Strange Attractors

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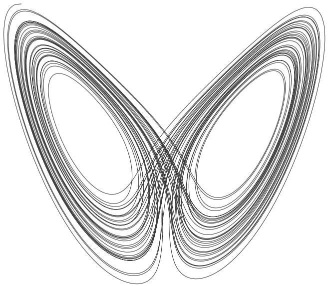
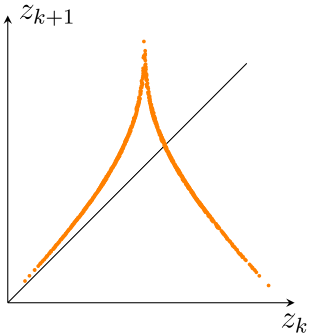
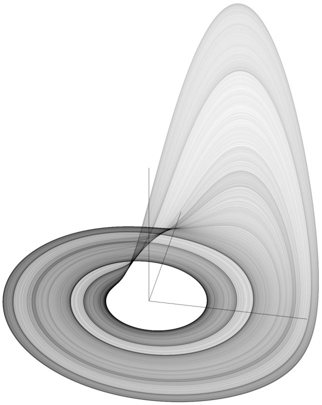
Arts & Métiers Institute of Technology, January 2022

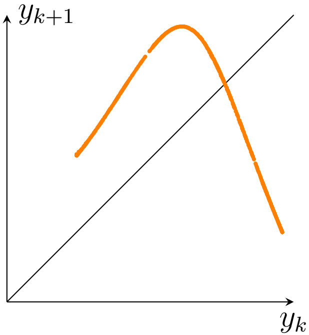
Lorenz system





Rössler system





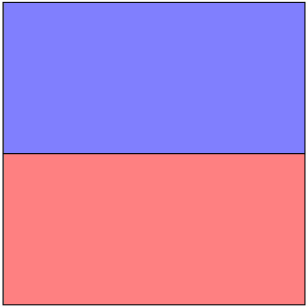
Making pastry

A lot can be understood about strange attractors by looking at how puff pastry is being made.

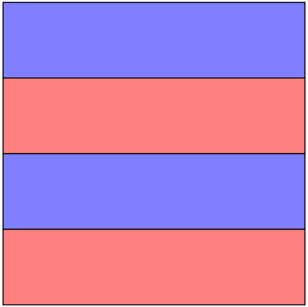
Baker's map

$$(x_{k+1}, y_{k+1}) = \begin{cases} (2x_k, ay_k) & \text{for } 0 \leq x_k \leq \dfrac{1}{2} \\ \\ (2x_k - 1, ay_k + 1 - a) & \text{for } \dfrac{1}{2} \leq x_k \leq 1. \end{cases}$$

Baker's map ($a = \frac{1}{2}$)



Baker's map ($a = \frac{1}{2}$)



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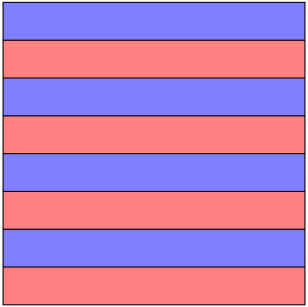

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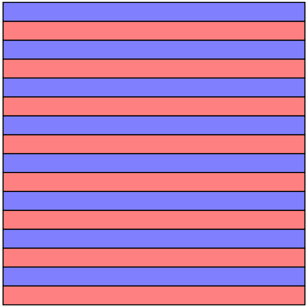
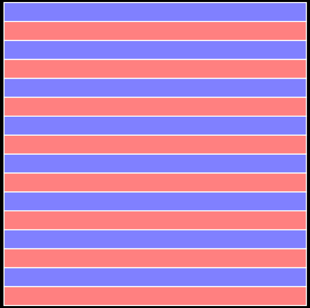


Baker's map ($a = \frac{1}{2}$)

