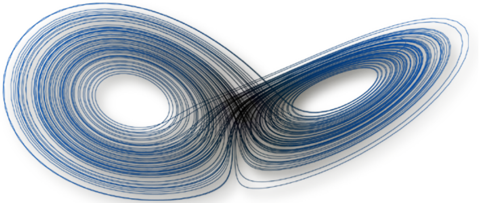

Nonlinear physics, dynamical systems and chaos theory

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One-dimensional maps

Introduction

- Let us focus our attention onto *discrete-time systems*.
  - ↪ Also known as difference equations, recursion relations, iterated maps or simply maps.
- Maps arise in various ways:
  - ↪ Tools for analyzing differential equations (see Poincarré and Lorenz maps).
  - ↪ Models of natural phenomena (e.g. digital electronics, finance).
  - ↪ As simple examples of chaos (what we'll do today).
- We'll focus on one-dimensional maps for now.
  - ↪ Easy and fast to simulate on a computer.

Logistic map

$$x_{k+1} = \mu x_k \left(1 - x_k\right)$$

Sine map

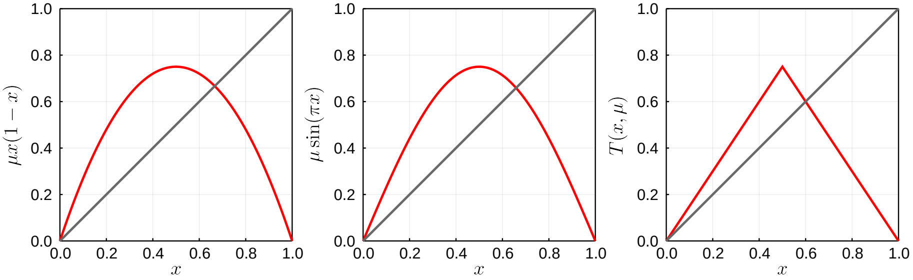
$$x_{k+1} = \mu \sin\left(\pi x_k\right)$$

Tent map

$$x_{k+1} = \begin{cases} \mu x & 0 \leq x \leq 1/2 \\ \mu - \mu x & 1/2 \leq x \leq 1. \end{cases}$$

One-dimensional maps

Introduction



One-dimensional maps

Fixed points and cobwebs

- Fixed points satisfy the equation

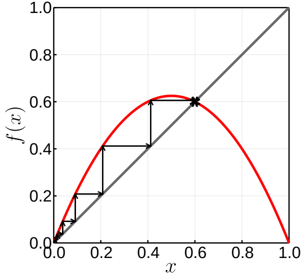
$$x^* = f(x^*).$$

- Their stability is governed by the asymptotic fate of an infinitesimal perturbation η whose dynamics are governed by

$$\eta_{k+1} = f'(x^*)\eta_k + \mathcal{O}\left(\eta_k^2\right).$$

- Neglecting the higher-order terms, we end up with the *linearized equation*

$$\eta_{k+1} = f'\left(x^*\right)\eta_k.$$



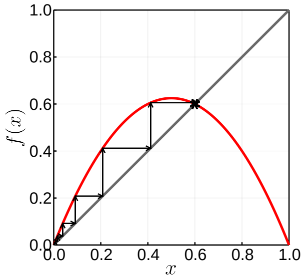
One-dimensional maps

Fixed points and cobwebs

- Introducing $\lambda = f'(x^*)$, the solution to the linearized equation is

$$\eta_{k+1} = \lambda^k \eta_0.$$

- If $|\lambda| < 1$, then $\eta_k \to 0$ as $k \to \infty$. The fixed point is **linearly stable**.
- If $|\lambda| > 1$, then $\eta_k \to \infty$ as $k \to \infty$. The fixed point is **linearly unstable**.
- If $|\lambda| = 1$, the fixed point is **marginally stable** and the higher-order terms can no longer be neglected.



Logistic map

Introduction

- Used by Robert May (1976) to illustrate how simple nonlinear maps can have very complicated dynamics.
- Article ends with *an evangelical plea for the introduction of these difference equations into elementary mathematics courses so that students' intuition may be enriched by seeing the wild thing that simple nonlinear equations can do.*

Logistic map

$$x_{k+1} = \mu x_k \left(1 - x_k\right)$$

Logistic map

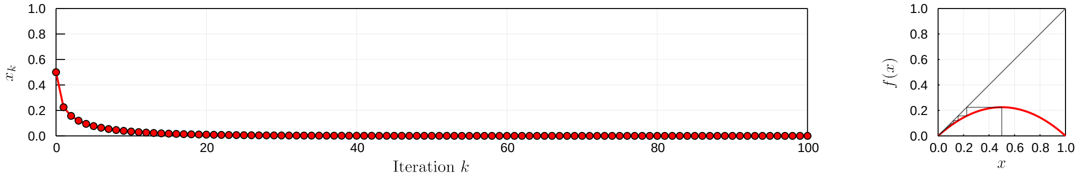
Numerics

```julia
using DynamicalSystems

# Define f(x, r, t).
f(x, r, t) = r*x*(1-x)

# Initial condition and parameter.
x, r = 0.1, 3.5

# Create dynamical system.
dynsys = DiscreteDynamicalSystem(f
    , x, r)

# Simulate system.
trajectory(dynsys, 100)
```

- Easy to simulate whichever programming language you use.
- In the rest, we'll use the `DynamicalSystems.jl` package in `Julia`.
  - ↪ 1st place in the 2018 Software Contest held by the dynamical systems division of SIAM.
  - ↪ Provides numerous tools to analyze dynamical systems.
- Check Youtube and Jupyter tutorials if you want to start using this library.

