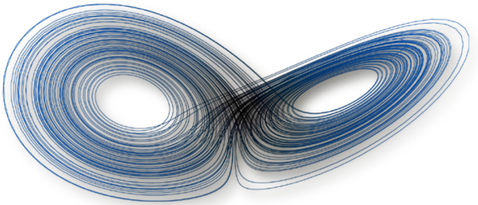


# Nonlinear physics, dynamical systems and chaos theory

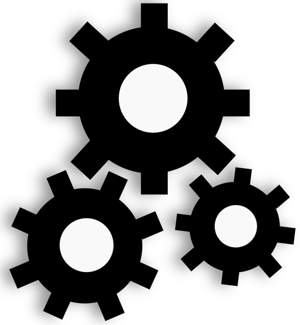

Jean-Christophe Loiseau

*jean-christophe.loiseau@ensam.eu*
*Laboratoire DynFluid*
*Arts et Métiers, France.*

## Basic information

Organization

- Lectures will take place every Tuesdays and Thurdays, from 3:30pm to 5:30pm until late February.
- Evaluation will be divided in two parts:
  - ↪ A two-hour long written exam late February.
  - ↪ A homework project involving mathematics and numerical simulations.
- Do not hesitate to go through your linear algebra notes during Christmas vacation to refresh a bit!



# Basic information

Homework project

- Please use **Python 3** or **Julia**.
  - ↪ Open-source programming languages with excellent scientific computing capabilities.
- You can install both of them using **Anaconda**.
  - ↪ Available for Windows, Mac OS and Linux.
- Numerous online resources to get familiar with both languages if needed, e.g.
  - ↪ `https://www.codeacademy.com`

# Basic information

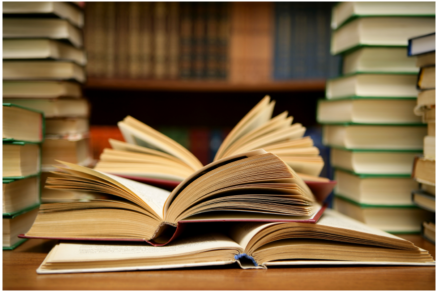
Useful references (in French)

## General knowledge

- I. Stewart. *Dieu joue-t'il au dés?* Flammarion (2004).
- J. Gleick. *La théorie du chaos.* Flammarion (2008).
- I. Prigogine. *Les lois du chaos.* Flammarion (2008).

## Textbooks

- P. Bergé *et al. L'ordre dans le chaos.* Hermann (1998).
- P. Manneville. *Instabilités, chaos et turbulence.* Ed. Ecole Polytechnique (2004).

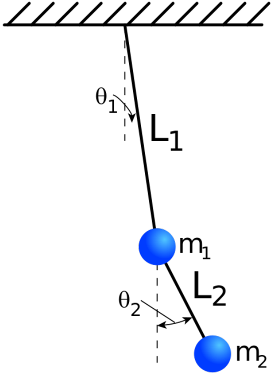


# What is a dynamical system?

## A few examples

# A few examples

The double pendulum




- Its Lagrangian is

$$\mathcal{L} = \overbrace{\frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \left( l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right)\right)}^{\text{Kinetic energy}}$$
$$\underbrace{-(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2}_{\text{Potential energy}}.$$

- Equations of motions are given by

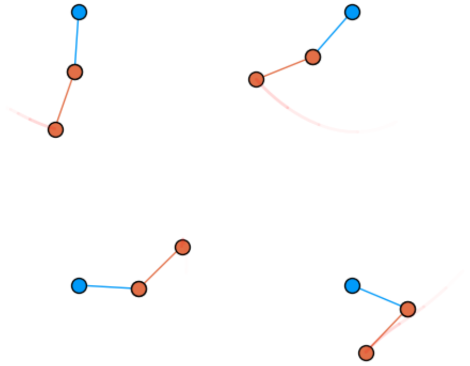
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0.$$

- Nonlinear system of ODEs.
  - ↪ Very few analytical solutions are known.

