A glimpse of fractal geometry

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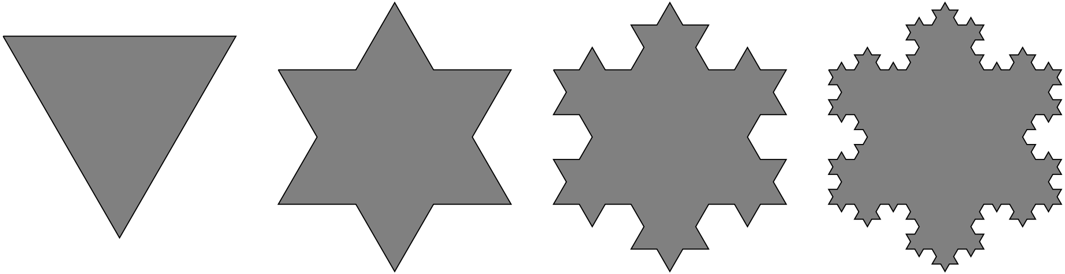
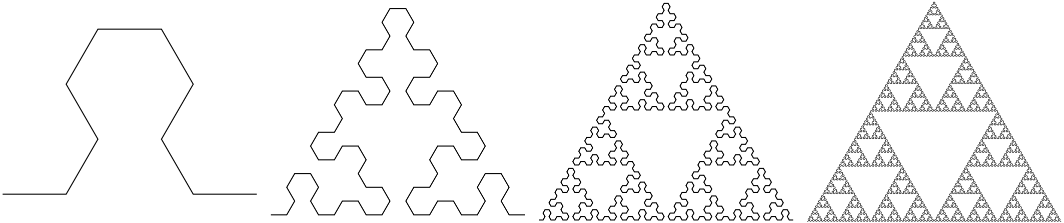
Arts & Métiers Institute of Technology, January 2022



von Koch snowflake (dim $\simeq 1.2619$)

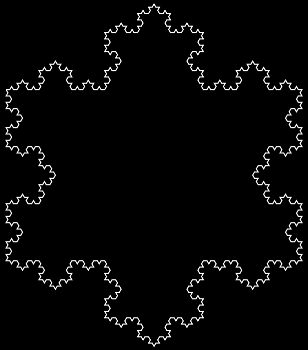


Sierpinski triangle $(\mathrm{dim} \simeq 1.585)$

von Koch snowflake

A self-similar fractal

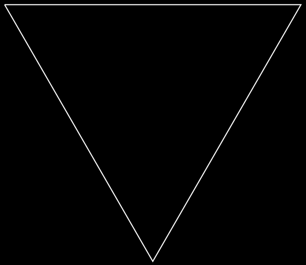
von Koch snowflake



On a continuous curve without tangents, constructible from elementary geometry.

Helge von Koch, *Ark. Mat. Astron. Fys.*, 1904.

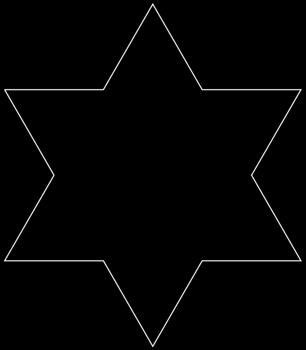
von Koch snowflake



<u>Iteration n°0</u>

Number of segment	$N_0 = 3$
Length of each segment	$L_0 = 1$
Perimeter	$P_0 = 3$

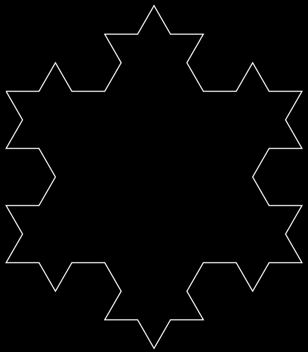
von Koch snowflake



<u>Iteration n°1</u>

Number of segment	$N_1 = 3 \times 4$
Length of each segment	$L_1 = \frac{1}{3}$
Perimeter	$P_1 = 3 \times \frac{4}{3}$

