A Very Impressive and Fancy Title for a Thesis

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Contents

Abstract

A really interesting and fascinating abstract

1 Introduction

NOTATION: we use a system of geometrized units $G = c = 1$.

2 From the Einstein's field equation to gravitational wave solutions

Albert Einstein developed the General Theory of Relativity between and 1907 and 1917, creating a new tool to observe the universe. The General Theory of Relativity has changed the way we describe and study the gravitational phenomena. In particular, the new theory of gravitation was not only able to solve unexplained observations, as for instance, anomalies in the newtonian description of planets' orbits as Mercury, but it also predictied new phenomena such as gravitational time dilation, gravitational lensing and gravitational waves. One of the most important test of the General Theory of Relativity was the discovery of the gravitational waves (GW).

2.1 Linearized Einstein's field equation

The Einstein's field equation represents how the geometry of space-time is related to the presence of masses and energy:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \frac{8\,\pi\, G}{c^4}\, T_{\mu\nu} \tag{1}$$

On the right hand side we have the energy-momentum tensor $T_{\mu\nu}$, which is interpreted as the flux of four momentum p^μ accross a surface of constant x^ν, and $G = 6.67408 \times 10^{-11}\,\mathrm{N\ m^2\ kg^{-2}}$ and $c = 299\,792\,458\,\mathrm{m/s}$ that are respectively the Newton constant of gravitation and the speed of light. On the left hand side the Einstein tensor $G_{\mu\nu}$ includes a measure of the curvature of spacetime through the Ricci tensor $R_{\mu\nu}$, the Ricci scalar $R = R_{\mu\nu} g^{\mu\nu}$ and the metric $g_{\mu\nu}$.
In order to solve the Einstein's equation we will make assumptions on the metric tensor $g_{\mu\nu}$ and we will derive the Einstein tensor $G_{\mu\nu}$ going through the following steps:

(a) Calculate the Christoffel symbol

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} \left(\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma}\right) \tag{2}$$

where ∂_μ means the partial derivative $\partial/\partial x^\mu$.

(b) Calculate the Riemann curvature tensor

$$R^{\alpha}{}_{\beta\gamma\sigma} = \Gamma^{\alpha}{}_{\gamma\lambda}\, \Gamma^{\lambda}{}_{\sigma\beta} - \Gamma^{\alpha}{}_{\sigma\lambda}\, \Gamma^{\lambda}{}_{\gamma\beta} + \partial_\gamma \Gamma^{\alpha}{}_{\sigma\beta} - \partial_\sigma \Gamma^{\alpha}{}_{\gamma\beta} \tag{3}$$

(c) Obtain the Ricci tensor and the Ricci scalar from the Riemann curvature tenor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad\qquad R = \eta^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} \tag{4}$$

$G_{\mu\nu}$ and $T_{\mu\nu}$ are symmetric tensors because $g_{\mu\nu}$ is symmetric as well. So the Einstein's field equation is a set of non-linear second-order partial differential equations with 10 linearly independent variables.
We show that the equation(1) leads to gravitational wave solutions if we consider a weak gravitational field, where the spacetime is nearly flat. Therefore, we assume the metric tensor $g_{\mu\nu}$ to be equal to the Minkwoski metric $\eta = \mathrm{diag}(-1, +1, +1, +1)$ plus a small perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5}$$

where perturbation is symmetric and $|h_{\mu\nu}| \ll 1$ for all μ and ν.
The metric $g_{\mu\nu}$ is also used to lower and raise indeces, however, in linearized theory we consider only the first order approximation in $h_{\mu\nu}$. So, it is possible to raise and lower indeces using the Minkoswkian metric $\eta_{\mu\nu}$.
Taking into account the mentioned approximations we follow the described procedure to calculate the so-called linearized Einstein's field equation. The Christoffel symbol is obtained keeping up to the first order in the perturbation $h_{\mu\nu}$

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\rho}(\partial_\beta h_{\gamma\rho} + \partial_\gamma h_{\rho\beta} - \partial_\rho h_{\beta\gamma})$$

The Riemann curvature tensor is

$$\begin{aligned} R^{\mu}{}_{\beta\gamma\nu} &= \partial_\gamma \Gamma^{\mu}{}_{\nu\beta} - \partial_\nu \Gamma^{\mu}{}_{\gamma\beta} \\ &= \frac{1}{2}[\eta^{\mu\rho}(\partial_\gamma\partial_\nu h_{\beta\rho} + \partial_\gamma\partial_\beta h_{\nu\rho} - \partial_\gamma\partial_\rho h_{\beta\nu}) - \eta^{\mu\sigma}(\partial_\nu\partial_\beta h_{\gamma\sigma} + \partial_\nu\partial_\gamma h_{\beta\sigma} - \partial_\nu\partial_\sigma h_{\beta\gamma})] \\ &= \frac{1}{2}\left(\partial_\gamma\partial_\beta h^{\mu}{}_{\nu} - \partial_\gamma\partial^\mu h_{\beta\nu} - \partial_\nu\partial_\beta h^{\mu}{}_{\gamma} + \partial_\nu\partial^\mu h_{\beta\gamma}\right) \end{aligned} \tag{6}$$

where we neglected the first two terms in eq(3) because they are second order terms. Contracting the first and the third indeces we get the Ricci tensor