# A few examples

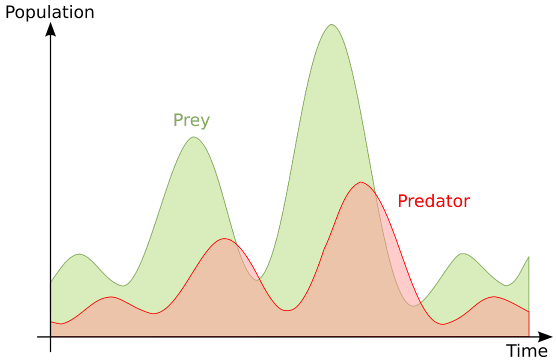
## The double pendulum

- Simple mechanical sytem exhibiting nonetheless complex dynamics.
- Evolutions of similar initial conditions diverge exponentially fast.
  - $\hookrightarrow$ Hallmark of chaotic dynamics.
- Limited prediction horizon despite its deterministic equations of motion.



# A few examples

## Prey-Predator system




- Dynamics of a prey-predator system can be modeled as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \alpha x - \beta xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\gamma y + \delta xy.\end{aligned}$$

- Describes the observations of hares and lynx populations in Canada in the early 1900's.

# A few examples

Chemical reaction-diffusion systems

- Spatio-temporal reaction-diffusion systems can be modeled as
$$\frac{\partial \boldsymbol{q}}{\partial t} = \boldsymbol{D}\nabla^2 \boldsymbol{q} + \mathcal{R}(\boldsymbol{q}),$$
where $\boldsymbol{D}$ describes the diffusion of each species and $\boldsymbol{R}(\boldsymbol{q})$ the inter-species reactions.

- Set of nonlinear partial differential equations exhibiting suprising physical phenomena!
  - ↪ Traveling waves, pattern formation, spatiotemporal chaos, etc.

# A few examples

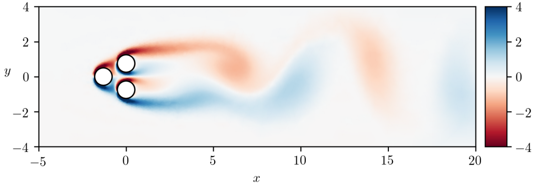
## Incompressible flow past cylinders

- Dynamics are governed by the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \boldsymbol{f}$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

- Set of nonlinear partial differential equations.

- Give rise to an extremely high-dimensional nonlinear system once discretized!
  $\hookrightarrow$ $10^5$ to $10^{10}$ degrees of freedom.



# How do we study dynamical systems?

A quick overview of what's coming.

# How do we study dynamical systems?

Mathematical frameworks

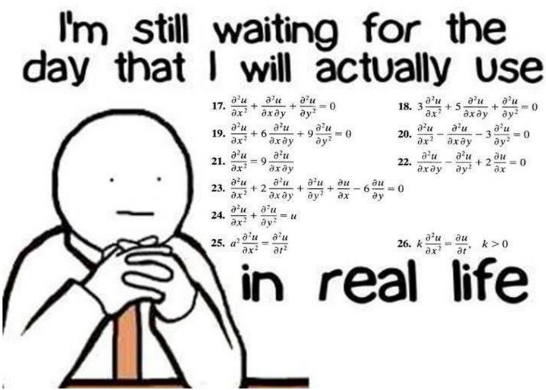
- All of the examples considered are governed by equations of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}),$$

  with $\boldsymbol{x}$ the state vector and $\boldsymbol{\mu}$ the vector of parameters.

- In general, the function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu})$ is a *nonlinear* function of the state.

- Skimming through your lecture notes on *ordinary differential equations* might be a good idea.



# What is a linear system?

Let us consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Under which condition(s) is it **linear**?

# What is a linear system?

A few definitions

- Consider $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ two solutions of our system.
- A system is said to be **linear** if:
  - ↪ $\boldsymbol{w}(t) = \alpha \boldsymbol{u}(t) + \beta \boldsymbol{v}(t)$ is also solution,
  - ↪ $\boldsymbol{f}\left(\alpha \boldsymbol{u} + \beta \boldsymbol{v}\right) = \alpha \boldsymbol{f}(\boldsymbol{u}) + \beta \boldsymbol{f}(\boldsymbol{v})$,
  - ↪ It satisfies the **superposition principle**.
- If so, it can be rewritten as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x},$$

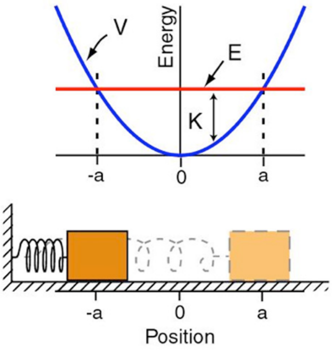
  where $\boldsymbol{A}$ is a linear operator (i.e. a matrix).

