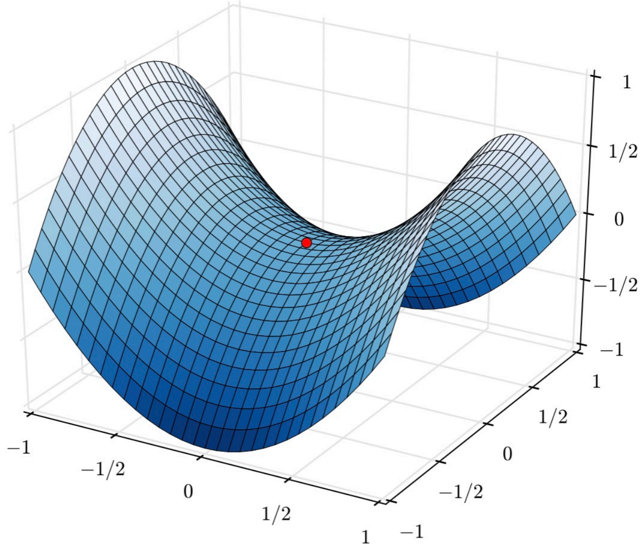

Metodi Matematici per Equazioni alle Derivate Parziali

Saddle point problems

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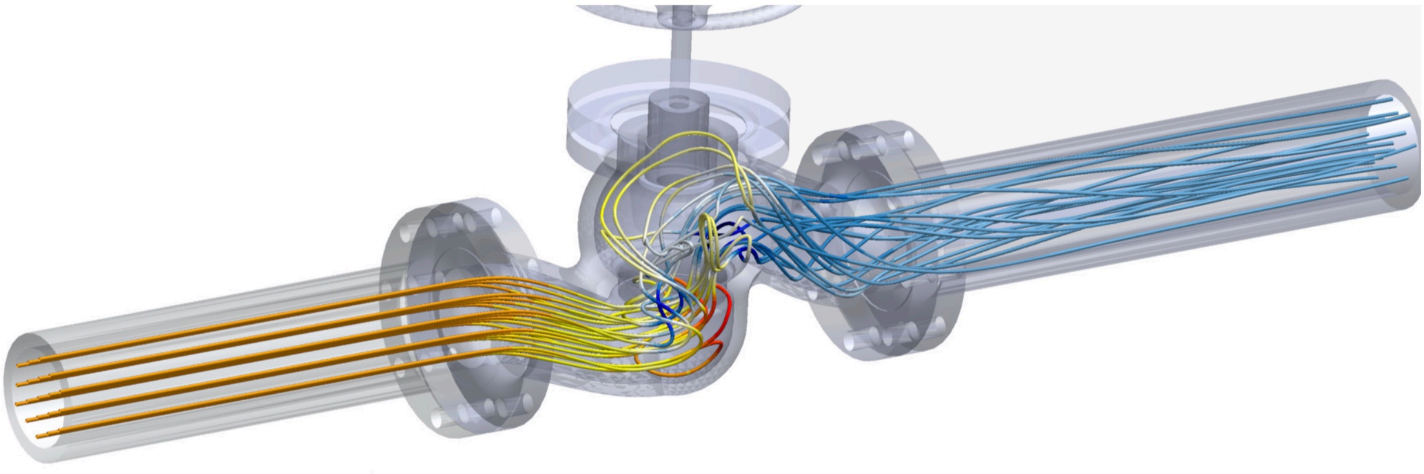

Mixed Problems

Two Hilbert spaces: V, Q, and two operators:

$$A : V \longrightarrow V' \qquad A \in \mathcal{L}(V, V')$$

$$B : V \longrightarrow Q' \qquad B \in \mathcal{L}(V, Q')$$

Given $f \in V'$, $g \in Q'$ find (u,p) in $V \times Q$ s.t.

$$\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}$$

1) $g \in \mathrm{Im}(B) \Rightarrow \exists\, u_g$ s.t. $Bu_g = g$

2) $\mathcal{Z} = \ker(B)$ $\qquad u = u_0 + u_g$, $u_0 \in \mathcal{Z}$

$$\Rightarrow \begin{cases} Au_0 + B^T p = f - Au_g = \tilde{f} & \text{in } V' \\ Bu_0 = 0 \end{cases}$$

