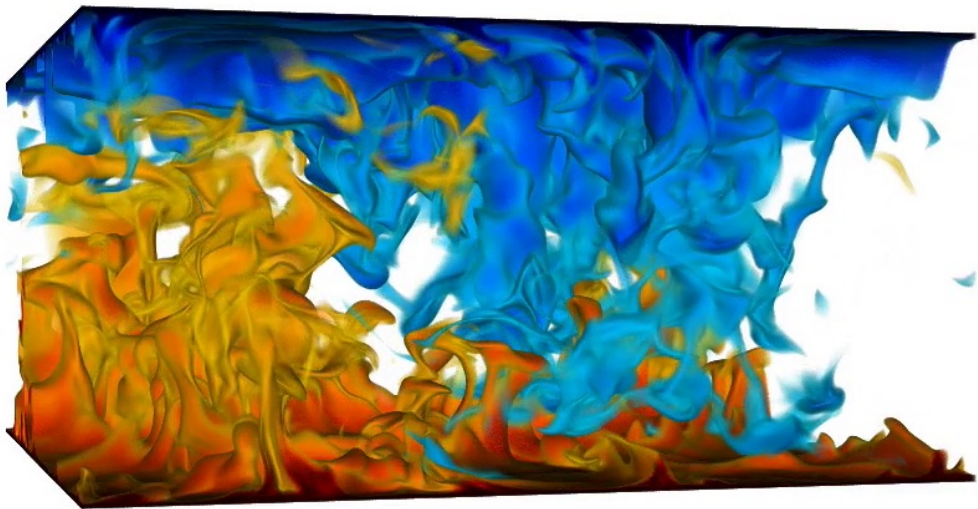


Lorenz system

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Arts & Métiers Institute of Technology, January 2022



Rayleigh-Bénard convection

Velocity equation

The diagram illustrates the velocity equation for Rayleigh-Bénard convection. The equation is presented as a sum of terms on the left of an equals sign, followed by a sum of terms on the right. Each term is enclosed in a colored box, and a label with a colored arrow points to it. The labels are: Inertia (red), Advection (blue), Pressure forces (purple), Viscous diffusion (dark blue), and Buoyancy (teal). The equation is:
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \delta \rho \mathbf{g}$$

Inertia points to $\frac{\partial \mathbf{u}}{\partial t}$ (red box).

Advection points to $(\mathbf{u} \cdot \nabla) \mathbf{u}$ (blue box).

Pressure forces points to $-\frac{1}{\rho} \nabla p$ (purple box).

Viscous diffusion points to $\nu \nabla^2 \mathbf{u}$ (dark blue box).

Buoyancy points to $\delta \rho \mathbf{g}$ (teal box).

Rayleigh-Bénard convection

Temperature equation

The diagram illustrates the temperature equation for Rayleigh-Bénard convection, with terms color-coded and labeled:

- Inertia:** A red label with an arrow pointing to the time derivative term $\frac{\partial T}{\partial t}$, which is enclosed in a red box.
- Convection:** A green label with an arrow pointing to the convective term $(\mathbf{u} \cdot \nabla) T$, which is enclosed in a green box.
- Conduction:** A purple label with an arrow pointing to the conductive term $\kappa \nabla^2 T$, which is enclosed in a purple box.

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T$$

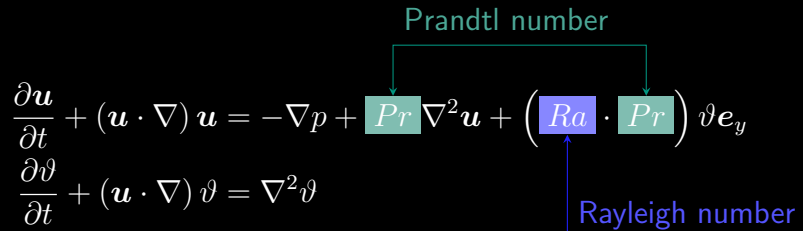
Rayleigh-Bénard convection

Non-dimensional equations

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \boxed{Pr} \nabla^2 \mathbf{u} + \left(\boxed{Ra} \cdot \boxed{Pr} \right) \vartheta \mathbf{e}_y \\ \frac{\partial \vartheta}{\partial t} + (\mathbf{u} \cdot \nabla) \vartheta &= \nabla^2 \vartheta\end{aligned}$$

