

## Metodi Matematici per Equazioni alle Derivate Parziali

Denis-Lions lemma, Bramble Hilbert lemma

Luca Heltai <luca.heltai@unipi.it>



Dipartimento  
di Matematica

UNIVERSITÀ DI PISA



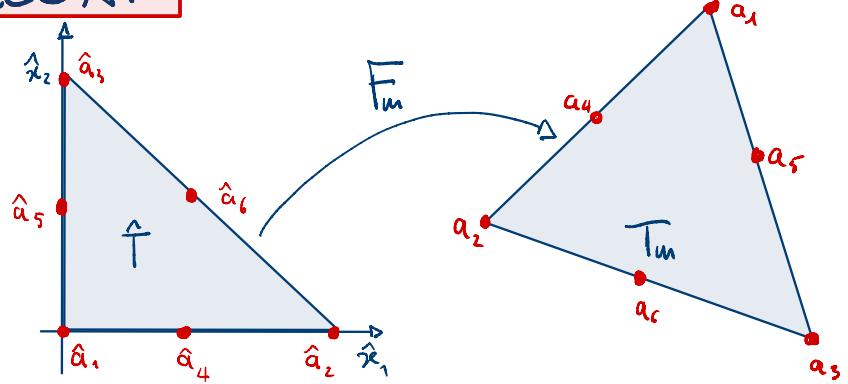
Lagrange:

$$\hat{\Sigma} = \text{Span}\{\hat{v}_i\}$$

$$\hat{v}_i = \delta(\hat{x} - \hat{a}_i)$$

$$\begin{aligned} \hat{v}_i(u) &= \int_{\Omega} \delta(\hat{x} - \hat{a}_i) u(\hat{x}) d\hat{x} \\ &= u(\hat{a}_i) \end{aligned}$$

RECAP



Reference FE

$$\begin{aligned} \hat{F}_E &= \left\{ \hat{T}, \hat{P}, \hat{\Sigma} \right\} \\ \hat{P} &= \text{Span} \left\{ \hat{v}_i \right\}_{i=1}^N \\ \hat{\Sigma} &= \text{Span} \left\{ \hat{v}_i \right\}_{i=1}^N \\ \hat{v}_i(\hat{v}_j) &= \delta_{ij} \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_m &= \frac{\hat{T}_m}{\left( \bigcup_{m=1}^M \hat{T}_m \right)} \\ \hat{v}_i &= \hat{v}_i \circ F_m^{-1} \end{aligned}$$

$$\begin{aligned} F_E &= \left\{ T_m, P_m, \Sigma_m \right\} \\ T_m &= F_m(\hat{T}) \\ v_i &= \hat{v}_i \circ F_m^{-1} \\ \hat{v}_i &= \hat{v}_i \circ F_m^{-1} \end{aligned}$$

Global Space

$\approx$  := extension by zero

$$P_{mIJ} = \begin{cases} 1 & \text{if } v_I|_{T_m} = \hat{v}_I \circ F_m^{-1} \\ 0 & \text{otherwise} \end{cases}$$

$$V_h = \text{Span} \left\{ v_I \right\}_{I=1}^{N_{\text{Dofs}}}, \quad v_I \in V,$$

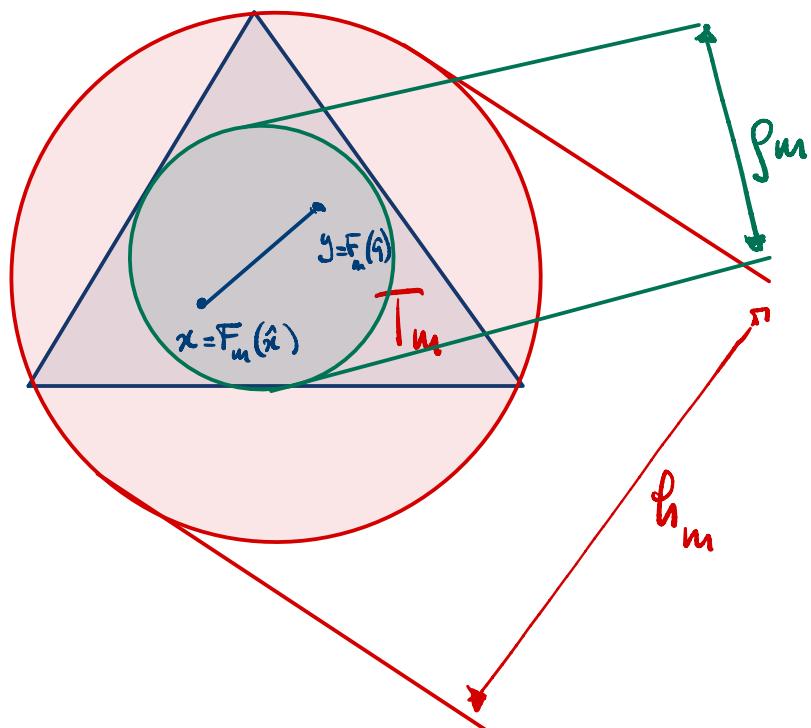
$$v_I = \sum_m \sum_j P_{mIJ} \hat{v}_j \circ F_m^{-1}$$

