

Poincaré-Lindstedt method for periodic dynamics

Jean-Christophe Loiseau

jean-christophe.loiseau@ensam.eu
Laboratoire DynFluid
Arts et Métiers, France.

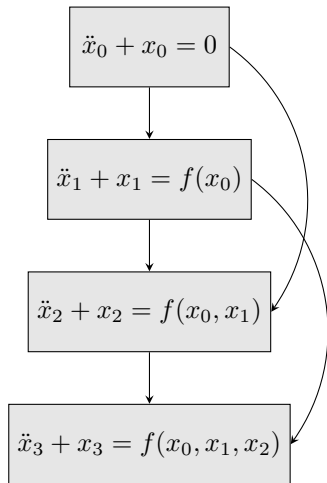
The Poincaré-Lindstedt method

Summary

The Poincaré-Lindstedt method is a powerful perturbative technique to approximate periodic solutions to ordinary differential equations.

By rescaling time as $\tau = \omega t$ and using power series expansions, it transforms a nonlinear system into a cascade of linear ones which we can easily solve.

The resulting approximation provides insights into the frequency shift phenomenon and harmonics generation induced by the nonlinearity.



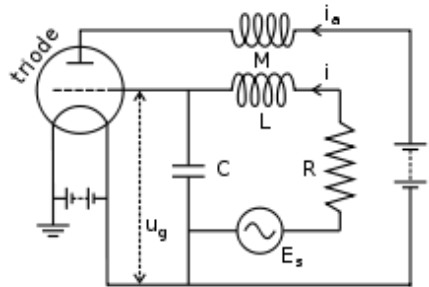
Beyond conservative systems

Application to the van der Pol oscillator

Let us consider the **van der Pol oscillator** whose governing equations are given by

$$\ddot{x} + x = \epsilon (1 - x^2) \dot{x}.$$

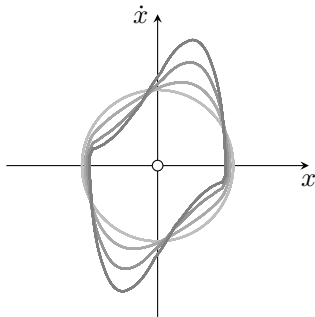
It is a canonical example of nonlinear oscillators proposed in 1927 by the Dutch electrical engineer Balthasar van der Pol.



Beyond conservative systems

Application to the van der Pol oscillator

For $\epsilon > 0$, its dynamics are characterized by a limit cycle that we wish to study analytically using the Poincaré-Lindstedt method.



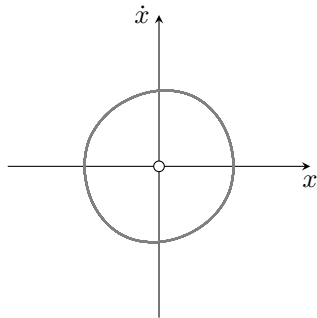
Poincaré-Lindstedt method

Application to the van der Pol oscillator

The problem to be studied is thus the following

$$\begin{cases} \ddot{x} + x = \epsilon (1 - x^2) \dot{x} \\ x(0) = x_{\text{in}} \\ \dot{x}(0) = 0 \end{cases}$$

where x_{in} is the unknown initial condition and $\epsilon \ll 1$ is our control parameter.



Asymptotic dynamics for
 $\epsilon = 0.1$

Poincaré-Lindstedt method

Application to the van der Pol oscillator

Let us rescale time as $\tau = \omega t$ such that

$$\frac{d}{dt} = \omega \frac{d}{d\tau}, \quad \text{and} \quad \frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2}$$

and use a power series expansion of the unknown frequency ω , solution $x(t)$ and initial condition x_{in}

$$\omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots$$

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \dots$$

$$x_{\text{in}} = A_0 + \epsilon A_1 + \epsilon^2 A_2 + \dots$$

where $x_1(\tau)$ and $x_2(\tau)$ are small corrections to the harmonic oscillator solution $x_0(\tau)$ and ω_1 and ω_2 are small corrections to its natural frequency.

Poincaré-Lindstedt method

Application to the van der Pol oscillator

Introducing these expansions into our equation and regrouping by power of ϵ yields

$$\mathcal{O}(\epsilon^0) : \ddot{x}_0 + x_0 = 0$$

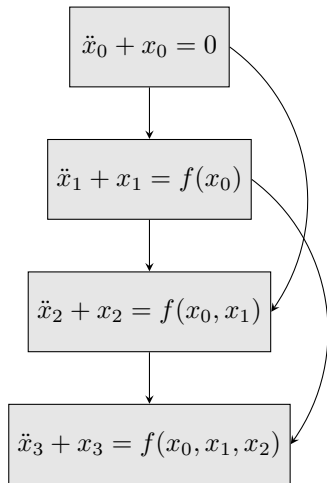
$$\mathcal{O}(\epsilon) : \ddot{x}_1 + x_1 = (1 - x_0^2) \dot{x}_0 - 2\omega_1 \ddot{x}_0$$

$$\mathcal{O}(\epsilon^2) : \ddot{x}_2 + x_2 = f(x_0, \dot{x}_0, \ddot{x}_0, x_1, \dot{x}_1, \ddot{x}_1)$$

supplemented with the initial conditions

$$x_i(0) = A_i, \quad \dot{x}_i(0) = 0 \quad \forall i.$$

Once again, we trade a nonlinear system for a cascade of linear ones.



