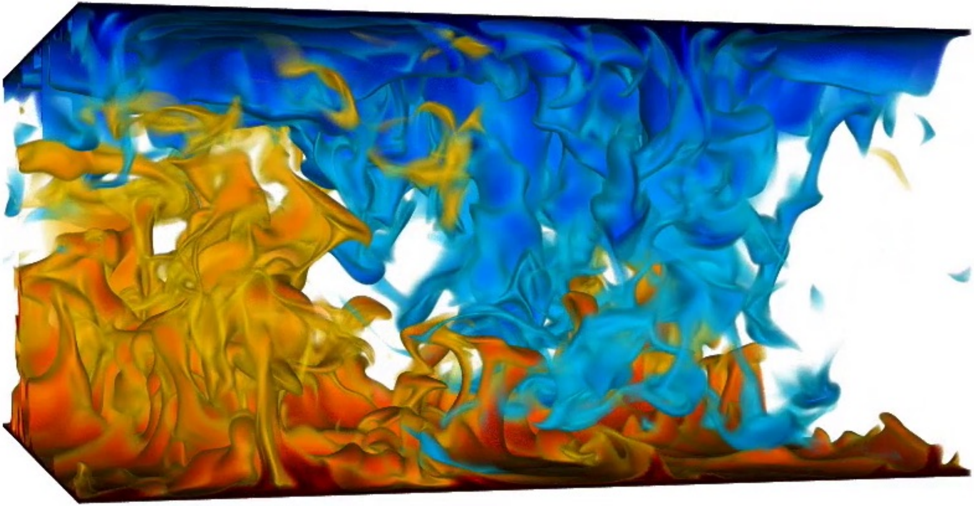
Lorenz system

Jean-Christophe LOISEAU

Arts & Métiers Institute of Technology, January 2022



Rayleigh-Bénard convection

Velocity equation

Inertia Pressure forces Buoyancy

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \delta\rho\boldsymbol{g}$$

Advection Viscous diffusion

Rayleigh-Bénard convection

Temperature equation

$$\underbrace{\boxed{\frac{\partial T}{\partial t}}}_{\text{Inertia}} + \underbrace{\boxed{(\boldsymbol{u} \cdot \nabla) T}}_{\text{Convection}} = \underbrace{\boxed{\kappa \nabla^2 T}}_{\text{Conduction}}$$

Rayleigh-Bénard convection

Non-dimensional equations

Prandtl number

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} &= -\nabla p + \boxed{Pr}\, \nabla^2 \boldsymbol{u} + \left(\boxed{Ra} \cdot \boxed{Pr} \right) \vartheta \boldsymbol{e}_y \\
\frac{\partial \vartheta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \vartheta &= \nabla^2 \vartheta
\end{aligned}
$$

Rayleigh number

Rayleigh-Bénard convection

Base flow

Conducting state

$$\begin{cases} \dfrac{d^2\Theta}{dy^2} = 0 \\ \Theta(0) = 1, \quad \Theta(1) = 0 \end{cases}$$

Rayleigh-Bénard convection

Base flow

Conducting state

$$\Theta(y) = 1 - y$$
$$\boldsymbol{U}_b = 0$$

Squire theorem : Only two-dimensional perturbations need to be considered.

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}'}{\partial t} &= -\nabla p' + Pr \nabla^2 \boldsymbol{u}' + (Ra \cdot Pr)\, \vartheta' \boldsymbol{e}_y \\
\frac{\partial \vartheta'}{\partial t} &= -v' + \nabla^2 \vartheta'
\end{aligned}
$$

with $\boldsymbol{u}' = \begin{bmatrix} u' & v' \end{bmatrix}^T$ and ϑ' the velocity and temperature fluctuations, respectively.

