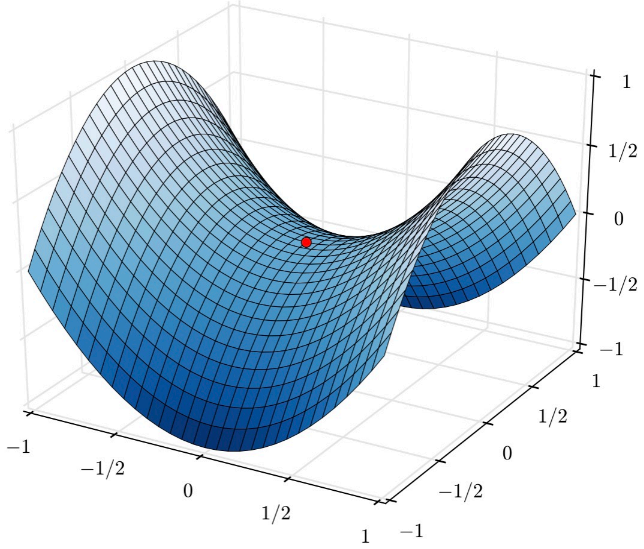

Metodi Matematici per Equazioni alle Derivate Parziali

Saddle point problems

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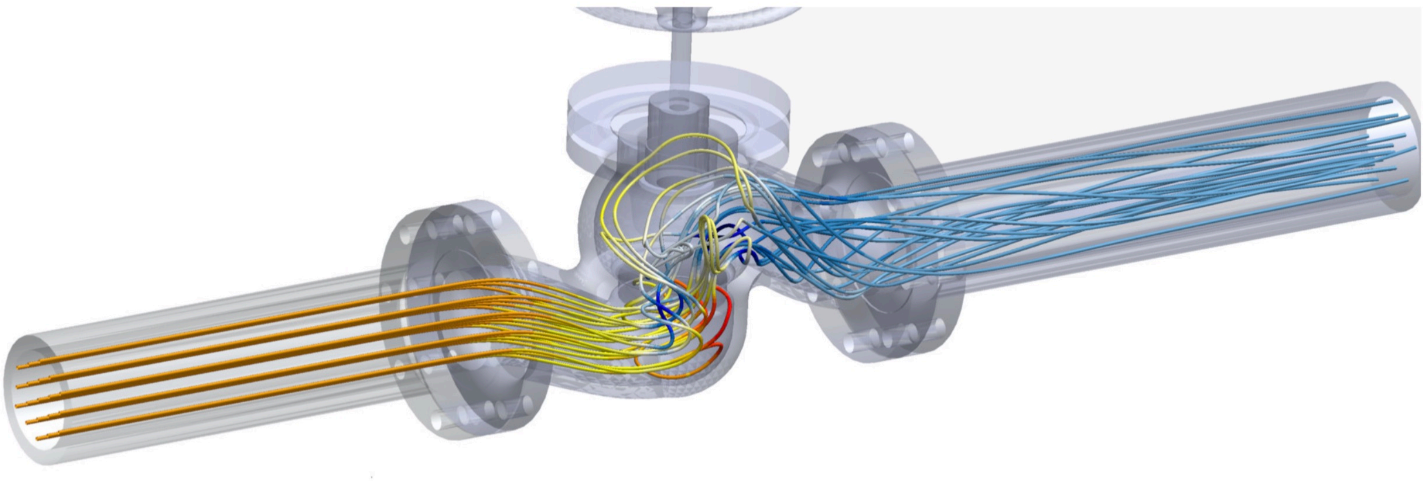


Mixed Problems

Two Hilbert spaces: V, Q, and two operators:

$$A : V \longrightarrow V' \qquad A \in \mathcal{L}(V, V')$$

$$B : V \longrightarrow Q' \qquad B \in \mathcal{L}(V, Q')$$

Given $f \in V'$, $g \in Q'$ find (u,p) in $V \times Q$ s.t.

