

Metodi Matematici per Equazioni alle Derivate Parziali

Quasi local a posteriori lower bound (optimality)

Luca Heltai <luca.heltai@unipi.it>



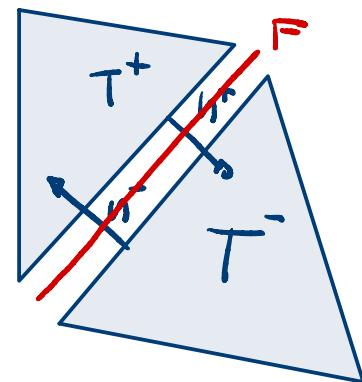
Dipartimento
di Matematica

UNIVERSITÀ DI PISA



Definition of Jump

$$[\![a]\!]_F := \begin{cases} a^+ n^+ + a^- n^- & \text{if } F \subset \mathcal{T}_h \\ a_n - g_N & \text{if } F \subset \partial \Omega_N \\ 0 & \text{if } F \subset \partial \Omega_D \end{cases}$$



Local error estimator

$$\eta_T := h_T \|f + \Delta u\|_{0,T} + \sum_{F \subset \partial T} \frac{1}{2} h_F^{\frac{1}{2}} \|[\![\Delta u]\!]\|_{0,F}$$

IDEALLY (GLOBAL PROPERTIES)

$$\sum_{T \in \mathcal{T}} \eta_T^2 \lesssim \|u - u_h\|_{1,\Omega}^2 \lesssim \sum_{T \in \mathcal{T}} \eta_T^2$$

(LOCAL PROPERTIES)

$$\underbrace{\eta_T^2 \lesssim \|u - u_h\|_{1,T}^2}_{\text{OPTIMALITY}} \lesssim \eta_T^2 \quad \underbrace{\qquad\qquad\qquad}_{\text{RELIABILITY}}$$

FALSE IN GENERAL

THEOREM : LOCAL OPTIMALITY

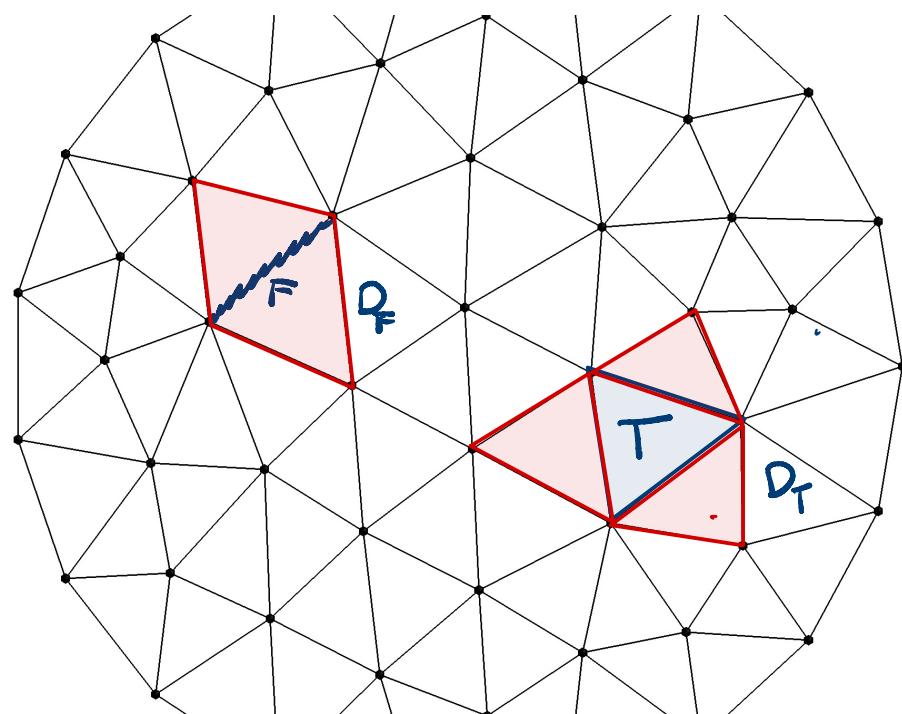
$$V_T := h_T \|f - \Delta u_h\| + \sum_{F \subset \partial T} \frac{1}{2} h_F^{\frac{1}{2}} \|[\partial \Delta u_h]\|_{0,F}$$