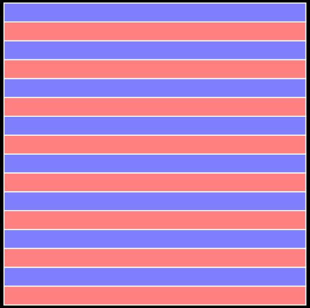


For $a = {}^1\!/_2$, Baker's map is **area-preserving**

$$\operatorname{area}(\mathcal{B}(R)) = \operatorname{area}(R)\,.$$

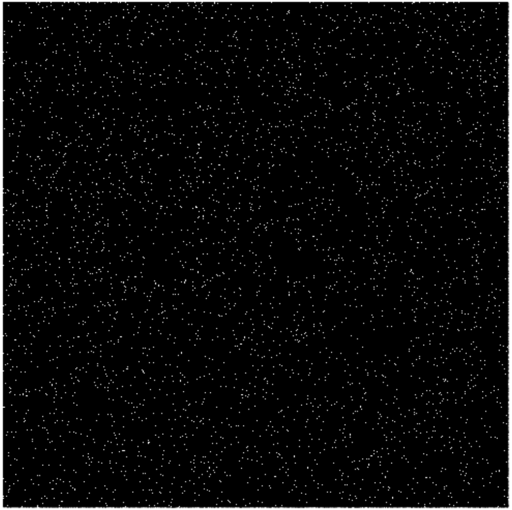
The square S is mapped *onto* itself and transients never die. Orbits never settle down to a lower-dimensional attractor.

Baker's map ($a = \frac{1}{2}$)



This type of chaos is known as **Hamiltonian chaos** and is beyond the scope of this class.

Baker's map ($a = \frac{1}{3}$)



Baker's map ($a = \frac{1}{3}$)

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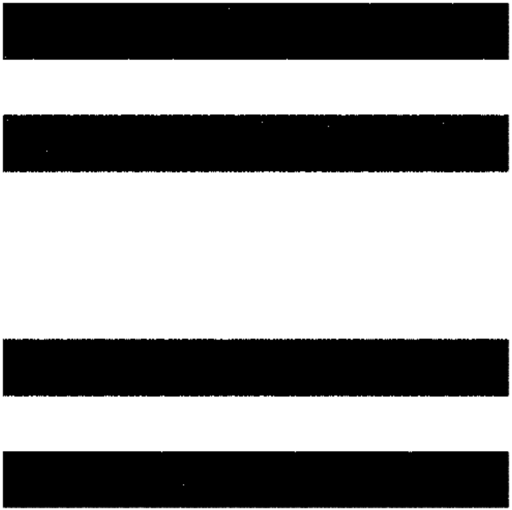

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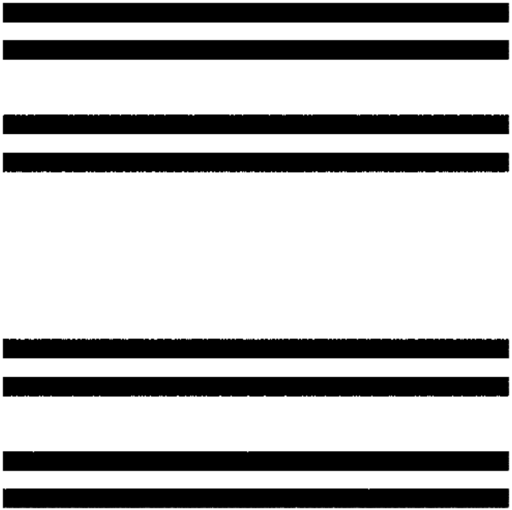
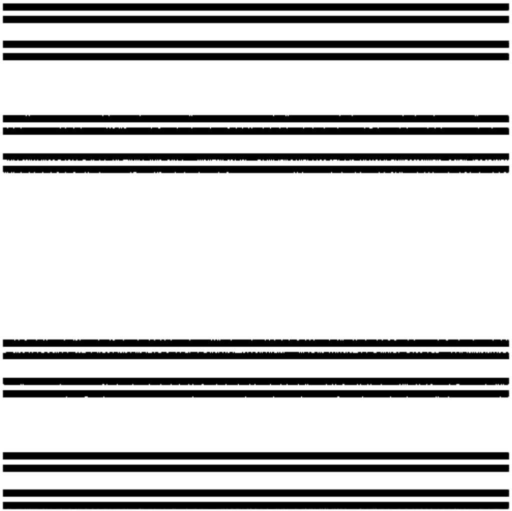
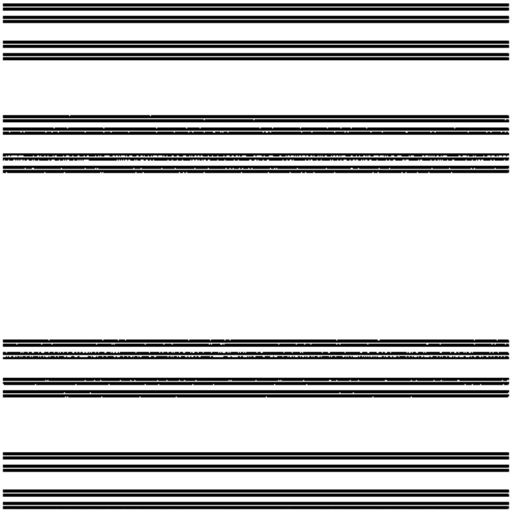


