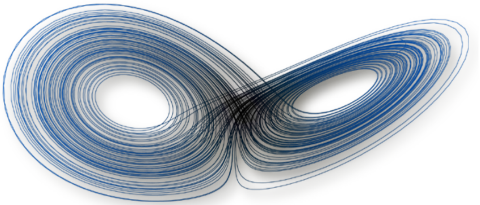


Weakly nonlinear oscillators

Jean-Christophe Loiseau

jean-christophe.loiseau@ensam.eu
Laboratoire DynFluid
Arts et Métiers, France.

Weakly nonlinear oscillators

Perturbation of the harmonic oscillator

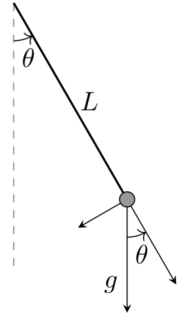
$$\boxed{\ddot{x} + x + \epsilon h(x, \dot{x}) = 0}$$

For $0 \leq \epsilon \ll 1$ and $h(x, \dot{x})$ an arbitrary smooth function, this is known as a **weakly nonlinear oscillator**. Because ϵ is small, they represent small perturbations of the **harmonic oscillator**.

Weakly nonlinear oscillators

Examples : the simple pendulum

Simple pendulum : $\ddot{\theta} + \sin(\theta)$



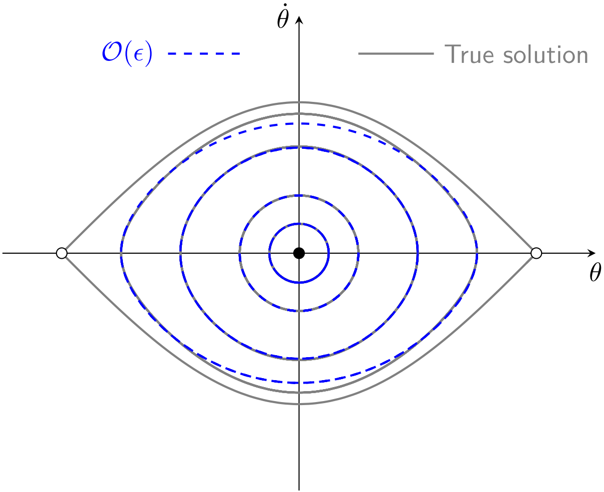
For small angles, $\sin$ can be replaced by its Taylor series. The equation of motion becomes

$$\ddot{\theta} + \theta - \frac{\theta^3}{6} = 0.$$

The **periodic dynamics** can be analyzed by means of the **Poincaré-Lindsted method**.

Weakly nonlinear oscillators

Examples : the simple pendulum

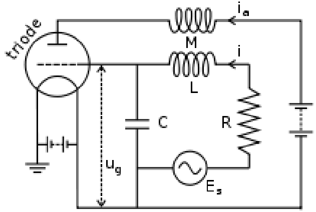


Weakly nonlinear oscillators

Examples : the van der Pol oscillator

$$\boxed{\textbf{van der Pol osc. :} \quad \ddot{x} + x + \epsilon\left(x^2 - 1\right)\dot{x} = 0}$$

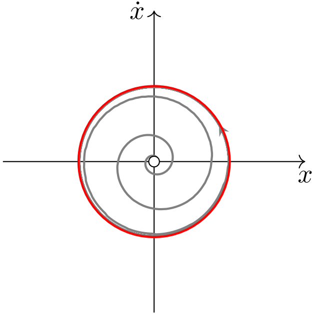
It is a canonical example of nonlinear oscillators proposed in 1927 by the Dutch electrical engineer Balthasar van der Pol.



Weakly nonlinear oscillators

From the fixed point to the limit cycle

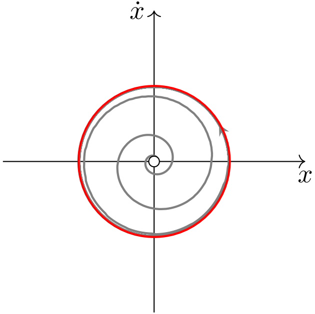
Periodic limit cycle can be approximated by means of the Poincaré-Lindstedt method, but what about the **transient behaviour**?



Weakly nonlinear oscillators

The failure of regular perturbation theory

Weakly nonlin. osc. : $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$



It is natural to seek an approximate solution in the form of a power series in ϵ, i.e.

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$$

This is known as **regular perturbation theory** but it is doomed to fail...

Weakly nonlinear oscillators

The failure of regular perturbation theory

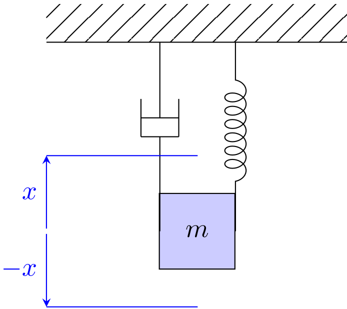
Consider $h(x, \dot{x}) = 2\dot{x}$. The system considered is thus the **weakly damped linear oscillator**

$$\ddot{x} + 2\epsilon\dot{x} + x = 0.$$

For $x(0) = 0$ and $\dot{x}(0) = 1$, its solution is

$$x(t, \epsilon) = \frac{1}{\sqrt{1-\epsilon^2}} \sin\left(\sqrt{1-\epsilon^2}\,t\right) e^{-\epsilon t}.$$

Let us now approximate this solution with regular perturbation theory.