For Affine mappings.

$$F_m = B_m \hat{x} + b_m = \sum_{i=1}^3 \hat{v}_i^{P'}(\hat{x}) \alpha_i$$

Given Lagrangian  $P'$

$$P'(\hat{T}) := \text{span}\{\hat{v}_i^{P'}\}_{i=1}^{\dim}$$



$$\begin{cases} \cdot \\ \cdot \end{cases} = y - x = F_m(\hat{y}) - F_m(\hat{x}) = B_m \hat{z}$$

$$\begin{cases} \cdot \\ \cdot \end{cases} = \hat{y} - \hat{x} = F_m^{-1}(y) - F_m^{-1}(x) = B_m^{-1} \hat{z}$$

$$\|B\| := \sup_{|\hat{z}|=\hat{s}} \frac{|B \hat{z}|}{|\hat{z}|} \leq \frac{h_m}{\hat{s}}$$

$$\|B^{-1}\| := \sup_{|\hat{z}|=S_m} \frac{|B^{-1} \hat{z}|}{|\hat{z}|} = \frac{\hat{h}}{S_m}$$

Transformation of Sobolev norms under  
affine mappings

$$\begin{aligned} \left\| \hat{V} \circ F_m^{-1} \right\|_{K,P,T_m}^P &:= \int_{T_m} \left| D^K (\hat{V} \circ F_m^{-1}) \right|^P dx \\ &= \int_{T_m} \left| \left[ (\hat{D}^K \hat{V}) \circ F_m^{-1} \right] \cdot B_m^{-K} \right|^P dx \\ &= \int_T \left| \hat{D}^K \hat{V} B_m^{-K} \right|^P J_m d\hat{x} \\ &\leq \|B_m\|^{-KP} \int \left| \hat{V} \right|_{K,P,\hat{T}}^P d\hat{x} \end{aligned}$$

Notation:

$$a \lesssim b := \exists c, \text{ind. of } a, b, \text{ s.t.}$$

$$a \leq c b, \quad a \sim b \Rightarrow a \lesssim b, b \lesssim a$$

$$|\hat{V} \circ F_m^{-1}|_{K,P,\hat{T}} \lesssim S_m^{-\kappa} J^{\frac{1}{P}} |\hat{V}|_{K,P,\hat{T}}$$

$$|\hat{V}|_{K,P,\hat{T}} \lesssim h_m^{-\kappa} J^{\frac{1}{P}} |\hat{V} \circ F_m^{-1}|_{K,P,T}$$

$$\text{for } 0 \leq \kappa, e \leq q$$

$$|V|_{K,P,T_m} \lesssim S_m^{-\kappa} h_m^e |V|_{e,P,T_m} \quad \text{fixe } P^q(T_m)$$

### Bramble - Hilbert

let  $\mathcal{Z} \in L(W^{k+1,p}(\Omega), W^{s,p}(\Omega))$ ,  $s \leq k$

If  $P^k \subset \text{Ker}(\mathcal{Z})$  then

$$\|\mathcal{Z}(u)\|_{s,p,\Omega} \lesssim \|\mathcal{Z}\|_{L(W^{k+1,p}(\Omega), W^{s,p}(\Omega))} \|u\|_{k+1,p,\Omega}$$

Think of  $\mathcal{Z}: I - \Pi^k$ .