$$\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}$$

1) $g \in \operatorname{Im}(B) \Rightarrow \exists\, u_g$ s.t. $Bu_g = g$

2) $\mathcal{Z} = \ker(B)$ $\qquad u = u_0 + u_g$, $u_0 \in \mathcal{Z}$

$$\Rightarrow \begin{cases} Au_0 + B^T p = f - Au_g = \tilde{f} & \text{in } V' \\ Bu_0 = 0 \end{cases}$$

Restrict our analysis to $g = 0$ $(\tilde{f} = f - Au_g)$

$$\langle Au_0, v_0 \rangle + \cancel{\langle B^T p, v_0 \rangle}^{\,0}_{\forall v_0 \in \mathcal{Z}} = \langle \tilde{f}, v_0 \rangle \qquad \forall v_0 \in \mathcal{Z}$$

$$\boxed{\begin{array}{l} u\text{-problem} \\ \langle Au_0, v_0 \rangle = \langle \tilde{f}, v_0 \rangle \end{array}} \iff \text{BNB is satisfied}$$

1) $$\boxed{\begin{aligned} &\inf_{u_0\in Z}\sup_{v_0\in Z}\frac{\langle Au_0, v_0\rangle}{\|u\|_V\|v\|_V}=\alpha \\ &\inf_{v_0\in Z}\sup_{u_0\in Z}\frac{\langle Au_0, v_0\rangle}{\|u\|_V\|v\|_V}=\alpha \end{aligned}}$$

ELL-KER on A

Then $\exists!\ u_0\in Z$ s.t.

$$\boxed{\|u_0\|_V\le\frac{1}{\alpha}\|\tilde f\|_{V'}\le\frac{1}{\alpha}\|f\|_{V'}+\frac{1}{\alpha}\|A\|_{V'}\|u_g\|_V}$$

Given u_0 solution to $\boxed{u\text{-problem}}$

Find p s.t. $(u=u_0+u_g)$

$$\boxed{\langle B^Tp, v\rangle = -\langle Au, v\rangle+\langle f, v\rangle} \quad p\text{-problem}$$

$$= -\langle Au_0, v\rangle+\langle\tilde f, v\rangle \qquad \forall v\in V$$

$BNB_1 + BNB_2 \to \ker(B^T)=0 \quad \operatorname{Im}(B)=\overline{\operatorname{Im}(B)}$

B is surjective

$\exists\,\beta>0 \quad \|B^Tp\|_{V'}\ge\beta\|p\|_Q \quad \forall p\in Q$

INF-SUP on B

$$\Leftrightarrow \quad \boxed{\exists\,\beta>0 \quad \text{s.t.} \quad \inf_{p\in Q}\sup_{v\in V}\frac{\langle Bv, p\rangle}{\|v\|_V\|p\|_Q}=\beta}$$

Notice that $Au - f =: L \in V'$

$\langle L, v_0 \rangle = 0 \quad \forall \, v_0 \in Z \Rightarrow L \in Z^0$

Summary: $\forall (f,g) \in V', Q' \quad \exists! \, (u,p) \in V, Q$ s.t.

$$\boxed{\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}} \qquad Z := \ker(B)$$

If and only if $\exists \, \alpha, \beta$ s.t.

$$\boxed{\begin{array}{ll}
1) & \inf\limits_{u \in Z} \sup\limits_{v \in Z} \dfrac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = \alpha > 0 \\[2em]
1) & \inf\limits_{v \in Z} \sup\limits_{u \in Z} \dfrac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = \alpha > 0 \\[2em]
3 & \inf\limits_{q \in Q} \sup\limits_{u \in V} \dfrac{\langle Bu, q \rangle}{\|u\|_V \, \|q\|} = \beta > 0
\end{array}}$$

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|\tilde{f}\|_{V'} \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{1}{\alpha} \|A\|_{V'} \|u_g\|_V}$$

$$\|g\|_{Q'} = \|Bu_g\|_{V'} \geq \beta \|u_g\|_V$$

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{\|A\|_{V'}}{\alpha \beta} \|g\|_{Q'}}$$

