

Metodi Matematici per Equazioni alle Derivate Parziali

A posteriori error analysis

Luca Heltai <luca.heltai@unipi.it>



Dipartimento
di Matematica

UNIVERSITÀ DI PISA



Short Recap

Inverse inequalities

on quasi uniform grids

$$\forall v \in P_T \subset P_n(\Gamma), \quad 0 \leq m, l \leq K \quad \text{real } m, l$$

$$\|v\|_{m,T} \lesssim h_T^{l-m} \|v\|_{e,T}$$

Trace Theorems

$$\text{For } s \in (\frac{1}{p}, 1]$$

bounded and linear

Σ Lip., $\exists \gamma: W^{s,p}(\Sigma) \rightarrow W^{s-\frac{1}{p}, p}(\Gamma)$

Hahn-Banach extension of cont. restriction

$$\gamma u = u \text{ for } u \in C^0(\Sigma)$$

$$1) \quad \|\gamma v\|_{s-\frac{1}{p}, p, \Gamma} \lesssim \|v\|_{s, p, \Sigma}$$

γ has bounded right inverse \square

$$\ker \gamma := W_0^{s,p}(\Sigma)$$

$$2) \quad \|Eg\|_{s, p, \Sigma} \lesssim \|g\|_{s-\frac{1}{p}, p, \Gamma}, \quad \gamma E g = g \quad \forall g \in W^{s,p}(\Gamma)$$

$$3) \quad \text{For } v \in W^{1,p}(\Sigma), \Sigma \text{ Lip}, \quad 1 \leq p \leq \infty$$

$$\|\gamma v\|_{0, p, \partial\Sigma}^p \lesssim \|v\|_{0, p, \Sigma}^{p-1} \|v\|_{1, p, \Sigma}^1 \quad \forall v \in W^{1,p}(\Sigma)$$

$$\|\gamma v\|_{0, \partial\Sigma}^2 \lesssim \|v\|_{0, \Sigma} \|v\|_{1, \Sigma} \quad \forall v \in H^1(\Sigma)$$

Interpolation of non smooth functions

Scott-Zhang interpolation

$$SZ: H^1(\Omega) \rightarrow V_h = P^{k,0}(u)$$

$$1) \forall v \in H_0^1(\Omega) \quad SZv \in H_0^1(\Omega)$$

$$2) SZu_h = u_h \quad \forall u_h \in V_h$$

In general ϕ^i is not unique. We construct S^i s.t. $S^i(u_k) = \phi^i(u_k)$

$$SZu \neq Tu := \sum \phi^i(u) \phi^i$$

$$V_h = \text{span} \{ \phi^i \}_{i=1}^N$$

$$SZu := \underbrace{S^i(u)}_{\phi^i} \phi^i \quad S^i(u) + \sum \phi^j(u) = \delta_{ij}$$

Make a choice: $\forall a_i$ support point of V_h

V_h : piecewise polynomial of order k on each T ,

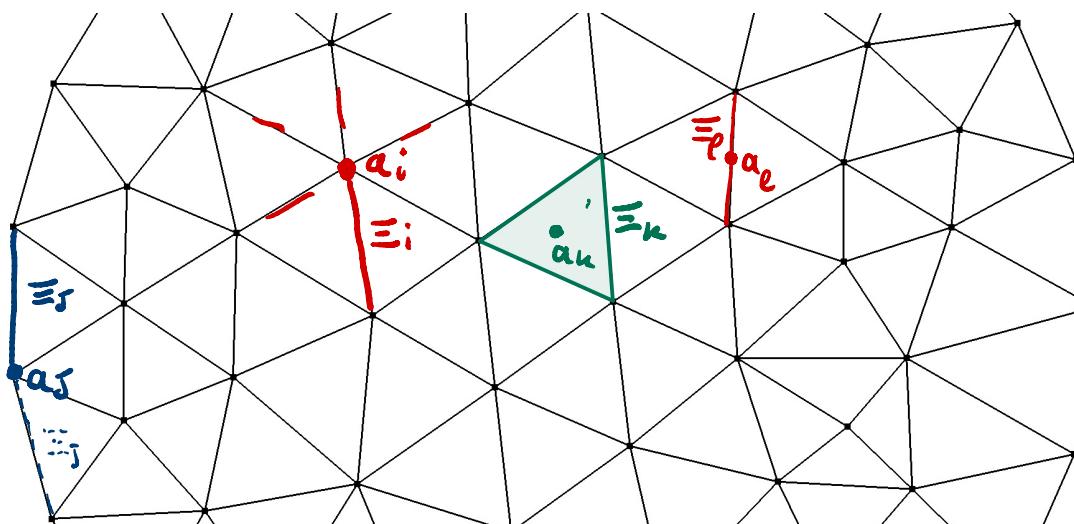
$$\phi^i(u) = u(a_i) \quad \forall u \in C^0(\bar{\Omega})$$

