

# Metodi Matematici per Equazioni alle Derivate Parziali

Transformations of local Finite Element Spaces

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## Short recap

Local finite element space: triple  $\{T_m, P_{T_m}, \Sigma_{T_m}\}$ .

- $T_m$ : simple geometrical domain (cell)  $\Omega = \bigcup_{m=1}^M T_m$
- $P_{T_m}$ : finite dimensional space on  $T_m$  ( $\dim(P_{T_m}) = N_{T_m}$ )
- $\Sigma_{T_m}$ : basis for  $(P_{T_m})'$  (nodal basis  $\Sigma_T := \{v_m^i\}_{i=1}^{N_{T_m}}$ )

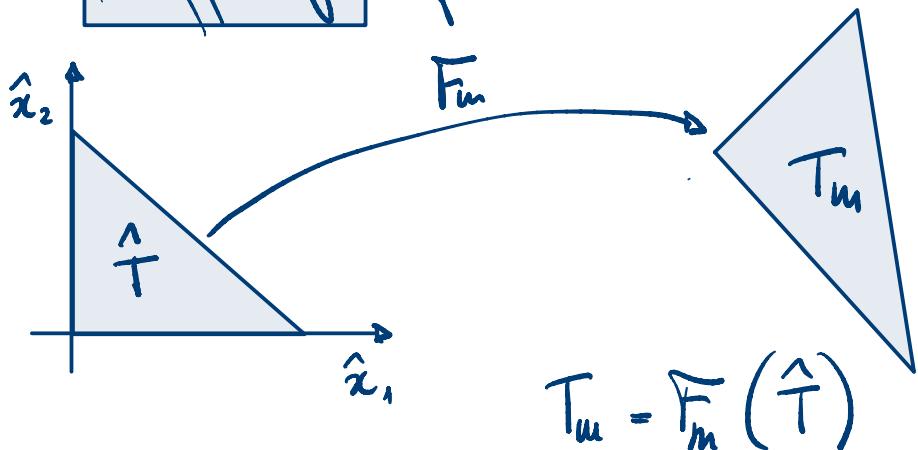
$$P_{T_m} = \text{span } \{v_{mi}\}_{i=1}^{N_T} \quad \text{where } v_{mi} \in P_{T_m}, v^j(v_i) = \delta_{ij}$$

1) Define triple  $\{\hat{T}, P_{\hat{T}} = P^k(\hat{T}), \Sigma_{\hat{T}}\}$

Finite Element

2) Define, for each  $T_m \in \mathcal{T}_h$  ( $\mathcal{T}_h = \overline{\bigcup_{m=1}^H T_m}$ )

**INVERTIBLE**  
a Mapping \* from  $\hat{T}$  to  $T_m$   $\mathcal{T}_h \equiv \Omega$



let's consider  
Affine Mapping  
for the theory

Define  $P_m$  as  $\bigtimes_{\hat{T}_m} P_{\hat{T}} = P^k(\hat{T})$ ,  $\Sigma_m := \bigtimes_{\hat{T}_m} \Sigma_{\hat{T}}$

$$P_m := \{v_{mi}\}_{i=1}^{N_{T_m}} = \left\{ \hat{v}_i \circ \bar{T}_m^{-1} \right\}$$

3) Make sure that each local dof maps to a unique global dof s.t. the resulting  $\mathcal{V}_h = \text{span} \{ v_i \}_{i=1}^N$  is conforming:  $\mathcal{V}_h \subset \mathcal{V}$

Exploit this in the assembly of linear systems

3)  $\rightarrow \nexists T_m, \exists$  local to global numbering  
s.t.  $\hat{v}_i \circ T_m^{-1} = v_{I_m}$

$$I_m \in (0, N) \text{ and}$$

$\{v_I\}_{I=0}^{N-1}$  is a basis for  $\mathcal{V}_h$

$$I_m = \ell_2 g_m$$

DOFHandler  
is responsible  
for the numbering

How to use all of this?

$$a(v_j, v_I) := A_{IJ}$$

$$\langle f, v_I \rangle := F_I$$

$$a(u^T v_j, v_I) = \langle f, v_I \rangle$$

$$\Rightarrow \underbrace{A_u = F}_{\Leftrightarrow} \quad \underbrace{a(u^T v_j, v_I) = \langle f, v_I \rangle}_{\Leftrightarrow}$$

$$\text{For } -\Delta u = f \Rightarrow a(v_j, v_I) := \int_{\Omega} \nabla v_j \cdot \nabla v_I$$

$$F_I = \int_{\Omega} f v_I$$

How to compute  $F_I$  ?

