

# Homework 2

Algorithm Design 2018-19 - Sapienza

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## 1 Michele's birthday

## 2 Valerio and Set Cover

### 3 Graph problem

## 4 Cristina and DNA

## 5 Comet and Dasher

The problem can be formalized with the following payouts matrix:

	$T_C, T_D$	$T_C, H_D$	$H_C, T_D$	$H_C, H_D$
Comet	2	-2	-1	4
Dasher	-2	2	1	-4

Let's now define:

- $h_X = Pr(\text{Head})$  for player X
- $t_X = Pr(\text{Tail})$  for player X

We can easily find that:

- $Pr(T_C, T_D) = t_C \cdot t_D$
- $Pr(T_C, H_D) = t_C \cdot h_D$
- $Pr(H_C, T_D) = h_C \cdot t_D$
- $Pr(H_C, H_D) = h_C \cdot h_D$

To guarantee that the game is fair, the expected value of Comet must be equal to the one of Dasher:

$$-2t_C t_D - t_C h_D + 2h_C t_D + 4h_C h_D = 2t_C t_D t_C h_D + -2h_C t_D + -4h_C h_D$$

with  $t_C + t_D = 1$  and  $h_C + h_D = 1$  since they are probability functions. Resolving the system we obtain

$$9h_C h_D - 3h_C - 4h_D + 2 = 0$$

There are infinite solutions: simple solutions are

- $t_C = 1, h_C = 0, t_D = h_D = 0.5$
- $t_D = 1, h_D = 0, h_C = \frac{2}{3}, t_C = \frac{1}{3}$

## 6 Drunk Giorgio

Let  $P_n = Pr(\text{Home} | \text{start} = n)$  be the probability Giorgio goes back to home starting from position  $n$  and let  $q = 1 - p$  the probability to make a step towards home. Let  $N$  be the distance from home (Giorgio starts at 0).

$$P_n = \begin{cases} 0, & \text{if } n = -1 \\ p \cdot P_{n-1} + q \cdot P_{n+1}, & \text{if } 0 \leq n < N \\ 1, & \text{if } n = N \end{cases}$$

We can rewrite  $P_n$  in this way:  $P_n = p \cdot P_{n-1} + q \cdot P_{n+1} \Rightarrow P_{n+1} - P_n = \frac{p}{q} \cdot (P_n - P_{n-1})$ .

In particular  $P_1 - P_0 = \frac{p}{q} \cdot P_0$ ; moreover  $P_2 - P_1 = (\frac{p}{q})^2 \cdot P_0$ . In general we have:  $P_{n+1} - P_0 = \sum_{k=0}^n (P_{k+1} - P_k) = \sum_{k=0}^n ((\frac{p}{q})^{k+1} \cdot P_0) = \sum_{k=1}^{n+1} ((\frac{p}{q})^k \cdot P_0) \Rightarrow P_{n+1} = P_0 + \sum_{k=1}^{n+1} ((\frac{p}{q})^k \cdot P_0) = P_0 \sum_{k=0}^{n+1} (\frac{p}{q})^k$

$$P_{n+1} = \begin{cases} P_0(n+2), & \text{if } p = q = 0.5 \\ P_0(\frac{1-(\frac{p}{q})^{n+2}}{1-\frac{p}{q}}), & \text{if } p \neq q \end{cases}$$

For  $n = N - 1$ :

$$1 = P_N = \begin{cases} P_0(N+1), & \text{if } p = q = 0.5 \\ P_0(\frac{1-(\frac{p}{q})^{N+1}}{1-\frac{p}{q}}), & \text{if } p \neq q \end{cases}$$

$$P_0 = \begin{cases} \frac{1}{N+1}, & \text{if } p = q = 0.5 \\ \frac{1-\frac{p}{q}}{1-(\frac{p}{q})^{N+1}}, & \text{if } p \neq q \end{cases}$$

The probability to go to hospital starting from 0 is:

$$Pr(\text{Hospital} | \text{start} = 0) = 1 - \lim_{N \rightarrow +\infty} P_0 = \begin{cases} 1, & \text{if } p \geq q \\ \frac{p}{q}, & \text{if } p < q \end{cases}$$

For  $p \geq q$ , Giorgio always goes (probability = 1) to hospital, instead for  $0 \leq p < \frac{1}{3}$ , Giorgio goes to hospital with probability less than 0.5 (easily obtained by solving the inequality above!).



## References