Solutions to Part A of Problem Sheet 2

Solution (2.1) We implement the function $f(x)$ in MATLAB as follows

```
  f = @(x)((x-1).^6);
 df = @(x)(6*(x-1).^5);
ddf = @(x)(30*(x-1).^4);
```

We then run the code (`pr21.m`) implementing Newton's method with the three stopping criteria and with tolerance $\varepsilon = 10^{-6}$. The results are:

(a) Iterations: 66, Solution found: $\overline{x} = 1.0000$;

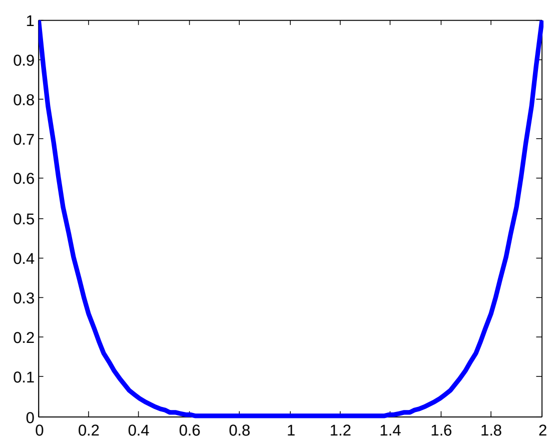
(b) Iterations: 24, Solution found: $\overline{x} = 1.0425$;

(c) Iterations: 100, Solution found: $\overline{x} = 1.0000$.

While (b) was the fastest, the accuracy was not so great. Stopping criterium (a) is reasonable, since in view of the quadratic convergence, the inequality

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| \geq \|\boldsymbol{x}_k - \boldsymbol{x}^*\| - \|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*\| \geq \|\boldsymbol{x}_k - \boldsymbol{x}^*\| - M\|\boldsymbol{x}_k - \boldsymbol{x}^*\|^2$$

shows that the difference of successive iterates gives a good bound on the error. Criterium (b) is problematic, as the function can have a very flat slope while still being far away from the minimizer, as the example shows. This issue is related to the



notion of conditioning of the function f. Criterium (c) always has the same running time, and 100 iterations would normally be more than enough for Newton's method. However, this is a very pessimistic bound, and takes much longer than necessary.

Solution (2.2) The code is contained in `pr22.m`, which makes use of the functions `newton2d.m`, `graddescco.m`, and `graddescbt.m`. The problems are solved with

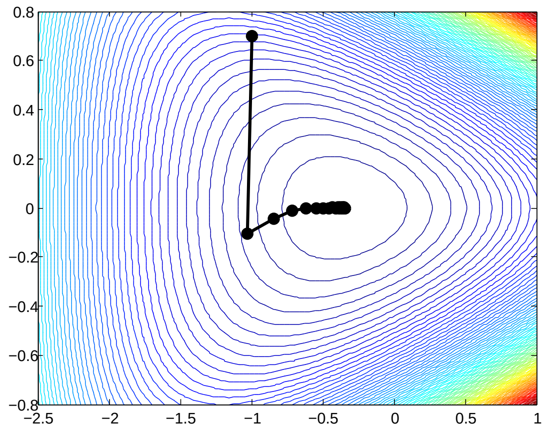


Figure 1: Gradient descent with step size 0.1

tolerance $\varepsilon = 10^{-6}$. Gradient descent with constant step length 0.1 takes 52 iterations and we get plot in Figure 1.

Gradient descent with backtracking takes 31 iterations and we get the trajectory of Figure 2.

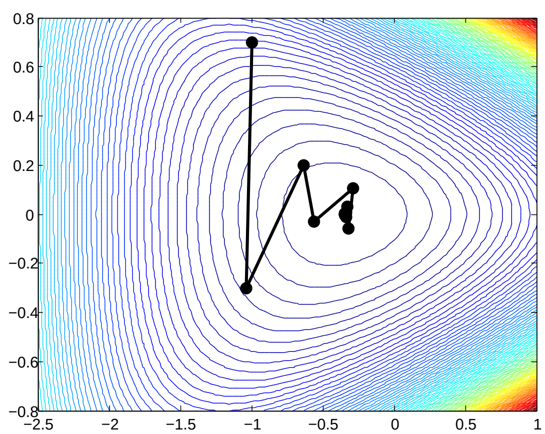


Figure 2: Gradient descent with backtracking

Finally, Newton's method only takes 6 iterations and the trajectory is pictured in Figure 3.

Solution (2.3)

(a) The verification follows from applying the definition of convexity.

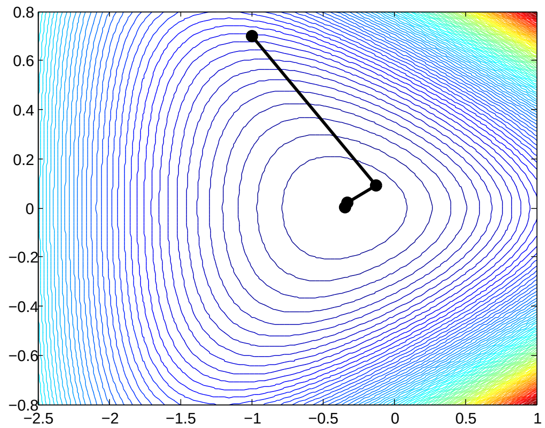


Figure 3: Newton's method

(b) The code `pr23.m` minimizes the function using Newton's method with starting point $(a,b) = (0.5, 0.5)$ and tolerance 10^{-6}. The number of iterations is 6 and the solution is $(a,b) = (1.2533, -3.2975)$. Plotting the probability of passing the exam as a function of the number of preparation hours, with the parameters (a,b), gives the curve in Figure 4.

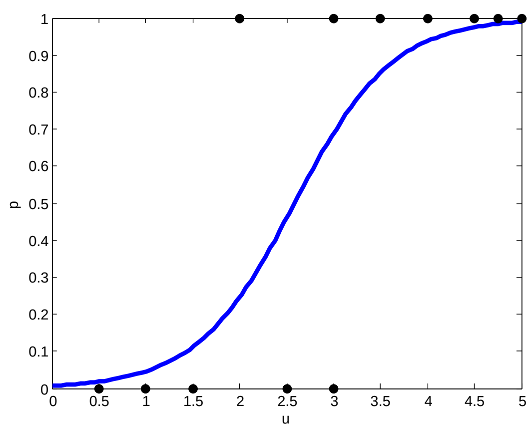


Figure 4: Probability of passing exam as function of preparation