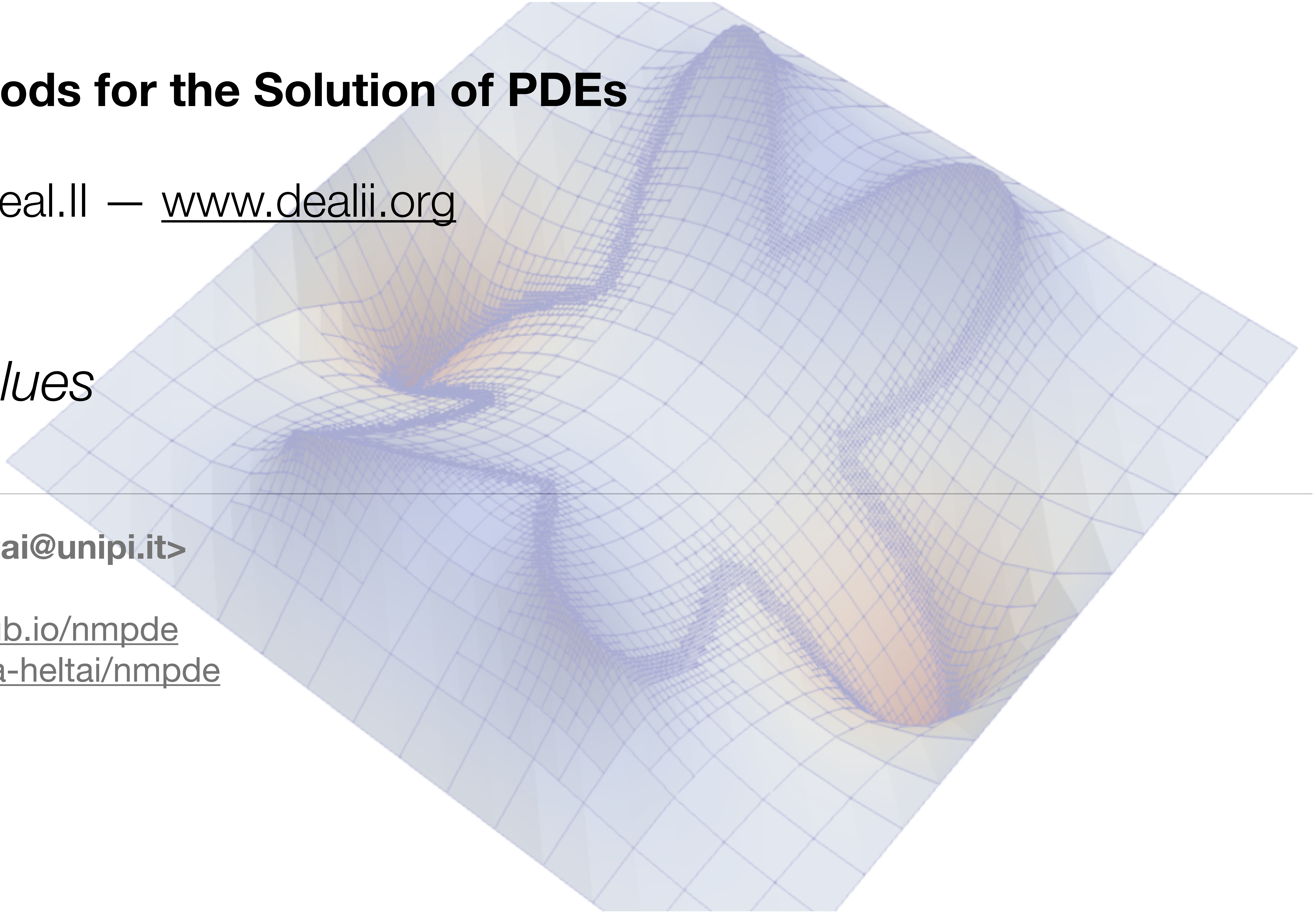
Numerical Methods for the Solution of PDEs

Laboratory with deal.II — www.dealii.org

LAB 3 — FEValues



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https://luca-heltai.github.io/nmpde
https://github.com/luca-heltai/nmpde

Aims for this Lecture

- Integration of Finite Element functions on cells (or faces)
    - Local numbering VS global numbering
    - Mapping
    - Basis function Values and Gradients
- Post-processing and visualisation
    - Error tables

