MATH 4338 Main Problem 6

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Proof. Suppose f is uniformly continuous on I_1 and I_2. Let $\epsilon > 0$. Then, $\exists\, \delta_1 > 0$ such that $\forall i, j \in I_1$ if $|i - j| < \delta_1$, then $|f(i) - f(j)| < \frac{\epsilon}{2}$. Similarly, $\exists\, \delta_2 > 0$ such that $\forall i, j \in I_2$ if $|i - j| < \delta_2$, then $|f(i) - f(j)| < \frac{\epsilon}{2}$.

Consider $I_1 \cup I_2$, where $I_1 = [a, b]$ and $I_2 = [c, d]$. If $c - b > 0$, pick $\delta = min\{\delta_1, \delta_2, (c - b)\}$. Otherwise, pick $\delta = min\{\delta_1, \delta_2\}$. As δ_1, $\delta_2 > 0$ and $c - b > 0$, $\delta > 0$. Note, $\delta \in \mathbb{R}$. Suppose $i, j \in I_1 \cup I_2$. Then there are 2 cases:

Case 1: $I_1 \cap I_2 \neq \emptyset$

Let k $\in I_1 \cap I_2$. Then,

$$\begin{aligned} |i - j| = & \\ &= |i - k + k - j| \\ &\leq |i - k| + |-1||j - k| && \text{[by Triangle Inequality]} \\ &= |i - k| + |j - k| \end{aligned}$$

As $k \in I_1$ and f is uniformly continuous on I_1, then $|i - k| < \delta_1$ and $|f(i) - f(k)| < \frac{\epsilon}{2}$. As $k \in I_2$ and f is uniformly continuous on I_2, then $|j - k| < \delta_2$ and $|f(j) - f(k)| < \frac{\epsilon}{2}$. Suppose $|i - k| < \delta$ and $|j - k| < \delta$, then,

$$\begin{aligned} |f(i) - f(k)| + |f(j) - f(k)| = & \\ &< \delta_1 + \delta_2 \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Case 2: $I_1 \cap I_2 = \emptyset$

Without loss of generality, let $i \in I_1$ and $j \in I_2$. Suppose $|i - j| < \delta$. As $|i - j| \geq c - b$, there are two cases:

Case 1: $\delta = c - b$

This is a contradiction because $|i - j| \geq c - b$ and $|i - j| < c - b$.

Case 2: $\delta < c - b$

This is a contradiction because $|i - j| \geq c - b$ and $|i - j| < \delta < c - b$.

Thus our hypothesis is false and the conclusion is vacuously true.

Thus, f is uniformly continuous on $I_1 \cup I_2$. □