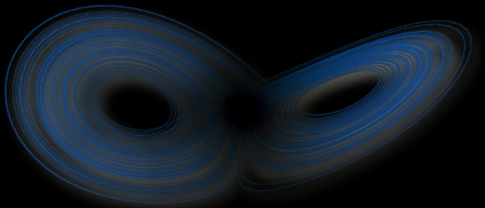

Nonlinear physics, dynamical systems and chaos theory

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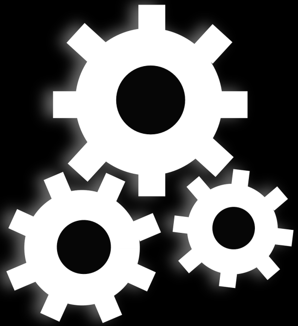
Basic information

Organization

- Lectures: Tuesdays and Thurdays, from 3:30pm to 5:30pm.
- Evaluation divided into two parts:
  - ↪ A two-hour long written exam late February.
  - ↪ A homework project.



Basic information

Homework project

- Ideally in **Python 3** or **Julia**.
  - $\hookrightarrow$ Open-source programming languages for scientific computing.
- You can install both of them from scratch or using **Anaconda**.
  - $\hookrightarrow$ Available for Windows, Mac OS and Linux.

Basic information

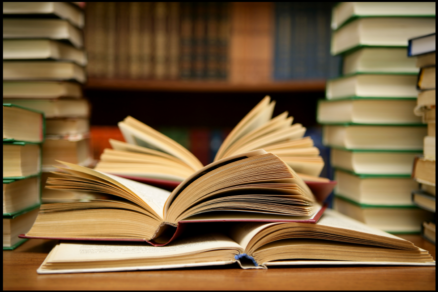
Useful references (in French)

General knowledge

- I. Stewart. *Dieu joue-t'il au dés?* Flammarion (2004).
- J. Gleick. *La théorie du chaos.* Flammarion (2008).
- I. Prigogine. *Les lois du chaos.* Flammarion (2008).

Textbooks

- P. Bergé *et al.* *L'ordre dans le chaos.* Hermann (1998).
- P. Manneville. *Instabilités, chaos et turbulence.* Ed. Ecole Polytechnique (2004).



Basic information

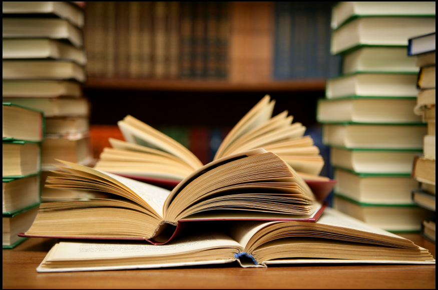
Useful references (in English)

Textbooks

- P. Manneville. *Instabilities, chaos and turbulence.* Ed. Ecole Polytechnique (2004).
- S. Strogatz. *Nonlinear dynamics and chaos.* 2nd edition, Avalon Publishing (2016).

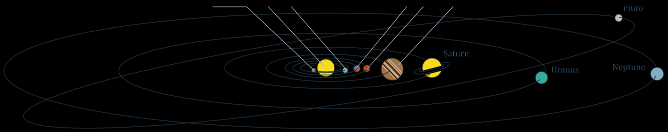
Online videos

- Steve Bruton : `https://www.youtube.com/c/Eigensteve/videos`
- Prof Gristh Math : `https://www.youtube.com/channel/UC5N5pRddyicAX1QJyJjIIdg`



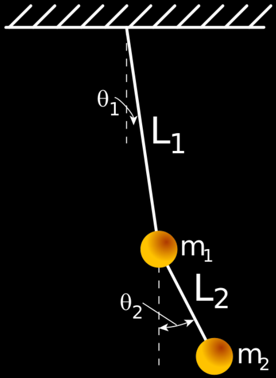
What is dynamical systems theory ?

It is the mathematics of behaviour and the classification of how systems evolve over time.



