MATH 4338 Main Problem 4

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Question 1. *2.3 #2*
Given,

$$f(x)\begin{cases} \frac{x^2-1}{x^4-1} & 1 < x < 2 \\ x^2+3x-2 & 2 \leq x < 5 \end{cases}$$

- *Show whether f is continuous on the left at 2 using the method from the book.*
- *Show whether f is continuous on the right at 2 using the ϵ-δ definition of continuity.*

Proof. Continuity on the left:

Start by multiplying the given $f(x)$ by 1, in the form of $\frac{x^{-4}}{x^{-4}}$:

$$(\frac{x^2-1}{x^4-1})(\frac{x^{-4}}{x^{-4}}) = \frac{x^{-2}-x^{-4}}{1-x^{-4}}$$

We can compute the limit of this function with the limit rules:

$$\lim_{x\to 2^-} \frac{x^{-2}-x^{-4}}{1-x^{-4}}$$

$$\begin{aligned}
&= \frac{\lim_{x\to 2^-}(x^{-2}-x^{-4})}{\lim_{x\to 2^-}(1-x^{-4})} && \text{[Thm 2.8]} \\
&= \frac{\lim_{x\to 2^-}(x^{-2}) - \lim_{x\to 2^-}(x^{-4})}{\lim_{x\to 2^-}(1) - \lim_{x\to 2^-}(x^{-4})} && \text{[Thm 2.5]} \\
&= \frac{\frac{1}{2^2}-\frac{1}{2^4}}{(1)-(2^{-4})} && [\lim_{x\to 2^-}(1) = 1 \text{ by Thm 2.2}] \\
&= \frac{\frac{1}{4}-\frac{1}{16}}{\frac{16}{16}-\frac{1}{16}} = \frac{3/16}{15/16} = \frac{3}{15} = \frac{1}{5} && \text{[by algebra]}
\end{aligned}$$

As 2 is in the domain of f, we can evaluate our function at $x = 2$:

$$f(2) = \frac{(2)^2-1}{(2)^4-1} = \frac{3}{15} = \frac{1}{5}$$

By the definition of continuous on the left, $f(x)$ ijs continuous on the left at 2. □

Proof. Continuity on the right:

Let $\epsilon > 0$. Pick $\delta = \frac{-7+\sqrt{49+4\epsilon}}{2}$.
Note:

$$\begin{aligned} \delta(\delta+7) &= \epsilon \\ \delta^2 + 7\delta - \epsilon &= 0 \\ \delta &= \frac{-7 \pm \sqrt{49+4\epsilon}}{2} \end{aligned}$$

Since $\delta > 0$, we ignore $\frac{-7-\sqrt{49+4\epsilon}}{2}$. As $\epsilon > 0$, then $49 + 4\epsilon > 49 > 0$. Taking the square root of all terms, $\sqrt{49+4\epsilon} > 7$. Thus, $\delta > 0$ and $\delta \in \mathbb{R}$. Let $x \in [2, 5)$. Suppose $0 \leq x - 2 < \delta$. Then,

$$\begin{aligned} &|f(x) - f(2)| \\ &= |(x^2 + 3x - 2) - (2^2 + 3(2) - 2)| \\ &= |x^2 + 3x - 2 - 8| \\ &= |(x+5)(x-2)| \\ &= |x+5||x-2| \\ &= |x-2+7||x-2| < \delta(\delta+7) = \epsilon \end{aligned}$$

Thus, by defintion, $\lim_{x\to 2+} f(x) = 2$, so $f(x)$ is continuous on the right at 2. $\square$