$$\|u - \Pi^k u\|_{s,p,\Omega} \lesssim \|\mathcal{Z}\|_* \|u\|_{k+1,p,\Omega}$$

Proof: Step 1: show that  $\forall z \in L(V, H)$ ,  
 s.t.  $Q_z = \ker z$  we have

$$\|z(u)\|_H = \|z(u+p)\|_H \leq \|z\|_* \|u+p\|_V \quad \forall p \in Q$$

$$\|z(u)\|_H \leq \|z\|_* \inf_{p \in Q} \|u+p\|_V \quad \forall u \in V$$

Step 2: (Denis-Lions lemma)

for  $u \in W^{k+1, p}(\Omega)$

$$|u|_{k+1, p, \Omega} \lesssim \inf_{q \in P^k(\Omega)} \|u+q\|_{k+1, p, \Omega} \lesssim |u|_{k+1, p, \Omega}$$

1st ineq.:

$$|u|_{k+1, p, \Omega} \lesssim \inf_{q \in P^k(\Omega)} \|u+q\|_{k+1, p, \Omega}$$

$$|u+q|_{k+1, p, \Omega} = |u|_{k+1, p, \Omega} \quad \forall q \in P^k$$

$$\text{because } |q|_{k+1, p, \Omega} = 0 \quad \forall q \in P^k$$

2nd ineq.:

We show that 2nd ineq. is equivalent to proving  
 the following result:

$$P^k = \text{Span} \left\{ v_i \right\}_{i=1}^{N_{pk}}, \quad V^j \in (P^k)^* \text{ s.t. } V^j(v_i) = \delta^j_i$$

By Hahn-Banach extension theorem,

$$\exists \underline{\tilde{v}^i} \in (W^{k+1,p}(\Omega))^* \text{ s.t. } \tilde{v}^i(q) = v^i(q) \forall q \in P^k$$

$$\begin{aligned} \Pi^k: W^{k+1,p}(\Omega) &\longrightarrow P^k & (\Pi^k)^2 = \Pi^k \\ u &\longmapsto \sum_i \tilde{v}^i(u) v_i \end{aligned}$$

2nd ineq. follows from proving that

$$\textcircled{1} \|u\|_{k+1,p,\Omega}^p \leq \|u\|_{k+1,p,\Omega}^p + \sum_i |\tilde{v}^i(u)|^p$$

\textcircled{1} implies that  $\exists c$ . s.t.  $\forall u \in W^{k+1,p}(\Omega)$  we have  $\|u\|_{k+1,p,\Omega}^p \leq \|u\|_{k+1,p,\Omega}^p + \sum_i |\tilde{v}^i(u)|^p$   
 $\Rightarrow$  prove by contradiction:

$$\forall c, \exists w_c \in W^{k+1,p}(\Omega) \text{ s.t. } \|w_c\|_{k+1,p,\Omega} = 1,$$

$$c \left( \|w_c\|_{k+1,p,\Omega}^p + \sum_i |\tilde{v}^i(w_c)|^p \right) \leq \|w_c\|_{k+1,p,\Omega}$$

Take  $c = i$  in  $[0, 1, \dots, N-1]$

$\Rightarrow$  There exists a seq reduce  $w_i$  s.t.

$$\|w_i\|_{k+1,p,\Omega} = 1 \text{ and } \lim_{i \rightarrow \infty} \left( \|w_i\|_{k+1,p,\Omega}^p + \sum_{j=1}^N |\tilde{v}^j(w_i)|^p \right) = 0$$

$$\cdot W^{k+1,p}(\Omega) \hookrightarrow W^k(\Omega)$$

Bounded sequence in  $W^{k+1,p}(\Omega) \Rightarrow \exists$  strongly convergent subsequence in  $W^k(\Omega)$

a)  $\exists w \in W^k(\Omega)$  s.t.

$$\|w_i - w\|_{k,p(\Omega)} \rightarrow 0$$

b) for hypothesis,  $|w_i|_{k+1,p} \rightarrow 0$

a) + b)  $\rightarrow w \in W^{k+1,p}(\Omega), \|w_i - w\|_{k+1,p} \rightarrow 0$

b) implies that  $w \in P^k(\Omega)$  ( $\Omega$  is connected)

$$|w|_{k+1,p} = 0 \Rightarrow w \in P^k(\Omega)$$

$$\|w\|_{k+1,p} = 1 \quad \text{for continuity of the norm}$$

$$(\|w_i\|_{k+1,p} = 1 \quad \forall i)$$

$\|w_i\|_{k+1,p}$  norm is zero because

$$|w|_{k+1,p,\Omega}^p + \sum_j |\tilde{v}^j(w)|^p = 0$$

$$\Rightarrow \tilde{v}^j(w) = v^j(w) = 0 \quad \forall j \Rightarrow w=0$$

$$\Rightarrow \|w\|_{k+1,p} = 0, |w|_{k+1,p} = 0, \|w\|_{k+1,p} = 1$$

impossible

$$\|u\|_{k+1, p, \Sigma}^p \lesssim \|u\|_{k+1, p, \Sigma}^p + \sum_i |\tilde{v}^i(u)|^p$$

