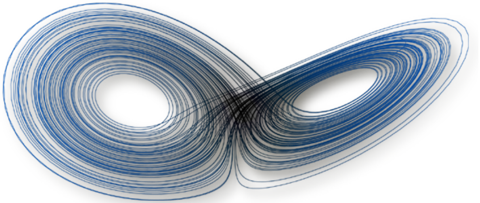

Poincaré-Lindstedt method for periodic dynamics

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Poincaré-Bendixson thereom

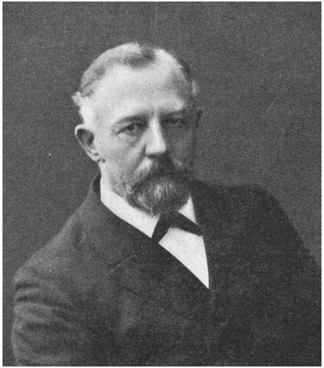
Periodic orbit or not ?

One of the fundamental theorem in dynamical system theory pertaining to the existence of periodic orbits in planar systems (i.e. two-dimensional systems).

How to study periodic dynamics ?

The Poincaré-Lindstedt method

Method developed by A. Lindstedt in the 1880's to approximate uniformly periodic solutions to ordinary differential equations in the context of celestial mechanics later refined by H. Poincaré.



How to study periodic dynamics ?

Example : the simple pendulum

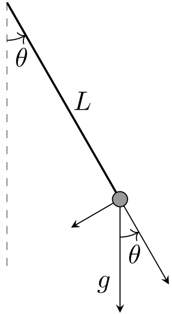
Starting from Newton's principles, the equations of motion are

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0.$$

Introducing the rescaled time $\tau = \sqrt{\frac{g}{L}}t$, the governing equations are reduce to

$$\ddot{\theta} + \sin(\theta) = 0,$$

where the dot notation now denotes derivation w.r.t. τ.



How to study periodic dynamics ?

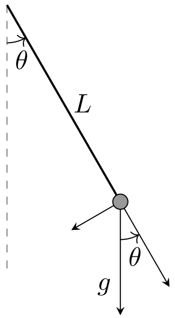
Example : the simple pendulum

The exact period of oscillation for arbitrary initial angle can be expressed as

$$T = 4\mathcal{F}\left(\frac{\pi}{2}, \sin\frac{\theta_0}{2}\right)$$

where $\mathcal{F}(\phi, k)$ is the incomplete elliptic integral of the first kind.

Although exact, this formula provides very limited insights into the dynamics...



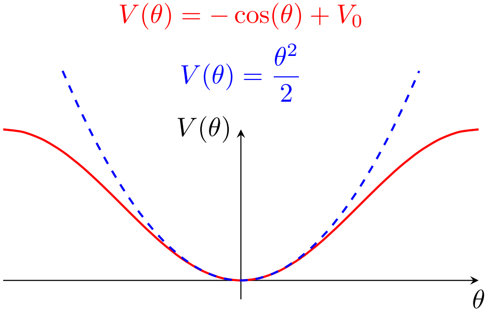
How to study periodic dynamics ?

Example : the simple pendulum

For infinitesimally small angles θ, $\sin(\theta) \simeq \theta$. The system reduces to the **harmonic oscillator**

$$\ddot{\theta} + \theta = 0$$

Its solution reads $\theta(t) = \theta_0 \cos(t)$ (assuming $\dot{\theta}(0) = 0$).

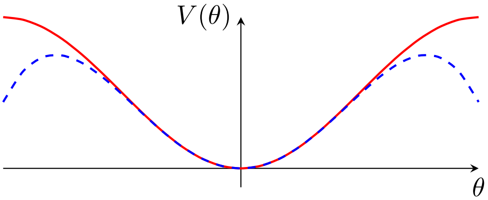


How to study periodic dynamics ?

Example : the simple pendulum

$$V(\theta) = -\cos(\theta) + V_0$$

$$V(\theta) = \frac{\theta^2}{2} - \frac{\theta^4}{24}$$



Tthe second-order Taylor expansion of $\sin(\theta)$ yields

$$\ddot{\theta} + \theta - \frac{1}{6}\theta^3 = 0.$$

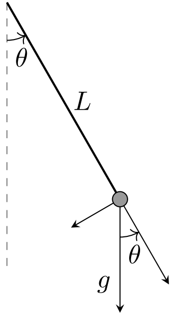
It is known as **Duffing oscillator**. Its exact solution can be expressed in terms of **elliptic Jacobi functions**.

