Driven oscillators and the resonance phenomenon

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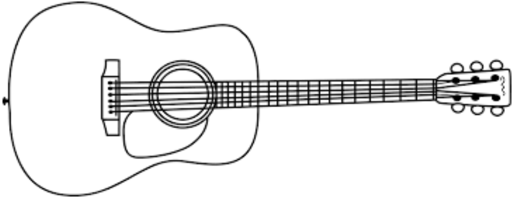
Collapse of the Tacoma bridge (1940).

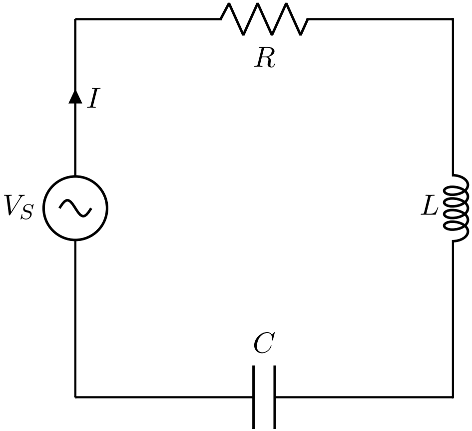


Forced Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = \gamma \cos(\omega t)$$

Forced Harmonic oscillator

$$\ddot{x} + 2\xi\dot{x} + x = \gamma\cos(\omega t)$$



R
I
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L
C

Forced linear oscillator (no damping)

$$\ddot{x} + x = \gamma \cos(\omega t)$$

Forced linear oscillator (no damping)

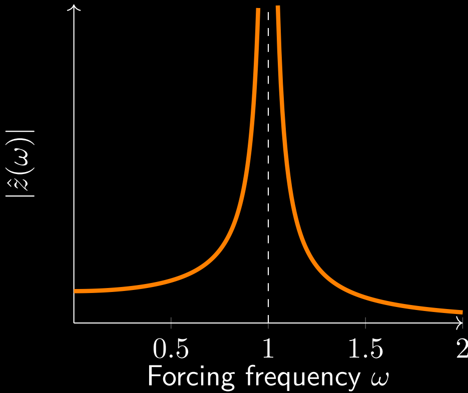
$$\ddot{z} + z = \gamma e^{i\omega t}$$

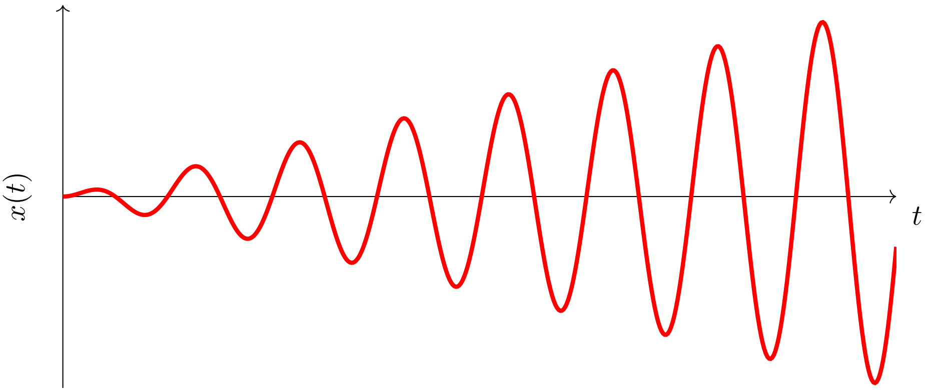
Forced linear oscillator (no damping)

$$\left(1-\omega^2\right)\hat{z}e^{i\varphi}=\gamma$$

Forced linear oscillator (no damping)

$$\hat{z} = \frac{\gamma}{1-\omega^2} e^{-i\varphi}$$





$x(t)$
t

Forced linear oscillator (linear damping)

$$\ddot{x} + 2\xi\dot{x} + x = \gamma\cos(\omega t)$$

Forced linear oscillator (linear damping)

