Nonlinear dynamical systems and data-driven modeling

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Introduction

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Introduction to dynamical systems

1. First-order systems

1.1 Dynamical systems on the real number line

1.1.1 List of problems

Problem 1.1 Consider the simple resistor-capacitor circuit shown in figure 1.1. The equation governing the evolution of the voltage $Q(t)$ accross the capacitor is given by

$$\dot{Q} = -\frac{Q}{RC} + \frac{V_0}{RC}$$

where R is the resistance, C is the capacitance and $V_0 > 0$ the voltage of the power supply. For $t < 0$, the voltage accross the capacitor is assumed to be 0. At $t = 0$, the switch is closed and current starts to flow in the circuit.

a) Show graphically that the system has a single fixed point.
b) Show that the equation above can be recast as

$$\frac{dx}{d\tau} = 1 - x$$

if one introduces the rescaled variable $Q(t) = V_0 x(t)$ and time $t = RC\tau$.
c) Solve the equation analytically and show that

$$\lim_{\tau \to +\infty} x(\tau) = 1$$

and give the expression for $Q(t)$.

Problem 1.2 Consider the following population growth model

$$\dot{P} = \mu P \left(1 - \frac{P}{N}\right)$$

known as the logistic equation (also sometimes called the Verhulst model after Pierre François Verhulst who proposed it during the 1840's). It can be used to model the growth of a population in

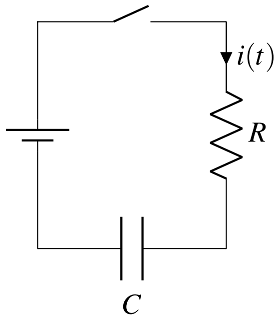


Figure 1.1: RC circuit for Problem 1.1

an environment with limited food supply such as fish in a small lake. In that context, $P(t)$ denotes the current population size, $\mu > 0$ its reproduction rate and N is the *carrying capacity*, i.e. the maximum population size that can be sustained in that specific environment.

a) Introducing the rescaled population size $x(t)$ such that $P(t) = Nx(t)$, show that the governing equation for $x(t)$ is

$$\dot{x} = \mu x (1 - x) .$$

b) Sketch the phase line of the system assuming $\mu > 0$. How does the population evolves if $x(0)$ is only slightly larger than 0 ? Same question for $x(0)$ much larger than unity.
c) This logistic model belongs to the class of *Bernoulli equations* which can be solved analytically. Introducing the change of variable $y = 1/x$, show that the equation for $y(t)$ is linear.
d) Solve the equation for $y(t)$ and thus for $x(t)$. Are the predictions of your solution for $x(0) \simeq 0$ and $x(0) \gg 1$ consistent with your intuition gained from the phase line ?

1.2 Elementary bifurcations

1.2.1 List of problems

Problem 1.3 Consider the following system

$$\dot{x} = -x \left(x^2 - 2x - \mu \right)$$

where $x \in \mathbb{R}$ is the state of the system and $\mu \in \mathbb{R}$ is our control parameter.

a) Compute all the branches of solutions and discuss their existence and stability properties.
b) Show that a saddle-node bifurcation occurs at $\mu = -1$ and a transcritical one at $\mu = 0$.
c) Sketch the bifurcation diagram of the system.

Problem 1.4 Consider a simple pendulum of mass m and length L in the gravitational field of earth (g) and driven by a constant torque Γ. Starting from Newton's principles, the equation of motion reads

$$mL^2 \ddot{\theta} + b\dot{\theta} + mgL \sin(\theta) = \Gamma.$$

Although it is a second-order equation (i.e. it involves a second derivative with respect to time), we'll see that under certain assumptions about the system it can be approximated by a first-order equation. Our aim will be to determine the type of bifurcation experienced by the system as the applied torque Γ is varied.

a) Introducing a time scale τ such that $t \mapsto \tau t$, show that we have the following choices for τ :

$$\tau = \sqrt{\frac{L}{g}} \quad \text{and} \quad \tau = \frac{b}{mgL}$$

and discuss their physical meaning.

b) If we are in the over-damped situation, the equation of motion can be reduced to

$$\dot{\theta} = \gamma - \sin(\theta)$$

where $\gamma = {}^{\Gamma}/_{mgL}$. Sketch the phase line of the system for different values of the control parameter γ.

c) As $\gamma \to 1$ from above, a meta-stable fixed point is created at $\theta = {}^{\pi}/_{2}$. Introducing $\mu = \gamma - 1$ and using a second-order Taylor expansion of the equation in the vicinity of this fixed point, show that the dynamics of a small perturbation can be approximated by

$$\dot{\eta} = \mu + \frac{1}{2}\eta^2.$$

What type of bifurcation is this ?

d) For $|\gamma| < 1$, the system admits two equilibrium solutions. Discuss their stability properties and their physical interpretation.

