Analyzing Gravitational Waves through Numerical Simulations

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Contents

What can Gravitational Waves tell us?

The speed of light is the speed of causality.

Gravity must be causal.

A change in a gravitating source is communicated to a distant observer not faster than light.

Communication of a change in spacetime $\rightarrow$ Gravitational Waves

Einstein's Field Equation

Einstein's Field Equation is a rather complicated tensor equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} (\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma})$$

$$R^{\alpha}{}_{\beta\gamma\sigma} = \Gamma^{\alpha}{}_{\gamma\lambda} \Gamma^{\lambda}{}_{\sigma\beta} - \Gamma^{\alpha}{}_{\sigma\lambda} \Gamma^{\lambda}{}_{\gamma\beta} + \partial_\gamma \Gamma^{\alpha}{}_{\sigma\beta} - \partial_\sigma \Gamma^{\alpha}{}_{\gamma\beta}$$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad R = \eta^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}$$

Linearized Einstein's Field Equation

- Weak field linearized approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

- Trace-reversed perturbation metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \qquad \bar{h} = \bar{h}^{\mu\nu}\eta_{\mu\nu} = -h$$

- Lorenz gauge $\partial_\mu h^{\mu\nu} = 0$

Linearized Einstein's field equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}$$

Linearized Einstein's Field Equation $T_{\mu\nu} = 0$

In vacuum $T_{\mu\nu} = 0$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = 0$$

Using the Transverse Traceless gauge

$$h^{\mathsf{TT}}_{0\nu} = 0 \qquad \eta^{\mu\nu} h^{\mathsf{TT}}_{\mu\nu} = 0 \qquad \partial^{\mu} h^{\mathsf{TT}}_{\mu\nu} = 0$$