- The general solution is given by

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0.$$

# What is a linear system?

Example: the harmonic oscillator




- Dynamics of a (damped) harmonic oscillator are governed by

$$\ddot{x} = -2k\dot{x} - \omega_0^2 x.$$

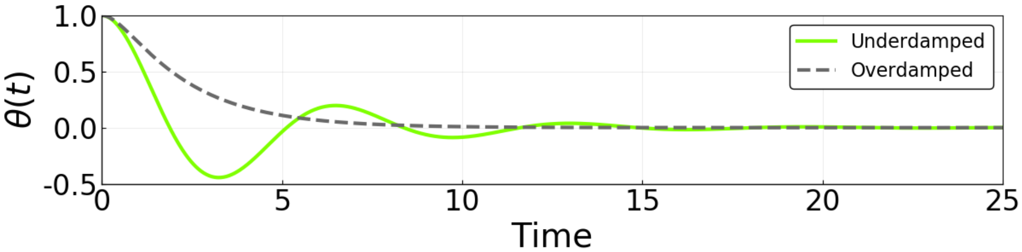
- Introducing $y = \dot{x}$, we can recast it as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Depending on the friction parameter $k$, the system can be overdamped, under-damped or criticaly damped.
  - $\hookrightarrow$ Each regime corresponds to different dynamics.

# What is alinear system?

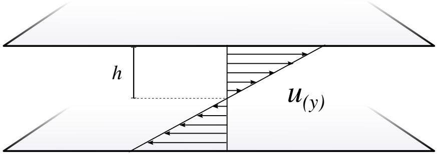
Example: the harmonic oscillator



# What is a linear system?

Example: the Orr-Sommerfeld-Squire equations

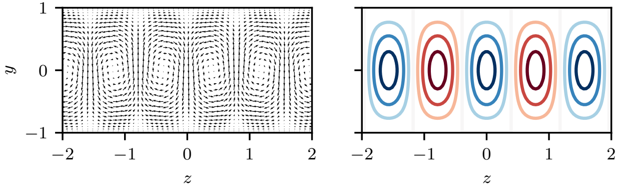
- OSS equations are a reformulation of the linearized Navier-Stokes equations.
  - ↪ See your hydrodynamic instability class for more details.




- In matrix form, they read

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}\end{bmatrix} = \underbrace{\begin{bmatrix}\boldsymbol{L}_{OS} & 0\\ \boldsymbol{C} & \boldsymbol{L}_{S}\end{bmatrix}}_{\mathcal{L}}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\eta}\end{bmatrix},$$

with $\boldsymbol{v}$ the wall normal velocity and $\boldsymbol{\eta}$ the vorticity.




- Describe the evolution of infinitesimal perturbations in parallel shear flows.

## Warning!

If linear systems have appeared so frequently during the course of your studies, it is solely because they are easy to study! Most linear systems are actually only an approximation of a more complex (but more realistic) nonlinear system.

# What is a nonlinear system?

Let us consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Under which condition(s) is it **nonlinear**?

# What is a nonlinear system?

Example: photon emission in laser

- Photon emission in a laser can be modeled by

$$\dot{n} = gn(N_0 - an) - kn$$

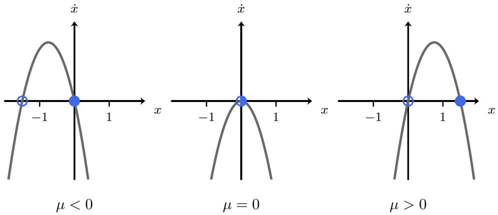
where $g$ is the gain of the laser, $k$ describe the loss and $N(t) = N_0 - an$ is the number of excited atoms.

- It can be recast as

$$\dot{x} = \mu x - x^2.$$

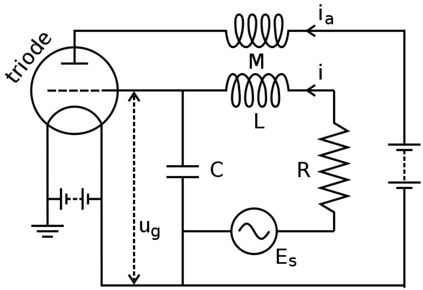
- It is a first-order nonlinear ordinary differential equations.

- We do not have a closed-form solution despite the apparent simplicity of the equation.