$$\begin{aligned} R_{\beta\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\beta h^{\mu}{}_{\nu} - \partial_\mu\partial^\mu h_{\beta\nu} - \partial_\nu\partial_\beta h^{\mu}{}_{\mu} + \partial_\nu\partial^\mu h_{\beta\mu}\right) \\ &= \frac{1}{2}(\partial_\mu\partial_\beta h^{\mu}{}_{\nu} - \Box h_{\beta\nu} - \partial_\nu\partial_\beta h + \partial_\nu\partial^\mu h_{\beta\mu}) \end{aligned}$$

where the trace of the perturbation is defined as $h = \eta^{\mu\nu}h_{\mu\nu} = h^{\mu}{}_{\mu}$, and the d'Alambertian opereator in flat space is $\Box = \partial_\mu\partial^\mu$.
Contracting again to obtain the Ricci scalar yields

$$\begin{aligned} R &= \frac{1}{2}\left(\partial_\mu\partial^\nu h^{\mu}{}_{\nu} - \Box h^{\beta}{}_{\beta} - \partial_\beta\partial^\beta h + \partial_\nu\partial^\mu h^{\nu}{}_{\mu}\right) \\ &= \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \end{aligned}$$

Therefore the Einstein tensor is

$$G_{\beta\nu} = \frac{1}{2}\left(\partial_\mu\partial_\beta h^{\mu}{}_{\nu} - \Box h_{\beta\nu} - \partial_\nu\partial_\beta h + \partial_\nu\partial^\mu h_{\beta\mu} - \eta_{\beta\nu}\partial_\mu\partial_\lambda h^{\mu\lambda} - \eta_{\beta\nu}\Box h\right) \tag{7}$$

If we define the **trace-reversed** perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \qquad\qquad \bar{h} = \bar{h}^{\mu\nu}\eta_{\mu\nu} = -h$$

we can simplify the equation(7). Thus,

$$R_{\beta\nu} = \frac{1}{2}\bigg(\partial_\mu\partial_\beta\bar{h}^\mu{}_\nu - \Box\bar{h}_{\beta\nu} - \cancel{\partial_\nu\partial_\beta h} + \partial_\nu\partial^\mu\bar{h}_{\beta\mu} + \cancel{\frac{1}{2}\eta_{\nu\mu}\partial^\mu\partial_\beta h} - \frac{1}{2}\eta_{\beta\nu}\Box h + \cancel{\frac{1}{2}\eta_{\beta\mu}\partial_\nu\partial^\mu h}\bigg)$$

$$= \frac{1}{2}\bigg(\partial_\mu\partial_\beta\bar{h}^\mu{}_\nu - \Box\bar{h}_{\beta\nu} + \partial_\nu\partial^\mu\bar{h}_{\beta\mu} - \frac{1}{2}\eta_{\beta\nu}\Box h\bigg)$$

If we contract the above tensor we obtain

$$R = \partial_\mu\partial_\beta\bar{h}^\mu{}_\nu + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu h - \Box h \quad = \partial_\mu\partial_\nu\bar{h}^{\mu\nu} - \frac{1}{2}\Box h$$

So the Einstein's tensor expressed as a function of $\bar{h}_{\mu\nu}$ is

$$G_{\beta\nu} = \frac{1}{2}\big(\partial_\mu\partial_\beta\bar{h}^\mu{}_\nu - \Box\bar{h}_{\beta\nu} + \partial_\nu\partial^\mu\bar{h}_{\beta\mu} - \eta_{\mu\nu}\partial_\mu\partial_\nu\bar{h}^{\mu\nu}\big) \tag{8}$$

This expression can be simplified further by choosing an appropriate gauge transformation. Using the **Lorenz gauge** condition

$$\partial_\mu\bar{h}^{\,\mu\nu} = 0 \tag{9}$$

the Einstein's tensor of equation(8) becomes

$$G_{\beta\nu} = \frac{1}{2}\big(\partial_\beta\partial_\mu\bar{h}^{\mu\alpha}\eta_{\alpha\nu} + \partial_\nu\partial_\mu\bar{h}^{\mu\alpha}\eta_{\alpha\beta} - \eta_{\mu\nu}\partial_\nu\partial_\mu\bar{h}^{\mu\nu} - \Box\bar{h}_{\beta\nu}\big) = -\frac{1}{2}\Box\bar{h}_{\beta\nu}$$

The linearized Einstein's field equation is

$$\Box\bar{h}_{\beta\nu} = -16\pi\,T_{\beta\nu} \tag{10}$$

we will solve the above equation with individual approximations in section(), whereas we work in vacuum. The energy-momentumn tensor $T_{\beta\nu}$ is null in vacuum so the linearized Einstein's equation in vacuum assumes the form of the wave equation in a tensorial form

$$\Box\bar{h}_{\beta\nu} = 0 \tag{11}$$

The above equation shows that the trace-reversed metric perturbation propagate as a wave distorting a flat spacetime.
The simplest solution to the linearized Einstein's equation(11) is a plane wave

$$\bar{h}_{\beta\nu} = A_{\beta\nu}\,\exp\left(i\,k_\alpha x^\alpha\right)$$

where $A_{\beta\nu}$ is called **amplitude tensor** and it is symmetric, since $\bar{h}_{\mu\nu}$ is symmetric.
Substitution of the plane wave solution into equation(11) implies that $k_\alpha k^\alpha = 0$, so k^α is a null four vector. Therefore, the plane wave solution is a gravitational wave which travels at the speed of light in the spatial direction $\mathbf{k} = (k^1, k^2, k^3)/k^0$ and with frequency $\omega = k^0$, i.e. $\bar{h}_{\beta\nu} = A_{\beta\nu}\,\exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]$. Furthermore, any $\bar{h}_{\mu\nu}$ satisfying the linearized Einstein's field equation(11) in vacuum describes a **gravitational wave** propagating at the speed of light, and it can be Fourier-expanded as a superposition of plane waves.