Logistic map

Dynamics

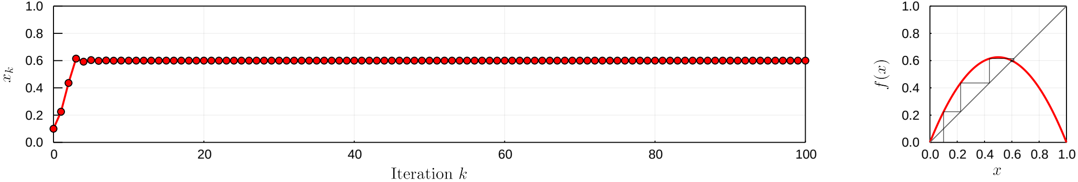


Dynamics for $\mu = 0.9$.

- For small growth rate ($r < 1$), the population goes extinct.

Logistic map

Dynamics

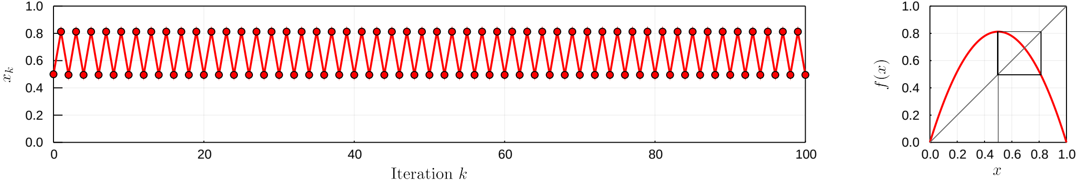


Dynamics for $\mu = 2.5$.

- For $1 < r < 3$, the population rows and eventually reaches a nonzero steady state.

Logistic map

Dynamics

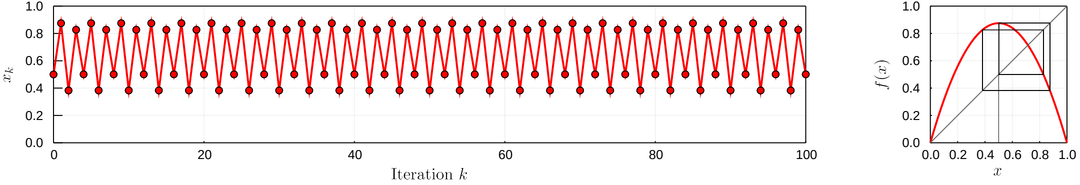


Dynamics for $\mu = 3.25$.

- For larger μ, the population oscillates periodically between two values. We have a **period-2** cycle.

Logistic map

Dynamics

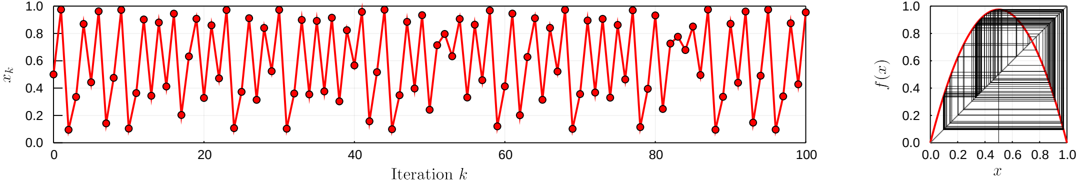


Dynamics for $\mu = 3.5$.

- Further increasing μ, the population oscillates between four values. We have a **period-4** cycle.

Logistic map

Dynamics



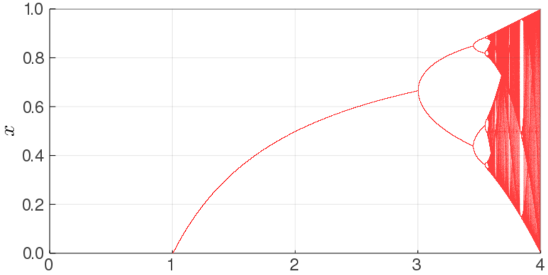
Dynamics for $\mu = 3.9$.

- Finally, for μ sufficiently large, the population behaves eratically. These are **chaotic dynamics**.

Logistic map

Orbit diagram

- The evolution of the system's behaviour as μ varies can be summarized in an **orbit diagram**.
- For the logistic map, the orbit diagram exhibits various features such as
  - ↪ Fixed points,
  - ↪ Bifurcations,
  - ↪ 2^n-cycles,
  - ↪ chaotic dynamics,
  - ↪ $3 \cdot 2^n$-cycles,
  - ↪ a fractal structure.
- In the rest, we'll try to explain each of these.