Newton's law of gravitation

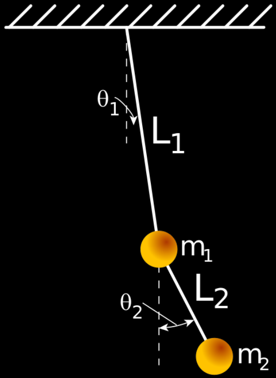
$$\ddot{\boldsymbol{x}}_j = \sum_{i \neq j}^{n} \frac{GM_i}{\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^3} (\boldsymbol{x}_i - \boldsymbol{x}_j)$$



Lagrangian Mechanics

$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \text{Kinetic Energy} - \text{Potential Energy}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$$

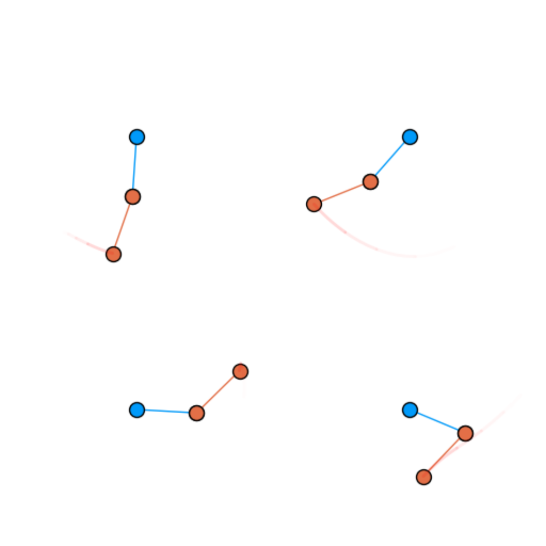


Hamiltonian Mechanics

$$\mathcal{H}(p_1, p_2, q_1, q_2) = \text{Kinetic Energy} + \text{Potential Energy}$$

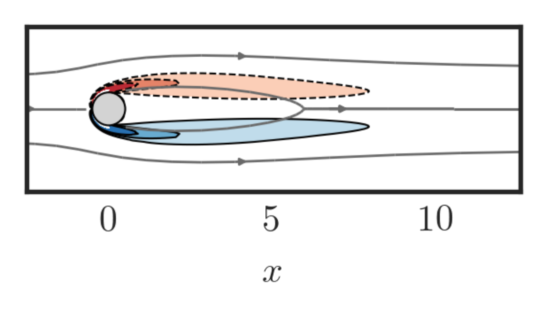
$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_i}, \qquad \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_i}$$

- Simple mechanical sytem exhibiting nonetheless complex dynamics.
- Evolutions of similar initial conditions diverge exponentially fast.
- Limited prediction horizon despite its deterministic equations of motion.

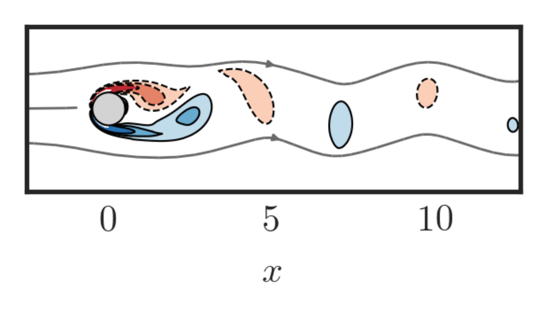


Some examples

Fluid dynamics



Below the critical Reynolds number



Above the critical Reynolds number

Some examples

Chemistry

- Spatio-temporal system described by

$$\frac{\partial \boldsymbol{q}_i}{\partial t} = \boldsymbol{D}\nabla^2 \boldsymbol{q}_i + \mathcal{R}(\boldsymbol{q}_i, \boldsymbol{q}_j)$$

  where $\boldsymbol{D}$ describes the diffusion of each species and $\mathcal{R}(\boldsymbol{q}_i, \boldsymbol{q}_j)$ the inter-species reactions.

- Can give rise to wonderful spatio-temporal patterns !

Some examples

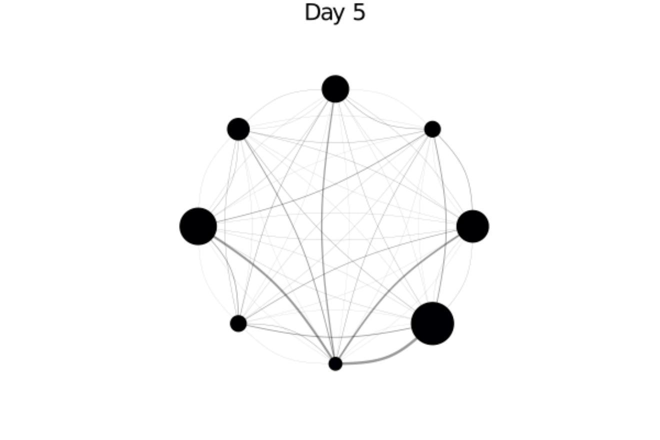
Biology

Synchronization occurs in numerous biological systems, e.g.