Rayleigh-Bénard convection

Linear stability

Fluctuation's streamfunction

$$\frac{\partial}{\partial t}\nabla^2 \boxed{\Psi} = -\left(Ra \cdot Pr\right)\frac{\partial \vartheta}{\partial x} + Pr\nabla^4 \boxed{\Psi}$$

$$\frac{\partial \vartheta}{\partial t} = -\frac{\partial \boxed{\Psi}}{\partial x} + \nabla^2 \vartheta$$

Rayleigh-Bénard convection
Dispersion relation

(1916) Assuming free-slip boundary conditions for the fluctuation leads to

$$\Psi(x,y,t) = \hat{\Psi}(t)\sin(n\pi y)\sin(kx),$$
$$\vartheta(x,y,t) = \hat{\vartheta}(t)\sin(n\pi y)\cos(kx),$$

the problem can be solved analytically. We'll also let $\gamma^2 = (n\pi)^2 + k^2$.

Rayleigh-Bénard convection

Dispersion relation

$$
\begin{aligned}
-\gamma^2 \frac{d\hat{\Psi}}{dt} &= Pr\gamma^4 \hat{\Psi} + (Ra \cdot Pr)\, k\hat{\vartheta} \\
\frac{d\hat{\vartheta}}{dt} &= -k\hat{\Psi} - \gamma^2 \hat{\vartheta}
\end{aligned}
$$

Rayleigh-Bénard convection

Dispersion relation

$$\begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix} = \begin{bmatrix} Pr\gamma^4 & (Ra \cdot Pr)\,k \\ -k & -\gamma^2 \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix}$$

Rayleigh-Bénard convection

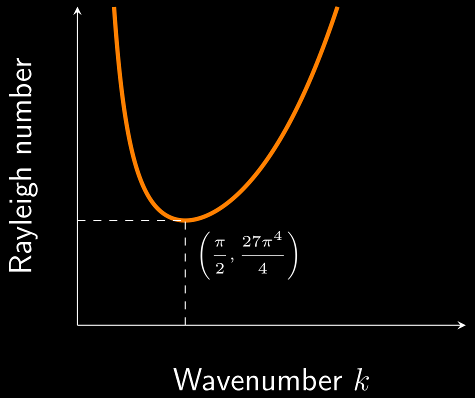
Dispersion relation

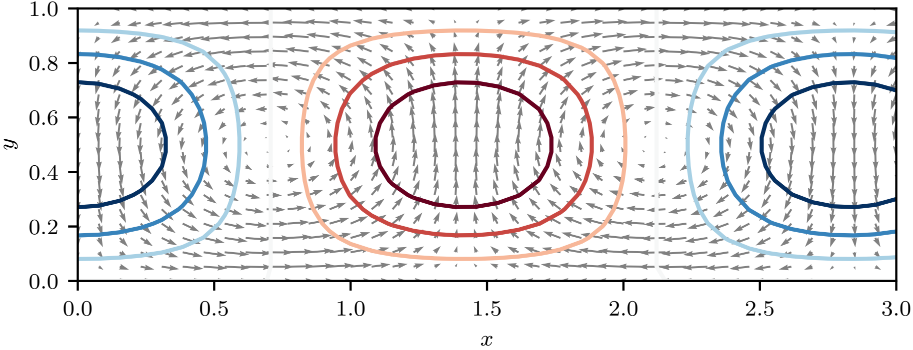
$$\lambda \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix} = \begin{bmatrix} Pr\gamma^4 & (Ra \cdot Pr)\, k \\ -k & -\gamma^2 \end{bmatrix} \begin{bmatrix} \hat{\Psi} \\ \hat{\vartheta} \end{bmatrix}$$

Rayleigh-Bénard convection
Dispersion relation

Neutral curve

$$Ra_c(n,k) = \frac{\left((n\pi)^2 + k^2\right)^3}{k^2}$$





Temperature and velocity field of the most unstable eigenmode.