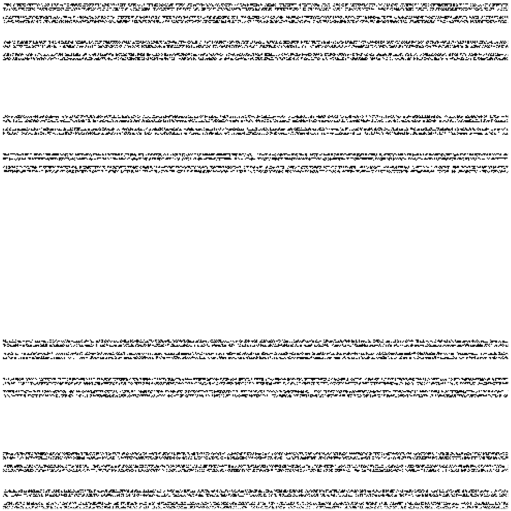
Baker's map ($a = \frac{1}{3}$)



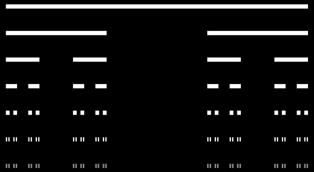
Baker's map ($a = \dfrac{1}{3}$)



Baker's map ($a = \frac{1}{3}$)



Baker's map ($a = \frac{1}{3}$)

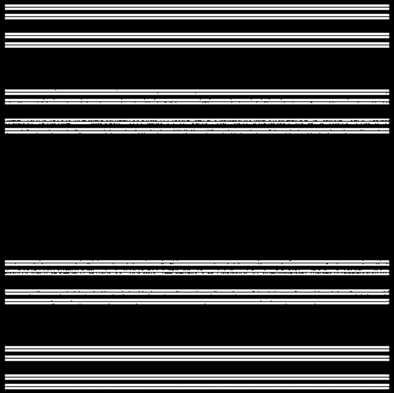


This particular fractal structure is known as the (uniform) **Cantor set**. Its dimension is

$$D = \frac{\log(2)}{\log(3)} \simeq 0.631$$

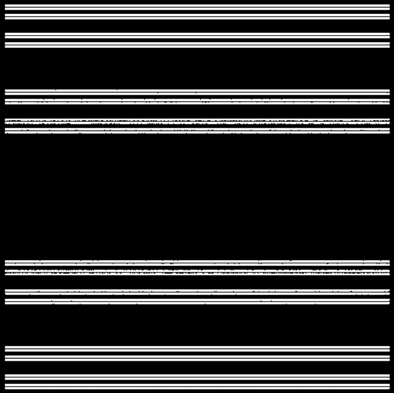
It less than a line but more than isolated points.

