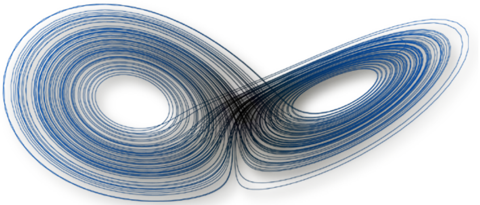


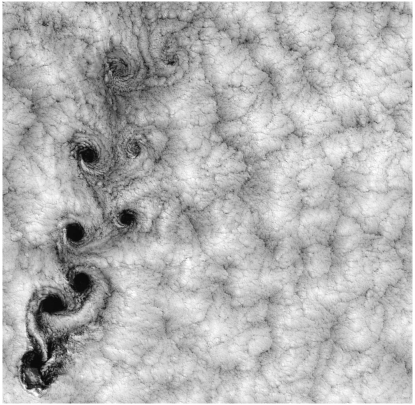
Two-dimensional cylinder flow at low Reynolds numbers

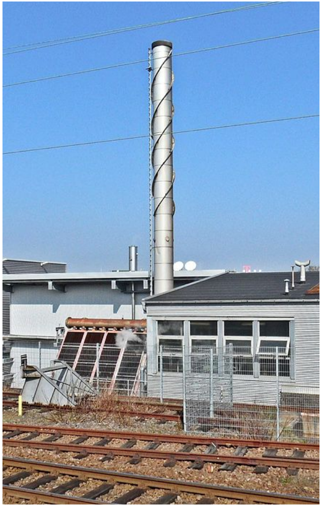
Jean-Christophe Loiseau

jean-christophe.loiseau@ensam.eu
Laboratoire DynFluid
Arts et Métiers, France.

Two-dimensional cylinder flow

A canonical example of flow oscillators





Two-dimensional cylinder flow

A canonical example of flow oscillators

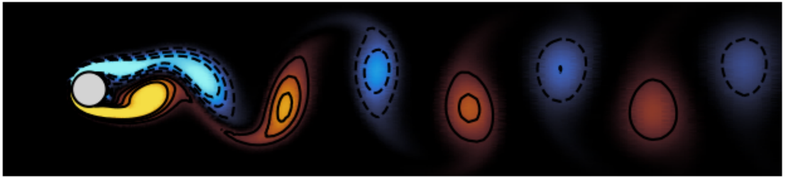
Its dynamics are governed by the **Navier-Stokes** equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity field and $p(\boldsymbol{x}, t)$ is the pressure field.

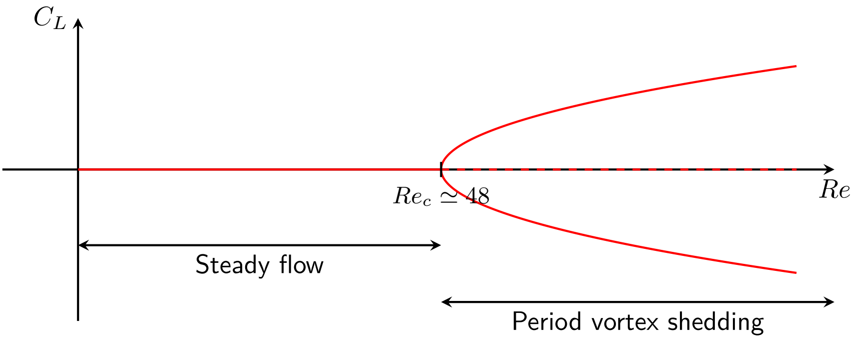
Two-dimensional cylinder flow

A canonical example of flow oscillators



Two-dimensional cylinder flow

Bifurcation diagram

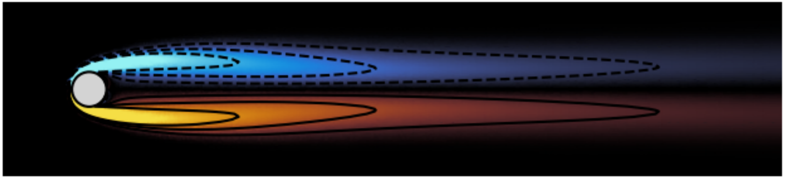


Two-dimensional cylinder flow

Finding fixed points

$$\boxed{\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u}}$$

Define the state vector $\boldsymbol{q} = (\boldsymbol{u}, p)^T$. Reformulate the problem as a root-finding problem $\mathcal{F}(\boldsymbol{q}, Re) = \mathbf{0}$ and use Newton's method (or variants) to solve it.