Logistic map

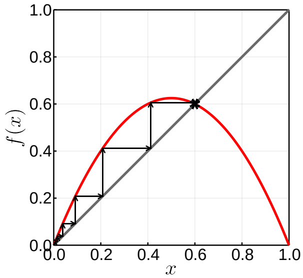
Analysis : Fixed points and stability

- Fixed points are solution to

$$x = \mu x \left(1 - x\right)$$
$$x \left(x - \frac{\mu - 1}{\mu}\right) = 0.$$

- It can easily be shown that the system admits the following fixed points:

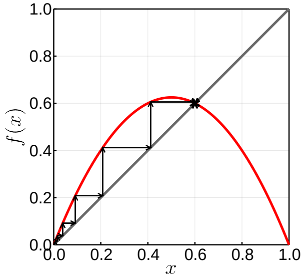
$$x^* = 0 \quad \forall \mu \in [0, 4[\,,$$
$$x^* = \frac{\mu - 1}{\mu} \quad \forall 1 \leq \mu < 4.$$



Logistic map

Analysis : Fixed points and stability

- Fixed points are linearly stable if $|f'(x^*, \mu)| < 1$ and linearly unstable if $|f'(x^*, \mu)| > 1$.
  - ↪ If $|f'(x^*, \mu)| = 1$, the fixed point is marginally stable and one needs to study the effect of the nonlinear terms.
  - ↪ If $|f'(x^*, \mu)| = 0$, the fixed point is said to be **super-stable**.
- For the logistic map, simply maths show that
  - ↪ $x^* = 0$ is stable for $0 \leq \mu < 1$ and unstable otherwise.
  - ↪ $x^* = {}^{\mu-1}/_{\mu}$ is stable for $1 \leq \mu \leq 3$ and unstable for $3 \geq \mu$.
  - ↪ $x^* = {}^{\mu-1}/_{\mu}$ is super-stable for $\mu = 2$ (and thus $x^* = {}^{1}/_{2}$).



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Logistic map

Analysis : Stable and super-stable fixed points

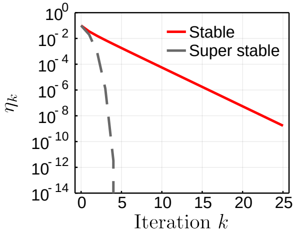
- For a regular fixed point, we have seen that

$$\begin{aligned} \eta_{k+1} &= \lambda \eta_k + \mathcal{O}(\eta_k^2) \\ &\simeq \lambda^k \eta_0. \end{aligned}$$

- For a superstable fixed point, $\lambda = 0$ and thus one can no longer neglect the quadratic term. Hence

$$\eta_{k+1} \sim \eta_0^{2^k}.$$

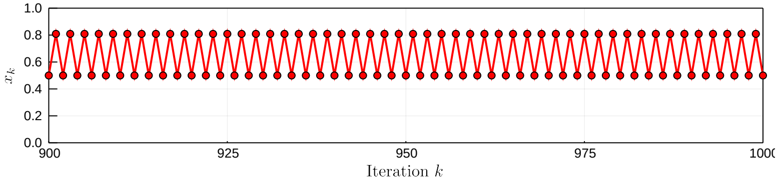
- Perturbations in the vicinity of a superstable fixed point decay much faster than they would in the vicinity of a regular fixed point.

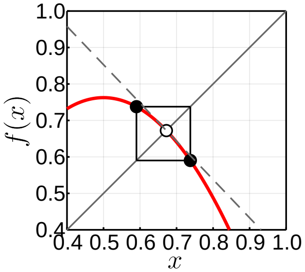


Logistic map

Analysis : Creation of a 2-cycle

- At $\mu = 3$, $x^* = {}^{\mu-1}/_{\mu}$ looses its stability and $f'(x^*, \mu) = -1$.
  - ↪ This is known as a **flip bifurcation** (or period-doubling).
  - ↪ It gives rise to a 2-cycle.





Logistic map

Analysis : Creation of a 2-cycle

- The equations for a 2-cycle are

$$
\begin{aligned}
p &= f(q) \\
q &= f(p).
\end{aligned}
$$

- Combining these two equations, p and q are thus fixed points of

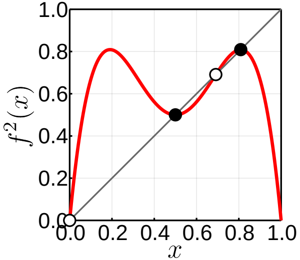
$$
f^2(x) - x = 0.
$$

- For the logistic equation, we thus have

$$
\mu^2 x \left(1 - x\right) \left(1 - \mu x \left(1 - x\right)\right) = 0.
$$

Factoring out x and $x - {}^{\mu-1}/_{\mu}$, we finally have

$$
p, q = \frac{\mu + 1 \pm \sqrt{\left(\mu - 3\right)\left(\mu + 1\right)}}{2\mu}.
$$



Logistic map

Stability of a 2-cycle

- p and q being fixed points of $x_{k+1} = f^2(x_k)$, their stability is dictated by

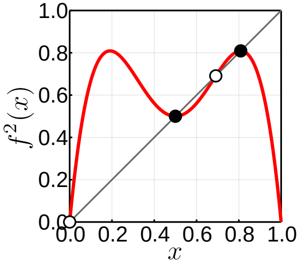
$$
\begin{aligned}
\lambda &= \left. \frac{\mathrm{d}f^2(x)}{\mathrm{d}x} \right|_{x=p} \\
&= f'(q)f'(p) \\
&= 4 + 2\mu - \mu^2.
\end{aligned}
$$

- This 2-cycle is thus stable for

$$
|4 + 2\mu - \mu^2| < 1,
$$

i.e. for $3 < \mu < 1 + \sqrt{6}$ $(= 3.449\ldots)$.