Baker map



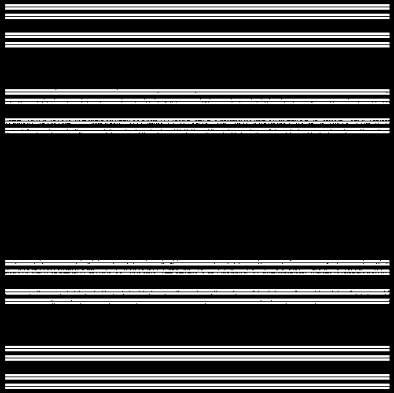
$$D = -\lim_{\epsilon \to 0} \frac{\log(N)}{\log(\epsilon)}$$

Baker map



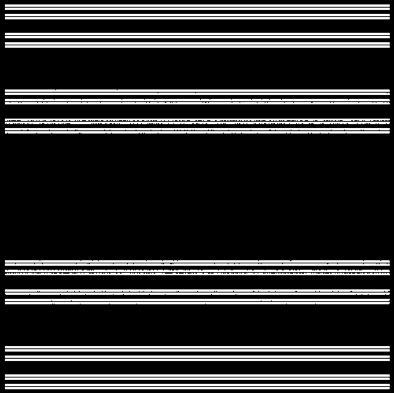
$$D = \lim_{n\to\infty} \frac{\log\left(2^n \times a^{-n}\right)}{\log\left(a^{-n}\right)}$$

Baker map



$$D = 1 + \frac{\log(2)}{\log(1/a)}$$

Baker map



$$D \simeq 1.63$$

Hénon Map

A somewhat analog to Lorenz in discrete-time

Hénon Map (1976)

$$x_{k+1} = y_k + 1 - \alpha x_k^2$$
$$y_{k+1} = \beta x_k$$

Michel Hénon (1931-2013)

