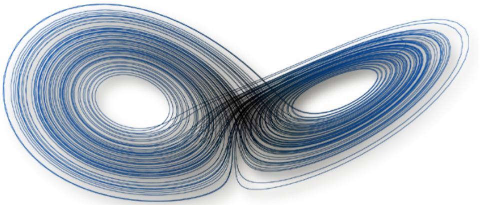

Nonlinear physics, dynamical systems and chaos theory

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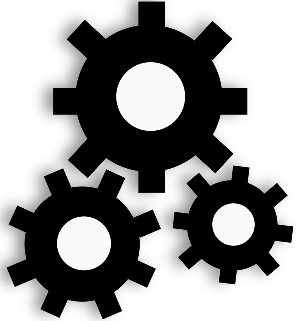
Basic information

Organization

- Lectures will take place every Tuesdays and Thurdays, from 3:30pm to 5:30pm until late February.

- Evaluation will be divided in two parts:
  - ↪ A two-hour long written exam late February.
  - ↪ A homework project involving mathematics and numerical simulations.

- Do not hesitate to go through your linear algebra notes during Christmas vacation to refresh a bit!



Basic information

Homework project

- Please use **Python 3** or **Julia**.
  - ↪ Open-source programming languages with excellent scientific computing capabilities.
- You can install both of them using **Anaconda**.
  - ↪ Available for Windows, Mac OS and Linux.
- Numerous online resources to get familiar with both languages if needed, e.g.
  - ↪ `https://www.codeacademy.com`

Basic information

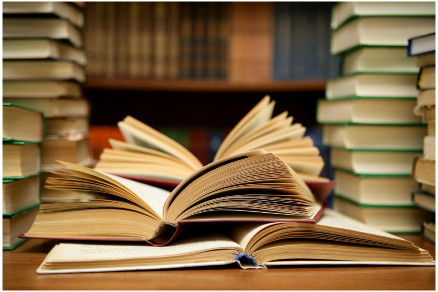
Useful references (in French)

General knowledge

- I. Stewart. *Dieu joue-t'il au dés?* Flammarion (2004).
- J. Gleick. *La théorie du chaos.* Flammarion (2008).
- I. Prigogine. *Les lois du chaos.* Flammarion (2008).

Textbooks

- P. Bergé *et al. L'ordre dans le chaos.* Hermann (1998).
- P. Manneville. *Instabilités, chaos et turbulence.* Ed. Ecole Polytechnique (2004).

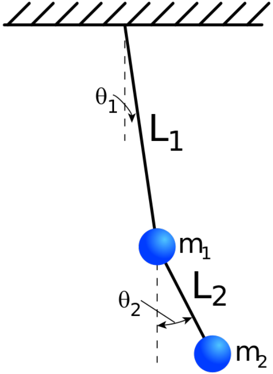


What is a dynamical system?

A few examples

A few examples

The double pendulum



- Its Lagrangian is

$$\mathcal{L} = \overbrace{\frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right)}^{\text{Kinetic energy}} \underbrace{-(m_1+m_2)g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2}_{\text{Potential energy}}.$$

- Equations of motions are given by

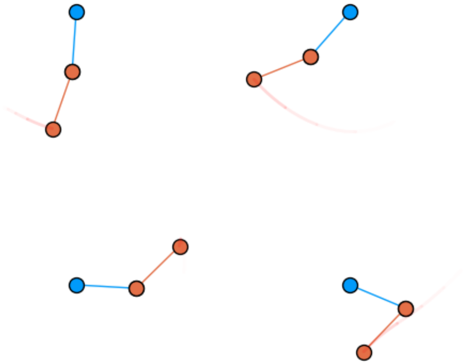
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0.$$

- Nonlinear system of ODEs.
  - $\hookrightarrow$ Very few analytical solutions are known.

A few examples

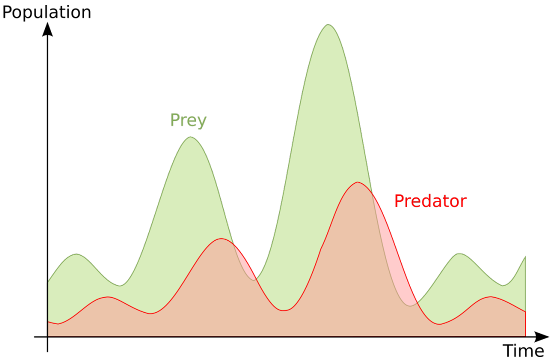
The double pendulum

- Simple mechanical sytem exhibiting nonetheless complex dynamics.
- Evolutions of similar initial conditions diverge exponentially fast.
  - ↪ Hallmark of chaotic dynamics.
- Limited prediction horizon despite its deterministic equations of motion.