\lesssim

$$V_T \lesssim \|u - u_h\|_{1,D_T} + h_T \inf_{v_h \in V_h} \|f - v_h\|_{0,D_T}$$

$D_T := \left\{ \overline{\bigcup_{T_m}^o} \text{ s.t. } \bar{T}_m \cap \bar{T} = F + \{\phi\} \text{ where } \dim(F) = d-1 \right.$
 All elements that share a face with T

$$D_F := \left\{ \overline{\bigcup_{T_m}^o} \text{ s.t. } \bar{T}_m \cap \bar{T}_{m_2} = F \right\}$$



Second ingredient

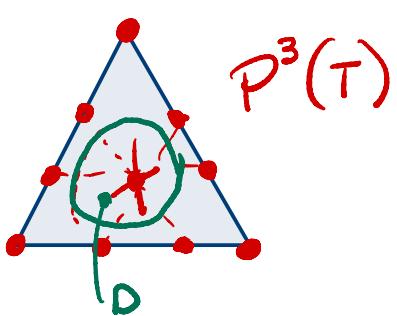
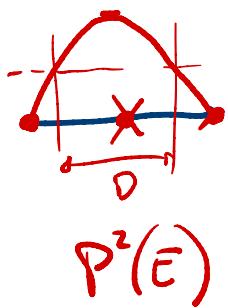
Bubble functions on T , or on F

b_T is a bubble function if:

a) $0 \leq b_T \leq 1$ in \bar{T}

b) $b_T \in H^1_0(T)$

c) $\exists D \subset \bar{T}$ s.t. $|D| > 0$
 s.t. $b_T|_D \geq \frac{1}{2}$



Valid for $T \in \mathbb{R}^d$
simplex or
hyper cube

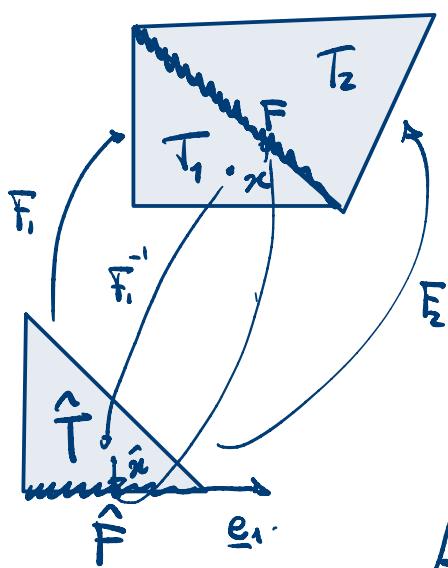
with a), b), c) as properties then

$$i) \| b_T \phi \|_{0,T} \lesssim \| \phi \|_{0,T} \lesssim \| b_T^{\frac{1}{2}} \phi \|_{0,T}$$

$$ii) \| b_T \phi \|_{1,T} \lesssim h_T^{-1} \| \phi \|_{0,T} \quad \forall v \in P^k(T) \quad k \geq 1$$

LIFTING OPERATOR

$RE : P^k(F) \longrightarrow P^k(D_F)$ element wise



$$T_1 = F_i(\hat{T})$$

$$T_2 = F_2(\hat{T})$$

Assume that \hat{T} is such that

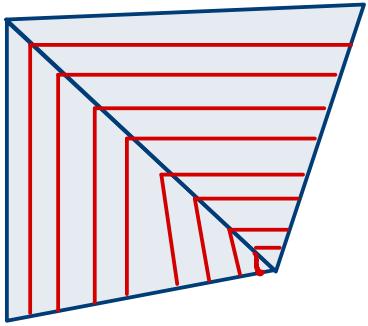
$$F_i(\hat{T}) = F_2(\hat{T}) = F$$

And assume that \hat{T} is aligned with \hat{e}_1
(simplifying assumption, but not necessary)

