MATH 4338 Main Problem 1

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Theorem 1. *If $a, b \in \mathbb{R}$ and $a \neq 0$, then there is one and only one number x s.t. $a \cdot x = b$. The number is given by $x = ba^{-1}$.*

Proof. Assume $a, b \in \mathbb{R}$ and $a \neq 0$. Let $x = ba^{-1}$. As $a \in \mathbb{R}$, a^{-1} is a number by the existence of a reciprocal axiom. By the closure property, $x \in \mathbb{R}$. Using the axioms,

$$\begin{aligned}
ax &= \\
&= a(ba^{1}) && \text{[by substitution]} \\
&= a(a^{-1}b) && \text{[by commutative property]} \\
&= (aa^{-1})b && \text{[by associativity]} \\
&= (1)b && \text{[since } a^{-1} \text{ is the reciprocal of } a\text{]} \\
&= b && \text{[since 1 is the multiplicative identity]}
\end{aligned}$$

Hence, $x = ba^{-1}$. Assume y is a number such that $a \cdot y = b$. Using the axioms,

$$\begin{aligned}
ay &= b && \text{[by assumption]} \\
(a^{-1})ay &= (a^{-1})b && \text{[multiply by} a^{-1}\text{]} \\
(a^{-1}a)y &= (a^{-1}b) && \text{[by associativity]} \\
(aa^{-1})y &= (ba^{-1}) && \text{[by commutativity]} \\
(1)y &= ba^{-1} && \text{[since } a^{-1} \text{ is the reciprocal of } a\text{]} \\
y &= ba^{-1} && \text{[since 1 is the multiplicative identity]}
\end{aligned}$$

Thus, $y = ba^{-1} = x$ and x is unique. $\square$