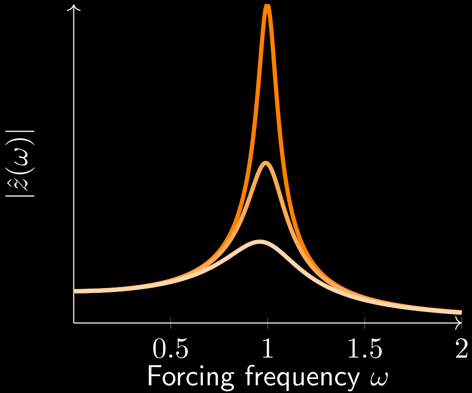
$$\ddot{z} + 2\xi\dot{z} + z = \gamma e^{i\omega t}$$

Forced linear oscillator (linear damping)

$$\left(2\xi i\omega + 1 - \omega^2\right)\hat{z}e^{i\varphi} = \gamma$$

Forced linear oscillator (linear damping)

$$\hat{z} = \frac{\gamma}{2\xi i\omega + 1 - \omega^2} e^{-i\varphi}$$



Conservative nonlinear oscillator

Forced Liénard equation

$$\ddot{x} + g(x) = \gamma \cos(\omega t)$$

Conservative nonlinear oscillator

Forced Duffing oscillator

$$\ddot{x} + x + \epsilon x^3 = \gamma \cos(\omega t)$$

Conservative nonlinear oscillator

$$\frac{d^2x}{d\tau^2} + \frac{1}{\omega^2}x + \frac{\epsilon}{\omega^2}x^3 = \frac{\gamma}{\omega^2}\cos(\tau)$$

Conservative nonlinear oscillator

Retaining only terms of order $\mathcal{O}(\epsilon)$ leads to

$$\frac{d^2x}{d\tau^2} + x = \epsilon\left(\Gamma\cos(\tau) + 2\omega_1 x - x^3\right)$$

where the parameters are defined as $\gamma = \epsilon\Gamma$.

Conservative nonlinear oscillator

Introducing the power series expansion $x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau)$ yields

$$\begin{aligned} \mathcal{O}(1): &\quad \ddot{x}_0 + x_0 = 0 \\ \mathcal{O}(\epsilon): &\quad \ddot{x}_1 + x_1 = 2\omega_1 x_0 - x_0^3 + \Gamma\cos(\tau) \end{aligned}$$

which we can now solve.

Conservative nonlinear oscillator

At leading order, the solution is given by $x_0(\tau) = A\cos(\tau)$. At the next order, we have

$$\ddot{x}_1 + x_1 = \underbrace{\left(2\omega_1 A - \frac{3}{4}A^3 + \Gamma\right)}_{=0}\cos(\tau) - \frac{A^3}{4}\cos(3\tau)$$

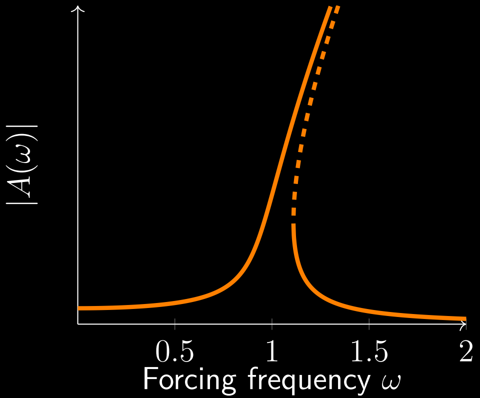
which leads to $\omega_1 = \dfrac{3}{8}A^2 - \dfrac{\Gamma}{2A}$ to avoid secular growth.

Conservative nonlinear oscillator

In the original variables, this leads to

$$\frac{3}{4}\epsilon A^3 + \left(1 - \omega^2\right) A - \gamma = 0$$

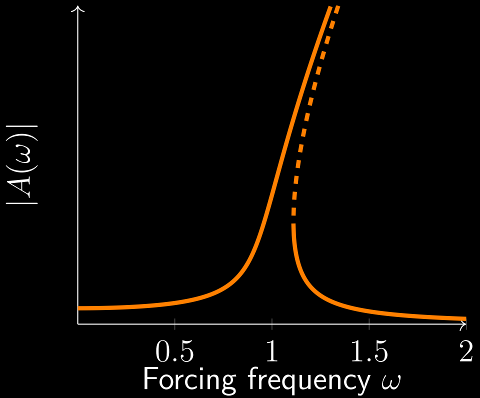
describing the **nonlinear response function** of the system.



