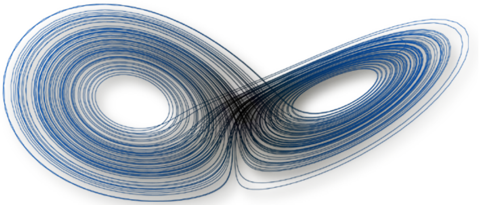


# Elementary bifurcations

Jean-Christophe Loiseau

jean-christophe.loiseau@ensam.eu
*Laboratoire DynFluid*
*Arts et Métiers, France.*

# First-order systems

Flow on the real number line

## First-order systems

$$\dot{x} = f(x)$$

- $x(t)$ a real-valued function of time $t$,
- $f(x) : \mathbb{R} \to \mathbb{R}$ a smooth real-valued function of $x$ and does not explicitely depend on time $t$.

$$\dot{x} = \sum_{k=0}^{N} a_k x^k, \quad \text{with } a_k \in \mathbb{R} \quad \forall k$$

$$\dot{x} = \sin x$$

$$\dot{x} = \frac{1}{x}$$

$$\vdots$$

# Parameterized first-order systems

Elementary bifurcations

## Parameterized first-order systems

$$\dot{x} = f(x, \mu)$$

- $x(t)$ a real-valued function of time $t$,
- $\mu$ a real-valued parameter (or vector of parameters),
- $f(x, \mu) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a smooth real-valued function of $x$ and $\mu$ which does not explicitely depend on time $t$.

$$\dot{x} = \mu - x^2$$

$$\dot{x} = \mu x - x^2$$

$$\dot{x} = \mu x + x^3$$

$$\dot{x} = \mu - \sin(x)$$

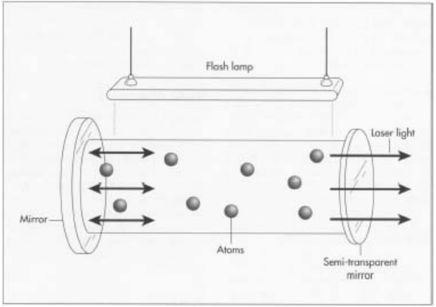
$$\vdots$$

# Parameterized first-order systems

Motivating example : Rudimentary model of solid-state lasers

**Haken's model** (1983)

$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss}\\ &= GnN(t) - kn\end{aligned}$$




In the model above, we have :

- $n(t)$ is the number of photons,
- $N(t)$ is the number of excited atoms,
- $G$ is the gain coefficient,
- $k$ is the rate at which photons escape.

# Parameterized first-order systems

Motivating example : Rudimentary model of solid-state lasers

Assuming that $N(t) = N_0 - \alpha n$ (with $\alpha > 0$), the model becomes

$$\begin{aligned} \dot{n} &= Gn(N_0 - \alpha n) - kn \\ &= (GN_0 - k)n - (\alpha G)n^2. \end{aligned}$$

The number of photons $n(t)$ in the laser thus appears to depend on four parameters $G$, $N_0$, $k$ and $\alpha$.

# Parameterized first-order systems

Motivating example : Rudimentary model of solid-state lasers

Rescaling time as $t \to \tau t$ and choose the time scale $\tau$ appropriately, the equation becomes

$$\begin{aligned}\dot{n} &= \frac{GN_0 - k}{\alpha G} n - n^2 \\ &= \mu n - n^2.\end{aligned}$$

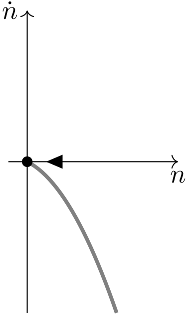
The dynamics of the system effectively depend on a single parameter $\mu$.

