MATH 4338 Main Problem 9

Andy Lu

Question. *Given the function f defined on $I = x : 0 \leq x \leq 1$ by the formula*

$$f(x) = \begin{cases} 1 & \textit{if } x \textit{ is rational} \\ 0 & \textit{if } x \textit{ is irrational} \end{cases}$$

Prove that $\underline{\int_0^1} f(x)\,dx = 0$ and $\overline{\int_0^1} f(x)\,dx = 1$.

Proof. Let f be defined on $I = x : 0 \leq x \leq 1$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Let Δ be a subdivision of I s.t. $0 = d_0 < d_1 < \ldots < d_{n-1} < d_n = 1$ with the corresponding subintervals denoted $I_1, I_2, \ldots I_n$. By the Archimedian Principle, $\forall\, I_k\ \exists$ a rational number and an irrational number in I_k, for $k = 1, 2, \ldots n$. Let m_k and M_k denote the g.l.b and l.u.b of f on I_k. Since the range of f on I_k contains only 2 values,

$$\inf_{I_k} f = 0 \qquad \sup_{I_k} f = 1 \tag{1}$$

With respect to Δ,

$$\begin{aligned}
S^+(f, \Delta) = &\sum_{i=1}^{n} M_i(d_i - d_{i-1}) && \text{(by definition of Upper Darboux Sum)} \\
= &\sum_{i=1}^{n} (d_i - d_{i-1}) && \text{(since } M_i = 1\text{)} \\
= &(d_1 - d_0) + (d_2 - d_1) + \ldots + (d_n - d_{n-1}) && \text{(by definition of sum)} \\
= &d_n - d_0 && \text{(by combining like terms)} \\
= &1 - 0 && \text{(by construction of } \Delta\text{)} \\
= &1
\end{aligned}$$

As $S^+(f, \Delta) = 1$, for f on $I = [0, 1]$ and subdivision Δ, the g.l.b. of $S^+(f, \Delta)$ is 1. Then by definition of upper Darboux integral, $\overline{\int_0^1} f(x)\,dx = 1$. Similarly, with respect to Δ,

$$\begin{aligned} S\_(f,\Delta) = &\sum_{i=1}^{n} m_i(d_i - d_{i-1}) && \text{(by definition of Lower Darboux Sum)} \\ =&\sum_{i=1}^{n}(0)(d_i - d_{i-1}) && \text{(since } m_i = 0\text{)} \\ =&0 \end{aligned}$$

As $S\_(f,\Delta) = 0$, for f on $I = [0,1]$ and subdivision Δ, the l.u.b. of $S\_(f,\Delta)$ is 0. Then by definition of the Lower Darboux Integral, $\underline{\int_0^1} f(x)\,dx = 0$. Hence, $\underline{\int_0^1} f(x)\,dx = 0$ and $\overline{\int_0^1} f(x)\,dx = 1$. □