Prandtl number

Rayleigh number



Rayleigh-Bénard convection

Base flow

Conducting state

$$\begin{cases} \frac{d^2\Theta}{dy^2} = 0 \\ \Theta(0) = 1, \quad \Theta(1) = 0 \end{cases}$$

Rayleigh-Bénard convection

Base flow

Conducting state

$$\Theta(y) = 1 - y$$

$$\mathbf{U}_b = 0$$

Rayleigh-Bénard convection

Linear stability

Squire theorem : Only two-dimensional perturbations need to be considered.

$$\begin{aligned}\frac{\partial \mathbf{u}'}{\partial t} &= -\nabla p' + Pr \nabla^2 \mathbf{u}' + (Ra \cdot Pr) \vartheta' \mathbf{e}_y \\ \frac{\partial \vartheta'}{\partial t} &= -v' + \nabla^2 \vartheta'\end{aligned}$$

with $\mathbf{u}' = [u' \quad v']^T$ and ϑ' the velocity and temperature fluctuations, respectively.

Rayleigh-Bénard convection

Linear stability

Fluctuation's streamfunction

$$\frac{\partial}{\partial t} \nabla^2 \Psi = - (Ra \cdot Pr) \frac{\partial \vartheta}{\partial x} + Pr \nabla^4 \Psi$$

$$\frac{\partial \vartheta}{\partial t} = - \frac{\partial \Psi}{\partial x} + \nabla^2 \vartheta$$

Rayleigh-Bénard convection

Dispersion relation

(1916) Assuming free-slip boundary conditions for the fluctuation leads to

$$\Psi(x, y, t) = \hat{\Psi}(t) \sin(n\pi y) \sin(kx),$$

$$\vartheta(x, y, t) = \hat{\vartheta}(t) \sin(n\pi y) \cos(kx),$$

the problem can be solved analytically. We'll also let $\gamma^2 = (n\pi)^2 + k^2$.

Rayleigh-Bénard convection

Dispersion relation

$$\begin{aligned} -\gamma^2 \frac{d\hat{\Psi}}{dt} &= Pr \gamma^4 \hat{\Psi} + (Ra \cdot Pr) k \hat{\vartheta} \\ \frac{d\hat{\vartheta}}{dt} &= -k \hat{\Psi} - \gamma^2 \hat{\vartheta} \end{aligned}$$

Rayleigh-Bénard convection

Dispersion relation

$$\begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix} = \begin{bmatrix} Pr\gamma^4 & (Ra \cdot Pr) k \\ -k & -\gamma^2 \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix}$$

Rayleigh-Bénard convection

Dispersion relation

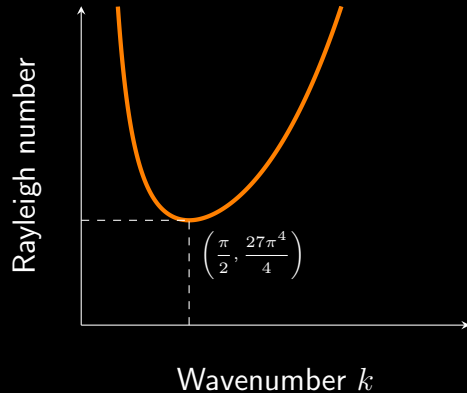
$$\lambda \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix} = \begin{bmatrix} Pr\gamma^4 & (Ra \cdot Pr) k \\ -k & -\gamma^2 \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix}$$