- Fireflies in South-East Asia,
- Pacemaker cells in the heart,
- Neurons during epilepsy,
- etc.

Some examples

Epidemiology



Combining epidemiology, dynamical systems and graph theory

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = -\beta s_i i_i$$

$$\frac{\mathrm{d}i_i}{\mathrm{d}t} = \beta s_i i_i - \gamma i_i + \sum_{i \neq j}^{n} \mathcal{R}_{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

$$\frac{\mathrm{d}r_i}{\mathrm{d}t} = \gamma i_i$$

How do we study them ?

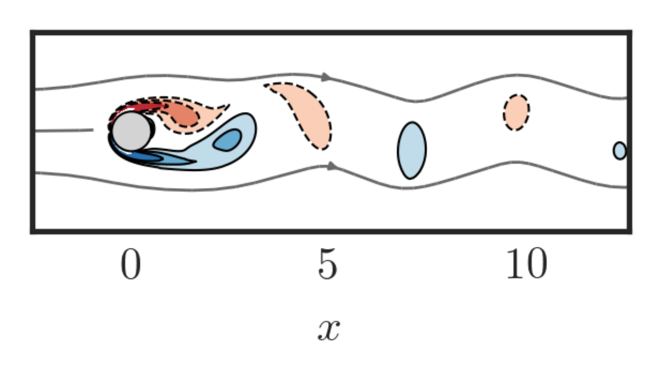
Find common patterns in the dynamics of seemingly different systems and distill them to their essence.

How do we study them ?

From a complex system to a simple model

Mean-field model

$$\dot{x} = \sigma x - \omega y - xz$$
$$\dot{y} = \omega x + \sigma y - yz$$
$$\dot{z} = -\beta z + x^2 + y^2$$

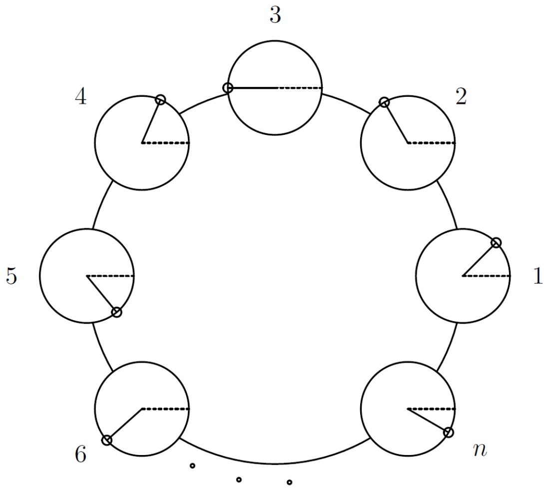


How do we study them ?

From a complex system to a simple model

Phase models

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j\neq i}^{N} K_{ij}\sin(\theta_j - \theta_j)$$



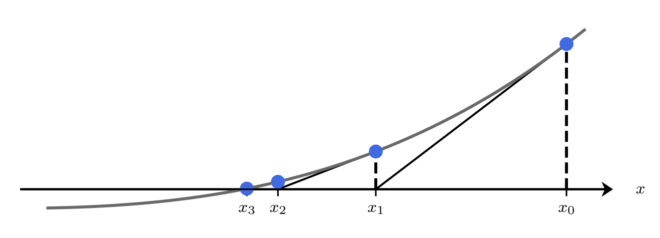
How do we study them ?

Looking for equilibrium solutions

Fixed points

Given a nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, find solutions to

$$\boldsymbol{f}(\boldsymbol{x}^*) = 0$$



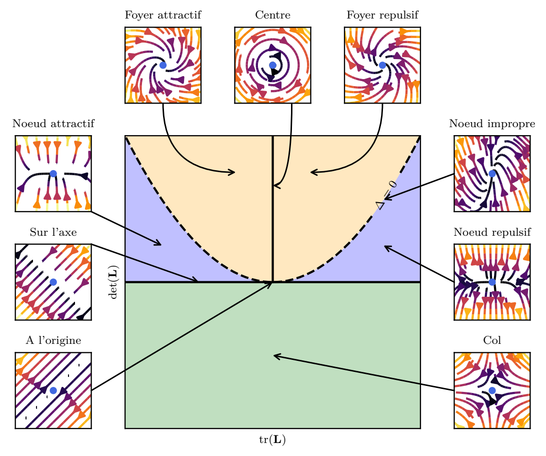
How de we study them ?

