Solutions to Part B of Problem Sheet 8

Solution (8.4) The largest eigenvalue of $\boldsymbol{A}$ can be written as

$$\lambda_{\max}(\boldsymbol{A}) = \max_{\boldsymbol{v}\in\mathbb{R}^n} \frac{\boldsymbol{v}^\top \boldsymbol{A}\boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}}.$$

Therefore, $t \geq \lambda_{\max}(\boldsymbol{A})$ is equivalent to the statement that for all $\boldsymbol{v}$,

$$t \geq \frac{\boldsymbol{v}^\top \boldsymbol{A}\boldsymbol{v}}{\boldsymbol{v}^\top \boldsymbol{v}} \;\Leftrightarrow\; t\boldsymbol{v}^\top \boldsymbol{v} - \boldsymbol{v}^\top \boldsymbol{A}\boldsymbol{v} \geq 0.$$

We can write $\boldsymbol{v}^\top \boldsymbol{v} = \boldsymbol{v}^\top \boldsymbol{I}\boldsymbol{v}$, with the itendity matrix $\boldsymbol{I}$, so that the above is equivalent to

$$t\boldsymbol{I} - \boldsymbol{A} \succeq \boldsymbol{0}.$$

The following semidefinite programming problem gives the largest eigenvalue:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & t\boldsymbol{I} - \boldsymbol{A} \succeq \boldsymbol{0}. \end{array}$$

Solution (8.5)

(a) A symmetric matrix can be diagonalized by orthogonal transformations, $\Sigma = \boldsymbol{Q}^\top \boldsymbol{A}\boldsymbol{Q}$, with $\boldsymbol{Q}$ orthogonal, and Σ is diagonal with the eigenvalues of $\boldsymbol{A}$ in the diagonal. The trace is invariant under similarity transformations, $\mathrm{tr}(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{Q}^\top \boldsymbol{A}\boldsymbol{Q})$, from which we get that the trace equals the sum of the eigenvalues.

(b) By Part (a), the problem is formally defined as

$$\text{minimize} \quad \boldsymbol{I} \bullet \boldsymbol{X} \quad \text{subject to} \quad x_{ij} = a_{ij} \text{ for } (i,j) \in \Omega, \quad \boldsymbol{X} \succeq \boldsymbol{0}.$$

We can formulate each of the constraints in the form $\boldsymbol{A}_{ij} \bullet \boldsymbol{X} = a_{ij}$, with $\boldsymbol{A}_{ij}$ the matrix with a 1 in entry (i,j) and 0 elsewhere. The dual problem would be of the form

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{(i,j)\in\Omega} y_{ij} a_{ij} \\ \text{subject to} & \displaystyle\sum_{(i,j)\in\Omega} y_{ij} \boldsymbol{A}_{ij} + \boldsymbol{S} = \boldsymbol{I} \\ & \boldsymbol{S} \succeq \boldsymbol{0}. \end{array}$$