Problem Sheet 9

Problems in Part A will be discussed in class. Problems in Part B come with solutions and should be tried at home.

Part A

(9.1) Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, formulate the first-order optimality conditions for the problem

$$f(\boldsymbol{x}) = -\sum_{i=1}^{m} \log(b_i - \boldsymbol{a}_i^\top \boldsymbol{x}),$$

with the constraints $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b}$ and $\boldsymbol{s} > \boldsymbol{0}$. Compute the Lagrange dual.

(9.2) Consider the optimization problem

$$\text{minimize} \quad x_1^2 + x_2^2 \qquad \text{subject to} \quad \frac{x_1}{1+x_2^2} \leq 0,\ (x_1+x_2)^2 = 0. \tag{1}$$

Show that this problem is not a convex optimization problem. Derive a convex optimization problem that has the same solution as (**??**)

(9.3) A quadratically constraint quadratic problem (QCQP) has the form

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^\top \boldsymbol{x} + r \\ \text{subject to} \quad & \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{P}_i\boldsymbol{x} + \boldsymbol{q}_i^\top \boldsymbol{x} + r_i \leq 0,\ 1 \leq i \leq m, \end{aligned}$$

with $\boldsymbol{P}$ symmetric positive definite and $\boldsymbol{P}_1, \dots, \boldsymbol{P}_m$ symmetric positive semidefinite. Derive the Lagrange dual of this problem.

Part B

(9.4) Consider the *Boolean* optimization problem

$$\begin{array}{ll} \text{minimize} & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & x_i \in \{0,1\},\ 1 \leq i \leq n. \end{array}$$

This problem requires the x_i to have integer values, and falls outside the scope of continuous optimization. Show that the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & x_i(1-x_i) = 0,\ 1 \leq i \leq n. \end{array}$$

While this problem is not convex (the equality constraints are quadratic), we can still formulate the Lagrange dual to this problem, whose optimal value gives a lower bound. Show that the Lagrange dual is a convex optimization problem, thus giving a way to *approximate* the solution of the discrete problem by solving a convex optimization problem.

(9.5) Consider the problem

$$\text{minimize} \quad \boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x} \qquad \text{subject to} \quad x_i^2 = 1,\ 1 \leq i \leq n \tag{2}$$

for a symmetric matrix $\boldsymbol{W}$. The feasible points are the sets of vectors $\boldsymbol{x} \in \{-1,1\}^n$, with each coordinate either -1 or 1. In principle, we can solve this problem by testing the objective function $\boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x}$ on all 2^n such problems, but it is computationally inefficient to do so. An interpretation of this problem is as follows: we want to group n elements into two groups, one labeled with -1 and one with 1. The entry w_{ij} of the matrix can be seen as the cost of having i and j in the same partition.

Using Lagrangian duality, show that the optimal value p^* of (**??**) satisfies

$$p^* \geq n \cdot \lambda_{\min}(\boldsymbol{W}),$$

where $\lambda_{\min}(\boldsymbol{W})$ is the smallest eigenvalue of $\boldsymbol{W}$.