From Navier-Stokes to Lorenz

Investigating the nonlinearities

From Navier-Stokes to Lorenz

Nonlinear equations

Nonlinear advection

$$\frac{\partial}{\partial t}\nabla^2\Psi = -\left(Ra \cdot Pr\right)\frac{\partial \vartheta}{\partial x} + Pr\nabla^4\Psi + \boxed{\mathcal{J}\left[\nabla^2\Psi, \Psi\right]}$$

$$\frac{\partial \vartheta}{\partial t} = -\frac{\partial \Psi}{\partial x} + \nabla^2\vartheta + \boxed{\mathcal{J}\left[\vartheta, \Psi\right]}$$

Nonlinear convection

From Navier-Stokes to Lorenz
Nonlinear equations

$$\mathcal{J}\left[f, g\right] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

Saltzmann (1962): The general solution to the partial differential equation can be expressed as a doubly infinite Fourier series

From Navier-Stokes to Lorenz
Deriving the reduced-order model

Close to the bifurcation point, the fluctuations are described by the unstable mode, hence

$$\Psi(x,y,t) \simeq \Psi_1(t)\sin(\pi y)\sin(kx) + \cdots$$
$$\vartheta(x,y,t) \simeq \vartheta_1(t)\sin(\pi y)\cos(kx) + \cdots$$

Which kind of harmonics to they generate due to the nonlinearity ?

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$
\begin{aligned}
\mathcal{J}\left[\Psi, \nabla^2 \Psi\right] &= \mathcal{J}\left[\Psi, -\gamma^2 \Psi\right] \\
&= -\gamma^2 \mathcal{J}\left[\Psi, \Psi\right] \\
&= 0
\end{aligned}
$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$
\begin{aligned}
\mathcal{J}\left[\Psi, \vartheta\right] &= \Psi_1 \vartheta_1 \mathcal{J}\left[\sin(kx)\sin(\pi y), \sin(kx)\cos(\pi y)\right] \\
&= \Psi_1 \vartheta_1 \frac{k\pi}{2}\sin(2\pi y)
\end{aligned}
$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\Psi(x,y,t) \simeq \underbrace{\boxed{\Psi_1(t)\sin(kx)\sin(\pi y)}}_{\text{Unstable eigenmode}}$$

$$\vartheta(x,y,t) \simeq \boxed{\vartheta_1(t)\sin(kx)\cos(\pi y)} + \underbrace{\boxed{\vartheta_2\sin(2\pi y)}}_{\text{Nonlinear distortion}}$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{\partial}{\partial t}\nabla^2\Psi = -\underbrace{\boxed{(Ra \cdot Pr)\,\frac{\partial \vartheta}{\partial x}}}_{\text{Buoyancy}} + \underbrace{\boxed{Pr\nabla^4\Psi}}_{\text{Viscous diffusion}}$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{d\Psi_1}{dt} = -\frac{k}{\gamma^2}\left(Ra \cdot Pr\right)\vartheta_1 - Pr\gamma^2\Psi_1$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

Convection

Nonlinear distortion

$$\frac{\partial \vartheta}{\partial t} = -\boxed{\frac{\partial \Psi}{\partial x}} + \boxed{\nabla^2 \vartheta} + \boxed{\mathcal{J}\left[\vartheta, \Psi\right]}$$

Thermal diffusion

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{d\vartheta_1}{dt} = k\Psi_1\left(1 + \pi\vartheta_2\right) - \gamma^2\vartheta_1$$
$$\frac{d\vartheta_2}{dt} = \frac{k\pi}{2}\Psi_1\vartheta_1 - 4\pi^2\vartheta_2$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\frac{d\Psi_1}{dt} = \frac{k}{\gamma^2}\left(Ra \cdot Pr\right)\vartheta_1 - Pr\gamma^2\Psi_1$$

$$\frac{d\vartheta_1}{dt} = \Psi_1\left(k + k\pi\vartheta_2\right) - \gamma^2\vartheta_1$$

$$\frac{d\vartheta_2}{dt} = \frac{k\pi}{2}\Psi_1\vartheta_1 - 4\pi^2\vartheta_2$$

From Navier-Stokes to Lorenz

Deriving the reduced-order model

$$\begin{aligned}\dot{x} &= \sigma\left(y - x\right)\\ \dot{y} &= x\left(\rho - z\right) - y\\ \dot{z} &= xy - \beta z\end{aligned}$$

Lorenz system

A case study

Equivariance : The equations remain unchanged if $(x, y, z) \mapsto (-x, -y, z)$.

$$\boldsymbol{S}\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{S}\boldsymbol{x})$$

If (x, y, z) is a solution, so is $(-x, -y, z)$. They come in pairs.