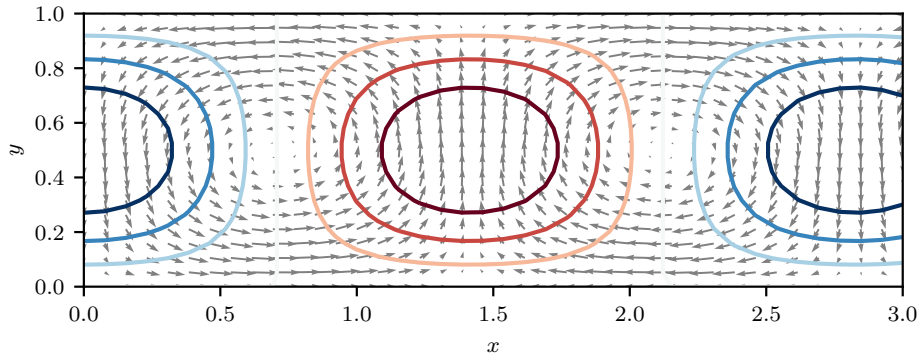
Rayleigh-Bénard convection

Dispersion relation

Neutral curve

$$Ra_c(n, k) = \frac{((n\pi)^2 + k^2)^3}{k^2}$$





Temperature and velocity field of the most unstable eigenmode.

From Navier-Stokes to Lorenz

Investigating the nonlinearities

From Navier-Stokes to Lorenz

Nonlinear equations

Nonlinear advection

$$\frac{\partial}{\partial t} \nabla^2 \Psi = - (Ra \cdot Pr) \frac{\partial \vartheta}{\partial x} + Pr \nabla^4 \Psi + \mathcal{J} [\nabla^2 \Psi, \Psi]$$

$$\frac{\partial \vartheta}{\partial t} = - \frac{\partial \Psi}{\partial x} + \nabla^2 \vartheta + \mathcal{J} [\vartheta, \Psi]$$

Nonlinear convection

From Navier-Stokes to Lorenz

Nonlinear equations

$$\mathcal{J}[f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}$$

From Navier-Stokes to Lorenz

Nonlinear equations

Saltzman (1962): The general solution to the partial differential equation can be expressed as a doubly infinite Fourier series

From Navier-Stokes to Lorenz

Deriving the reduced-order model

Close to the bifurcation point, the fluctuations are described by the unstable mode, hence

$$\Psi(x, y, t) \simeq \Psi_1(t) \sin(\pi y) \sin(kx) + \dots$$

$$\vartheta(x, y, t) \simeq \vartheta_1(t) \sin(\pi y) \cos(kx) + \dots$$

Which kind of harmonics do they generate due to the nonlinearity ?

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\begin{aligned}\mathcal{J} [\Psi, \nabla^2 \Psi] &= \mathcal{J} [\Psi, -\gamma^2 \Psi] \\ &= -\gamma^2 \mathcal{J} [\Psi, \Psi] \\ &= 0\end{aligned}$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\begin{aligned}\mathcal{J} [\Psi, \vartheta] &= \Psi_1 \vartheta_1 \mathcal{J} [\sin(kx) \sin(\pi y), \sin(kx) \cos(\pi y)] \\ &= \Psi_1 \vartheta_1 \frac{k\pi}{2} \sin(2\pi y)\end{aligned}$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

Unstable eigenmode

$$\Psi(x, y, t) \simeq \Psi_1(t) \sin(kx) \sin(\pi y)$$

$$\vartheta(x, y, t) \simeq \vartheta_1(t) \sin(kx) \cos(\pi y) + \vartheta_2 \sin(2\pi y)$$

Nonlinear distortion

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{\partial}{\partial t} \nabla^2 \Psi = - \underbrace{(Ra \cdot Pr)}_{\text{Buoyancy}} \frac{\partial \vartheta}{\partial x} + \underbrace{Pr \nabla^4 \Psi}_{\text{Viscous diffusion}}$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{d\Psi_1}{dt} = -\frac{k}{\gamma^2} (Ra \cdot Pr) \vartheta_1 - Pr\gamma^2 \Psi_1$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

The diagram shows the Lorenz equation with three terms highlighted in colored boxes and labeled with arrows:

- Convection:** A red arrow points from the label "Convection" to the red box containing $\frac{\partial \Psi}{\partial x}$.
- Thermal diffusion:** A blue arrow points from the label "Thermal diffusion" to the blue box containing $\nabla^2 \vartheta$.
- Nonlinear distortion:** A green arrow points from the label "Nonlinear distortion" to the green box containing $\mathcal{J}[\vartheta, \Psi]$.