**Solution**

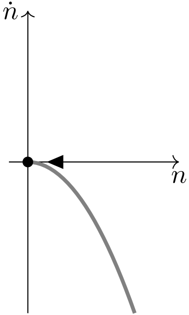
$$n(t) = \frac{\mu n_0}{(\mu - n_0)e^{-\mu t} + n_0}$$

# Parameterized first-order systems

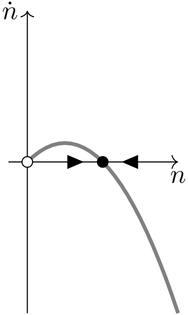
Motivating example : Rudimentary model of solid-state lasers




$\mu = -\frac{1}{2}$




$\mu = 0$

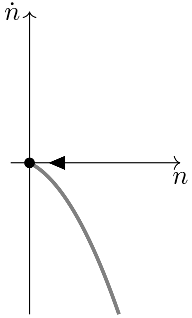



$\mu = 1$

# Parameterized first-order systems

Motivating example : Rudimentary model of solid-state lasers

For $\mu < 0$, $n = 0$ is a stable fixed point. No stimulated emission happens and the laser acts as a lamp.




$\mu < 0$

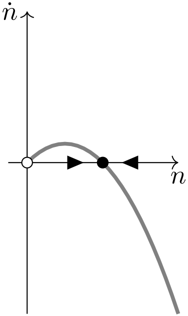
# Parameterized first-order systems

Motivating example : Rudimentary model of solid-state lasers

For $\mu < 0$, $n = 0$ is a stable fixed point. No stimulated emission happens and the laser acts as a lamp.

For $\mu > 0$, $n = 0$ is no longer stable. The process of stimulated emission sets in and the laser behave as expected.

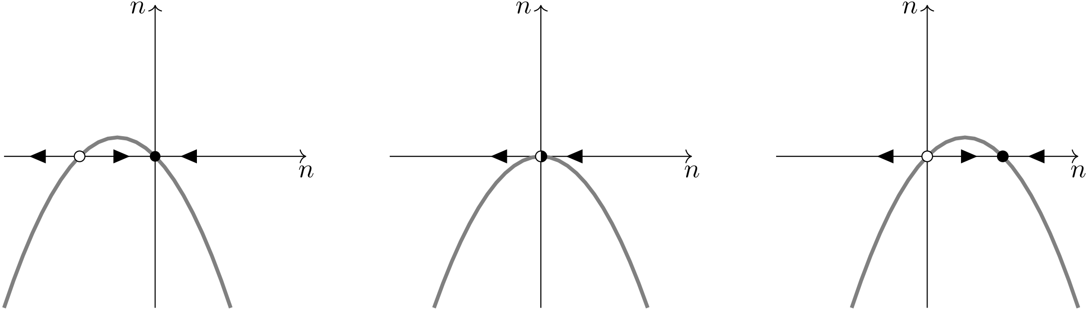
This drastic change of dynamics as the parameter $\mu$ exceeds a critical threshold is known as a **bifurcation**.




$\mu > 0$

# Transcritical bifurcation

The law of equivalent exchange




$\mu = -1$ $\mu = 0$ $\mu = 1$

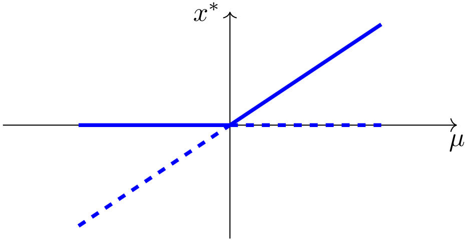
# Transcritical bifurcation

The law of equivalent exchange

## Transcritical bifurcation

$$\dot{x} = \mu x - x^2$$

- Two fixed points given by $x_1 = 0$ and $x_2 = \mu$ exist for all $\mu$.
- At $\mu = 0$ they collide and exchange their stability properties for $\mu > 0$.