von Koch snowflake



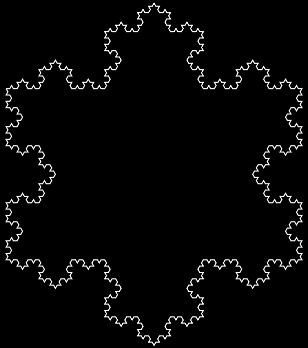
Iteration n°2

Number of segment $$N_2 = 3 \times 4^2$$

Length of each segment $$L_2 = \frac{1}{3^2}$$

Perimeter $$P_2 = 3 \times \left(\frac{4}{3}\right)^2$$

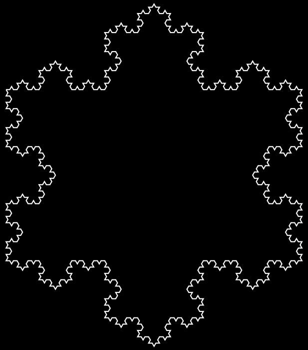
von Koch snowflake



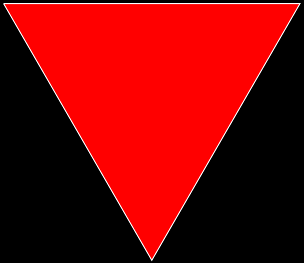
Iteration n°n

Number of segment	$N_n = 3 \times 4^n$
Length of each segment	$L_n = \dfrac{1}{3^n}$
Perimeter	$P_n = 3 \times \left(\dfrac{4}{3}\right)^n$

von Koch snowflake



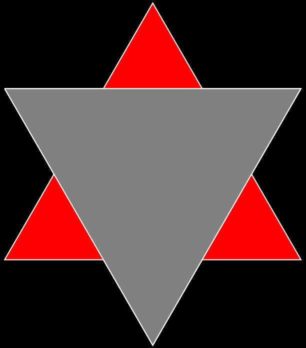
$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} 3 \times \left(\frac{4}{3}\right)^n$$



Iteration n°0

Number of triangles $T_0 = 1$

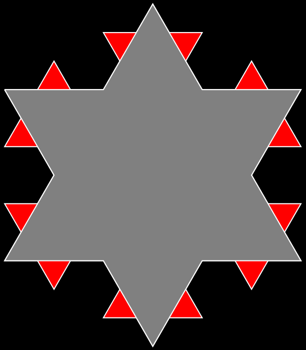
Surface A_0



Iteration n°1

Number of triangles $T_1 = 3$

Surface $A_1 = A_0 \left(1 + 3 \times \frac{1}{9}\right)$

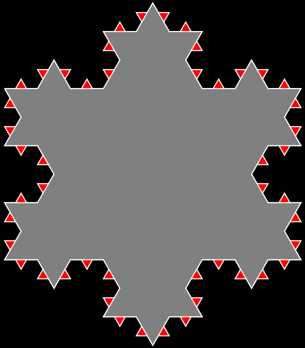


Iteration n°2

Number of triangles $T_2 = 3 \times 4$

Surface $A_2 = A_1 \left(1 + \frac{3}{4} \times \left(\frac{4}{9}\right)^2\right)$

von Koch snowflake

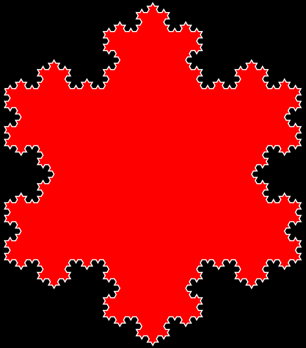


Iteration n°n

Number of triangles $$T_n = \frac{3}{4} \times 4^n$$

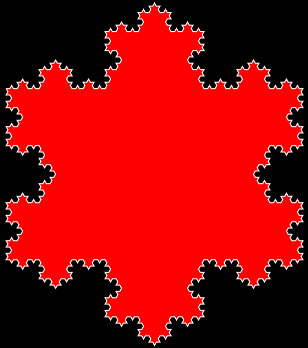
Surface $$A_n = A_{n-1} \left(1 + \frac{3}{4} \times \left(\frac{4}{9}\right)^n\right)$$

von Koch snowflake



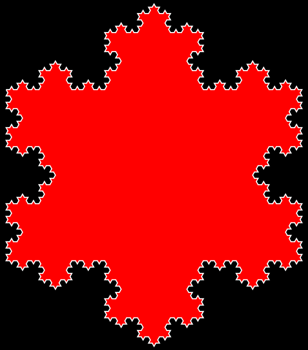
$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} A_{n-1}\left(1+\frac{3}{4}\left(\frac{4}{9}\right)^n\right)$$

