Analyzing Gravitational Waves through Numerical Simulations

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Why Gravitational Waves?

The speed of light is the speed of causality.

Gravity must be casual.

A change in a gravitating source is communicated to a distant observer not faster than light.

Communication of a change in spacetime $\rightarrow$ Gravitational Waves

Einstein's Field Equation

Einstein's Field Equation is a rather complicated tensor equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} (\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma})$$

$$R^{\alpha}{}_{\beta\gamma\sigma} = \Gamma^{\alpha}{}_{\gamma\lambda} \Gamma^{\lambda}{}_{\sigma\beta} - \Gamma^{\alpha}{}_{\sigma\lambda} \Gamma^{\lambda}{}_{\gamma\beta} + \partial_\gamma \Gamma^{\alpha}{}_{\sigma\beta} - \partial_\sigma \Gamma^{\alpha}{}_{\gamma\beta}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad R = \eta^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}$$

Linearized Einstein's Field Equation

- Weak field linearized approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad |h_{\mu\nu}| \ll 1$$

- Trace-reversed perturbation metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \qquad\qquad \bar{h} = \bar{h}^{\mu\nu}\eta_{\mu\nu} = -h$$

- Lorenz gauge $\partial_\mu h^{\mu\nu} = 0$

leads to the linearized Einstein's field equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}$$

Linearized Einstein's Field Equation $T_{\mu\nu} = 0$

In vacuum $T_{\mu\nu} = 0$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = 0$$

Plane wave solution

$$\bar{h}_{\mu\nu} = A_{\mu\nu}\cos\left[i\omega(t - z/c)\right]$$

Radiative degrees of freedom

The conditions imposed by the Transverse Traceless gauge

$$h_{0\nu}^{\mathsf{TT}} = 0 \qquad \eta^{\mu\nu} h_{\mu\nu}^{\mathsf{TT}} = 0 \qquad \partial^{\mu} h_{\mu\nu}^{\mathsf{TT}} = 0$$

allow us to obtain the only two radiative degrees of freedom

$$\bar{h}_{\mu\nu}^{\mathsf{TT}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$h_+ = A_{11} \cos\left(\omega(t - z/c)\right)$$

$$h_\times = A_{12} \cos\left(\omega(t - z/c)\right)$$

Geodesic Deviation Equation

For slowly-moving free-falling particles the geodesic deviation equations become

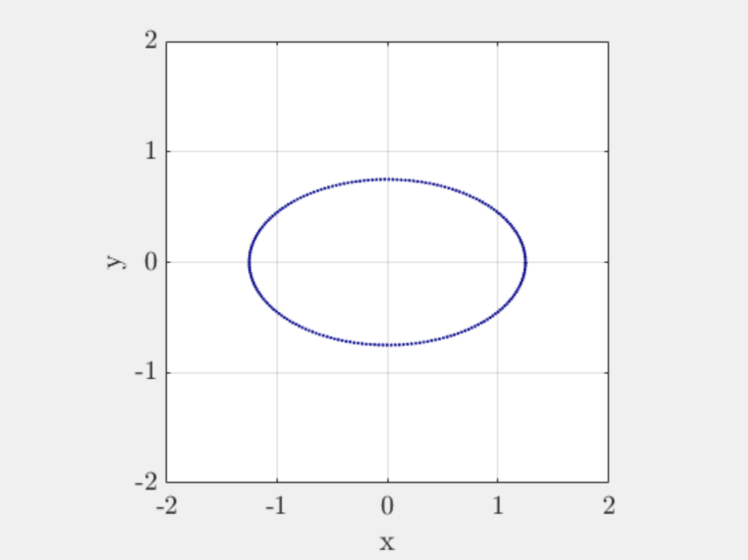
$$\begin{aligned}
\frac{\partial^2}{\partial t^2} S^1 &= \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} h_+ + \frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} h_\times \\
\frac{\partial^2}{\partial t^2} S^2 &= \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} h_\times - \frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} h_+
\end{aligned}$$