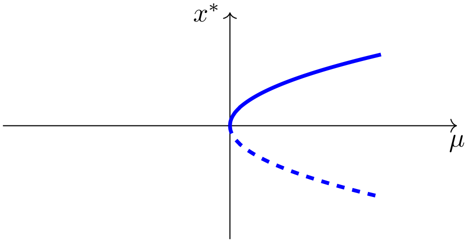
# Saddle-node bifurcation

Fixed points appearing "out of the clear blue sky"

## Saddle-node bifurcation

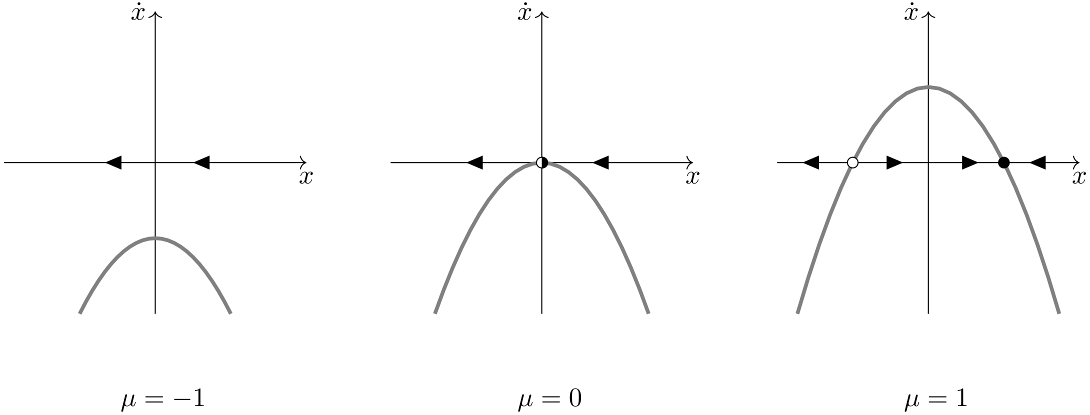
$$\dot{x} = \mu - x^2$$

- For $\mu < 0$, no fixed points exist.
- As $\mu$ becomes positive, two fixed points given by $x_{1,2} = \pm\sqrt{\mu}$ appear out of thin air.
- One of these fixed points is linearly stable while the other is linearly unstable.



# Saddle-node bifurcation

Fixed points appearing "out of the clear blue sky"



# Saddle-node bifurcation

Motivating example : Over-damped pendulum driven by a constant torque

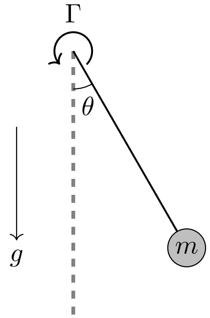
- Starting from Newton's principles, the equation of motion is given by

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) = \Gamma.$$

- Dividing by $mgL$ and rescaling time as $t \to \tau t$ yields

$$\frac{L}{g\tau^2}\ddot{\theta} + \frac{b}{mgL\tau}\dot{\theta} + \sin(\theta) = \frac{\Gamma}{mgL}.$$

- We are now left with two possible choices for the time scale $\tau$.



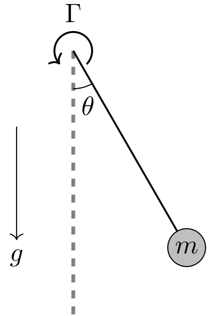
# Saddle-node bifurcation

Motivating example : Over-damped pendulum driven by a constant torque

If friction is by far the dominant force (over-damped situation), we can approximately reduce our equation to

$$\dot{\theta} = \gamma - \sin(\theta)$$

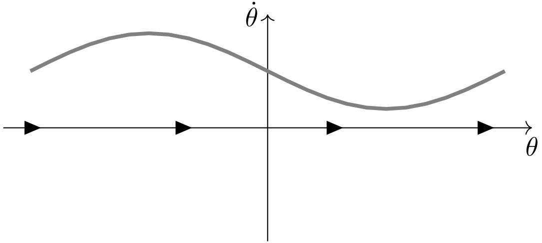
where $\gamma = {}^{\Gamma}/_{mgL}$ is our control parameter.