2.2 Gauge transformations and GW polarizations

A **gauge transformation** in linearized theory is defined as a transformation of the perturbation $h_{\mu\nu}$ into a new metric perturbation $h'_{\mu\nu}$

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{12}$$

for a given vector field ξ^μ. Gauge transformations are particularly important because they leave the Riemann curvature tensor unchanged (up to the first order in $h_{\mu\nu}$), indeed, the physical spacetime is unchanged. The invariance of the curvature under such transformations is analogous to the traditional gauge invariance of electromagnetism.
Assuming that the Einstein's field equation(10) are valid everywhere the metric perturbation $h_{\mu\nu}$ contains: gauge degrees of freedom; physical, radiative degrees of freedom; and physical, non-radiative degrees of freedom tied to the matter source of the GW.
It is possible to show that the linearized Einstein's equation can be written as 5 Poisson-type quations for certain combinations of spacetimemetric, plus a wave equation for the transverse-traceless components of the metric perturbation, which represents the radiative degrees of freedom.
Neverthless this procedure will manifestly demonstrate that the radiative degrees of freedom in spacetime are two, it is a cumbersome and long derivation. Instead, we ignore the degrees of freedom tied to the matter and we consider only solutions of equation(11):

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp\left(i\, k_\alpha x^\alpha\right)$$

By using the Lorenz gauge and the transverse traceless gauge, we reduce progressively the number degrees of freedom of a plane wave from 10 to 2.

We want now to find the conditions on the parameter ξ_μ in order to satisfy the Lorenz gauge condition, that we used in the previous section. The initial metric perturbation $h_{\mu\nu}$ tranforms into $h'_{\mu\nu}$ if a gauge transformation is used. However, the new trace reversed metric $\bar{h}'_{\mu\nu}$ transforms as

$$\begin{aligned}
\bar{h}'_{\mu\nu} &= h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}\left(h + \partial_\alpha \xi^\alpha + \partial_\alpha \xi^\alpha\right) \\
\bar{h}'_{\mu\nu} &= \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\, \partial_\alpha \xi^\alpha
\end{aligned} \tag{13}$$

Imposing the Lorenz gauge $\partial_\mu \bar{h}'^{\,\mu\nu} = \partial_\mu \bar{h}^{\,\mu\nu} = 0$ we obtain

$$\begin{aligned}
\partial_\mu \bar{h}'^{\,\mu\nu} &= \partial_\mu \bar{h}^{\,\mu\nu} + \partial_\mu \partial^\mu \xi^\nu + \partial_\mu \partial^\nu \xi^\mu - \partial_\mu \eta^{\mu\nu}\, \partial_\alpha \xi^\alpha \\
&= 0 + \Box \xi^\nu + \partial^\nu \partial_\mu \xi^\mu - \partial^\nu \partial_\alpha \xi^\alpha = 0
\end{aligned}$$

Any metric perturbation $h_{\mu\nu}$ can therefore be put into a Lorenz gauge by using transformations that satisfy

$$\Box \xi_\mu = 0$$

The plane wave $\xi_\mu = C_\mu \exp[ik_\alpha x^\alpha]$ is a solution of the above equation and it generates a gauge transformation through the four arbitrary constants C_μ.
The **Transverse-Traceless (TT) gauge** is the most convinient gauge for the analysis of the gravitational waves, and it is defined for a plane wave by the following conditions:

a) The Lorenz gauge condition fixes four components of $A_{\mu\nu}$

$$\partial^\mu \bar{h}_{\mu\nu} = A_{\mu\nu} k^\nu = 0$$

The amplitude tensor $A_{\mu\nu}$ and the four vector k^μ are orthogonal.

b) Three components of the aplitude tensor can be eliminated selecting $\xi_\mu = C_\mu \exp[ik_\alpha x^\alpha]$ so that $A^{\mu\nu} u_\mu = 0$ for some chosen four velocity u_μ. Three and not four components are fixed, since one firther constraint $k^\mu A_{\mu\nu} u^\nu$ needs to be satisfied.

c) One component of the aplitude tensor can be eliminated selecting $\xi_\mu = C_\mu \exp[ik_\alpha x^\alpha]$ so that $A^\mu_\mu = 0$.

This means that we have suffecient freedom to fix the values of 8 components of $A_{\mu\nu}$ from a), b) and c), hence, reducing the number of independent component from 10 to 2. Note that $\bar{h}^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu}$ from c).
What does the TT gauge tell us about gravitational radiation ?
Let use consider a test a particle at rest with four-velocity $u^\alpha = (1, 0, 0, 0)$ in a nearly flat spacetime. If we orient our spatial coordinate axes so that the a plane gravitational wave is travelling in the positive z-direction $k^\sigma = (\omega, 0, 0, \omega)$ the transverse traceless conditions becomes

$$\left.\begin{aligned} A^{\mathrm{TT}}_{\mu 0}\,\omega + A^{\mathrm{TT}}_{\mu 3}\,\omega = 0 \\ A^{\mathrm{TT}}_{0\nu} = 0 \\ A^{\mathrm{TT}}_{00} + A^{\mathrm{TT}}_{11} + A^{\mathrm{TT}}_{22} + A^{\mathrm{TT}}_{33} = 0 \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{\mathrm{TT}}_{11} & A^{\mathrm{TT}}_{12} & 0 \\ 0 & A^{\mathrm{TT}}_{12} & -A^{\mathrm{TT}}_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As a consequence of the transverse traceless gauge the only non-zero component of the metric perturbation $\bar{h}^{\mathrm{TT}}_{\mu\nu}$ are, respectively, the plus (+) and the cross ($\times$) polarization of the gravitational wave

$$\begin{aligned} \bar{h}^{\mathrm{TT}}_{11} &= -\bar{h}^{\mathrm{TT}}_{22} \equiv h_+ \\ \bar{h}^{\mathrm{TT}}_{12} &= \bar{h}^{\mathrm{TT}}_{21} \equiv h_\times \end{aligned}$$

