Problem Sheet 6

Problems in Part A will be discussed in class. Problems in Part B come with solutions and should be tried at home.

Part A

(6.1) Given a linear programming problem,

$$\text{minimize} \quad \langle \boldsymbol{c}, \boldsymbol{x} \rangle \qquad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0} \tag{P}$$

recall the feasible sets

$$\begin{aligned} \mathcal{F} &= \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) : \boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0},\ \boldsymbol{s} \geq \boldsymbol{0}\} \\ \mathcal{F}^\circ &= \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) : \boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} > \boldsymbol{0},\ \boldsymbol{s} > \boldsymbol{0}\} \end{aligned}$$

The long-step primal dual interior point method restricts to steps in the neighbourhood

$$\mathcal{N}_{-\infty}(\gamma) = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) \in \mathcal{F}^\circ : x_i s_i \geq \gamma\mu,\ 1 \leq i \leq n\}.$$

Show that $\mathcal{N}_{-\infty}(1)$ coincides with the central path. In particular, in the extreme case $\gamma = 1$ we would force the trajectory to be exactly on the central path.

(6.2) Show that if $f(\boldsymbol{x})$ is a convex function, then the set $\{\boldsymbol{x} : f(\boldsymbol{x}) \leq 0\}$ is a convex set. Conclude that the feasible set

$$\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^n : g_1(\boldsymbol{x}) \leq 0, \dots, g_m(\boldsymbol{x}) \leq 0, h_1(\boldsymbol{x}) = 0, \dots, h_\ell(\boldsymbol{x}) = 0\},$$

with g_i convex and h_j linear, is a convex set.

(6.3) Given a constrained optimization problem with equality constraints

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) \\ \text{subject to} \quad & g_1(\boldsymbol{x}) = \dots = g_m(\boldsymbol{x}) = 0, \end{aligned}$$

the **Lagrangian** function is defined as the function in $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{\lambda} \in \mathbb{R}^m$,

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{i=1}^m \lambda_i g_i(\boldsymbol{x}) = f(\boldsymbol{x}) - \langle \boldsymbol{\lambda}, \boldsymbol{g}(\boldsymbol{x}) \rangle.$$

A point $\boldsymbol{x}$ is a local minimimum of the equality constrained optimization problem, if there exist **Lagrange multipliers** $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that the Lagrangian satisfies $\nabla \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{0}$, where the gradient is with respect to both sets of variables $(\boldsymbol{x}, \boldsymbol{\lambda})$. If f is convex and the g_i linear, this is a necessary and sufficient condition for a global minimum.

Use the method of Lagrange multipliers to find a closed-form solution for the minimum of an equality constrained quadratic optimization problem

$$\text{minimize} \quad \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

Part B

(6.4) Consider the following linear programming problem

$$\begin{aligned} \text{maximize} \quad & y_1 + y_2 \\ \text{subject to} \quad & 0.2p \cdot y_1 + y_2 + s_p = 1 + 0.01p^2, \\ & s_p \geq 0, 0 \leq p \leq 10. \end{aligned}$$

(a) Formulate the primal version of this problem, and determine the matrix $\boldsymbol{A}$ and the vectors $\boldsymbol{b}$, $\boldsymbol{c}$.

(b) Using a computing system such as Python or MATLAB, solve this problem using the long-step primal-dual method with parameters $\sigma = 0.1, 0.5, 0.9$. Plot the corresponding trajectories in the $x_2 s_2 - x_5 s_5$ plane and in the $y_1 - y_2$ plane.

(c) Describe the central path in the $y_1 - y_2$ plane for this problem.

(6.5) Consider the following portfolio optimization problem.

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{x}^\top \Sigma \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{r}^\top \boldsymbol{x} = \mu \\ & \boldsymbol{e}^\top \boldsymbol{x} = 1. \end{aligned} \tag{1}$$

where

- $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is a positive semidefinite symmetric matrix;
- $\boldsymbol{e} = (1, \ldots, 1)^\top$;
- $\boldsymbol{r} \in \mathbb{R}^n$ is a vectors of estimated returns.

The interpretation is that $\boldsymbol{\Sigma}$ is an estimated covariance matrix, and the goal is to find an investment strategy that minimizes the risk for a given return level. Using the method of **Lagrange multipliers**, show that the solution is characterized by:

$$x = \frac{1}{ac - b^2} \left(c\boldsymbol{\Sigma}^{-1} r - b\boldsymbol{\Sigma}^{-1} e \right) + \mu \cdot \left(a\boldsymbol{\Sigma}^{-1} e - b\boldsymbol{\Sigma}^{-1} r \right),$$

where $a = e^\top \boldsymbol{\Sigma}^{-1} e$, $b = e^\top \boldsymbol{\Sigma}^{-1} r$ and $c = r^{-1} \boldsymbol{\Sigma}^{-1} r$.

Given the covariance matrix and expected returns as follows,

$$\boldsymbol{r} = \begin{pmatrix} 14 \\ 12 \\ 15 \\ 7 \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 185 & 86.5 & 80 & 20 \\ 86.5 & 196 & 76 & 13.5 \\ 80 & 76 & 411 & -19 \\ 20 & 13.5 & -19 & 25 \end{pmatrix},$$

Compute the *efficient frontier*, i.e., the plot that relates the solution of (1) to target returns μ for μ varying between 5 and 35.

Repeat the same exercise, but this time with the additional constraint $\boldsymbol{x} \geq 0$. You can use CVXPY for that. Give an interpretation of this additional constraint.