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \frac{\|A\|_{V'}}{\alpha\beta}\|g\|_{Q'} + \frac{1}{\beta}\|g\|_{Q'}$$

$$\boxed{\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\|g\|_{Q'}}$$

$$\|p\|_Q \leq \frac{1}{\beta}\|\angle\|_{V'} \leq \frac{1}{\beta}\left(\|A\|\,\|u\| + \|f\|_{V'}\right)$$

$$\|p\|_Q \leq \frac{\|A\|}{\beta}\left(\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\right)\|g\|_{Q'} + \frac{1}{\beta}\|f\|_{V'}$$

$$\boxed{\|p\|_Q \leq \frac{(\|A\|+1)}{\beta}\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\frac{\|A\|}{\beta}\|g\|_{Q'}}$$

1), 2), 3) $\Longleftrightarrow$ a), b) $\qquad V \times Q = \mathbb{V}$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \mathbb{A} \qquad \mathbb{A} : \mathcal{L}(V \times Q \to V' \times Q')$$

equivalent to 1,2,3

$$\exists\ \bar{\alpha}\ \text{s.t.}$$

a) $$\inf_{\psi \in V} \sup_{\theta \in \mathbb{V}} \frac{\langle \mathbb{A}\psi, \theta \rangle}{\|\psi\| \, \|\theta\|} = \bar{\alpha}$$

b) $$\inf_{\theta \in \mathbb{V}} \sup_{\psi \in V} \frac{\langle \mathbb{A}\psi, \theta \rangle}{\|\psi\| \, \|\theta\|} = \bar{\alpha}$$

1) ELL-KER in Z_h $\Rightarrow$ $\exists\, \alpha_h$ s.t. inf sup on Z_h

2) INF SUP on V_h, Q_h for B_h

$$Z_h := \{ v_h \in V_h \ \text{ s.t. } \ b(v_h, q_h) = 0 \ \forall q_h \in Q_h \}$$

$$\begin{matrix} \ker B \subset V \\ V_h \subset V \end{matrix} \ \not\Rightarrow \ \ker B \cap V_h \neq \{\emptyset\}$$

$$\ker B_h \not\subset \ker B$$

$$a(u-u_h, v_h) = -b(p-p_h, v_h) \qquad \forall v_h \in V_h$$

Restrict to $w_h \in Z_h$

$$a(u-u_h, w_h) = -b(p, w_h) \qquad \forall w_h \in Z_h$$

$$= -b(p-q_h, w_h) \qquad \forall w_h \in Z_h$$

$$\langle A u_h, w_h\rangle = \langle A u, w_h\rangle + \langle B^T(q_h - p), w_h\rangle \ \forall q_h \in Q_h$$

$$\|u-u_h\| \leq \|u - v_h + v_h - u_h\|$$

$$\leq \|u - v_h\| + \frac{1}{\alpha_h} \|A(v_h - u_h)\|_{*, Z_h'}$$

$$\|u-u_h\| \leq \|u-v_h\| + \frac{\|A\|}{\alpha_h}\|u-v_h\| + \frac{1}{\alpha_h}\|B\|\|q_h - p\| \qquad \begin{matrix}\forall v_h \in V_h \\ \forall q_h \in Q_h\end{matrix}$$

$$\boxed{\|u-u_h\| \leq \left(1+\frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in Z_h} \|u-v_h\| + \frac{\|B\|}{\alpha_h} \inf_{q_h \in Q_h} \|q_h - p\|}$$

$$B : V \longrightarrow Q'$$

$$B^T : Q \longrightarrow V'$$

$$\|p-p_h\| \leq \|p-q_h+q_h-p_h\|$$

$$\leq \|p-q_h\| + \frac{1}{\beta_h} \|B_h^T(q_h-p_h)\|_{*,V_h}$$

$$\|B_h^T q_h\|_{*,V_h} \geq \beta_h \|q_h\|$$

$$\langle B^T(q_h-p), v_h\rangle = \langle A(u_h-u), v_h \rangle$$

$$\leq \|p-q_h\| + \frac{\|A\|}{\beta_h}\|u_h-u\| + \frac{\|B\|}{\beta_h}\|p-q_h\|$$