How to study periodic dynamics ?

Example : the simple pendulum

Introducing the change of variable $\theta(t) = \theta_0 x(t)$, the equation of motion can be written as

$$\begin{cases} \ddot{x} + x = \epsilon x^3 \\ x(0) = 1 \\ \dot{x}(0) = 0 \end{cases}$$



where $\epsilon = \dfrac{\theta_0^2}{6}$. For ϵ sufficiently small, the nonlinear term can be understood as a small perturbation of the harmonic oscillator.

The Poincaré-Lindstedt method

Example : the simple pendulum

Let us rescale time as $\tau = \omega t$ such that

$$\frac{d}{dt} = \omega \frac{d}{d\tau}, \quad \text{and} \quad \frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2}.$$

and use a power series expansion of the unknown frequency ω and solution $x(t)$

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$
$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots$$

where $x_1(\tau)$ and $x_2(\tau)$ are small corrections to the harmonic oscillator solution $x_0(\tau)$ and ω_1 and ω_2 are small corrections to its natural frequency.

The Poincaré-Lindstedt method

Example : the simple pendulum

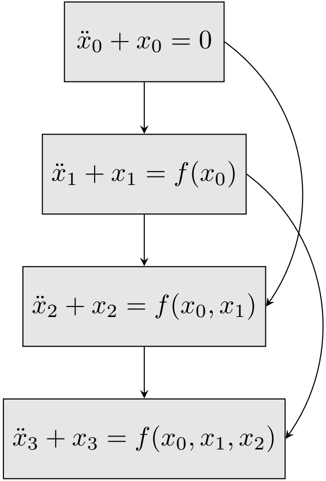
Introducing these expansions into our equation and regrouping by power of ϵ yields

$$
\begin{aligned}
\mathcal{O}(\epsilon^0): &\quad \ddot{x}_0 + x_0 = 0, \\
\mathcal{O}(\epsilon^1): &\quad \ddot{x}_1 + x_1 = x_0^3 - 2\omega_1 \ddot{x}_0, \\
\mathcal{O}(\epsilon^2): &\quad \ddot{x}_2 + x_2 = 3x_0^2 x_1 - 2\omega \ddot{x}_1 - (2\omega_2 + \omega_1^2)\ddot{x}_0
\end{aligned}
$$

supplemented with the initial conditions

$$x_0(0) = 1, \quad \dot{x}_0(0) = 0, \quad x_i(0) = \dot{x}_i(0) = 0 \quad \forall i > 0.$$

We have thus traded a nonlinear system for a cascade of linear systems we can solve sequentially !



The Poincaré-Lindstedt method

Example : the simple pendulum

The zeroth-order solution is simply given by $x_0(\tau) = \cos(\tau)$.
Injecting $x_0(\tau)$ into the equation for $x_1(\tau)$ yields

$$\ddot{x}_1 + x_1 = \cos^3(\tau) + 2\omega_1 \cos(\tau)$$

which can be simplified to

$$\ddot{x}_1 + x_1 = \left(\frac{3}{4} + 2\omega_1\right)\cos(\tau) + \frac{1}{4}\cos(3\tau)$$

using trigonometric identities.

Trig. identity

$$\cos^3(\tau) = \frac{1}{4}\left(3\cos(\tau) + \cos(3\tau)\right)$$

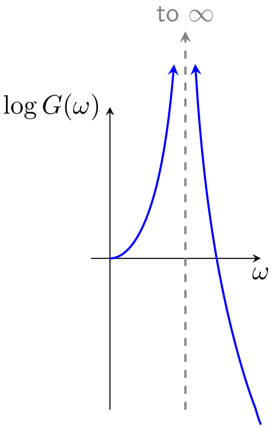
The Poincaré-Lindstedt method

Example : the simple pendulum

If left unchecked, the $\cos(\tau)$ forcing term will lead to **secular growth** (i.e. $x_1(\tau) \to \infty$ as $\tau \to \infty$) which is unphysical. We can however set the first-order frequency correction to

$$\omega_1 = -\frac{3}{8}$$

to avoid this unphysical behaviour (see the **Fredholm theorem** for a theoretical justification).