A few examples

Prey-Predator system



- Dynamics of a prey-predator system can be modeled as

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \alpha x - \beta x y \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\gamma y + \delta x y.
\end{aligned}
$$

- Describes the observations of hares and lynx populations in Canada in the early 1900's.

A few examples

Chemical reaction-diffusion systems

- Spatio-temporal reaction-diffusion systems can be modeled as
$$\frac{\partial \boldsymbol{q}}{\partial t} = \boldsymbol{D}\nabla^2 \boldsymbol{q} + \mathcal{R}(\boldsymbol{q}),$$
where $\boldsymbol{D}$ describes the diffusion of each species and $\boldsymbol{R}(\boldsymbol{q})$ the inter-species reactions.

- Set of nonlinear partial differential equations exhibiting suprising physical phenomena!
  - $\hookrightarrow$ Traveling waves, pattern formation, spatiotemporal chaos, etc.

A few examples

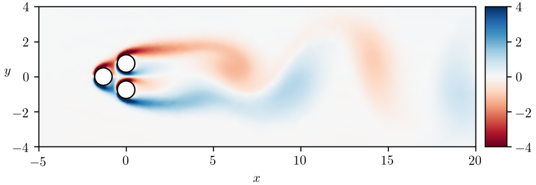
Incompressible flow past cylinders

- Dynamics are governed by the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

- Set of nonlinear partial differential equations.

- Give rise to an extremely high-dimensional nonlinear system once discretized!
  $\hookrightarrow$ 10^5 to 10^{10} degrees of freedom.



How do we study dynamical systems?

A quick overview of what's coming.

How do we study dynamical systems?

Mathematical frameworks

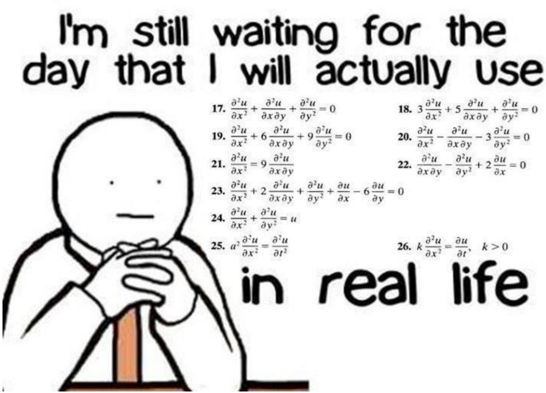
- All of the examples considered are governed by equations of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{\mu}\right),$$

  with $\boldsymbol{x}$ the state vector and $\boldsymbol{\mu}$ the vector of parameters.

- In general, the function $\boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{\mu}\right)$ is a *nonlinear* function of the state.

- Skimming through your lecture notes on *ordinary differential equations* might be a good idea.



What is a linear system?

Let us consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Under which condition(s) is it **linear**?

What is a linear system?

A few definitions

- Consider $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ two solutions of our system.
- A system is said to be **linear** if:
  - ↪ $\boldsymbol{w}(t) = \alpha \boldsymbol{u}(t) + \beta \boldsymbol{v}(t)$ is also solution,
  - ↪ $\boldsymbol{f}\left(\alpha \boldsymbol{u} + \beta \boldsymbol{v}\right) = \alpha \boldsymbol{f}(\boldsymbol{u}) + \beta \boldsymbol{f}(\boldsymbol{v})$,
  - ↪ It satisfies the **superposition principle**.
- If so, it can be rewritten as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x},$$

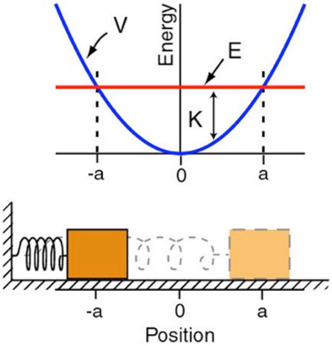
  where $\boldsymbol{A}$ is a linear operator (i.e. a matrix).

- The general solution is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

What is a linear system?

