



# Analyzing Gravitational Waves through Numerical Simulations

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# Contents

1. Introduction
2. Einstein's Field Equation
3. Effects of Gravitational Waves
4. Production of Gravitational Waves
5. Numerical Evolution of Compact Binaries



# Why Gravitational Waves?

The speed of light is the speed of causality.

Gravity must be causal.

A change in a gravitating source is communicated to a distant observer not faster than light.

Communication of a change in spacetime → Gravitational Waves



# Einstein's Field Equation

Einstein's Field Equation is a rather complicated tensor equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

where

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2}g^{\alpha\rho}(\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma})$$

$$R^\alpha_{\beta\gamma\sigma} = \Gamma^\alpha_{\gamma\lambda}\Gamma^\lambda_{\sigma\beta} - \Gamma^\alpha_{\sigma\lambda}\Gamma^\lambda_{\gamma\beta} + \partial_\gamma\Gamma^\alpha_{\sigma\beta} - \partial_\sigma\Gamma^\alpha_{\gamma\beta}$$

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \quad R = \eta^{\mu\nu}R_{\mu\nu} = R^\mu_{\mu}$$



# Linearized Einstein's Field Equation

- Weak field linearized approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1$$

- Trace-reversed perturbation metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \quad \bar{h} = \bar{h}^{\mu\nu}\eta_{\mu\nu} = -h$$

- Lorenz gauge  $\partial_\mu h^{\mu\nu} = 0$

leads to the linearized Einstein's field equation

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}$$



# Linearized Einstein's Field Equation $T_{\mu\nu} = 0$

In vacuum  $T_{\mu\nu} = 0$

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{h}_{\mu\nu} = 0$$

Plane wave solution

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \cos [i\omega(t - z/c)]$$



# Radiative degrees of freedom

The conditions imposed by the Transverse Traceless gauge

$$h_{0\nu}^{\text{TT}} = 0 \quad \eta^{\mu\nu} h_{\mu\nu}^{\text{TT}} = 0 \quad \partial^\mu h_{\mu\nu}^{\text{TT}} = 0$$

allow us to obtain the only two radiative degrees of freedom

$$\bar{h}_{\mu\nu}^{\text{TT}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$h_+ = A_{11} \cos(\omega(t - z/c))$$

$$h_\times = A_{12} \cos(\omega(t - z/c))$$



## Geodesic Deviation Equation

For slowly-moving free-falling particles the geodesic deviation equations become

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} h_+ + \frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} h_\times$$

$$\frac{\partial^2}{\partial t^2} S^2 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} h_\times - \frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} h_+$$

first order solution

$$S^1 = S^1(t=0) \left( 1 + \frac{1}{2} h_+ \right) + \frac{1}{2} h_\times S^2(t=0)$$

$$S^2 = S^2(t=0) \left( 1 - \frac{1}{2} h_+ \right) + \frac{1}{2} h_\times S^1(t=0)$$



+ polarization  $h_x = 0$



× polarization  $h_+ = 0$



# Left-handed polarization



# Right-handed polarization



# Linearized Einstein's Field Equation $T_{\mu\nu} \neq 0$

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}$$

- far field approximation
- slowly moving source
- isolated system



## Quadrupole Formula

A solution of the Linearized Einstein's Field equation is the Quadrupole Formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 r} \frac{d^2 \mathcal{I}_{kl}(t_r)}{dt^2} \left( P_i^k P_j^l - \frac{1}{2} P_{ij} P^{kl} \right)$$

where

$$\mathcal{I}_{kj} = \int \rho(\mathbf{y}) \left( y_k y_j - \frac{1}{3} \delta_{kj} y^l y_l \right) d^3 y$$

and

$$P_{ij} = \delta_{ij} - n_i n_j$$



## slowly-moving equal-mass binary system

For an equal-mass binary system rotating at angular frequency  $\omega$  and orbital radius  $R$ , the quadrupole formula becomes

$$h_{ij}^{\text{TT}} = \frac{8 GM R^2 \omega^2}{c^4 r} \begin{bmatrix} -\cos 2\omega t_r & -\sin 2\omega t_r & 0 \\ -\sin 2\omega t_r & \cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

estimate of the amplitude

$$\frac{8 GM R^2 \omega^2}{c^4 r} = 1.6 \times 10^{-22}$$

setting  $M = M_\odot = 2 \times 10^{30}$  kg,  $\omega = \sqrt{GM/(4R^3)}$ ,  
 $R = 3r_s = 8.86$  km and  $r = 100$  Mpc.