first order solution

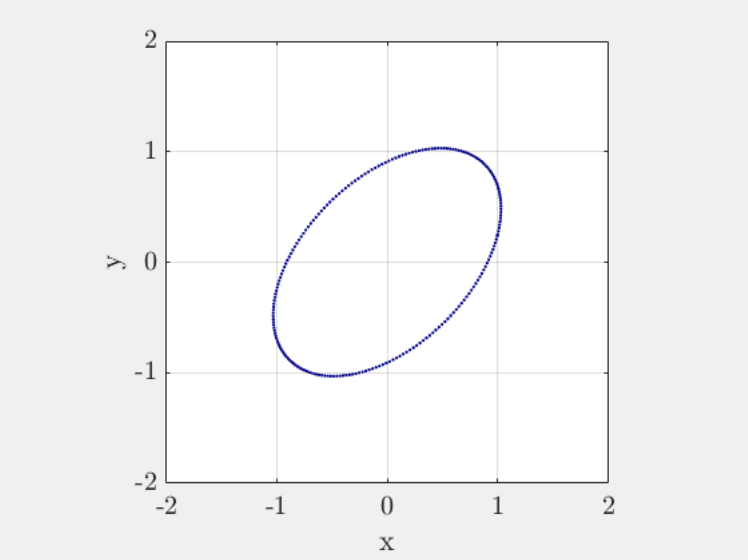
$$S^1 = S^1(t=0)\left(1 + \frac{1}{2} h_+\right) + \frac{1}{2} h_\times S^2(t=0)$$

$$S^2 = S^2(t=0)\left(1 - \frac{1}{2} h_+\right) + \frac{1}{2} h_\times S^1(t=0)$$

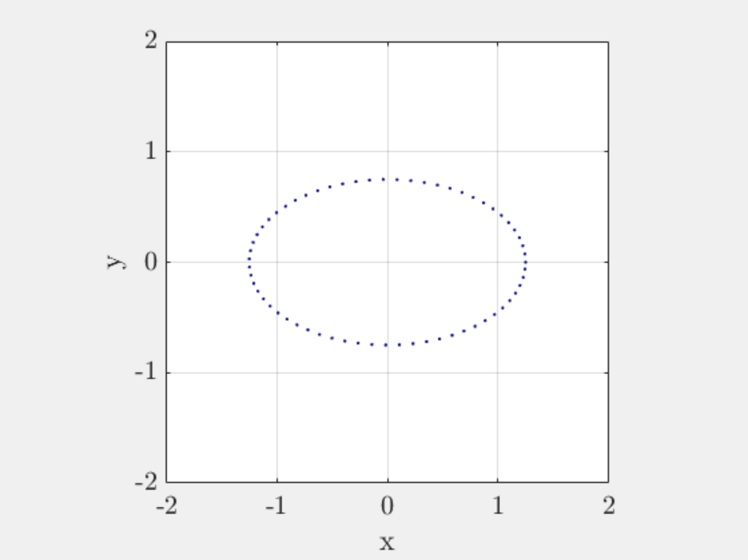
$+$ polarization $h_\times = 0$



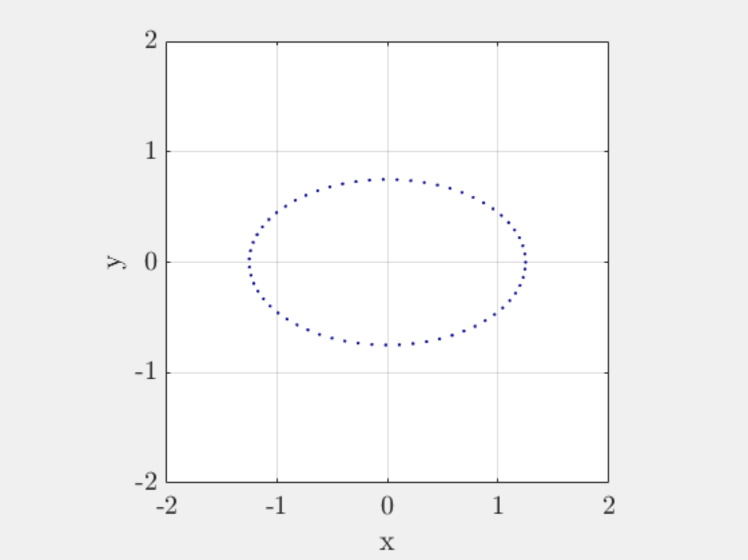
$\times$ polarization $h_+ = 0$



Left-handed polarization



Right-handed polarization



Linearized Einstein's Field Equation $T_{\mu\nu} \neq 0$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = \frac{16\pi G}{c^4}T_{\mu\nu}$$