Two-dimensional cylinder flow

Linear stability analysis

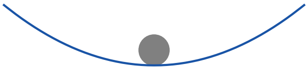
Denote by $\boldsymbol{U}_b(\boldsymbol{x}, Re)$ the base flow and linearized around it to obtain the linearized system

$$\boldsymbol{B}\frac{d\boldsymbol{q}}{dt} = \boldsymbol{L}\boldsymbol{q}$$

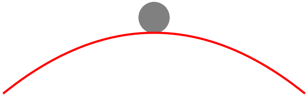
and look for the eigenvalues and eigenvectors of the generalized eigenvalue problem

$$\lambda \boldsymbol{B}\hat{\boldsymbol{q}} = \boldsymbol{L}\hat{\boldsymbol{q}}$$

using numerical eigensolvers.



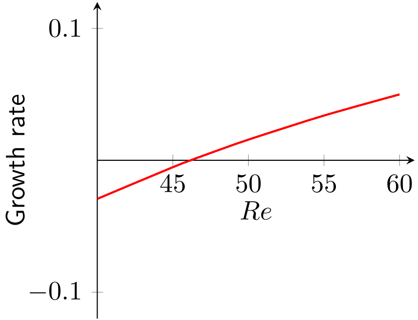
Stable equilibrium



Unstable equilibrium

Two-dimensional cylinder flow

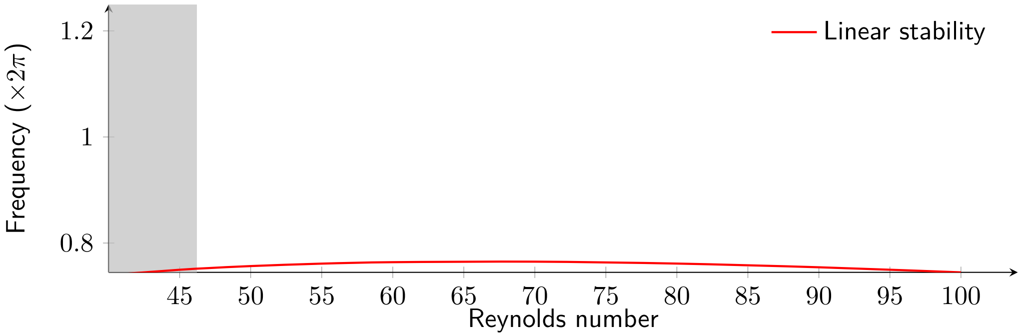
Linear stability analysis



- Leading eigenvalues come in complex-conjugate pairs.
- **Hopf bifurcation** at $Re \simeq 46.27$.