Hénon Map

$$
\begin{aligned}
x' &= x_k \\
y' &= 1 + y_k - \alpha x_k^2
\end{aligned}
$$

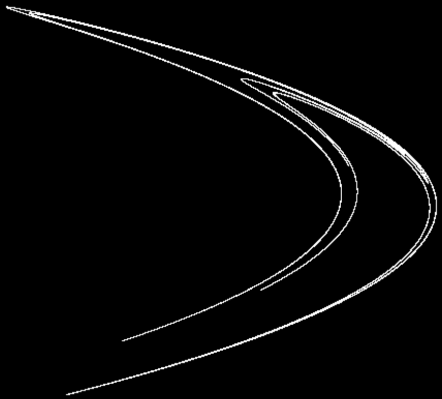
Hénon Map

$$
\begin{aligned}
x'' &= \beta x' \\
y'' &= y'
\end{aligned}
$$

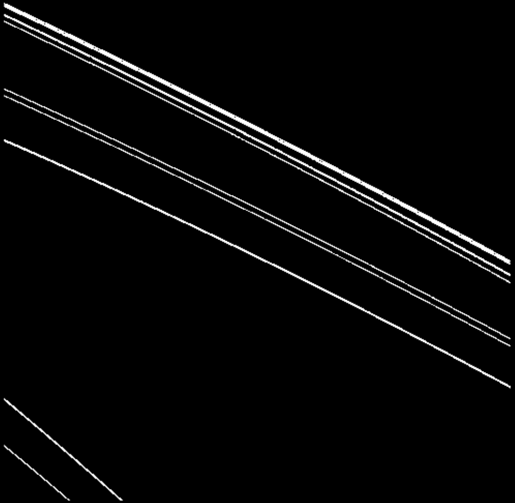
Hénon Map

$$x_{k+1} = y''$$
$$y_{k+1} = x''$$

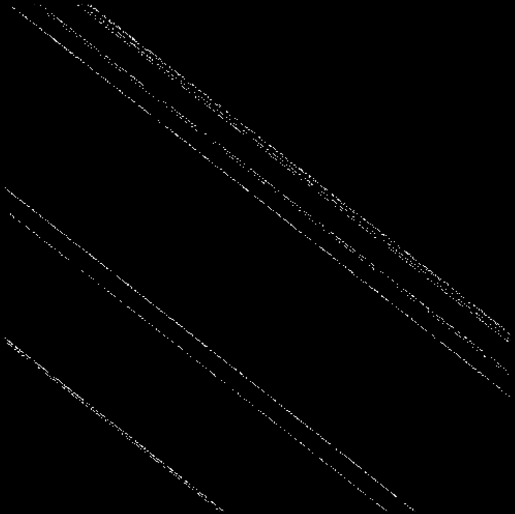
Hénon Map $(a, b) = (1.4, 0.3)$



Hénon Map $(a, b) = (1.4, 0.3)$



Hénon Map $(a, b) = (1.4, 0.3)$



Rössler system

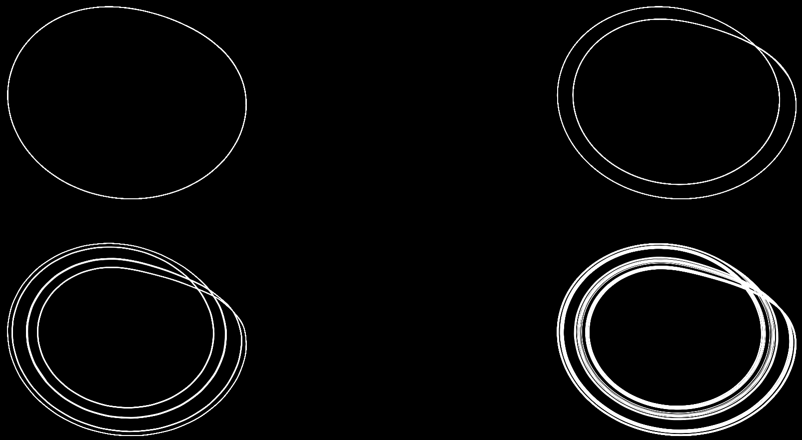
Folding and stretching in continuous time

Rössler system (1976)

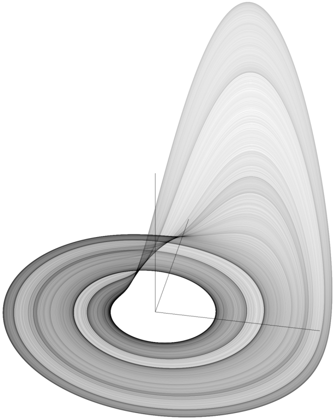
$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

Otto Rössler (81 years old)

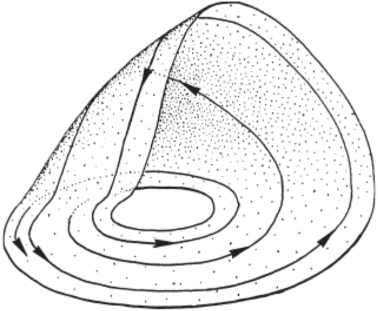
Rössler system



Rössler system

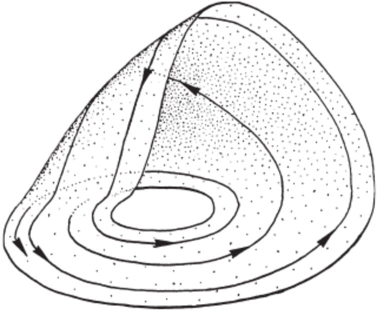


Rössler system



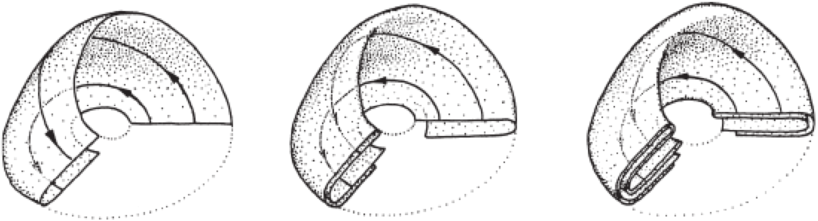
Stretching occurs along the two-dimensional unstable manifold of the spiral-saddle point close to the origin.