Reference material

- Documentation
  - https://www.dealii.org/current/doxygen/deal.II/group__FE__vs__Mapping__vs__FEValues.html
  - https://www.dealii.org/current/doxygen/deal.II/group__UpdateFlags.html

What we want to compute today:

Error in the finite element space interpolation:

$$\|u - u_h\|_{1,\Omega} = \sqrt{\int_\Omega (u - u_h)^2 + (\nabla u - \nabla u_h)^2}$$

Split integration on cells:

Integrate g on all cells T_m:

$$b := \int_\Omega g = \sum_m b_m := \sum_m \int_{T_m} g dT_m$$

Transform integral on T_m to $\hat{T}$

$$\int_{T_m} g dT_m = \int_{\hat{T}} (g \circ F_m) J_m d\hat{T} \qquad J_m := \det(DF_m)$$

Use quadrature formula on $\hat{T}$

$$\int_{\hat{T}} (g \circ F_m) J d\hat{T} \simeq \sum_{q=0}^{n_q-1} g(F_m(\hat{x}_q)) J_m \hat{w}_q$$

Split integration on cells:

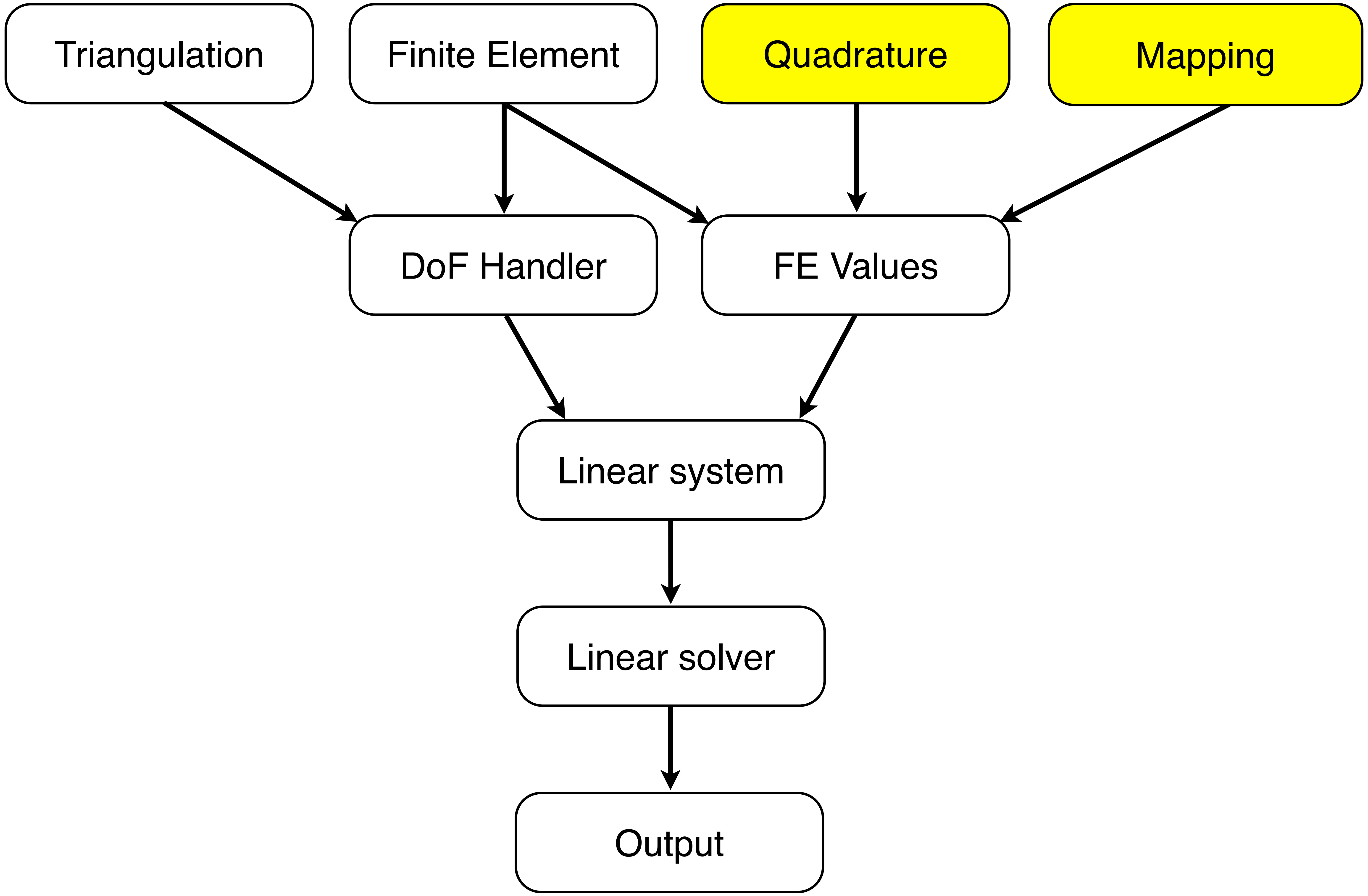
For gradients:

