Homework 2

Algorithm Design 2018-19 - Sapienza

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Contents

1 Michele's birthday

2 Valerio and Set Cover

3 The ”k min-cut” problem

Let F^* be an optimal solution for the problem and let F_i^* be the isolating cut in the optimal solution for s_i. Since F_i is a minimum cut for s_i,

$$\sum_{e \in F_i} c_e \leq \sum_{e \in F_i^*} c_e$$

The cost of our solution is at most

$$\sum_{i=1}^{k} \sum_{e \in F_i} c_e \leq \sum_{i=1}^{k} \sum_{e \in F_i^*} c_e$$

Since each edge in an optimal solution $F*$ can be present in at most 2 different F_i^*, we have that our solution is bounded by:

$$\sum_{i=1}^{k} \sum_{e \in F_i} c_e \leq \sum_{i=1}^{k} \sum_{e \in F_i^*} c_e \leq 2 \cdot \sum_{e \in F*} c_e \leq 2 \cdot OPT$$

and this shows the 2-approximation.

4 Cristina and DNA

5 Comet and Dasher

The problem can be formalized with the following payouts matrix:

	T_C, T_D	T_C, H_D	H_C, T_D	H_C, H_D
Comet	2	-2	-1	4
Dasher	-2	2	1	-4

Let's now define:

- $h_X = Pr(\text{Head})$ for player X
- $t_X = Pr(\text{Tail})$ for player X

  We can easily find that:

- $Pr(T_C, T_D) = t_C \cdot t_D$
- $Pr(T_C, H_D) = t_C \cdot h_D$
- $Pr(H_C, T_D) = h_C \cdot t_D$
- $Pr(H_C, H_D) = h_C \cdot h_D$

To guarantee that the game is fair, the expected value of Comet must be equal to the one of Dasher:

$$-2t_c t_D - t_C h_D + 2h_C t_D + 4h_C h_D = 2t_c t_D t_C h_D + -2h_C t_D + -4h_C h_D$$

with $t_C + t_D = 1$ and $h_C + h_D = 1$ since they are probability functions. Resolving the system we obtain

$$9h_C h_D - 3h_C - 4h_D + 2 = 0$$

There are infinite solutions: simple solutions are

- $t_C = 1, h_C = 0, t_D = h_D = 0.5$
- $t_D = 1, h_D = 0, h_C = \frac{2}{3}, t_C = \frac{1}{3}$

6 Drunk Giorgio

Let $P_n = Pr(Home|start = n)$ be the probability Giorgio goes back to home starting from position n and let q = 1 - p the probability to make a step towards home. Let N be the distance from home (Giorgio starts at 0).

$$P_n = \begin{cases} 0, & \text{if } n = -1 \\ p \cdot P_{n-1} + q \cdot P_{n+1}, & \text{if } 0 \leq n < N \\ 1, & \text{if } n = N \end{cases}$$

We can rewrite P_n in this way: $P_n = p \cdot P_n + q \cdot P_n = p \cdot P_{n-1} + q \cdot P_{n+1}$ $\Rightarrow P_{n+1} - P_n = \frac{p}{q} \cdot (P_n - P_{n-1})$.
In particular $P_1 - P_0 = \frac{p}{q} \cdot P_0$; moreover $P_2 - P_1 = (\frac{p}{q})^2 \cdot P_0$. In general we have: $P_{n+1} - P_0 = \sum_{k=0}^{n}(P_{k+1} - P_k) = \sum_{k=0}^{n}((\frac{p}{q})^{k+1} \cdot P_0) = \sum_{k=1}^{n+1}((\frac{p}{q})^k \cdot P_0)$ $\Rightarrow P_{n+1} = P_0 + \sum_{k=1}^{n+1}((\frac{p}{q})^k \cdot P_0) = P_0 \sum_{k=0}^{n+1}(\frac{p}{q})^k$

$$P_{n+1} = \begin{cases} P_0(n+2), & \text{if } p = q = 0.5 \\ P_0(\frac{1-(\frac{p}{q})^{n+2}}{1-\frac{p}{q}}), & \text{if } p \neq q \end{cases}$$

For $n = N - 1$:

$$1 = P_N = \begin{cases} P_0(N+1), & \text{if } p = q = 0.5 \\ P_0(\frac{1-(\frac{p}{q})^{N+1}}{1-\frac{p}{q}}), & \text{if } p \neq q \end{cases}$$

$$P_0 = \begin{cases} \frac{1}{N+1}, & \text{if } p = q = 0.5 \\ \frac{1-\frac{p}{q}}{1-(\frac{p}{q})^{N+1}}, & \text{if } p \neq q \end{cases}$$

The probability to go to hospital starting from 0 is:

$$Pr(Hospital|start = 0) = 1 - \lim_{N \to +\infty} P_0 = \begin{cases} 1, & \text{if } p \geq q \\ \frac{p}{q}, & \text{if } p < q \end{cases}$$

For $p \geq q$, Giorgio always goes (probability = 1) to hospital, instead for $0 \leq p \leq \frac{1}{3}$, Giorgio goes to hospital with probability less than 0.5 (easily obtained by solving the inequality above!).

References