# slowly-moving equal-mass binary system



# Numerical Evolution of Compact Binaries

Quasi-equilibrium initial conditions of compact objects

- **Binary Black Holes (BBH)**
- **Binary Neutron Stars (BNS)**

Gravitational Radiation  $\Leftrightarrow$  Dynamical Evolution



## BBH-b3



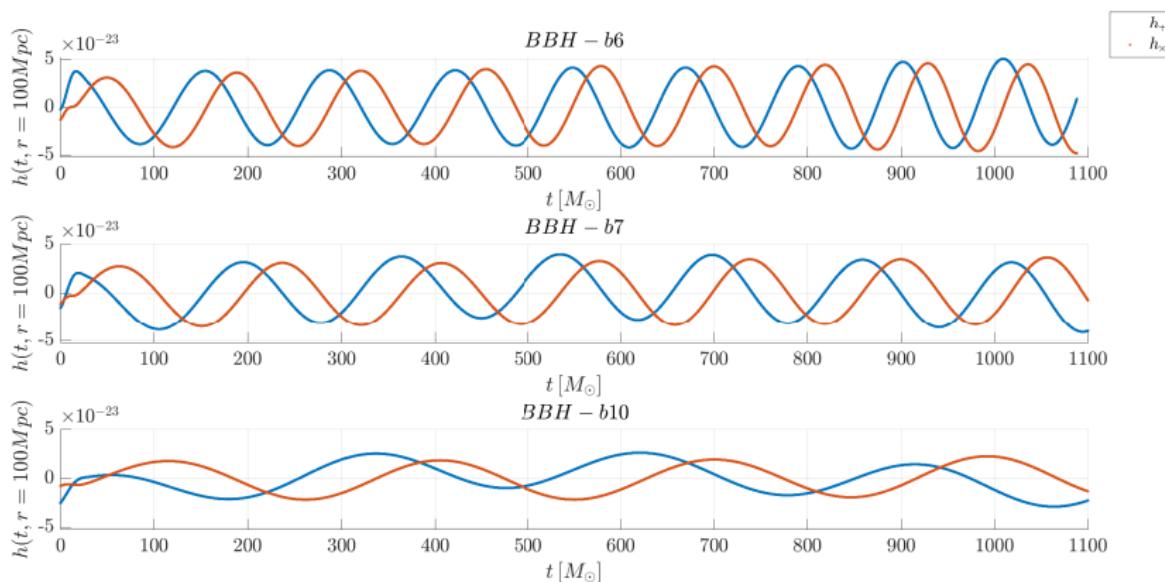
## BBH-b4



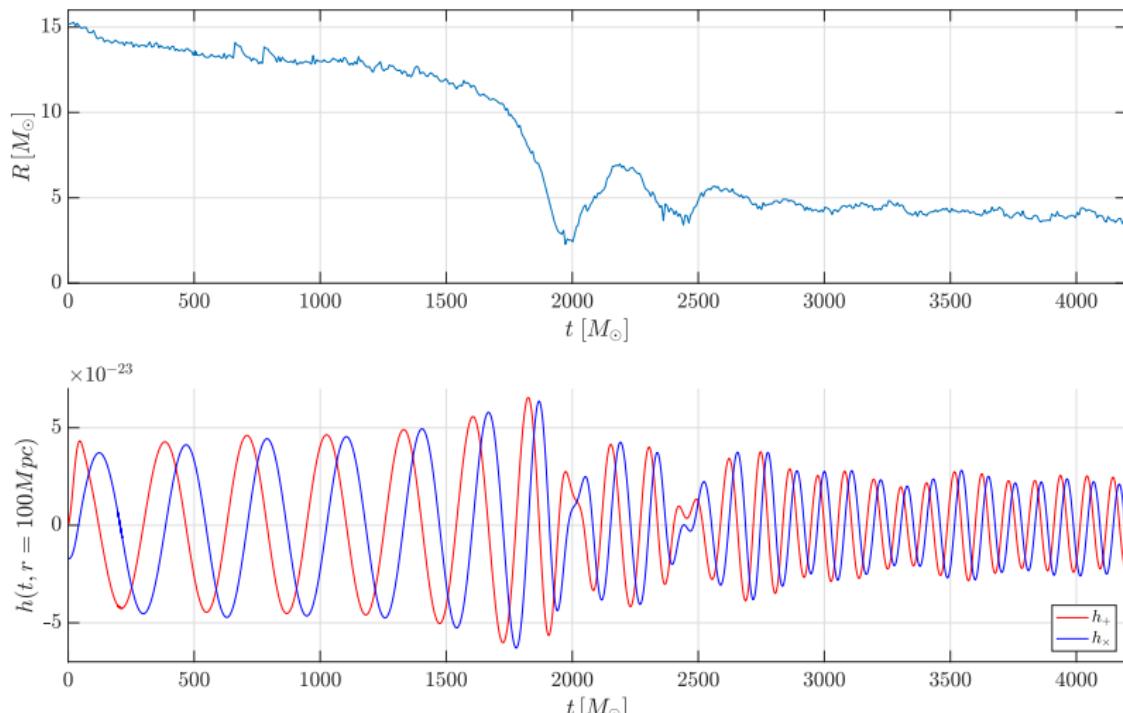
# BBH-b5



# BBH-b6, BBH-b7, BBH-b10



# BNS



# Rest Mass Density Evolution



## Conclusions

*"Gravitational waves will show us details of the bulk motion of dense concentrations of energy"*

Kip S. Thorne

