Problem Sheet 3

Problems in Part A will be discussed in class. Problems in Part B come with solutions and should be tried at home.

Part A

(3.1) Recall that a norm was a function $\|\cdot\|$ on $\mathbb{R}^n$ such that

- $\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = \mathbf{0}$;
- $\|\lambda \boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$ for $\lambda \in \mathbb{R}$;
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

The most prominent examples are

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}, \quad \|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|, \quad \|\boldsymbol{x}\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

(a) Show that for any norm, the *norm balls*

$$B(\boldsymbol{p}, r) := \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{p}\| \leq r\}$$

are convex sets.

(b) To a convex set C with $\mathbf{0} \in \operatorname{int} C$ we can associate the *polar* set

$$C^* := \{\boldsymbol{y} \in \mathbb{R}^n : \forall \boldsymbol{x} \in C, \langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq 1\}.$$

Show that the polar of a convex set is again convex.

(c) Describe the polar sets of the norm balls $B(\mathbf{0}, 1)$ for the 1, 2, and ∞ norms.

(3.2) Let $C, D \subseteq \mathbb{R}^n$ be disjoint, non-empty, closed, bounded convex sets. Show that there exists an affine hyperplane H strictly separating C and D (i.e., $C \subset \operatorname{int} H_-$ and $D \subset \operatorname{int} H_+$). Hint: consider the set $C - D = \{\boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x} \in C, \boldsymbol{y} \in D\}$ and use the separation theorem for a convex set and a point from the lecture. Give an example of closed convex sets C and D that cannot be strictly separated by a hyperplane.

(3.3) (Facility location.) Facility location problems concern the location of points (facilities or devices on a circuit) in a region, some of which are required to be connected by links (streets or wires). The objective is to locate some of the points that are not fixed in a way that minimizes the transport cost along the links. The distance can be measured with respect to the usual Euclidean norm or the 1-norm.

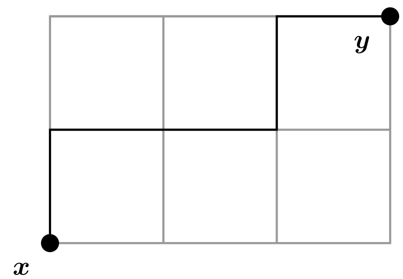


Figure 1: Manhattan distance

(a) Given a rectangular grid, show that the length of a path from coordiante $\boldsymbol{x}$ to $\boldsymbol{y}$ is given by

$$d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|_1.$$

This is an example of the so-called *Manhattan distance*.

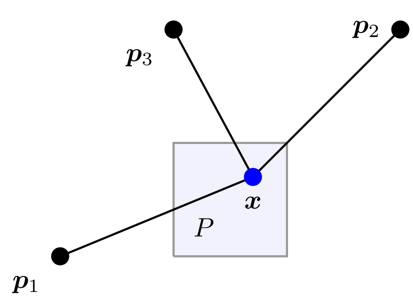
(b) Suppose now that we are given N fixed points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_N$ in on a fine grid in $\mathbb{R}^2$ and would like to find a point $\boldsymbol{x}$ that minimizes the total 1-norm distance

$$\text{minimize} \quad \sum_{i=1}^{N} \|\boldsymbol{p}_i - \boldsymbol{x}\|_1 = \sum_{i=1}^{N} |p_{i,1} - x_1| + |p_{i,2} - x_2|.$$

Assume in addition that the point $\boldsymbol{x}$ is constrained to be in a polyhedral region

$$P = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}.$$

Formulate this optimal placement problem as a linear programming problem.

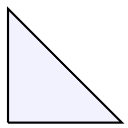


Part B

(3.4) A *polyhedron* P is a convex set defined by a system of linear inequalities

$$P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{x} \in \mathbb{R}^d$ and $\boldsymbol{b} \in \mathbb{R}^m$. A vertex $\boldsymbol{z}$ of P is a point of P that can not be expressed as a convex combination of distinct points in P. For $\boldsymbol{z} \in P$, let $\boldsymbol{A}_{\boldsymbol{z}}$ denote the matrix consisting of those rows $\boldsymbol{a}_i^\top$ of $\boldsymbol{A}$ where $\boldsymbol{a}_i^\top \boldsymbol{z} = b_i$. For example, in Figure 2 the matrix corresponding to the vertex $\boldsymbol{z} = (1,0)^\top$ is given by

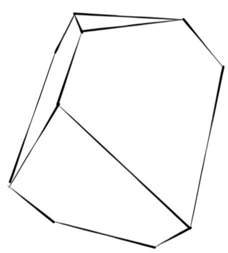


$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{A}_{\boldsymbol{z}} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Figure 2: A triangle

Show that the vertices of P are precisely the points $\boldsymbol{z}$ for which $\boldsymbol{A}_{\boldsymbol{z}}$ has rank n. In particular, every polyhedron has only finitely many vertices.