choose one Ξ_i for each a_i —

$$\text{case 1: } a_i \in T_m \Rightarrow \Xi_i := T_m$$

$$\text{case 2: } a_i \in \partial T_m \setminus \partial \Omega \Rightarrow \Xi_i := F_i \in \partial T_m \text{ one of the faces not on } \partial \Omega$$

$$\text{case 3: } a_i \in \partial T_m \cap \partial \Omega \Rightarrow \Xi_i := F_i \in \partial T_m \cap \partial \Omega$$



Ξ_i is a d -simplex if $a_i \in \overline{T_m}$, otherwise it is a $(d-1)$ -simplex

- n_i is number of support points that belong to Ξ_i
- $\{\phi_{i,q}\}_{q=1}^{n_i}$ are the basis functions associated to the support points in Ξ_i $\phi_{i,1} \equiv \phi_i$

- $\{\chi_{i,q}\}_{q=1}^{n_i} \in \text{span}\{\phi_{i,q}\}_{q=1}^{n_i}$ s.t.

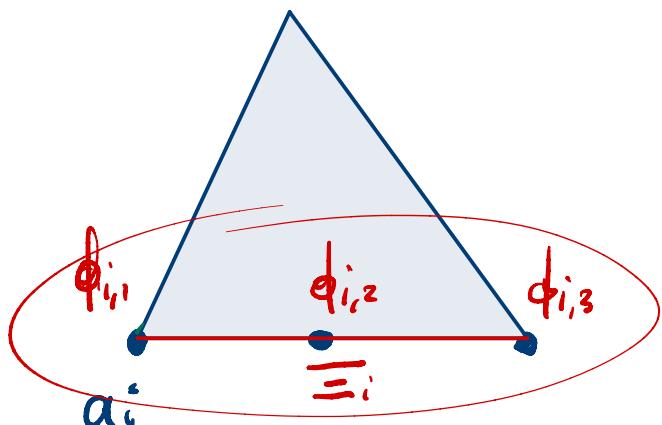
$$\int_{\Xi_i} \chi_{i,q} \phi_{i,x} = \delta_{qr}$$

$$S^i(u) := \int_{\Xi_i} \chi_{i,1} u$$

$$\begin{aligned} S^i_L u &:= \sum_{i=1}^N \phi_i \int_{\Xi_i} \chi_{i,1} u \\ &= S^i(u) \phi_i \end{aligned}$$

$$\forall u \in V_h \text{ st. } u = u^J \phi_J$$

$$\begin{aligned} S^i_L u &= S^i(u^J \phi_J) \phi_i = \sum_{i=1}^N \phi_i \int_{\Xi_i} \chi_{i,1} u^J \phi_J \\ &\quad \underbrace{\qquad\qquad\qquad}_{\delta_{ij}} \\ &= u^i \phi_i = u \end{aligned}$$



By construction $\sum S^i_L u = 0 \quad \forall u \in H_0^1(\Omega)$

Properties of Scott-Zhang

$$1 \leq p \leq +\infty \quad \ell \geq 1 \text{ if } p=1 \quad \ell > \frac{1}{p} \text{ otherwise}$$