Conservative nonlinear oscillator

Nonlinearity prevents the unbounded growth of the oscillations by generating higher-order harmonics.

As the amplitude grows, the resonant forcing frequency changes.



Dissipative nonlinear oscillator

Forced Duffing oscillator with damping

$$\ddot{x} + \delta\dot{x} + x + \epsilon x^3 = \gamma\cos(\omega t)$$

Dissipative nonlinear oscillator

Let's illustrate another technique to derive the nonlinear response function, namely **Harmonic Balance**. Assume the solution is of the form $x(t) = A\cos(\omega t) + B\sin(\omega t)$ and inject into the equations.

Dissipative nonlinear oscillator

$$\left(-\omega^2 A + \omega\delta B + A + \frac{3}{4}\epsilon A^3 + \frac{3}{4}\epsilon A B^2 - \gamma\right)\cos(\omega t)$$
$$+\left(-\omega^2 B - \omega\delta A + \frac{3}{4}\epsilon B^3 + B + \frac{3}{4}\epsilon A^2 B\right)\sin(\omega t)$$
$$+\left(\frac{1}{4}\epsilon A^3 - \frac{3}{4}\epsilon A B^3\right)\cos(3\omega t)$$
$$+\left(\frac{3}{4}\epsilon A^2 B - \frac{1}{4}\epsilon B^3\right)\sin(3\omega t) = 0$$

Dissipative nonlinear oscillator

Neglecting superharmonics at 3ω leads to the balance equations

$$\left(1-\omega^2\right)A+\omega\delta B+\frac{3}{4}\epsilon A^3+\frac{3}{4}\epsilon AB^2=\gamma$$
$$\left(1-\omega^2\right)B-\omega\delta A+\frac{3}{4}\epsilon B^3+\frac{3}{4}\epsilon A^2B=0.$$

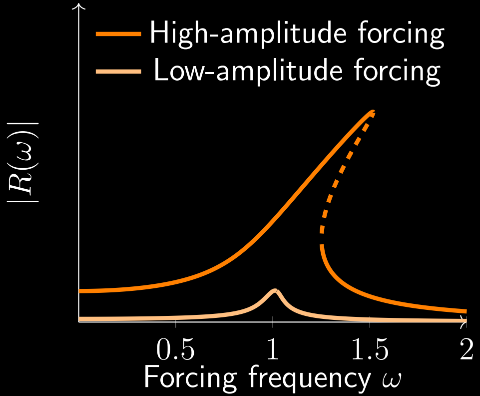
If $B=0$ and $\delta=0$, we recover the response function derived for the undamped case.

Dissipative nonlinear oscillator

These conditions can be combined into

$$\left[\left(\frac{3}{4}\epsilon R^2 + 1 - \omega^2\right)^2 + (\delta\omega)^2\right] R^2 = \gamma^2$$

with $R = \sqrt{A^2 + B^2}$ the amplitude of the oscillation.

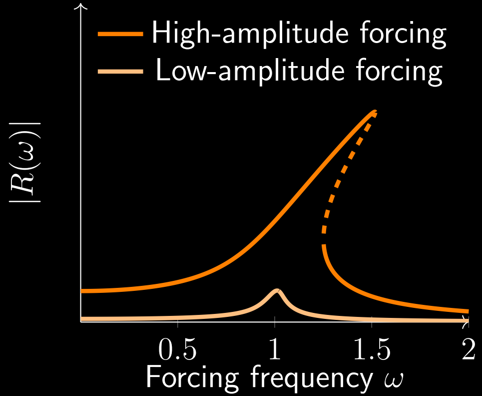


Dissipative nonlinear oscillator

It can be simplified to

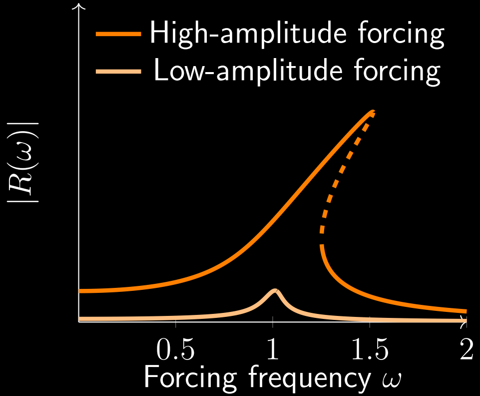
$$\left[\left(\frac{3}{4}R^2 - 2\omega_1\right)^2 + \Delta^2\right] R^2 = \Gamma^2$$

with $1 - \omega^2 = -2\epsilon\omega_1$, $\Delta = \delta/\epsilon$ and $\Gamma = \gamma/\epsilon$.