$$\frac{\partial \vartheta}{\partial t} = - \frac{\partial \Psi}{\partial x} + \nabla^2 \vartheta + \mathcal{J}[\vartheta, \Psi]$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\begin{aligned}\frac{d\vartheta_1}{dt} &= k\Psi_1(1 + \pi\vartheta_2) - \gamma^2\vartheta_1 \\ \frac{d\vartheta_2}{dt} &= \frac{k\pi}{2}\Psi_1\vartheta_1 - 4\pi^2\vartheta_2\end{aligned}$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{d\Psi_1}{dt} = \frac{k}{\gamma^2} (Ra \cdot Pr) \vartheta_1 - Pr\gamma^2\Psi_1$$

$$\frac{d\vartheta_1}{dt} = \Psi_1 (k + k\pi\vartheta_2) - \gamma^2\vartheta_1$$

$$\frac{d\vartheta_2}{dt} = \frac{k\pi}{2}\Psi_1\vartheta_1 - 4\pi^2\vartheta_2$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\dot{x} = \sigma (y - x)$$

$$\dot{y} = x (\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

Lorenz system

A case study

Lorenz model

Some fundamental properties

Equivariance : The equations remain unchanged if $(x, y, z) \mapsto (-x, -y, z)$.

$$S\dot{x} = f(Sx)$$

If (x, y, z) is a solution, so is $(-x, -y, z)$. They come in pairs.

Lorenz model

Some fundamental properties

Invariant axis : If $x(0) = y(0) = 0$, then $x(t) = y(t) = 0$ at all time.

$$\dot{z} = -\beta z \quad \Rightarrow \quad z(t) = \exp(-\beta t) z_0$$

The z -axis is an **invariant manifold** of the system.

Lorenz model

Some fundamental properties

Strongly dissipative : Every in phase space, we have that

$$\nabla \cdot \mathbf{f}(\mathbf{x}) = -\sigma - 1 - \beta < 0.$$

Any given volume V of initial conditions will eventually tend to 0 as $t \rightarrow \infty$.

Lorenz model

Exercise

Exercise : Onset of the convection cells

1. Compute the fixed points of the system as a function of ρ .
2. When does the conducting state ($x = 0$) lose its stability?
3. Using a symmetry argument, what type of bifurcation can it be?

$$\dot{x} = \sigma (y - x)$$

$$\dot{y} = x (\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

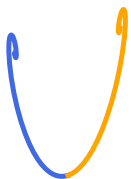
Lorenz system

Dynamics for $1 \leq \rho \leq 14$

$$\rho = 1.10$$



$$\rho = 2.50$$



$$\rho = 5.00$$



$$\rho = 10.00$$



$$\rho = 13.90$$



Lorenz system

Homoclinic connection ($\rho \simeq 13.926$)

$$\rho = 13.926$$

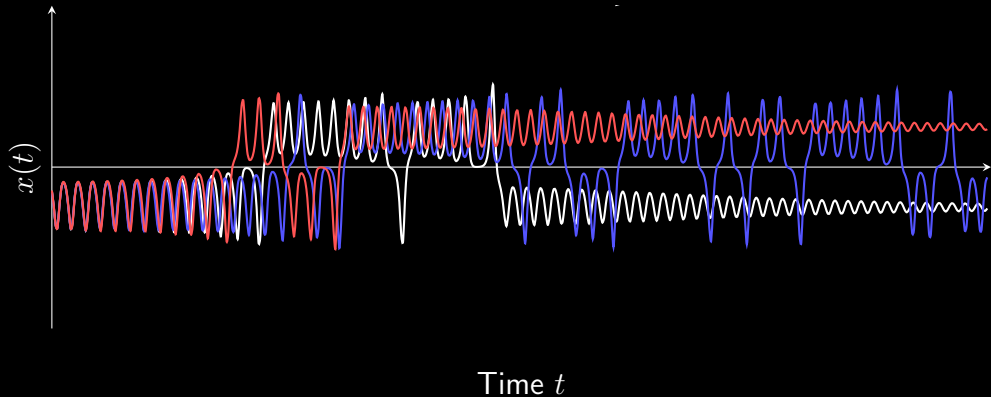
Perturbation leaves the conducting state along its unstable manifold and returns to it along its stable one.



