1. (AoPS) For any triangle with angles A, B, C, prove that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

Solution: We can apply Jensen's inequality on the function $f(x) = \sin x$, which is concave (flip the inequality sign) on $[0, \pi]$. Plugging in the three angles above, which are all less than π since we have a triangle, we get

$$\lambda_1 \sin A + \lambda_2 \sin B + \lambda_3 \sin C \leq \sin(\lambda_1 A + \lambda_2 B + \lambda_3 C).$$

This is true for any $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that $\sum_{i=1}^{3} \lambda_i = 1$. So we can choose all three equal to $1/3$ to obtain

$$\frac{1}{3}(\sin A + \sin B + \sin C) \leq \sin\left(\frac{1}{3}(A + B + C)\right).$$

Since we have a triangle, $A + B + C = \pi$, and $\sin(\pi/3) = \sqrt{3}/2$. Plugging this in we get the desired result

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

2. Prove the "AM-GM inequality", that the arithmetic mean of non-negative real numbers $x_1, x_2, \cdots, x_n$ is always greater than or equal to their geometric mean. That is

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}.$$

We can apply Jensen's inequality to the function $f(x) = \log x$, which is also concave (flip the inequality sign). Again we will choose the weights to all be equal, so in this case $\lambda_i = 1/n$, for all $i = 1, 2, \cdots, n$. Therefore

$$\log\left(\frac{x_1}{n} + \frac{x_2}{n} + \cdots + \frac{x_n}{n}\right) \geq \frac{1}{n}\log x_1 + \frac{1}{n}\log x_2 + \cdots + \frac{1}{n}\log x_n$$

$$\log\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \geq \log\left((x_1 \cdot x_2 \cdots x_n)^{1/n}\right)$$

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n},$$

as desired.