$\Rightarrow$  split  $\int_{\Omega}$   $\Leftrightarrow \sum_{m=1}^M \int_{T_m}$   $d\Gamma_m$

$\hat{J} = \det(DF_m)$

$$\int_{T_m} f v_I m_i d\Gamma_m = \int_{\hat{T}} f \circ F_m \hat{v}_i \hat{J} d\hat{T}$$

$$\approx \sum_{q=1}^{N_q} (f \circ F_m)(\hat{x}_q) \hat{v}_i(\hat{x}_q) \hat{J}(\hat{x}_q) w_q$$

$f(F_m(\hat{x}_q))$

$$f_{mi} = \sum_{i=1}^{N_m} \sum_{q=1}^{N_q} f(x_{mq}) \hat{v}_i(\hat{x}_q) J_x W_{mq}$$

$$F = 0 \rightarrow F(I_{mi}) += f_{mi}$$

$$F_i = \int_{\Omega} v_I f = \int_{\text{supp}(v_I)} v_I f$$

FOR EACH CELL  $T_m$

- COMPUTE LOCAL INTEGRALS ON  $T_m$

- STORE RESULTS ON LOCAL RHS, and MATRIX

$$\begin{array}{ll} \cdot & f_{m,i} \\ \cdot & a_{m,ij} \end{array} \quad \begin{array}{l} \in \mathbb{R}^{N_{T_m}} \\ \in \mathbb{R}^{N_{T_m} \times N_{T_m}} \end{array}$$

- Distribute Local integrals to global matrices :

For  $i = [0, \dots, N_{T_m}]$

For  $j = [0, \dots, N_{T_m}]$

$$A_{I_m i, I_m j} = A_{I_m i, I_m j} + a_{m,ij}$$

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$$F_{I_m i} = F_{I_m i} + f_{m,i}$$

How about  $a_{m,ij}$  ?

$$a_{m,ij} := \int_{T_m} \nabla(\hat{v}_j \circ F_m^{-1}) \cdot \nabla(\hat{v}_i \circ F_m^{-1}) d\Omega_m$$

$$= \int_T \underbrace{\hat{D}F_m^{-T} \hat{\nabla} \hat{v}_j}_{\nabla \hat{v}_j \circ F_m^{-1}} \cdot \underbrace{\hat{D}F_m^{-T} \hat{\nabla} \hat{v}_i}_{\nabla \hat{v}_i \circ F_m^{-1}} d\Omega$$

$$(\nabla v)_i := \frac{\partial}{\partial x_i} (\hat{v} \circ F_m^{-1}) = \left[ \left( \frac{\partial}{\partial \hat{x}_j} \hat{v} \right) \circ F_m^{-1} \right] \frac{\partial F_m^{-1}}{\partial \hat{x}_j}$$

$$\nabla r = \underbrace{DF_m}_{-T} \circ \hat{\nabla} \rightarrow \text{does not depend on } m$$

depends on  $m$  ( $F_m : \hat{T} \rightarrow T_m$ )

only need evaluation  $\star$  on quadrature points

Given : 1) Quadrature formula

2) Finite Element  $\Rightarrow \hat{v}_i, \hat{D}\hat{v}_i \quad i=1, \dots, N_f$

3) Mapping  $\star : F_m : \hat{T} \rightarrow T$

We compute FE Values  $\star$  on quadrature points.

1) values  $\hat{v}_i(\hat{x}_q)$

2) quadrature points  $x_{mq} = F_m(\hat{x}_q)$

3)  $J \times W$   $J_{\times}W_{mq} := \det(DF_m) \cdot w_q$

3) gradients  $(\nabla r)(F_m(\hat{x}_q)) =$   
 $DF_m^{-T} \hat{D}\hat{v}(\hat{x}_q)$

With these three, we can assemble

$$f_{m,i} := \sum_j f(F_m(\hat{x}_j)) J \times W_{mj}$$

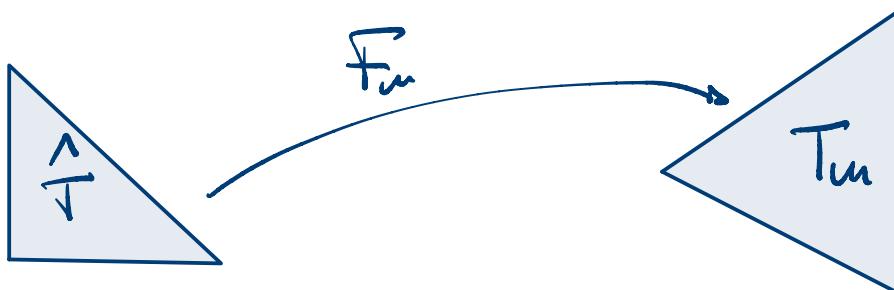
$$\alpha_{m,ij} := \sum_j \underbrace{DF_m^{-T}(\hat{x}_j) (\hat{\nabla}_{V_j})(\hat{x}_j)}_{\nabla_{V_j}|_{F_m(\hat{x}_j)}} \cdot \underbrace{DF_m^{-T}(\hat{x}_j) \hat{\nabla}_{V_i}(\hat{x}_j)}_{\nabla_{V_i}|_{F_m(\hat{x}_j)}} J \times W_{mj}$$

