

# Metodi Matematici per Equazioni alle Derivate Parziali

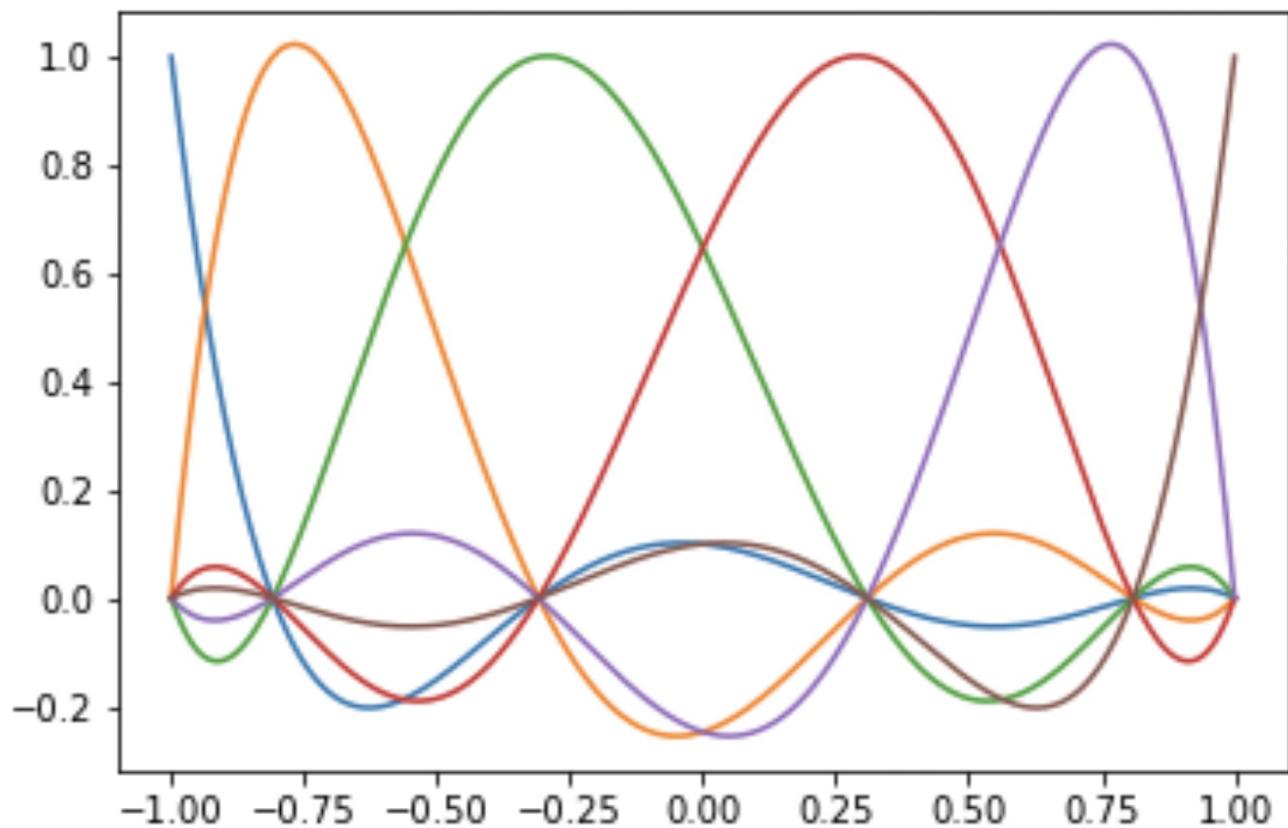
Lagrangian finite elements

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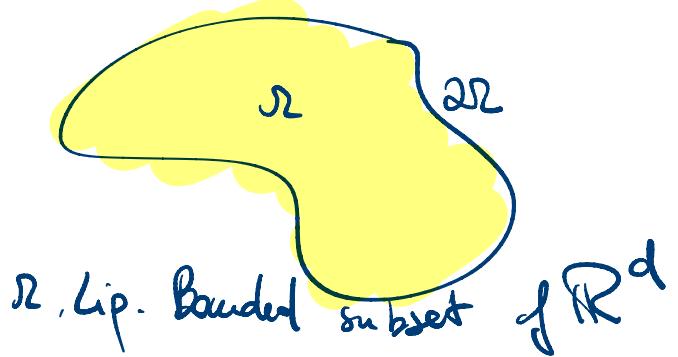
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# Prototypical Problem

$$i) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



Set the problem in a Hilbert Space  $V$

$$V = H_0(\Omega) := \left\{ v \in L^2(\Omega), \nabla v \in L^2(\Omega), \underbrace{v|_{\partial\Omega}}_{} = 0 \right\}$$