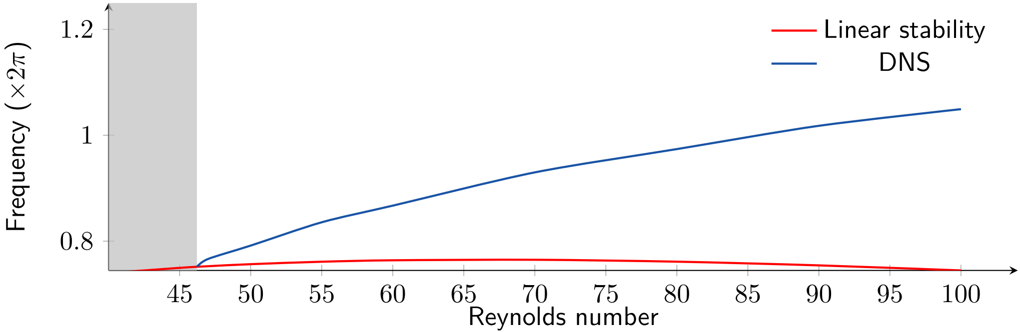
Two-dimensional cylinder flow

Linear stability vs. real life



Two-dimensional cylinder flow

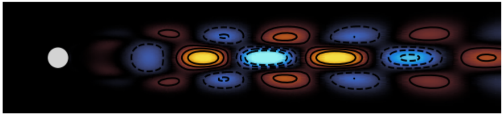
Linear stability vs. real life



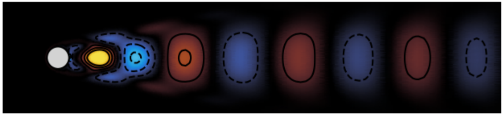
Two-dimensional cylinder flow

Linear stability vs. Nonlinear evolution

Stability mode



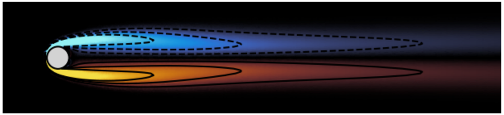
POD mode



Two-dimensional cylinder flow

Base flow vs. mean flow

Base flow solution

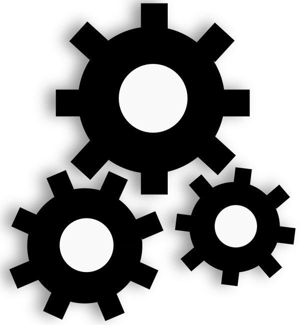


Time-averaged solution

Two-dimensional cylinder flow

Modeling objectives

Objective : Simple model capturing the essence of the problem.

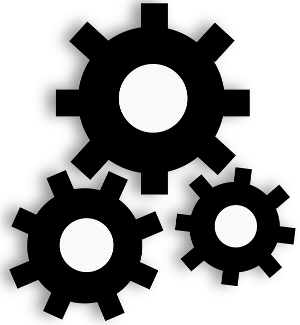


1. Linearly unstable nature of the fixed point.
2. Captures the transition to the limit cycle.
3. Explains why the base flow and mean flow are so different.
4. Explains why the frequency predictions are bad.
5. Capture the Reynolds number dependence.

Two-dimensional cylinder flow

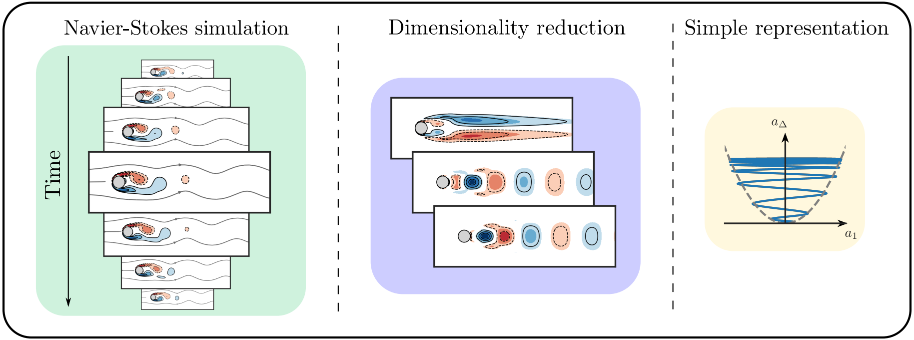
Modeling strategy

- Transform PDE into a handful of ODE.
  - $\hookrightarrow$ Dimensionality reduction, reduced-order modeling, ...
- Statistical inference of the parameters.
  - $\hookrightarrow$ Least-squares, calibration techniques, interpolation, ...
- Mathematical analysis of the model's properties.
  - $\hookrightarrow$ Linear and weakly nonlinear analyses, comparison with ground truth, ...



Two-dimensional cylinder flow

Dimensionality reduction



Two-dimensional cylinder flow

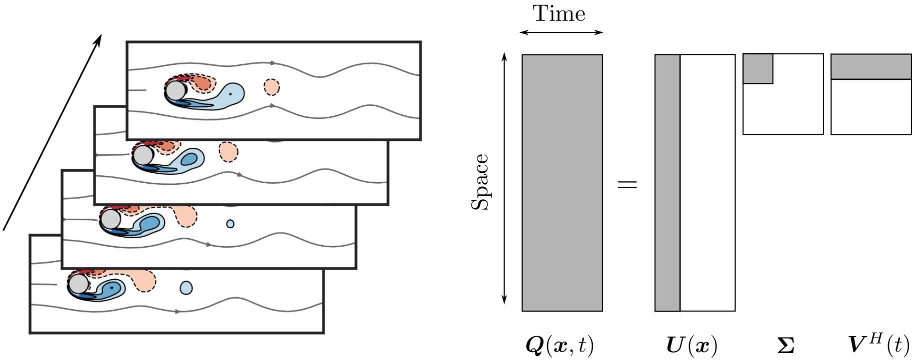
Dimensionality reduction

Objective : Find proxies for the vortex shedding's amplitude, phase and distortion between the base flow and the mean flow.