- far field approximation
- slowly moving source
- isolated system

Quadrupole Formula

A solution of the Linearized Einstein's Field equation is the Quadrupole Formula

$$h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 r}\frac{\mathrm{d}^2\mathcal{I}_{kl}(t_r)}{\mathrm{d}t^2}\left(P_i^{\ k}P_j^{\ l} - \frac{1}{2}P_{ij}P^{kl}\right)$$

where

$$\mathcal{I}_{kj} = \int \rho(\mathbf{y})\left(y_k y_j - \frac{1}{3}\delta_{kj}y^l y_l\right)\mathrm{d}^3y$$

and

$$P_{ij} = \delta_{ij} - n_i n_j$$

slowly-moving equal-mass binary system

For an equal-mass binary system rotating at angular frequency ω and orbital radius R, the quadrupole formula becomes

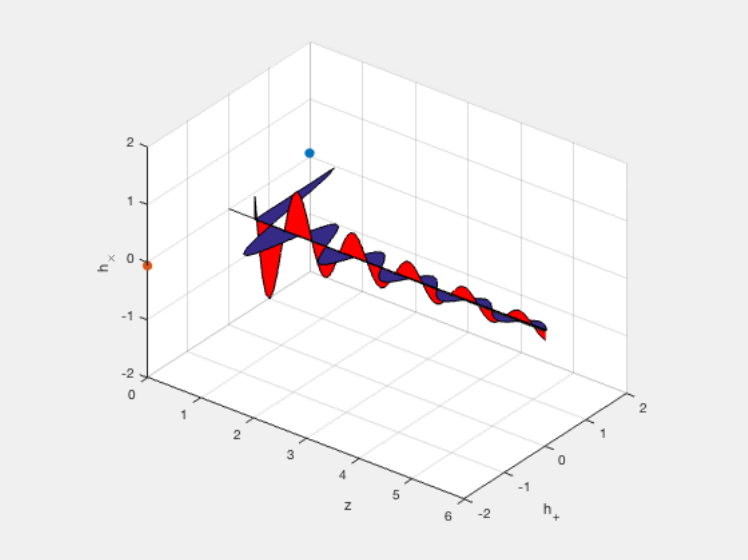
$$h_{ij}^{\mathsf{TT}} = \frac{8\,GM\,R^2\omega^2}{c^4\,r} \begin{bmatrix} -\cos 2\omega t_r & -\sin 2\omega t_r & 0 \\ -\sin 2\omega t_r & \cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

estimate of the amplitude

$$\frac{8\,GM\,R^2\omega^2}{c^4\,r} = 1.6 \times 10^{-22}$$

setting $M = M_\odot = 2 \times 10^{30}$ kg, $\omega = \sqrt{GM/(4R^3)}$,
$R = 3r_s = 8.86$ km and $r = 100$ Mpc.