Problem 1.5 Consider the following mechanical system : an inverted pendulum made of a massless rigid rod is pivoted about its lower end with a torsion spring providing a restoring torque proportional to the angular displacement from the vertical equilibrium. A load, in the form of an attached mass, is applied vertically at the top of the rod. Starting from Newton's principle, the equation of motion is

$$I\ddot{\theta} + \beta\dot{\theta} + k\theta - PL\sin(\theta) = 0$$

where I the moment of inertia of the system, k is the torsional spring constant, P is the applied load, L is the length of the rod and β is the damping coefficient. Our goal will be to determine when the system bifurcates and what type of bifurcation it experiences as the applied load P is varied.

a) Show that the equation of motion is invariant with respect to the transformation $\theta \mapsto -\theta$. What does this invariance tells you about the properties of the system ?

b) As before, we'll consider the over-damped situation (i.e. friction dominates). Once time has been rescaled using $\tau = {}^{\beta}/_{PL}$, the equation of motion reduces to

$$\dot{\theta} + \frac{k}{PL}\theta - \sin(\theta) = 0.$$

Using a third-order Taylor expansion of $\sin(\theta)$ around $\theta = 0$, show that the system experiences a bifurcation for ${}^{k}/_{PL} = 1$. Sketch the bifurcation diagram. What type of bifurcation is this ?

c) In a realistic system, one may have some imperfections breaking the symmetry. Redo the same analysis for the perturbed system

$$I\ddot{\theta} + \beta\dot{\theta} + k\theta - PL(h + \sin(\theta)) = 0$$

where h models a small imbalance in the applied load. You can assume that $h > 0$. How does the bifurcation change ?

Problem 1.6 Let consider once more the evolution of a fish population model by the following logistic growth equation

$$\dot{P} = rP\left(1 - \frac{P}{N}\right)$$

where $P(t) \in \mathbb{R}$ is the fish population, r is the growth rate due to reproduction and N is the carrying capacity of the environment. In this exercise, we'll investigate the influence of different harvesting strategies on the long time evolution of the fish population. Two different strategies will be considered.

Constant harvesting

The first strategy considered is constant harvesting. In this strategy, fish are harvested at a constant rate irrespective of the current population level. If we denote by H this rate, our model becomes

$$\dot{P} = rP\left(1 - \frac{P}{N}\right) - H$$

supplemented with the initial condition $P(0) = P_0$.

a) The model above has three parameters : the reproduction rate r, the carrying capacity N and the harvesting rate H. Using a suitable rescaling, show that one of the parameter can be eliminated so that our model can be recast as

$$\dot{x} = rx(1-x) - h.$$

b) Assuming r is a positive constant, compute the fixed points of the model and study their linear stability properties.
c) Sketch the bifurcation diagram. What type of bifurcation is this ? From a biological point of view, what happens to the fish population when h is too large ?

Proportional harvesting

In this second strategy, fish are harvested at a rate proportional to the current population level. Our population model then becomes

$$\dot{P} = rP\left(1 - \frac{P}{N}\right) - HP.$$

a) As for the previous model, show that the current one can be written as

$$\dot{x} = rx(1-x) - hx$$

using a suitable rescaling.
b) Assuming r is a positive constant, compute the fixed points of the model and study their linear stability properties.
c) Sketch the bifurcation diagram. What type of bifurcation is this ? From a biological point of view, what happens to the fish population when h is too large ? How is this different from the previous strategy ?

2. Second-order systems

2.1 Linear systems

2.1.1 List of problems

Linear algebra

Problem 2.1 Given a matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$, show that its characteristic polynomial can be expressed as

$$\lambda^2 - \mathrm{tr}(\boldsymbol{A})\lambda + \det(\boldsymbol{A}) = 0$$

where $\mathrm{tr}(\boldsymbol{A})$ denotes the trace of the matrix and $\det(\boldsymbol{A})$ its determinant.

Problem 2.2 Show that, if $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a symmetric matrix (i.e. $\boldsymbol{A}^T = \boldsymbol{A}$), then all of its eigenvalues λ are real and its eigenvectors $\boldsymbol{v}$'s form an orthonormal basis for $\mathbb{R}^n$.

Problem 2.3 Show that if $\boldsymbol{A}$ is a symmetric negative definite matrix (i.e. $\boldsymbol{A}^T = \boldsymbol{A}$ and all of its eigenvalues are negative) then

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

implies

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{x}\|_2^2 = -\|\boldsymbol{R}\boldsymbol{x}\|_2^2$$

where $\boldsymbol{R}$ is the Cholesky factor of $-\boldsymbol{A}$ (i.e. $\boldsymbol{A} = -\boldsymbol{R}^T\boldsymbol{R}$ where $\boldsymbol{R}^T\boldsymbol{R}$ is a symmetric positive definite matrix). What does the equation above implies for the evolution of the energy in the system ?

Problem 2.4 Show that any matrix of the form

$$\boldsymbol{A} = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$$

with $a \neq 0$ has only a one-dimensional eigenspace corresponding to the eigenvalue λ. Then solve the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ and sketch the phase portrait.

Problem 2.5 Show that the eigenvalues of a skew-symmetric matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

are complex conjugate imaginary numbers.

Problem 2.6 Consider the linear dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ given by

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

a) Show that its eigenvalues are given by $\lambda = 1 \pm i$ and compute its eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.
b) The general solution is $\boldsymbol{x}(t) = \alpha e^{\lambda_1 t}\boldsymbol{v}_1 + \beta e^{\lambda_2 t}\boldsymbol{v}_2$. If $\boldsymbol{x}(0) \in \mathbb{R}^2$ then $\boldsymbol{x}(t) \in \mathbb{R}^2\ \forall t$. In such a case, express $\boldsymbol{x}(t)$ purely in terms of real-valued functions.
c) Which aspect of the dynamics do the real and imaginary parts of λ characterize ?