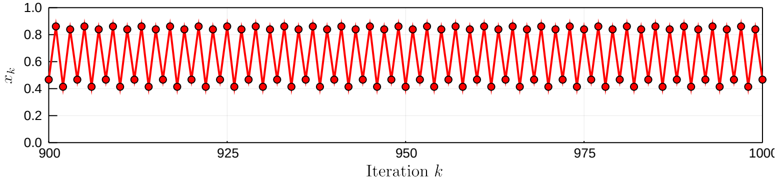
- It is superstable for $\mu = 1 + \sqrt{5}$ $(= 3.236\ldots)$.

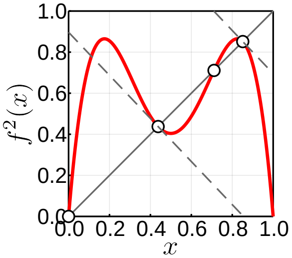


Logistic map

Analysis : Creation of a 4-cycle

- At $\mu = 1 + \sqrt{6}$, our 2-cycle looses its stability and $\lambda = -1$.
  - $\hookrightarrow$ As before, this is a period-doubling bifurcation.
  - $\hookrightarrow$ The system now exhibits a 4-cycle.





Logistic map

Analysis : Subharmonic cascade

- We can generalize to 2^n-cycles. Points belonging to such cycles are fixed points of

$$f^{2^n}(x) - x = 0.$$

- Their stability is dictated by the modulus of

$$\lambda_n = \left. \frac{\mathrm{d}f^{2^n}(x)}{\mathrm{d}x} \right|_{x=p}.$$

- The critical parameter μ_n for which $|\lambda_n| = 1$ needs to be evaluated numerically (for $n > 2$).

- The sequence $\{\mu_n\}_{n=1,\infty}$ seems to converge as $n \to \infty$ but to what value?

μ	$\mu_n - \mu_{n-1} / \mu_{n+1} - \mu_n$	Period
3		2
$1+\sqrt{6}$	4.751...	4
$3.54409\ldots$	4.657...	8
$3.564407\ldots$	4.659...	16
$3.568750\ldots$	4.620...	32
$\vdots$	$\vdots$	$\vdots$
$\mu_\infty = ?$	$\delta = ?$	∞

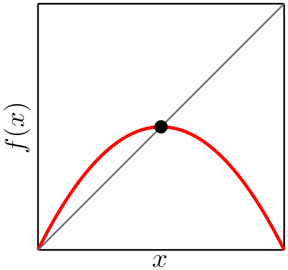
Logistic map

Analysis : Renormalization

- Let us try to explain graphically this convergence of the sequence of $\{\mu_n\}$ as $n \to \infty$.
- Compare the graph of $f(x, \mu_1)$ and $f^2(x, \mu_2)$ when $x = 0.5$ is a superstable fixed point.
- In the vicinity of the superstable fixed point, these two graphs are very similar so that we have

$$f(x, \mu_1) \simeq \alpha f^2(x/\alpha, \mu_2),$$

where $\alpha < 0$ (and $|\alpha| > 1$) is a scaling parameter.



Logistic map

Analysis : Renormalization

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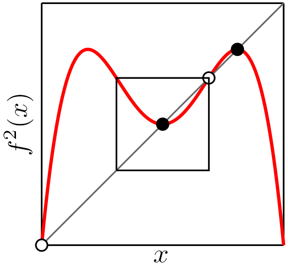


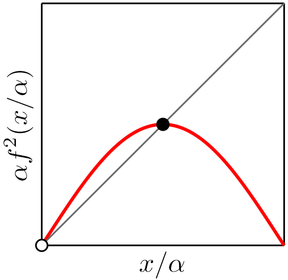
Logistic map

Analysis : Renormalization

- Let us try to explain graphically this convergence of the sequence of $\{\mu_n\}$ as $n \to \infty$.
- Compare the graph of $f(x, \mu_0)$ and $f^2(x, \mu_1)$ when $x = 0.5$ is a superstable fixed point.
- In the vicinity of the superstable fixed point, these two graphs are very similar so that we have

$$f(x, \mu_0) \simeq \alpha f^2({}^{x}/_{\alpha}, \mu_1),$$

where $\alpha < 0$ (and $|\alpha| > 1$) is a scaling parameter.



Logistic map

Analysis : Renormalization

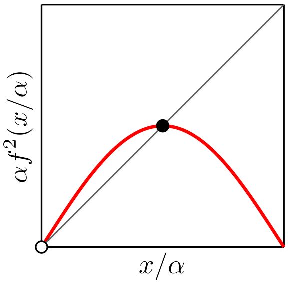
- $f(x, \mu_1)$ has been **renormalized** by taking its second iterate, rescaling $x \to {}^{x}/_{\alpha}$ and shifting μ to the next superstable value.