$$\hat{\nabla}(g \circ F_m) = [\nabla g \circ F_m] DF_m^T \Longrightarrow \nabla g = [\hat{\nabla}(g \circ F_m)] DF_m^{-T}$$

On each quadrature point we need:

$$x_q := F_m(\hat{x}_q), \quad B_m := DF_m(\hat{x}_q), \quad J_q w_q = \det(B_m), \quad B_m^{-T}$$

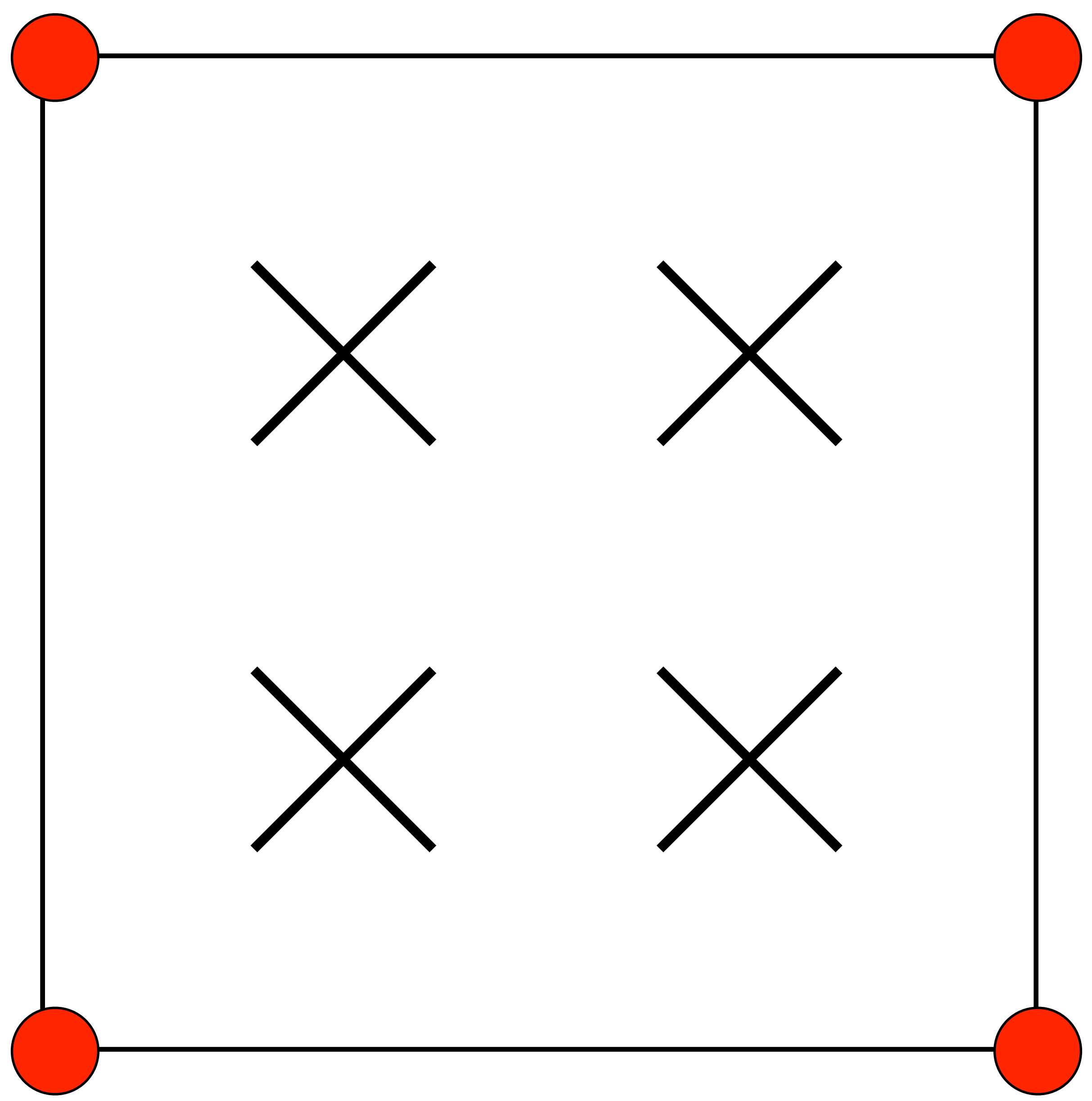
New classes


Triangulation
Finite Element
Quadrature
Mapping
DoF Handler
FE Values
Linear system
Linear solver
Output

