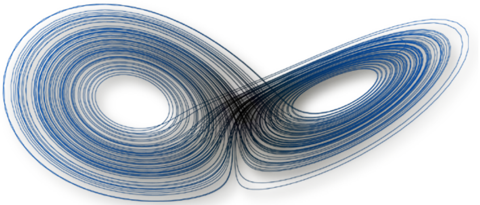

Two-dimensional nonlinear systems

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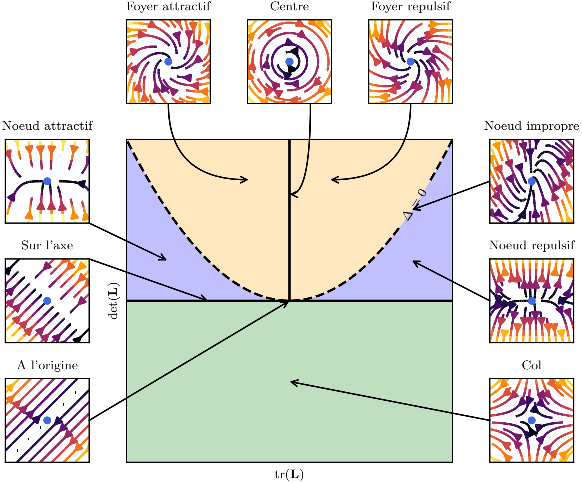
Two-dimensional systems

Linearity and fixed points classification



System : $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$

$$\mathsf{tr}(\boldsymbol{A}) = a_{11} + a_{22}$$

$$\det(\boldsymbol{A}) = a_{11}a_{22} - a_{21}a_{12}$$

$$\Delta = \mathsf{tr}^2(\boldsymbol{A}) - 4\det(\boldsymbol{A})$$

Two-dimensional systems

Nonlinearity

Nonlinear systems

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

- $x(t)$ and $y(t)$ are real-valued functions of time t.
- $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are smooth real-valued functions of x and y only.

$$\ddot{x} = -\sin(x)$$

$$\ddot{x} = -x - x^3$$

$$\dot{x} = x - y - (x^2 + y^2)x$$
$$\dot{y} = x + y - (x^2 + y^2)y$$

$$\vdots$$

Two-dimensional systems

Example : Rabbits vs. Sheeps

- Two species fighting from the same limited food supply.
- Each species can grow to its carrying capacity in the absence of the other.
- When they encounter (at a rate proportional to the size of each population), the conflicts reduce the growth rate of each species.

Lotka-Voltera model of competition

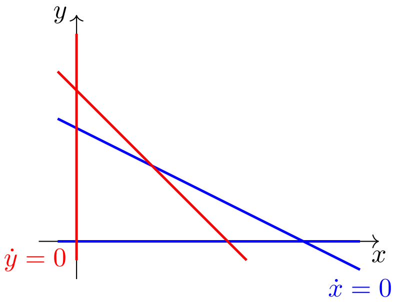
$$\begin{cases} \dot{x} = x\,(3 - x) - 2xy \\ \dot{y} = y\,(2 - y) - xy \end{cases}$$

$$\boldsymbol{J} = \begin{bmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{bmatrix}$$

Two-dimensional systems

Example : Rabbits vs. Sheeps

Fixed points can be identified visually as the intersections of the **nullclines**, i.e. the set of points satisfying $\dot{x} = 0$ or $\dot{y} = 0$.



Two-dimensional systems

Example : Rabbits vs. Sheeps

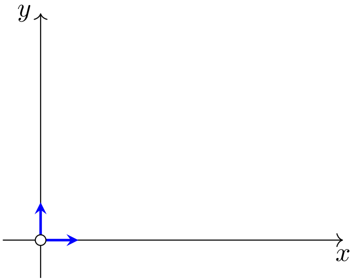
At the origin, we have

$$\boldsymbol{J} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$ while the eigenvectors are

$$\boldsymbol{v}_1 = \boldsymbol{e}_1, \quad \boldsymbol{v}_2 = \boldsymbol{e}_2.$$

It is an **unstable node**.