$$RE(v)(x) := v \left(F_i \left(\hat{\pi}_{\hat{F}}(F_i^{-1}(x)) \right) \right) = v \circ F_{1/2} \circ \hat{\pi}_{\hat{F}} \circ F_{1/2}^{-1}$$

$$\hat{\pi}_{\hat{F}}(\hat{x}) = e_1 \otimes e_1 \hat{x} = (\hat{x} \cdot e_1) e_1$$

orthogonal projection
of \hat{x} onto \hat{F}



Contour plots of $\text{RE}(\phi)$ for $\phi \in P(F)$

Define a bubble function on F and on D_F
 (It is a bubble function for both F as $d-1$ -manifold
 and for D_F as d -manifold)

a) $0 \leq b_F \leq 1$

b) $b_F|_F \in H^1_o(F), \quad b_F \in H^1_o(D_F)$

c) $\exists \tilde{D}_F \subset F$ s.t. $|\tilde{D}_F| > 0$ (measure of dim. $(d-1)$)
 when $b_F|_{\tilde{D}_F} \geq \frac{1}{2}$

d) $\exists \tilde{D}_{D_F} \subset D_F$ s.t. $|\tilde{D}_{D_F}| > 0$ (measure of (d))

when $b_F|_{\tilde{D}_{D_F}} \geq \frac{1}{2}$

Then

i) $\|b_F \phi\|_{0,F} \lesssim \|\phi\|_{0,F} \lesssim \|b_F^{\frac{1}{2}} \phi\|_{0,F} \quad \forall \phi \in P(F)$

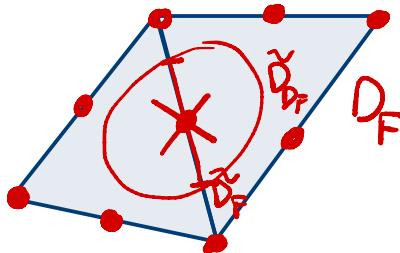
ii) $b_F^{\frac{1}{2}} \|\phi\|_{0,F} \lesssim \|b_F \text{RE}(\phi)\|_{0,D_F} \lesssim b_F^{\frac{1}{2}} \|\phi\|_{0,F}$

iii) $|b_F \text{RE}(\phi)|_{1,D_F} \lesssim b_F^{-\frac{1}{2}} \|\phi\|_{0,F}$

Possible b_F :

basis function
of $\tilde{P}(T)$

associated with center of F .



THEO:

$$b_T \lesssim \|u - u_h\|_{1, D_T} + h_T \inf_{v_h \in V_h} \|f - v_h\|_{0, D_T}$$

$$b_T := h_T \|f + \Delta u_h\|_{0,T} + \sum_{F \subset \partial T} \frac{1}{2} h_F^{\frac{1}{2}} \|(\Delta u_h)\|_{0,F}$$

$$\textcircled{1} \quad \|f + \Delta u_h\|_{0,T} \lesssim \|f - v_h\|_{0,T} + \|v_h + \Delta u_h\|_{0,T}$$

$$\textcircled{3} \quad \|v_h + \Delta u_h\|_{0,T}^2 \lesssim \|b_T^{\frac{1}{2}} (v_h + \Delta u_h)\|_{0,T}^2$$

$$\lesssim \int_T \underbrace{(v_h - f - \Delta u_h + \Delta u_h)}_{=0} b_T (v_h + \Delta u_h) dT$$

$$b_T \Big|_{\partial T} = 0 \quad \text{by construction}$$

$$\lesssim \int_T (v_h - f) b_T (v_h + \Delta u_h) dT + \int_T \nabla(u - u_h) \nabla(b_T (v_h + \Delta u_h)) dT$$

$$\lesssim \|v_h + \Delta u_h\|_{0,T} \quad \lesssim h_T^{-1} \|v_h + \Delta u_h\|_{0,T}$$

$$\lesssim \|v_h - f\|_{0,T} \|b_T (v_h + \Delta u_h)\|_{0,T} + \|u - u_h\|_1 |b_T (v_h + \Delta u_h)|_1$$

$$\|v_\theta + \Delta u_{\text{null}}\|_{0,T}^2 \lesssim \|v_\theta + \Delta u_{\text{null}}\|_{0,T} \left(\|u - u_\theta\|_1 h_T^{-1} + \|v_\theta - f\|_{0,T} \right)$$

