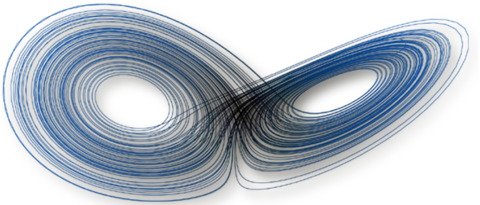

Linear systems

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First-order systems

Flow on the real number line

First-order systems

$$\dot{x} = f(x, \mu)$$

- $x(t)$ a real-valued function of time t,
- $f(x, \mu) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a smooth real-valued function of x and μ and does not explicitely depend on time t.

$$\dot{x} = \mu \pm x^2$$

$$\dot{x} = \mu x \pm x^2$$

$$\dot{x} = \mu x \pm x^3$$

Two-dimensional systems

Flow on the plane

Two-dimensional systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

- $\boldsymbol{x}(t) \in \mathbb{R}^2$ is a vector-valued function of time t.
- $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth vector-valued function of $\boldsymbol{x}$.
- Again, $\boldsymbol{f}$ is autonomous, i.e. it does not depend explicitely on time.

$$\ddot{\theta} = -\dot{\theta} - \sin(\theta)$$

$$\begin{cases} \dot{x} = x - y - (x^2 + y^2)x \\ \dot{y} = x + y - (x^2 + y^2)y \end{cases}$$

Two-dimensional systems

Linear systems

- For today, we'll restrict our attention to **linear time invariant systems** of the form
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$
where $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$.

- These typically result from linearizing the nonlinear system in the vicinity of a fixed point.

- Our goal for today is to **classify** the possible dynamics in such simple systems.

Linear systems

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear systems

Analytic solution

- Given the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, the analytical solution is given by

$$\boldsymbol{x}(t) = e^{t\boldsymbol{A}}\boldsymbol{x}_0$$

where $e^{t\boldsymbol{A}}$ is the matrix exponential.

- Once again, the existence of an analytical solution provides little insights into what the dynamics look like.

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

$$e^{t\boldsymbol{A}} = \boldsymbol{I} + t\boldsymbol{A} + \frac{(t\boldsymbol{A})^2}{2!} + \frac{(t\boldsymbol{A})^3}{3!} + \cdots$$

Linear systems

Eigenvalues and eigenvectors

Eigenvalues and eigenvectors are solution to the following equation

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

which can be rewritten as

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\, \boldsymbol{v} = \boldsymbol{0}.$$

Geometrically speaking, eigenvectors are left unchanged when multiplied by $\boldsymbol{A}$ except for a scaling λ.

Example

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Linear systems

Eigenvalues and eigenvectors

The equation $(\lambda \boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{v} = \boldsymbol{0}$ has non-trivial solutions if

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0.$$

For a 2×2 matrix, this polynomial equation reduces to

$$\lambda^2 - \mathsf{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = 0$$

where $\mathsf{tr}(\boldsymbol{A}) = a_{11} + a_{22}$ is the **trace** of $\boldsymbol{A}$ and $\det(\boldsymbol{A}) = a_{11}a_{22} - a_{21}a_{12}$ its **determinant**.

$$\lambda_1 + \lambda_2 = \mathsf{tr}(\boldsymbol{A})$$

$$\lambda_1 \lambda_2 = \det(\boldsymbol{A})$$

Linear systems

Eigenvalues and eigenvectors : tips and tricks

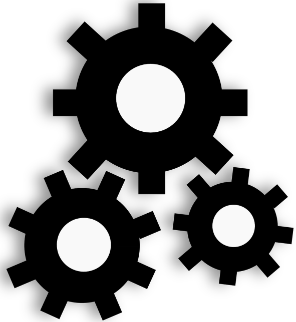
- If $\boldsymbol{A}$ is symmetric (i.e. $\boldsymbol{A} = \boldsymbol{A}^T$), then all of its eigenvalues are real.