von Koch snowflake



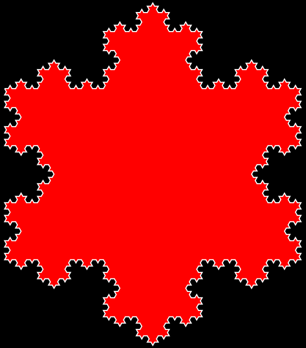
$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} A_0 \left(1 + \frac{3}{4}\sum_{i=1}^{n}\left(\frac{4}{9}\right)^{n}\right)$$

von Koch snowflake



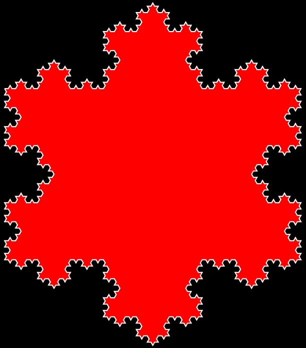
$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} A_0 \left(1 + \frac{1}{3}\sum_{i=0}^{n-1}\left(\frac{4}{9}\right)^n\right)$$

von Koch snowflake



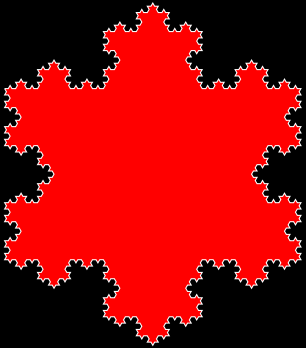
$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{A_0}{5}\left(8 - 3\left(\frac{4}{9}\right)^n\right)$$

von Koch snowflake



$$\lim_{n\to\infty} A_n = \frac{8}{5} A_0$$

von Koch snowflake

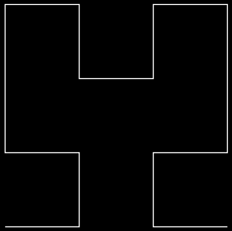


The snowflake encloses a **finite area**, but has an **infinite perimeter**. How is this possible ?

Hilbert curve

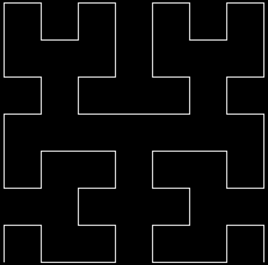
A space-filling curve

Hilbert curve



Hilbert curve was first described by German mathema-cian David Hilbert in 1891.

Hilbert curve



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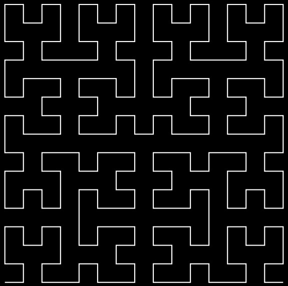
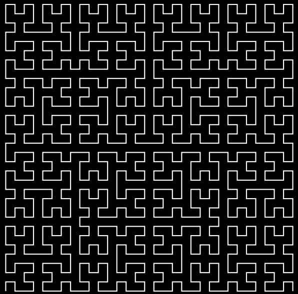
Hilbert curve



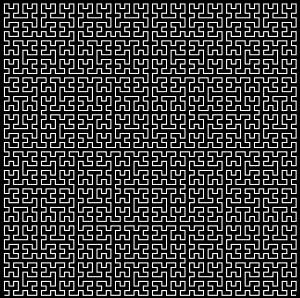
$$L_n = 2^n - \frac{1}{2^n}$$

Hilbert curve



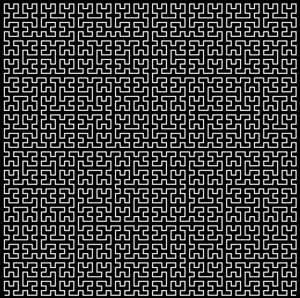
$$L_n = 2^n - \frac{1}{2^n}$$

Hilbert curve



As $n \to \infty$, it has infinite length yet remains enclosed in the unit square. It actually goes through every single point of $\mathbb{R}^2$.