So, the plane wave solution in the TT gauge is:

$$\bar{h}^{\mathrm{TT}}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

where we express the real part of the solution with

$$h_+ = A^{\mathrm{TT}}_{11} \cos\left(\omega(t - z)\right)$$
$$h_\times = A^{\mathrm{TT}}_{12} \cos\left(\omega(t - z)\right)$$

h_+ and $h_\times$ are the two independent polarizations of a gravitational wave and they completely characterize the gravitational wave solution. We finally found that the radiative degrees of freedom are only two and they are represented by h_+ and $h_\times$.

Generally, within any finite vacuum region it is always possible to find a gauge which is locally transverse and traceless, that is, a guage which satisfies the following general conditions

$$h^{\mathrm{TT}}_{0\nu} = 0$$
$$\eta^{\mu\nu} h^{\mathrm{TT}}_{\mu\nu} = 0$$
$$\partial_\mu h^{\mu\nu}_{\mathrm{TT}} = 0$$

The transverse traceless gauge does not only simplify the expression of the perturbation metric, but it also gives an important relation between the Riemann curvature tensor and the metric perturbation. Since we have already calculated the Riemann curvature tensor in equation(6) we recall the result taking into account the TT gauge conditions

$$\begin{aligned} R^{\mu}{}_{00\sigma} &= \frac{1}{2}\left(\partial_0\partial_0 h^{\mathrm{TT}\mu}{}_{\sigma} - \partial_0\partial^\mu h^{\mathrm{TT}}_{0\sigma} - \partial_\sigma\partial_0 h^{\mathrm{TT}\,\mu}{}_{0} + \partial_\sigma\partial^\mu h^{\mathrm{TT}}_{00}\right) \\ &= \frac{1}{2}\partial_0\partial_0 h^{\mathrm{TT}\mu}{}_{\sigma} \qquad\qquad \text{using } h^{\mathrm{TT}}_{\mu 0} = 0 \end{aligned} \tag{15}$$

The above result tells us that the curvature of spacetime is proportional to the 'acceleration' of the gravitational wave. Considering a plane wave we have

$$R^{\mu}{}_{00\sigma} = -\frac{1}{2}\omega^2 A^{\mathrm{TT}\mu}{}_{\sigma} \cos(\omega(t - z))$$

where ω is the frequency of the plane wave. The curvature is proportional to the square of the frequency, in fact we expect a bigger curvature if the wave oscillates more times per second. Naively, the spacetime is more curved if the ripples of the GW are squeezed in a narrow time interval. Analogously, the electromagnetic field of an oscillating electric dipole is proportional to the square of the frequency, in fact we expect a more intense field if the charge oscillates more times per second.

3 Effects of Gravitational Waves

We have shown how the gravitational waves are obtained from the Einstein's field equation. By using the transverse traceless gauge we introduced crucial semplifications. We want now to explain the important physical consequences of the theoretical results obtained in the previous part. Throughout the next sections we will use the linearized theory of gravitatinal waves and we consider our metric to be in the TT gauge.

3.1 Free particles and detetction principles

In general relativity the trajectory of a free falling particle is described by the **geodesic equation**

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} + \Gamma^\beta{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{16}$$

where the coordinates of the particle are represented x^β and τ is the proper time.
We choose a frame in which a test particle is initially at rest, i.e. with initial four-velocity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = (1, 0, 0, 0)$$

We consider a plane wave in the TT gauge propagating towards the test particle.
Equation(16) can be used to express the initial acceleration of the particle

$$\left(\frac{\mathrm{d}u^\beta}{\mathrm{d}\tau}\right)_0 = -\Gamma^\beta{}_{00} = -\frac{1}{2}\eta^{\beta\alpha}(\partial_0 h_{\alpha 0} + \partial_0 h_{0\alpha} + \partial_\alpha h_{00})$$

However, we recall from the TT gauge that

$$h^{\mathrm{TT}}_{0\alpha} = 0 \qquad\qquad h^{\mathrm{TT}}_{\mu\nu} = \bar{h}^{\mathrm{TT}}_{\mu\nu}$$

for all α. Hence, the initial acceleration of the particle is zero and a free particle, initally at rest, will remain at rest indefinitely.
In this constext "being at rest" means that the coordinates of the particle do not change, so the TT gauge is a good choice of coordinate. As the gravitational waves propagates, the coordinate system moves with the ripples of the spacetime, in order to keep the particle in the initial position.
In the TT gauge free falling bodies are not influenced by GWs, and their coordinate separation is constant. However, the proper separation is not constant, so let us calculate it.
Consider two free falling test particles located at $z = 0$ and separated on the x axis by a coordinate distance L_c. We still consider a plane wave in the TT gauge propagating in

the z direction.
The proper distance between the particles is

$$\begin{aligned} L &= \int_0^{L_c} |g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu|^{1/2} = \int_0^{L_c} \sqrt{g_{11}}\mathrm{d}x = \int_0^{L_c} \sqrt{1+h_+(t,z=0)}\mathrm{d}x \\ &\approx \int_0^{L_c} \left(1+\frac{1}{2}h_+(t,z=0)\right)\mathrm{d}x = L_c\left(1+\frac{1}{2}h_+(t,z=0)\right) \end{aligned}$$