1) $\forall h, \forall v \in W^{\ell,p}(2)$

$$\|S\mathcal{L}v\|_{e,p,\Omega} \leq \|v\|_{e,p,\Omega}$$

polynomial degree of v stability property

2) $\ell \leq k+1$ $0 \leq m \leq \ell$

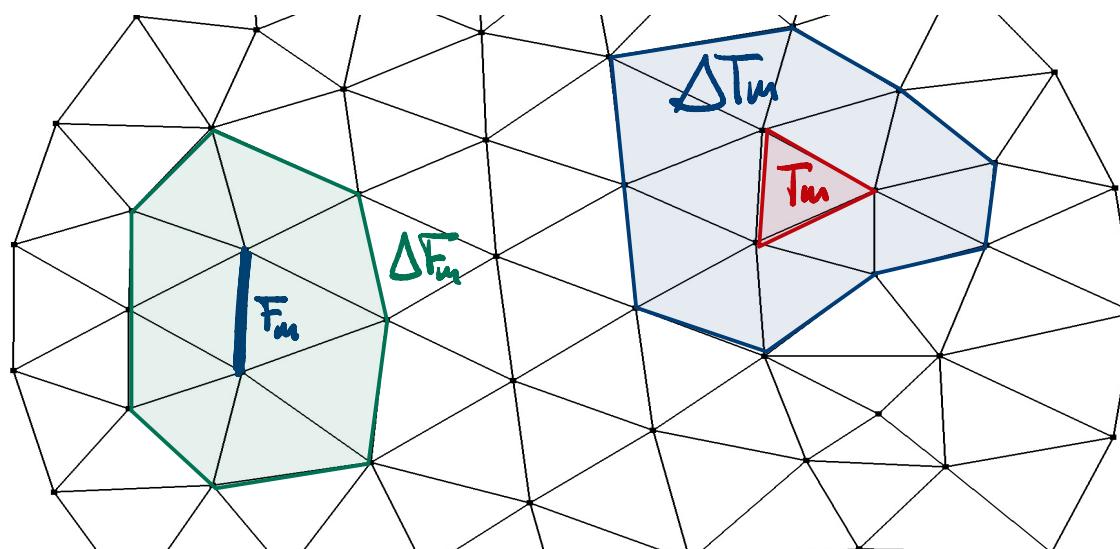
$$\forall h, \forall T \in \mathcal{T}_h, \forall v \in W^{\ell,p}(\Delta T)$$

$$\|v - S\mathcal{L}v\|_{m,p,T} \lesssim h_T^{\ell-m} |v|_{e,p,\Delta T}$$

3) $\|v - S\mathcal{L}v\|_{m,p,F} \lesssim h_F^{\ell-m} |v|_{e,p,\Delta F}$

$$\Delta T_m: \left(\overline{\bigcup_j T_j} \right) \text{ s.t. } \exists i,j,k \text{ s.t. } a_{mi} = a_{jk}$$

$$\Delta F_m: \left(\overline{\bigcup_j T_j} \right) \text{ s.t. } \exists i,j,k \text{ s.t. } a_{mi} = a_{jk}$$



Orthonormal projections

$$\Pi^{\circ} : L^2(\Omega) \longrightarrow V_h \cap H^1(\Omega) \quad \leftarrow L^2 \text{ projection}$$

$$\Pi^1 : H^1(\Omega) \longrightarrow V_h \cap H^1(\Omega) \quad \begin{matrix} \leftarrow \text{Riesz projection} \\ \text{Elliptic projection} \end{matrix}$$