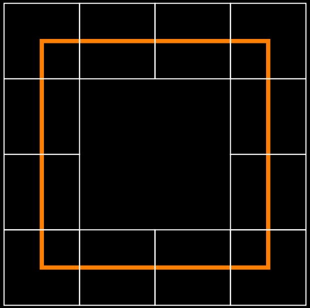
Hilbert curve

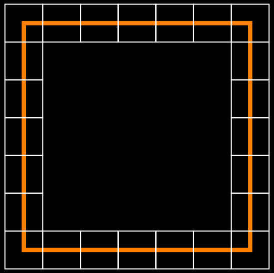


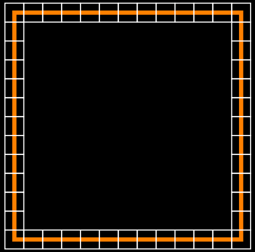
How come the range of a 1D object contains the whole 2D unit square ?

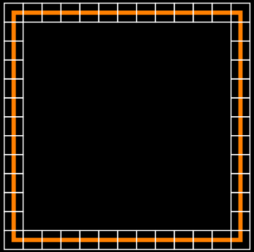
Questionning the concept of dimension

The dimension of an object

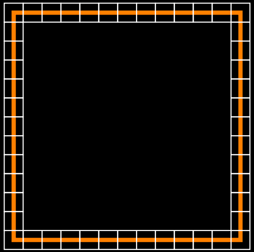




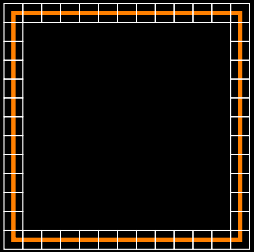




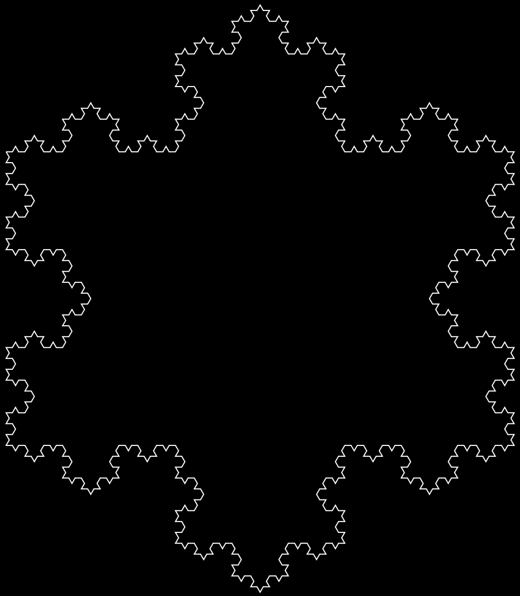
$$N \propto \left(\frac{1}{\varepsilon}\right)^D$$



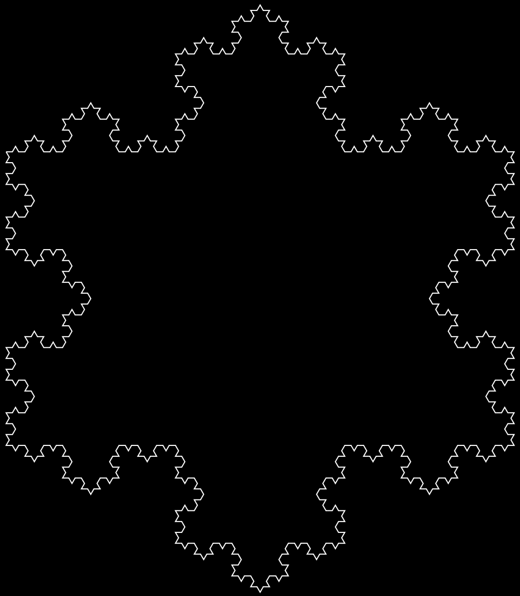
$$D = -\lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon}$$