$$\boxed{\|p-p_h\| \leq \left(1+\frac{\|B\|}{\beta_h}\right) \inf_{q_h \in Q_h} \|p-q_h\| + \frac{\|A\|}{\beta_h}\|u-u_h\|}$$

$$\|p-p_h\| \leq \left(1+\frac{\|B\|}{\beta_h} + \frac{\|A\|\,\|B\|}{\alpha_h \beta_h}\right) \inf_{q_h \in Q_h} \|p-q_h\| +$$

$$\frac{\|A\|}{\beta_h}\left(1+\frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in Z_h} \|u-v_h\|$$

Mixed Laplacian

$V := H_{div}(\Omega) \qquad Q := L^2(\Omega)$

Given $g \in L^2(\Omega)$ find $u, p \in V \times Q$ s.t.

$$\boxed{\begin{aligned} (u,v) + (\operatorname{div} v, q) &= 0 \quad \forall v \in V \\ (\operatorname{div} u, q) &= (g,q) \quad \forall q \in Q \end{aligned}}$$

Darcy flow

$B :=$ "div"

$$\ker B := \{ v \in V \text{ s.t. } (\operatorname{div} v, q) = 0 \ \forall q \in L^2(\Omega) \}$$
$$\operatorname{div} v = 0 \quad \text{in } L^2(\Omega)$$

ELL-KER (INF-SUP conditions on A):

- $a(u,u) \geq \alpha \|u\|_V^2 \quad \forall u \in K$
- $\|Au\|_{V'} \geq \alpha \|u\|_V \quad \forall u \in K$
  $\|A^T u\|_{V'} \geq \alpha \|u\|_V \quad \forall u \in K$

$$\|u\|_V^2 := \|u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2$$

$$a(u,u) = (u,u) + \underbrace{(\operatorname{div} u, \operatorname{div} u)}_{= 0 \text{ in } K} \geq \alpha \|u\|_V^2 \quad \forall u \in K$$

$\Rightarrow$ ELL ker is OK with $\boxed{\alpha = 1}$

Is INF SUP condition satisfied?

$\forall g \in L^2(\Omega)$, can we build $v_g \in V$ s.t. $B v_g = g$

surjectivity?

$-\operatorname{div}(v_g) = g \quad \text{in } L^2 \qquad v_g = \nabla \phi$

$$\boxed{v_g = -\nabla \phi}$$

$(\nabla \phi, \nabla v) = (g, v) \qquad \forall v \in H_0^1(\Omega)$

$\Rightarrow \exists!\ \phi \in H_0^1(\Omega) \cap H^2(\Omega) \Rightarrow -\nabla \phi \in L^2(\Omega), \quad -\nabla\phi \in H_{div}$

$\Rightarrow$ B satisfies the inf sup

Stokes?

$V = (H_0^1(\Omega))^d \qquad Q = L_0^2(\Omega) := \{ q \in L^2(\Omega), (q,1)=0 \}$

$(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f, v) \quad \forall v \in V$

$(\operatorname{div} u, q) = 0 \quad \forall q \in L_0^2(\Omega)$

ELL KER : $a(u,u) := \|\nabla u\|_0^2 \geq \alpha \|u\|_1^2 \quad \forall v \in V$

poincaré inequality

INF SUP : $B(V) \equiv L_0^2(\Omega)$ (long proof : Temam)

Discrete version of Abstract saddle point problems.