The Poincaré-Lindstedt method

Example : the simple pendulum

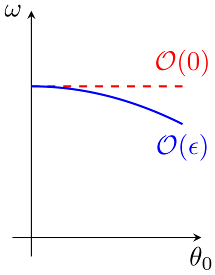
At order $\mathcal{O}(\epsilon)$, the equation then reduces to

$$\ddot{x}_1 + x_1 = \frac{1}{4}\cos(3\tau)$$

whose solution is given by $x_1(\tau) = \dfrac{1}{32}\left(\cos(\tau) - \cos(3\tau)\right)$. The first-order approximation to the true solution is thus given by

$$x(\tau) = \cos(\tau) + \frac{\epsilon}{32}\left(\cos(\tau) - \cos(3\tau)\right) + \mathcal{O}(\epsilon^2)$$

with $\tau = \left(1 - \dfrac{3\epsilon}{8} + \cdots\right) t$ our rescaled time.

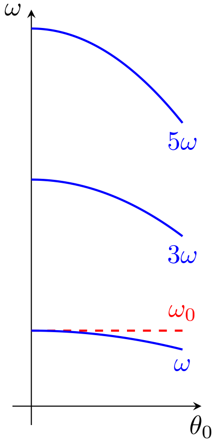


The Poincaré-Lindstedt method

Example : the simple pendulum

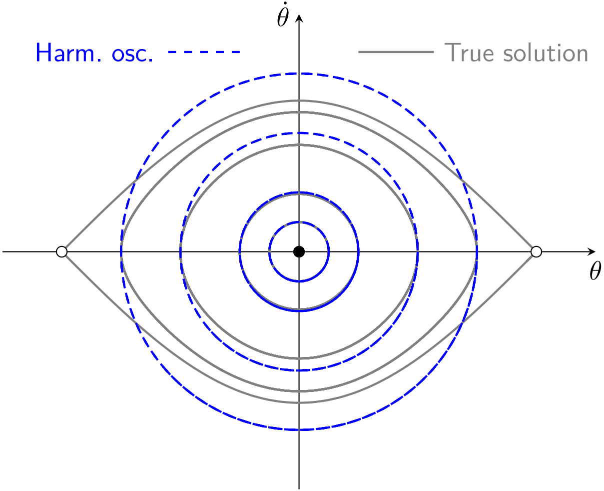
Continuing this process for $\mathcal{O}(\epsilon^2)$ would yield the second-order frequency correction and an expression for $x(t)$ involving $\cos(\omega t)$, $\cos(3\omega t)$ and $\cos(5\omega t)$.

As ϵ is increased, the evolution of $x(t)$ is no longer a pure sine wave. The nonlinearity causes a **frequency shift** as well as the generation of **higher-order harmonics** !



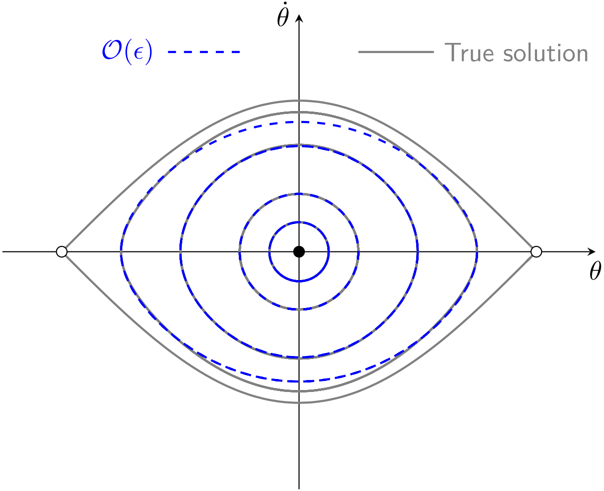
The Poincaré-Lindstedt method

Example : the simple pendulum



The Poincaré-Lindstedt method

Example : the simple pendulum



The Poincaré-Lindstedt method

Summary

The Poincaré-Lindstedt method is a powerful perturbative technique to approximate periodic solutions to ordinary differential equations.

By rescaling time as $\tau = \omega t$ and using power series expansions, it transforms a nonlinear system into a cascade of linear ones which we can easily solve.

The resulting approximation provides insights into the frequency shift phenomenon and harmonics generation induced by the nonlinearity.