- If $\boldsymbol{A}$ is skew-symmetric (i.e. $\boldsymbol{A} = -\boldsymbol{A}^T$), then all of its eigenvalues are imaginary numbers.

- Given the eigenvalues and eigenvectors, the matrix exponential $e^{t\boldsymbol{A}}$ is given by

$$e^{t\boldsymbol{A}} = \boldsymbol{V} e^{t\boldsymbol{\Lambda}} \boldsymbol{V}^{-1}$$

  where $\left(e^{t\boldsymbol{\Lambda}}\right)_{ii} = e^{t\lambda_i}$.



Linear systems

Asymptotic fate of $\boldsymbol{x}(t)$

Starting from $e^{t\boldsymbol{A}} = \boldsymbol{V} e^{t\boldsymbol{\Lambda}} \boldsymbol{V}^{-1}$, one can provide lower and upper bounds for $\|\boldsymbol{x}(t)\|_2^2 = \|e^{t\boldsymbol{A}}\boldsymbol{x}_0\|_2^2$.

$$\|e^{t\boldsymbol{\Lambda}}\|_2^2 \leq \|e^{t\boldsymbol{A}}\|_2^2 \leq \|\boldsymbol{V}\|_2^2 \|\boldsymbol{V}^{-1}\|_2^2 \|e^{t\boldsymbol{\Lambda}}\|_2^2$$

Vector-induced matrix norm

$$\begin{aligned} \|\boldsymbol{G}\|_2 &= \max_{\boldsymbol{x}} \frac{\|\boldsymbol{G}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \\ &= \sigma_1(\boldsymbol{G}) \end{aligned}$$

where $\sigma_1(\boldsymbol{G})$ is the leading singular value of $\boldsymbol{G}$.

Linear systems

Asymptotic fate of $\boldsymbol{x}(t)$

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$$e^{2t\Re(\lambda_1)} \leq \|e^{t\boldsymbol{A}}\|_2^2 \leq \kappa(\boldsymbol{A}) e^{2t\Re(\lambda_1)}$$

The fate of $\boldsymbol{x}(t)$ is thus dictated by the real part of the leading eigenvalue.

- If $\Re(\lambda_1) > 0$, then $\|\boldsymbol{x}(t)\|_2 \to \infty$ as $t \to \infty$.
- If $\Re(\lambda_1) < 0$, then $\|\boldsymbol{x}(t)\|_2 \to 0$ as $t \to \infty$.

Vector-induced matrix norm

If $\boldsymbol{G}$ is diagonal, then its 2-norm is

$$\|\boldsymbol{G}\|_2 = \max_i(|G_{ii}|).$$

where G_{ii} is the diagonal entries of $\boldsymbol{G}$.

Linear systems

What if the eigenvalues are imaginary ?

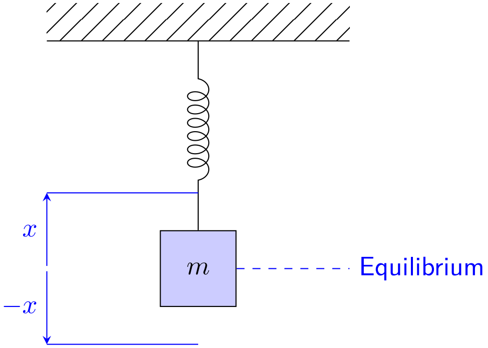
Consider the harmonic oscillator

$$\ddot{x} + x = 0.$$

Introducing the change of variable $y = \dot{x}$, it can be recast in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\boldsymbol{A}$ is skew-symmetric.



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Solution

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \quad \Leftrightarrow \quad \lambda = \pm i$$

$$\begin{aligned} x(t) &= A \cos t + B \sin t \\ &= A \cos(t + \phi) \end{aligned}$$

Oscillatory dynamics !

Linear systems

What is the eigenvalues are complex ?

