Definition. *An* $\boxed{X\text{–}tree}$ $\mathcal{T} = (T, \varphi)$ *is an ordered pair where* $T = (V, E)$ *is a tree and* $\varphi : X \to V$ *is a labeling map such that* $\varphi(X)$ *contains all vertices of* T *with degree* ≤ 2.

Definition. *A* $\boxed{phylogenetic\ X\text{–}tree}$ $\mathcal{T} = (T, \varphi)$ *is an* X*-tree where* φ *is a bijection (1-to-1) from* X *to the leaves of* T.

Definition. *A* $\boxed{dissimilarity\ map}$ *on* X *is a function* $\delta : X \times X \to \mathbb{R}$ *with* $\delta(x, x) = 0$ *and* $\delta(x, y) = \delta(y, x)$ *for all* $x, y \in X$.

Definition. *A dissimilarity map* δ *on* X *is a* $\boxed{tree\ metric}$ *if there exists an* X*-tree* $\mathcal{T} = (T, \varphi)$ *and edge weight function* $w : E \to \mathbb{R}_+$ *such that the induced metric* $d_{(\mathcal{T},w)}$ *is equal to* δ.

Definition. *A dissimilarity map* δ *on* X *satisfies the* $\boxed{4\ point\ condition\ (4PC)}$ *if for all* $w, x, y, z \in X$, $\delta(w, x) + \delta(y, z) \leq \max\{\delta(w, y) + \delta(x, z), \delta(w, z) + \delta(x, y)\}$.

Definition. *A dissimilarity map* δ *on* X *is an* $\boxed{ultrametric}$ *if for distinct* $x, y, z \in X$, $\delta(x, y) \leq \max\{\delta(x, z), \delta(y, z)\}$.

Definition. *An* $\boxed{equidistant\ X\text{–}tree}$ *is a phylogenetic* X*-tree* $\mathcal{T} = (T, \varphi)$ *rooted at* ρ*, and an edge weight function* $w : E \to \mathbb{R}$ *such that for all* $x, y \in X$, $d_{(\mathcal{T},w)}(\rho, \varphi(x)) = d_{(\mathcal{T},w)}(\rho, \varphi(y))$*, and* $w(e) \geq 0$ *for all* $e \in E$.

Definition. *Given a dissimilarity map* δ *on* X *and some* $r \in X$*, the* $\boxed{Gromov\ product}$ *is a function* $\delta_r : X \backslash \{r\} \to \mathbb{R}$ *defined*

$$\delta_r(x, y) = \begin{cases} \frac{1}{2}(\delta(x, y) - \delta(x, r) - \delta(y, r)), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Lemma. *If* δ *is an ultrametric, then there exists an equidistant* X*-tree* $(\mathcal{T}, w)$ *with* $\delta = d_{(\mathcal{T},w)}$.

Lemma. *A dissimilarity map* δ *satisfies the 4PC if and only if for all* $r \in X$*, the Gromov product* δ_r *is an ultrametric on* $X \backslash \{r\}$.

Theorem. *A nonnegative dissimilarity map* δ *satisfies the 4PC if and only if it is a tree metric.*