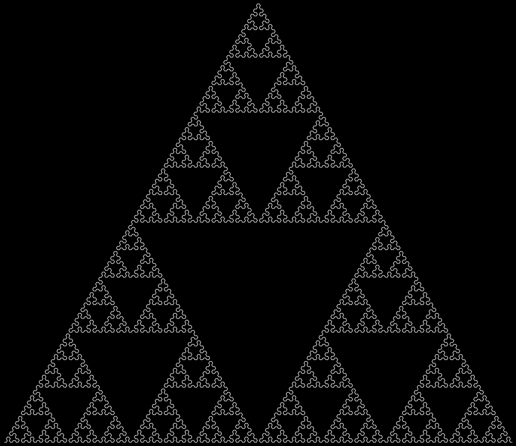
$$D = 2$$



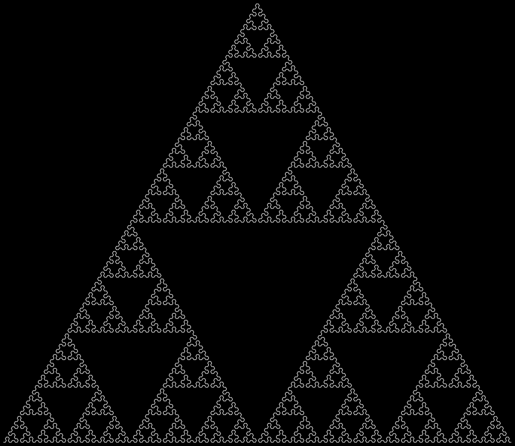
$$D = \frac{\log 4}{\log 3}$$



$D \simeq 1.26$



$$D = \frac{\log 3}{\log 2}$$

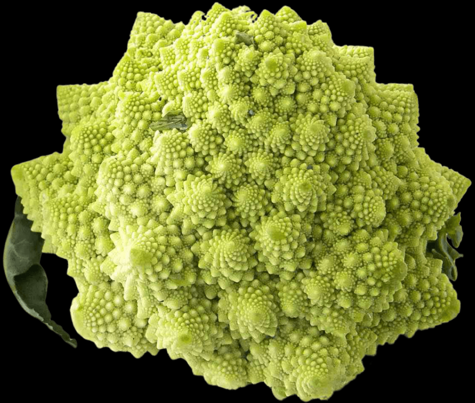


$$D \simeq 1.585$$

Fractals in Nature and Engineering

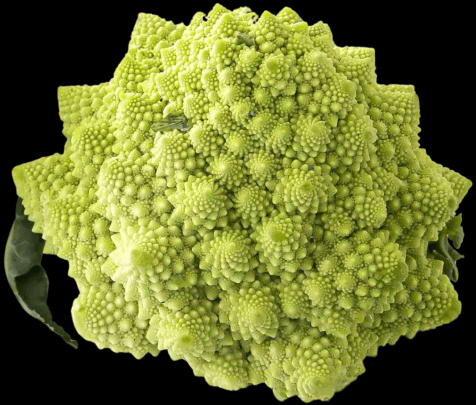
Some illustrations

In Nature



A power-law relationship exists between the rate at which the plant grows and the rate at which it produces buds.

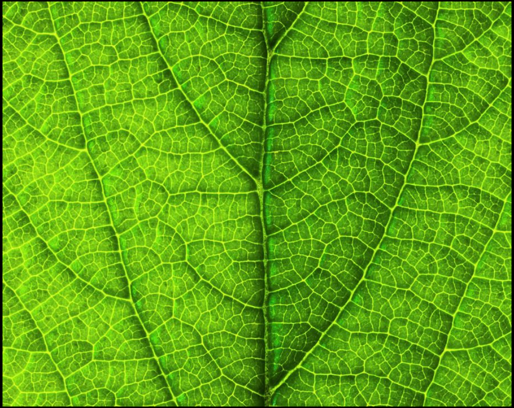
In Nature



Spirals are closely related to the Fibonacci sequence and the golden ratio.

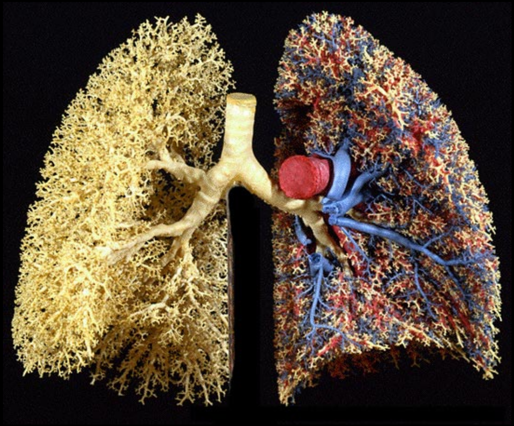
In Nature

In Nature



Fractal structure of the veins network ensures an almost optimal coverage the surface of the leaf.