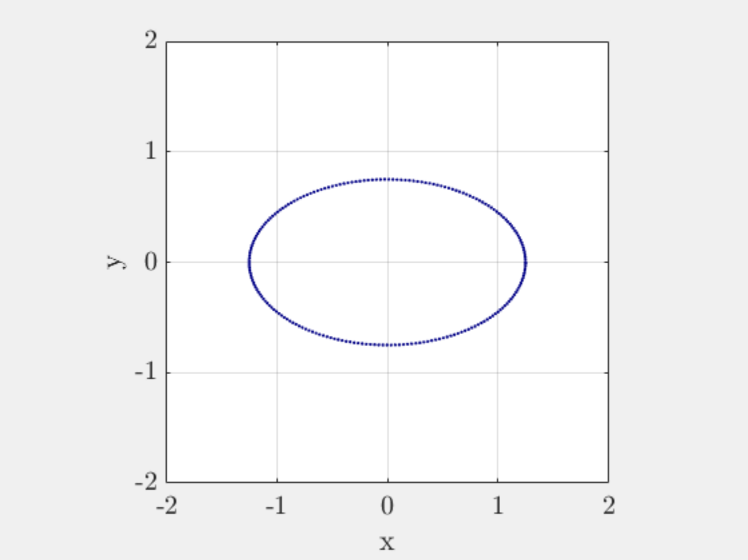
A solution of the above equation is

$$\bar{h}^{\mathsf{TT}}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

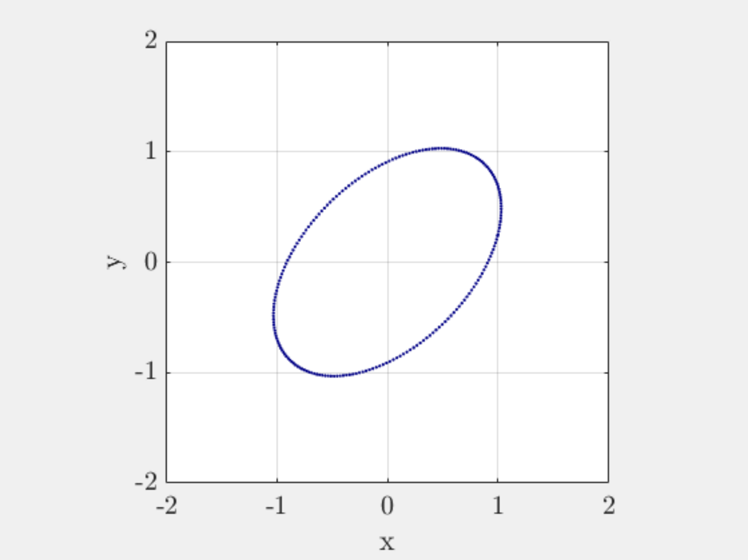
where

$$h_+ = A_{11}\cos\left(\omega(t - z/c)\right) \qquad h_\times = A_{12}\cos\left(\omega(t - z/c)\right)$$

