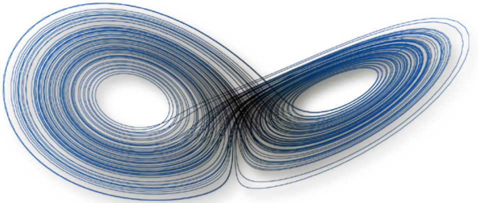

# Poincaré-Lindstedt method for periodic dynamics


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# The Poincaré-Lindstedt method

Summary

The Poincaré-Lindstedt method is a powerful perturbative technique to approximate periodic solutions to ordinary differential equations.

By rescaling time as $\tau = \omega t$ and using power series expansions, it transforms a nonlinear system into a cascade of linear ones which we can easily solve.

The resulting approximation provides insights into the frequency shift phenomenon and harmonics generation induced by the nonlinearity.

$$
\begin{tikzcd}
\boxed{\ddot{x}_0 + x_0 = 0} \arrow[d] \arrow[dd, bend left] \\
\boxed{\ddot{x}_1 + x_1 = f(x_0)} \arrow[d] \arrow[dd, bend left] \\
\boxed{\ddot{x}_2 + x_2 = f(x_0, x_1)} \arrow[d] \\
\boxed{\ddot{x}_3 + x_3 = f(x_0, x_1, x_2)}
\end{tikzcd}
$$

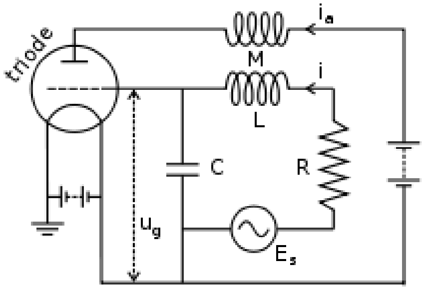
# Beyond conservative systems

Application to the van der Pol oscillator

Let us consider the **van der Pol oscillator** whose governing equations are given by

$$\ddot{x} + x = \epsilon \left(1 - x^2\right) \dot{x}.$$

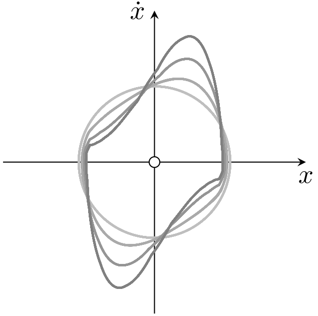
It is a canonical example of nonlinear oscillators proposed in 1927 by the Dutch electrical engineer Balthasar van der Pol.



# Beyond conservative systems

Application to the van der Pol oscillator

For $\epsilon > 0$, its dynamics are characterized by a limit cycle that we wish to study analytically using the Poincaré-Lindstedt method.



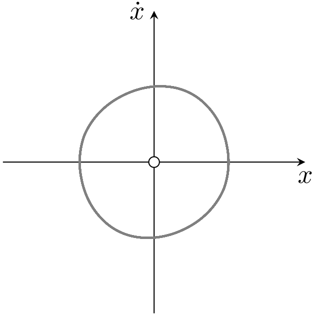
# Poincaré-Lindstedt method

Application to the van der Pol oscillator

The problem to be studied is thus the following

$$\begin{cases} \ddot{x} + x = \epsilon\left(1 - x^2\right)\dot{x} \\ x(0) = x_{\text{in}} \\ \dot{x}(0) = 0 \end{cases}$$

where $x_{\text{in}}$ is the unknown initial condition and $\epsilon \ll 1$ is our control parameter.




Asymptotic dynamics for $\epsilon = 0.1$

# Poincaré-Lindstedt method

Application to the van der Pol oscillator

Let us rescale time as $\tau = \omega t$ such that

$$\frac{d}{dt} = \omega \frac{d}{d\tau}, \quad \text{and} \quad \frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2}$$

and use a power series expansion of the unknown frequency $\omega$, solution $x(t)$ and initial condition $x_{\text{in}}$

$$\begin{aligned}
\omega &= 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \\
x(\tau) &= x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \cdots \\
x_{\text{in}} &= A_0 + \epsilon A_1 + \epsilon^2 A_2 + \cdots
\end{aligned}$$

where $x_1(\tau)$ and $x_2(\tau)$ are small corrections to the harmonic oscillator solution $x_0(\tau)$ and $\omega_1$ and $\omega_2$ are small corrections to its natural frequency.

# Poincaré-Lindstedt method

Application to the van der Pol oscillator

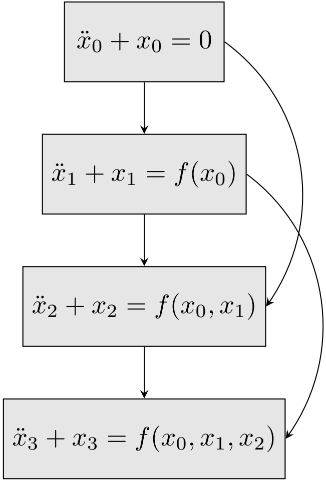
Introducing these expansions into our equation and regrouping by power of $\epsilon$ yields

$$\begin{aligned}
\mathcal{O}(\epsilon^0): &\quad \ddot{x}_0 + x_0 = 0 \\
\mathcal{O}(\epsilon): &\quad \ddot{x}_1 + x_1 = \left(1 - x_0^2\right)\dot{x}_0 - 2\omega_1 \ddot{x}_0 \\
\mathcal{O}(\epsilon^2): &\quad \ddot{x}_2 + x_2 = f(x_0, \dot{x}_0, \ddot{x}_0, x_1, \dot{x}_1, \ddot{x}_1)
\end{aligned}$$

supplemented with the initial conditions

$$x_i(0) = A_i, \quad \dot{x}_i(0) = 0 \quad \forall i.$$

Once again, we trade a nonlinear system for a cascade of linear ones.