- We can keep this process going to renormalize $f^2(x, \mu_1)$ to $f^4(x, \mu_2)$ yielding

$$
\begin{aligned}
f^2(x, \mu_1) &\simeq \alpha f^4({}^{x}/_{\alpha^2}, \mu_2), \\
f(x, \mu_0) &\simeq \alpha^2 f^4({}^{x}/_{\alpha^2}, \mu_2).
\end{aligned}
$$

- After n iterations, we have

$$
f(x, \mu_0) \simeq \alpha^n f^{(2^n)}({}^{x}/_{\alpha^n}, \mu_n)
$$



Logistic map

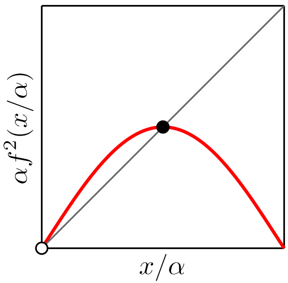
Analysis : Renormalization

- It has been shown by Feigenbaum (1978) that

$$\lim_{n\to\infty} \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, \mu_n\right) = g_0(x),$$

where $g_0(x)$ is a **universal function** with a superstable fixed point.

- This limit exists only when the scale parameter α is chosen appropriately, namely $\alpha = -2.5029\ldots$



Logistic map

Analysis : Renormalization

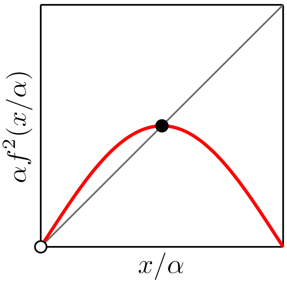
- To obtain the universal function $g_1(x)$ with a superstable 2-cycle, start the renormalization process with $f(x, \mu_1)$ instead of $f(x, \mu_0)$ so that

$$\lim_{n \to \infty} \alpha^n f^{(2^n)} \left(\frac{x}{\alpha^n}, \mu_{n+1} \right) = g_1(x).$$

- Of particular interest is the case where we start with $\mu = \mu_\infty$ (at the onset of chaos). When then have

$$f(x, \mu_\infty) \simeq \alpha^2 f^2 \left(\frac{x}{\alpha}, \mu_\infty \right).$$

Note that we longer need to shift μ to the next superstable value.



Logistic map

Analysis : Renormalization

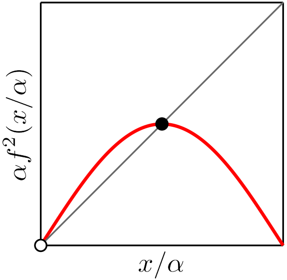
- The corresponding universal function $g(x)$ satisfies the **functional equation**

$$g(x) = \alpha g^2 \left(\frac{x}{\alpha} \right)$$

with appropriate boundary conditions (e.g. $g(0) = 1$, $g'(0) = 0$ and $g''(0) < 0$).

- The scaling parameter α is then given by

$$\alpha = \frac{1}{g(g(0))} = \frac{1}{g(1)}.$$



Logistic map

Analysis : Renormalization

- No closed-form solutions have been found so far and we thus need to approximate $g(x)$ using power series for instance, i.e.

$$g(x) \simeq \sum_{k=0}^{n} a_i x^{2k}$$

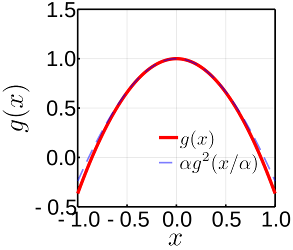
for n sufficiently large.

- Feigenbaum used a seven-term expansion and found

$$c_2 \simeq -1.5276, \quad c_4 \simeq 0.1048$$

along with $\alpha = -2.5029$.

- The theory also explain the geometric factor δ but it requires more sophisticated mathematics which are beyond the scope of this course.



Logistic map

Exercise : Renormalization

- Let us approximate $g(x)$ as

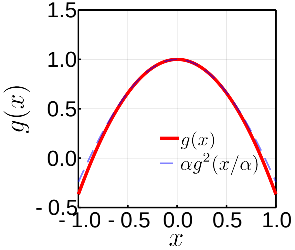
$$g(x) \simeq 1 + a_1 x^2.$$

- Neglecting the quartic term, our functional equation becomes

$$1 + a_1 x^2 = \alpha(1 + a_1) + \frac{2a_1^2}{\alpha} x^2.$$

- We thus have

$$1 = \alpha(1 + a_1) \quad \text{and} \quad a_1 = \frac{\alpha}{2}.$$



Logistic map

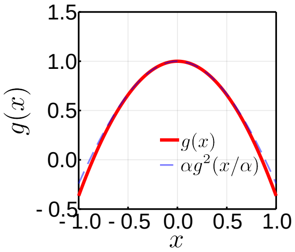
Exercise : Renormalization

- Finally, using the condition $g''(0) < 0$, we obtain

$$a_1 = \frac{-1-\sqrt{3}}{2} \quad \text{and} \quad \alpha = -1-\sqrt{3} \ (\simeq -2.732\ldots)$$

- Despite our two-term expansion of $g(x)$, our approximate value of the scale parameter α differs only by 9% from its exact value.

- Let us now look at this problem once more, but considering the bifurcation points rather than the superstable points.



Logistic map

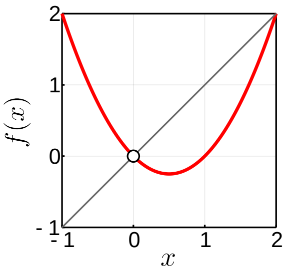
Exercise : Renormalization

- Let us consider the system

$$x_{k+1} = f(x_k, \epsilon)$$

with $f(x, \epsilon) = -(1+\epsilon)\, x_k + x_k^2$.

- This system corresponds to a rescaling of the logistic map.
- In what follows, we will compute approximation of α and δ, illustrating some of the basic ideas of renormalization.



