Some observations on the Column-Row game

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ABSTRACT

In this paper we study a new combinatorial game played on Young diagrams, called *Column-Row*. We devise a dynamic-programming algorithm for computing winning positions, or, more generally, Sprague-Grundy values. In turn, we identify winning strategies for several infinite families of starting positions. We prove those results formally, and conclude with a conjecture arising from this work.

KEYWORDS

Sprague-Grundy theory, Young diagrams, Combinatorial games, Dynamic programming

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1 INTRODUCTION

Combinatorial game theory is a large and growing field that includes in its scope a wide range of game types, generally focusing on two-player games in which both players have perfect information and there are no moves of chance. The main question in combinatorial game theory is, given some position and optimal play, which player has a winning strategy? Sprague [9] and Grundy [6] introduced a method of quantifying game positions for normal play impartial games. These Sprague-Grundy values are a generalization of winning and losing positions. Furthermore, the Sprague-Grundy values are instrumental in the theory of disjunctive sums of impartial games; see [1, 3, 6, 8–10]. This is why recent research on impartial games is predominantly focused on the Sprague-Grundy values rather than on winning/losing positions. A more detailed introduction to combinatorial game theory, including impartial games, can be found in [2, 3, 8].

In this paper we study winning/losing positions (and more generally, Sprague-Grundy values), of a new game called *Column-Row*. We proceed with some preliminary definitions concerning partitions, and Young diagrams. In Section 2, we describe the game and the methods used to verify the results, which we proceed to present in Sections 3 to 5. Each of these sections presents the Sprague-Grundy values of the *Column-Row* game on a given partition family. We conclude the paper by describing some open problems and discussing their difficulty.

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1.1 Preliminaries

We begin by defining some important concepts. A *winning position*, $\mathcal{N}$-position, is a state in which the next player can guarantee a win. A *losing position*, $\mathcal{P}$-position, is a state from which the previous player can guarantee a win. Let S be a set of integers. Then *minimal excluded value* of S, denoted by $\text{mex}(S)$, is the smallest integer which is not in S. Let G be a short impartial combinatorial game under normal play. We say $\widetilde{G}$ is a *subposition* of G if it can be obtained as a result of a single move from G. Let G have k subpositions, $G_1, \dots, G_k$. Then, we calculate the Sprague-Grundy value x of G recursively as $\text{mex}(\{x_1, \dots, x_k\})$, where x_i represents the Sprague-Grundy value of G_i, and write $SG(G) = x$. The Sprague-Grundy value of an empty partition is $\text{mex}(\emptyset) = 0$. This definition allows for computing Sprague-Grundy values recursively and represents an essential part of an algorithm presented in Section 2. A Young

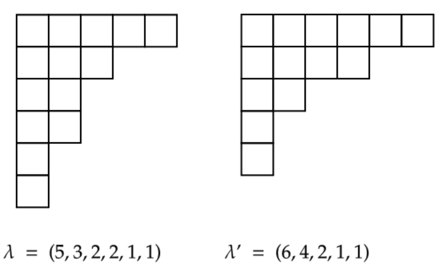

Figure 1: A Young diagram λ and its conjugate λ'.

Diagram is a diagram representing a partition λ of a non-negative integer n, $n = x_1 + \cdots + x_k$ where $x_1 \geq \cdots \geq x_k \geq 0$ is a sequence of non-decreasing integers, as a left-justified array of boxes, such that row i has x_i boxes. We write $\lambda = (x_1, \dots, x_k)$. We make no distinction between a partition λ and its Young diagram. The only partition of 0 is the empty sum, denoted by $\emptyset$. The *conjugate*, λ' of λ is a partition $\lambda' = (x'_1, \dots, x'_{x_1})$ where x'_i is the largest integer such that $x_{x'_i} \geq i$. An example of a Young diagram and its corresponding conjugate can be found on Fig. 1. A *subpartition* $\lambda[i, j]$ of λ rooted at the box (i, j) is a partition $(x_i - j,\ x_{i+1} - j,\ \dots,\ x_1 - j)$, as shown in Fig. 2. We proceed to define families of partitions. All partition parameters are strictly positive. A *rectangle* partition $R_{a,b}$ is a partition of the form (b^a), with a rows and b columns. A *gamma* partition $\Gamma_{a,b}$ is a partition of the form $(b, 1^{a-1})$, with a rows and b columns, such that the first row has b boxes and every subsequent row consists of only one box. An *almost gamma* partition $\Gamma^{1,d}_{a,b}$, is