$\forall v_\theta \in V_\theta$

$$\textcircled{1} \Rightarrow h_T \|v_\theta + \Delta u_{\text{null}}\|_{0,T} \lesssim \|u - u_\theta\|_{1,T} + h_T \|v_\theta - f\|_{0,T}$$

$$h_T \|f + \Delta u_{\text{null}}\| \lesssim h_T \|f - v_\theta\| + \|u - u_\theta\|_{1,T}$$

$$\textcircled{2} \quad \|\llbracket \nabla u_\theta \rrbracket\|_{0,F}^2 \lesssim \|b_F^{\frac{1}{2}} \llbracket \nabla u_\theta \rrbracket\|_0^2$$

$$\lesssim \int_F \llbracket \nabla u_\theta - \nabla u \rrbracket b_F \llbracket \nabla u_\theta \rrbracket dF$$

$$\sum_{T \in \mathcal{D}_F} \int_T -\Delta u v = \sum_{T \in \mathcal{D}_F} \left(\int_T \nabla u \nabla v - \int_{\partial T} n \cdot \nabla u v \right) \quad \forall v, u \text{ smooth enough}$$

$$\sum_{T \in \mathcal{D}_F} \int_T -\Delta u v = \int_{D_F} \nabla u \nabla v - \int_F \llbracket \nabla u \rrbracket v - \int_{\partial D_F} n \cdot \nabla u \cdot v \quad \forall v, u \text{ smooth enough}$$

$$\int_{\partial D_F} v \Big|_{\partial D_F} = 0 \quad \Rightarrow \quad \int_F \llbracket \nabla u \rrbracket v = \int_{D_F} \Delta u v + \int_F \nabla u \nabla v$$

$$\text{Apply this to } \int_F \nabla u b_F \llbracket \nabla u_\theta \rrbracket \\ \equiv \llbracket \nabla u_\theta \rrbracket \text{ on } F$$

$$\|\llbracket \nabla u_\theta \rrbracket\|_{0,F}^2 \lesssim \int_F \llbracket \nabla u_\theta - \nabla u \rrbracket b_F \operatorname{Re}(\llbracket \nabla u_\theta \rrbracket) dF$$

$$\lesssim \sum_{T \in D_F} \int_T (\Delta u_\eta - \Delta u) b_F \operatorname{RE}([\Gamma \nabla u_\eta]) dT$$

$$+ \int_{D_F} (\nabla u_\eta - \nabla u) \nabla (b_F \operatorname{RE}([\Gamma \nabla u_\eta])) dT$$

$$\lesssim \sum_{T \in D_F} \left(\|\Delta u_\eta + f\|_{0,T} h_F^{-\frac{1}{2}} \|[\Gamma \nabla u_\eta]\|_{0,F} \right)$$

$$+ \|u - u_\eta\|_{1,D_F} \|b_F \operatorname{RE}([\Gamma \nabla u_\eta])\|_{1,D_F}$$

$$\lesssim h_F^{-\frac{1}{2}} \|[\Gamma \nabla u_\eta]\|_{0,F}$$

$$\|[\Gamma \nabla u_\eta]\|_{0,F} \lesssim \left(\|u - u_\eta\|_{1,D_F} h_F^{-\frac{1}{2}} + \|\Delta u_\eta + f\|_{0,D_F} h_F^{\frac{1}{2}} \right)$$

$h_F \sim h_T$ for shape regular \mathcal{T}

$$\textcircled{2} \quad \frac{1}{2} h_F^{\frac{1}{2}} \|[\Gamma \nabla u_\eta]\|_{0,F} \lesssim \|u - u_\eta\|_{1,D_F} + h_T \|\Delta u_\eta + f\|_{0,D_F}$$