Rössler system



Folding and re-injection is induced by along the stable and unstable manifolds of the second fixed point.

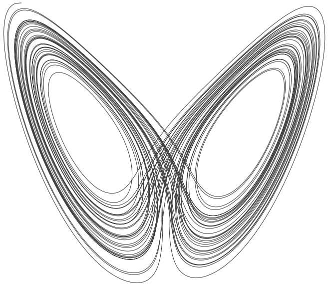
Rössler system

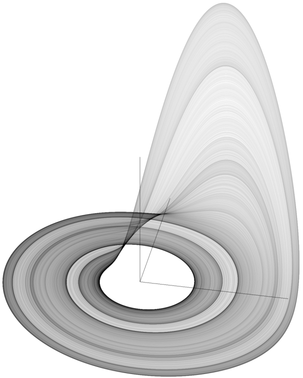


(a) (b) (c)

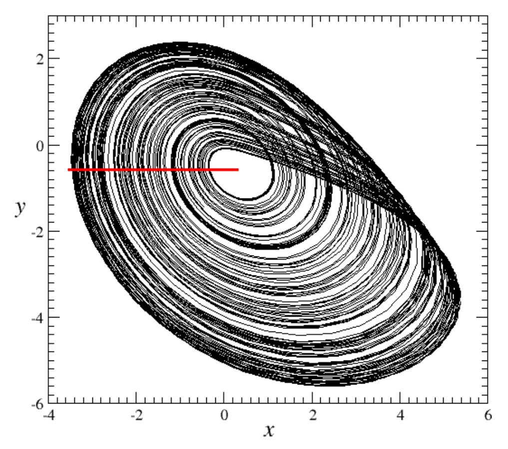
Four basic mechanisms

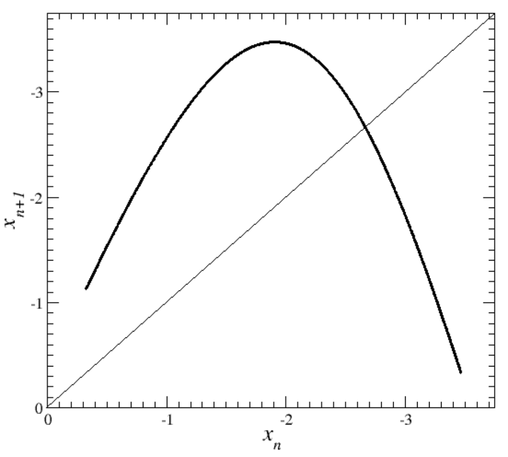
Folding, inverted folding, tearing and half-inverted tearing