The proper distance is streched by the passing gravitational wave and the two particles oscillate with a fractional length change given by

$$\frac{\delta L}{L} \approx \frac{1}{2}h_+(t,z=0) \tag{17}$$

The proper distance is a very important quantity for a laser interferometer gravitational wave detector, because the phase $\delta\phi$ accumualated by a photon that travels down and back the arm of a laser interferometer is

$$\delta\phi = \frac{4\pi\delta L}{\lambda}$$

where λ is the wavelength and δL is the distance the mirror moves relative to the beam splitter. Although we used TT gauge to calculate the above formula, it can be shown that this result is gauge independent.
If we had considered two particles on the y axis separated by the same coordinate distance, the proper distance would have been

$$L \approx L_c\left(1-\frac{1}{2}h_+(t,z=0)\right)$$

Therefore, recalling the expression of the plus polarization for a plane wave

$$h_+ = A_{11}^{\mathrm{TT}}\cos\left(\omega(t-z)\right)$$

we notice that the particles along x axis are streched, whereas the particles along the y axis are squeezed. This is one of the features of the plus polarization, that we will thoroughly analyze in the next section.

ADD DETECTION PRINCIPLES

3.2 From the geodesic deviation equation to the + and × polarizations

Since free-falling bodies obey to the geodesic equation(16), in this section we study the physical effects of the two polarizations plus + and times × of the gravitational waves

through relative motions of geodesics.
The **geodesic deviation equation** expresses the relative acceleration between two neighboring geodesics belonging to a one-parameter geodesics $\gamma_s(\tau)$:

$$\frac{D^2}{d\tau^2}S^\mu = R^\mu{}_{\nu\rho\sigma}T^\nu T^\rho S^\sigma \tag{18}$$

where $S^\mu = \partial x^\mu/\partial s$ is the deviattion from the geodesic, $T^\nu = \partial x^\mu/\partial\tau$ is the tangent to the geodesic and the directional covariant derivative is

$$\frac{D}{d\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\nabla_\mu$$

A non-zero acceleration of the deviation bewteen between neighbouring geodesics is a signature of spacetime curvature. In fact, geodesic deviation cannot distinguish between a zero gravitational field and a uniform grabitational field. Only tidal gravitational fields give rise to an acceleration in the geodesic deviation.
Let us consider some nearby particles with four-velocities described by a single vector field u^μ and separation vector field S^μ, the geodesic deviation equation(18) becomes

$$\frac{D^2}{d\tau^2}S^\mu = R^\mu{}_{\nu\rho\sigma}u^\nu u^\rho S^\sigma \tag{19}$$

The four-velocity vector can be approximated with a unit vector in the time direction plus corrections of order $h^{\mathrm{TT}}_{\mu\nu}$ and higher, however the Reimann curvature tensor is already first order. Therefore, we ignore the corrections of the four-velocity vector and we approximate $u^\nu = (1,0,0,0)$.
Since we have already calculated the Riemann curvature tensor in equation(15) taking into account the TT gauge conditions we recall the result

$$R^\mu{}_{00\sigma} = \frac{1}{2}\partial_0\partial_0 h^{\mathrm{TT}\mu}{}_\sigma$$

In the lowest order approximation the free-falling particles are slowly moving, then we have $\tau = x^0 = t$, so the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2}S^\mu = \frac{1}{2}S^\sigma\frac{\partial^2}{\partial t^2}h^{\mathrm{TT}\mu}{}_\sigma \tag{20}$$

The above equation is a set of differential equations that can be rewritten using the two polarizations of the metric perturbation (equation(14))

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}S^1 &= \frac{1}{2}S^1\frac{\partial^2}{\partial t^2}h_+ + \frac{1}{2}S^2\frac{\partial^2}{\partial t^2}h_\times \\
\frac{\partial^2}{\partial t^2}S^2 &= \frac{1}{2}S^1\frac{\partial^2}{\partial t^2}h_\times - \frac{1}{2}S^2\frac{\partial^2}{\partial t^2}h_+
\end{aligned}$$

Time evolution of the + polarization

Figure 1: Effect of the h_+ mode on a ring of free-falling test particles at $\omega t = n\pi/6$ with $n = 0, ..., 12$.

Time evolution of the $\times$ polarization.

Figure 2: Effect of the $h_\times$ mode on a ring of free-falling test particles at $\omega t = n\pi/6$ with $n = 0, ..., 12$.

These can be solved to yield, to lowest order,

$$S^1 = S^1(t=0)\left(1 + \frac{1}{2}h_+\right) + \frac{1}{2}h_\times S^2(t=0)$$

$$S^2 = S^2(t=0)\left(1 - \frac{1}{2}h_+\right) + \frac{1}{2}h_\times S^1(t=0)$$

Let us study the effects of the two polarizations h_+ and $h_\times$ of a gravitational wave, which propagates through the center of a ring of free-falling test particles. So, let us use consider a plane wave travelling along the z axis, and let us place a ring of free-falling test particles on the x-y plane with its center in $(0, 0, 0)$. The ring is initially parametrized by $(\cos\theta, \sin\theta)$ with $\theta \in (0, 2\pi]$ and the separation vector S^μ measures the deformation of the ring from its center.

