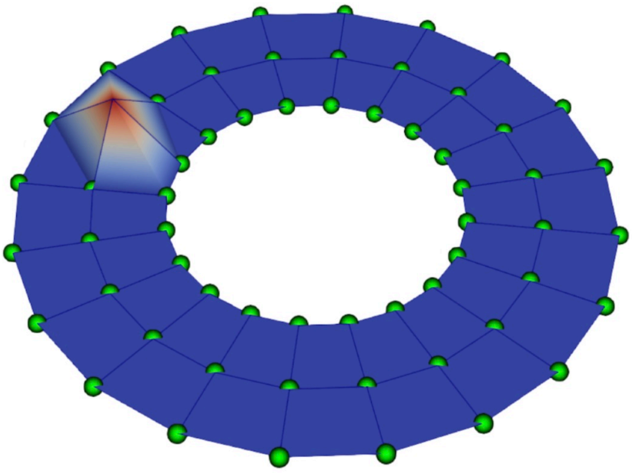

Metodi Matematici per Equazioni alle Derivate Parziali

Degrees of freedom and Finite Dimensional Spaces

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Consider cases where $\boxed{V \equiv H^1(\Omega)}$

$\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$

- Construct $V_h \subset H^1(\Omega)$ according to
  - $\mathcal{T}_h$: $\boxed{\text{Triangulation}}$ of Ω
  - Reference $\boxed{\text{Finite Element}}$ $\{\hat{T}, P_{\hat{T}}, \hat{\Sigma}\}$
  - $\boxed{\text{Mapping}}$ between $\boxed{\hat{T}}$ and $\boxed{T_k}$ $\forall\, T_k \in \mathcal{T}_h$
  - Satisfy continuity conditions for $V_h \subset V$

    $\boxed{\text{DoF Handler}}$

Red boxes coincide with Classes in deal.II

Start with $\hat{T}$:

d-simplex

Choose $P_{\hat{T}}$ finite dimensional.

- Lagrangian case $P_{\hat{T}} := \mathbb{P}^k(\hat{T})$ polynomials of order k
- Construct a basis for $(P_{\hat{T}})'$

  do this through a selection of n_{FE} support points

  $(n_{FE} := \dim(P_{\hat{T}}) = \dim(\mathbb{P}^k(\hat{T})))$

N_d where d is dimension

$N_0 = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k \geq 1 \end{cases}$ by convention (points)

$N_1 = k+1$ (lines)

$N_2 = \frac{1}{2}(k+1)(k+2)$ (triangles)

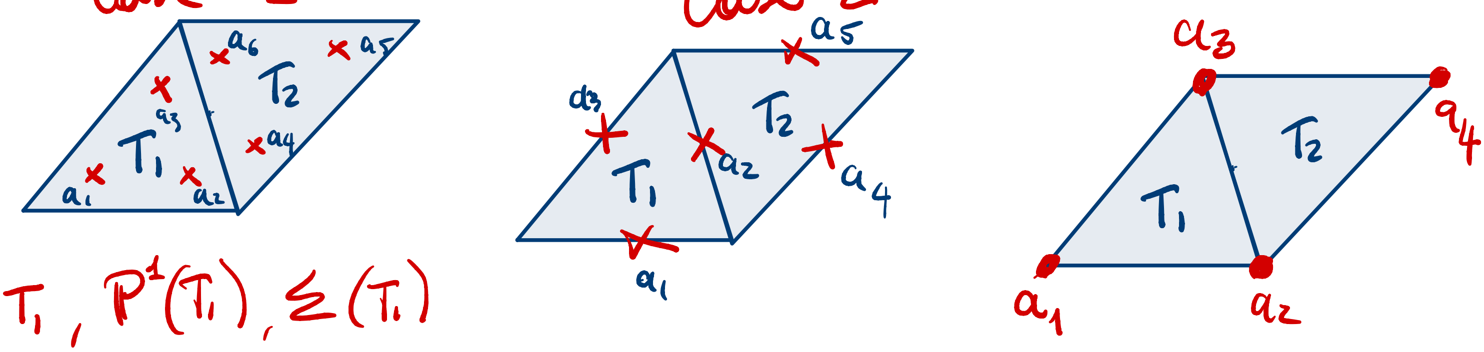
$N_3 = \frac{1}{6}(k+1)(k+2)(k+3)$ (tetrahedra)