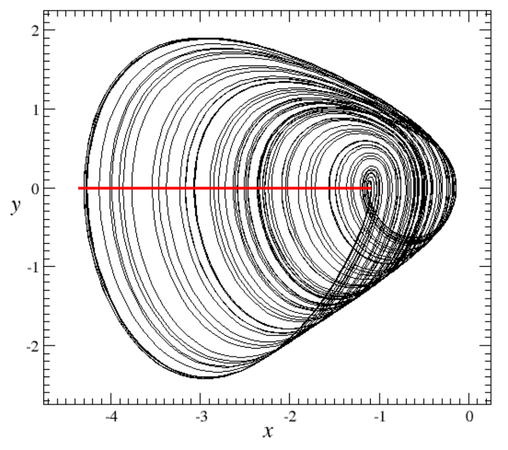


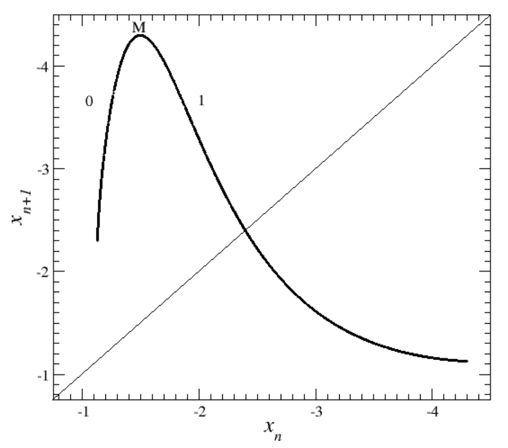
Folding



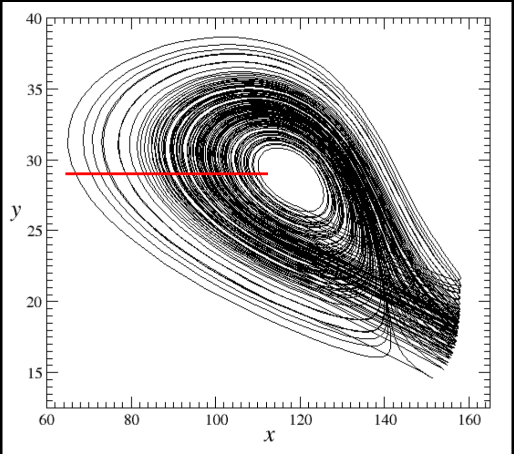


Inverted folding

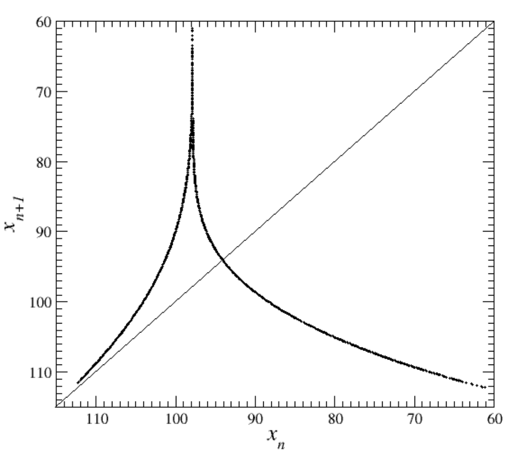




Tearing

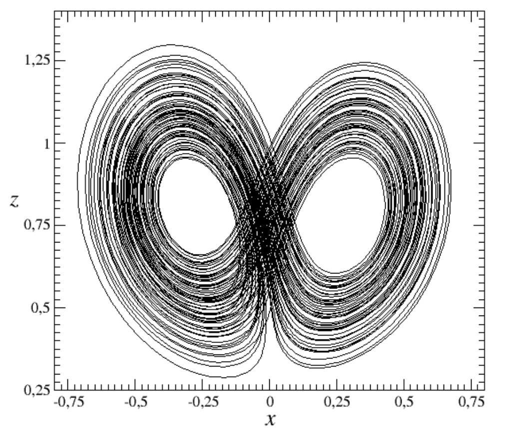


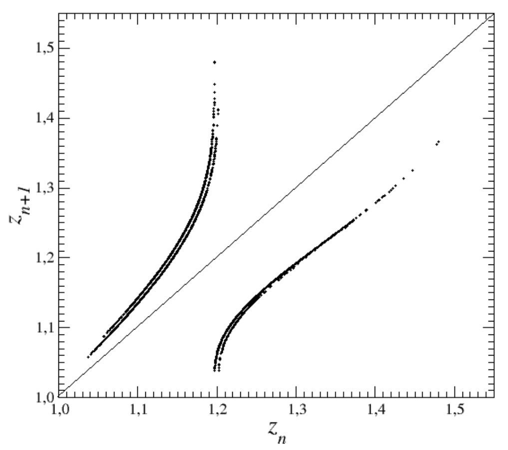
40
35
30
25
20
15
y
60
80
100
120
140
160
x



60
70
80
90
100
110
x_{n+1}
110
100
90
80
70
60
x_n

Half-inverted tearing





What did we not cover in this class ?

An awful lot !

Thank you for your attention

Any question ?