Harmonic oscillator (with and without damping)

Problem 2.7 Consider the canonical harmonic oscillator whose equation of motion is given by

$$\ddot{x} + \omega^2 x = 0.$$

a) Introducing a suitable potential $V(x) : \mathbb{R} \to \mathbb{R}$, show that the equation for the harmonic oscillator can be rewritten as

$$\ddot{x} + \frac{dV}{dx} = 0.$$

b) Show that the equation above implies

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + V(x)\right) = 0.$$

What is the meaning of this equation ? What does the quantity $H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$ represent from a physical point of view ?
c) Given the expression for the potential $V(x)$ found in the previous question, show that the trajectories of the harmonic oscillator in phase space correspond to ellipses of the form $\omega^2 x^2 + y^2 = C$ with $C > 0$.

Problem 2.8 Consider the simple mechanical system shown in figure 2.1. Starting from Newton's principles, the equation of motion is given by

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = 0$$

where x denotes the deviation from the equilibrium position, $k > 0$ is the friction coefficient and ω_0 is the natural frequency of the system in the absence of friction. In this exercise, we aim at classifying all of the possible dynamics exhibited by the system as we vary its parameters.

a) In its current form, our model depends explicitely on two parameters, namely the friction coefficient k and the natural frequency of oscillation ω_0. Using a suitable rescaling of time, show that the model can be rewritten as

$$\ddot{x} + 2\mu\dot{x} + x = 0$$

and give the expression of μ.

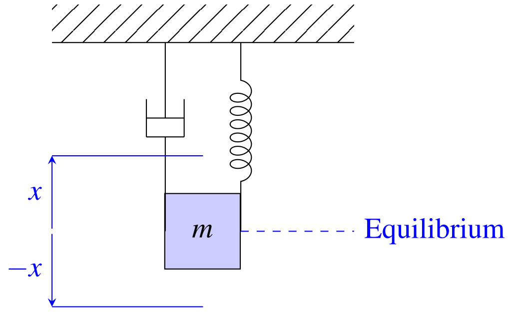


Figure 2.1: Mechanical system considered in Problem 2.2

b) Rewrite the model above as a two-dimensional linear system of the form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.
c) Compute the characteristic polynomial of the matrix $\boldsymbol{A}$ and classify the type of fixed point for all the possible cases. For all cases, sketch the phase portrait of the system.
d) How do these results relate to the standard notions of overdamped, critically damped and underdamped oscillations ?

Problem 2.9 Consider the electric circuit shown in figure 2.2. The Kirchoff's voltage law implies

$$V_R(t) + V_L(t) + V_C(t) = V_{\text{in}}(t)$$

where V_R, V_L and V_C are the voltages accross R, L, and C respectively and V_{in} is the (possibly time-varying) voltage from the source. Using Ohm's law, the voltage accross the resistor can be expressed as

$$V_R(t) = Ri(t).$$

Similarly, the voltage V_L accross the inductor is related to the intensity of the current by

$$V_L(t) = L\frac{di}{dt}.$$

Finally, the intensity $i(t)$ of the current is related to the voltage accross the capacitor by

$$i = C\frac{dV_C}{dt}.$$

Combining all of these consitutive equations together yields the following second-order differential equation

$$\frac{d^2V_C}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_C}{LC} = \frac{V_{\text{in}}}{LC}$$

where we'll assume that the source voltage V_{in} is constant.

a) Show that a natural time scale for the oscillatory behaviour of the system is given by $\tau = \sqrt{LC}$ while the damping due to the resitor occurs on a time scale $\tau = R/L$.
b) Show that the system admits a single fixed point given by $V_C = V_{\text{in}}$.
c) If $V_{\text{in}} = 0$, show that we have

$$\frac{d}{dt}\left(\frac{1}{2}Li^2 + \frac{1}{2}CV_C^2\right) = -Ri^2$$

where $Li^2/2$ is the magnetic energy in the coil and $CV_C^2/2$ is the electric energy in the condensator. From a physical point of view, what does the term $-Ri^2$ represent ?

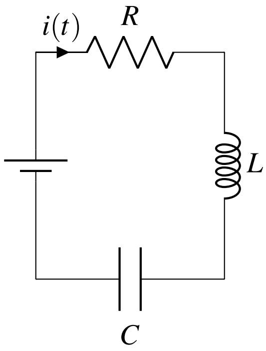


Figure 2.2: RLC circuit for Problem 2.2

2.2 Nonlinear systems

2.2.1 List of problems

Conservative systems

Problem 2.10 Consider the Duffing oscillator whose equation is given by

$$\ddot{x}+\varepsilon x+x^3=0$$

where ε can either be positive or negative.

a) Show that this system is conservative and give the expression of its Hamiltonian.
b) For $\varepsilon>0$, show that the system has a nonlinear center at its origin and sketch the phase portrait.
c) For $\varepsilon<0$, show that the system admits three fixed points and classify them. Sketch the phase portrait.