$$(\Pi^{\circ} u, v)_o = (u, v)_o \quad \forall v \in V_h$$

$$(\Pi^1 u, v)_1 = (u, v)_1 \quad \forall v \in V_h$$

$$(u, v)_o := \int_{\Omega} u v \, d\Omega$$

$$(u, v)_1 := \int_{\Omega} (uv + \nabla u \cdot \nabla v) \, d\Omega$$

$$M_{ij} u^J = U_i^o \quad U_i^o := (u, v)_o$$

$$M_{ij} := (v_j, v_i)_o$$

$$K_{ij} u^J = U_i^1 \quad U_i^1 := (u, v_i)_1$$

$$K_{ij} := (v_j, v_i)_1$$

$$M^{iJ} M_{JK} = S_K^i$$

↑ inverse matrix

$$u^J = M^{Jk} U_k^o$$

$$u = K^{jk} U_k^o$$

$$\Pi^{\circ} u := M^{J_k}(u, v_k) \circ \nu_j$$

$$\Pi^1 u := K^{J_k}(u, v_k)_1 \nu_j$$

$$\|\Pi^{\circ} v\|_{0,\Omega} \lesssim \|v\|_{0,\Omega} \quad \forall v \in L^2(\Omega)$$

$$\|\Pi^1 v\|_{1,\Omega} \lesssim \|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega)$$

If Σ is quasi uniform:

$$\|\Pi^{\circ} v\|_{1,\Omega} \lesssim \|v\|_{1,\Omega} \quad \text{is also stable}$$

For "smooth" functions: $K \geq 1 \quad 1 \leq e \leq k$

$$\|v - \Pi^{\circ} v\|_{0,\Omega} \lesssim h^{e+1} |v|_{e+1,\Omega}$$

$$\|v - \Pi^1 v\|_{1,\Omega} \lesssim h^e |v|_{e+1,\Omega}$$

If Σ is quasi uniform

$$\|v - \Pi^{\circ} v\|_{1,\Omega} \lesssim h^e |v|_{e+1,\Omega}$$

A posteriori error analysis

$$\begin{cases} \Delta u = f \text{ with} \\ u=0 \text{ on } \partial\Omega \end{cases}$$

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h \subset H_0^1(\Omega)$$

Instead of exploiting coercivity, we exploit inf-sup:

$$H_0^1(\Omega) \quad H^{-1}(\Omega)$$

$$\langle A_h v, v \rangle = \langle f, v \rangle \quad A: V \rightarrow V^*$$

inf-sup: A is bounded: $\|A u\|_{V^*} \geq \alpha \|u\|_V$

Ideally we'd like to be able to build e_T s.t.

$$e_T(u_h, f) \lesssim \|u - u_h\|_{1,T} \lesssim e_T(u_h, f)$$

Then you could ask FALSE IN GENERAL

$$1) \left(\sum_{T \in \mathcal{T}_h} e_T^2 \right)^{\frac{1}{2}} \leq \text{tol}$$

2) e_T is balanced over all $T \in \mathcal{T}_h$

$$\frac{\text{tol}}{M} \leq e_T \leq \frac{\text{tol}}{M} \quad M: \text{number of elements}$$

Easy part: Global upper bound

$$\sum_T \|u - u_h\|_{1,T}^2 \lesssim \left(\sum_T e_h^2 \right)$$

Difficult part: "Almost local" lower bound

$$e_T(u_e, f) \lesssim \|u - u_e\|_{1, \Delta T} + \Pi(h_T, \Delta T, f)$$

ΔT : patch of all elements around T

Π : perturbation negligible or same order
of error $\|u - u_e\|$

Easy part

$$\|u - u_e\|_{1, \Omega} \lesssim \frac{\|A\|}{\alpha} \inf_{v \in V_h} \|u - v\|_1 \lesssim \left(\sum_{T \in \mathcal{T}_h} h_T^{2K} \|u\|_{k+1, T}^2 \right)^{\frac{1}{2}}$$

