

## Problem Sheet 1

Problems in Part A will be discussed in class. Problems in Part B come with solutions and should be tried at home, they will be discussed in class if time permits.

### Part A

(1.1) Find examples of

- (a) A function  $f \in C^2(\mathbb{R})$  with a strict minimizer  $x$  such that  $f''(x) = 0$  (that is, the second derivative is not positive definite).
- (b) A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with a strict minimizer  $x^*$  that is not an isolated local minimizer.

(1.2) For this problem you might want to recall some linear algebra.

- (a) Let  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be a symmetric matrix,  $\mathbf{b} \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ . Show that the quadratic function

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \quad (1)$$

with symmetric  $\mathbf{A}$  is convex if and only if  $\mathbf{A}$  is positive semidefinite.

- (b) Now let  $\mathbf{A} \in \mathbb{R}^{m \times n}$  be an arbitrary matrix. Show that the function

$$f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$

is convex (the 2-norm is defined as  $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}}$ ). Moreover, if  $m \geq n$  and the matrix  $\mathbf{A}$  has rank  $m$ , then it is strictly convex and the unique minimizer is

$$\mathbf{x}^* = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \mathbf{b}.$$

(1.3) A set  $S \subseteq \mathbb{R}^n$  is called *convex*, if for any  $\mathbf{x}, \mathbf{y} \in S$  and  $\lambda \in [0, 1]$ ,

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in S.$$

In words, for any two points in  $S$ , the line segment joining them is also in  $S$ . Which of the following sets are convex?

- (a)  $S = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2 = 1\}$  (the unit sphere);
- (b)  $S = \{\mathbf{x} \in \mathbb{R}^2 : 1 \leq x_1 - x_2 < 2\}$ ;
- (c)  $S = \{\mathbf{x} \in \mathbb{R}^n : |x_1| + \cdots + |x_d| \leq 1\}$ ;
- (d)  $S = \mathcal{S}_+^n \subset \mathbb{R}^{n \times n}$ , the set of symmetric, positive semidefinite matrices.

## Part B

(1.4) In engineering applications<sup>1</sup> one sometimes encounters a problem of the form

$$\text{minimize } \|A\mathbf{x} - \mathbf{b}\|_\infty, \quad (2)$$

with  $A \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$  and  $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$  is the  $\infty$ -norm.

- (a) Draw the “unit circle”  $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_\infty \leq 1\}$ .
- (b) Formulate a linear programming problem  $\mathcal{P}$  with decision variables  $(\mathbf{x}, t)$ , such that if  $(\mathbf{x}^*, t^*)$  is the unique minimizer of  $\mathcal{P}$ , then  $\mathbf{x}^*$  is the unique minimizer of (2).

Even though (2) is not a linear programming problem (the objective is not linear), it is *equivalent* to one, in the sense that a minimizer can be read off the solution of a linear programming problem.

(1.5) For this problem we generalize the notion of convexity to function not necessarily defined on all of  $\mathbb{R}^n$ . Denote by  $\text{dom} f$  the *domain* of  $f$ , i.e., the set of  $\mathbf{x}$  on which  $f(\mathbf{x})$  attains a finite value. A function  $f$  is called *convex*, if  $\text{dom} f$  is a convex set and for all  $\mathbf{x}, \mathbf{y} \in \text{dom} f$  and  $\lambda \in [0, 1]$ ,

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}).$$

Which of the following functions are convex?

- (a)  $f(x) = \log(x)$  on  $\mathbb{R}_{++}$  (the positive real numbers);
- (b)  $f(x) = x^4$  on  $\mathbb{R}$ ;
- (c)  $f(\mathbf{x}) = x_1 x_2$  on  $\mathbb{R}_{++}^2$ ;
- (d)  $f(\mathbf{x}) = x_1 / x_2$  on  $\mathbb{R}_{++}^2$ ;
- (e)  $f(x) = e^x - 1$  on  $\mathbb{R}$ ;
- (f)  $f(\mathbf{x}) = \max_i x_i$  on  $\mathbb{R}^n$ .

(1.6) Using Python or another computing system, compute and plot the sequence of points  $\mathbf{x}_k$ , starting with  $\mathbf{x}_0 = (0, 0)^\top$ , for the gradient descent algorithm for the problem

$$\text{minimize } \|A\mathbf{x} - \mathbf{b}\|_2^2$$

with data

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 10 \\ -1 \\ 0 \end{pmatrix}.$$

<sup>1</sup>For example in the synthesis of linear time-invariant dynamical systems.