Lorenz system

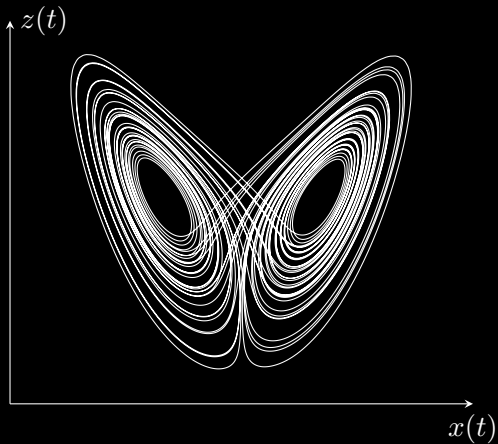
Transient chaos

For $\rho > 14$, the system exhibits **sensitive dependence on initial conditions**.



Lorenz system

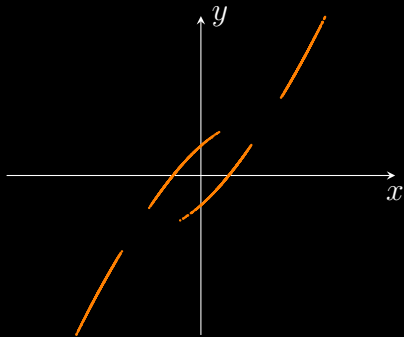
Strange attractor for $(\sigma, \rho, \beta) = (10, 28, 8/3)$



Lorenz system

Strange attractor

The attractor is not a volume, but not a surface either. It is something in-between: a **fractal object**.



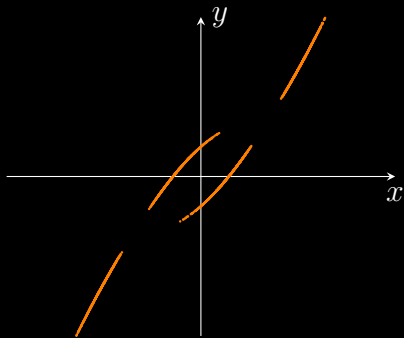
Lorenz system

Strange attractor

Its dimension is estimated to be

$$2.06 \pm 0.01,$$

i.e. it is not an integer.



Lorenz system

Strange attractor

$$t_{\text{horizon}} \sim \mathcal{O} \left(\frac{1}{\lambda} \log \frac{a}{\delta_0} \right)$$

Accuracy of the prediction

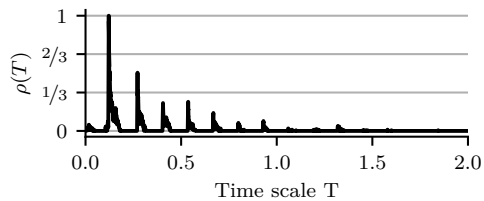
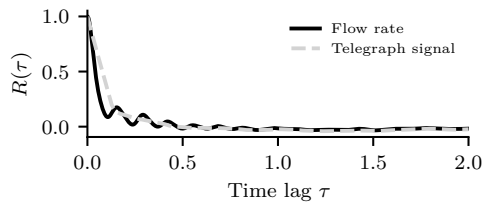
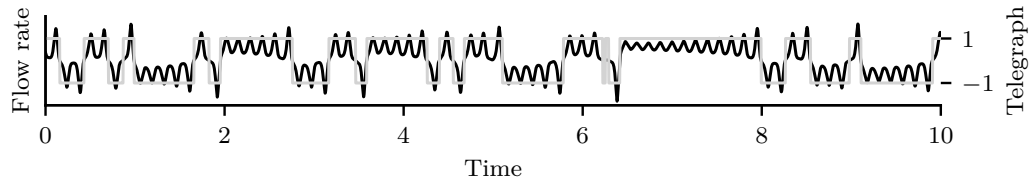
Lyapunov exponent

Uncertainty on the initial condition

The diagram illustrates the formula for the horizon time t_{horizon} in the Lorenz system. The formula is $t_{\text{horizon}} \sim \mathcal{O} \left(\frac{1}{\lambda} \log \frac{a}{\delta_0} \right)$. The variables are highlighted with colored boxes: λ is in a red box, a is in a blue box, and δ_0 is in a green box. Arrows point from the labels to these variables: a red arrow from 'Lyapunov exponent' to λ , a green arrow from 'Uncertainty on the initial condition' to δ_0 , and a blue arrow from 'Accuracy of the prediction' to a .

