exponentially fast. The fixed point is said to be **linearly stable**.
- If $f'(x^*) > 0$, $\eta(t)$ decays exponentially fast. The fixed point is said to be **linearly unstable**.
- If $f'(x^*) = 0$, one cannot conclude and nonlinear analyses are required.



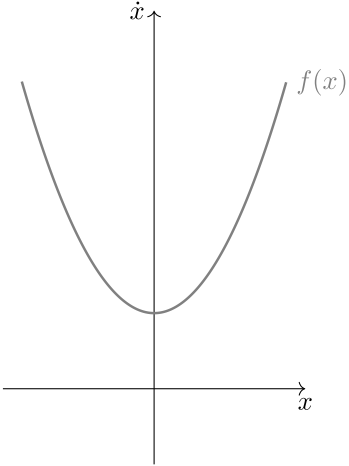
First-order systems

Some cavats : Finite-time blow up

Consider the system $\dot{x} = 1 + x^2$ with initial condition $x(0) = 0$. Its solution is given by

$$\int \frac{dx}{1+x^2} = \int dt$$
$$\tan^{-1} x = t + C$$
$$x(t) = \tan t.$$

As $t \to \pm \pi/2$, $x(t) \to \pm\infty$. The system has solutions reaching infinity *in finite time*. This phenomenon is called **blow-up** and is relevant in e.g. combustion models.



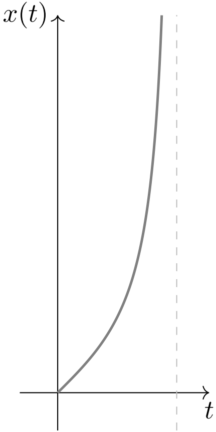
First-order systems

Some cavats : Finite-time blow up

Consider the system $\dot{x} = 1 + x^2$ with initial condition $x(0) = 0$. Its solution is given by

$$\begin{aligned} \int \frac{dx}{1+x^2} &= \int dt \\ \tan^{-1} x &= t + C \\ x(t) &= \tan t. \end{aligned}$$

As $t \to \pm\pi/2$, $x(t) \to \pm\infty$. The system has solutions reaching infinity *in finite time*. This phenomenon is called **blow-up** and is relevant in e.g. combustion models.

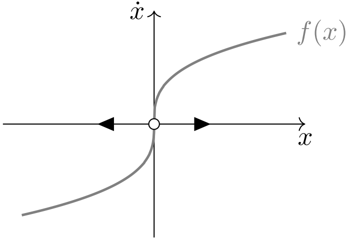


First-order systems

Some caveats : Non-uniqueness of the solution

Consider the system $\dot{x} = \sqrt[3]{x}$ with initial condition $x(0) = 0$. $x = 0$ is obviously a fixed point and so $x(t) = 0$ is the solution. But is it ?

$$\frac{dx}{dt} = \sqrt[3]{x}$$
$$\int \frac{dx}{\sqrt[3]{x}} = \int dt$$
$$\frac{3}{2} x^{2/3} = t + C$$



Given that $x(0) = 0$, then $x(t) = \left(\frac{2}{3}t\right)^{3/2}$ is also solution !

First-order systems

Impossibility of oscillations

Warning

First-order systems defined on the real number line cannot exhibit oscillatory behaviour !

Trajectories of first-order systems can only vary monotically. They either end up on a stable fixed point or diverge to $\pm\infty$.