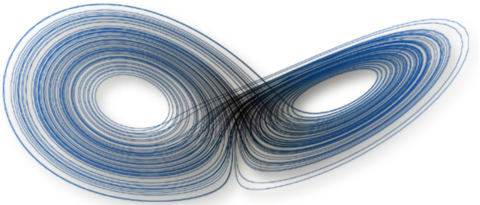

Conservative, reversible and dissipative dynamical systems

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Two-dimensional systems

Two major classes of dynamical systems

Conservative

$$\dot{x} = y$$
$$\dot{y} = -\sin(x)$$

Dissipative

$$\dot{x} = y$$
$$\dot{y} = -y - x - x^3$$

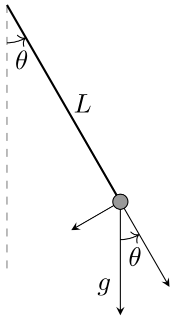
Two-dimensional systems

Conservative systems

The equations of motion of many physical systems can be derived from Newton's principles. They are of the form

$$m\ddot{x} = F(x)$$

where $F(x)$ is independent of both $\dot{x}$ and t, i.e. the system is autonomous and frictionless.



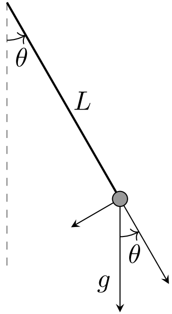
Two-dimensional systems

Conservative systems

Under this assumption, we can show that $F(x)$ derives from a potential, i.e.

$$F(x) = -\frac{dV}{dx}$$

such as the gravitational potential energy $V(x) = mgx$ or the elastic potential energy of a spring $V(x) = \frac{1}{2}kx^2$.



Two-dimensional systems

Conservative systems

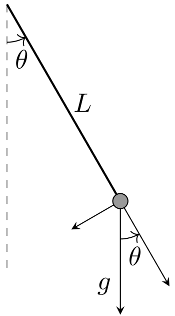
From there, the equations of motion can be rewritten as

$$m\ddot{x} + \frac{dV}{dx} = 0.$$

One can then show that

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) = 0.$$

The quantity $\frac{1}{2}m\dot{x}^2 + V(x)$ is thus **conserved** over time.

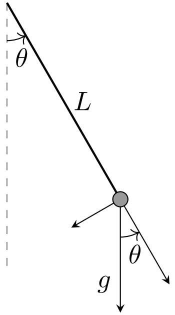


Two-dimensional systems

Conservative systems

For the pendulum, the equations of motions are

$$\underbrace{mL\ddot{\theta}}_{\text{Acceleration}} + \underbrace{mg\sin(\theta)}_{\text{-Weight}} = 0$$



Multiplying by the velocity $L\dot{\theta}$ yields

$$\frac{d}{dt}\left(\frac{1}{2}m(L\dot{\theta})^2 - mgL\cos(\theta)\right) = 0.$$

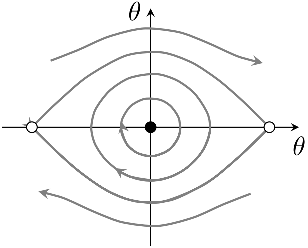
We thus recover that the **total energy** of the system is conserved (in the absence of friction).

Two-dimensional systems

Conservative systems

For a conservative system, the energy remains constant along a given trajectory.

Fixed points cannot be attractive and the trajectories are closed orbits.



Conservative systems

Hamiltonian formalism

Conservative systems are crucially important in physics and gave rise to **Hamiltonian mechanics** (in opposition to Lagrangian mechanics).

In a suitable reference frame $(\boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{p}$ and $\boldsymbol{q}$ are the **generalized coordinates**, the equations of motion take a special form known as the **Hamilton equations**.

Hamilton equations

$$\dot{\boldsymbol{p}} = -\frac{d\mathcal{H}(\boldsymbol{p}, \boldsymbol{q})}{d\boldsymbol{q}}$$

$$\dot{\boldsymbol{q}} = +\frac{d\mathcal{H}(\boldsymbol{p}, \boldsymbol{q})}{d\boldsymbol{p}}$$

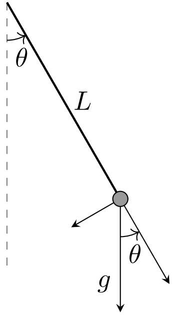
Two-dimensional systems

Dissipative systems

Let us reconsider the pendulum example but now taking into damping. The equations of motion now read

$$\ddot{\theta} + 2k\dot{\theta} + \sin(\theta) = 0$$

where the term $2k\dot{\theta}$ models friction at the joint or viscous drag due to the surrounding air.



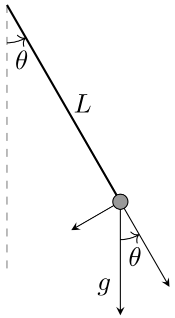
Two-dimensional systems

Dissipative systems

Multiplying by $\dot{\theta}$ to obtain the energy equation yields

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \cos(\theta)\right) = -k\dot{\theta}^2 < 0.$$

The total energy is no longer conserved. It decreases over time. We'll say it gets **dissipated** due to friction.

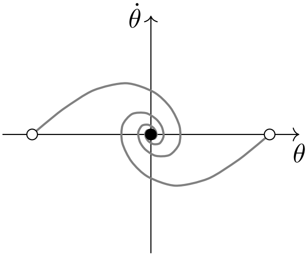


Two-dimensional systems

Dissipative systems – Attractors

As $t \to \infty$, the dynamics of a dissipative systems settle on an **attractor**.

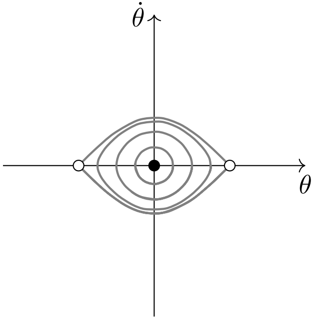
For a two-dimensional system, these attractors can only be **fixed points** or **limit cycles**.



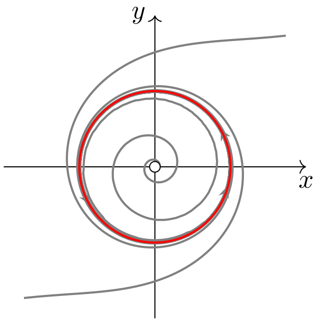
Two-dimensional systems

Periodic orbits in conservative systems vs. limit cycles in dissipative systems

Conservative system



Dissipative system



Two-dimensional systems

Dissipative systems – Volume contraction in phase space

Thinking of the dynamics $\boldsymbol{f}(\boldsymbol{x})$ as a vector field, dissipativity can be understood as volume contraction in phase space.

Using the divergence theorem, it can easily be shown that a volume $V(t)$ of initial conditions evolves as

$$\dot{V} = \int_V \nabla \cdot \boldsymbol{f}(\boldsymbol{x}) \; dV.$$

For a conservative system, we thus have $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = 0 \; \forall \boldsymbol{x}$. For a dissipative system, we have that on average $\nabla \cdot \boldsymbol{f}(\boldsymbol{x}) < 0$.