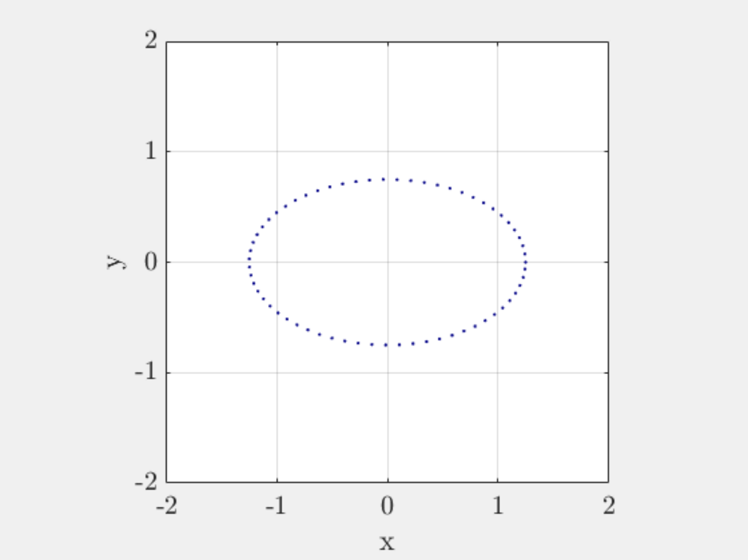
$+$ polarization $h_\times = 0$



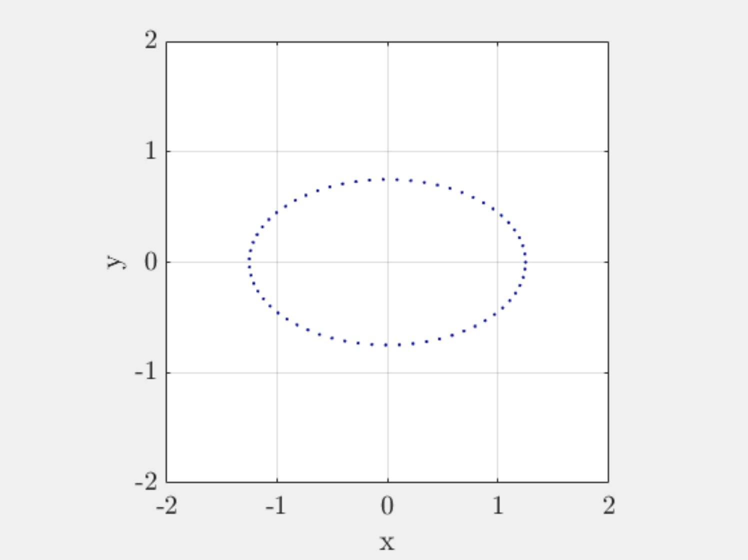
$\times$ polarization $h_+ = 0$



Left-handed polarization



Right-handed polarization



Linearized Einstein's Field Equation $T_{\mu\nu} \neq 0$

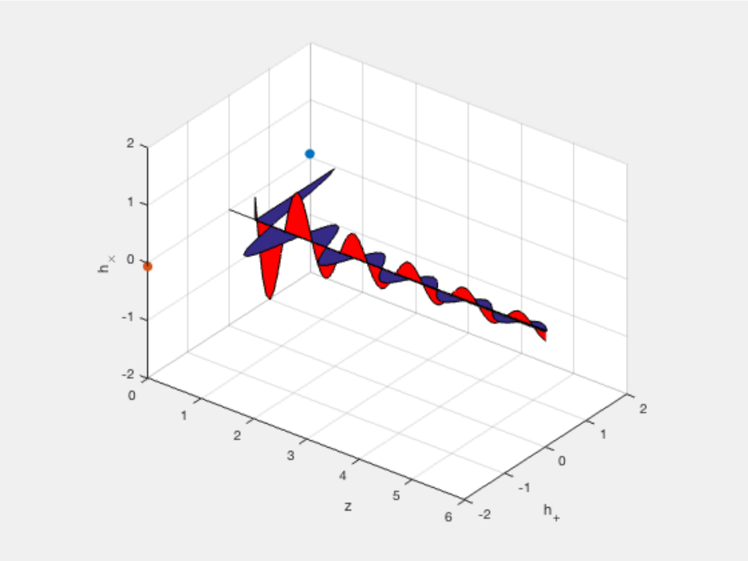
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\bar{h}_{\mu\nu} = \frac{16\pi G}{c^4}T_{\mu\nu}$$

- **far field approximation**
- **slowly moving source**
- **isolated system**

For an equal-mass binary system rotating at angular frequency ω and orbital radius R, the solution is

$$h_{ij}^{\mathsf{TT}} = \frac{8\,GM\,R^2\omega^2}{c^4\,r}\begin{bmatrix} -\cos 2\omega t_r & -\sin 2\omega t_r & 0 \\ -\sin 2\omega t_r & \cos 2\omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

slowly-moving equal-mass binary system



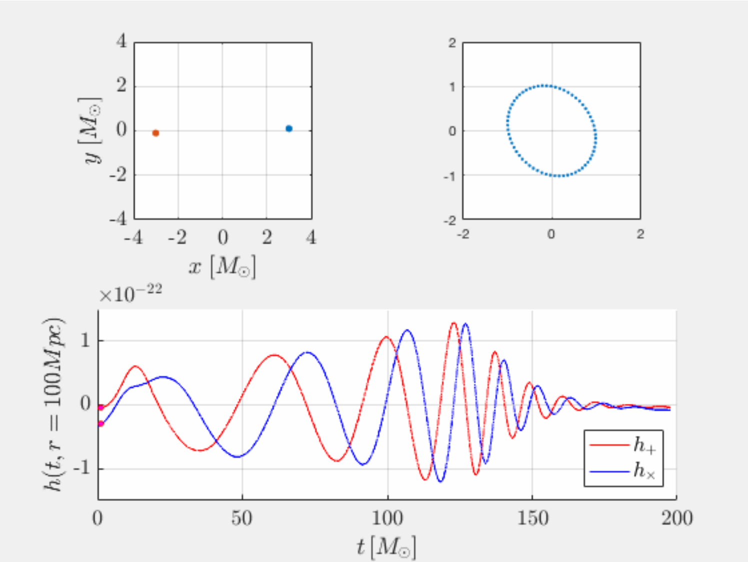
Numerical Evolution of Compact Binaries

Quasi-equilibrium initial conditions of compact objects

- **Binary Black Holes (BBH)**
- **Binary Neutron Stars (BNS)**

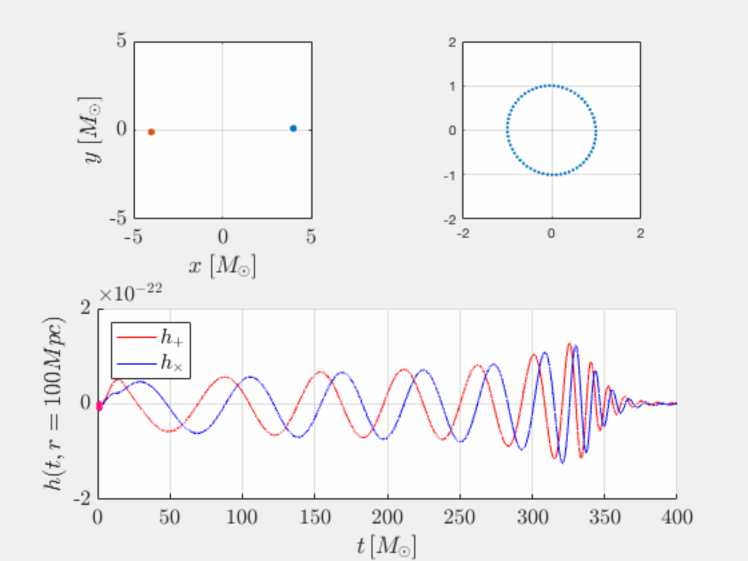
Gravitational Radiation $\Leftrightarrow$ Dynamical Evolution