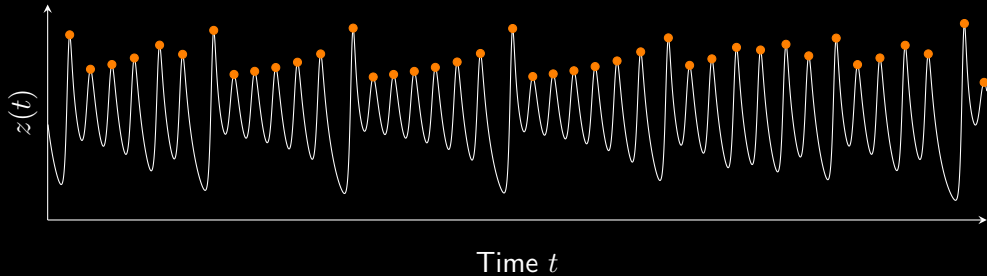
Lorenz system
Strange attractor

$$t_{\text{Lorenz}} \sim \mathcal{O} \left(\log \frac{a}{\delta_0} \right)$$



Lorenz system

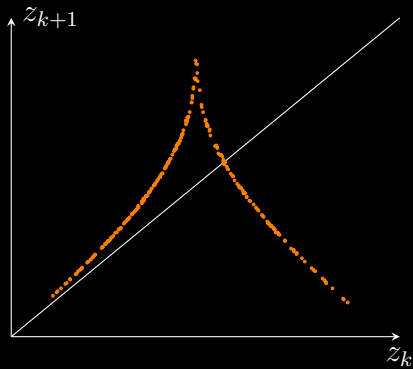
Strange attractor



Lorenz system
Strange attractor

Lorenz map

$$z_{k+1} = f(z_k)$$

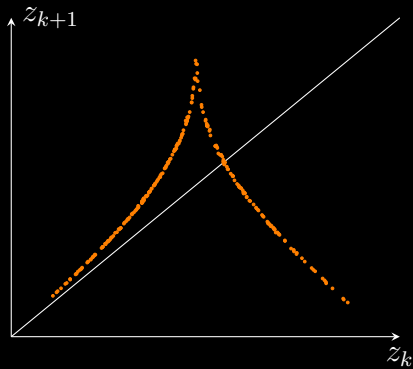


Lorenz system

Strange attractor

Lorenz map

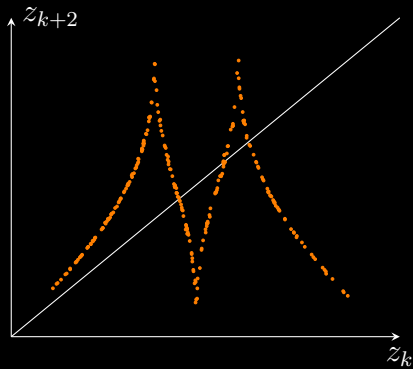
$$|f'(z)| > 1 \quad \forall z$$



Lorenz system
Strange attractor

Period-2 orbit

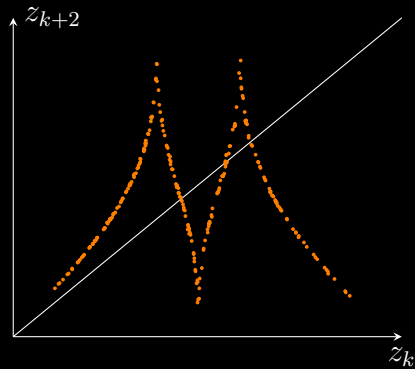
$$z = (f \circ f)(z)$$



Lorenz system
Strange attractor

Period-2 orbit

$$\eta_{k+2} \simeq (f'(p)f'(q)) \eta_k$$



Lorenz system
Strange attractor

Period-n orbit

$$z = (f \circ f \circ \cdots \circ f)(z)$$

Lorenz system
Strange attractor

Period-n orbit

$$\eta_{k+1} \simeq \left(\prod_{i=1}^n f'(z_i) \right) \eta_k$$

Lorenz system

Strange attractor

Period-n orbit

$$\eta_{k+1} \simeq \left(\prod_{i=1}^n f'(z_i) \right) \eta_k$$

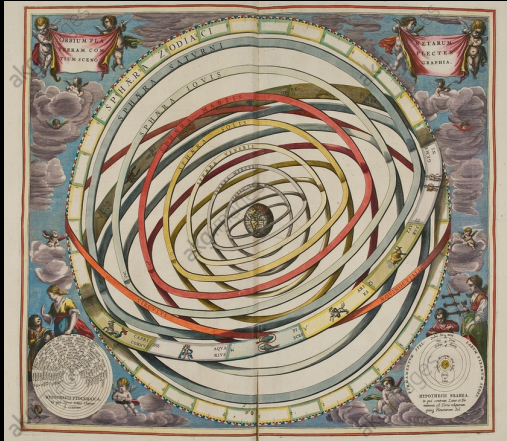
The skeleton of the attractor is made of an infinite number of **unstable periodic orbits**.

On the importance of chaos in Science

A (very) brief history

On the importance of chaos in Science

The clockwork Universe



- Ptolemy
- Copernicus
- Gallileo
- Kepler
- Newton
- Leibniz
- ...

On the importance of chaos in Science

Mathematical determinism

Cauchy-Lipschitz (1920): Under suitable regularity conditions on $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the problem

$$\begin{aligned}\dot{x} &= f(x) \\ x(0) &= x_0\end{aligned}$$

admits a unique solution fully determined by f and the initial condition.

On the importance of chaos in Science

