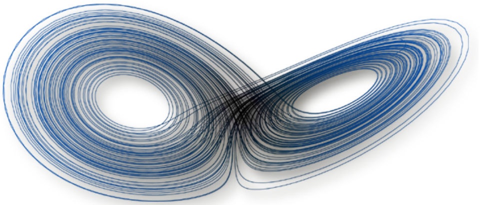
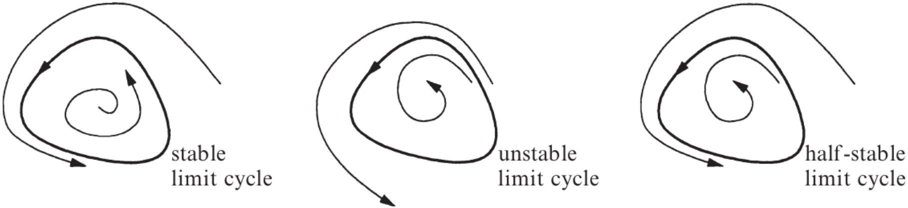


Limit cycles cannot exist

Jean-Christophe Loiseau

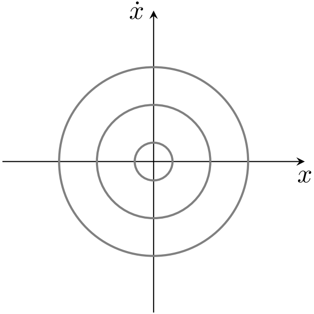
jean-christophe.loiseau@ensam.eu
Laboratoire DynFluid
Arts et Métiers, France.

Limit cycles



Closed orbits are not always limit cycles!

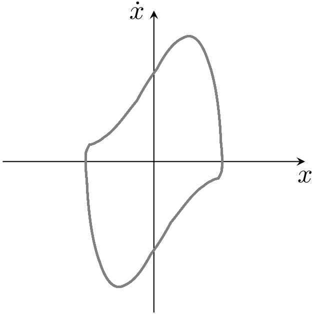
Linear systems $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ can have periodic solutions if $\mathrm{eig}(\boldsymbol{A}) = \pm i\omega$. They are not limit cycles though. If $\boldsymbol{x}(t)$ is a periodic solution, so is $\alpha\boldsymbol{x}(t) \quad \forall\alpha \neq 0$. These closed orbits are not **isolated**.



Limit cycles at not always circle in phase space

van der Pol oscillator

$$\ddot{x} + \mu \left(x^2 - 1 \right) \dot{x} + x = 0$$



When are periodic dynamics **impossible**?

Gradient systems

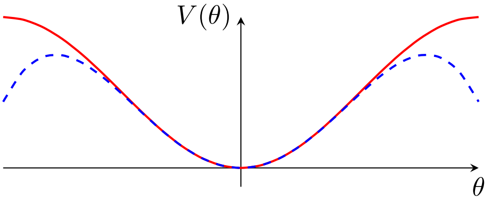
Consider the single-valued scalar function $V(\boldsymbol{x})$ such that the system we consider can be written as

$$\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x}).$$

This is a **gradient system** with **potential function** $V(\boldsymbol{x})$.

$$V(\theta) = -\cos(\theta) + V_0$$

$$V(\theta) = \frac{\theta^2}{2} - \frac{\theta^4}{24}$$

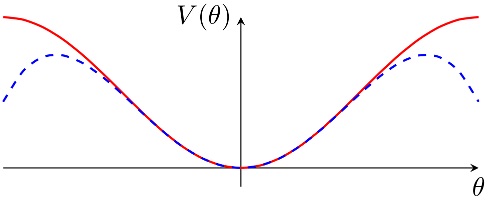


Gradient systems

Theorem: Closed orbits are impossible in gradient systems.

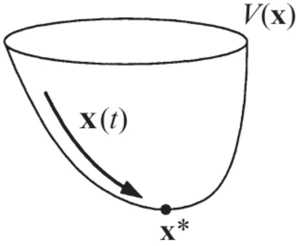
$$V(\theta) = -\cos(\theta) + V_0$$

$$V(\theta) = \frac{\theta^2}{2} - \frac{\theta^4}{24}$$



Lyapunov functions

It is sometimes possible to construct an energy-like function that decreases along trajectories. This is known as a **Lyapunov function**.

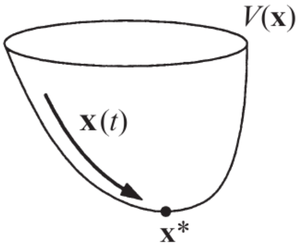


Lyapunov functions

Consider a system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ with a fixed point at $\boldsymbol{x}^*$. A **Lyapunov function** $V(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ needs to satisfy the following properties

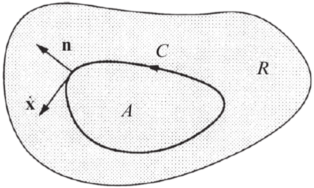
1. It is continuously differentiable.
2. $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^*$ and $V(\boldsymbol{x}^*) = 0$.
3. $\dot{V} < 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^*$.

If such a function function exists, $\boldsymbol{x}^*$ is globally asymptotically stable.



Dulac's criterion

Let $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla \cdot (g\dot{\boldsymbol{x}})$ has one sign throughout R, then there are no closed orbits lying entirely in R.



Divine inspiration

Checking if the system is a **gradient system** is relatively simple. Unfortunately, there is no systematic way to construct **Lyapunov functions** or the $g(\boldsymbol{x})$ function in **Dulac's criterion**. If you can find such functions for your problem, these are however very powerful results.