- Snapshots of the full state vector $\{\boldsymbol{q}(\boldsymbol{x}, t_k)\}$ are available :
  $\hookrightarrow$ use linear dimensionality reduction techniques such as POD/PCA or DMD.
- If only limited sensor measurements are available :
  $\hookrightarrow$ Use time-delay embeddings to construct the proxies.

Two-dimensional cylinder flow

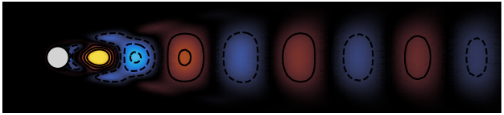
Dimensionality reduction

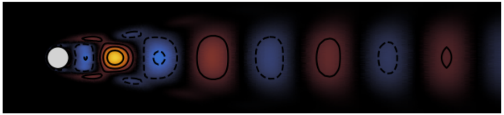


Two-dimensional cylinder flow

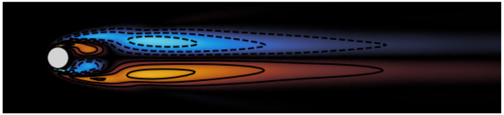
Dimensionality reduction

Modes 1 and 2 : Spatial structure of the vortex shedding. Their time-dependant amplitudes provide our proxy variables to describe the evolution of the oscillations.



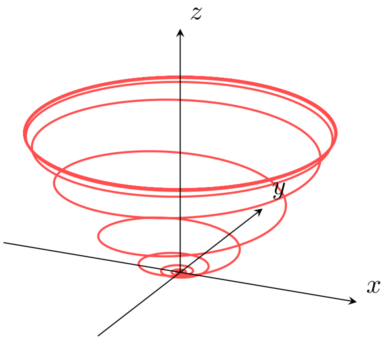


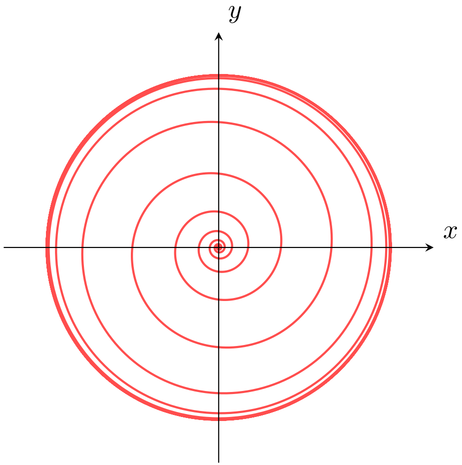
Mode 3 : Distortion between the base flow and the mean flow. Its amplitude provides the remaining proxy variable for our model.



Two-dimensional cylinder flow

Low-dimensional representation





Two-dimensional cylinder flow

Low-order model

Objective : Obtain a dynamical system describing the evolution of our proxy variables.

- If the original equations are known :
  - ↪ Use classical reduced-order modeling techniques (e.g. Galerkin or Petrov-Galerkin projections)
- If the original equations are unknown :
  - ↪ Use data-driven techniques (e.g. system identification or machine learning)

Model

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, Re)$$

Two-dimensional cylinder flow

POD-Galerkin projection

Step 1 : Galerkin expansion of the velocity field as

$$\boldsymbol{u}(\boldsymbol{x}, t) \simeq \boldsymbol{U}_b(\boldsymbol{x}) + \boldsymbol{u}_1(\boldsymbol{x})a_1(t) + \boldsymbol{u}_2(\boldsymbol{x})a_2(t) + \boldsymbol{u}_\Delta a_\Delta(t) + \cdots$$

Step 2 : Inject the Galerkin expansion into the Navier-Stokes equations and project onto the span of the POD modes.

$$\boldsymbol{U}^T \boldsymbol{U} \frac{d\boldsymbol{a}}{dt} = \boldsymbol{U}^T \boldsymbol{f}(\boldsymbol{U}\boldsymbol{a}, Re)$$

Step 3 : Inspect the model and use for rapid (approximate) simulations of the original system.