Integration on a cell: the Quadrature classes

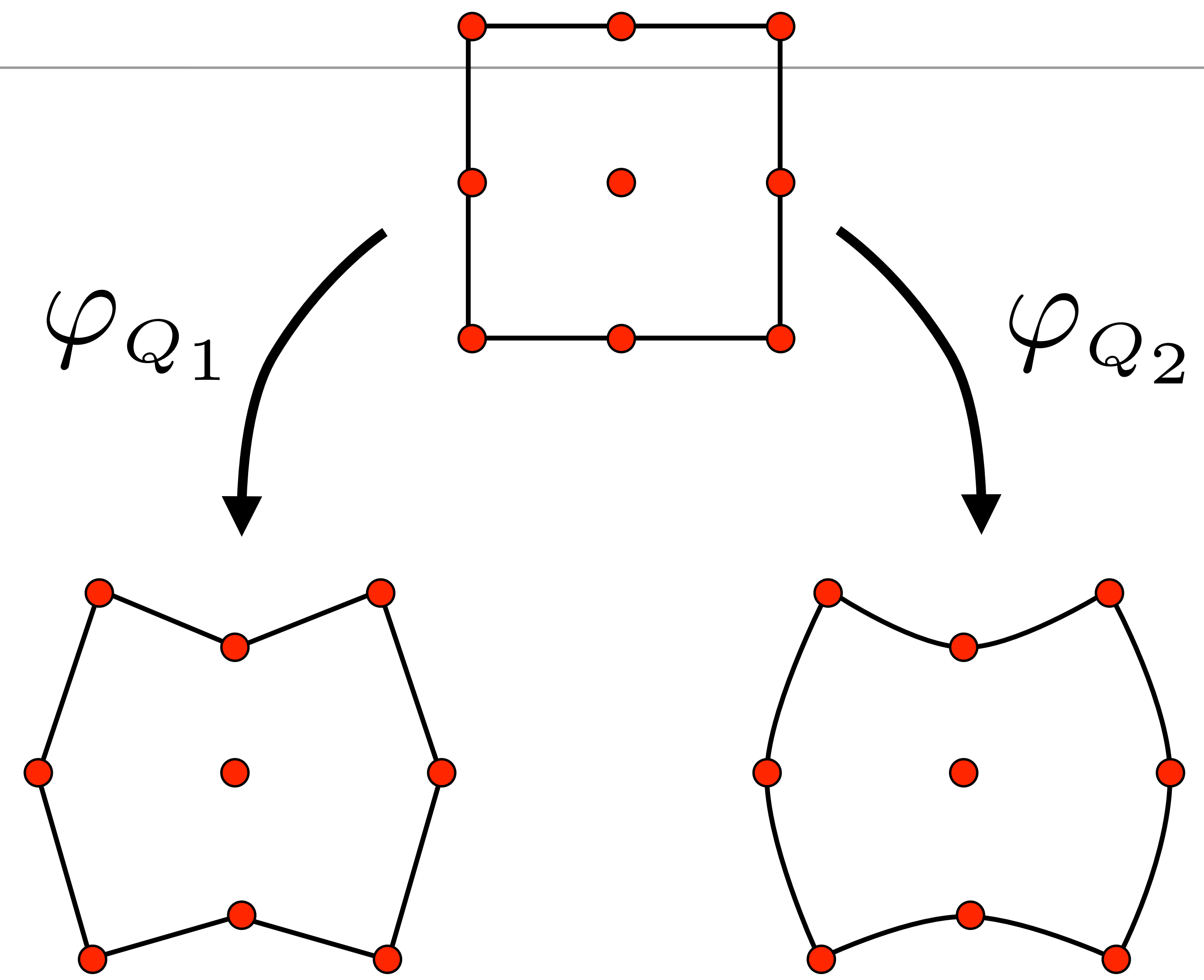
- n-Order Gauss quadrature
- Other rules
  - Gauss Lobatto
  - Simpson
  - Trapezoidal
  - Midpoint
  - A few others
- Anisotropic
  - Tensor product