Multiply i) by  $v$ , integrate by parts:

$$\int_{\Omega} -\Delta u v = \int_{\Omega} f v + \nabla v \cdot \nabla u \quad \forall v \in V \equiv H_0(\Omega)$$

$$\boxed{\int_{\Omega} \nabla u \cdot \nabla v} \quad \xrightarrow{\text{by linear on } V \times V} \quad \boxed{- \int_{\partial\Omega} \frac{\partial u}{\partial n} v} \quad \boxed{= \int_{\Omega} f v} \quad \forall v \in V \equiv H_0(\Omega)$$

$$a(u, v) \quad \text{by linear on } V \times V$$

$$\langle F, v \rangle := \int_{\Omega} f v \quad F \text{ is linear on } V \quad \mathcal{L}(V, \mathbb{R})$$

$$F \in V'$$

$$\nexists a: V \times V \rightarrow \mathbb{R} \quad \exists! A: V \rightarrow V'$$

$$a(u, v) = \langle Au, v \rangle$$

Variational formulation: Given  $F \in V'$ , find  $u \in V$  s.t.

$$a(u, v) = \langle F, v \rangle \quad \forall v \in V \quad \Leftrightarrow$$

$$Au = F \quad \text{in } V'$$

if  $a$  is coercive (and bounded) then  
 $\nexists F \in V'$ ,  $\exists ! u \in V$  s.t.  $Au = F$  and

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$$

recall : a coercive means:  $\exists \alpha > 0$  s.t.

$$a(u, u) \geq \alpha \|u\|^2 \quad \forall u \in V$$

Construct  $V_h$  finite dimensional, s.t.  $V_h \subset V$

Search for  $u_h \in V_h$  s.t.

$$② \quad a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_h$$

$$Au_h = F \quad \text{in } V_h'$$

By Lax Milgram,  $\exists ! u_h$  that satisfies ②

$$\Rightarrow a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \quad \text{orthogonality of error}$$

$$\alpha \|u - u_h\|^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h)$$

$$\leq \|A\| \|u - u_h\| \|u - v_h\| \quad \forall v_h \in V_h$$

$$\|u - u_h\| \leq \frac{\|A\|}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$$

Cea's Lemma

How to construct  $V_h$ ?

$$V_h = \text{span} \left\{ v_i \right\}_{i=1}^n \quad \text{where } n = \text{dimension}(V_h)$$

$$\Rightarrow \forall u_h \in V_h \quad \exists! \left\{ u^i \right\}_{i=1}^n \in \mathbb{R}^n \text{ st.}$$

$$u_h(x) = \sum_{i=1}^n u^i v_i(x) \quad \text{sum is implied}$$

$$\exists n \text{ dual basis functions } \left\{ v^i \right\}_{i=1}^n \in V' \text{ s.t.}$$

$$\forall u_h(x) \quad \langle v^i, u_h \rangle = u^i \quad \forall i=1, \dots, n$$

$$\Leftrightarrow v^i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

think of  $\{v^i\}_{i=1}^n$  as a "basis for  $V_h'$ "

$$\Rightarrow A_u = F \quad A \in \mathbb{R}^{n \times n} \quad F \in \mathbb{R}^n$$

$$u = \{u^i\}_{i=1}^n \in \mathbb{R}^n$$

$$A_{ij} u^j = F_i \quad A_{ij} := \int_{\Omega} v_j \cdot \nabla v_i = \langle Av_j, v_i \rangle_{\mathbb{R}^n}$$