If we want to have continuity in V_h, then

$\forall v_h \in V_h$ if $\overline{T_1} \cap \overline{T_2} \neq \{\phi\}$

then $v_h \big|_{\overline{T_1} \cap \overline{T_2}}$ has to be single valued

Case 1 Case 2 Case 3



$T_1, \mathbb{P}^1(T_1), \Sigma(T_1)$

$T_2, \mathbb{P}^1(T_2), \Sigma(T_2)$

Support points $\{a_{1i}\}_{i=1}^{3}, \{a_{2i}\}_{i=1}^{3}$ $\quad v^{ki} := \delta(x - a_{ki})$

Case 1 : $a_{1,i} \neq a_{2,j}$ $\quad \forall i,j$ in $1,2,3$

(a total of 6 support points) 3 for each triangle

$u_h\big|_{T_1} \rightarrow \exists! \{u^i\}_{i=1}^{3}$ s.t. $u_h\big|_{T_1} = u^i v_{1i} = \langle v^{1i}, u_h \rangle v_{1i}$

$= \sum_i u_h(a_{ki}) v_{1i}$

Select 6 real numbers :

$\exists \; u^{ki} \in \mathbb{R}^{2\times 3}$ st. $u_h = u^{ki} v_{kj} \notin C^0(\overline{T_1} \cup \overline{T_2})$

This is ok if $V_h \subset L^2(\Omega)$

Not ok if you want $V_h \subset H^1(\Omega)$

dim case 1 : 6 (3 x n triangles)

Case 2 is still not continuous.

dim case 2 : 5 (1 × n edges)

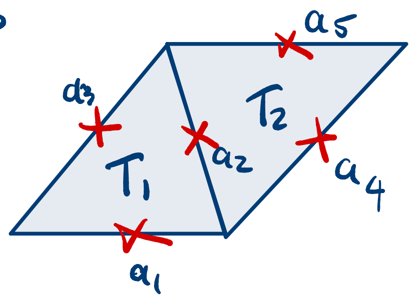
local to global indexing : vector of vectors.

$l2g[k][i] \rightsquigarrow$ index of support point i on element k

Similarly to list of vertices and elements for triangulat. we need a list of support points, and a list of indices that refers to them for the definition of

$FE_T : \{ T, \mathbb{P}^k(T), \Sigma_T \}$

$l2g_{ki}$ $\quad k = 1,2 \quad i = 1,2,3$



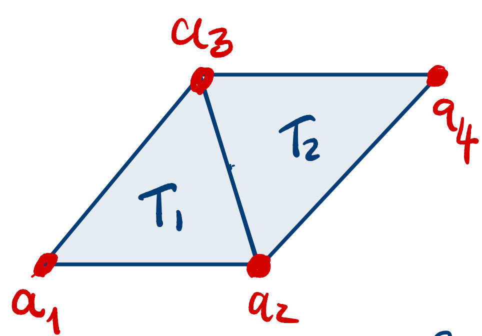
$l2g_1 = \{1,2,3\}$

$l2g_2 = \{2,4,5\}$

$$\boxed{\Sigma_{T_k} := \{ \delta(x - a_{l2g_{ki}}) \}_{i=1}^{3}}$$

for case 1 $\quad l2g_1 = \{1,2,3\} \quad l2g_2 = \{4,5,6\}$

Case 3

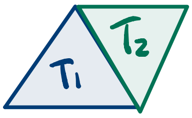


$l2g_1 = \{1,2,3\}$

$l2g_2 = \{3,2,4\}$

This space IS continuous : $u_h = \sum_{k=1}^{2} \sum_{i=1}^{3} u_h(a_{ki}) v_{ki}$

$$u_h = \sum_{k=1}^{2} \sum_{i=1}^{3} u_h(a_{ki}) v_{ki}$$ is cont. only in case 3



$$u_h(a_{11}) V_{11} + \dots + u_h(a_{21}) V_{21} + \dots +$$

We identify v_i with one single index with the degree of freedom associated to a_i with one single index

