

# Numerical Methods for the Solution of PDEs

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Handling systems of PDEs

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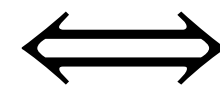
# Vector valued problems

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- In reality, most problems are not scalar, i.e., they have more than one solution variable. We call them vector-valued.
- Example: The mixed Poisson equation:

$$K^{-1}\mathbf{u} + \nabla p = 0$$

$$\nabla \mathbf{u} = g$$



$$-\nabla \cdot (K \nabla p) = g$$

$$-K \nabla p = \mathbf{u}$$



# Vector valued problems

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- In reality, most problems are not scalar, i.e., they have more than one solution variable. We call them vector-valued.
- Example: The Stokes equation

$$\begin{aligned} -\nabla \cdot (\eta \nabla \mathbf{u}) + \nabla p &= f \\ \nabla \mathbf{u} &= 0 \end{aligned}$$



# Vector valued problems

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- A systematic way to treat vector-valued problems:
  - Write the solution in the product space, using the graph norm, i.e.,  $\mathbf{u} \in V$ ,  $p \in Q$ , and  $\psi \in V \times Q \equiv \mathbb{V}$
  - Write the test functions as  $\mathbf{v} \in V$ ,  $q \in Q$ , and collect them as  $\phi \in V \times Q \equiv \mathbb{V}$
  - Write the functionals  $f \in V'$  and  $g \in Q'$  as  $\mathbb{F} \in V' \times Q'$
  - Write the operator as  $\mathbb{A} : \mathbb{V} \mapsto \mathbb{V}'$

$$\psi = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \quad \mathbb{A} = \begin{pmatrix} \mathbb{A}_{uu} & \mathbb{A}_{up} \\ \mathbb{A}_{pu} & \mathbb{A}_{pp} \end{pmatrix} \quad \phi = \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix} \quad \langle \mathbb{A}\psi, \phi \rangle = \langle \mathbb{F}, \phi \rangle$$



# Stokes problem

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$$V = H_0^1(\Omega)^d \quad Q = L_0^2(\Omega)$$

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= (f, \mathbf{v}) & \forall \mathbf{v} \in V \\ (\nabla \cdot \mathbf{u}, q) &= 0 & \forall q \in Q \end{aligned} \quad \Longleftrightarrow \quad \langle \mathbb{A}\psi, \phi \rangle = \langle \mathbb{F}, \phi \rangle$$

$$\psi = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \quad \mathbb{A} = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \quad \phi = \begin{pmatrix} \mathbf{v} \\ q \end{pmatrix}$$

$$\langle Au, v \rangle := (\nabla \mathbf{u}, \nabla \mathbf{v}) \quad \langle Bv, q \rangle := (\nabla \cdot \mathbf{v}, q)$$



# Accessing subspaces

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- Use symbolic names (*not indices*: extractor objects) to index subvectors/subspaces:
  - $\phi_u, \phi_p$  translate into `phi[velocity], phi[pressure]`
  - $\psi_u, \psi_p$  translate into `psi[velocity], psi[pressure]`



# Assembly of vector valued problems

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- Assemble systems and rhs using the following identities:

- $\phi_{i,u}(x_q) \implies \text{fe\_values}[\text{velocity}].\text{value}(i,q)$

- $\phi_{i,p}(x_q) \implies \text{fe\_values}[\text{pressure}].\text{value}(i,q)$

- $\nabla \phi_{i,u}(x_q) \implies \text{fe\_values}[\text{velocity}].\text{gradient}(i,q)$

- $\nabla \cdot \phi_{i,u}(x_q) \implies \text{fe\_values}[\text{velocity}].\text{divergence}(i,q)$





# Defining the finite element

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- We defined solution, shape functions, and test functions as having multiple components.
- Each component is usually built from a simpler element.
- Example 1: The Taylor-Hood element for 2d Stokes

```
FESystem<2> stokes_element (FE_Q<2>(2), 1,    // one copy of FE_Q(2) for  $u_x$ 
```

```
FE_Q<2>(2), 1,    // one copy of FE_Q(2) for  $u_y$ 
```

```
FE_Q<2>(1), 1);  // one copy of FE_Q(1) for  $p$ 
```





# Describing logical connections

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- You know which components logically form a vector or are scalars
- The visualization program doesn't.
- Solution:
  - You need to describe it to the DataOut class when adding a solution vector
  - DataOut can then represent this information in the output file



# Describing logical connections

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```
std::vector<std::string> solution_names (dim, "velocity");
```

```
solution_names.push_back ("pressure");
```

```
std::vector<DataComponentInterpretation::DataComponentInterpretation>
```

```
    data_component_interpretation(dim, DataComponentInterpretation::component_is_part_of_vector);
```

```
data_component_interpretation.push_back (DataComponentInterpretation::component_is_scalar);
```

```
data_out.add_data_vector(solution, solution_names, DataOut<dim>::type_dof_data,data_component_interpretation);
```



# Back to pc

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- Today's lecture: solve linear elasticity problem with mixed boundary conditions

$$-\nabla \cdot (\sigma(u)) = f$$

$$\sigma(u) := 2\mu\varepsilon(u) + \lambda I(\nabla \cdot u) = \mu(\nabla u + (\nabla u)^T) + \lambda I(\nabla \cdot u)$$

- Weak form:

given  $f \in (H_0^1(\Omega)^d)'$ , find  $u \in (H_{D,g}^1(\Omega))^d := \{u \in H^1(\Omega)^d \text{ s.t. } \gamma u = g \text{ on } \partial\Omega_D\}$

$$(\mu\varepsilon(u), \varepsilon(v)) + (\lambda \nabla \cdot u, \nabla \cdot v) = (f, v) + \int_{\partial\Omega_N} g_N v \, d\partial\Omega \quad \forall v \in (H_0^1(\Omega))^d$$