Logistic map

Exercise : Renormalization

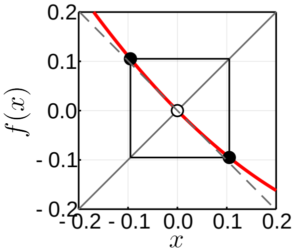
- At $\epsilon = 0$, a flip bifurcation occurs. The fixed point $x = 0$ becomes unstable and a 2-cycle is created.

- Let us denote by p and q the two points of this cycle such that

$$p = -(1+\epsilon)q + q^2 \quad \text{and} \quad q = -(1+\epsilon)p + p^2.$$

- These points are thus fixed points of $f^2(x) - x$ and are given by

$$p = \frac{\epsilon + \sqrt{\epsilon^2 + 4\epsilon}}{2} \quad \text{and} \quad q = \frac{\epsilon - \sqrt{\epsilon^2 + 4\epsilon}}{2}.$$

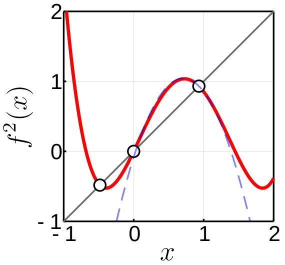


Logistic map

Exercise : Renormalization

- The dynamics of a perturbation η in the vicinity of p are governed by
$$p + \eta_{k+1} = f^2(p + \eta_k, \epsilon).$$
- Using a Taylor expansion, we have
$$\eta_{k+1} = \left(1 - 4\epsilon - \epsilon^2\right)\eta_k + C\eta_k^2 + \cdots,$$
where $C = {}^{\mathrm{d}^2 f(f(x))}/_{\mathrm{d}x^2}$ evaluated at $x = p$.
- Introducing the scaled variable $\tilde{x} = C\eta$, we obtain
$$\tilde{x}_{k+1} = \left(1 - 4\epsilon - \epsilon^2\right)\tilde{x}_k + \tilde{x}_k^2 + \cdots$$

Note that C plays the same rescaling role as α!



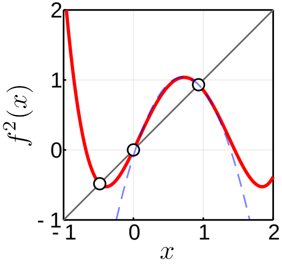
Logistic map

Exercise : Renormalization

- Finally, introducing the renormalized parameter $\tilde{\epsilon} = \epsilon^2 + 4\epsilon - 2$, our system reads

$$\tilde{x}_{k+1} = -\left(1 + \tilde{\epsilon}\right)\tilde{x}_k + \tilde{x}_k^2 + \cdots$$

- We thus recover a scaled version of our original system. It has been renormalized.
- When $\tilde{\epsilon} = 0$, the renormalized map undergoes a flip bifurcation.
  - ↪ the 2-cycle of the original map loses stability and creates a 4-cycle.
- We can then repeat this process *ad infinitum*.



Logistic map

Exercise : Renormalization

- It can easily be shown that ϵ_k satisfies

$$\epsilon_{k-1} = \epsilon_k^2 + 4\epsilon_k - 2,$$

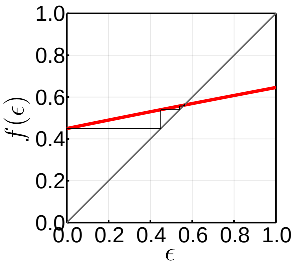
or equivalently

$$\epsilon_k = -2 + \sqrt{6 + \epsilon_{k-1}}.$$

- You can now easily show that

$$\begin{aligned}\epsilon^* &= \lim_{k\to\infty} \epsilon_k \\ &= \frac{1}{2}\left(-3 + \sqrt{17}\right) \\ &\simeq 0.56\end{aligned}$$

- At $\epsilon = \epsilon^*$, a cycle of infinite period emerge. This is the onset of chaos.



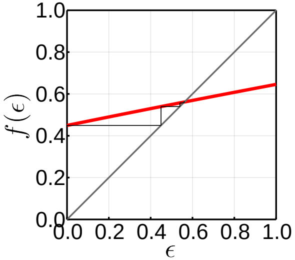
Logistic map

Exercise : Renormalization

- Recalling that $\epsilon = 0 \iff \mu = 3$ for the logistic map (i.e. creation of the 2-cycle), we thus predict $\mu_\infty = 3.56$ while numerical computations show $\mu_\infty = 3.5699\ldots$

- For $k \gg 1$, the sequence $\{\epsilon_k\}$ converges toward ϵ^* at a rate given by the Feigenbaum constant δ, hence

$$\begin{aligned}
\delta &= \lim_{k\to\infty} \frac{\epsilon_{k-1} - \epsilon^*}{\epsilon_k - \epsilon^*} \\
&= \left.\frac{\mathrm{d}\epsilon_{k-1}}{\mathrm{d}\epsilon_k}\right|_{\epsilon=\epsilon^*} \\
&= 1 + \sqrt{17} \simeq 5.12.
\end{aligned}$$

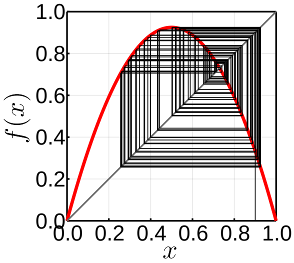


- Our estimate is only 10% larger than the true value of δ.