$V_h \subset V \qquad Q_h \subset Q$

1) A_h is invertible on K_h ($\equiv \ker B_h$)

2) B_h is surjective

$\langle A_h v_h, w_h \rangle := a(v_h, w_h) \quad \forall v_h, w_h \in V_h$

$\langle B_h v_h, q_h \rangle := b(v_h, q_h) \quad \forall v_h, q_h \in V_h \times Q_h$

$K_h := \{ \ker(B_h) : u_h \in V_h, \text{ s.t. } b(u_h, q_h) = 0 \ \forall q_h \in Q_h \}$

$K_h \subset K$

ELL-KER (2 INF SUP) on A

$\Longleftrightarrow$

A_h is invertible on K_h

INF-SUP on B_h

B_h is full rank (B_h surjective ...)

Must hold uniformly w.r.t. h

ELL-KER

$$\boxed{\begin{array}{ll} \circ\ \|A_h u_h\| \geq \alpha_h \|u_h\| \quad \forall u_h \in K_h & \\ & \alpha_h \geq \alpha_0 > 0 \quad \forall h \\ \circ\ \|A_h^T u_h\| \geq \alpha_h \|u_h\| \quad \forall u_h \in K_h & \end{array}}$$

INF SUP

$$\boxed{\cdot\ \|B_h^T p_h\| \geq \beta_h \|p_h\| \quad \forall p_h \in Q_h \qquad \beta_h \geq \beta_0 > 0 \quad \forall h}$$

Error analysis

$$a(u,v) + b(v,p) = F(v) \qquad \forall v \in V \qquad \text{pick } v \equiv v_h \in V_h$$

$$a(u_h,v_h) + b(v_h,p_h) = F(v_h) \qquad \forall v_h \in V_h$$

$$\boxed{a(u-u_h, v_h) + b(v_h, p-p_h) = 0}$$

$$b(u,q) = G(q) \qquad \forall q \in Q \qquad \text{pick } q \equiv q_h \in Q_h$$

$$b(u_h,q_h) = G(q_h) \qquad \forall q_h \in Q_h$$

$$\boxed{b(u-u_h, q_h) = 0 \qquad \forall q_h \in Q_h}$$

Goal: estimate $\|u - u_h\|$ and $\|p - p_h\|$

Idea: prove an estimate for $\|u - u^I\|$ and $\|p - p^I\|$ and then use triangle inequality:

$$\boxed{a(u_h - u^I, v_h) + b(v_h, p_h - p^I) = a(u - u^I, v_h) + b(v_h, p - p^I)}$$

$$\boxed{b(u_h - u^I, q_h) = b(u - u^I, q_h)}$$

$$\mathcal{F}(v_h) := a(u - u^I, v_h) + b(v_h, p - p^I)$$

$$G(q_h) := b(u - u^I, q_h)$$

depends on α, β

$$\|u_h - u^I\|_V + \|p_h - p^I\| \leq c\left(\|\mathcal{F}\| + \|G\|\right)$$

$$\lesssim c\left(\|u - u^I\| + \|p - p^I\|\right)$$

$$\|u - u_h\| \leq \|u - u^I\| + \|u^I - u_h\|$$

$$\leq \|u - u^I\| + c_2\left(\|u - u^I\| + \|p - p^I\|\right)$$

$$\|p - p_h\| \leq \|p - p^I\| + \|p^I - p_h\|$$

$$\leq \|p - p^I\| + c_3\left(\|u - u^I\| + \|p - p^I\|\right)$$

$\Downarrow$

$$\|u - u_h\| + \|p - p_h\| \leq c\left(\|u - u^I\| + \|p - p^I\|\right) \quad \forall u^I, p^I \in V_h \times Q_h$$

very similar to: $\displaystyle \|u - u_h\| \leq c \inf_{v_h \in V_h} \|u - v_h\|$

If $K_h \subset K$ then

$$\boxed{\begin{aligned} \|u - u_h\| &\leq c\,\|u - u^I\| \quad \forall\, u^I \in K_h \subset K \\ \|p - p_h\| &\leq c\left(\|u - u^I\| + \|p - p^I\|\right) \quad \forall\, u^I, p^I \in K_h \times Q_h \end{aligned}}$$