# What is a nonlinear system?

Example: the van der Pol oscillator




- The equations governing this eletrical circuit are

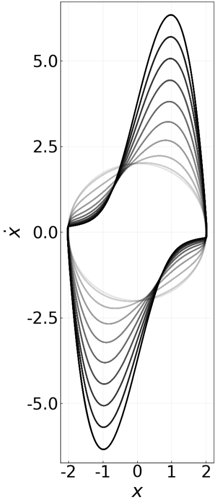
$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0.$$

- It is known as the *Van der Pol oscillator*.
  - ↪ It is a non-conservative oscillator with nonlinear damping.
- It serves to model a large variety of physical systems.
  - ↪ Action potentials of neurons, interaction between two plates in a geological fault, . . .

# What is a nonlinear system?

Example: the van der Pol oscillator

- Standard model to study the properties of simple nonlinear oscillators.
- The shape of the *limit cycle* in the $(x, \dot{x})$ plane varies drastically as $\mu$ increases.
- When forcing the system at certain frequencies, the dynamics become chaotic.



# Case study I

## The simple pendulum

# The simple pendulum

## Governing equations

- The governing equations read

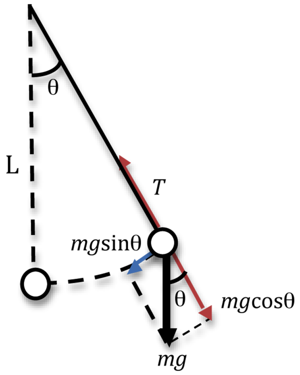
$$\ddot{\theta} + 2k\dot{\theta} + \omega_0^2 \sin(\theta) = 0,$$

with $k$ the friction coefficient and $\omega_0^2 = g/L$ the natural frequency of the pendulum.

- Introducing $x = \theta$ and $y = \dot{\theta}$, we can recast it as a system of first-order ODEs

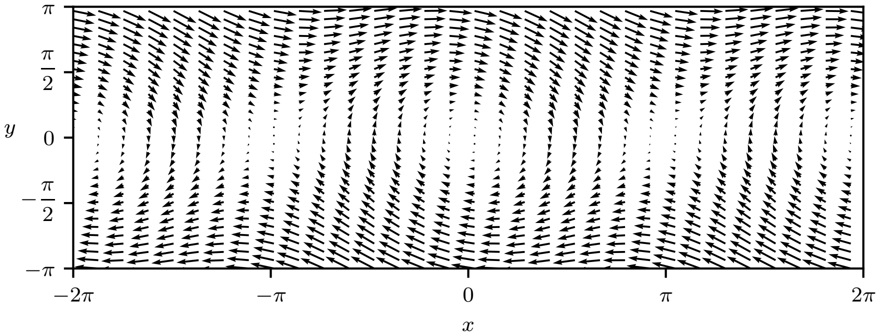
$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -2ky - \omega_0^2 \sin(x).\end{aligned}$$

- Let's study in details the properties of this dynamical system.



# The simple pendulum

Phase plane representation



# The simple pendulum

## Fixed points

- There are a few points of the phase plane where the system is in equilibrium. These are known as **fixed points**. They satisfy the equations

$$\dot{x} = 0 \text{ and } \dot{y} = 0.$$

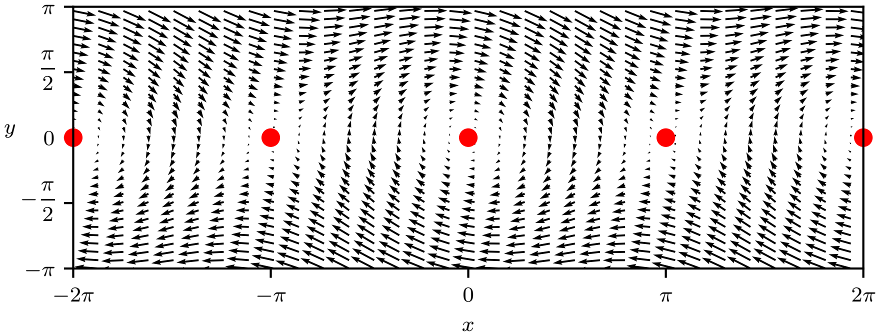
- For the simple pendulum, these fixed points are solution to

$$y = 0 \text{ and } \sin(x) = 0.$$

- Physically, they correspond to situation where the pendulum is pointing either downward ($x = 0 \mod 2\pi$ ) or upward ($x = \pi \mod 2\pi$).