Perturbing equilibrium solutions

Linear stability

$$\dot{\boldsymbol{\eta}} \simeq \boldsymbol{f}'(\boldsymbol{x}^*)\boldsymbol{\eta}$$

for $\|\boldsymbol{\eta}\| \ll 1$



How de we study them ?

Perturbing limit cycle solutions

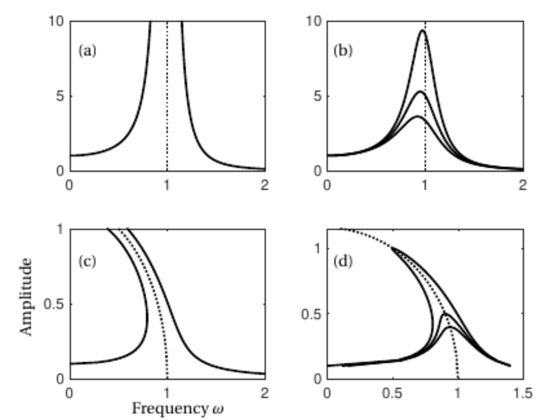
Poincaré-Lindsted

Rescale the solution and time according to

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots$$
$$\tau = \left(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\right) t$$

and solve a hierarchy of linear problems

$$\mathcal{O}(0): \quad \ddot{x}_0 - L(x_0) = 0$$
$$\mathcal{O}(1): \quad \ddot{x}_1 - L(x_0, \omega_0)x_1 = F_1(x_0, \omega_0, \omega_1)$$
$$\mathcal{O}(2): \quad \ddot{x}_2 - L(x_0, \omega_0)x_2 = F_2(x_0, x_1, \omega_0, \omega_1, \omega_2)$$

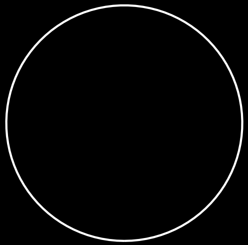


How do we study them ?

Introduce new tools and new concepts

Equation of a circle

$$x^2 + y^2 = R^2$$

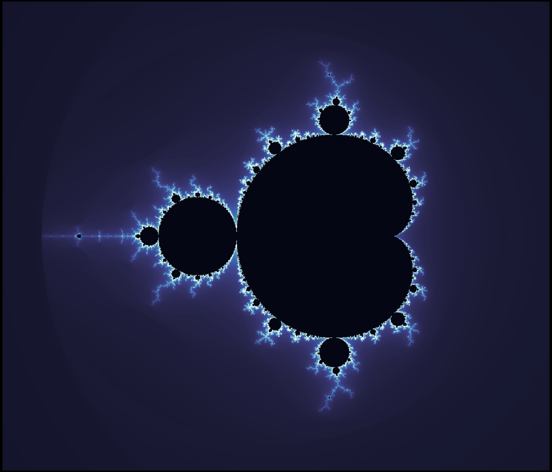


How do we study them ?

Introduce new tools and new concepts

Quadratic map

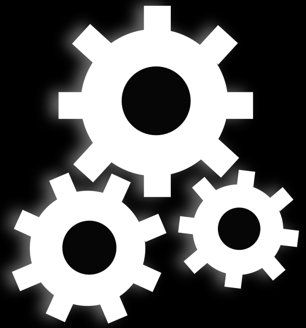
$$x_{k+1} = x_k^2 + c,$$
$$x_0 = 0,$$
$$c \in \mathbb{C}$$



Syllabus

What you'll see in this course

- L2 : Fixed points and their linear stability
- L3 : Elements of bifurcation theory
- L4 : Limit cycles
- L5 : Synchronization and phase dynamics
- L6-L9 : Chaos and strange attractors
- L10 : Cellular automata
- L11-L15 : Data-driven methods for dynamical systems



Syllabus

What you'll need

This is an 'informal' course on dynamical systems. We'll focus on intuition and liberal use of computers rather than proper mathemetical proofs.

We'll still need a bit of maths though :

- **Linear algebra** : eigenvalues and eigenvectors
- **Calculus** : Ordinary differential equations, Power series, Fourier series, Taylor series
- **Numerical analysis** : Temporal integration, root finding