slowly-moving equal-mass binary system



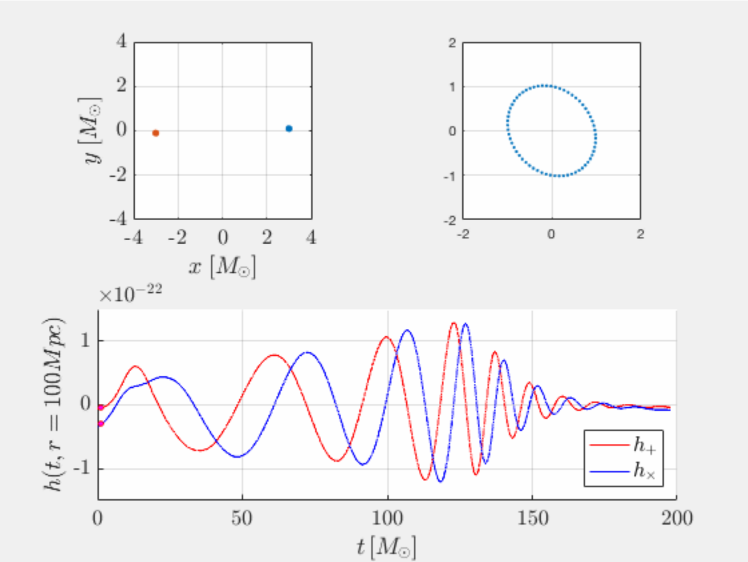
Numerical Evolution of Compact Binaries

Quasi-equilibrium initial conditions of compact objects

- **Binary Black Holes (BBH)**
- **Binary Neutron Stars (BNS)**

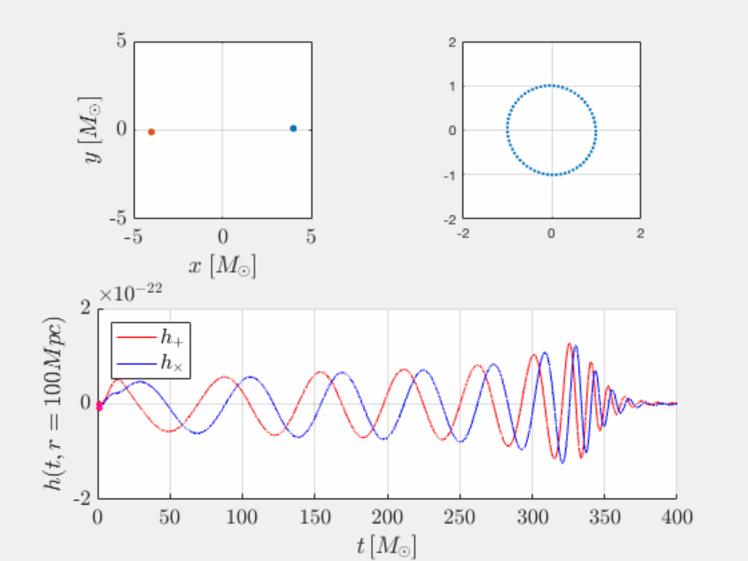
Gravitational Radiation $\Leftrightarrow$ Dynamical Evolution

BBH-b3

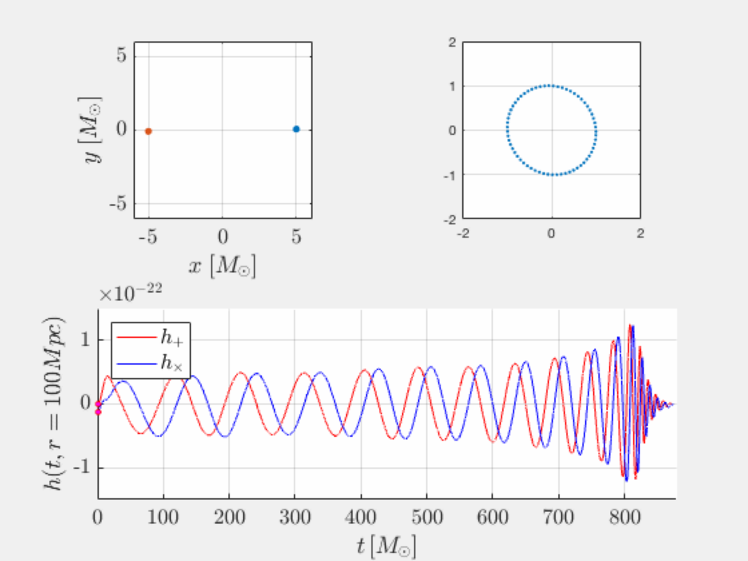


4
2
0
-2
-4
$y\ [M_\odot]$
-4 -2 0 2 4
$x\ [M_\odot]$
2
1
0
-1
-2
-2 0 2
$\times 10^{-22}$
1
0
-1
$h(t, r = 100Mpc)$
0 50 100 150 200
$t\ [M_\odot]$
h_+
$h_\times$