# Saddle-node bifurcation

Motivating example : Over-damped pendulum driven by a constant torque

- When $\gamma$ is sufficiently large, the system has no fixed point.
- Physically, the applied torque is large enough to cause the pendulum to spin indefinitely.

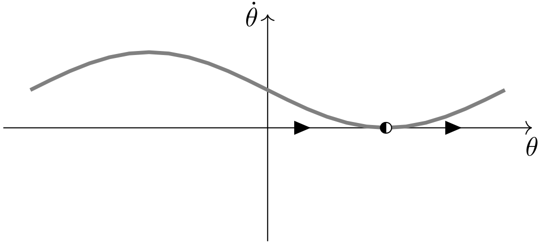



$\gamma = 1.5$

# Saddle-node bifurcation

Motivating example : Over-damped pendulum driven by a constant torque

- For $\gamma = 1$, the torque can barely counter-balance friction and gravity.
- A meta-stable fixed point is created.




$\gamma = 1$

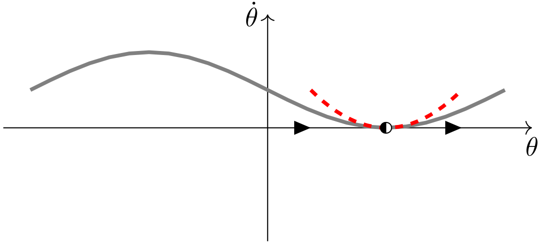
# Saddle-node bifurcation

Motivating example : Over-damped pendulum driven by a constant torque

In the vicinity of this point, the system can be approximated by

$$\dot{\eta} = \mu + \eta^2$$

with $\mu = \gamma - 1$, hence the name **normal form**.

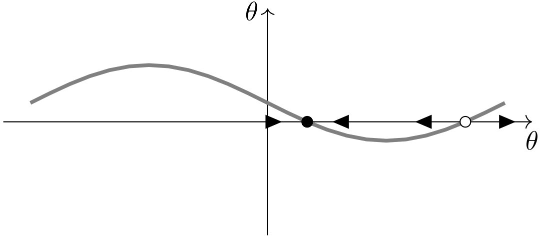



$\gamma = 1$

# Saddle-node bifurcation

Motivating example : Over-damped pendulum driven by a constant torque

- For $|\gamma| < 1$, torque is no longer able to overcome gravity and friction.
- Two equilibrium positions co-exist, a stable and an unstable one.




$\gamma = 0.5$

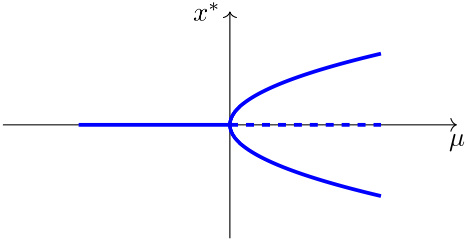
# Pitchfork bifurcation

Breaking the symmetry

## Supercritical pitchfork bifurcation

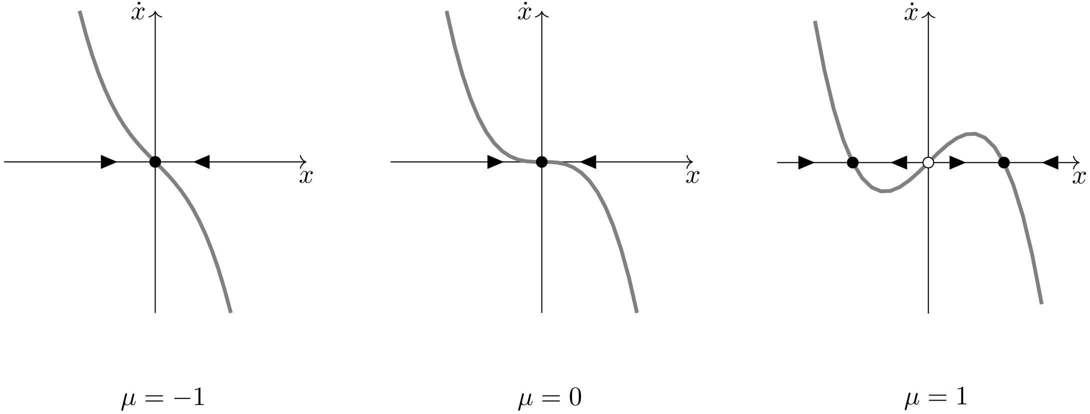
$$\dot{x} = \mu x - x^3$$

- For $\mu < 0$, a single stable fixed point exist.
- As $\mu$ becomes positive, two stable fixed points are created while the original one becomes unstable.