Logistic map

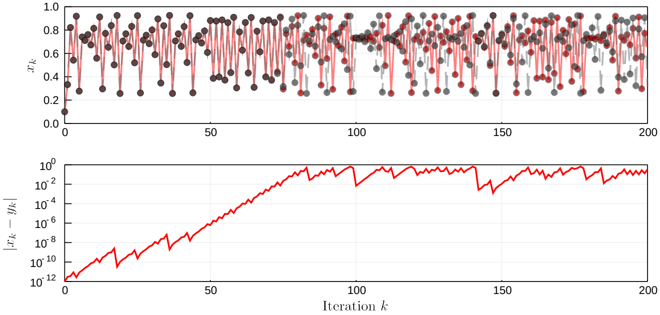
Into the chaos

- At $\epsilon = \epsilon^*$, a cycle of infinite period emerge. This is the onset of chaos.
- Although renormalization theory was helpful to determine the onset of chaos, it provides no information about how to characterize chaotic dynamics (for $\mu > \mu_\infty$).
- We need to introduce new mathematical concepts!



Logistic map

Analysis : Lyapunov exponent



- In the chaotic regime, the system exhibits **sensitive dependence on the initial condition**.
- Two nearby trajectories eventually diverge exponentially fast.
- The rate at which they diverge is given by the so-called **Lyapunov exponent**.

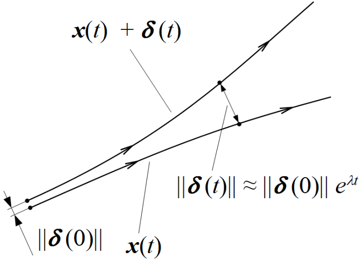
Logistic map

Analysis : Lyapunov exponent

- Consider two initial points given by x_0 and $x_0 + \delta_0$. After n iterations, we have $|\delta_n| \simeq |\delta_0| e^{\lambda n}$ where λ is the Lyapunov exponent.
  - $\hookrightarrow$ $\lambda > 0$ is the signature of chaos.
- A more precise and computational useful formula can be derived by noting that $\delta_n = f^n(x_0 + \delta_0) - f^n(x_0)$. Then

$$
\begin{aligned}
\lambda &= \frac{1}{n} \log \left| \frac{\delta_n}{\delta_0} \right| \\
&= \frac{1}{n} \log \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \\
&= \frac{1}{n} \log \left| \left(f^n\right)'(x_0) \right|.
\end{aligned}
$$

when $\delta_0 \to 0$.



Logistic map

Analysis : Lyapunov exponent

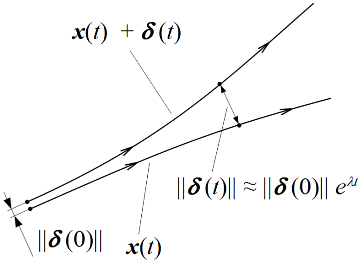
- Noting that $(f^n)'(x_0) = \prod_{i=0}^{n-1} f'(x_i)$, we finally have

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|.$$

- When $n \to \infty$, we define the limit to be the **Lyapunov exponent** for the orbit starting at x_0

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|.$$

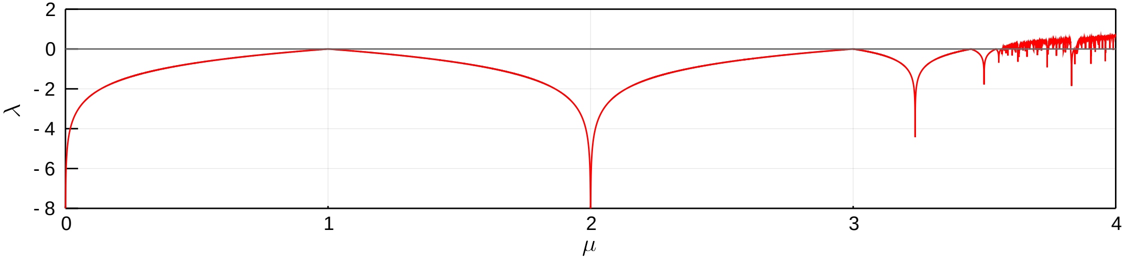
- Although λ appears to depend on x_0, it is actually the same for x_0 in the basin of attraction of a given attractor.



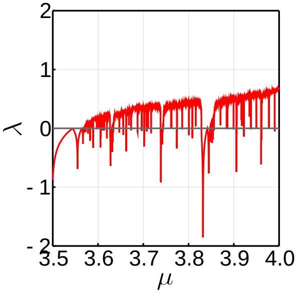
Logistic map

Analysis : Lyapunov exponent

- When the system is stable, $\lambda < 0$. At each bifurcation point, we have $\lambda = 0$, while $\lambda > 0$ once the system is chaotic.



- Zooming in the range $3.5 \leq \lambda \leq 4$, small windows of non-chaotic regimes can be seen.
- These correspond to so-called **periodic windows** over which the system exhibits $3 \cdot 2^n$-cycles, $5 \cdot 2^n$-cycles, etc.
- In what follows, we will look more closely at what happens in the range $3.8 \leq \lambda \leq 3.9$ during which the system exhibits such $3 \cdot 2^n$-cycles.