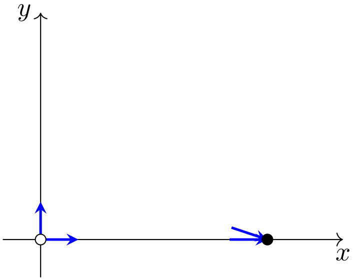
Two-dimensional systems

Example : Rabbits vs. Sheeps

At $(x, y) = (3, 0)$, we have

$$\boldsymbol{J} = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. This fixed point is thus a **stable node**.



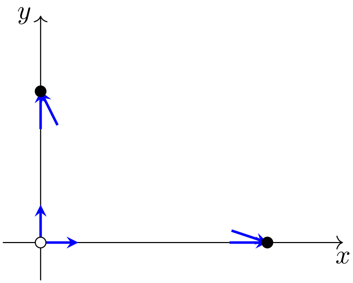
Two-dimensional systems

Example : Rabbits vs. Sheeps

At $(x, y) = (0, 2)$, we have

$$\boldsymbol{J} = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -1$. This fixed point is thus a **stable node**.



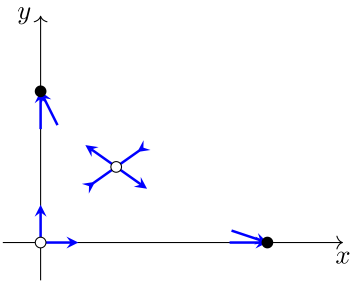
Two-dimensional systems

Example : Rabbits vs. Sheeps

At $(x, y) = (1, 1)$, we have

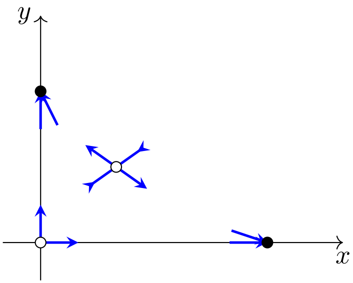
$$\boldsymbol{J} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$

whose eigenvalues are $\lambda_{1,2} = -1 \pm \sqrt{2}$. This fixed point is thus a **saddle**.



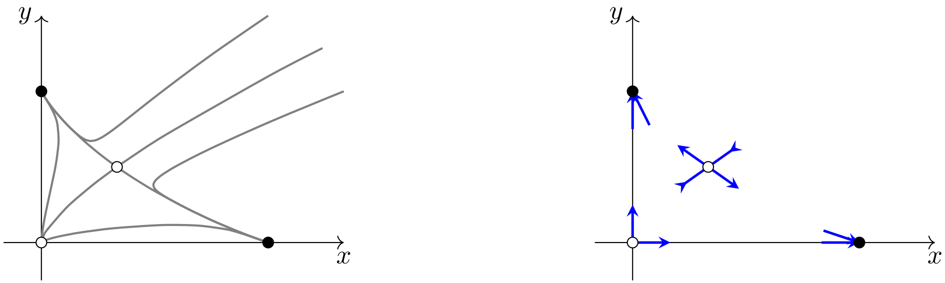
Two-dimensional systems

Example : Rabbits vs. Sheeps



Two-dimensional systems

Example : Rabbits vs. Sheeps

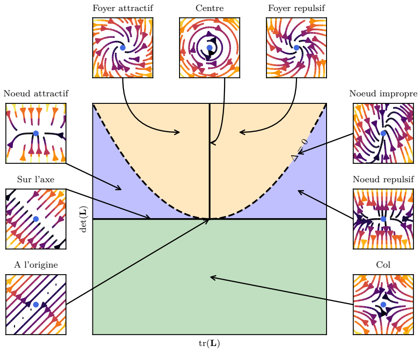


Two-dimensional systems

Hartman-Grobman theorem

Theorem : The behaviour of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behaviour of its linearization near this equilibrium point.