<u>Invariant axis</u> : If $x(0) = y(0) = 0$, then $x(t) = y(t) = 0$ at all time.

$$\dot{z} = -\beta z \quad \Rightarrow \quad z(t) = \exp(-\beta t) z_0$$

The z-axis is an **invariant manifold** of the system.

Strongly dissipative : Every in phase space, we have that

$$\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = -\sigma - 1 - \beta < 0.$$

Any given volume V of initial conditions will eventually tend to 0 as $t \to \infty$.

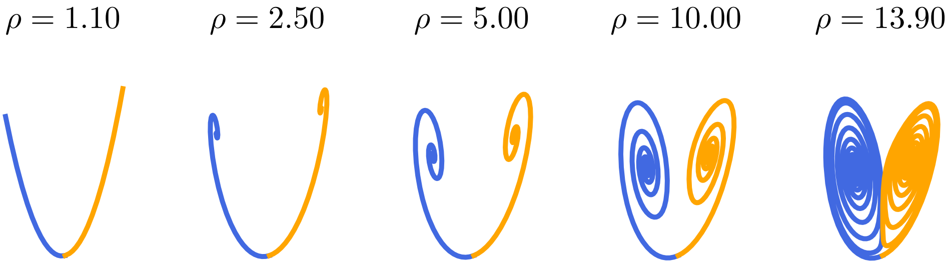
Exercise : Onset of the convection cells

1. Compute the fixed points of the system as a function of ρ.
2. When does the conducting state $(\boldsymbol{x} = 0)$ loose its stability?
3. Using a symmetry argument, what type of bifurcation can it be?

$$\begin{aligned}
\dot{x} &= \sigma\,(y - x) \\
\dot{y} &= x\,(\rho - z) - y \\
\dot{z} &= xy - \beta z
\end{aligned}$$

Lorenz system

Dynamics for $1 \leq \rho \leq 14$



Lorenz system

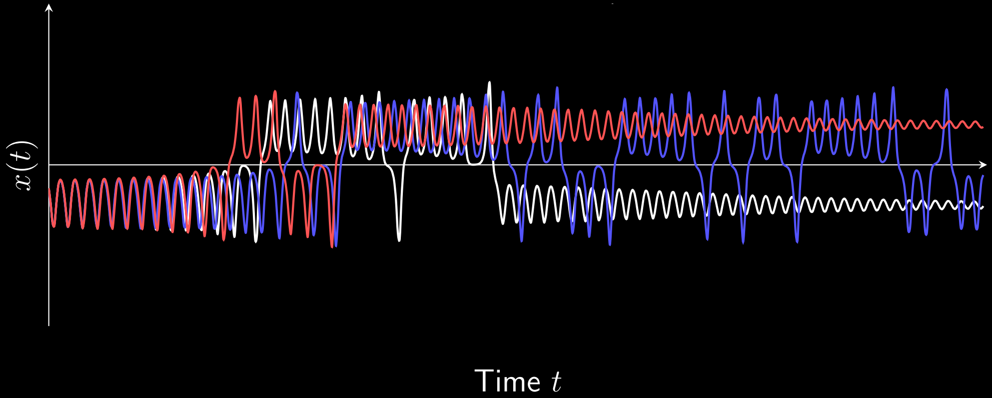
Homoclinic connection ($\rho \simeq 13.926$)

Perturbation leaves the conducting state along its unstable manifold and returns to it along its stable one.