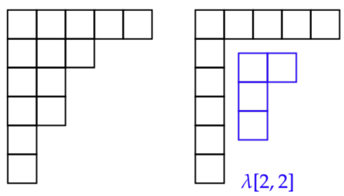

Figure 2: A subpartition $\lambda[2,2]$ (in blue) of a Young diagram λ.

a gamma partition such that column 0 is repeated $d-1$ times. A *staircase* partition S_n is a partition of the form $(n, n-1, \ldots, 1)$. A *quadrated* partition is a partition of n of the form $(x_1^{m_1}, \ldots, x_k^{m_k})$ where x_t appears m_t times, such that $\sum_{t=1}^{k} x_t \cdot m_t = n$ and all x_t, m_t are even. The examples of those partitions can be found on Fig. 3. For the family of rectangles, the allowable moves correspond to

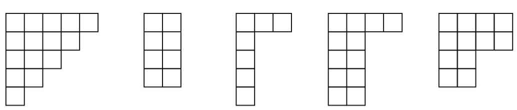

Figure 3: A staircase parition - S_5, rectangle parititon - $R_{4,2}$, gamma parition - $\Gamma_{5,3}$, almost gamma parition - $\Gamma_{5,4}^{1,3}$, and a quadrated partition with $n = 12$.

a simplified variant of the Column-Row game called LCTR (see [4, 5, 7]). In particular, the results by Gottlieb et. al [5] imply the following results for the Column-Row game.

Lemma 1.1. *For positive integers a and b, we have*

$$SG(R_{a,b}) = \begin{cases} 0 & \text{if } b > 1 \text{ and } a > 1 \text{ and } a+b \text{ is even} \\ 2 & \text{if } b \leq 2 \text{ or } a \leq 2 \text{ and } a+b \text{ is odd} \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 1.2 (LCTR [5]). *Let λ be a partition. Then we have*

(1) $\lambda'[i,j] = \lambda[j,i]'$ *and*
(2) $SG(\lambda') = SG(\lambda)$.

2 THE COLUMN-ROW GAME

Column-Row game (CR for short) is an impartial, combinatorial game played on a Young Diagram. A move in CR consists of removing any single row or any single column from the diagram. We do not make the distinction between the CR game and the corresponding Young diagram, thus the notations CR(λ) and λ are equivalent. We denote the row and column moves as i_r and j_c, where i and j are the indices of the row and column which we remove from the diagram, respectively, and write $\lambda \xrightarrow{i_r} \widetilde{\lambda}$ for a row move and $\lambda \xrightarrow{j_c} \widetilde{\lambda}$ for a column move. Refer to Fig. 4 for a visual representation of the moves of CR.

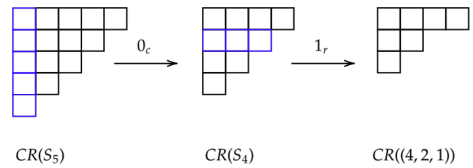

Figure 4: An example of both types of moves.

2.1 Verifying Sprague-Grundy values by aid of computer

As a consequence of the definition of Sprague-Grundy value of a game, we can use a computer to verify the results. By utilizing dynamic programming approach to store the Sprague-Grundy values of subpositions, this can be done efficiently. We outline our approach in Algorithm 1.

Function ***SG*** $(\lambda, \mathcal{D})$
  Input: Partition λ, dictionary $\mathcal{D}$.
  Output: SG value of the game CR played on λ.
  if $\lambda \notin \mathcal{D}$ **then**
    $\mathcal{M} \leftarrow \emptyset$;
    for *i-th column of λ* **do**
      let $\lambda \xrightarrow{i_c} \widetilde{\lambda}$ and add $SG(\widetilde{\lambda}, \mathcal{D})$ to $\mathcal{M}$;
    for *j-th row of λ* **do**
      let $\lambda \xrightarrow{i_r} \widetilde{\lambda}$ and add $SG(\widetilde{\lambda}, \mathcal{D})$ to $\mathcal{M}$;
    $\mathcal{D}[\lambda] \leftarrow \text{mex}(\mathcal{M})$;
  return $\mathcal{D}[\lambda]$;

Algorithm 1: An outline of the code which computes the Sprague-Grundy value SG(λ), for any input partition λ. In the code, $\mathcal{D}$ is assumed to be a dictionary of partitions as keys, stored together with the corresponding Sprague-Grundy values. $\mathcal{D}$ is initially empty.

Algorithm 1 is particularly useful for identifying (most of the times infinite) patterns of winning positions. As we will see in the subsequent sections, those patterns are then proved formally, or posed as an open conjecture. Without the aid of such an algorithm, computing Sprague-Grundy values would involve exhaustive case-analysis even for a single starting position.