Mixed Laplacian in 1D

$P^u_{c/d}$

$$\int_a^b u v - \int_a^b v' p = 0 \quad \forall\, v \in V \equiv H^1([a,b])$$

$$\int_a^b u' q = \int_a^b g q \quad \forall\, q \in Q \equiv L^2([a,b])$$

$$\boxed{P^1_c - P^1_c}$$

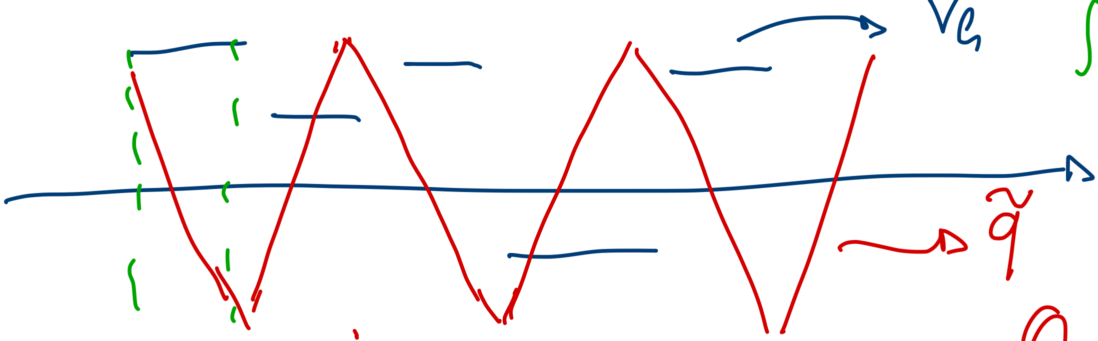
?? is it ok?

ELL-KER : $\int_a^b u^2 \geq \alpha\left(\|u\|_0^2 + \|u'\|_0^2\right) \quad \forall\, u \in K_h$

$$K := \left\{ v \in H^1([a,b]) \;\; \int_a^b v' q = 0 \;\; \forall\, q \in L^2 \Rightarrow v' = 0 \right\}$$

global constants.

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{\int_a^b v_h' q_h}{\|v_h\|_1 \, \|q_h\|_0} \overset{?}{\geq} \beta_h \geq \beta_0 > 0$$



$\int v_h' \tilde{q} = 0 \quad \forall\, v_h$

$Q_h = \operatorname{span}\{v_i\}$

take $\tilde{q} := \sum (-1)^i v_i$

$$\int_a^b \tilde{q}\, v_h' = 0 \quad \forall\ v_h \in V_h \Rightarrow \boxed{\beta_h = 0}$$

INF SUP is violated

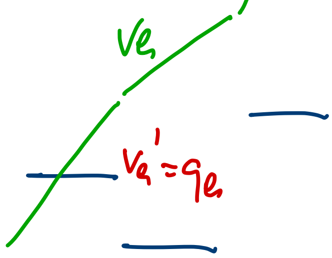
$$\boxed{P_c^1 - P_d^0}$$??

Given $q_h \in \boxed{P_d^0}$, look for $v_h \in \boxed{P_c^1}$ s.t.

- $b(v_h, q_h) = \|q_h\|^2$
- $\|v_h\|_1 \leq c \|v_h\|_0$

$$\Rightarrow \quad \beta = \frac{1}{c}$$

$$\int_a^b v_h' q_h \quad \Rightarrow \quad v_h' = q_h \qquad v_h = \int_a^t q_h(x)\,dx$$



$\|v_h\|_1 \leq c \|v_h\|_0$ poincaré ineq. $\qquad \|v\|_0 \leq c \|v'\|_0$

$$\frac{b(v_h, q_h)}{\|q_h\|_0 \|v_h\|_1} = \frac{\|q_h\|_0^2}{\cancel{\|q_h\|_0} \|v_h\|_1} = \frac{\|q_h\|_0}{\|v_h\|_0 + \|v_h'\|_0} \leq \frac{\cancel{\|q_h\|_0}}{c\|\cancel{v_h'}\|_0 + \|\cancel{v_h'}\|_0} = \frac{1}{c+1}$$

$$\boxed{\text{ELL-KER}} \qquad K_h \subset K$$

$$\boxed{P_c^2 - P_d^0}$$??

