Lecture 8

Linear programming is about problems of the form

$$\begin{array}{ll} \text{maximize} & \langle \boldsymbol{c}, \boldsymbol{x} \rangle \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \end{array}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$, and the inequality sign means inequality in each row. The *feasible set* is the set of all possible candidates,

$$\mathcal{F} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}.$$

This set can be empty (example: $x \leq 1,\ -x \leq -2$), unbounded (example: $x \leq 1$) or bounded (example: $x \leq 1, -x \leq 0$). In any case, it is a convex set. To understand linear programming it is of paramount importance to understand the geometry of the feasible sets of linear programming, also called *polyhedra*.

8.1 Linear Programming Duality: a first glance

Suppose we are faced with a linear programming problem and would like to know if the feasible set $\mathcal{F}$ is empty or not, i.e., if $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ has a solution. If it is not empty, we can certify that by producing a vector from $\mathcal{F}$. To verify that the feasible set is empty is more tricky: we are asked to show that *no* vector lives in $\mathcal{F}$. What we can try to do, however, is to show that the assumption of a solution to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ would lead to a contradiction. Denote by $\boldsymbol{a}_i^\top$ the rows of $\boldsymbol{A}$. Assuming $\boldsymbol{x} \in \mathcal{F}$, then given a vector $\boldsymbol{0} \neq \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)^\top$ with $\lambda_i \geq 0$, the linear combination satisfies

$$\sum_{i=1}^m \lambda_i \boldsymbol{a}_i^\top \boldsymbol{x} \leq \sum_{i=1}^m \lambda_i b_i = \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle. \tag{8.1}$$

If we can find parameters $\boldsymbol{\lambda}$ such that the left-hand side of (8.1) is identically 0 and the right-hand side is strictly negative, then we have found a contradition and can conclude that no $\boldsymbol{x}$ satisfies $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. A condition that ensures this is

$$\sum_{i=1}^m \lambda_i \boldsymbol{a}_i = \boldsymbol{0}, \quad \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle < 0. \tag{8.2}$$

In matrix form,

$$\exists \boldsymbol{\lambda} \geq \mathbf{0},\ \boldsymbol{A}^\top \boldsymbol{\lambda} = \mathbf{0},\ \langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle < 0.$$

This condition will still be satisfied if we normalise the vector $\boldsymbol{\lambda}$ such that $\sum_{i=1}^m \lambda_i = 1$, so the statement says that $\mathbf{0}$ is a convex combination of the vectors defining the equations.

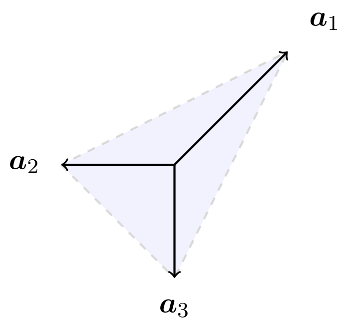
Example 8.1. Consider the system

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ -x_1 &\leq -1 \\ -x_2 &\leq -1.5. \end{aligned} \tag{8.3}$$

The transpose matrix $\boldsymbol{A}^\top$ and the vector $\boldsymbol{b}$ are

$$\boldsymbol{A}^\top = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ -1.5. \end{pmatrix}$$

Drawing the columns of $\boldsymbol{A}^\top$ we get We can get the origin as a convex combination



of the vectors $\boldsymbol{a}_i$ (drawing a rope around them encloses the origin), and such a combination is given by $\boldsymbol{\lambda} = (1/3, 1/3, 1/3)$. Taking the scalar product with the vector $\boldsymbol{b}$ we get

$$\langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle = \frac{1}{3}(2 - 1 - 1.5) = -\frac{1}{6} < 0.$$

This shows that the system (8.3) does not have a solution (a fact that in this simple example can also be seen by drawing a picture).

It turns out that Condition (8.2) is not only sufficient but also necessary, and the separating hyperplane theorem is an essential part of this. We first make a detour in order to better understand the feasible sets, the polyhedra.

8.2 Polyhedra

Definition 8.1. A polyhedron (plural: polyhedra) is a set defined as the solution of linear equalities and inequalities,

$$P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}, \tag{8.1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.

More classically, we can write out the equations.

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_d &\leq b_1, \\ &\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_d &\leq b_m. \end{aligned} \tag{8.2}$$

We now introduce some useful terminology and concepts associated to polyhedra, and illustrate them with a few examples. A supporting hyperplane H of a polyhedron P is a hyperplane such that $P \subseteq H_-$, where H_- is a halfspace associated to H. If H is a supporting hyperplane, then a set of the form $F = H \cap P$ is called a *face* of P. In particular, the polyhedron P is a face. Each of the inequalities $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i$ in (8.1) defines a supporting hyperplane, and therefore a face. The *dimension* of a face F, $\dim F$, is the smallest dimension of an affine space containing F. Faces of dimension $\dim F = \dim P - 1$ are called *facets*, faces of dimension 0 are vertices, and of dimension 1 edges. A vertex can equivalently be characterised as a point $\boldsymbol{x} \in P$ that can not be written as a convex combination of two other points in P.