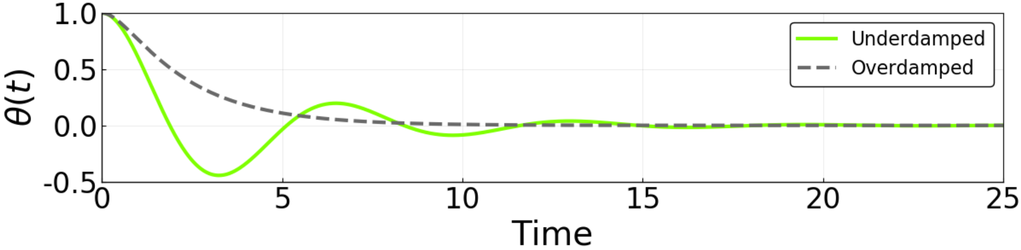
Example: the harmonic oscillator



- Dynamics of a (damped) harmonic oscillator are governed by
$$\ddot{x} = -2k\dot{x} - \omega_0^2 x.$$
- Introducing $y = \dot{x}$, we can recast it as
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$
- Depending on the friction parameter k, the system can be overdamped, under-damped or criticaly damped.
  - $\hookrightarrow$ Each regime corresponds to different dynamics.

What is alinear system?

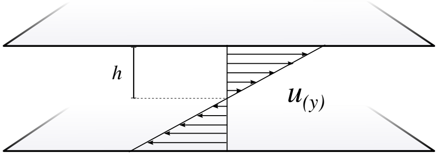
Example: the harmonic oscillator



What is a linear system?

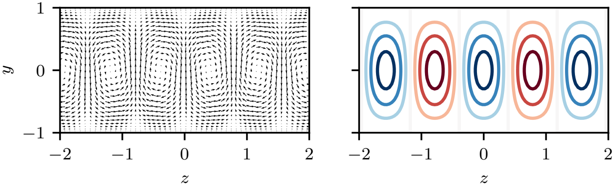
Example: the Orr-Sommerfeld-Squire equations

- OSS equations are a reformulation of the linearized Navier-Stokes equations.
  - ↪ See your hydrodynamic instability class for more details.



- In matrix form, they read

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}\end{bmatrix} = \underbrace{\begin{bmatrix}\boldsymbol{L}_{OS} & 0\\ \boldsymbol{C} & \boldsymbol{L}_S\end{bmatrix}}_{\mathcal{L}}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}\end{bmatrix},$$



with $\boldsymbol{v}$ the wall normal velocity and $\boldsymbol{\eta}$ the vorticity.

- Describe the evolution of infinitesimal perturbations in parallel shear flows.

Warning!

If linear systems have appeared so frequently during the course of your studies, it is solely because they are easy to study! Most linear systems are actually only an approximation of a more complex (but more realistic) nonlinear system.

What is a nonlinear system?

Let us consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Under which condition(s) is it **nonlinear**?

What is a nonlinear system?

Example: photon emission in laser

- Photon emission in a laser can be modeled by

$$\dot{n} = gn(N_0 - an) - kn$$

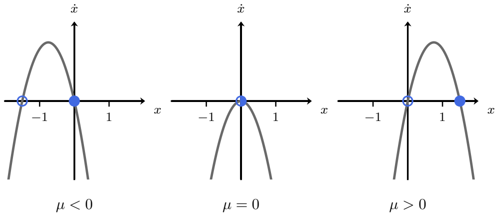
  where g is the gain of the laser, k describe the loss and $N(t) = N_0 - an$ is the number of excited atoms.

- It can be recast as

$$\dot{x} = \mu x - x^2.$$

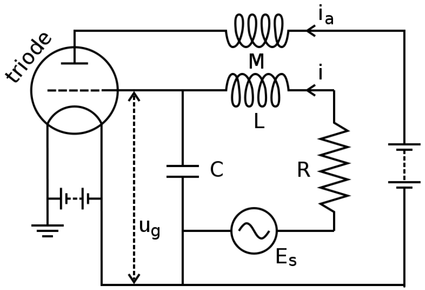
- It is a first-order nonlinear ordinary differential equations.

- We do not have a closed-form solution despite the apparent simplicity of the equation.



What is a nonlinear system?

Example: the van der Pol oscillator



- The equations governing this eletrical circuit are

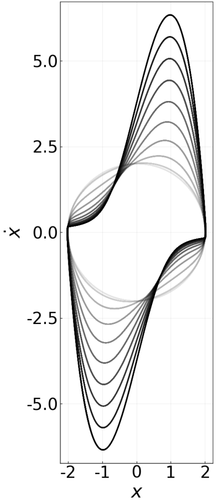
$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0.$$

- It is known as the *Van der Pol oscillator*.
  - ↪ It is a non-conservative oscillator with nonlinear damping.
- It serves to model a large variety of physical systems.
  - ↪ Action potentials of neurons, interaction between two plates in a geological fault, ...