Problem 2.11 Consider the *Lotka-Volterra predator-prey model* given by

$$\dot{R}=aR-bRF$$
$$\dot{F}=-cF+dRF$$

where $R(t)$ denotes the population of rabbits and $F(t)$ that of foxes. We'll assume that all constants are positive as well as $R(t)$ and $F(t)$.

a) Discuss the biological meaning of all the terms in the model as well as its limitations.
b) Introducing a suitable rescaling of the variables R and F as well as time, show that the model can be recast as

$$\dot{x}=x-xy$$
$$\dot{y}=-\mu y+\mu xy.$$

c) Show that the system is conservative.
d) Compute the fixed points of the system and classify them.
e) Assuming that both $x(0)$ and $y(0)$ are non-zero, show that model predicts cycles in the populations of both species and sketch the phase portrait.
f) Sketch the evolution of $x(t)$ and $y(t)$ as a function of time. Explain from a biological point of view what you observe.

Dissipative systems

Problem 2.12 Consider once again the Duffing oscillator but let us now include the influence of friction. The equation becomes

$$\ddot{x} + \dot{x} - x + x^3 = 0$$

a) Show that the system is no longer conservative.
b) Recast the system as two first-order equations, compute the fixed points of the system and classify them.
c) In the vicinity of the origin, the stable manifold can be represented as $y = h(x)$. Assuming that $h(x) = ax + bx^2 + cx^3$, determine the coefficients of this third-order approximation.

Modelling a pandemic

Problem 2.13 The recent years have been marked by the COVID-19 pandemic. Since the begining of the pandemic, numerous models have been proposed to forecast its evolution. One of the simplest, dating back to the seminal work of Kermack and McKendrick in the late 1920's and early 1930's, is known as the SIR model. It is given by

$$\begin{aligned}
\frac{dS}{dt} &= -\beta \frac{SI}{N} \\
\frac{dI}{dt} &= \beta \frac{SI}{N} - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}$$

where N is the total population and S, I and R denote the number of suceptible (i.e. not yet infected), infected and removed (i.e. having recovered or died from the disease) persons in the population. The parameter β characterizes the probability for susceptible individuals to become infected whenever they meet an infectious person. The parameter γ models how fast an infected individual recovers (or dies) from the disease.

a) Show that $S(t) + I(t) + R(t)$ is a conserved quantity.
b) Introducing a suitable rescaling of the variables and time, show that the model can be recast as

$$\begin{aligned}
\dot{x} &= -R_0 xy \\
\dot{y} &= R_0 xy - y \\
\dot{z} &= y
\end{aligned}$$

where R_0 is known as the *basic reproduction number*. From an epidemiological point of view, what does this number represent ?
c) The equation for z being decoupled from the rest, the dynamics of x and y indeed forms a two-dimensional system. Find and classify all of the fixed points of the system.
d) Sketch the nullclines and the phase portrait of the system.
e) Let us assume that, as the disease first emerges, most of the population is susceptible (i.e. $x_0 \simeq 1$). An epidemic is said to occur if $y(t)$ initially increases. Under what condition on the basic reproduction number R_0 does an epidemic can occur ?
f) At the begining of a pandemic, $x(t) \simeq 1$. Show that the number of infected individuals initially grows exponentially fast.
g) If an outbreak happens, it is considered that *herd immunity* has been achieved once the number of infected individuals starts to decrease. What fraction of the population needs to have been infected at one point or another before such a decrease happens ?

h) At the begining of the COVID-19, the basic reproduction number was estimated to be approximately equal to 3. Given that France has a population of roughly 70 millions inhabitants, how many people would have had to contract the disease before herd immunity had been achieved ?

Problem 2.14 Several strategies have been proposed to mitigate the pandemic. Two of the most widely used strategies are social distancing and isolating infected individuals. A simple model for the influence of these two strategies is

$$\dot{x} = -(R_0 - a)xy$$
$$\dot{y} = (R_0 - a)xy - by$$

where $a \geq 0$ characterizes by how much we reduced our social interactions and $b \geq 1$ how fast infected individuals are effectively removed from the population.

a) Show that social distancing leads to a slower growth of the number of infected individuals at the begining of the pandemic.
b) Rescaling time, show that isolation effectively leads to a modified basic reproduction number $R_b < R_0$.

2.3 Oscillators

2.3.1 The Poincaré-Lindstedt method

2.3.2 Weakly nonlinear self-sustained oscillators

2.3.3 Strongly nonlinear oscillators and relaxation oscillations

2.3.4 List of problems

Poincaré-Lindstedt method

Problem 2.15 Consider the conservative Duffing oscillator whose governing equation is given by

$$\ddot{x} + x + \varepsilon x^3 = 0 \quad \text{with initial conditions} \quad x(0) = 1, \quad \dot{x}(0) = 0$$

where ε is a small real number. The properties of this sytem and its fixed points have already been studied qualitatively in the previous section. We now aim for a more quantitative analysis.

a) Introducing a strained time variable $\tau = \omega t$ with $\omega = (1 + \varepsilon\omega_1 + \cdots)$ and assuming that $x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \cdots$, compute the zeroth and first order approximation of the periodic solution.
b) Sketch the evolution of the frequencies in the system as a function of ε (consider both positive and negative values of ε).
c) Write down a computer program that simulate the system above for different values of ε. For each value, compare the computer-generated trajectories in phase plane with your Poincaré-Lindstedt approximation. Compare the frequencies extracted from your simulation using the Fourier transform with the results from your theoretical analysis.

Problem 2.16 When discussing the Poincaré-Lindstedt method for the van der Pol oscillator

$$\ddot{x} + \varepsilon\left(x^2 - 1\right)\dot{x} + x = 0$$

we have seen that, when ε is sufficiently small, the oscillation frequency was given by $\omega = 1 + O(\varepsilon^2)$. Assuming now that

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)$$

and

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3),$$

show that the frequency is actually given by $\omega = 1 - \dfrac{1}{16}\varepsilon^2 + O(\varepsilon^3)$.