In Nature



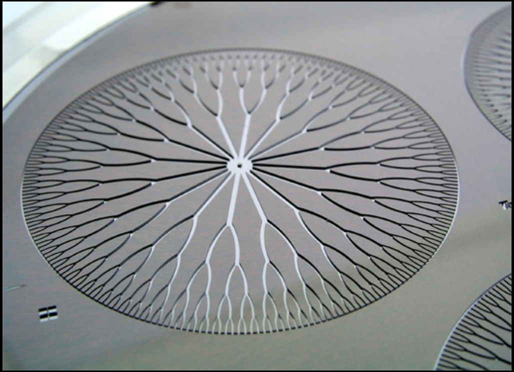
The fractal structure of the lungs encloses a extremely large exchange surface in an otherwise fairly small volume.

In Nature

In Engineering
Material science

The fractal nature of aggregates used for asphalt directly impact its mechanical properties.

In Engineering
Thermal systems

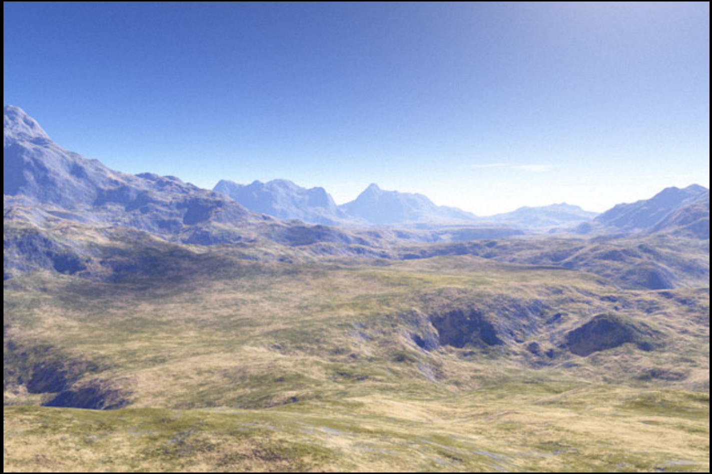


Heat exchanges with self-similar structures have increased efficiency.

In Engineering

Urbanism

In Engineering
Computer graphics

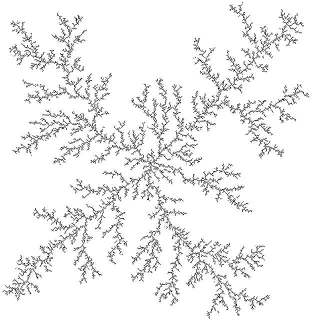
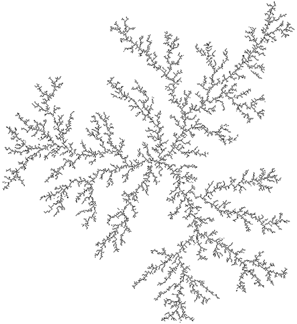
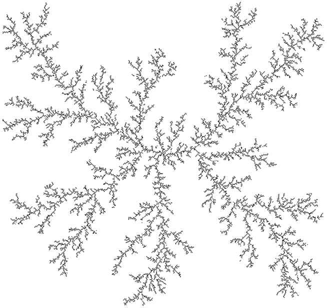


Fractal emergence in simple simulations

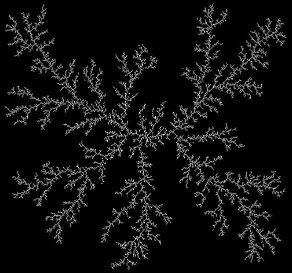
Some examples

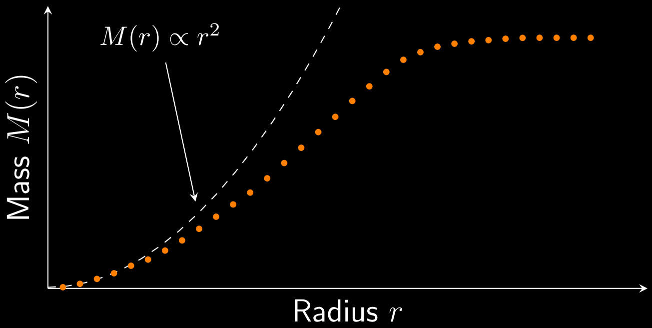
Diffusion-Limited Aggregation

Numerous natural phenomena exhibit fractal scaling laws. Some of these can be captured by simple **Diffusion-Limited Aggregation** models.