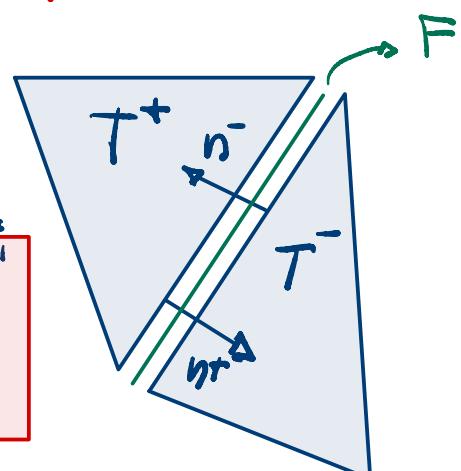
Simplest a posteriori estimate

$$\alpha \|u - u_e\|_{1, \Omega} \lesssim \|A(u - u_e)\|_{-1, \Omega} \lesssim \|f + \Delta u_e\|_{-1, \Omega}$$

issue: $\|\cdot\|_{-1, \Omega}$ is NOT a local norm

$$[\![a]\!]_F := a^+ n^+ + a^- n^-$$

$$\forall F \in \bigcup_{T \in \mathcal{T}_h} \partial T \setminus \partial \Omega := \Sigma$$



- Exploit definition of $\|\cdot\|_{-1, \Omega}$
- • the orthogonality of error

• Estimate results using S_L int. properties.

(1)

$$\|u - u_h\|_{1, \Omega} \lesssim \frac{1}{2} \|A(u - u_h)\|_{-1, \Omega} = \frac{1}{2} \sup_{v \in V} \frac{|\langle A(u - u_h), v \rangle|}{\|v\|_{1, \Omega}}$$

$$= \frac{1}{2} \sup_{v \in V} \frac{|\langle A(u - u_h), v - v_h \rangle|}{\|v\|_{1, \Omega}} \quad \forall v_h \in V$$

$$\langle A(u - u_h), v - v_h \rangle = \int_{\Omega} D(u - u_h) D(v - v_h) = \int_{\Omega} -\Delta u (v - v_h) - \int_{\Omega} \nabla u \cdot \nabla (v - v_h)$$

$$\lesssim \sum_{T \in \mathcal{T}_h} \left(\int_T (f + \Delta u_h)(v - v_h) - \int_T \nabla u_h \cdot (v - v_h) \right)$$

$$\lesssim \sum_{T \in \mathcal{T}_h} \left(\|f + \Delta u_h\|_{0,T} \|v - v_h\|_{0,T} + \underbrace{\frac{1}{FC\partial T \setminus \partial \Omega} \frac{1}{2} \|[\nabla u]\|_{0,F} \|v - v_h\|_F}_{\text{Trace inequality}} \right)$$

* S_L properties.

$$\|v - S_L^1 v\|_{0,T} \lesssim h_T^{-1} |v|_{1, \Delta T}$$

$$\|v - S_L^1 v\|_{0,F} \lesssim h_F^{-\frac{1}{2}} |v|_{\frac{1}{2}, \Delta F} \lesssim h_F^{-\frac{1}{2}} |v|_{1, \Delta T}$$

$$|\langle A(u - u_h), v - S_L^1 v \rangle| \lesssim \sum_{T \in \mathcal{T}_h} \left[\left(\|f + \Delta u_h\|_{0,T} h_T + \sum_F \frac{1}{2} h_F^{\frac{1}{2}} \|[\nabla u]\|_{0,F} \right) \|v\|_{1, \Delta T} \right]$$

$\tilde{N} := \max$ number of neighbours of T

$$\sum_{T \in \mathcal{Z}_0} \|v\|_{1,\Delta T}^2 \lesssim \tilde{N} \|v\|_{1,R}^2$$

$$(\sum (a_i \cdot b_i))^2 \lesssim (\sum a_i^2) (\sum b_i^2)$$



$$\|\Lambda(u - u_\alpha)\|_{-1,R} = \|\mathbf{f} + \Delta u_\alpha\|_{1,R} \lesssim \tilde{N} \left(\sum_{T \in \mathcal{Z}_0} e_T^2 \right)^{\frac{1}{2}}$$

$$e_T := \|\mathbf{f} + \Delta u_\alpha\|_{\partial T} h_T + \frac{1}{2} \sum_{F \subset \partial T \setminus \partial R} h_F^{\frac{1}{2}} \|[D u_\alpha]\|_F$$

$$\|u - u_\alpha\|_{1,R} \lesssim \frac{1}{\alpha} \left(\sum_T e_T^2 \right)^{\frac{1}{2}}$$