$\rightarrow$ INF SUP is ok for $P_c^1 - P_d^0$

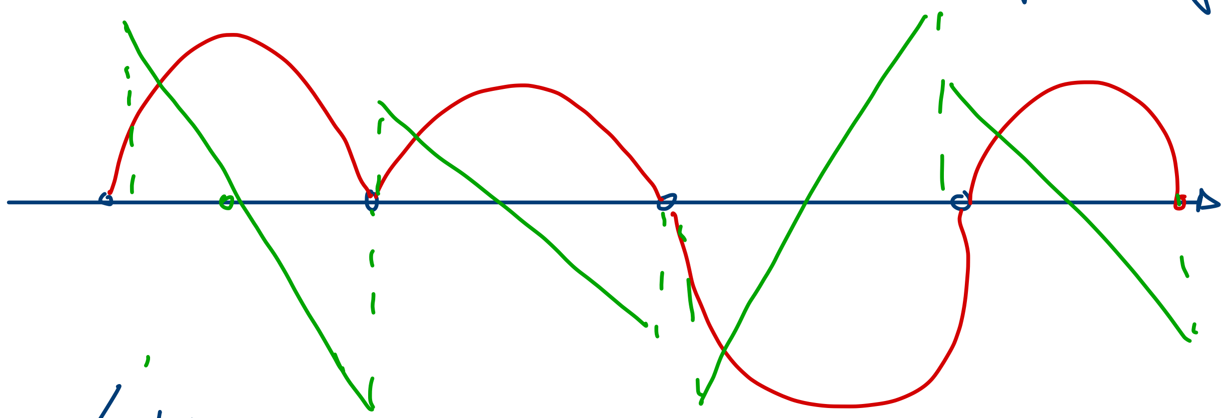
same reason

$\boxed{\text{ELL-KER}}$ $\quad K_h := \{ v_h \in P_c^2 \text{ s.t. } \int_a^b v_h' q_h = 0 \ \forall q_h \in P_d^0 \}$

$\Leftrightarrow \quad v_h \in K_h \Rightarrow \int_{x_i}^{x_{i+1}} v_h' = 0 \Rightarrow v_h(x_{i+1}) - v_h(x_i) = 0$

$\forall i \in [0, N)$

N points of the grid.

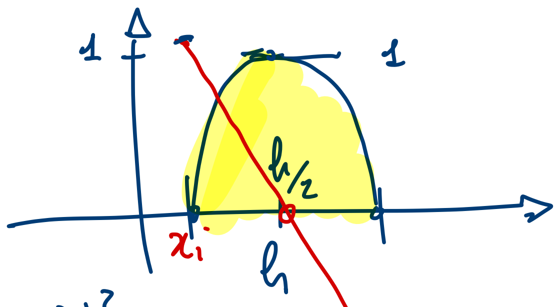


$K_h \not\subset K$

can we find $\alpha_h \geq \alpha_0 > 0$ s.t.

$a(v_h, v_h) \overset{?}{\geq} \alpha_h \|v_h\|_1^2 \overset{?}{\geq} \alpha_0 \|v_h\|_1^2 \quad \forall v_h \in K_h$?

$\alpha_h \geq \alpha_0 > 0$



$a(v_h, v_h) = \|v_h\|_0^2$

$\|v_h\|_0^2 = \int_{x_i}^{x_i+h} \left(c_2 (x - x_i)(x - x_i - h)\right)^2 \sim c_1 h$

$\|v_h'\|_0^2 \simeq c_1 \frac{1}{h} \Rightarrow \boxed{\alpha_h \sim h}$

$\lim_{h \to 0} \alpha_h = 0$ NO ELL-KER

$\boxed{P_c^k - P_d^{k-1}}$ 1D it works