Weakly nonlinear oscillators

Problem 2.17 — **A simple model for the Bénard-von Kàrmàn vortex street.** Consider the following third-order system

$$\begin{aligned}
\dot{x} &= \sigma x - y - xz - \alpha yz \\
\dot{y} &= x + \sigma y - yz + \alpha xz \\
\dot{z} &= -z + x^2 + y^2.
\end{aligned}$$

This model arises as a simplified representation of the Bénard-von Kàrmàn vortex street in the wake of a two-dimensional cylinder at low Reynolds numbers. Here, x and y represent the two-degrees of freedom needed to describe the oscillatory behaviour of the flow while z is known as the *shift-mode amplitude*. It describes how different the linearly unstable base flow is from the time-averaged mean flow. The term $x^2 + y^2$ in the equation for z models the influence of the fluctuation's Reynolds stresses on the mean flow.

a) Show that this dynamical system is equivariant with respect to the transformation

$$\gamma = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where γ represents a rotation around the z-axis.

b) Introducing the complex variable $\eta = x + iy = re^{i\phi}$, where r is the radius of oscillation and ϕ the phase, express the system in the cylindrical coordinates system (r, ϕ, z).

c) Compute the fixed points in the (r, z) plane. You should obtain $(r_0, z_0) = (0, 0)$ and $(r_1, z_1) = (\sqrt{\sigma}, \sigma)$. From a physical point of view, to what state does each of these fixed points correspond to ?

d) Show that the origin (r_0, z_0) is linearly unstable while the mean flow (r_1, z_1) is linearly stable.

e) The equation for the phase reads

$$\dot{\phi} = 1 + \alpha z.$$

How does the oscillation frequency $\omega(t) = 1 + \alpha z(t)$ evolves as the system transitions from its unstable equilibrium towards its limit cycle ?

f) Sketch a three-dimensional phase portrait in the coordinate system (x, y, z).

Problem 2.18 — **The Bénard-von Kàrmàn vortex street revisited.** Let us revisit the simple model for the Bénard-von Kàrmàn vortex street. Our goal will now be to derive an amplitude equation for the oscillation and the phase using a weakly nonlinear expansion in the vicinity of the fixed point $(x, y, z) = (0, 0, 0)$.

a) Introducing the complex variable $\eta = x + iy$, write down the model for η and z.

b) Let us introduce a slow time scale $\tau = \varepsilon^2 t$ with $\varepsilon^2 = \sigma$. Expanding the solution (η, z) in the vicinity of $(\eta_0, z_0) = (0, 0)$ as

$$\begin{aligned}
\eta(t, \tau) &= \varepsilon \eta_1(t, \tau) + \varepsilon^2 \eta_2(t, \tau) + \varepsilon^3 \eta_3(t, \tau) + O(\varepsilon^4) \\
z(t, \tau) &= \varepsilon z_1(t, \tau) + \varepsilon^2 z_2(t, \tau) + \varepsilon^3 z_3(t, \tau) + O(\varepsilon^4),
\end{aligned}$$

show that you obtain the following set of equations for each order

$$\begin{aligned}
O(\varepsilon): \quad & \frac{d\eta_1}{dt} = i\eta_1 \\
& \frac{dz_1}{dt} = -z_1 \\
O(\varepsilon^2): \quad & \frac{d\eta_2}{dt} = i\eta_2 - (1-i\alpha)\eta_1 z_1 \\
& \frac{dz_2}{dt} = -z_2 + |\eta_1|^2 \\
O(\varepsilon^3): \quad & \frac{d\eta_3}{dt} = i\eta_3 + \eta_1 - (1-i\alpha)(\eta_1 z_2 + \eta_2 z_1) - \frac{d\eta_1}{d\tau} \\
& \frac{dz_3}{dt} = -z_3 + \overline{\eta}_1\eta_2 + \eta_1\overline{\eta}_2 - \frac{dz_1}{d\tau}.
\end{aligned}$$

c) The solution for η_1 is $\eta_1(t,\tau) = A(\tau)e^{it}$. Show that $z_2(t,\tau) = |A(\tau)|^2(1-e^{-t})$ (assuming $z_2(0,0) = 0$).
d) Introducing these two expressions in the equation for η_3, show that invoking the Fredholm theorem to kill the resonant terms leads to

$$\frac{dA}{d\tau} = A - (1-i\alpha)|A|^2 A.$$

Expressing the complex amplitude as $A(\tau) = R(\tau)e^{i\phi(\tau)}$, show that we finally have

$$\begin{aligned}
\frac{dR}{d\tau} &= R - R^3 \\
\frac{d\phi}{d\tau} &= \alpha R^2.
\end{aligned}$$

e) Given that $\tau = \varepsilon^2 t = \sigma t$, show that the solution $\eta(t,\tau)$ to first order is

$$\eta(t,\tau) = \sqrt{\sigma}R(\tau)\exp(i\omega(\tau)t) + O(\varepsilon^2)$$

where $\omega(\tau) = 1 + \alpha\sigma R^2(\tau)$. Similarly, assuming that $z_1(0) = 0$, show that the solution for $z(t,\tau)$ up to second order is given by

$$z(t,\tau) = \sigma R^2(\tau)\left(1-e^{-t}\right) + O(\varepsilon^3).$$

Given that $R(\tau) \to 1$ when $\tau \to +\infty$, are your results regarding the amplitude and frequency of the oscillations and the amplitude of the distortion $z(t)$ consistent with the results from the previous exercise ?