BBH-b3

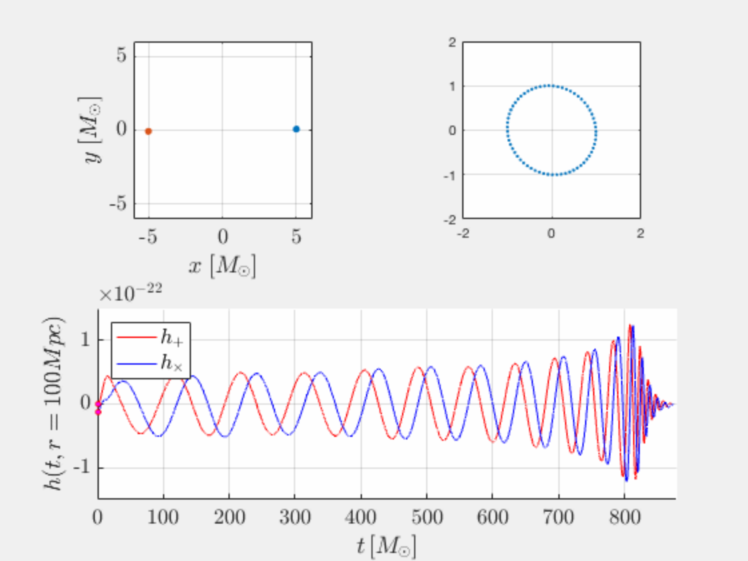


$y\ [M_\odot]$
$x\ [M_\odot]$
4
2
0
-2
-4
-4
-2
0
2
4
2
1
0
-1
-2
-2
0
2
$\times 10^{-22}$
$h(t, r = 100Mpc)$
1
0
-1
0
50
100
150
200
$t\ [M_\odot]$
h_+
$h_\times$

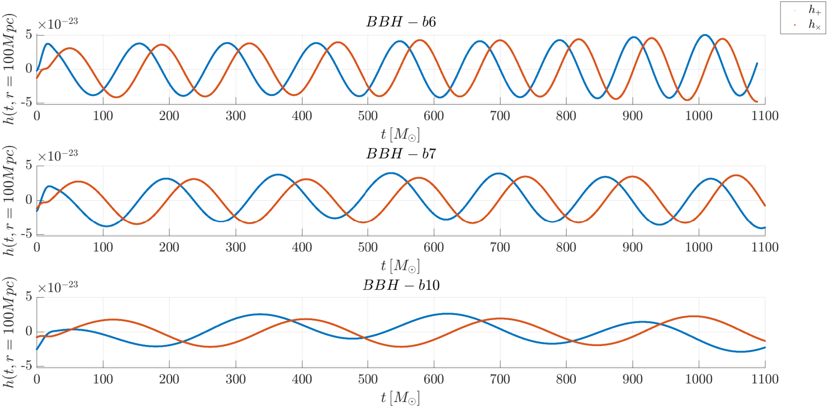
BBH-b4


5
0
-5
$y\ [M_\odot]$
-5
0
5
$x\ [M_\odot]$
2
1
0
-1
-2
-2
0
2
$\times 10^{-22}$
2
0
-2
$h(t, r = 100Mpc)$
h_+
$h_\times$
0
50
100
150
200
250
300
350
400
$t\ [M_\odot]$