Sum \textcircled{1} and \textcircled{2}, take inf on v_h :

$$q_T \lesssim \|u - u_\eta\|_{1,D_T} + h_T \inf_{v_h \in V_h} \|f - v_h\|_{0,D_T}$$

Constructing a new mesh

Error balancing : Reliability implies that

$$|u - u_{\text{ref}}|_1 \lesssim \left(\sum_k (\eta_k^3)^{\frac{1}{2}} \right)^{\frac{1}{2}} \lesssim \text{Tol}$$

if $\left(\sum_k \eta_k^2 \right)^{\frac{1}{2}} \leq \text{tol} \Rightarrow |u - u_{\text{ref}}| \leq \text{tol}$.

$M^i := \# \mathcal{E}^i$ number of elements of \mathcal{E}^i

\mathcal{E}^{i+1} is such that $\nabla T^i \subset \mathcal{E}^{i+1}$

$$\eta_{T^{i+1}}^2 \leq \frac{\text{Tol}^2}{M^{i+1}}$$

Bulk chasing algorithm, or Dörfler algorithm
or "fixed fraction" algorithm, bulk
criterion -

→ Given $0 < \theta < 1$ Define

$$E_{\text{tot}} = \left(\sum_{k \in \mathcal{E}^i} (\eta_k^i)^2 \right)^{\frac{1}{2}}$$

Mark for refinement smallest subset $M \subset \mathcal{E}^i$ s.t.

$$\theta E_{\text{tot}} \leq \left(\sum_{k \in M} (\eta_k^i)^2 \right)^{\frac{1}{2}}$$

1. order T_k^i s.t. $\eta_{T_k^i} \leq \eta_{T_j^i}$ when

$$j \leq k$$

2. Compute $E_{\text{tot}} = \left(\sum_{T \in \mathcal{T}} \left(\eta_T^2 \right) \right)^{\frac{1}{2}}$

3. Mark for refinement up to element \bar{k}

s.t.

$$\left(\sum_{j=1}^{\bar{k}} \eta_{T_j}^2 \right)^{\frac{1}{2}} \geq \theta E_{\text{tot}}$$

and

$$\left(\sum_{j=1}^{\bar{k}-1} \eta_{T_j}^2 \right)^{\frac{1}{2}} < \theta E_{\text{tot}}$$

Quasi uniform partition

$$\min_{T \in \mathcal{T}} h_T \lesssim \max_{T \in \mathcal{T}} h_T \lesssim \max_{T \in \mathcal{T}} h_T^{-1} \lesssim \min_{T \in \mathcal{T}} h_T^{-1}$$

$$\frac{|S_2|}{M} \sim h_T^{-d}$$

$$h_T \sim M^{-\frac{1}{d}}$$

$$\|\mu - \mu_\ell\|_{M, S} \lesssim \left(\sum_{T \in \mathcal{T}_\ell} h_T^{2(\ell-m)} \|u\|_e^2 \right)^{\frac{1}{2}}$$

$$\|u - u_h\|_{m, \Omega} \lesssim h_T^{e-m} \|u\|_e \quad \text{for } e \text{ such that } m=0,1$$

$$\|u - u_h\|_{m, \Omega} \lesssim M^{-(e-m)/d} \|u\|_e$$

$$k=1 \Rightarrow u \in H^2 \cap H_0'$$

$$\|u - u_h\|_{m, \Omega} \lesssim M^{(2-m)/d} \|u\|_2$$

what happens if $u \notin H^2$?

If you equidistribute error ($\eta_k \sim \delta + h_k$)

$$\Rightarrow \|u - u_h\|_{m, \Omega} \lesssim \underline{\underline{M}}^{(2-m)/d} \|u\|_1$$

no longer true that $\min_h \sim \max_h$

but it is true that

$$\eta_k \sim \delta + h_k$$

\Rightarrow locally error is constant