What is a nonlinear system?

Example: the van der Pol oscillator

- Standard model to study the properties of simple nonlinear oscillators.
- The shape of the *limit cycle* in the $(x, \dot{x})$ plane varies drastically as μ increases.
- When forcing the system at certain frequencies, the dynamics become chaotic.



Case study I

The simple pendulum

The simple pendulum

Governing equations

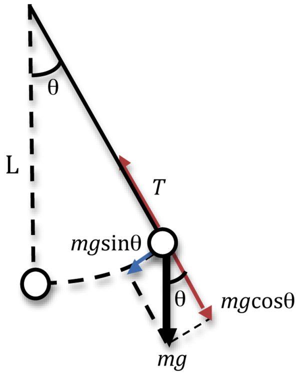
- The governing equations read

$$\ddot{\theta} + 2k\dot{\theta} + \omega_0^2 \sin(\theta) = 0,$$

with k the friction coefficient and $\omega_0^2 = {}^{g}/_{L}$ the natural frequency of the pendulum.

- Introducing $x = \theta$ and $y = \dot{\theta}$, we can recast it as a system of first-order ODEs

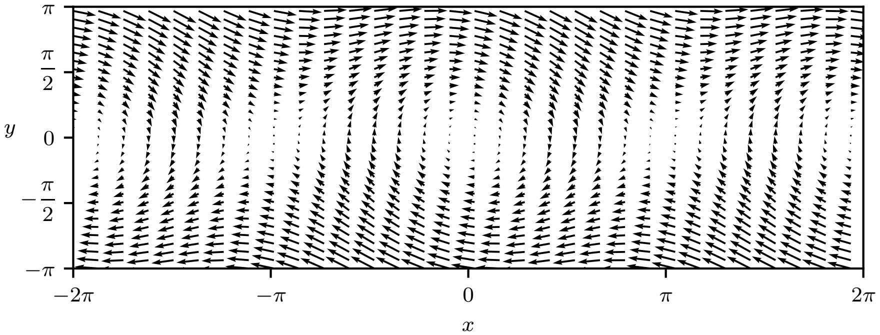
$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -2ky - \omega_0^2 \sin(x).\end{aligned}$$



- Let's study in details the properties of this dynamical system.

The simple pendulum

Phase plane representation



The simple pendulum

Fixed points

- There are a few points of the phase plane where the system is in equilibrium. These are known as **fixed points**. They satisfy the equations

$$\dot{x} = 0 \text{ and } \dot{y} = 0.$$

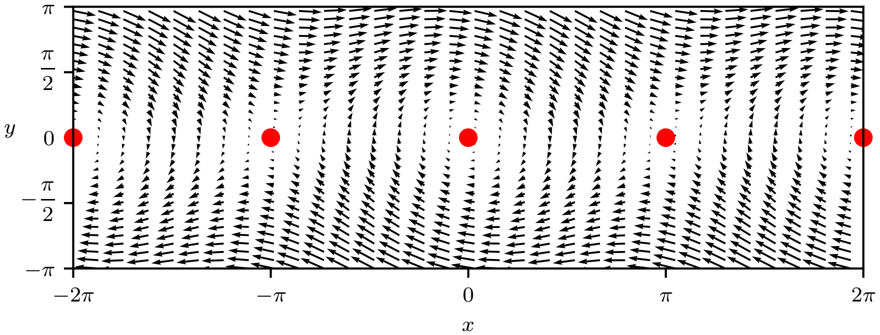
- For the simple pendulum, these fixed points are solution to

$$y = 0 \text{ and } \sin(x) = 0.$$

- Physically, they correspond to situation where the pendulum is pointing either downward ($x = 0 \mod 2\pi$) or upward ($x = \pi \mod 2\pi$).