Beginning with the case $h_\times = 0$ and $h_+ \neq 0$, the solutions of the geodesic deviation equation are

$$S^1_+ = \cos\theta\left(1 + \frac{1}{2}A^{\mathrm{TT}}_{11}\cos(\omega t)\right) \tag{21}$$

$$S^2_+ = \sin\theta\left(1 - \frac{1}{2}A^{\mathrm{TT}}_{11}\cos(\omega t)\right) \tag{22}$$

where $h_+ = A^{\mathrm{TT}}_{11}\cos(\omega t)$ for a plane wave. The time evolution of the ring is shown in Figure(1).

When the plus polarized gravitational wave propagates through the ring, it increases the proper distance between the ring and its center along the x axis when the phase of the

wave is close to $\omega t = 0, 2\pi$, meanwhile it squeezes the test particles along the y axis. If the phase of the gravitational wave is close to $\omega t = \pi/2, 3\pi/2$ the ring is strecthed along the y axis and the test particles move inwards, therefore, the proper distance from the center of the ring is reduced. As the wave passes, the test particles bounce back and forth in the shape of + as shown in Figure(4a).
On the ohter hand, the case where $h_\times \neq 0$ and $h_+ = 0$ yields the geodesic deviation solutions to be

$$S^1_\times = \cos\theta + \frac{1}{2}\sin\theta A^{\mathrm{TT}}_{12}\cos(\omega t)$$

$$S^2_\times = \sin\theta + \frac{1}{2}\cos\theta A^{\mathrm{TT}}_{12}\cos(\omega t)$$

where $h_\times = A^{\mathrm{TT}}_{12}\cos(\omega t)$ for a plane wave. The relationship between these solutions and those for $h_+ \neq 0$ can be easily found if we rotate the x and y axis through an angle of $-\pi/4$, so that the new coordinate axis are

$$x' = \frac{1}{\sqrt{2}}(x - y)$$

$$y' = \frac{1}{\sqrt{2}}(x + y)$$

Then, the geodesic deviations S_+ of equations (21) and (22) with $h_+ \neq 0$ and $h_\times = 0$ become

$$S'^{\,1}_+ = (S^1_+ - S^2_+)/\sqrt{2} = \cos(\theta + \pi/4) + \frac{1}{2}\sin(\theta + \pi/4)h_+$$

$$S'^{\,2}_+ = (S^1_+ + S^2_+)/\sqrt{2} = \sin(\theta + \pi/4) + \frac{1}{2}\cos(\theta + \pi/4)h_+$$

The above equations are similiar to those with $h_+ = 0$ and $h_\times \neq 0$, in fact the deviations $S^1_\times$ and $S^2_\times$ are nothing but the plus polarization rotated of an angle $-\pi/4$. So, in this case ($h_+ = 0$ and $h_\times \neq 0$) the circle of particles bounce back and forth in the shape of $\times$ as we can see from Figures (2) and (4b).

We could consider also right- and left-handed cricularly polarized modes by defining

$$h_R = \frac{1}{\sqrt{2}}(h_+ - ih_\times) \tag{23}$$

$$h_L = \frac{1}{\sqrt{2}}(h_+ + ih_\times) \tag{24}$$

The effect of a pure h_R wave would be to rotate the particles in a right-handed sense and similiarly for the left-handed mode h_L. It is important to stress that the particles do not

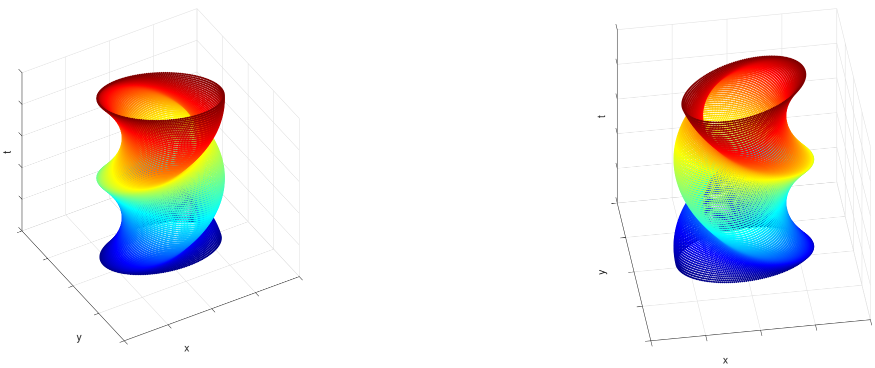
+ Polarization and × Polarization

(a) + polarized gravitational wave. (b) × polarized gravitational wave.

Figure 3: Spatial positions occupied by a ring of free-falling test particles disturbed by a gravitational wave.

travel around the ring, they just move in little epicycles.

Right- and Left-handed cricularly polarized modes



(a) Right-handed polarized gravitational wave.

(b) Left-handed polarized gravitational wave.

Figure 4: Time evoluition of a ring of free-falling test particles on the x-y plane.