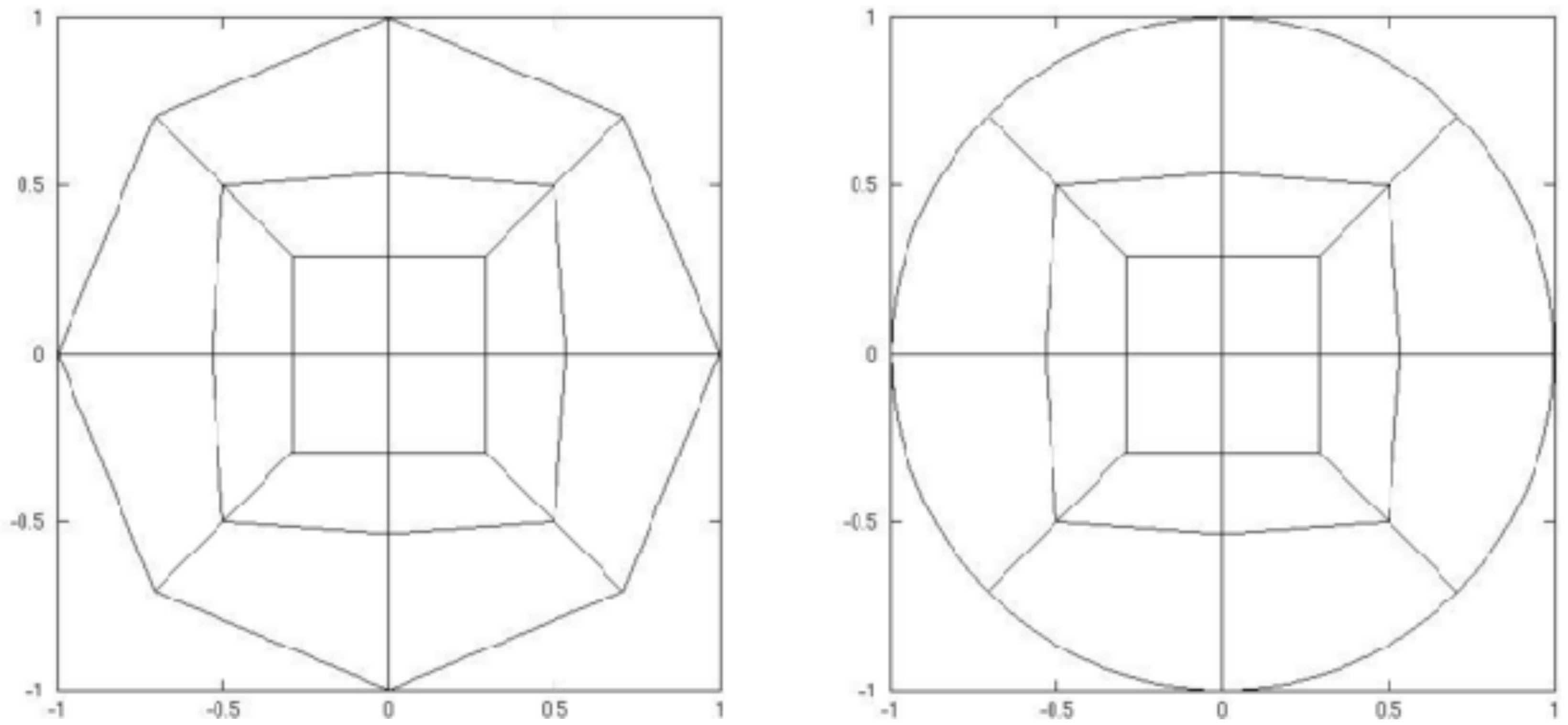
`FE_Q<2>(1)`



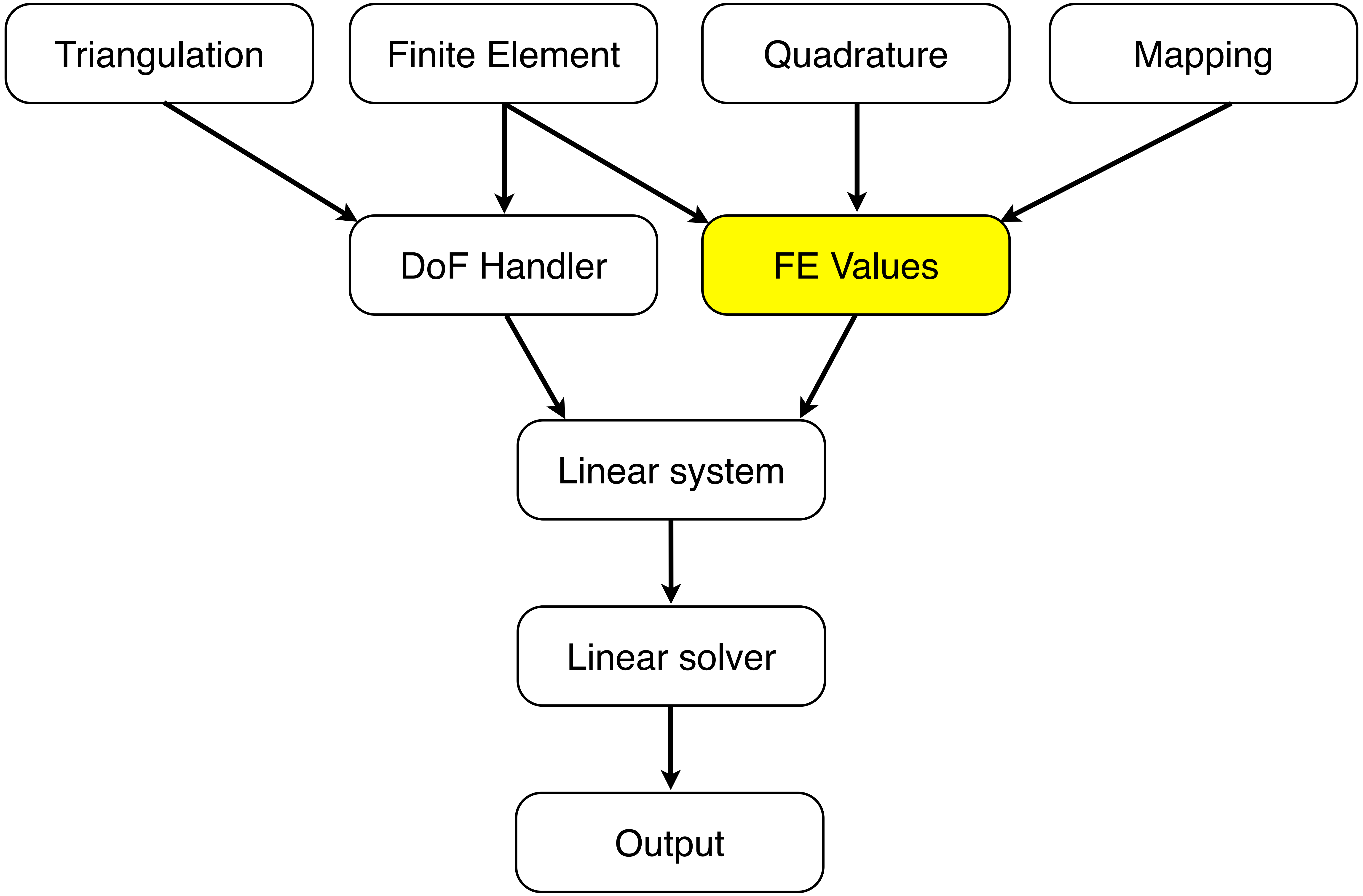
Integration on a cell: the Mapping classes

- n-order mappings
  - Increase accuracy of:
    - Integration schemes
    - Surface basis vectors
- Boundary and interior manifolds





New class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Integration on a cell: the FEValues class

- Object that helps perform integration
- Combines information of:
  - Cell geometry
  - Finite-element system
  - Quadrature rule
  - Mappings
- Can provide:
  - Shape function data
  - Quadrature weights and mapping jacobian at a point
  - Normal on face surface
  - Covariant/contravariant basis vectors
- More ways it can help:
  - Object to extract shape function data for individual fields
  - Natural expressions when coding
- Low level optimisations

$$\boxed{b_m := \sum_{q=0}^{n_q-1} g(F_m(\hat{x}_q))J_m\hat{w}_q}$$

```
b(m) +=

    g(fe_values.quadrature_point(q_point)) *

    fe_values.JxW (q_point);
```