Restrict our analysis to $g = 0$ $(\tilde{f} = f - Au_g)$

$$\langle Au_0, v_0 \rangle + \underbrace{\cancel{\langle B^T p, v_0 \rangle}}_{\forall v_0 \in \mathcal{Z}} {}^{\to 0} = \langle \tilde{f}, v_0 \rangle \qquad \forall v_0 \in \mathcal{Z}$$

$$\boxed{\begin{array}{l} u\text{-problem} \\ \langle Au_0, v_0 \rangle = \langle \tilde{f}, v_0 \rangle \end{array}}$$

$\Longleftrightarrow$ BNB is satisfied

1)
$$\boxed{\begin{aligned} &\inf_{u_0 \in Z} \sup_{v_0 \in Z} \frac{\langle A u_0, v_0 \rangle}{\|u\|_V \|v\|_V} = \alpha \\ &\inf_{v_0 \in Z} \sup_{u_0 \in Z} \frac{\langle A u_0, v_0 \rangle}{\|u\|_V \|v\|_V} = \alpha \end{aligned}}$$

ELL-KER
on A

Then $\exists!\ u_0 \in Z$ s.t.

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|\tilde{f}\|_{V'} \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{1}{\alpha} \|A\|_{V'} \|u_g\|_V}$$

Given u_0 solution to $\boxed{u\text{-problem}}$

Find p s.t. $(u = u_0 + u_g)$

$$\boxed{\langle B^T p, v \rangle = -\langle A u, v \rangle + \langle f, v \rangle}$$

p-problem

$$= -\langle A u_0, v \rangle + \langle \tilde{f}, v \rangle \quad \forall v \in V$$

$BNB_1 + BNB_2 \rightarrow \ker(B^T) = 0 \quad \mathrm{Im}(B) = \overline{\mathrm{Im}(B)}$

B is surjective

$\exists\ \beta > 0 \quad \|B^T p\|_{V'} \geq \beta \|p\|_Q \quad \forall p \in Q$

INF-SUP on B

$$\Leftrightarrow \quad \boxed{\exists\ \beta > 0 \quad \text{s.t.} \quad \inf_{p \in Q} \sup_{v \in V} \frac{\langle B v, p \rangle}{\|v\|_V \|p\|_Q} = \beta}$$

Notice that $Au - f =: L \in V'$

$\langle L, v_0 \rangle = 0 \quad \forall v_0 \in Z \Rightarrow L \in Z^\circ$

Summary: $\forall (f,g) \in V', Q' \quad \exists! (u,p) \in V, Q$ s.t.

$$\boxed{\begin{cases} Au + B^T p = f \\ Bu = g \end{cases}}$$

$Z := \ker(B)$

If and only if $\exists \alpha, \beta$ s.t.

$$\boxed{\begin{array}{ll}
1) & \inf_{u \in Z} \sup_{v \in Z} \frac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = \alpha > 0 \\[2ex]
1) & \inf_{v \in Z} \sup_{u \in Z} \frac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = \alpha > 0 \\[2ex]
3 & \inf_{q \in Q} \sup_{u \in V} \frac{\langle Bu, q \rangle}{\|u\|_V \|q\|} = \beta > 0
\end{array}}$$

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|\tilde{f}\|_{V'} \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{1}{\alpha} \|A\|_{V'} \|u_g\|_V}$$

$$\|g\|_{Q'} = \|Bu_g\|_{V'} \geq \beta \|u_g\|_V$$

$$\boxed{\|u_0\|_V \leq \frac{1}{\alpha} \|f\|_{V'} + \frac{\|A\|_{V'}}{\alpha \beta} \|g\|_{Q'}}$$

