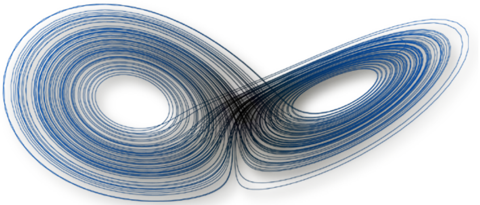

Limit cycles are all you have

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Poincaré-Bendixson theorem

Henri Poincaré (1854-1912)

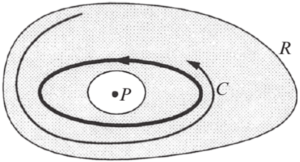
Ivar Otto Bendixson (1861-1935)

Poincaré-Bendixson theorem

Poincaré-Bendixson theorem: Let us suppose that

1. R is a closed, bounded subset of the plane.
2. $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is a continuously differentiable vector field on an open set containing R.
3. R does not contain any fixed points.
4. There exists a trajectory C "confined" in R (i.e. it starts in R and stays in R for all times).

Then, either C is a closed orbit or it spirals toward a closed orbit as $t \to \infty$. In either case, R *contains a closed orbit*!

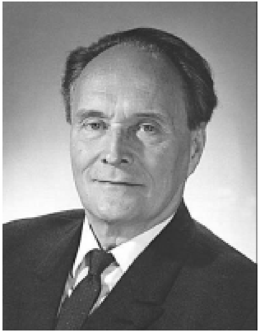


A predator-prey model

Consider the following predator-prey model

$$\begin{aligned}\dot{x} &= g(x)x - p(x)y \\ \dot{y} &= (q(x) - d)\, y,\end{aligned}$$

with $d > 0$, while $x(t)$ and $y(t)$ are the prey and predator densities.



Georgii Frantsevich Gause
(1910-1986)

A predator-prey model

$$\dot{x} = g(x)x - p(x)y$$
$$\dot{y} = (q(x) - d)\, y$$

A predator-prey model

The function $g(x)$ describes the evolution of the prey population in the absence of predation. Self-regulation in the prey implies there exists a $K > 0$ so that

- $g(x) > 0$ for $x < K$,
- $g(K) = 0$,
- $g(x) < 0$ for $x > K$.

The constant K is known as the **prey carrying capacity**.

A predator-prey model

The function $p(x)$ is the **predator trophic function**. It describes the number of prey killed by one predator. Its fundamental properties are

- $p(0) = 0$ and $p(x) > 0$ for all $x \in \mathbb{R}_+$.
- Reasonable to assume that $\lim_{x \to +\infty} p(x) = C$.

Three types of predator trophic functions exist, depending on extra assumptions made about $p(x)$.

A predator-prey model

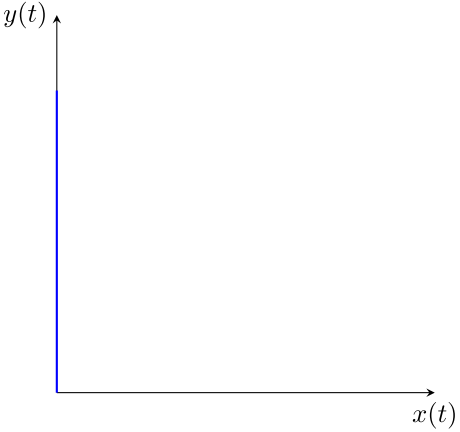
The function $q(x)$ describes the consumption of prey and conversion into predator individuals. It can be independent of $p(x)$ albeit we often have $p(x) = q(x)$ (e.g. Lotka-Volterra). We require that $q(0) = 0$ and $q'(x) > 0$ for $x > 0$.

A predator-prey model

Isoclines

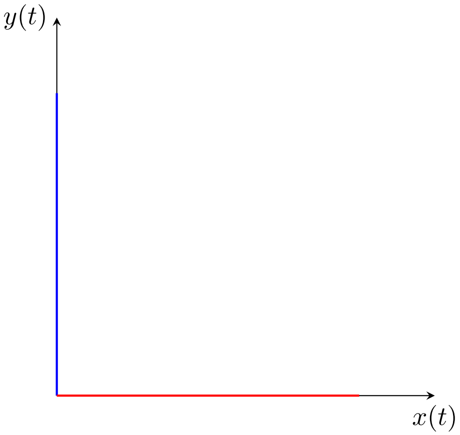
Prey equation

$$x = 0, \quad l_1 = \left\{ (x, y) : y = \frac{xg(x)}{p(x)} \right\}.$$



A predator-prey model

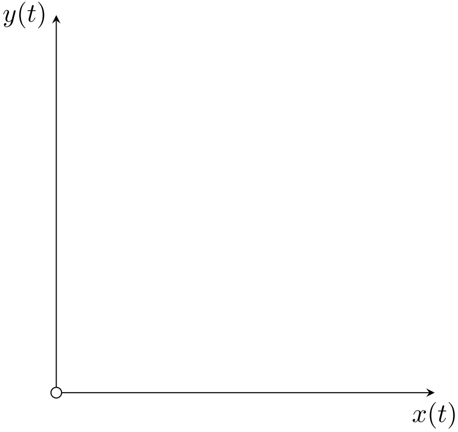
Isoclines



Predator equation

$$y = 0, \quad l_2 = \{(x, y) : x = \hat{x}, q(\hat{x}) = d\}.$$

A predator-prey model



Why is this theorem so important?

It is one of the central results of nonlinear dynamics. Dynamical possibilites in the phase plane are very limited: if a trajectory is confined to a closed bounded region with no fixed points, it must eventually approach a **closed orbit**. Nothing more complicated is possible!

No chaos in phase plane!