Example 8.2. Polyhedra in one dimension are the sets $[a, b]$, $[a, \infty)$, $(-\infty, b]$, $\mathbb{R}$ or $\emptyset$, where $a \leq b$. Each of them is clearly convex.

Example 8.3. The set

$$P = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 + x_2 \leq 1,\ x_1 \geq 0,\ x_2 \geq 0\}.$$

is the polyhedron shown in Figure 8.1. We can write the defining inequalities in standard form $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ by setting

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This polyhedron has one face of dimension 2 (itself), three facets of dimension 1 (the sides, corresponding to the three equations), and three vertices of dimension 0 (the corners, corresponding any two of the defining equations).

Example 8.4. The polyhedron in Albrecht Dürer's famous "Melencolia I", aka the "truncated triangular trapezohedron", can be described using eight inequalities, see Figure 8.2.

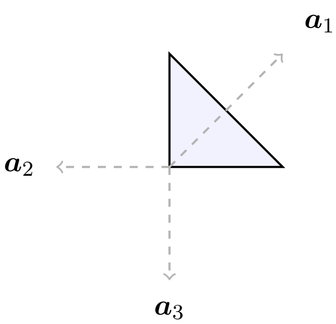


Figure 8.1: A two-dimensional polyhedron and defining equations.

$$\begin{aligned}
0.7071x_1 - 0.4082x_2 + 0.3773x_3 &\leq 1\\
-0.7071x_1 + 0.4082x_2 - 0.3773x_3 &\leq 1\\
0.7071x_1 + 0.4082x_2 - 0.3773x_3 &\leq 1\\
-0.7071x_1 - 0.4082x_2 + 0.3773x_3 &\leq 1\\
0.8165x_2 + 0.3773x_3 &\leq 1\\
-0.8165x_2 - 0.3773x_3 &\leq 1\\
0.6313x_3 &\leq 1\\
-0.6313x_3 &\leq 1
\end{aligned}$$

Figure 8.2: Dürer's Melencolia and equations defining the polyhedron

We now move to a different characterization of bounded polyhedra. The main result of this lecture is that bounded polytopes can be described completely from knowing their vertices. A polyhedron P is called bounded, if there exists a ball $B(\mathbf{0}, r)$ with $r > 0$ such that $P \subset B(\mathbf{0}, r)$. For example, halfspaces are not bounded, but the polytope from Examples 8.3 and 8.4 are.

We first observe the nontrivial fact that a polyhedron has only fin

Definition 8.5. A *polytope* is the convex hull of finitely many points,

$$P = \operatorname{conv}(\{x_1, \dots, x_k\}) = \{\sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i,\ \lambda_i \geq 0, \sum_i \lambda_i = 1\}.$$

Theorem 8.6. *A bounded polyhedron P is the convex hull of its vertices.*

Example 8.7. The triangle in Example 8.3 is the convex hull of the points $(0,0)^\top$, $(0,1)^\top$, and $(1,0)^\top$.

Example 8.8. The Dürer polytope is the convex hull of the following 12 vertices:

$$\boldsymbol{v}_1 = \begin{pmatrix} -1.4142 \\ -0.8165 \\ -0.8835 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -1.4142 \\ 0.8165 \\ 0.8835 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -0.8536 \\ -0.4928 \\ -1.5840 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} -0.8536 \\ 0.4928 \\ 1.5840 \end{pmatrix},$$
$$\boldsymbol{v}_5 = \begin{pmatrix} -0.0000 \\ -1.6330 \\ 0.8835 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} 0.0000 \\ -0.9856 \\ 1.5840 \end{pmatrix}, \boldsymbol{v}_7 = \begin{pmatrix} -0.0000 \\ 0.9856 \\ -1.5840 \end{pmatrix}, \boldsymbol{v}_8 = \begin{pmatrix} 0.0000 \\ 1.6330 \\ -0.8835 \end{pmatrix},$$
$$\boldsymbol{v}_9 = \begin{pmatrix} 0.8536 \\ -0.4928 \\ -1.5840 \end{pmatrix}, \boldsymbol{v}_{10} = \begin{pmatrix} 0.8536 \\ 0.4928 \\ 1.5840 \end{pmatrix}, \boldsymbol{v}_{11} = \begin{pmatrix} 1.4142 \\ -0.8165 \\ -0.8835 \end{pmatrix}, \boldsymbol{v}_{12} = \begin{pmatrix} 1.4142 \\ 0.8165 \\ 0.8835 \end{pmatrix}.$$

The converse of Theorem 8.6 is also true.

Theorem 8.9. *A polytope is a bounded polyhedron.*

The equivalence between polytopes and bounded polyhedra gives a first glimpse into linear programming duality theory, a topic of central importance in both modeling and algorithm design.