Strongly nonlinear oscillators

2.4 Bifurcations revisited

2.4.1 List of problems

Problem 2.19 — **The Bénard-von Kàrmàn problem revisited (again).**

3. Resonance and synchronization

4. Chaotic dynamics

4.1 Driven oscillators

4.2 The Lorenz system

4.2.1 List of problems

The Lorenz system

Problem 4.1 — In-depth analysis of the Lorenz system. The Lorenz system reads

$$\begin{aligned}
\dot{x} &= \sigma(y-x)\\
\dot{y} &= x(\rho - z) - y\\
\dot{z} &= xy - \beta z
\end{aligned}$$

where σ, ρ and β are all positive parameters. In this exercise, you'll dive deep down the rabbit hole and explore this incredible system both from the theoretical and numerical point of view using all of the tools we've discussed so far.

Simple properties of the Lorenz system

a) Show that the Lorenz system is equivariant with respect to the transformation $(x,y,z) \mapsto (-x,-y,z)$. What does this properties tell you about the solutions of the system ?

b) Show that the z-axis is an invariant line, i.e. if $x(0) = y(0) = 0$ then $x(t) = y(t) = 0\ \forall t$.

c) Show that the Lorenz system is dissipative and that any volume V of initial conditions evolves according to

$$\dot{V} = -(\sigma + 1 + b)V.$$

Can the Lorenz system exhibit quasi-periodic dynamics ? Explain your reasonning from a geometrical point of view. Using a similar argument, explain why the Lorenz system cannot have repelling fixed points (i.e. unstable nodes) or repelling orbits.

d) Compute the fixed points of the Lorenz system and give their domain of existence. In analogy to the Rayleigh-Bénard convection problem, to what type of flow do these fixed points correspond to ?

The origin $(x,y,z) = (0,0,0)$

a) Given the Lyapunov function

$$V(x,y,z,t) = \frac{x^2}{\sigma} + y^2 + z^2 \geq 0,$$

show that the origin is the only attractor in the whole phase space (i.e. it is globally stable) for $0 \leq \rho \leq 1$. (Hint : Show that $\dot{V} < 0 \; \forall x,y,z$ and that $V = 0$ for $x = y = z = 0$)

b) Compute the Jacobian matrix of the system and show that the origin experiences a bifurcation at $\rho_c = 1$. Using a symmetry argument, what type of bifurcation could it be ?

c) Let us now start our analysis to determine precisely what type of bifurcation it is. For that purpose, let us write ρ as

$$\begin{aligned} \rho &= \rho_c + \varepsilon \\ &= 1 + \varepsilon. \end{aligned}$$

The equation for y then reads

$$\dot{y} = (1+\varepsilon)x - y - xz.$$

For $\varepsilon = 0$ (i.e. $\rho = \rho_c$), the matrix of eigenvectors $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \begin{bmatrix} 1 & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Introducing the change of variable $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{u}$, with $\boldsymbol{x} = (x,y,z)$ and $\boldsymbol{u} = (u,v,w)$, show that the Lorenz system can be recast as

$$\begin{aligned} \dot{u} &= \frac{\sigma}{1+\sigma}(\varepsilon - w)(u + \sigma v) \\ \dot{v} &= -(1+\sigma)v - \frac{1}{1+\sigma}(\varepsilon r - w)(u + \sigma v) \\ \dot{w} &= -\beta w + (u + \sigma v)(u - v). \end{aligned}$$

d) Let us now supplement the above system with

$$\dot{\varepsilon} = 0.$$

The center manifold W_c is given by

$$W_c = \{(u,v,w,\varepsilon) : v = h_1(u,\varepsilon), w = h_2(u,\varepsilon), h_i(0,0) = 0, Dh_i(0,0) = 0\}$$

where Dh_i denotes the Jacobian matrix of the nonlinear parametrization $h_i(u,\varepsilon)$. Assume now that

$$\begin{aligned} h_1(u,\varepsilon) &= a_{00} + a_{10}u + a_{01}\varepsilon + a_{20}u^2 + a_{11}u\varepsilon + a_{22}\varepsilon^2 + \cdots \\ h_2(u,\varepsilon) &= b_{00} + b_{10}u + b_{01}\varepsilon + b_{20}u^2 + b_{11}u\varepsilon + b_{22}\varepsilon^2 + \cdots \end{aligned}$$

show that the condition $h_i(0,0)$ implies

$$a_{00} = b_{00} = 0.$$

Similarly, show that $Dh_i(0,0) = 0$ implies

$$\begin{aligned} a_{10} &= a_{01} = 0 \\ b_{10} &= b_{01} = 0. \end{aligned}$$

e) Our parametrizations reduce to

$$h_1(u,\varepsilon) = a_{20}u^2 + a_{11}u\varepsilon + a_{22}\varepsilon^2$$
$$h_2(u,\varepsilon) = b_{20}u^2 + b_{11}u\varepsilon + b_{22}\varepsilon^2.$$

Using the fact that

$$\frac{dh_i}{dt} = \frac{\partial h_i}{\partial u}\dot{u} + \frac{\partial h_i}{\partial \varepsilon}\dot{\varepsilon},$$

show that the governing equation for u can be expressed as

$$\dot{u} = \mu u + \lambda u^3$$

where μ and λ are to be determined.

f) Given the expressions of μ and λ obtained at the previous question, conclude about the nature of the bifurcation happening at $\rho = 1$.