$$F_i := \int_{\Omega} f \cdot v_i = \langle F, v_i \rangle_{\mathbb{R}^n}$$

If I know  $\{v^i\}_{i=1}^n$  then I can define

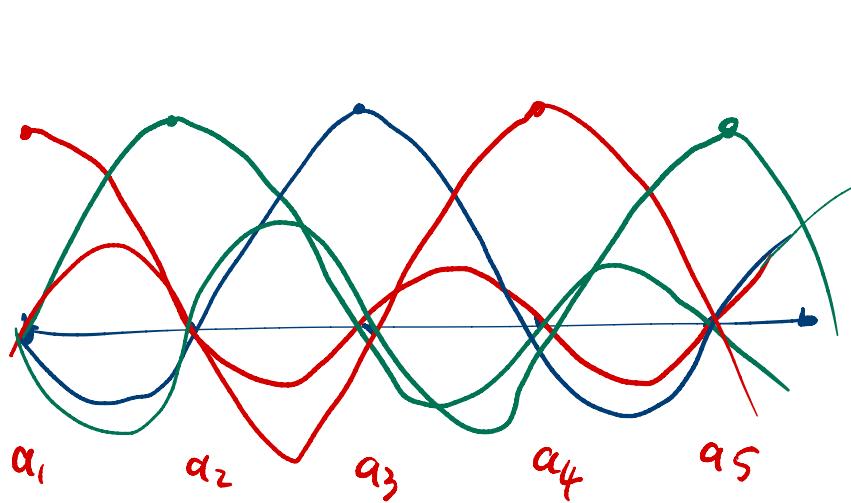
$$\Pi_{V_h} : V \longrightarrow V_h$$

$$u \longrightarrow \underbrace{\langle v^i, u \rangle}_{\bar{u}^i} v_i$$

1D:  $S_2 = [0,1]$   $V_h = P^{n-1}([0,1])$

- Select a set of support points  $\{a_i\}_{i=1}^n$ ,  $a_1 < a_2 < \dots < a_n$
- Define  $v^i(u) = u(a_i)$   $\forall u \in H^1(0,1)$
- Construct  $v_i(x)$  s.t.  $v_i(x) \in P^{n-1}$ , and

$$v^j(v_i) = v_i(a_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$



$$v_i \in P^{n-1}$$

Lagrange  
Polynomials  
for  $\{a_i\}_{i=1}^n$

$$v_i := \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(x - a_j)}{(a_i - a_j)}$$

The set  $\{v^i\}_{i=1}^n$   
is called "nodal  
basis functions"

Lagrange basis functions:

$$v_i = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(x - a_j)}{(a_i - a_j)}$$

$$v^i := \delta(x - a_i) \Rightarrow \langle v^i, u \rangle = \int_2 v^i u = u(a_i)$$

$$\Pi_{V_h} : V \rightarrow V_h$$

$$u \rightarrow \langle v^i, u \rangle \quad v_i = u^i \quad v_i = \sum_{i=1}^n u(a_i) v_i$$

$\{v^i\}$  Allows you to put  $V_h$  in one to one relation with  $\mathbb{R}^n$

For  $H_0([0,1])$  we choose  $\{a_i\}_{i=0}^{nn}$ ,  $\dim = n$

$$\text{we set } \{v^i\}_{i=1}^n = \delta(x - a_i) \quad a_0 = 0, a_{nn} = 1$$

$$v_i = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(x - a_j)}{(a_i - a_j)} \in \mathcal{P}^{n+1}$$

and define  $\Pi_{V_h} u$ :  $v^i(u) v_i$

sum is between 1 and n

Split the domain into  $T_h$  which is a collection of simple domains

Triangulation \*

$$\mathcal{Z}_h = \left( \bigcup_{k=1}^m \overline{T_k} \right)$$

1D 

choose  $\{x_i\}_{i=1}^{m+1}$  st.  $x_i \neq x_j$  if  $i \neq j$   
 define  $T_i = (x_i, x_{i+1})$   $i=1, \dots, m$

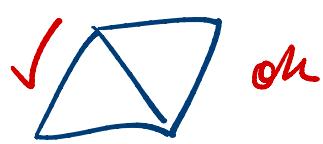
$$\mathcal{Z}_h = (0, 1)$$

2D Assume that  $\Sigma$  is Polygonal

$$\mathcal{Z}_h = \left( \bigcup_{k=1}^m \overline{T_k} \right)$$

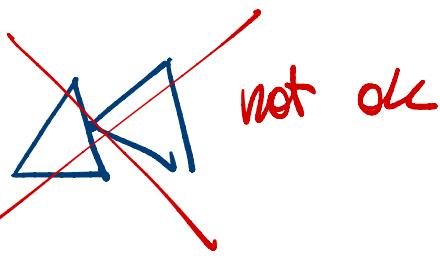
$T_k$  is triangle with vertices  
 $v_{ki}$   $k=1, \dots, m$ ,  $i=1, \dots, 3$

•  $\overline{T_i} \cap \overline{T_j} = \{\emptyset\}$



= } edge of triangle }

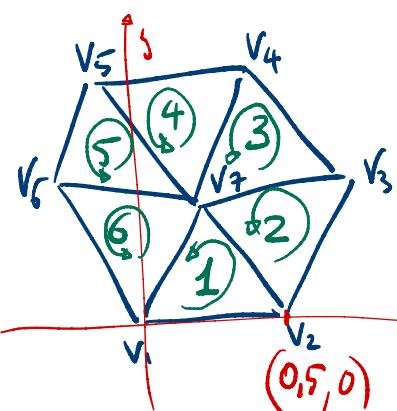
= } common vertex }



In higher dimensions the same

Two lists:

- vertices  $V \in \mathbb{R}^{N_v \times d}$
- elements  $E \in \mathbb{N}^{M \times n_r(d)}$



$V_{id} :=$  coordinate  $\alpha$  of vertex  $i$

$E_{id} :=$  index of vertex  $\alpha$  that belongs to element  $i$

For simplices  $N_r(d) = d+1$

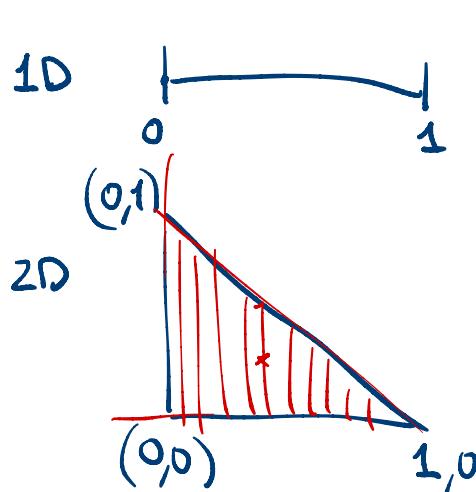
$$V_{11} = 0 \quad V_{12} = 0 \quad V_{21} = 0.5 \quad V_{22} = 0$$

$$E_{11} = 1 \quad E_{12} = 2 \quad E_{13} = 7$$

For every  $T_k$ , we create a local basis.

How?

We start with "Reference Element" \*



Reference element  $\hat{T}$

$$\hat{T} := \{ \hat{x} \mid 0 < \sum_{i=1}^d \hat{x}_i < 1 \}$$

Finite dimensional space on  $\hat{T}$ :

Polynomials in  $\mathbb{R}^d$  of order  $k$

$$\dim(P^k(\hat{T})) = \binom{d+k}{k} = \hat{n}_k$$

$$d=1 \Rightarrow \hat{n}_k = k+1$$

$$d=2 \Rightarrow \hat{n}_k = \frac{1}{2}(k+1)(k+2)$$

$$d=3 \Rightarrow \hat{n}_k = \frac{1}{6}(k+1)(k+2)(k+3)$$

$k=1$

$2$

$3$

$4$

How do we choose a basis for  $P^k(\hat{T})$

1) basis on  $P^k(\bar{\hat{T}})$  | Face of  $\hat{T}$  to be a basis for  $P^k(F)$

2) define the basis in terms of support points that

live as much as possible on  $\hat{T}$

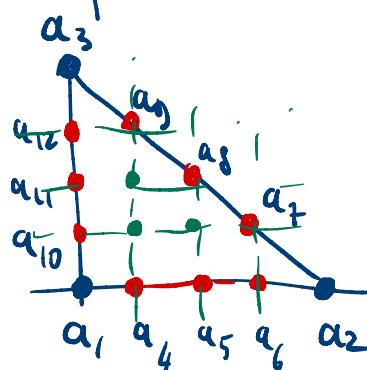
Start with 1D: We need  $k+1$  support points

→ 2 support points are taken by vertices

∴ the rest are internal

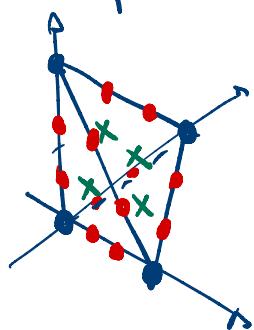
... ) Lagrangian basis on  $a_i$

Go up to 2D Remember  $\hat{n}_k = \frac{1}{2}(k+1)(k+2)$



$$\begin{array}{ll} 1 \text{ per vertex (3)} & k=4 \\ 3 \text{ per edge (9)} & \underline{\underline{15}} \\ 3 \text{ per triangle} & \end{array}$$

Go up to 3D (take  $k=3$ )



$$n_k \text{ 1D} = 4$$

$$n_k \text{ 2D} = 10$$

$$n_k \text{ 3D} = 20 \rightarrow 4 \text{ vertices} + 6 \text{ edges} \times 6 + 1 \text{ triangle}^{\times 4}$$

no support point inside Tetrahedra

Now define  $\hat{v}_i$  st.  $\hat{v}_i(\hat{a}_j) = \delta_{ij}$   $\hat{v}_i \in \hat{P}^k(\hat{T})$

Formal definition of **Finite Element** \*

Triple  $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$   $\hat{\Sigma} = \{\hat{v}_i\}_{i=1}^{n_p}$

i).  $\hat{T}$ : Reference cell

ii).  $P(\hat{P}^k(\hat{T}))$ : Finite dimensional space (Polynomial)

iii)  $\Sigma$ : a basis for  $(\hat{P})'$  (dual basis or nodal basis)

iii) equivalent to saying that  $\{\hat{v}_i(p)\}_{i=1}^{n_p} \in \mathbb{R}^n$  is a bijective operation on  $\hat{P}$

