Homework 2

Algorithm Design 2018-19 - Sapienza

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The problem can be formalized with the following payouts matrix:

	T_C, T_D	T_C, H_D	H_C, T_D	H_C, H_D
Comet	2	-2	-1	4
Dasher	-2	2	1	-4

Let's now define:

- $h_X = Pr(\text{Head})$ for player X
- $t_X = Pr(\text{Tail})$ for player X

We can easily find that:

- $Pr(T_C, T_D) = t_C \cdot t_D$
- $Pr(T_C, H_D) = t_C \cdot h_D$
- $Pr(H_C, T_D) = h_C \cdot t_D$
- $Pr(H_C, H_D) = h_C \cdot h_D$

To guarantee that the game is fair, the expected value of Comet must be equal to the one of Dasher:

$$-2t_c t_D - t_C h_D + 2h_C t_D + 4h_C h_D = 2t_c t_D t_C h_D + -2h_C t_D + -4h_C h_D$$

with $t_C + t_D = 1$ and $h_C + h_D = 1$ since they are probability functions. Resolving the system we obtain

$$9h_C h_D - 3h_C - 4h_D + 2 = 0$$

There are infinite solutions: simple solutions are

- $t_C = 1, h_C = 0, t_D = h_D = 0.5$
- $t_D = 1, h_D = 0, h_C = \frac{2}{3}, t_C = \frac{1}{3}$

6 Drunk Giorgio

Let D_t be a random variable that denotes the position of Giorgio at time t. Let $P_n = Pr(H|D_0 = n)$ be the probability Giorgio goes to hospital starting from position n.

$$P_n = \begin{cases} 1 & \text{if } n = -1 \\ p \cdot P_{n-1} + (1-p) \cdot P_{n+1} & \text{if } n \geq 0 \end{cases}$$

Thus we obtain the following recurrence equation:

$(1-p) \cdot P_{n+1} - P_n + p \cdot P_{n-1} = 0$

For $n = -1$ it is true since Giorgio has already touched the highway. Now, be E the event to make a step towards the highway.

$P_n = Pr(H|D_0 = n)$
$= Pr(H \cap E|D_0 = n) + Pr(H \cap \neg E|D_0 = n)$
$= Pr(E|D_0 = n) \cdot Pr(H|E \cap D_0 = n) + Pr(\neg E|D_0 = n) \cdot Pr(H|\neg E \cap D_0 = n)$
$= p \cdot Pr(H|D_1 = n-1) + (1-p) \cdot Pr(H|D_1 = n+1)$
$= p \cdot Pr(H|D_0 = n-1) + (1-p) \cdot Pr(H|D_0 = n+1)$
$= p \cdot P_{n-1} + (1-p) \cdot P_{n+1}$

and this justifies the recurrence equation defined above. We can now solve the characteristic equation: $(1-p) \cdot r^2 - r + p = 0$ and find the roots: (i) $\frac{p}{1-p}$ and (ii) 1. We can thus write:

$P_n = A \cdot (\frac{p}{1-p})^n + B \cdot 1^n = A \cdot (\frac{p}{1-p})^n + B$, where A and B are two constants. Since $P_0 = p$ (Giorgio is at beginning one step away from the highway!) $= A + B \Rightarrow B = p - A$ and $P_{-1} = 1 = A \cdot \frac{1-p}{p} + B \Rightarrow A = \frac{p-1}{1-\frac{1-p}{p}}$, we can state that Giorgio goes to hospital with probability:

$Pr(H) = p - \frac{p-1}{1-\frac{1-p}{p}} + \frac{p-1}{1-\frac{1-p}{p}} \cdot \sum_{n=0} (\frac{p}{1-p})^n$

The series converges for $p \leq \frac{1}{2}$, and diverges for $p > \frac{1}{2}$, where diverging means that the event H always happens after an infinite amount of steps! If converges (its sum is a well-known one since it is a series of powers), we can also bound $Pr(H) \leq \frac{1}{2} \Leftrightarrow p \leq \frac{1}{2}$ (obtained by solving the inequality)

References