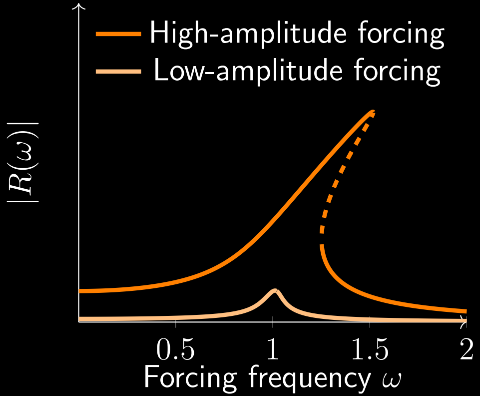
Dissipative nonlinear oscillator

As Γ increases, $R(\omega)$ switches from being single-valued to multi-valued because of a **saddle-node bifurcation**.

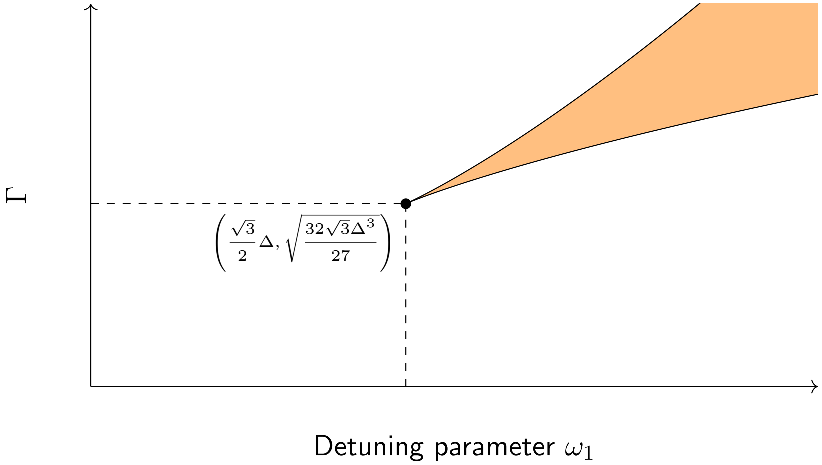


Dissipative nonlinear oscillator

What are the critical values of Γ and ω_1 at which this bifurcation happen ?