4 Production of Gravitational Waves

We have studied how the gravitational waves propagate in vacuum. We now want to understand the relation between the gravitational waves and their source. In most of the physical scenarios the system that produces the gravitational waves is small compared to the distances with the detector. Therefore, In order to study the solutions of the linearized Einstein's field equation(10) we make a crucial approximation

4.1 Solution of the linearized Einstein's field equation

The generation of gravitational radiation depends on the movements of objects in space-time. So far, we have neglected the presence of matter and we solved the linearized the Einstein's field equation in vaccum. However, if we want to analyze the relation between sources and gravitational waves we need to consider $T_{\mu\nu} \neq 0$ and solve equation(10):

$$\Box \bar{h}_{\mu\nu} = -16\pi \, T_{\mu\nu}$$

It is possible to solve this equation using a Green function $G(x^\sigma - y^\sigma)$, such that

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma) \tag{25}$$

And the general solution is, then, given by

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) \, \mathrm{d}^4 y \tag{26}$$

$$\Box_x \bar{h}_{\mu\nu}(x^\sigma) = -16\pi \int \Box_x G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) \, \mathrm{d}^4 y = -16\pi \, T_{\mu\nu}(x^\sigma)$$

There are two solutions of equation(25): one solution represents a wave travelling forward in time and, the other represents a wave travelling backward in time. The two solutions are called, respectively, retarded and advanced. We are interested in the **retarded Green function**, which represents the accumulated effect of signals received at (x^0, x^1, x^2, x^3) from a source at (y^0, y^1, y^2, y^3):

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \, \delta\left[|\mathbf{x} - \mathbf{y}| - (x^0 - y^0)\right] \theta(x^0 - y^0)$$

where have used boldface to denote the patial vectors $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$, with norm $|\mathbf{x} - \mathbf{y}| = \left[\delta_{ij}(x^i - y^i)(x^j - y^j)\right]^{1/2}$. The Heaviside step function $\theta(x^0 - y^0)$ is 1 when $x^0 > y^0$, and zero otherwise.
Plugging the retarded Green function into equation(26)

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y}) \, \mathrm{d}^3 y \tag{27}$$

where $t = x^0$ and the integration is made over the spatial coordinates. From equation(27) we notice that the metric perturbation is influenced by the matter and energy distribution, $T_{\mu\nu}$, at time $t - |\mathbf{x} - \mathbf{y}|$. Since the gravitational radiation travels at the speed of light $c = 1$, the metric perturbation at $(t, \mathbf{x})$ is influenced by the radiation that was produced by the source at the retarded time $t_r = t - |\mathbf{x} - \mathbf{y}|$.
We have obtained a general solution, however it possible to derive a formula that reaveals the quadrupole nature of the gravitational radiation if we assume:

- **far field approximation**: the metric perturbation (27) is evaluated at large distances from the source
$$|\mathbf{x} - \mathbf{y}| \approx |x| \equiv r \tag{28}$$
The fractional error of this approximation scales as $\sim L/r$, where L is the size of the source.
- **slowly moving source**: the light traverses the source much faster than the components of the cource itself do. Therefore, the source moves at non relativistic speeds.
- **isolated system**: the source of the gravitational radiation is an isolated and compact. We assume that our system and the radiation are not gravitationally influenced by other bodies.

So we rewrite equation(27) using the far field approximation:
$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4}{r} \int T_{\mu\nu}(t - r, \mathbf{y})\, \mathrm{d}^3 y$$
Using the Fourier trassform and inverse with respect to time
$$\begin{aligned} \phi(t, \mathbf{x}) &= \mathcal{F}^{-1}[\tilde{\phi}(\omega, \mathbf{x})] \equiv \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\omega e^{-i\omega t} \tilde{\phi}(\omega, \mathbf{x}) \\ \tilde{\phi}(\omega, \mathbf{x}) &= \mathcal{F}[\phi(t, \mathbf{x})] \equiv \frac{1}{\sqrt{2\pi}} \int \mathrm{d}t e^{i\omega t} \phi(\omega, \mathbf{x}) \end{aligned}$$
applied to the metric perturbation
$$\begin{aligned} \mathscr{H}_{\mu\nu}(\omega, t) &\equiv \mathcal{F}[\bar{h}_{\mu\nu}(t, \mathbf{x})] = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} \bar{h}_{\mu\nu}(t, \mathbf{x})\, \mathrm{d}t \\ &= \frac{4}{\sqrt{2\pi}} \int e^{i\omega t} \frac{T_{\mu\nu}(t_r, \mathbf{y})}{r}\, \mathrm{d}t\, \mathrm{d}^3 y \\ &= \frac{4}{\sqrt{2\pi}\, r} \int e^{i\omega(t_r + r)}\, T_{\mu\nu}(t_r, \mathbf{y})\, \mathrm{d}t_r\, \mathrm{d}^3 y \\ &= \frac{4\, e^{i\omega r}}{r} \int \mathscr{T}_{\mu\nu}(\omega, \mathbf{y})\, \mathrm{d}^3 y \end{aligned} \tag{29}$$

where we used a change of variable and we defined the Fourier transform of the energy momentum tensor as $\mathscr{T}_{\mu\nu} \equiv \mathcal{F}[T_{\mu\nu}]$.
The Lorenz gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$ in the Fourier space becomes

$$\mathcal{F}\big[\partial_0 \bar{h}^{0\nu} + \partial_j \bar{h}^{j\nu}\big] = 0$$

$$\mathscr{H}^{0\nu} = \frac{i}{\omega}\partial_j \mathscr{H}^{j\nu}$$

As a consequence, we only need to calculate the spacelike components $\mathscr{H}^{j\nu}$. We set $\nu = k$ in order to find $\mathscr{H}^{0k}$ from $\mathscr{H}^{jk}$, afterwards we use $\mathscr{H}^{k0}$ to get h^{00}. The integration by parts of the spacelike components of equation(29) is

$$\int \mathscr{T}^{jk} \mathrm{d}^3 y = \int \partial_m\big(\mathscr{T}^{mk} y^j\big)\mathrm{d}^3 y - \int \partial_m\big(\mathscr{T}^{mk}\big) y^j \mathrm{d}^3 y$$

