MATH 4338 Main Problem 2

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Question 1. *1.2 #10*

- *Define the operations of addition and multiplication for the set* $\mathbb{Q}[\sqrt{5}]$
- *State the additive and multiplicative identities (no proof needed)*
- *For any element* $a + b\sqrt{5} \in \mathbb{Q}$ *that is not the additive identity, show what its muliplicative inverse is and prove that it is unique*

Proof. Let $a + b\sqrt{5}$, $c + d\sqrt{5} \in \mathbb{Q}[\sqrt{5}]$. Then we define the following:

Addition Let $e + f\sqrt{5} \in \mathbb{Q}[\sqrt{5}]$, where

$$\begin{aligned} e + f\sqrt{5} &= (a + b\sqrt{5}) + (c + d\sqrt{5}) \\ &= (a + c) + (b + d)\sqrt{5} \end{aligned}$$

So $e = a + c$ and $f = b + d$.

Multiplication Let $g + h\sqrt{5} \in \mathbb{Q}[\sqrt{5}]$, where

$$\begin{aligned} g + h\sqrt{5} &= (a + b\sqrt{5}) \cdot (c + d\sqrt{5}) \\ &= (a)(c + d\sqrt{5}) + (b\sqrt{5})(c + d\sqrt{5}) \\ &= (a)(c) + (a)(d\sqrt{5}) + (c)(b\sqrt{5}) + ((d\sqrt{5}))((b\sqrt{5})) \\ &= (a)(c) + (5)(b)(d) + (a)(d)(\sqrt{5}) + (c)(b)(\sqrt{5}) \\ &= [(a)(c) + (5)(b)(d)] + [(a)(d) + (b)(c)]\sqrt{5} \end{aligned}$$

So $g = a \cdot c + 5 \cdot b \cdot d$ and $h = a \cdot d + b \cdot c$.

Additive Identity : $0 + 0\sqrt{5}$ where $0 \in \mathbb{Q}$.

Multiplicative Identity : $1 + 0\sqrt{5}$ where $1, 0 \in \mathbb{Q}$.

Multiplicative Inverse : Let $x \in \mathbb{Q}[\sqrt{5}]$ and suppose $x = a + b\sqrt{5}$ and $y = c + d\sqrt{5}$, where $a, b \neq 0$. Pick

$$c = \frac{-a}{-a^2 + 5b^2} \qquad d = \frac{b}{-a^2 + 5b^2}$$

Since $x \in \mathbb{Q}[\sqrt{5}]$, then $a, b \in \mathbb{Q}$. Suppose $a = \frac{p}{q}$ and $b = \frac{r}{s}$ Note that $p, q, r, s \in \mathbb{Z}$, by definition of $\mathbb{Q}$.

The denominator of c and d then, by arithmetic, is:

$$\begin{aligned}(-a^2 + 5b^2) &= (\frac{-p}{q} + \frac{5 \cdot r \cdot r}{s \cdot s}) \\ &= \frac{-p \cdot q \cdot s \cdot s + 5 \cdot r \cdot r \cdot q}{q \cdot s \cdot s}\end{aligned}$$

Because $a, b \in \mathbb{Q}$, q and s cannot be equal to zero. Then, the product $q \cdot s \cdot s$ is the product of three non-zero integers. Similarly in the numerator, because $a, b \neq 0$, then $p, r \neq 0$. Thus the numerator is a sum of two integer products. Then $\exists u, v \in \mathbb{Z}$ such that

$$\begin{aligned}(-a^2 + 5b^2) &= \frac{-p \cdot q \cdot s \cdot s + 5 \cdot r \cdot r \cdot q}{q \cdot s \cdot s} \\ &= \frac{u}{v}\end{aligned}$$

Thus, by the definition of $\mathbb{Q}$, $c, d \in \mathbb{Q}$. Furthermore, by definition of $\mathbb{Q}[\sqrt{5}]$, $y \in \mathbb{Q}[\sqrt{5}]$. Using the definition of multiplication for two elements of $\mathbb{Q}[\sqrt{5}]$,

$$\begin{aligned}(x \cdot y) &= (a + b\sqrt{5})(c + d\sqrt{5}) \\ &= (a)(c) + (a)(d\sqrt{5}) + (c)(b\sqrt{5}) + (b\sqrt{5})(d\sqrt{5})\end{aligned}$$

Note:

$$\begin{aligned}(a)(d\sqrt{5}) &= (a)(\frac{b}{-a^2 + 5b^2}) \\ &= \frac{ab}{-a^2 + 5b^2} \\ (c)(b\sqrt{5}) &= (\frac{-a}{-a^2 + 5b^2})(b\sqrt{5}) \\ &= \frac{-ab}{-a^2 + 5b^2}\end{aligned}$$

Thus $(a)(d\sqrt{5}) + (c)(b\sqrt{5}) = 0$.

Continuing $(x \cdot y)$:

$$
\begin{aligned}
(x \cdot y) &= (a)(c) + (a)(d\sqrt{5}) + (c)(b\sqrt{5}) + (b\sqrt{5})(d\sqrt{5}) \\
&= (a)(c) + (0) + (b\sqrt{5})(d\sqrt{5}) \\
&= (a)(c) + (b\sqrt{5})(d\sqrt{5}) \\
&= (a)(c) + (b)(d)(\sqrt{5} \cdot \sqrt{5}) \\
&= (a)(c) + (5)(b)(d) \\
&= (a)(\frac{-a}{-a^2 + 5b^2}) + (5b)(\frac{b}{-a^2 + 5b^2}) \\
&= (\frac{-a^2}{-a^2 + 5b^2}) + ((\frac{5b^2}{-a^2 + 5b^2})) \\
&= (\frac{-a^2 + 5b^2}{-a^2 + 5b^2}) \\
&= 1
\end{aligned}
$$

Thus, the multiplicative inverse y exists. Suppose $z \in \mathbb{Q}[\sqrt{5}]$ and $x \cdot z = 1$. Then $x \cdot y = 1 = x \cdot y$. Therefore $y = z$. Thus, y is unique.

$\square$