Weakly nonlinear oscillators

The failure of regular perturbation theory

Introducing our power series expansion into the equation leads to

$$\frac{d^2}{dt^2}\left(x_0 + \epsilon x_1 + \cdots\right) + 2\epsilon \frac{d}{dt}\left(x_0 + \epsilon x_1 + \cdots\right) + \left(x_0 + \epsilon x_1 + \cdots\right) = 0.$$

Grouping terms by powers of ϵ, we get

$$\boxed{\begin{aligned} \mathcal{O}(1): &\quad \ddot{x}_0 + x_0 = 0 \\ \mathcal{O}(\epsilon): &\quad \ddot{x}_1 + 2\dot{x}_0 + x_1 = 0. \end{aligned}}$$

with initial conditions $x_0(0) = x_1(0) = 0$ along with $\dot{x}_0 = 1$ and $\dot{x}_1(0) = 0$.

Weakly nonlinear oscillators

The failure of regular perturbation theory

$$\boxed{\begin{aligned} \mathcal{O}(1): \quad & \ddot{x}_0 + x_0 = 0 \\ & x_0(0) = 0, \quad \dot{x}_0(0) = 1. \end{aligned}}$$

The zeroth-order solution is simply given by

$$x_0(t) = \sin(t).$$

In the absence of friction, the dynamics are simply periodic in time.

Weakly nonlinear oscillators

The failure of regular perturbation theory

$$
\boxed{\begin{aligned} \mathcal{O}(\epsilon): \quad & \ddot{x}_1 + x_1 = -2\cos(t) \\ & x_1(0) = 0, \quad \dot{x}_1(0) = 0. \end{aligned}}
$$

⚠ The right-hand side is a **resonant** forcing ! The first-order correction is given by

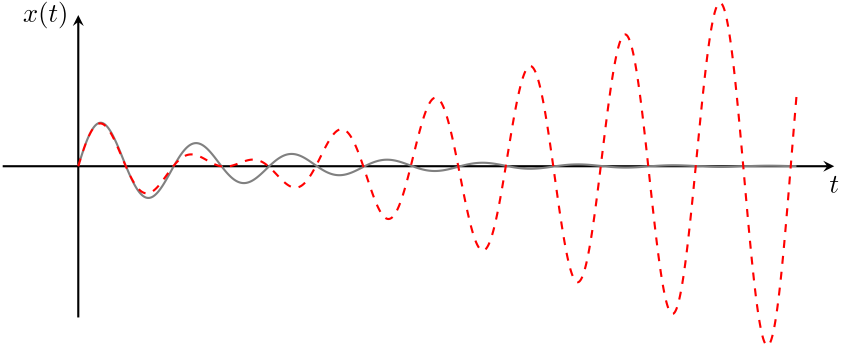
$$x_1(t) = -t\sin(t)$$

which is a **secular** term, i.e. it grows without bound as $t \to \infty$.

Weakly nonlinear oscillators

The failure of regular perturbation theory

Perturbation theory : $x(t,\epsilon) = \sin(t) - \epsilon t \sin(t) + \mathcal{O}(\epsilon^2)$



Weakly nonlinear oscilators

The failure of regular peturbation theory

Question 1 : How is the regular perturbation theory solution related to the true solution ?

Question 2 : Why does regular perturbation theory fail to capture the correct behaviour ?

Weakly nonlinear oscillators

Regular perturbation theory vs. true solution

Analytic solution : $x(t,\epsilon) = \dfrac{1}{\sqrt{1-\epsilon^2}} \sin(\sqrt{1-\epsilon^2}\,t) e^{-\epsilon t}$

Taylor expansion

$$\frac{1}{\sqrt{1-\epsilon^2}} \simeq 1 + \frac{1}{2}\epsilon^2, \quad \sin(\sqrt{1-\epsilon^2}\,t) \simeq \sin(t), \quad e^{-\epsilon t} \simeq 1 - \epsilon t$$

Weakly nonlinear oscillators

Regular perturbation theory vs. true solution

Analytic solution : $x(t,\epsilon) = \dfrac{1}{\sqrt{1-\epsilon^2}}\sin(\sqrt{1-\epsilon^2}\,t)e^{-\epsilon t}$

Hence, the first-order Taylor expansion of the analytic solution is $x(t,\epsilon) = \sin(t) - \epsilon t\sin(t) + \mathcal{O}(\epsilon^2)$.
This is precisely the solution obtained using regular perturbation theory !