The cusp catastrophe

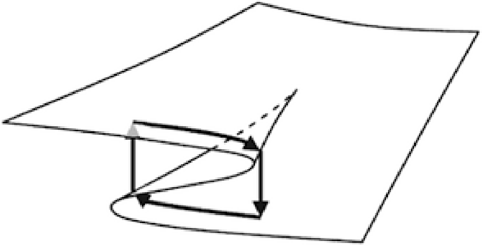


The cusp catastrophe

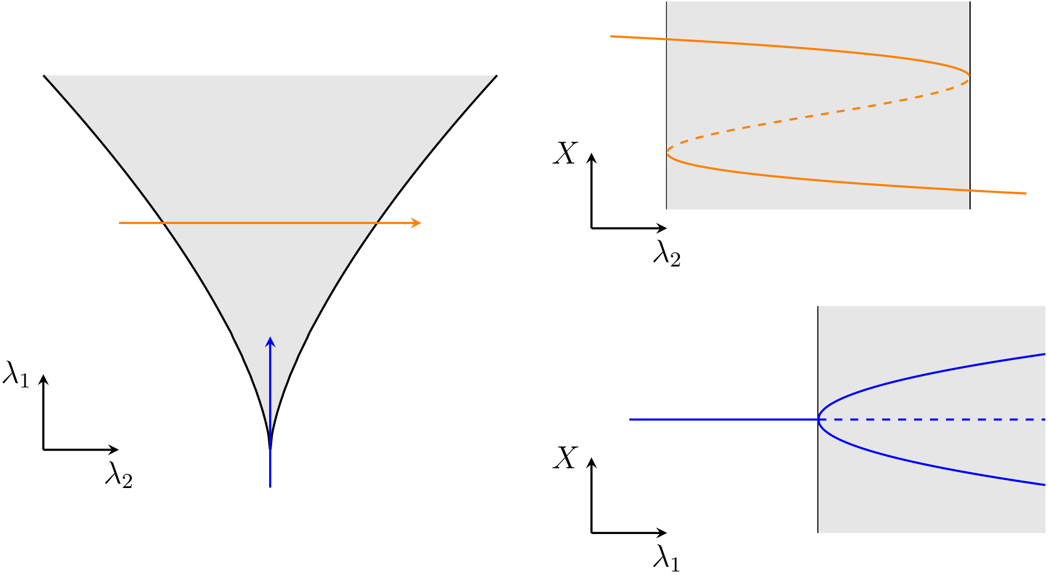
The response function is of the form

$$X^3 + \lambda_1 X + \lambda_2 = 0.$$

This is the canonical form of the **cusp catastrophe**.



The cusp catastrophe



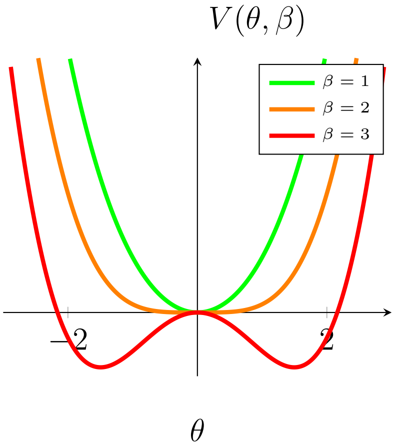
The cusp catastrophe

Add schematic

The cusp catastrophe

Total energy in the system

$$V = 2\mu\theta^2 + m\sin(\theta) - 2\beta\,(1 - \cos(\theta))$$

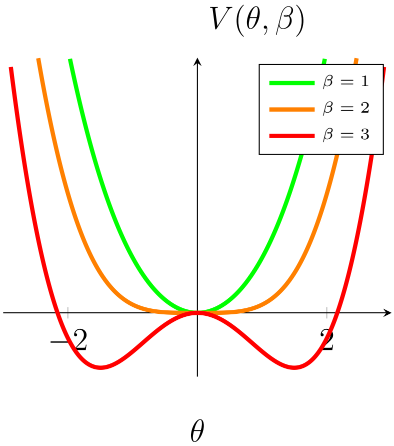


Potential for $m = 0$

The cusp catastrophe

Taylor expansion (and $\beta = 2\mu + b$)

$$V = \frac{2\mu + b}{12}\theta^4 - \frac{m}{6}\theta^3 - b\theta^2 + m\theta$$

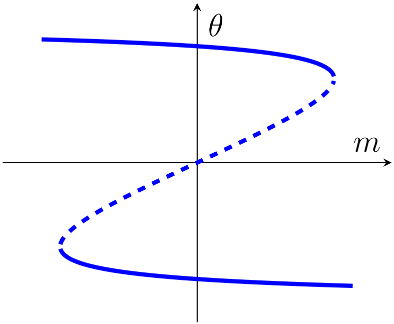


Potential for $m = 0$

The cusp catastrophe

Equilibria are stationary points

$$\frac{2\mu + b}{3}\theta^3 - \frac{m}{3}\theta^2 - 2b\theta + m = 0$$

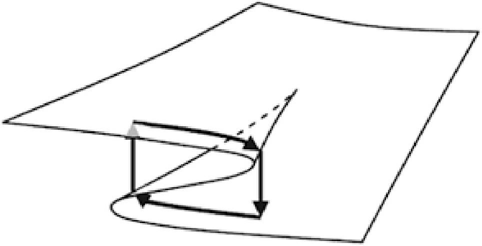


Fixed points for $(\mu, b) = (1, 1.1)$

The cusp catastrophe

Canonical form of the cusp catastrophe

$$X^3 + \lambda_1 X + \lambda_2 = 0$$



Thank you for your attention

Any question ?