(3.5) Consider the following polyhedron.



$$
\begin{aligned}
0.7071x_1 - 0.4082x_2 + 0.3773x_3 &\leq 1 && (1)\\
-0.7071x_1 + 0.4082x_2 - 0.3773x_3 &\leq 1 && (2)\\
0.7071x_1 + 0.4082x_2 - 0.3773x_3 &\leq 1 && (3)\\
-0.7071x_1 - 0.4082x_2 + 0.3773x_3 &\leq 1 && (4)\\
0.8165x_2 + 0.3773x_3 &\leq 1 && (5)\\
-0.8165x_2 - 0.3773x_3 &\leq 1 && (6)\\
0.6313x_3 &\leq 1 && (7)\\
-0.6313x_3 &\leq 1 && (8)
\end{aligned}
$$

Figure 3: Truncated triangular trapezohedron and defining equations

(a) Print the foldout in Figure 4 and assemble the polyhedron.

(b) On the foldout, label each of the eight faces with the number of a corresponding equation (due to symmetry, such an assignment is not unique).

(c) Each of the 12 vertices arises as the intersection of three affine hyperplanes, given as the set of points where three of the equations in Figure 3 are equalities. Using a computing system such as Python, determine the vertices.

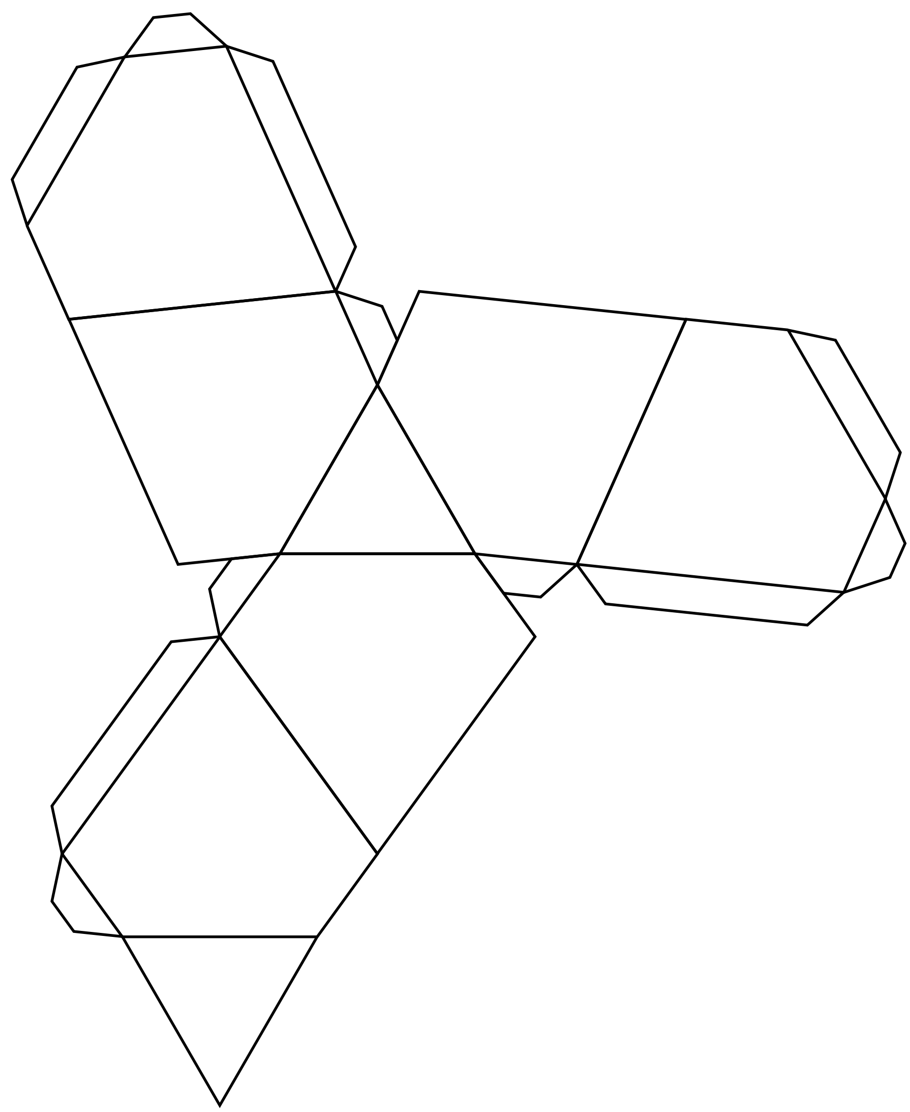


Figure 4: Foldout of a polyhedron