Since we assumed that the source is isolated, the first term, which is a surface integral, vanishes. Whereas, the conservation of the energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$ yields in the Fourier space

$$-\partial_m\big(\mathscr{T}^{mk}\big) = i\omega \mathscr{T}^{0k}$$

Thus,

$$\begin{aligned}
\int \mathscr{T}^{jk}(\omega, \mathbf{y})\mathrm{d}^3 y \quad &= \quad i\omega \int y^j \mathscr{T}^{0k}\, \mathrm{d}^3 y \\
\text{symmetry of } \mathscr{T}_{k\nu} \rightarrow \quad &= \quad \frac{i\omega}{2}\int \big(y^j \mathscr{T}^{0k} + y^k \mathscr{T}^{0j}\big)\mathrm{d}^3 y \\
\partial_l\big(y^k y^j \mathscr{T}^{0l}\big) = \delta^k_l y^j \mathscr{T}^{0l} + \delta^j_l y^k \mathscr{T}^{0l} + y^k y^j \partial_l \mathscr{T}^{0l} \rightarrow \quad &= \quad \frac{i\omega}{2}\int \big[\partial_l\big(y^k y^j \mathscr{T}^{0l}\big) - y^k y^j \partial_l \mathscr{T}^{0l}\big]\, \mathrm{d}^3 y \\
\partial_l \mathscr{T}^{0l} = \partial_l \mathscr{T}^{l0} = -i\omega \mathscr{T}^{00} \rightarrow \quad &= \quad -\frac{\omega^2}{2}\int y^k y^j \mathscr{T}^{00}(\omega, \mathbf{y})\, \mathrm{d}^3 y
\end{aligned}$$

Then, equation(29) becomes

$$\begin{aligned}
\mathscr{H}_{kj} \quad &= \quad -\frac{4\, e^{i\omega r}}{r}\frac{\omega^2}{2}\int y^k y^j \mathscr{T}^{00}(\omega, \mathbf{y})\, \mathrm{d}^3 y \\
\mathcal{F}\left[\frac{\partial^2 T^{00}}{\partial t^2}\right] = -\omega^2 \mathcal{F}[T^{00}] \rightarrow \quad &= \quad \frac{2}{r}\int y^k y^j \mathcal{F}\left[\frac{\partial^2}{\partial t^2} T^{00}(t_r, \mathbf{y})\right]\, \mathrm{d}^3 y \\
&= \quad \mathcal{F}\left[\frac{2}{r}\frac{\partial^2}{\partial t^2}\left(\int y^k y^j T^{00}(t_r, \mathbf{y})\, \mathrm{d}^3 y\right)\right]
\end{aligned}$$

We can transform back the above result to obtain the original metric perturbation

$$\bar{h}_{kj} = \frac{2}{r}\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_{kj}(t_r) \tag{30}$$

where we defined the **quadrupole moment tensor**

$$I_{kj}(t) = \int y_k y_j T^{00}(t, \mathbf{y})\ \mathrm{d}^3 y \tag{31}$$

To complete the derivation we need to express the metric perturbation in the TT gauge, so we must make the right hand side of equation(30) traceless and transverse.
We begin by introducing the spatial projection tensor

$$P_{ij} = \delta_{ij} - n_i n_j \tag{32}$$

which projects the components of a tensor (with rank 2) into a surface orthogonal to the unit vector n^i

$$(P_{ij} X^{il}) n^j = X^{jl} n_j - n_i n_j X^{il} n^j = 0$$

We can us the **projection tensor** to construct the transverse-traceless version of a symmetric spatial tensor X_{ij} via

$$X_{ij}^{TT} = \left(P_i^{\ k} P_j^{\ l} - \frac{1}{2} P_{ij} P^{kl} \right) X_{kl} \tag{33}$$

where the first and second terms make the tensor, respectively, transverse and tracless.
In addition, we define the **reduced quadrupole moment tensor** as

$$\mathcal{I}_{kj} = I_{kj} - \frac{1}{3}\delta_{kj} I \qquad\qquad \text{where } I = \eta^{lm} I_{lm} = I_m^m \tag{34}$$

which is traceless, and, for $T^{00} = \rho$, it assume the expression

$$\mathcal{I}_{kj} = \int \rho(\mathbf{y}) \left(y_k y_j - \frac{1}{3}\delta_{kj} y^l y_l \right)\ \mathrm{d}^3 y$$

We now have all the concepts to write down the **quadrupole formula**

$$h_{ij}^{\mathrm{TT}} = \frac{2}{r} \frac{\mathrm{d}^2 \mathcal{I}_{kl}(t_r)}{\mathrm{d}t^2} \left(P_i^{\ k} P_j^{\ l} - \frac{1}{2} P_{ij} P^{kl} \right) \tag{35}$$

which represents the metric perturbation of equation(30) in the TT gauge, since $h_{\mu\nu}^{\mathrm{TT}} = \bar{h}_{\mu\nu}^{\mathrm{TT}}$.

5 Numerical Evolution of Compact Binaries

5.1 Binary Black holes

5.2 Binary Neutron stars

6 Conclusion

7 Appendices

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