Lorenz system

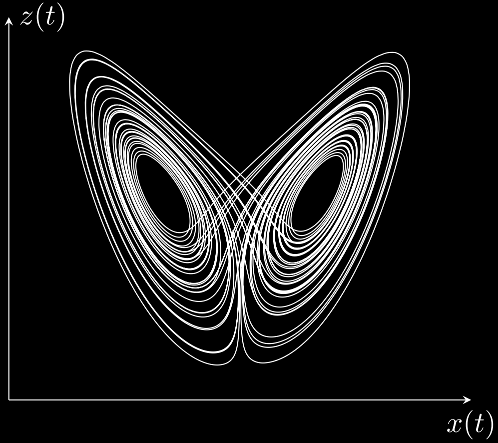
Transient chaos

For $\rho > 14$, the system exhibits **sensitive dependence on initial conditions**.



Lorenz system

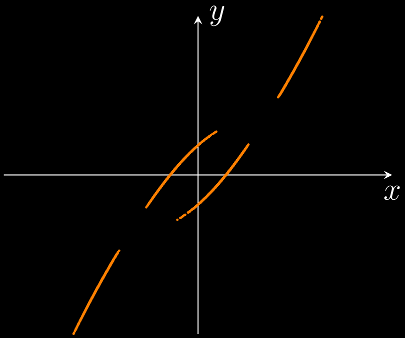
Strange attractor for $(\sigma, \rho, \beta) = (10, 28, 8/3)$



Lorenz system

Strange attractor

The attractor is not a volume, but not a surface either. It is something in-between: a **fractal object**.

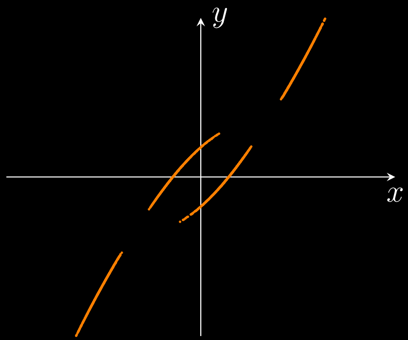


Lorenz system
Strange attractor

Its dimension is estimated to be

$$2.06 \pm 0.01,$$

i.e. it is not an integer.



Lorenz system

Strange attractor

Accuracy of the prediction

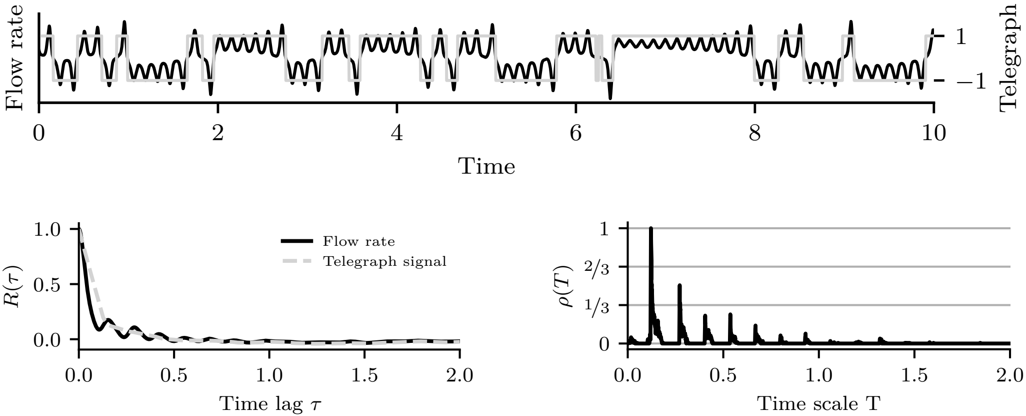
$$t_{\text{horizon}} \sim \mathcal{O}\left(\frac{1}{\boxed{\lambda}} \log \frac{\boxed{a}}{\boxed{\delta_0}}\right)$$