A hyperbolic fixed point is a point for which the associated Jacobian matrix has no eigenvalue with zero real part.

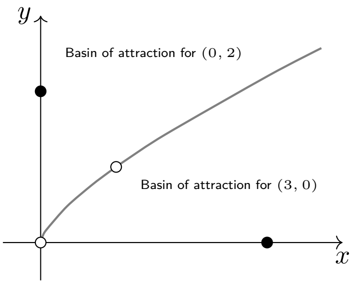


Two-dimensional systems

Basins of attraction and separatrices

Given a fixed point $\boldsymbol{x}^*$, its **basin of attraction** is the set of initial conditions $\boldsymbol{x}_0$ such that $\boldsymbol{x}(t) \to \boldsymbol{x}^*$ as $t \to \infty$.

The **stable manifold** of the saddle point form the **separatrix** between two different basins.



Two-dimensional systems

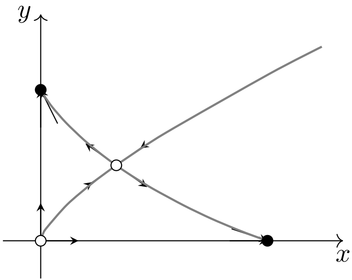
Approximating the stable and unstable manifolds

In the vicinity of $(x^*, y^*) = (1, 1)$, the stable and unstable manifolds can be approximated using a power series expansion.

Given the change of variable $\xi = x^* - x$ and $\eta = y^* - y$, our system becomes

$$\dot{\xi} = -\xi(1 + 2\eta + \xi) - 2\eta$$
$$\dot{\eta} = -\eta(1 + \xi + \eta) - \xi$$

corresponding to a shift of the origin of our phase space to the saddle point $(x^*, y^*) = (1, 1)$.



Two-dimensional systems

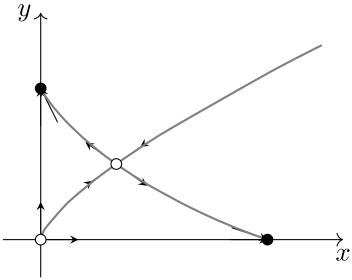
Approximating the stable and unstable manifolds

In a second step, assume that $\eta = h(\xi)$ so that

$$\dot{\eta} = \frac{dh}{d\xi}\frac{d\xi}{dt}.$$

Replacing η by $h(\xi)$ in our original system yields

$$\underbrace{-h(\xi)\left(1+\xi+h(\xi)\right)-\xi}_{\dot{\eta}} = \underbrace{\frac{dh}{d\xi}\left(-\xi(1+2h(\xi)+\xi)-2h(\xi)\right)}_{h'(\xi)\dot{\xi}}.$$



Two-dimensional systems

Approximating the stable and unstable manifolds

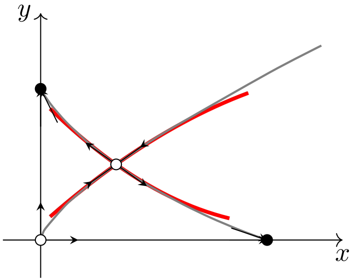
Assuming that $h(\xi) = a\xi + b\xi^2$, expanding both sides of the equation and regrouping like powers yields the following system for a and b

$$\begin{aligned} \mathcal{O}(\xi): \qquad & a^2 = \frac{1}{2} \\ \mathcal{O}(\xi^2): \qquad & a^2 + (1+6a)b = 0. \end{aligned}$$

Two sets of solutions exists giving rise to

$$h_\pm(\xi) = a_\pm \xi + b_\pm \xi^2.$$

Here, $h_+(\xi)$ (resp. $h_-(\xi)$) is the approximation of the stable (resp. unstable) manifold.