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \frac{\|A\|_{V'}}{\alpha\beta}\|g\|_{Q'} + \frac{1}{\beta}\|g\|_{Q'}$$

$$\boxed{\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\|g\|_{Q'}}$$

$$\|p\|_Q \leq \frac{1}{\beta}\|L\|_{V'} \leq \frac{1}{\beta}\left(\|A\|\,\|u\| + \|f\|_{V'}\right)$$

$$\|p\|_Q \leq \frac{\|A\|}{\beta}\left(\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\right)\|g\|_{Q'} + \frac{1}{\beta}\|f\|_{V'}$$

$$\boxed{\|p\|_Q \leq \frac{(\|A\|+1)}{\beta}\|f\|_{V'} + \left(\frac{\|A\|_{V'}+\alpha}{\alpha\beta}\right)\frac{\|A\|}{\beta}\|g\|_{Q'}}$$

1), 2), 3) $\Longleftrightarrow$ a), b) $\qquad V \times Q = \mathbb{V}$

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \mathbb{A} \qquad \mathbb{A} : \mathcal{L}(V\times Q \to V'\times Q')$$

equivalent to 1,2,3

$\exists\ \bar{\alpha}$ s.t.

a) $$\inf_{\psi \in V} \sup_{\theta \in Y} \frac{\langle A\psi, \theta \rangle}{\|\psi\| \|\theta\|} = \bar{\alpha}$$

b) $$\inf_{\theta \in Y} \sup_{\psi \in V} \frac{\langle A\psi, \theta \rangle}{\|\psi\| \|\theta\|} = \bar{\alpha}$$

1) ELL-Ker in $Z_h \Rightarrow \exists\, \alpha_h$ s.t. Zinfsup on Z_h

2) INF SUP on V_h, Q_h for B_h

$$Z_h := \{ v_h \in V_h \text{ s.t. } b(v_h, q_h) = 0 \ \forall q_h \in Q_h \}$$

$$\begin{array}{l} \ker B \subset V \\ V_h \subset V \end{array} \not\Rightarrow \ker B \cap V_h \neq \{\emptyset\}$$

$$\ker B_h \not\subset \ker B$$

$$a(u - u_h, v_h) = -b(p - p_h, v_h) \quad \forall v_h \in V_h$$

Restrict to $w_h \in Z_h$

$$a(u - u_h, w_h) = -b(p, w_h) \quad \forall w_h \in Z_h$$

$$= -b(p - q_h, w_h) \quad \forall w_h \in Z_h$$

$$\langle A u_h, w_h \rangle = \langle A u, w_h \rangle + \langle B^T(q_h - p), w_h \rangle \ \forall q_h \in Q_h$$

$$\|u - u_h\| \leq \| u - v_h + v_h - u_h\|$$

$$\leq \|u - v_h\| + \frac{1}{\alpha_h} \|A(v_h - u_h)\|_{*, Z_h'}$$

$$\|u - u_h\| \leq \|u - v_h\| + \frac{\|A\|}{\alpha_h} \|u - v_h\| + \frac{1}{\alpha_h} \|B\| \|q_h - p\| \quad \begin{array}{l} \forall v_h \in V_h \\ \forall q_h \in Q_h \end{array}$$

$$\boxed{\|u-u_h\| \leq \left(1+\frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in Z_h} \|u-v_h\| + \frac{\|B\|}{\alpha_h} \inf_{q_h \in Q_h} \|q_h-p\|}$$

$$B : V \longrightarrow Q'$$
$$B^T : Q \longrightarrow V'$$

$$\|p-p_h\| \leq \|p-q_h+q_h-p_h\|$$

$$\leq \|p-q_h\| + \frac{1}{\beta_h}\|B_h^T(q_h-p_h)\|_{*,V_h}$$

$$\|B_h^T q_h\|_{*,V_h} \geq \beta_h \|q_h\|$$

$$\langle B^T(q_h-p), v_h \rangle = \langle A(u_h-u), v_h \rangle$$

$$\leq \|p-q_h\| + \frac{\|A\|}{\beta_h}\|u_h-u\| + \frac{\|B\|}{\beta_h}\|p-q_h\|$$

$$\boxed{\|p-p_h\| \leq \left(1+\frac{\|B\|}{\beta_h}\right) \inf_{q_h \in Q_h} \|p-q_h\| + \frac{\|A\|}{\beta_h}\|u-u_h\|}$$

$$\|p-p_h\| \leq \left(1+\frac{\|B\|}{\beta_h}+\frac{\|A\|\,\|B\|}{\alpha_h \beta_h}\right) \inf_{q_h \in Q_h} \|p-q_h\| +$$
$$\frac{\|A\|}{\beta_h}\left(1+\frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in Z_h} \|u-v_h\|$$