# Poincaré-Lindstedt method

Application to the van der Pol oscillator

The zeroth-order solution is given by $x_0(\tau) = A_0 \cos(\tau)$. Injecting $x_0(\tau)$ into the equation for $x_1(\tau)$ yields

$$\ddot{x}_1 + x_1 = A_0 \left(A_0^2 \cos^2(\tau) - 1\right) \sin(\tau) + 2\omega_1 A_0 \cos(\tau)$$

which can be simplified to

$$\ddot{x}_1 + x_1 = 2A_0\omega_1 \cos(\tau) + A_0 \left(\frac{A_0^2}{4} - 1\right) \sin(\tau) + \frac{A_0^3}{4} \sin(3\tau)$$

using trigonometric identities.

**Trig. identity**

$$\cos^2(\tau) \sin(\tau) = \frac{1}{4} \left(\sin(\tau) + \sin(3\tau)\right)$$

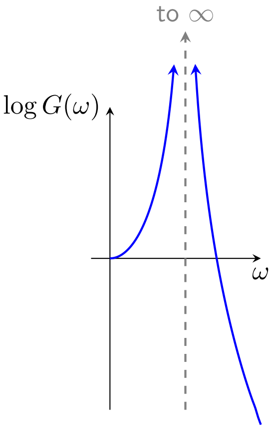
# Poincaré-Lindstedt method

Application to the van der Pol oscillator

If left unchecked, the terms $\cos(\tau)$ and $\sin(\tau)$ will lead to **secular growth**. We thus need to set $\omega_1$ and $A_0$ such that

$$
\begin{aligned}
2A_0\omega_1 &= 0 \\
A_0\left(\frac{A_0^2}{4} - 1\right) &= 0
\end{aligned}
$$

to avoid this unphysical behaviour. This leads to $\omega_1 = 0$ and $A_0 = 2$.



# Poincaré-Lindstedt method

Application to the van der Pol oscillator

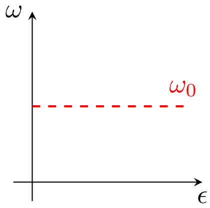
At order $\mathcal{O}(\epsilon)$, the equation reduces to

$$\ddot{x}_1 + x_1 = 2\sin(3\tau)$$

whose general solution is given by

$$x_1(\tau) = A_1\cos(\tau) + \frac{1}{4}\left(3\sin(\tau) - \sin(3\tau)\right).$$

Note that $A_1$ is still undetermined and one needs to go to order $\mathcal{O}(\epsilon^2)$ to determine it.



# Poincaré-Lindstedt method

Application to the van der Pol oscillator

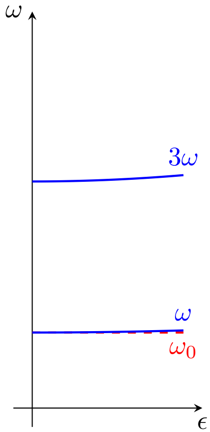
Continuing this process to $\mathcal{O}(\epsilon)$ leads to the second-order frequency correction

$$\omega_2 = \frac{7}{16}$$

as well as to the condition $A_1 = 0$. Our first-order approximation thus reads

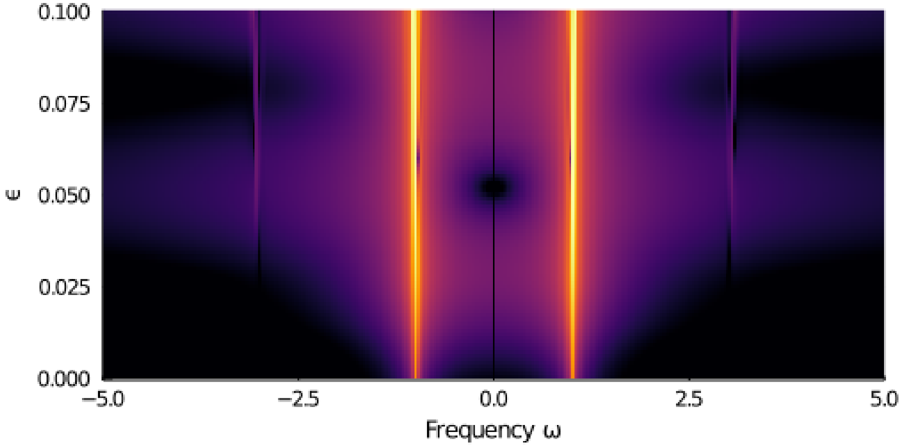
$$x(t) = 2\cos(\omega t) + \epsilon\left(\frac{3}{4}\sin(\omega t) - \frac{1}{4}\sin(3\omega t)\right) + \mathcal{O}(\epsilon^2)$$

with $\omega = 1 + \frac{7\epsilon^2}{16} + \mathcal{O}(\epsilon^4)$. Once again, nonlinearity causes a **frequency shift** and the generation of **high-order harmonics**.



# Poincaré-Lindstedt method

Application to the van der Pol oscillator



# Poincaré-Lindstedt : the van der Pol oscillator

## Summary

The Poincaré-Lindstedt method is a powerful perturbative technique to approximate periodic solutions to ordinary differential equations.

For $\epsilon < 0$, the only attractor is a stable fixed point. As $\epsilon$ becomes positive, the dynamics settles on a constant-amplitude limit cycle.

By design, the Poincaré-Lindstedt method cannot however capture transient effects. We'll need another technique for that purpose.