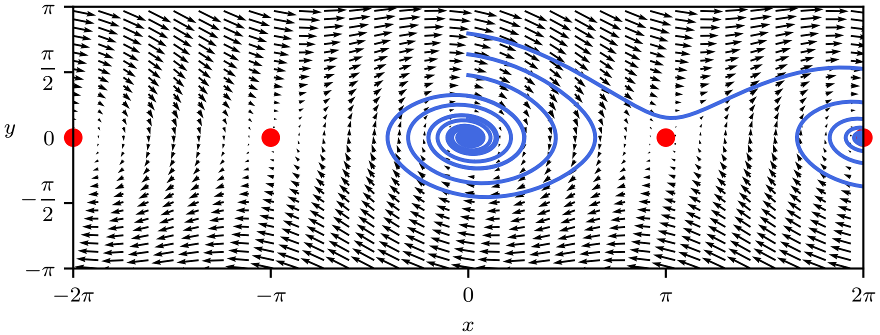
The simple pendulum

Fixed points



The simple pendulum

Trajectories



The simple pendulum

Linearizing the system in the downward position

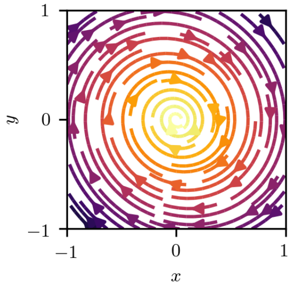
- Let us linearize the equations in the vicinity of the downward position. The equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2k \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- In this small oscillation limit, we recover the damped harmonic oscillator model.
- For all (positive) values of the friction parameter k, the eigenvalues λ of $\boldsymbol{A}$ are characterized by

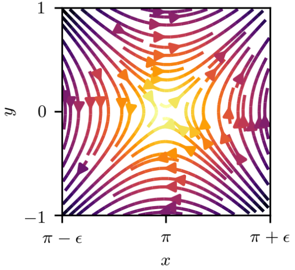
$$\Re(\lambda) < 0$$

indicating that this equilibrium position is **linearly stable**.



The simple pendulum

Linearizing the system in the upward position



- Let us linearize the equations in the vicinity of the upward position. The equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 2k \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- For all (positive) values of the friction parameter k, the eigenvalues λ of $\boldsymbol{A}$ are characterized by

$$\Re(\lambda) > 0$$

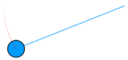
indicating that this equilibrium position is **linearly unstable**.

The simple pendulum

Conclusion

- The simple pendulum by itself is a fairly boring system.
  - ↪ Two fixed points, one linearly stable and one linearly unstable.
  - ↪ For $t \to \infty$, the system eventually settles in the stable fixed point due to the dissipative nature of the system.

- Its brother, the double pendulum, has far richer dynamics!
  - ↪ Harder to study though because of its four-dimensional phase space.
  - ↪ In the Hamiltonian limit (i.e. no friction), special numerical techniques need to be used to verify the physics.



Case study II

The Lorenz system

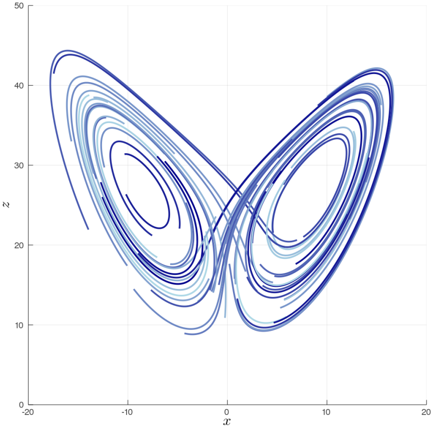
The Lorenz system

A simplified model of atmospheric convection

- It is a (very) simplified model of atmospheric convection which can be derived analytically from the Navier-Stokes equations.
- In its most common form, it reads

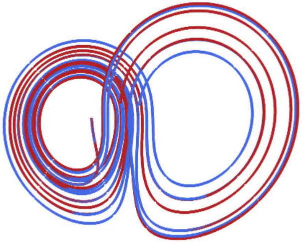
$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z.\end{aligned}$$

- In what follows, we set $\sigma = 10$ and $\beta = 8/3$ and study the evolution of the system as ρ varies.



The Lorenz system

A (popular) chaotic system



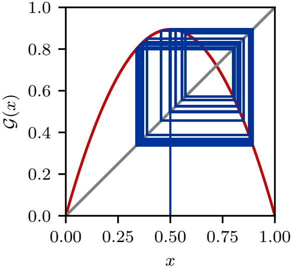
- For certain range of parameters, the Lorenz system exhibits chaotic dynamics.
- The corresponding *attractor* is known as a *strange attractor*.
  - ↪ It cannot be described by standard geometry.
  - ↪ One instead needs to use *fractal geometry* (its dimension is 2.06 for instance).
- Its double-winged structure is one of the reason why the sensitivity to initial condition is called the *butterfly effect* in mainstream media.