Poincaré-Lindstedt method

Application to the van der Pol oscillator

The zeroth-order solution is given by $x_0(\tau) = A_0 \cos(\tau)$.

Injecting $x_0(\tau)$ into the equation for $x_1(\tau)$ yields

$$\ddot{x}_1 + x_1 = A_0 (A_0^2 \cos^2(\tau) - 1) \sin(\tau) + 2\omega_1 A_0 \cos(\tau)$$

which can be simplified to

$$\ddot{x}_1 + x_1 = 2A_0\omega_1 \cos(\tau) + A_0 \left(\frac{A_0^2}{4} - 1 \right) \sin(\tau) + \frac{A_0^3}{4} \sin(3\tau)$$

using trigonometric identities.

Trig. identity

$$\cos^2(\tau) \sin(\tau) = \frac{1}{4} (\sin(\tau) + \sin(3\tau))$$

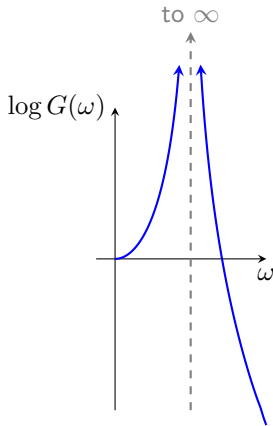
Poincaré-Lindstedt method

Application to the van der Pol oscillator

If left unchecked, the terms $\cos(\tau)$ and $\sin(\tau)$ will lead to **secular growth**. We thus need to set ω_1 and A_0 such that

$$\begin{aligned}2A_0\omega_1 &= 0 \\ A_0 \left(\frac{A_0^2}{4} - 1 \right) &= 0\end{aligned}$$

to avoid this unphysical behaviour. This leads to $\omega_1 = 0$ and $A_0 = 2$.



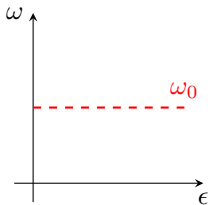
Poincaré-Lindstedt method

Application to the van der Pol oscillator

At order $\mathcal{O}(\epsilon)$, the equation reduces to

$$\ddot{x}_1 + x_1 = 2 \sin(3\tau)$$

whose general solution is given by $x_1(\tau) = ??$. Note that A_1 is still undetermined and one needs to go to order $\mathcal{O}(\epsilon^2)$ to determine it.



Poincaré-Lindstedt method

Application to the van der Pol oscillator

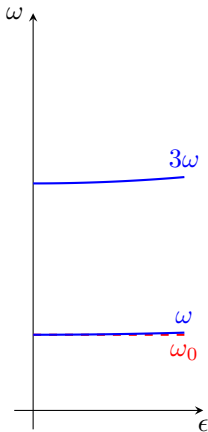
Continuing this process to $\mathcal{O}(\epsilon)$ leads to the second-order frequency correction

$$\omega_2 = \frac{7}{16}$$

as well as to the condition $A_1 = 0$. Our first-order approximation thus reads

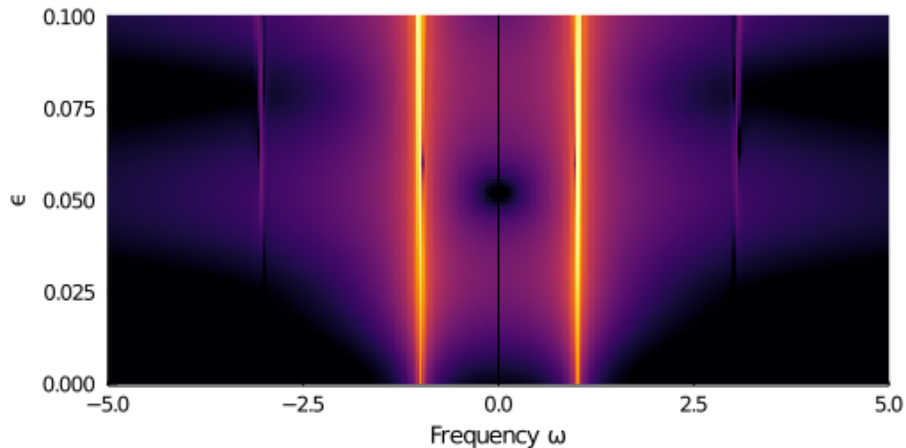
$$x(t) = 2 \cos(\omega t) + \epsilon \left(\frac{3}{4} \sin(\omega t) - \frac{1}{4} \sin(3\omega t) \right) + \mathcal{O}(\epsilon^2)$$

with $\omega = 1 + \frac{7\epsilon^2}{16} + \mathcal{O}(\epsilon^4)$. Once again, nonlinearity causes a **frequency shift** and the generation of **high-order harmonics**.



Poincaré-Lindstedt method

Application to the van der Pol oscillator



Poincaré-Lindstedt : the van der Pol oscillator

Summary

The Poincaré-Lindstedt method is a powerful perturbative technique to approximate periodic solutions to ordinary differential equations.

For $\epsilon < 0$, the only attractor is a stable fixed point. As ϵ becomes positive, the dynamics settles on a constant-amplitude limit cycle.

By design, the Poincaré-Lindstedt method cannot however capture transient effects. We'll need another technique for that purpose.