The fixed points C^+ and C^-

Let us now turn our attention to the two symmetric fixed points newly created hereafter denoted as

$$C^+ = (\sqrt{\beta(\rho-1)}, \sqrt{(\beta(\rho-1)}, \rho-1)$$
$$C^- = (-\sqrt{\beta(\rho-1)}, -\sqrt{(\beta(\rho-1)}, \rho-1).$$

a) Show that the characteristic polynomial of the Jacobian matrix at C^+, C^- is

$$\lambda^3 + (\sigma+\beta+1)\lambda^2 + (\rho+\sigma)\beta\lambda + 2\beta\sigma(\rho-1) = 0.$$

(Hint : Use the fact that, for a 3×3 matrix $\boldsymbol{A}$, its characteristic polynomial is given by $\lambda^3 - \operatorname{tr}(\boldsymbol{A})\lambda^2 - \frac{1}{2}\left(\operatorname{tr}(\boldsymbol{A})^2 - \operatorname{tr}(\boldsymbol{A}^2)\right)\lambda - \det(\boldsymbol{A}) = 0$.)

b) As ρ is increased, C^+ and C^- eventually experience a subcritical Hopf bifurcation. Assuming that $\lambda = \pm i\omega$, show that this bifurcation happens for

$$\rho_H = \sigma\frac{\sigma+\beta+3}{\sigma-\beta-1}.$$

In doing so, why do we need to assume $\sigma > \beta + 1$?

c) The bifurcation being subcritical, there exists no stable limit cycle in the direct vicinity of C^+ and C^-. As discussed in § **??**, the trajectories nonetheless remain bounded in phase space. Show that there exist a ellipsoidal region Ω of the form

$$\rho x^2 + \sigma y^2 + \sigma(z-2\rho)^2 \leq C$$

where C is finite positive number such that all trajectories of the Lorenz system eventually enter Ω and stay in there forever.

d) Given all of the results obtained so far, sketch the current status of the bifurcation diagram of the Lorenz system.

Chaotic dynamics

Let us now explore the chaotic dynamics of the Lorenz system numerically. For that purpose, we will hereafter consider the classical parameters

$$\rho = 28, \quad \sigma = 1, \quad \text{and} \quad \beta = \frac{8}{3}.$$

a) Using your favorite programming language, simulate the Lorenz system for this set of parameters and plot its phase portrait. You can discard any initial transients.

b) Having computed a sufficiently long time-series of $z(t)$, extract the successive maxima $z_0, z_1, z_2, \cdots$ and plot the Lorenz map $z_{k+1} = f(z_k)$ along with the bisectrice $z_{k+1} = z_k$.

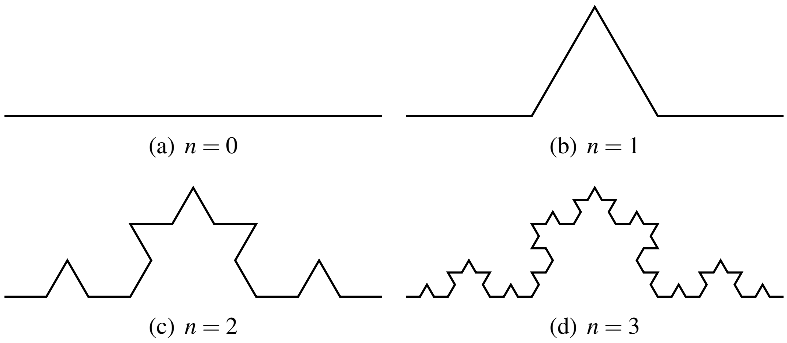


Figure 4.1: First four iterations for the construction of the Koch curve.

Other Lorenz-like systems

Problem 4.2 — **Laser model.** As discussed in [**??**] and [**??**], the Maxwell-Block equations for a laser are

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - p)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

where ...

a) Find a change of variables that transform this system of equations into the Lorenz system.
b) Show that the supercritical pitchfork happening for $\rho = 1$ in the Lorenz system corresponds to the non-lasing state ($E^* = 0$) in the Maxwell-Block loosing its stability.

Problem 4.3 — **The Malkus waterwheel.** As discussed in [**??**], the equations governing the dynamics of the Malkus waterwheel shown in figure **??** are given by

$$\dot{a} = \omega b - Ka$$
$$\dot{b} = -\omega a + q - Kb$$
$$\dot{\omega} = -\frac{\nu}{I}\omega + \frac{\pi g r}{I}a,$$

where ....

a) Find a change of variables transforming the waterwheel equations into the Lorenz system.
b) Show that, for the water wheel, $\beta = 1$ in the corresponding Lorenz system.
c) As shown when discussing the Lorenz system, a supercritical pitchfork bifurcation occurs at $\rho = 1$. To what kind of motion do the newly created fixed points correspond to for the waterwheel ?