3 SPRAGUE-GRUNDY VALUES OF COLUMN-ROW GAME ON A QUADRATED PARTITION

In this section, we present a solution to the *Column-Row game* on a family of quadrated partitions in terms of traditional $\mathcal{N}/\mathcal{P}$-positions.

Lemma 3.1. *Let $Q = (x_1^{m_1}, \ldots, x_k^{m_k})$ be quadrated, and let G be a subposition of Q. Then*

(1) *G is not quadrated.*
(2) *There exists a quadrated subposition of G.*

(3) *G is an $\mathcal{N}$-position, Q is a $\mathcal{P}$-position.*

Proof. As a consequence of Lemma 1.2, it is enough to consider only the row moves of Q. Suppose G is obtained by $Q \xrightarrow{i_r} G$ for some i. We show each item in turn. Item 1 and Item 2 are an immediate consequence of the properties of quadrated partitions, while Item 3 is shown by induction.

(1) Removing a row reduces the number of occurrences m_t of some x_t in G. Since Q is quadrated, $m_t - 1$ is odd G cannot be quadrated.
(2) Suppose i is odd and consider the move $G \xrightarrow{(i-1)_r} \widetilde{Q}$. It is easy to see that $\widetilde{Q}$ is quadrated. Similarly, suppose i is even. Then the move $G \xrightarrow{(i+1)_r} \widetilde{Q}$ gives us the desired quadrated subposition of G.
(3) We proceed by induction on n. For the base case, we consider $n = 0$. Notice that the only such partition is $\emptyset$, which is quadrated and trivially a $\mathcal{P}$-position. Let Q be a quadrated partition of n, let G be its subposition, and assume the claim holds for any quadrated partition of k for $k < n$. By Item 1, G is not quadrated, and, by Item 2, it has a quadrated subposition $\widetilde{Q}$, a partition of $k < n$. By the induction hypothesis, $\widetilde{Q}$ is a $\mathcal{P}$-position, and therefore G is an $\mathcal{N}$-position. Consequently, since every subposition of Q is an $\mathcal{N}$-position, Q is a $\mathcal{P}$-position.

□

4 SPRAGUE-GRUNDY VALUES OF COLUMN-ROW GAME ON A GAMMA PARTITION

Recall that a gamma partition $\Gamma_{a,b}$ is a partition of the form $(b, 1^{a-1})$, with a rows and b columns, such that the first row has b boxes and every subsequent row consists of only one box. In this section we resolve the Sprague-Grundy values of Column-Row game on any gamma partition.

Theorem 4.1. *The Sprague-Grundy value of a gamma partition $\Gamma_{a,b}$ is given by:*

$$SG(\Gamma_{a,b}) = \begin{cases} (a+b) \bmod 2 + 1 & \text{if } \min(a,b) = 1 \\ 0 & \text{if } a \text{ and } b \text{ have the same parity} \\ 3 & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on $a + b$. Given $\Gamma_{a,b}$, we have four possible types of moves and thus, four possibly distinct subpositions $G_1, \ldots, G_4$:

$$G_1\colon\ \Gamma_{a,b} \xrightarrow{0_r} \Gamma_{a-1,1},$$

$$G_2\colon\ \Gamma_{a,b} \xrightarrow{0_c} \Gamma_{1,b-1},$$

$$G_3\colon\ \Gamma_{a,b} \xrightarrow{j_c,\ j>0} \Gamma_{a,b-1},$$

$$G_4\colon\ \Gamma_{a,b} \xrightarrow{i_r,\ i>0} \Gamma_{a-1,b}.$$

Notice that G_3 and G_4 are uniquely defined. For the base case we consider $\Gamma_{a,b}$ such that $a + b \leq 4$. We have two possibilities for a and b, namely, either $\min(a, b) = 1$ or $a = b = 2$. In the former case, the Sprague-Grundy values for $\Gamma_{1,3}$ and $\Gamma_{3,1}$ are given directly by Lemma 1, since $\min(a, b) = 1$, thus we have

$$SG(\Gamma_{1,3}) = SG(\Gamma_{3,1}) = 1 = (3+1) \bmod 2 + 1.$$

In the latter case, removing the first row or the first column results in $\Gamma_{1,1}$, while the other two moves result in subpositions $\Gamma_{2,1}$ and $\Gamma_{1,2}$ respectively. In both cases, $\min(a, b) = 1$, hence