↓

$$\inf_{p \in P^k} \|u+p\|_{k+1, p, \Sigma}^p \leq \|u - \Pi^k u\|_{k+1, p, \Sigma}^p$$

$$\lesssim \|u\|_{k+1, p, \Sigma}^p + \sum_i |\tilde{v}^i(u - \Pi^k u)|^p$$

by construction

$$\begin{aligned} \tilde{v}^i(u - \Pi^k u) &= \tilde{v}^i(u) - \tilde{v}^j(u) \tilde{v}^i(v_j) \\ &= \tilde{v}^i(u) - \tilde{v}^i(u) = 0 \end{aligned}$$

$\underbrace{\delta_j^i}$

Take  $\Sigma$  as a single element of our triang.  $T$

$$\|u - \Pi u\|_{S, p, T} \lesssim \|I - \Pi\|_* \|u\|_{k+1, p, T} \quad T = F(\hat{T})$$

$$\hat{u} = u \circ F, \quad \hat{\Pi} \hat{u} = (\Pi u) \circ F$$

$$\|u - \Pi u\|_{S, p, T} \lesssim S_T^{-s} J^{\frac{1}{p}} \|\hat{u} - \hat{\Pi} \hat{u}\|_{S, p, \hat{T}} \lesssim S_T^s J^{\frac{1}{p}} |\hat{u}|_{k+1, p, \hat{T}}$$

$$\|u - \Pi u\|_{S, P, T} \lesssim \int_T^{-s} h_T^{k+1} |u|_{k+1, P, T}$$

For quasi uniform triangulations:

$$h := \max_m h_m \quad , \quad S = \min_m S_m$$

$$f_m \Rightarrow S_m \geq \delta h_m \quad \delta \in (0, 1)$$

$$\|u - \Pi u\|_{S, P, T} \lesssim \delta^{-s} h^{k+1-s} |u|_{k+1, P, T}$$

For example:

$$\langle A u, v \rangle := \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

$$V = H_0^1(\Omega)$$

$$\langle F, v \rangle := \int_{\Omega} f v \, dx \quad *$$

$$f \in L^2(\Omega) \quad , \quad \Omega \text{ Lip. convex} \Rightarrow u \in H^2(\Omega) \cap H_0^1(\Omega)$$

$$* \Rightarrow \langle A(u - u_e), v_e \rangle = 0 \quad \text{2-regular:}$$

$$f \in H^s \Rightarrow u \in H^{s+2}$$

$$\Rightarrow \|u - u_e\|_{1, \Omega} \leq \frac{\|A\|_*}{\alpha} \inf_{v \in V_e} \|u - v_e\|_{1, \Omega}$$

$$k+1-s=1$$

$$\leq \frac{\|A\|_*}{\alpha} \|u - \Pi u\|_{1, \Omega} \leq \frac{\|A\|_*}{\alpha} \delta^{-1} h |u|_{2, \Omega}$$

$$\text{Thm } \forall u \in V_h \rightarrow \underbrace{\Pi u = \sum_m \sum_{\Sigma} \sum_J P_{mIJ} \tilde{v}_J^{\delta} (u \circ F_m)}_{\text{Helmholtz Extension}} \xrightarrow{\text{extension by zero}} \hat{v}_J \circ F_m^{-1}$$

$$\sum_m \|u - \Pi u\|_{1, T_m} \leq \tilde{c}^{-1} h^{1-s} \sum_m \|u\|_{k, T_m}$$

In general you can say

$$\left( \sum_m \|u - \Pi u\|_{s, T_m} \right) \leq \tilde{c}^{-s} h^{k+1-s} \|u\|_{k+1, \Omega} \quad \forall u \in H^k(\Omega) \quad k \leq s \leq k+1$$

In general  $\sum_m \|u - \Pi u\|_{s, T_m} \neq \|u - \Pi u\|_{s, \Omega}$

The equality holds for all  $H^s$  s.t.  $V_h \subset H^s$

For example :  $V_h \subset H_0^1(\Omega)$ , then

$$\|u - \Pi u\|_{s, \Omega} \leq \tilde{c}^{-s} h^{k+1-s} \|u\|_{k+1, \Omega}$$

for  $s=0, 1$ ,  $k \geq 1$  arbitrary