# Pitchfork bifurcation

Breaking the symmetry



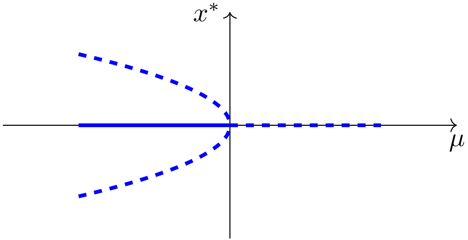
# Pitchfork bifurcation

Breaking the symmetry

## Subcritical pitchfork bifurcation

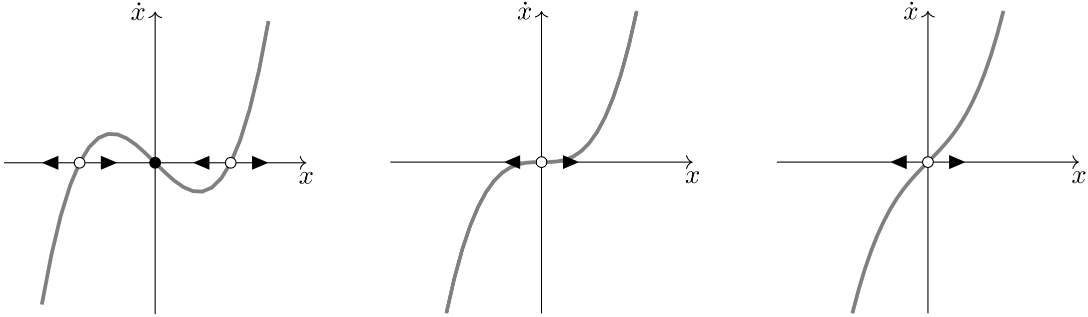
$$\dot{x} = \mu x + x^3$$

- For $\mu < 0$, three fixed points exist, two unstable and one stable.
- As $\mu$ becomes positive, two unstable fixed points disappear while the central one becomes unstable.



# Pitchfork bifurcation

Breaking the symmetry




$\mu = -1$ $\mu = 0$ $\mu = 1$

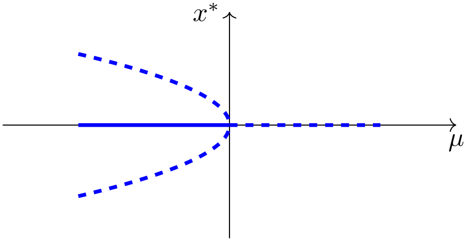
# Subcritical pitchfork bifurcation

Break the symmetry and the hysteresis phenomenon

- For $\mu > 0$, $x(t) \to \pm\infty$ in finite-time. It is the **finite blow-up** phenomenon we mentionned in L2.
- In most real systems, this is unphysical and our simple model needs to be modified.
- Assuming the system is still symmetric under $x \to -x$, the simplest modification is

$$\dot{x} = \mu x + x^3 - x^5.$$

The analysis of this sytem and of the resulting hysteresis phenomenon is left as an exercise.



# One system – multiple bifurcations

Example by J. Nathan Kutz

**Example :** $\dfrac{dx}{dt} = -x\left(x^2 - 2x - \mu\right)$

A **saddle-node bifurcation** occurs at $\mu = -1$ while a transcritical one occurs at $\mu = 0$.

For $\mu \geq -1$, the system can exhibit **hysteresis** and **bi-stability**.