$$SG(\Gamma_{1,1}) = 1,$$

while

$$SG(\Gamma_{2,1}) = SG(\Gamma_{1,2}) = 2.$$

Then, the Sprague-Grundy value of $\Gamma_{2,2}$ is given by

$$SG(\Gamma_{2,2}) = \operatorname{mex}\{1, 2\} = 0,$$

thus the claim holds. Assume now, that the claim holds for $\Gamma_{a,b}$ such that $a + b < k$, with $k > 4$ and consider $\Gamma_{a,b}$ such that $a + b = k$. We consider the cases where a and b are of the same and distinct parity separately. Firstly, let a and b have the same parity, and consider the subpositions

$$G_1 = \Gamma_{a-1,1} \qquad G_2 = \Gamma_{1,b-1} \qquad G_3 = \Gamma_{a,b-1} \qquad G_4 = \Gamma_{a-1,b}.$$

We notice that G_1 and G_2 have the same Sprague-Grundy value. In particular, because a and b have the same parity and $\min(a, b) = 1$,

$$\begin{aligned} SG(\Gamma_{a-1,1}) &= SG(\Gamma_{1,b-1}) \\ &= ((a-1)+1) \bmod 2 + 1 \\ &= (1+(b-1)) \bmod 2 + 1, \end{aligned}$$

that is, the value is 2 if a and b are even and 1 otherwise. Similarly, since both G_3 and G_4 have parameters of distinct parity whose sum is strictly smaller than k, by the induction hypothesis we have

$$SG(\Gamma_{a,b-1}) = SG(\Gamma_{a-1,b}) = 3.$$

Then, the Sprague-Grundy value of $\Gamma_{a,b}$ where a and b are of the same parity is either $\operatorname{mex}\{2, 3\} = 0$, if both are even, or $\operatorname{mex}\{1, 3\} = 0$ if they are odd. Finally, let a and b be of distinct parity. Consequently, as the parities of $a - 1$ and $b - 1$ are distinct as well, the Sprague-Grundy values of G_1 and G_2 cannot be equal, and as they are given by $((a-1)+1) \bmod 2 + 1$ and $(1+(b-1)) \bmod 2 + 1$, respectively, we are able to reach both a position with Sprague-Grundy value of 2, and a position with Sprague-Grundy value of 1, depending on whether a or b is odd. As for G_3 and G_4, since both a and $b - 1$ as well as $a - 1$ and b have the same parity, the induction hypothesis gives us

$$SG(\Gamma_{a-1,b}) = SG(\Gamma_{a,b-1}) = 0.$$

Thus, the Sprague-Grundy value of $\Gamma_{a,b}$ where a and b are of distinct parity is therefore

$$SG(\Gamma_{a,b}) = \operatorname{mex}\{0, 1, 2\} = 3,$$

as desired. □

5 ALMOST-GAMMA

Recall an *almost gamma* partition is a partition of form (b, d^{a-1}). An example of such a partition $\Gamma_{5,4}^{1,3}$ is shown on Fig. 3. The Sprague-Grundy values of any almost gamma partition is given by the following theorem:

THEOREM 5.1. *Let $\Gamma_{a,b}^{1,d}$ be an almost-gamma partition, such that $a \geq 4,\ b \geq 3,\ d \geq 3$ and $d \leq b$. Then*

$$SG(\Gamma_{a,b}^{1,d}) = \begin{cases} a+b \pmod 2 & \textit{if } b \textit{ and } d \textit{ have the same party;} \\ 2 & \textit{if } a \textit{ and } b \textit{ have the same parity} \\ & \textit{distinct from } d; \\ 3 & \textit{otherwise.} \end{cases}$$

6 CONCLUSION

While analyzing and verifying Sprague-Grundy values for certain partition families is significantly easier with the aid of dynamic programming, doing so in general is far from trivial. In particular, depending on the structure of the partition and the resulting subpositions, the problem could pose a significant computational challenge. Such is the case with the staircase partition defined in Section 1. The following conjecture is given by running Algorithm 1 on S_n for $n \in 1, \cdots 16$

CONJECTURE 1. *Let S_n be a straircase partition. Then*

$$SG(S_n) = \begin{cases} 0 & \textit{if } n \textit{ is even} \\ 1 & \textit{if } n = 5 \\ 2 & \textit{otherwise.} \end{cases}$$

In general, solving a game, like LCTR, see [5, 7] on a staircase partition is, computationally, a relatively easy task. However, when it comes to the *Column-Row*, each staircase partition S_n has exactly n subpositions. Only one of these subpositions is a staircase partition. This significantly limits the usefulness of the algorithm as the remaining $n-1$ positions will have to be calculated with every increase of n.

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