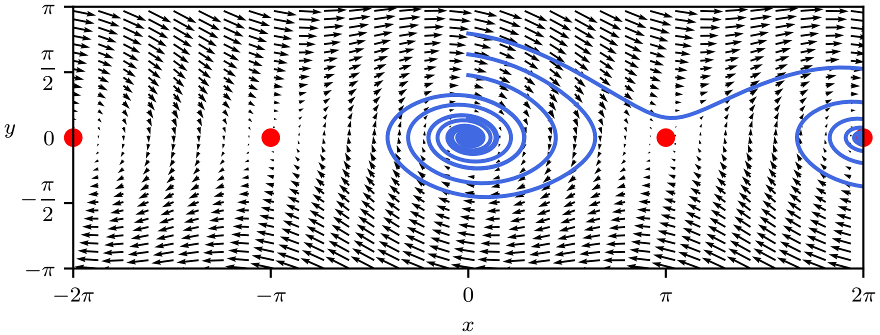
# The simple pendulum

Fixed points



# The simple pendulum

Trajectories



# The simple pendulum

Linearizing the system in the downward position

- Let us linearize the equations in the vicinity of the downward position. The equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2k \end{bmatrix}}_{\boldsymbol{A}}\begin{bmatrix} x \\ y \end{bmatrix}.$$

- In this small oscillation limit, we recover the damped harmonic oscillator model.
- For all (positive) values of the friction parameter $k$, the eigenvalues $\lambda$ of $\boldsymbol{A}$ are characterized by

$$\Re(\lambda) < 0$$

indicating that this equilibrium position is **linearly stable**.

## The simple pendulum

Linearizing the system in the upward position

- Let us linearize the equations in the vicinity of the upward position. The equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 2k \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- For all (positive) values of the friction parameter $k$, the eigenvalues $\lambda$ of $\boldsymbol{A}$ are characterized by

$$\Re(\lambda) > 0$$

indicating that this equilibrium position is **linearly unstable**.

# The simple pendulum

## Conclusion

- The simple pendulum by itself is a fairly boring system.
  - ↪ Two fixed points, one linearly stable and one linearly unstable.
  - ↪ For $t \to \infty$, the system eventually settles in the stable fixed point due to the dissipative nature of the system.
- Its brother, the double pendulum, has far richer dynamics!
  - ↪ Harder to study though because of its four-dimensional phase space.
  - ↪ In the Hamiltonian limit (i.e. no friction), special numerical techniques need to be used to verify the physics.

# Case study II

## The Lorenz system

# The Lorenz system

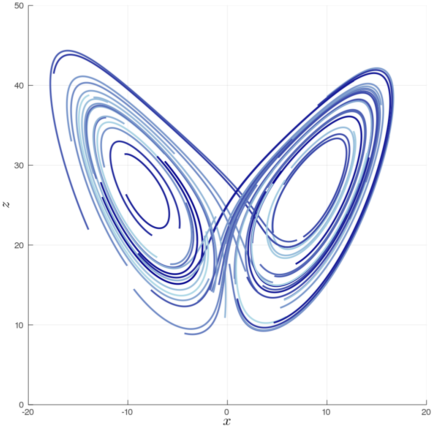
A simplified model of atmospheric convection

- It is a (very) simplified model of atmospheric convection which can be derived analytically from the Navier-Stokes equations.

- In its most common form, it reads

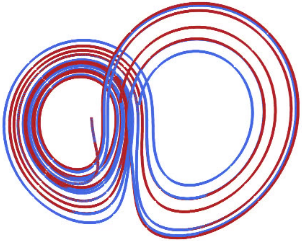
$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z.\end{aligned}$$

- In what follows, we set $\sigma = 10$ and $\beta = {}^{8}/{3}$ and study the evolution of the system as $\rho$ varies.



# The Lorenz system

A (popular) chaotic system



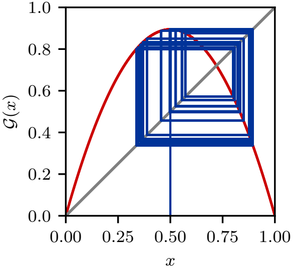
- For certain range of parameters, the Lorenz system exhibits chaotic dynamics.
- The corresponding *attractor* is known as a *strange attractor*.
  - ↪ It cannot be described by standard geometry.
  - ↪ One instead needs to use *fractal geometry* (its dimension is 2.06 for instance).
- Its double-winged structure is one of the reason why the sensitivity to initial condition is called the *butterfly effect* in mainstream media.