Two-dimensional cylinder flow

POD-Galerkin reduced-order model

Reduced-order model

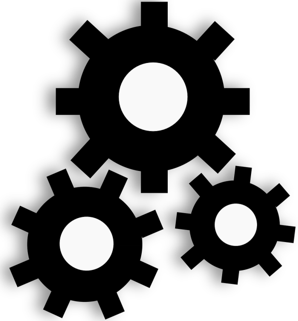
$$
\begin{aligned}
\dot{x} &= \sigma x - \omega y - xz - \alpha yz \\
\dot{y} &= \omega x + \sigma y - yz + \alpha xz \\
\dot{z} &= -z + x^2 + y^2
\end{aligned}
$$

Two-dimensional cylinder flow

Physical analysis

Objective : What can this model tell us about the physics of the problem and to what extent is it correct?

- **Physical consistency :**
  - ↪ Does it respect the known physics?
  - ↪ Are its predictions consistent with the observations?
- **Improved understanding :**
  - ↪ What does it tells us about the problem which was not directly obvious?
  - ↪ What insights are to be gained?



Two-dimensional cylinder flow

Reduced-order model consistency

Property : The quadratic nonlinear term in Navier-Stokes equations is energy-preserving.

The kinetic energy is given by $E(t) = \|\boldsymbol{a}(t)\|_2^2$ and we thus have

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\boldsymbol{a}^T\dot{\boldsymbol{a}} \\
&= \frac{1}{2}\left(x\dot{x} + y\dot{y} + z\dot{z}\right) \\
&= \sigma(x^2 + y^2) - z^2
\end{aligned}
$$

Only the linear terms in our model contribute to this energy budget.

Two-dimensional cylinder flow

Reduced-order model consistency

Property : The fixed point has a two-dimensional unstable subspace characterized by complex-conjugate eigenvalues.

The Jacobian matrix of the system reads

$$\boldsymbol{J} = \begin{bmatrix} \sigma - z & -\omega - \alpha z & -x - \alpha y \\ \omega + \alpha z & \sigma - z & -y + \alpha x \\ 2x & 2y & -1 \end{bmatrix}$$

For $(x, y, z) = (0, 0, 0)$ and $\sigma > 0$, its eigenvalues are $\sigma \pm i\omega$ and -1.

Two-dimensional cylinder flow

What insights are to be gained?

What can this reduced-order model actually tell me about the two-dimensional cylinder flow ?

$$\dot{x} = \sigma x - \omega y - xz - \alpha yz$$
$$\dot{y} = \omega x + \sigma y - yz + \alpha xz$$
$$\dot{z} = -z + x^2 + y^2$$

Two-dimensional cylinder flow

What insights are to be gained?

$$
\begin{aligned}
\dot{x} &= \sigma x - \omega y - xz - \alpha yz \\
\dot{y} &= \omega x + \sigma y - yz + \alpha xz \\
\dot{z} &= -z + x^2 + y^2
\end{aligned}
$$

Two-dimensional cylinder flow

What insights are to be gained?

$$\dot{\eta} = (\sigma + i\omega)\,\eta - \eta z + i\alpha z$$
$$\dot{z} = -z + |\eta|^2$$

Two-dimensional cylinder flow

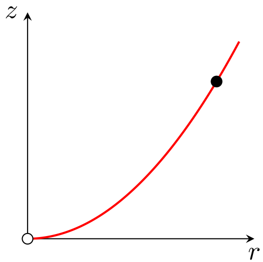
What insights are to be gained?

$$
\begin{aligned}
\dot{r} &= (\sigma - z)\, r \\
\dot{\varphi} &= \omega + \alpha z \\
\dot{z} &= -z + r^2
\end{aligned}
$$

Two-dimensional cylinder flow

What insights are to be gained?

Both the linearly unstable **baseflow** $(r, z) = (0, 0)$ and linearly stable **mean flow** $(r, z) = (\bar{r}, \bar{z})$ are fixed points of the **phase-averaged** equations.



Two-dimensional cylinder flow

What insights are to be gained?

Distortion eq : $\dot{z} = -z + r^2$

As r increases, the amplitude z of the distortion increases. It does so until a balance is met where $z = r^2$. Here, r^2 plays the role of the **Reynolds stesses** in the Navier-Stokes eqn.