Lyapunov exponent

Uncertainty on the initial condition

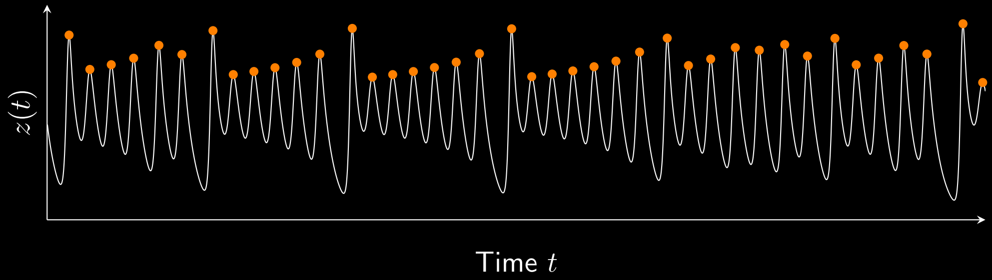
Lorenz system
Strange attractor

$$t_{\mathsf{Lorenz}} \sim \mathcal{O}\left(\log \frac{a}{\delta_0}\right)$$


Flow rate
Telegraph
1
−1
0
2
4
6
8
10
Time
R(τ)
1.0
0.5
0.0
Flow rate
Telegraph signal
0.0
0.5
1.0
1.5
2.0
Time lag τ
ρ(T)
1
2/3
1/3
0
0.0
0.5
1.0
1.5
2.0
Time scale T