$$\nabla_{V_j}|_{F_m(\hat{x}_j)} \quad \nabla_{V_i}|_{F_m(\hat{x}_j)}$$

$$\alpha_{m,ij} \simeq \int_{T_m} \nabla_{V_{I_{mi}}} \nabla_{V_{I_{mj}}} dT_m$$

$\Rightarrow$  Distribute to  $A_{I_{mi} I_{mj}}$

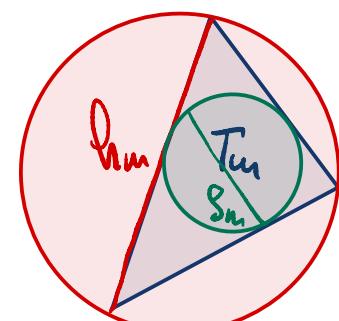
## AFFINE MAPPINGS



$$F_m = B\hat{x} + b \quad B \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d$$

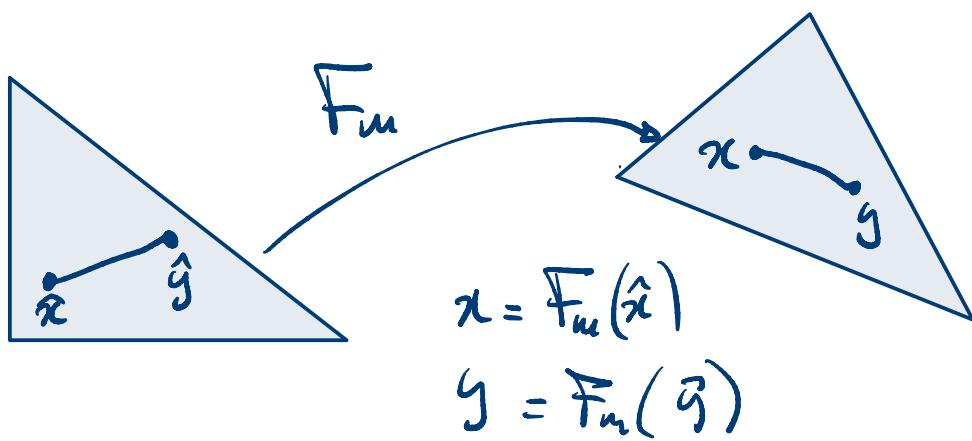
$$h_m := \max_{x, y \in T_m} |x - y|$$

$$S_m := \sup_{B_x \subseteq T_m} \operatorname{diam}(B_x)$$



$$\|B\| := \sup_{|\xi|=\hat{s}} \frac{|B\xi|}{|\xi|} \leq \frac{h_m}{\hat{s}}$$

$$\vec{z} := \hat{x} - \hat{y} \quad F_m(\hat{x}) - F_m(\hat{y})$$



$$F_m(\hat{x}) - F_m(\hat{y}) = B(\hat{x} - \hat{y}) \leq h_m$$

In the opposite direction :

$$\|B^{-1}\| := \sup_{|x-y|=S_m} \frac{|B^{-1}(x-y)|}{|x-y|} \leq \frac{\hat{h}}{\hat{S}_m} = \hat{h} S_m^{-1}$$

$$\Rightarrow \|B\| \leq c_1 h_m \quad \|B^{-1}\| \leq c_2 S_m^{-1}$$

Let's see how  $F_m$  transforms Sobolev norm:

$$\|v\|_{k,p,T}^p := \int_T |D^k v|^p dT$$

$k$  is a multi-index in  $\mathbb{N}_0^d$

$$\hat{D}^k(v \circ F_m) = [(D^k_v) \circ F_m] (\underbrace{DF_m}_{B})^{k-1}$$

$$|v|_{k,p,T}^p := \int_T (D^k v) B^{-k} B^{-1} )^p J J^{-1} d\Gamma$$

$$\leq \|B^{-1}\|^{kp} \int_T (\hat{D}^k \hat{v})^p d\hat{\Gamma}$$

$$|v|_{k,p,T} \leq C \|\underbrace{B^{-1}}_{g^{-k}}\|^{kp} \int_T |v \circ F_m|_{k,p,\hat{\Gamma}}$$

$$|v \circ F_m|_{k,p,\hat{\Gamma}} \leq C \|\underbrace{B^{-1}}_{D_m^{-k}}\|^{kp} |v|_{k,p,T}$$

For  $v \in P^e(T)$  for  $0 \leq k, s \leq e$

$\exists c_1, c_2$  st.

$$c_1 |v \circ F_m|_{k,p,\hat{\Gamma}} \leq |v \circ F_m|_{s,p,\hat{\Gamma}} \leq c_2 |v \circ F_m|_{k,p,\hat{\Gamma}}$$

$$\|v\|_{k,p,T_m} \leq C g_m^{-k} h_m^s \|v\|_{s,p,T_m}$$