Two-dimensional systems

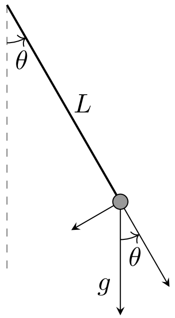
Oscillatory dynamics

The main interest of two-dimensional systems is their ability to model **oscillatory** behaviours, the canonical example being that of the simple pendulum.

Starting from Newton's principles, the equation of motion is given by

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

where θ is the angle of pendulum with respect to the vertical axis.



Two-dimensional systems

Oscillatory dynamics

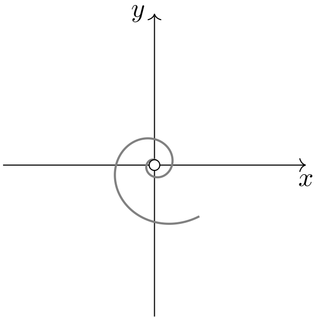
Let us consider instead the following system

$$\dot{x} = \mu x - y - (x^2 + y^2)x$$
$$\dot{y} = x + \mu y - (x^2 + y^2)y.$$

Its Jacobian matrix reads

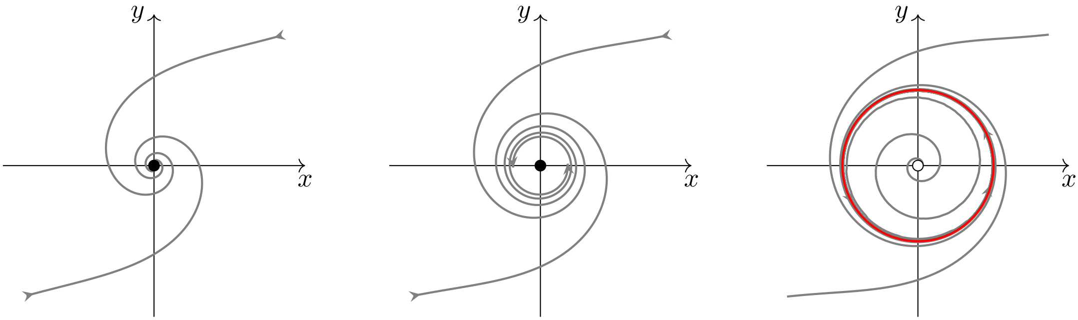
$$\boldsymbol{J} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$

and its eigenvalues are $\lambda = \mu \pm i$. The origin is thus a stable ($\mu < 0$) or unstable ($\mu > 0$) spiral.



Two-dimensional systems

Oscillatory dynamics



$\mu < 0$ $\mu = 0$ $\mu > 0$

Two-dimensional systems

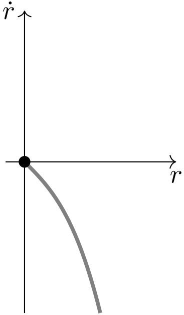
Introducing the complex variable $z = x + iy$, our system can be recast as

$$\dot{z} = (\mu + i)z + |z|^2 z.$$

Turning to polar coordinates $z = re^{i\theta}$, the equation becomes

$$\dot{r} = \mu r - r^3$$
$$\dot{\theta} = 1$$

which simplifies the analysis quite a lot.



$\mu < 0$

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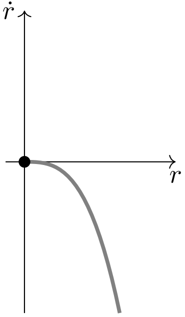

$\mu = 0$

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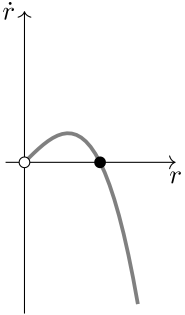

$\mu > 0$

Two-dimensional systems

Oscillatory dynamics

As μ becomes positive, the system exhibits periodic dynamics known as a **limit cycle**.

For this particular system, this limit cycle is created through a **supercritical Hopf bifurcation** at $\mu = 0$ which will be the subject of an upcoming lecture.