Case study III

The logistic map

The logistic map

A discrete-time nonlinear system



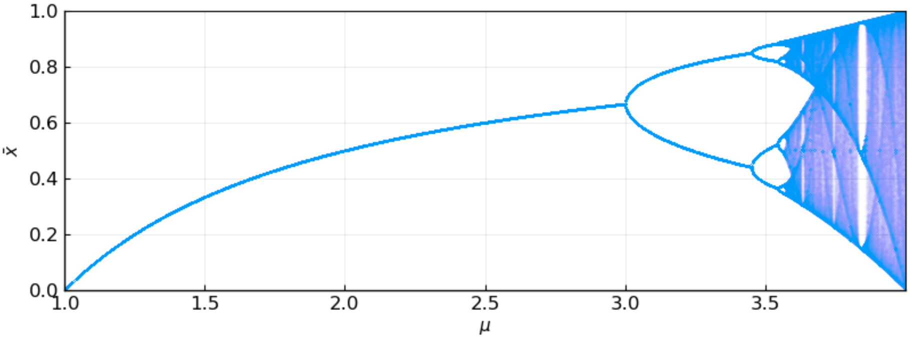
- Very simple model for population dynamics popularized during the 1970's by Robert May.
  - ↪ Played a crucial role in the development of chaos theory.
- The model reads

$$x_{k+1} = \mu x_k (1 - x_k).$$

- Despite its simplicity, the system can exhibit complex dynamics.
  - ↪ Fixed points, periodic orbits, chaotic dynamics.
- You are strongly encourage to implement this model yourself and play with it!
  - ↪ Only a handful lines of code are needed.

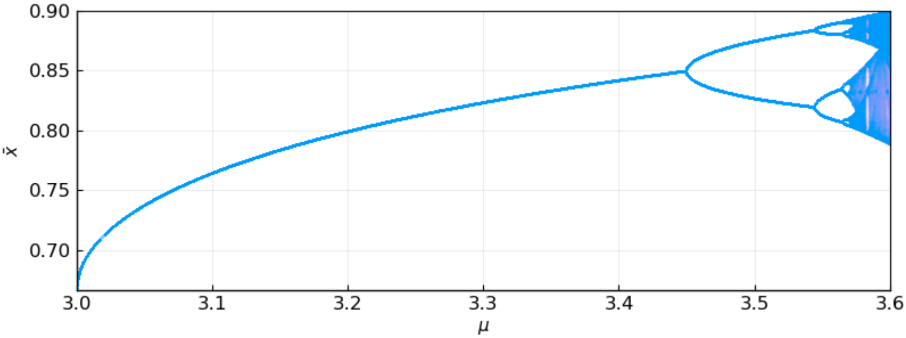
The logistic map

Bifurcation diagram



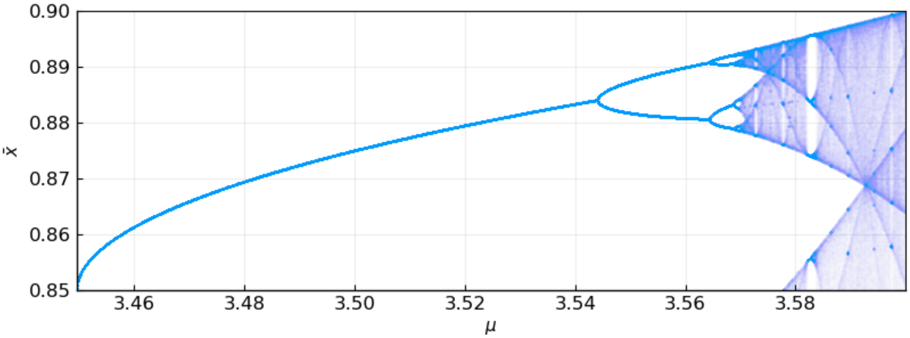
The logistic map

Bifurcation diagram



The logistic map

Bifurcation diagram



The logistic map

Connection with the Mandelbrot set

- The logistic map is closely related to the Mandelbrot set, a famous fractal object.
- Given the map $z_{k+1} = z_k^2 + c$ with $c \in \mathbb{C}$, the Mandelbrot set is defined as

$$c \in \mathcal{M} \iff \lim_{k \to \infty} |z_{k+1}| \leq 2.$$

- The last course of this class will be dedicated to a quick introduction to fractal geometry.