Two concrete examples:

Mixed Poisson with nat. b.c.

$$-\Delta p = g \qquad -\nabla p = u$$
$$\Downarrow \qquad \nabla p\cdot n = 0$$
$$v\cdot n = 0$$

$$\begin{cases} u + \nabla p = 0 \\ \operatorname{div} u = g \end{cases}$$

$$(u,v) - (\operatorname{div} v, p) = 0 \quad \forall v \in H^{\operatorname{div}}_0(\Omega)$$

$$(\operatorname{div} u, q) = (g,q) \quad \forall q \in L^2(\Omega)$$

Stokes

$$-\Delta u + \nabla p = f$$
$$\operatorname{div} u = 0$$
$$\Downarrow$$
$$(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f,v) \quad \forall v \in H^1_0(\Omega)^d$$
$$(\operatorname{div} u, q) = 0 \quad \forall q \in L^2_0(\Omega)$$

$$\langle Bv, q\rangle := \int_\Omega \operatorname{div} v \, q \qquad \forall v \in H^1(\Omega) \subset H^{\operatorname{div}}(\Omega)$$
$$\forall q \in L^2(\Omega)$$

$$H^{\operatorname{div}} := \{ v \in (L^2)^d \text{ s.t. } \operatorname{div} v \in L^2 \} \qquad \|u\|^2_{\operatorname{div}} := \|u\|_0^2 + \|\operatorname{div} u\|_0^2$$

$$H^{\operatorname{div}}_0 := \{ v \in H^{\operatorname{div}} \text{ s.t. } v\cdot n = 0 \text{ on } \gamma \}$$

$$M : H^{\operatorname{div}}_0 \longrightarrow (H^{\operatorname{div}}_0)'$$
$$\langle Mu, v\rangle \longrightarrow \int uv$$

$$A : (H^1_0(\Omega))^d \longrightarrow (H^1_0(\Omega)^d)'$$
$$\langle Au, v\rangle \longrightarrow \int_\Omega \nabla u : \nabla v$$
$$\sum_{i,j} u_{i,j} v_{i,j}$$

M satisfies ELL-KER ?

$\langle Mu,u\rangle = \|u\|_0^2$ ~ No control on div / ∇.

M is ellipte on <u>ker B</u>

$\ker B := \{ v \in H_0^{div}(\Omega) \text{ s.t. } \operatorname{div} v = 0 \}$

$\forall v \in \ker B \qquad \|v\|_{div} = \|v\|_0 + \underbrace{\|\operatorname{div} v\|_0}_{0}$

$\Rightarrow$ M is coercive on ker B $\quad \alpha = 1$

$H_0^1(\Omega)^d$, $\langle Au,v\rangle := \int_\Omega \nabla u : \nabla v$

A is coercive on $H_0^1(\Omega)^d$ not just on ker B

$\langle Au,u\rangle \geq \|u\|_1$ By poincaré

Simpler case : Stokes : $B : H_0^1(\Omega)^d \longrightarrow L_0^2(\Omega)$

inf sup of B ?

$$\inf_{q \in L_0^2(\Omega)} \sup_{v \in H_0^1(\Omega)^d} \frac{\int \operatorname{div} v \, q}{\|v\| \, \|q\|} \geq \beta \quad ?$$

If $\forall q \in L^2_0(\Omega) \quad \exists \quad v_q \in H^1_0(\Omega) \quad$ s.t.

$$\operatorname{div} v_q = q \quad \Longrightarrow \quad \int \operatorname{div} v_q \, q = \int q^2 = \|q\|^2$$

$$\longrightarrow \quad \inf_{q \in L^2_0(\Omega)} \; \sup_{v \in H^1_0(\Omega)^d} \frac{\int \operatorname{div} v \, q}{\|v\| \; \|q\|} \geq \inf_{q \in L^2_0(\Omega)} \frac{\|q\|^{\cancel{2}}}{\|v_q\| \cancel{\|q\|}}$$