Logistic map

Analysis : Periodic windows

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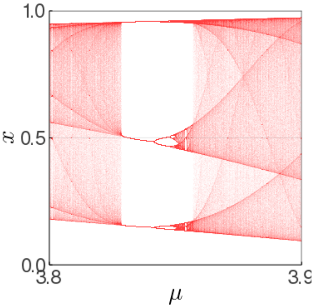
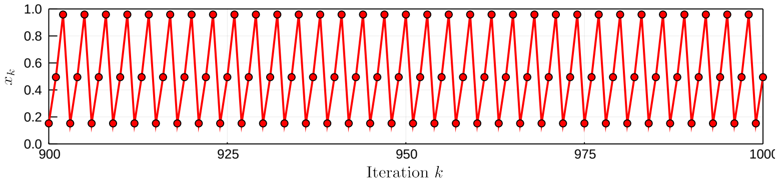
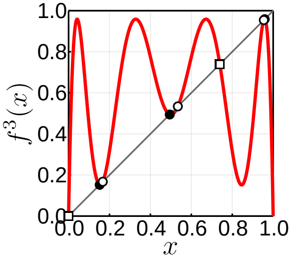


Logistic map

Analysis : $3 \cdot 2^n$-cycles

- At $\lambda = 1 + \sqrt{8}$, a **tangent bifurcation** (or saddle-node bifurcation) occurs, giving rise to a 3-cycle.
- Further increasing λ, a new subharmonic cascade happens giving rise to a sequence of $3 \cdot 2^n$-cycles.

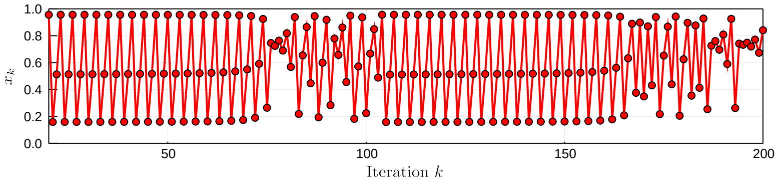


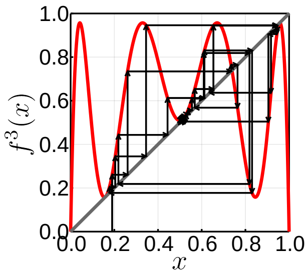


Logistic map

Analysis : Intermittency

- For λ slightly smaller than $1+\sqrt{8}$, the system exhibits almost periodic dynamics separated by random bursts of chaos.
  - $\hookrightarrow$ The system sees the "ghost" of the 3-cycle.
- It is better understood by looking at the cobweb and zooming in the vicinity of the ghost of one of the fixed points of $f^3(x) - x$.





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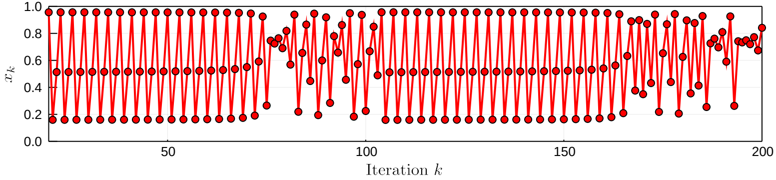
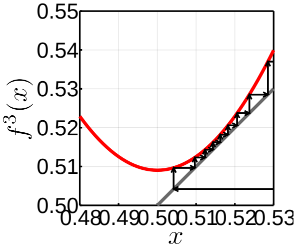




Type-I intermittency according to Pomeau-Manneville.

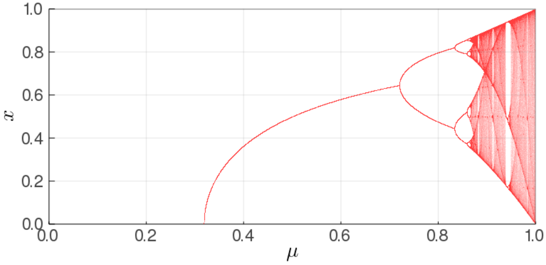
Universality of the subharmonic cascade

Discrete system : Sine map

- The sine map is another one-dimensional unimodal map defined as

$$x_{k+1} = \mu \sin\left(\pi x_k\right).$$

- Its orbit diagram is suprisingly similar to that of the logistic map.
  - ↪ Can be explained by the renormalization theory.



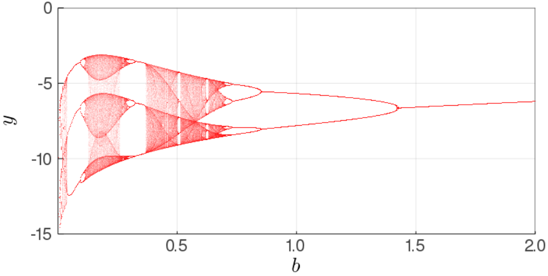
Universality of the subharmonic cascade

Continuous system : Rössler system

- The Rössler system is a continuous-time three-dimensional system given by

$$\begin{aligned}\dot{x} &= -y - z\\ \dot{y} &= x + ay\\ \dot{z} &= b + z\,(x - c)\,.\end{aligned}$$

- Tts orbit diagram (for $a = 0.2$ and $c = 5.7$) is also similar to that of the logistic map.
- The mechanisms causing the emergence of chaos in the logistic map also exist here.
  - $\hookrightarrow$ This will be the subject of another course.



Thank you for your attention.

Any questions?