Lorenz system

Strange attractor

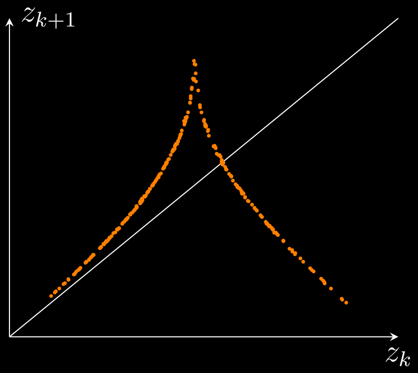


Lorenz system

Strange attractor

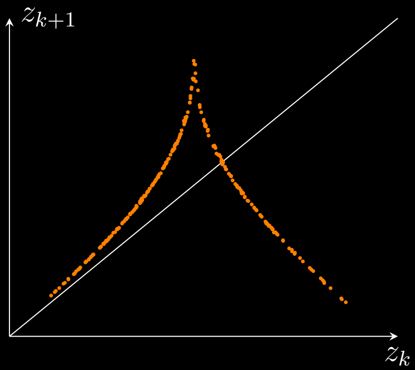
Lorenz map

$$z_{k+1} = f(z_k)$$



Lorenz map

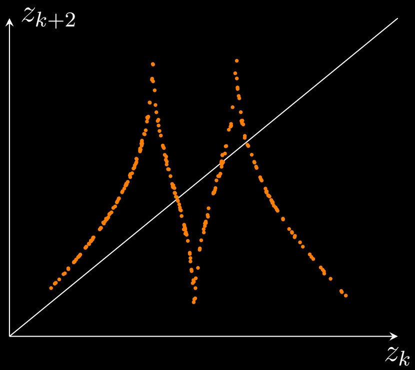
$$|f'(z)| > 1 \quad \forall z$$



Lorenz system
Strange attractor

Period-2 orbit

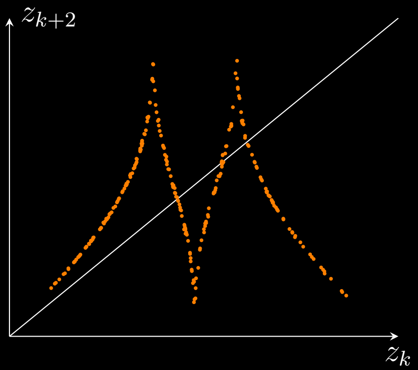
$$z = (f \circ f)(z)$$



Lorenz system
Strange attractor

Period-2 orbit

$$\eta_{k+2} \simeq \left(f'(p)\,f'(q)\right)\eta_k$$



Period-n orbit

$$z = (f \circ f \circ \cdots \circ f)(z)$$

Period-n orbit

$$\eta_{k+1} \simeq \left(\prod_{i=1}^{n} f'(z_i) \right) \eta_k$$

Lorenz system
Strange attractor

<u>Period-n orbit</u>

$$\eta_{k+1} \simeq \left(\prod_{i=1}^{n} f'(z_i) \right) \eta_k$$

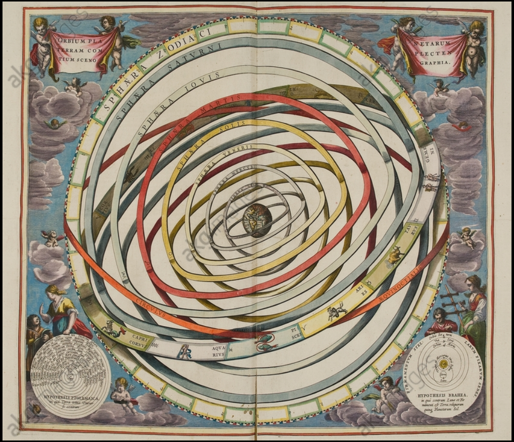
The skeleton of the attractor is made of an infinite number of **unstable periodic orbits**.

On the importance of chaos in Science

A (very) brief history

On the importance of chaos in Science

The clockwork Universe



- Ptolemy
- Copernicus
- Gallileo
- Kepler
- Newton
- Leibniz
- ...

Cauchy-Lipschitz (1920): Under suitable regularity conditions on $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$, the problem

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$
$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

admits a unique solution fully determined by $\boldsymbol{f}$ and the initial condition.

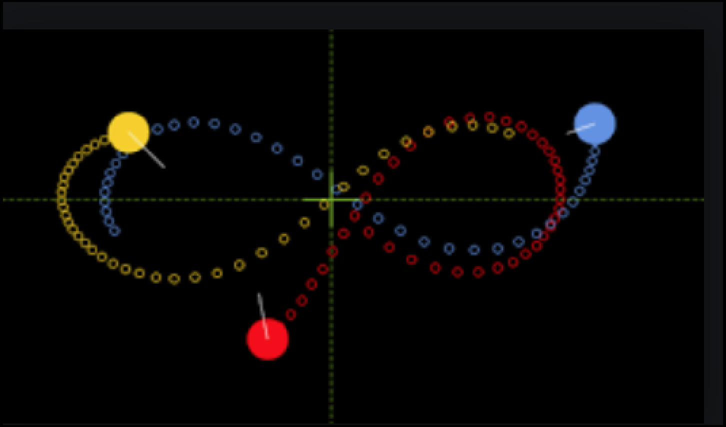
On the importance of chaos in Science
Laplace determinism (1814)

Nous devons [...] envisager l'état présent de l'Univers comme l'effet de sont état antérieur, et comme la cause de celui qui va suivre. Une intelligence qui pour un instant donné connaîtrait toutes les forces dont la nature est animée et la situation respective des êtes qui la compose [...] embrasserait dans la même formule les mouvements des plus grands corps de l'Univers comme ceux du plus léger atome : rien ne serait incertain pour elle, et l'avenir comme le passé serait présent à ses yeux.

Essai philosophie sur les probabilité.
Pierre Simon de Laplace, 1814.

On the importance of chaos in Science

The three body problem (circa 1890)



On the importance of chaos in Science

The three body problem (circa 1890)

> *Si nous connaissions exactement les lois de la nature et la situation de l'univers à l'instant initial, nous pourrions prédire exactement la situation de ce même univers à un instant ultérieur. Mais, alors même que les lois naturelles n'auraient plus de secret pour nous, nous ne pourrions connaître la situation qu'approximativement. Si cela nous permet de prévoir la situation ultérieure avec la même approximation, c'est tout ce qu'il nous faut, nous disons que le phénomène a été prévu, qu'il est régi par des lois ; mais il n'en est pas toujours ainsi, il peut arriver que de petites différences dans les conditions initiales en engendrent de très grandes dans les phénomènes finaux.*

Calcul des probabilités
Henri Poincaré, 1912.

On the importance of chaos in Science
Lorenz (1963)

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

Edward N. Lorenz (1917-2008)

Thank you for your attention

Any question ?