Laplace determinism (1814)

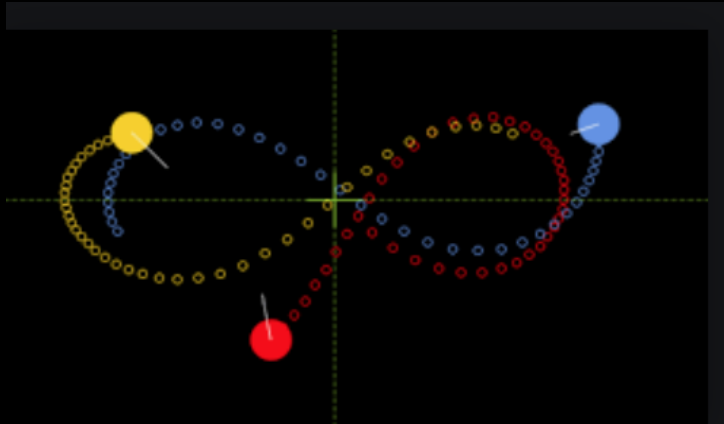
Nous devons [...] envisager l'état présent de l'Univers comme l'effet de son état antérieur, et comme la cause de celui qui va suivre. Une intelligence qui pour un instant donné connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la compose [...] embrasserait dans la même formule les mouvements des plus grands corps de l'Univers comme ceux du plus léger atome : rien ne serait incertain pour elle, et l'avenir comme le passé serait présent à ses yeux.

Essai philosophie sur les probabilité.

Pierre Simon de Laplace, 1814.

On the importance of chaos in Science

The three body problem (circa 1890)



On the importance of chaos in Science

The three body problem (circa 1890)

Si nous connaissions exactement les lois de la nature et la situation de l'univers à l'instant initial, nous pourrions prédire exactement la situation de ce même univers à un instant ultérieur. Mais, alors même que les lois naturelles n'auraient plus de secret pour nous, nous ne pourrions connaître la situation qu'approximativement. Si cela nous permet de prévoir la situation ultérieure avec la même approximation, c'est tout ce qu'il nous faut, nous disons que le phénomène a été prévu, qu'il est régi par des lois ; mais il n'en est pas toujours ainsi, il peut arriver que de petites différences dans les conditions initiales en engendrent de très grandes dans les phénomènes finaux.

Calcul des probabilités
Henri Poincaré, 1912.

On the importance of chaos in Science

Lorenz (1963)

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

Edward N. Lorenz (1917-2008)

Thank you for your attention

Any question ?