Amplitude eq : $\dot{r} = (\sigma - z)\, r$

$\sigma - z$ is the **effective growth rate** of the instability. The amplitude of the vortex shedding grows until a balance is met where $z = \sigma$.

Two-dimensional cylinder flow

Two-timing approximate solution

Assume that $\sigma = \epsilon^2$ and introduce a multiple time-scale expansion with $\tau = \epsilon^2 t$. Expanding the solution in the vicinity of $(r_0, z_0) = (0, 0)$ yields

$$
\begin{aligned}
r(t,\epsilon) &= \epsilon r_1(t,\tau) + \epsilon^2 r_2(t,\tau) + \epsilon^3 r_3(t,\tau) + \cdots \\
z(t,\epsilon) &= \epsilon z_1(t,\tau) + \epsilon^2 z_2(t,\tau) + \epsilon^3 z_3(t,\tau) + \cdots
\end{aligned}
$$

We can now use **regular perturbation theory** to obtain an approximation of the evolution of $r(t)$ and $z(t)$.

Two-dimensional cylinder flow

Two-timing approximate solution

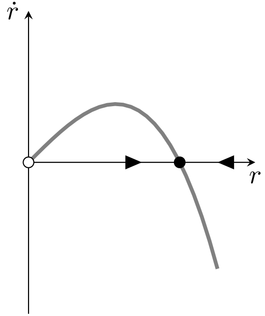
After some algebraic manipulations, we obtain that the vortex shedding's amplitude obeys

$$\frac{dr}{dt} = \sigma r - r^3$$

while the evolution of the distortion is given by

$$z(t) = Ae^{-t} + r^2(t)\left(1 - e^{-t}\right) + \mathcal{O}(\epsilon^3).$$

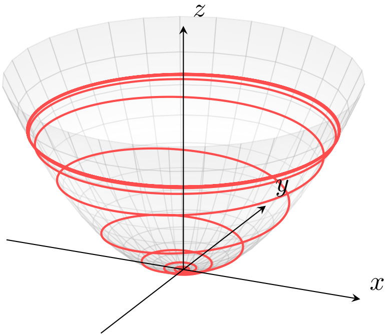
and so, very rapidly we have $z(t) \approx r^2(t)$.



Two-dimensional cylinder flow

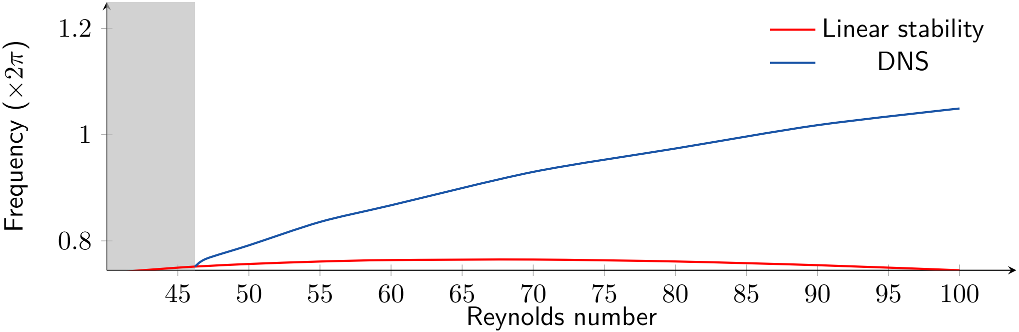
Further reducing the model's complexity...

$$\dot{r} = \sigma r - r^3$$
$$\dot{\varphi} = \omega + \alpha z$$
$$z = r^2$$



Two-dimensional cylinder flow

Piecing everything together



Two-dimensional cylinder flow

The power of mathematical modeling

Our model explains most of the dynamics observed in the flow :

- Saturation mechanism for the vortex shedding's amplitude,
- The baseflow gets distorted into the mean flow through the Reynold stresses,
- This distortion simultaneously induces a frequency shift.

To date, this is the simplest yet most accurate reduced-order model of the cylinder flow.

Two-dimensional cylinder flow

What next ?

Despite its accuracy and interpretability, our model leaves some questions unanswered, e.g.

- What is the physical mechanism responsible for the instability in the first place ?
- How exactly does the spatial support of the different structures evolves as their amplitude grows ?
- To what extent is our model generic and applicable to other flows ?

This is where we leave the realm of dynamical systems and enter that of classical fluid mechanics.