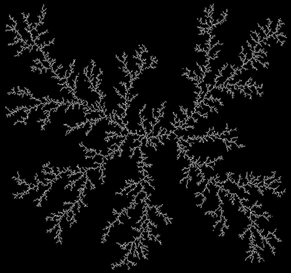




Diffusion-Limited Aggregation





Diffusion-Limited Aggregation



$$M(r) \propto r^{1.71}$$

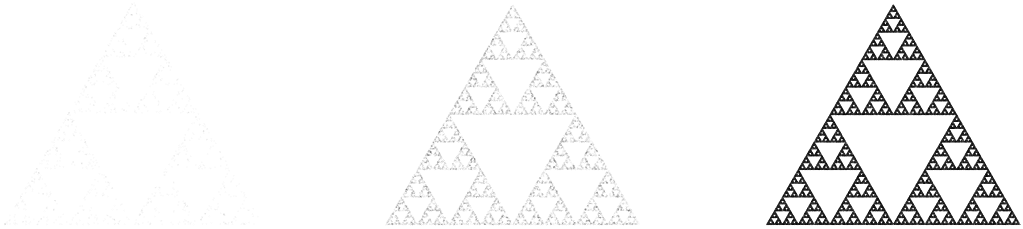

Iterated Fractals Systems

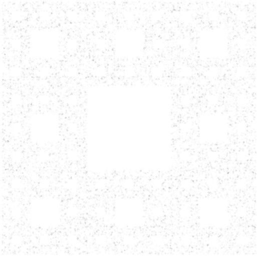
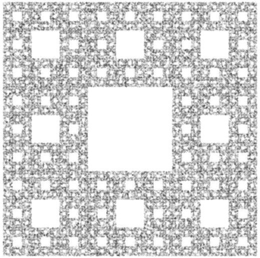
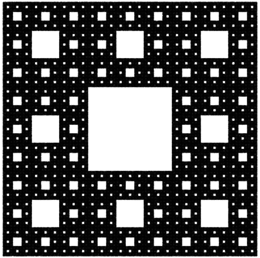
The Chaos Game

$$f_1(x, y) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f_2(x, y) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} w \\ 0 \end{bmatrix}$$

$$f_3(x, y) = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{4} \begin{bmatrix} w \\ 2h \end{bmatrix}$$



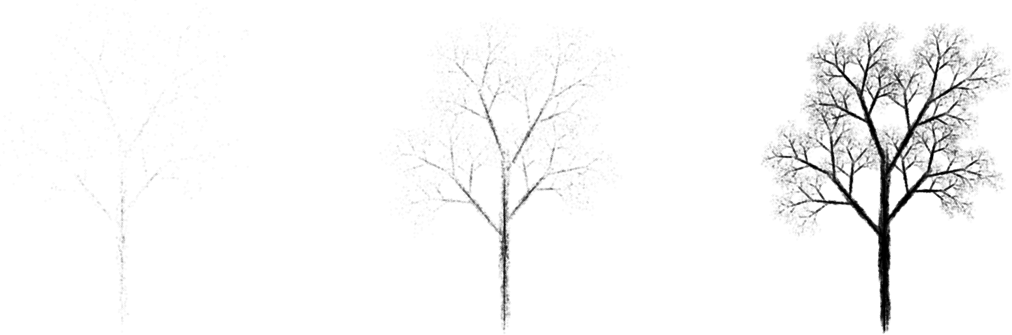
$$f_1(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f_2(x,y) = \begin{bmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

$$f_3(x,y) = \begin{bmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6 \end{bmatrix}$$

$$f_4(x,y) = \begin{bmatrix} -0.15 & 0.28 \\ -0.26 & 0.24 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0.44 \end{bmatrix}$$

$$\boldsymbol{f}_i(x, y) = \begin{bmatrix} r\cos(\theta) & -s\sin(\varphi) \\ r\sin(\theta) & s\cos(\varphi) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



Thank you for your attention

Any question ?