Solutions to Part B of Problem Sheet 7

Solution (7.4) The Lagrangian of this problem is

$$\begin{aligned}\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu}) &= \langle \boldsymbol{c},\boldsymbol{x}\rangle + \boldsymbol{\lambda}^\top(\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}) + \sum_{i=1}^d \mu_i x_i(1-x_i)\\ &= \boldsymbol{x}^\top \operatorname{diag}(\boldsymbol{\mu})\boldsymbol{x} + (\boldsymbol{c}+\boldsymbol{A}^\top\boldsymbol{\lambda}-\boldsymbol{\mu})^\top\boldsymbol{x} - \boldsymbol{b}^\top\boldsymbol{\lambda}.\end{aligned}$$

We want to minimize this over $\boldsymbol{x}$. If $\boldsymbol{\mu} < 0$, then this function is unbounded below. Otherwise, we compute the minimum by setting the gradient to zero,

$$2\operatorname{diag}(\boldsymbol{\mu})\boldsymbol{x} + \boldsymbol{c} + \boldsymbol{A}^\top\boldsymbol{\lambda} - \boldsymbol{\mu} = \boldsymbol{0},$$

and get the expression (after resolving the above for $\boldsymbol{x}$ and plugging into the Lagrangian)

$$g(\boldsymbol{\lambda},\boldsymbol{\mu}) = \begin{cases} -\boldsymbol{b}^\top\boldsymbol{\lambda} - \frac{1}{4}\sum_{i=1}^d (c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda} - \mu_i)^2/\mu_i, & \boldsymbol{\mu} \geq \boldsymbol{0}\\ -\infty & \text{else.}\end{cases}$$

The dual problem is

$$\begin{aligned}&\text{maximize} \quad -\boldsymbol{b}^\top\boldsymbol{\lambda} - \frac{1}{4}\sum_{i=1}^d (c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda} - \mu_i)^2/\mu_i\\ &\text{subject to} \quad \boldsymbol{\mu} \geq \boldsymbol{0},\ \boldsymbol{\lambda} \geq \boldsymbol{0}.\end{aligned}$$

The problem can be simplified a bit further by noting that for each summand

$$-\frac{1}{4}(c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda} - \mu_i)^2/\mu_i,$$

if $c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda} \geq 0$ we can set $\mu_i = c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda}$ and the summand disappears, and if $c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda} < 0$, then we can maximize the summand by computing the derivative, and get the value $c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda}$. Summarising, we have

$$\begin{aligned}&\text{maximize} \quad -\boldsymbol{b}^\top\boldsymbol{\lambda} + \sum_{i=1}^d \min\{0, c_i + \boldsymbol{a}_i^\top\boldsymbol{\lambda}\}\\ &\text{subject to} \quad \boldsymbol{\lambda} \geq \boldsymbol{0}.\end{aligned}$$

Note that the objective function is nonsmooth, but is still concave. One can reverse the signs (and replace the min with a max in the process) to obtain a convex problem. The optimal value of this new problem will be a lower bound, and approximation, to the optimal value of the problem we started out with.

Solution (7.5) The idea here is to *relax* the problem: we replace the individual constraints $x_i^2 = 1$ with the weaker constraint that the sum of the squares is m,

$$\text{minimize} \quad \boldsymbol{x}^\top\boldsymbol{W}\boldsymbol{x} \quad \text{subject to} \quad \sum_{i=1}^n x_i^2 = n. \tag{1}$$

This constraint includes the previous one, but we have more options. Therofore, the minimizer of (1) is smaller or equal to minimizer of the original problem, and any lower bound on this minimizer will be a lower bound on the minimizer of the original problem.

The Lagrangian of (1) is

$$\mathcal{L}(\boldsymbol{x}, \mu) = \boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x} + \mu(\boldsymbol{x}^\top \boldsymbol{x} - n),$$

and computing the gradient, we see that the minimizer satisfies

$$\boldsymbol{W}\boldsymbol{x} = -\mu \boldsymbol{x}.$$

This is precisely the equation for eigenvalues and eigenvectors of $\boldsymbol{W}$! If we multiply the first equation with $\boldsymbol{x}^\top$ and replace the term $\boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x}$ in the Lagrangian with $-\mu \boldsymbol{x}^\top \boldsymbol{x}$, then we get

$$g(\mu) = -\mu n.$$

The largest possible such μ is arrived at when $\lambda = -\mu$ is the smallest eigenvalue of $\boldsymbol{W}$. Therefore, $g(\mu) \geq n\lambda_{\text{min}}(\boldsymbol{W})$, and in particular the optimal value of (1) (and of the original problem) is bounded by this.