Solutions to Part A of Problem Sheet 7

Solution (7.1) First of all, note that the function is only defined for $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. We introduce new variables $\boldsymbol{y}$ and derive the dual to the problem

$$\begin{aligned} &\text{minimize} \quad -\sum_{i=1}^{m} \log(y_i) \\ &\text{subject to} \quad \boldsymbol{y} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}. \end{aligned}$$

The Lagrangian to this problem is

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\mu}) = -\sum_{i=1}^{m} \log(y_i) + \boldsymbol{\mu}^\top(\boldsymbol{y} - \boldsymbol{b} + \boldsymbol{A}\boldsymbol{x}).$$

The dual function is

$$g(\boldsymbol{\mu}) = \inf_{\boldsymbol{x},\boldsymbol{y}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\mu}).$$

The infimum is $-\infty$ if $\boldsymbol{\mu}^\top \boldsymbol{A} \neq \boldsymbol{0}$. If $\boldsymbol{\mu}$ has negative terms, then the infimum is also $-\infty$ (we could then choose an arbitrary large value for the corresponding y variable). If $\boldsymbol{\mu} > 0$, then we get a minimum by setting $y_i = \frac{1}{\mu_i}$. It follows that the dual function is

$$g(\boldsymbol{\mu}) = \begin{cases} -\sum_{i=1}^{m} \log(\mu_i) + m - \boldsymbol{b}^\top \boldsymbol{\mu} & \boldsymbol{A}^\top \boldsymbol{\mu} = \boldsymbol{0},\ \boldsymbol{\mu} > 0, \\ -\infty & \text{else.} \end{cases}$$

Solution (7.2) The problem is not convex since the equality constraint is not linear. We can formulate an equivalent convex optimization problem as

$$\text{minimize} \quad x_1^2 + x_2^2 \qquad \text{subject to} \quad x_1 \leq 0,\ x_1 + x_2 = 0.$$

Solution (7.3) Write

$$\boldsymbol{P}(\lambda) = \boldsymbol{P} + \sum_{i=1}^{m} \lambda_i \boldsymbol{P}_i,\ \boldsymbol{q}(\lambda) = \boldsymbol{q} + \sum_{i=1}^{m} \lambda_i \boldsymbol{q}_i,\ \boldsymbol{r}(\lambda) = \boldsymbol{r} + \sum_{i=1}^{m} \lambda_i \boldsymbol{r}_i.$$

With this notation, we can express the Lagrangian as

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{P}(\lambda)\boldsymbol{x} + \boldsymbol{q}(\lambda)^\top \boldsymbol{x} + \boldsymbol{r}(\lambda).$$

For $\boldsymbol{\lambda} \geq \boldsymbol{0}$ we have $\boldsymbol{P}(\lambda) > \boldsymbol{0}$, and

$$g(\boldsymbol{\lambda}) = \inf_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = -\frac{1}{2}\boldsymbol{q}(\boldsymbol{\lambda})^\top \boldsymbol{P}(\boldsymbol{\lambda})^{-1}\boldsymbol{q}(\boldsymbol{\lambda}) + \boldsymbol{r}(\boldsymbol{\lambda}).$$

The Lagrange dual is then given by

$$\begin{aligned} &\text{maximize} \quad -\frac{1}{2}\boldsymbol{q}(\boldsymbol{\lambda})^\top \boldsymbol{P}(\boldsymbol{\lambda})^{-1}\boldsymbol{q}(\boldsymbol{\lambda}) + \boldsymbol{r}(\boldsymbol{\lambda}) \\ &\text{subject to} \quad \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned}$$