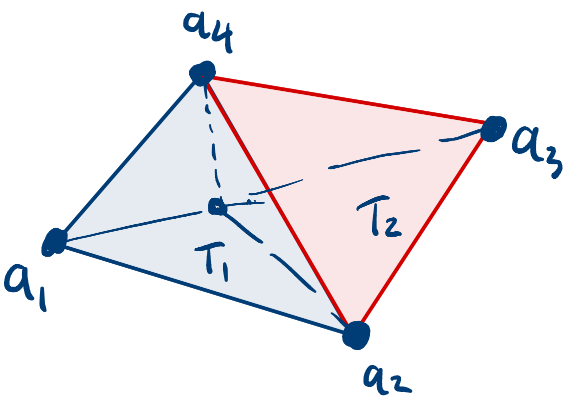
If V_{kJ} has same support point of V_{mi} for $k \neq m$ and some pair i,J Then we identify V_{kJ} and V_{mi} with the SAME global basis function, which is $V_{g2l_{kJ}} = V_{g2l_{mi}}$

$$V_{g2l_{kJ}} \Big|_{T_k} = V_{kJ}$$ ← local basis on T_k

$$V_h \Big|_{T_k} = \text{span} \{ V_{kJ} \}_{J=1}^{N_{T_k}} = P_{T_k}$$

$$\Sigma_{T_k} := \{ V^{kJ} \}_{J=1}^{N_k}$$

$$\langle V^{kJ}, V_{ki} \rangle = \delta_{iJ}$$



$$V_4 \Big|_{T_1} = V_{13} \qquad V_4 \Big|_{T_2} = V_{21}$$

For arbitrary orders, we need to ensure that the number of support points on the subelement $\boxed{X}$ of T_k (faces, edges, vertices) coincides with $\boxed{N_d = \dim\left(\mathbb{P}^k\big|_X\right)}$

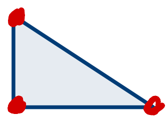
X is manifold of intrinsic dimension d:

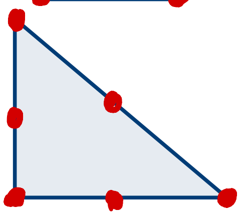
$$N_0 = \begin{cases} 0 & k=0 \\ 1 & k\geq 1 \end{cases}$$

$$N_d = \binom{d+k}{k}$$

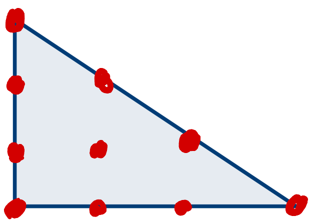
P^1 continuous: need 1 support point per vertex.

P^2 " " 1 support point per vertex and 1 " point per line





for P^2



for P^3

Concept of Degrees of freedom per object $\boxed{\text{dpo}}$

$$DPO \in \mathbb{N}^{d+1}$$

DPO_i number of support points on objects of intrinsic dimension i excluding subobjects.

$$DPO_i \neq N_i$$

DPO for $P^1(T_k)$

$DPO_0 = 1 \quad DPO_i = 0 \quad i>0$

$$\boxed{N_i = \sum_{j=0}^{i} DPO_j \cdot \left(n_{obj} \text{ of dim } j\right)}$$

$$N_{T_k} = \sum_{j=0}^{d} \boxed{DPO_j \cdot \left(n_{obj} \text{ of } \dim(j) \text{ in } T_k\right)}$$