4.3 Unimodal maps

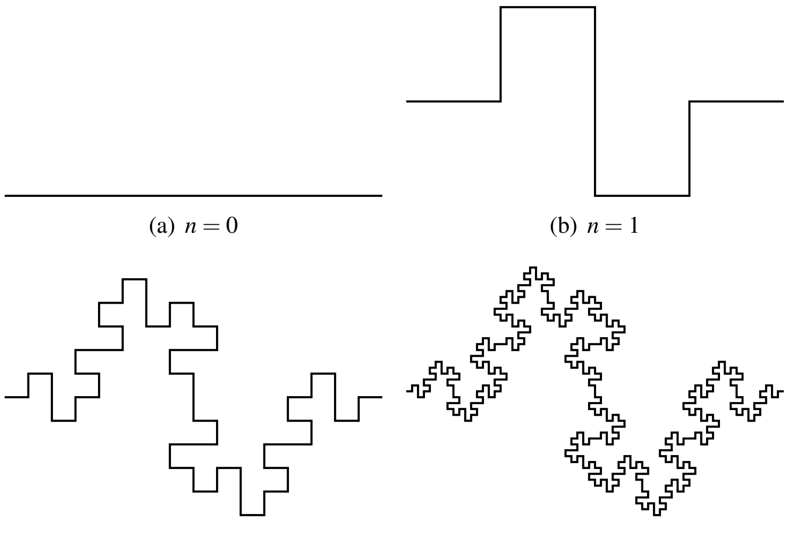
4.4 A primer in fractal geometry

4.4.1 List of problems

Self-similar fractals

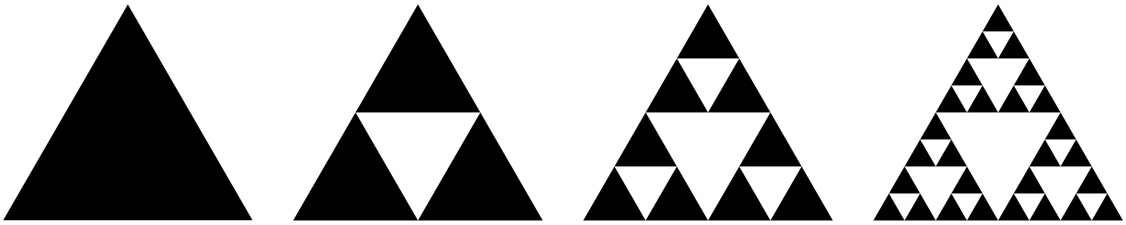
Problem 4.4 Consider the fractal object known as the Koch curve. The first four iterations to construct this curve are shown in figure 4.1. It can be generated by the same four affine transformations again and again.

a) Find all four affine transformations needed to construct this object and show that they are contractive (i.e. all of their eigenvalues are inside the unit circle).



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$ (d) $n = 3$

Figure 4.2: First four iterations for the construction of the $\frac{3}{2}$ curve.



(a) $n = 0$ (b) $n = 1$ (c) $n = 2$ (d) $n = 3$

Figure 4.3: First four iterations for the construction of the Sierpinsky triangle.

b) Show that its perimeter goes to infinity as the number of iterations $n \to \infty$.

Problem 4.5 Consider the curve shown in figute 4.2. This curve is simply known as the $\frac{3}{2}$ curve. Why is it call that way ?

Problem 4.6 Consider the fractal object known as the Sierpinsky triangle whose first four iterations are shown in figure 4.3. This fractal can be generated by applying repeatdly the same three affine transformations.

a) Find all three affine transformations needed to construct this figure and show that they are are contractive (i.e. all of their eigenvalues are inside the unit circle).
b) Using simple geometric arguments, show that the perimeter of the Sierpinsky triangle goes to infinity as $n \to \infty$ (where n is the number of iterations) while its area goes to zero.
c) What is the Haussdorf dimension of this fractal ?

The Mandelbrot set

Problem 4.7 — Logistic map and the Mandelbrot set. As discussed in § **??**, the logistic map

$$x_{k+1} = \mu x_k (1 - x_k), \quad \text{with} \quad \mu \in [0,4]$$

is actually the Mandelbrot set generating system $z_{k+1} = z_k^2 + c$ in disguise. Show that it is indeed the case. (Hint : use a simple linear transformation $z = ax + b$)

Problem 4.8 Prove by induction that any point $c \in \mathbb{C}$ such that $|c| > 2$ is not in the Mandelbrot set (i.e. the iteration $z_{k+1} = z_k^2 + c$ eventually diverges).

Problem 4.9 — Escape radius. Prove by induction that if $|z_k| > 2$ for any point $c \in \mathbb{C}$ such that $|c| < 2$, then c is not in the Mandelbrot set.

Problem 4.10 — Approximating π in a terribly inefficient way. π has the tendancy of appearing in surprising ways in numerous problems where, at first, it seems like it shouldn't. Let us consider $c = -\dfrac{3}{4} + i\varepsilon$ where ε is a small positive real number. Note that $c = -\dfrac{3}{4}$ is in the Mandelbrot set. It is thus expected that, as $\varepsilon \to 0$, an increasing number of steps is required for the iteration to reach the escape radius $|z_k| > 2$. Let us denote this number by $N(\varepsilon)$. Write down a small program that compute $N(\varepsilon)$ for $\varepsilon = 1, 0.1, 0.01, 0.001, \cdots$ and see what happens !

4.5 Back to strange attractors

5. Other types of dynamical systems

II Data-driven methods for dynamical systems

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