$\Sigma$  is also called "set of degrees of freedom"

iii)  $\Rightarrow \exists! \{v_i\}_{i=1}^{n_p}$  s.t.  $v^j(v_i) = \delta_{ij}$   
 s.t.  $P = \text{span}\{v_i\}_{i=1}^{n_p}$

$\{T, P, \Sigma\}$  is a Lagrangian FE space \*

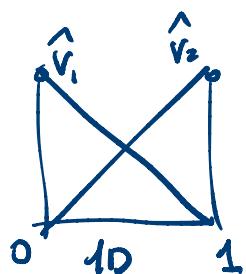
if  $\exists \{a_i\}_{i=1}^{n_p}$  s.t.  $\Sigma = \{\delta(x - a_i)\}_{i=1}^p$   
 $a_i$  are "support points"

$\Rightarrow \Pi_p: C^0(\bar{T}) \rightarrow P$

$u \xrightarrow{} \langle v_i, u \rangle \quad v_i = u(\hat{a}_i) v_i$

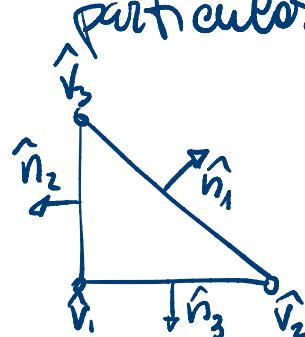
is called Lagrange interpolation \*

General



case:

$$\begin{aligned} \hat{v}_1 &= \hat{x} \\ \hat{v}_2 &= 1 - \hat{x} \end{aligned}$$



$$\hat{a}_i = \hat{v}_i$$

particular basis:  $P^1(\hat{T})$

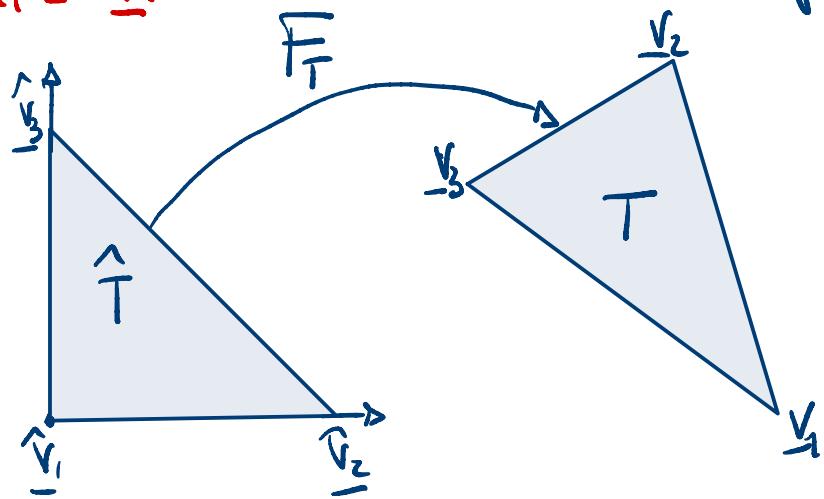
$$v_i = \frac{1 - (\hat{x} - \hat{a}_i) \hat{n}_i}{(\hat{a}_j - \hat{a}_i) \cdot n_j}$$

any  
J+i

20

$$\begin{aligned}V_1 &= 1 - x - y \\V_2 &= x \\V_3 &= y\end{aligned}$$

$$\hat{\alpha}_i \equiv \hat{V_i}$$



Any triangle  $T = \underline{V_1}, \underline{V_2}, \underline{V_3}$   
can be written as a push  
forward of  $\hat{T}$ .

$$\bar{F}_T : \hat{k} \longrightarrow k$$

$$\bar{F}_T(\hat{x}) = \sum_{i=1}^{n_v} \underline{V_i} \quad V_i(\hat{x})$$

$$\bar{F}_T(\hat{V_i}) = \underline{V_i}$$

Given  $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$  we can write  $\forall T$

$\{T, P, \Sigma\}$  as the push forward

through the mapping  $F_T$  of  $\{\hat{T}, \hat{P}, \hat{\Sigma}\}$

$$\Rightarrow V_i \circ F_T := \hat{V_i}$$

$$T = \bar{F}_T(\hat{T})$$

$$\Sigma := \left\{ \delta(x - \bar{F}_T(\hat{\alpha}_i)) \right\}_{i=1}^{n_k}$$