BBH-b4

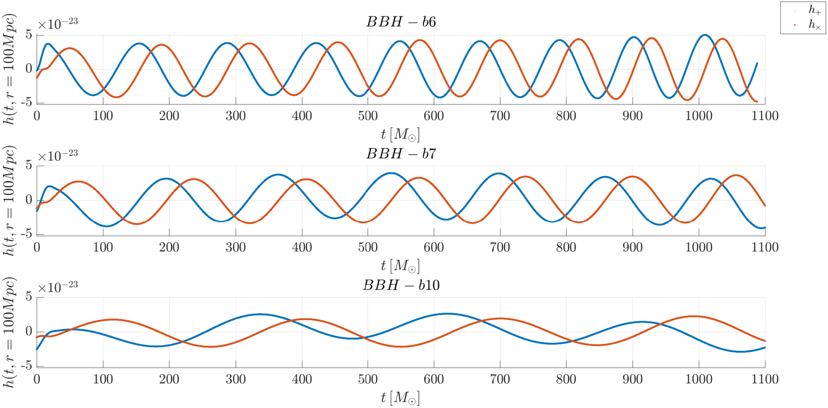

5
0
-5
-5
0
5
$y\ [M_\odot]$
$x\ [M_\odot]$
2
1
0
-1
-2
-2
0
2
$\times 10^{-22}$
2
0
-2
0
50
100
150
200
250
300
350
400
h_+
$h_\times$
$h(t, r = 100Mpc)$
$t\ [M_\odot]$

BBH-b5

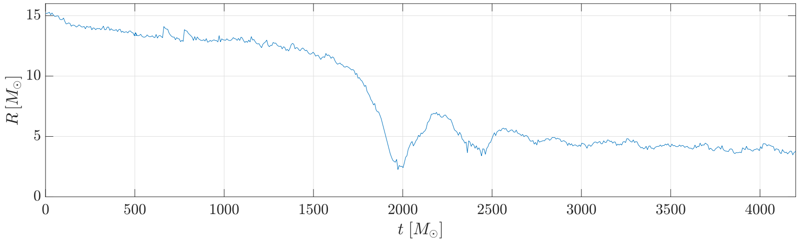


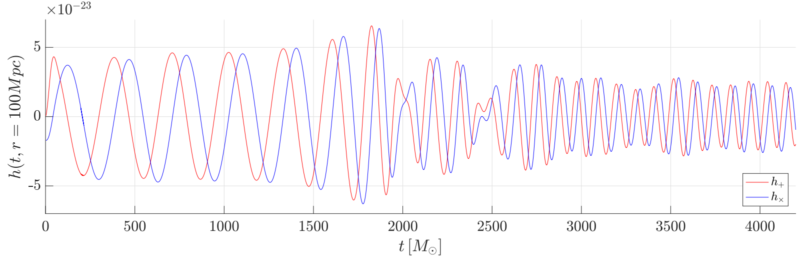
5
0
-5
-5
0
5
$y\ [M_\odot]$
$x\ [M_\odot]$
2
1
0
-1
-2
-2
0
2
$\times 10^{-22}$
h_+
$h_\times$
1
0
-1
0
100
200
300
400
500
600
700
800
$h(t, r = 100Mpc)$
$t\ [M_\odot]$

BBH-b6, BBH-b7, BBH-b10

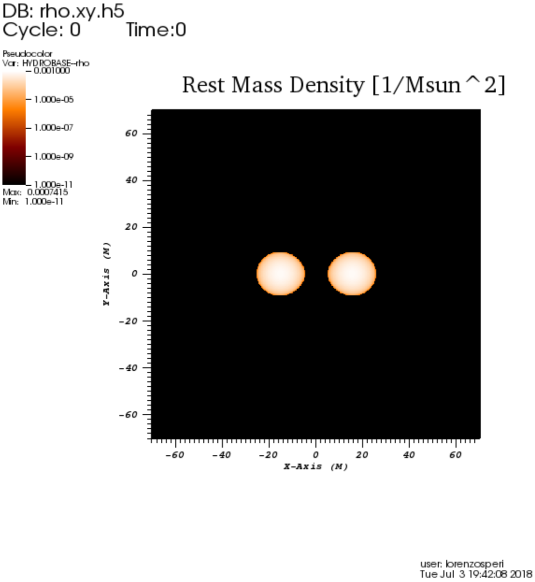


BNS


15
10
5
0
$R\,[M_\odot]$
0 500 1000 1500 2000 2500 3000 3500 4000
$t\,[M_\odot]$


$\times 10^{-23}$
5
0
-5
$h(t, r = 100Mpc)$
h_+
$h_\times$
0 500 1000 1500 2000 2500 3000 3500 4000
$t\,[M_\odot]$

Rest Mass Density Evolution



Conclusions

"Gravitational waves will show us details of the bulk motion of dense concentrations of energy"

Kip S. Thorne