Weakly nonlinear oscillators

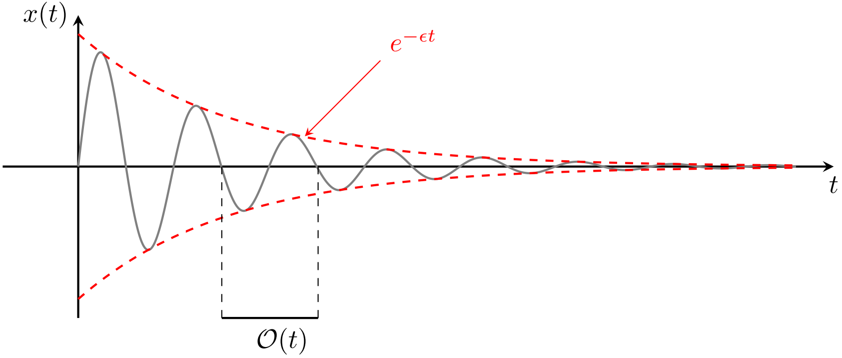
Why does it fail ?

Analytic solution : $x(t,\epsilon) = \dfrac{1}{\sqrt{1-\epsilon^2}} \sin(\sqrt{1-\epsilon^2}t)e^{-\epsilon t}$

The true solution exhibits **two time scales** : a *fast time* $t \sim \mathcal{O}(1)$ for the sinusoidal oscillations and a *slow time* $t \sim {}^{1}/_{\epsilon}$ over which the amplitude decays. This slow time scale is completely misrepresented by the regular perturbation theory solution.

Weakly nonlinear oscillators

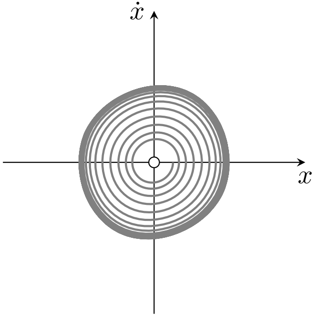
Two time scales



Weakly nonlinear oscillators

A two-time scales approach

Multiple time scales are ubquituous in nonlinear oscillators. The phase tends to evolves at a much faster rate than the oscillation's amplitude. We thus need to explicitly take into account these two time scales when approximating the solution.



Weakly nonlinear oscillators

Two-timing approach

Idea : Treat the two time scales as if they were independent.

Let $\tau = t$ denote the $\mathcal{O}(1)$ time scale and $T = \epsilon t$ the slow one. Functions of the slow time T will be regarded as *constants* on the fast time scale τ. Hard to justify rigorously but it works !

Weakly nonlinear oscillators

Two-timing approach

As before, let us expand our solution as a power series in ϵ

$$x(t,\epsilon) = x_0(\tau,T) + \epsilon x_1(\tau,T) + \mathcal{O}(\epsilon^2).$$

The time-derivatives in our governing equations are transformed using the chain rule.

$$\frac{dx}{dt} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T}$$

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial \tau^2} + 2\epsilon \frac{\partial^2 x}{\partial \tau \partial T} + \epsilon^2 \frac{\partial^2 x}{\partial T^2}$$

Weakly nonlinear oscillators

Two-timing approach

Introducing our power series expansion into the equation leads to

$$\frac{\partial^2 x_0}{\partial \tau^2} + \epsilon \left(\frac{\partial^2 x_1}{\partial \tau^2} + 2 \frac{\partial^2 x_0}{\partial \tau \partial T} \right) + 2\epsilon \frac{\partial x_0}{\partial \tau} + x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2) = 0.$$

Grouping by powers of ϵ yields

$$\boxed{\begin{aligned} \mathcal{O}(1): &\quad \frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0 \\ \mathcal{O}(\epsilon): &\quad \frac{\partial^2 x_1}{\partial \tau^2} + 2\frac{\partial^2 x_0}{\partial T \partial \tau} + 2\frac{\partial x_0}{\partial \tau} + x_1 = 0 \end{aligned}}$$

along with appropriate initial conditions.

Weakly nonlinear oscilllators

Two-timing approach

$$\boxed{\mathcal{O}(1): \quad \frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0}$$

The general solution of the zeroth-order approximation is simply

$$x_0(\tau, T) = A(T)\sin(\tau) + B(T)\cos(\tau).$$

To determine the functions $A(T)$ and $B(T)$, one needs to go to the next order in ϵ.

Weakly nonlinear oscillators

Two-timing approach

$$\boxed{\mathcal{O}(\epsilon): \quad \frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\left(\frac{dA}{dT} + A\right)\cos(\tau) + 2\left(\frac{dB}{dT} + B\right)\sin(\tau)}$$

The right-hand side is a **resonant** forcing that will lead to **secular growth** unless A and B satisfy

$$\frac{dA}{dT} = -A, \quad \text{and} \quad \frac{dB}{dT} = -B$$

whose general solutions are given by $A(T) = A(0)e^{-T}$ and $B(T) = B(0)e^{-T}$.

Weakly nonlinear oscillators

Two-timing approach

Two-timing solution : $x(t) = \sin(t)e^{-\epsilon t}$