# Case study III

## The logistic map

# The logistic map

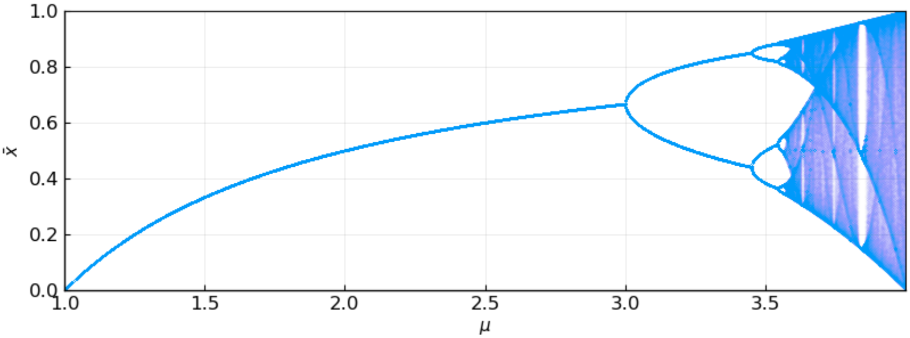
## A discrete-time nonlinear system




- Very simple model for population dynamics popularized during the 1970's by Robert May.
  - ↪ Played a crucial role in the development of chaos theory.
- The model reads
$$x_{k+1} = \mu x_k (1 - x_k).$$
- Despite its simplicity, the system can exhibit complex dynamics.
  - ↪ Fixed points, periodic orbits, chaotic dynamics.
- You are strongly encourage to implement this model yourself and play with it!
  - ↪ Only a handful lines of code are needed.

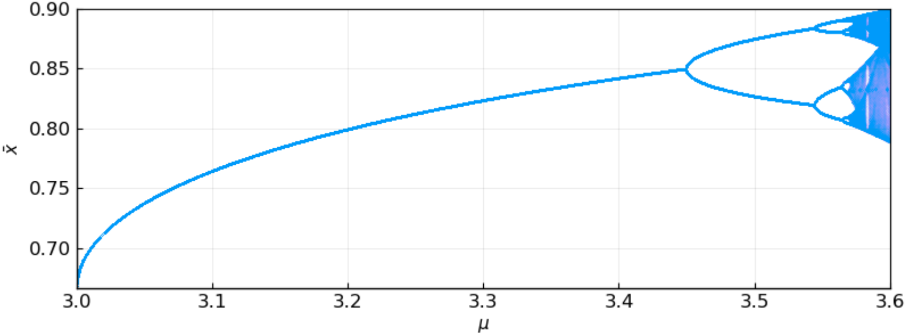
# The logistic map

## Bifurcation diagram



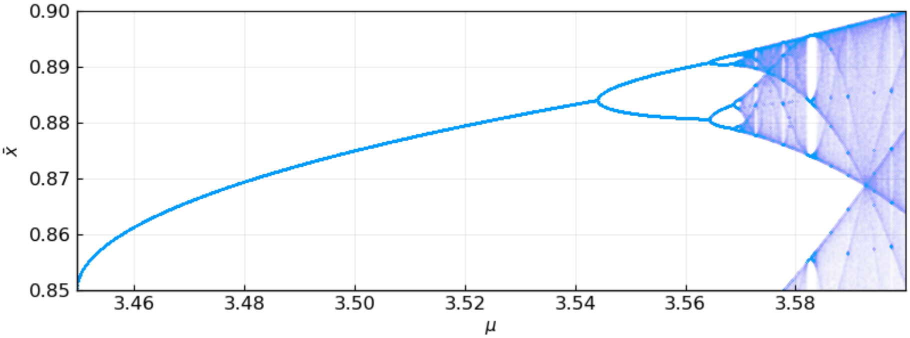
# The logistic map

## Bifurcation diagram



# The logistic map

Bifurcation diagram



# The logistic map

## Connection with the Mandelbrot set

- The logistic map is closely related to the Mandelbrot set, a famous fractal object.
- Given the map $z_{k+1} = z_k^2 + c$ with $c \in \mathbb{C}$, the Mandelbrot set is defined as

$$c \in \mathcal{M} \iff \lim_{k \to \infty} |z_{k+1}| \leq 2.$$

- The last course of this class will be dedicated to a quick introduction to fractal geometry.