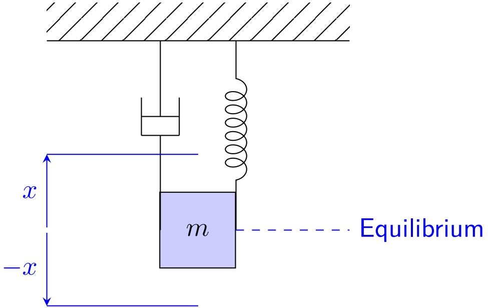
Consider the harmonic oscillator with a small damping (i.e. $k < 1$)

$$\ddot{x} + 2k\dot{x} + x = 0.$$

Introducing the change of variable $y = \dot{x}$, it can be recast in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2k \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $\boldsymbol{A}$ is no longer skew-symmetric.



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where $\boldsymbol{A}$ is no longer skew-symmetric.

Solution

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \quad \Leftrightarrow \quad \lambda = -k \pm i\sqrt{1-k^2}$$

$$x(t) = A\cos(\sqrt{1-k^2}t + \phi)e^{-kt}$$

Exponentially decaying oscillations !

Linear systems

What about the eigenvectors ?

Vectors are geometrical objects and so eigenvectors will help us understand the solutions from a geometric point of view.

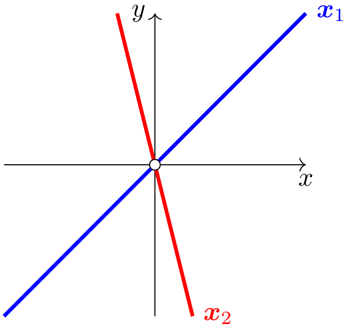
Consider the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Its eigenvalues and eigenvectors are given by

$$\lambda_1 = 2, \quad \lambda_2 = -3$$
$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

so the solutions to the system are $\boldsymbol{x}(t) = \alpha \boldsymbol{x}_1 e^{2t} + \beta \boldsymbol{x}_2 e^{-3t}$.



Linear systems

What about eigenvectors ?

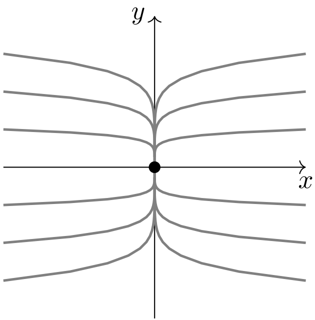
Let us explore the different kind of dynamics and phase portraits possible. For that, consider the system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

and vary the parameter μ.

In all cases, the eigenpairs are given by

$$\lambda_1 = \mu, \quad \lambda_2 = -1$$
$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



$|\mu| \gg 1$

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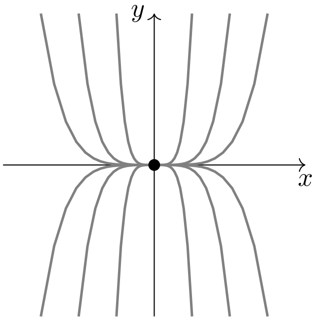

$|\mu| \ll 1$

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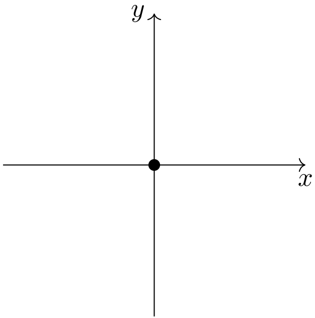

$|\mu| \simeq 1$

Linear systems

Classifying the fixed points

Given the matrix $\boldsymbol{A}$, compute

$$\begin{aligned} \operatorname{tr}(\boldsymbol{A}) &= a_{11} + a_{22} \\ \det(\boldsymbol{A}) &= a_{11}a_{22} - a_{21}a_{12} \\ \Delta &= \operatorname{tr}^2(\boldsymbol{A}) - 4\det(\boldsymbol{A}) \end{aligned}$$

and classify the dynamics.