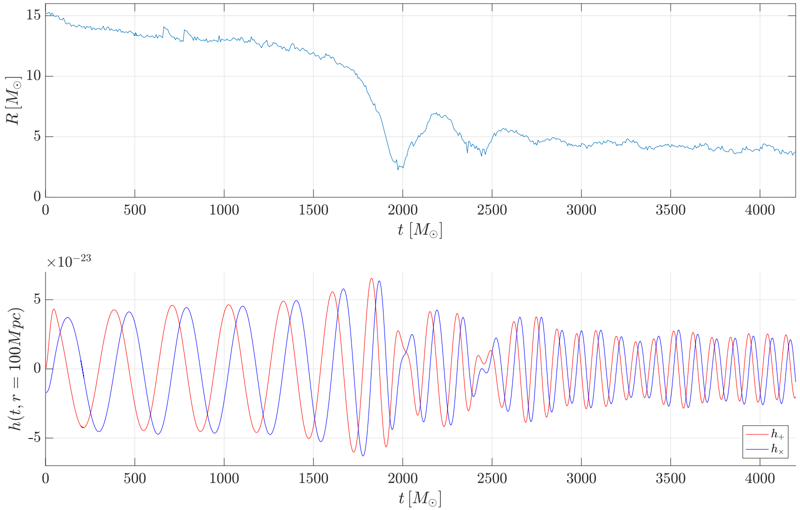
BBH-b5


5
0
-5
$y\,[M_\odot]$
-5
0
5
$x\,[M_\odot]$
2
1
0
-1
-2
-2
0
2
$\times 10^{-22}$
h_+
$h_\times$
1
0
-1
$h(t, r = 100Mpc)$
0
100
200
300
400
500
600
700
800
$t\,[M_\odot]$

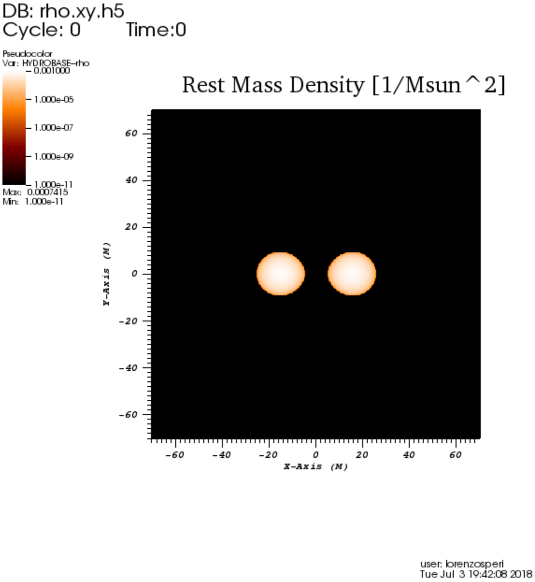
BBH-b6, BBH-b7, BBH-b10



BNS


15
10
5
0
$R\,[M_\odot]$
0
500
1000
1500
2000
2500
3000
3500
4000
$t\,[M_\odot]$
$\times 10^{-23}$
5
0
-5
$h(t, r = 100Mpc)$
h_+
$h_\times$
0
500
1000
1500
2000
2500
3000
3500
4000
$t\,[M_\odot]$

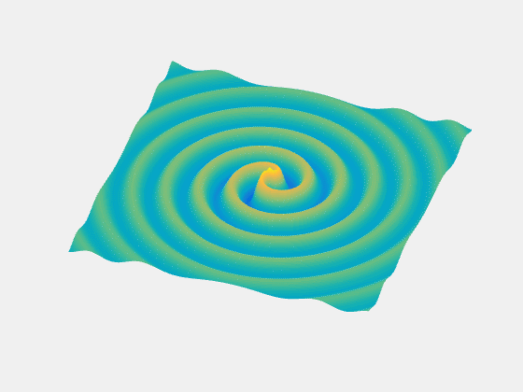
Rest Mass Density Evolution



Conclusions

"Gravitational waves will show us details of the bulk motion of dense concentrations of energy"

Kip S. Thorne



References and Codes

The codes used for the numerical simulations, the scripts for the data analysis, figures and animations are available at the website

`https://github.com/lorenzsp/thesis`