Metodi Matematici per Equazioni alle Derivate Parziali

Introduction

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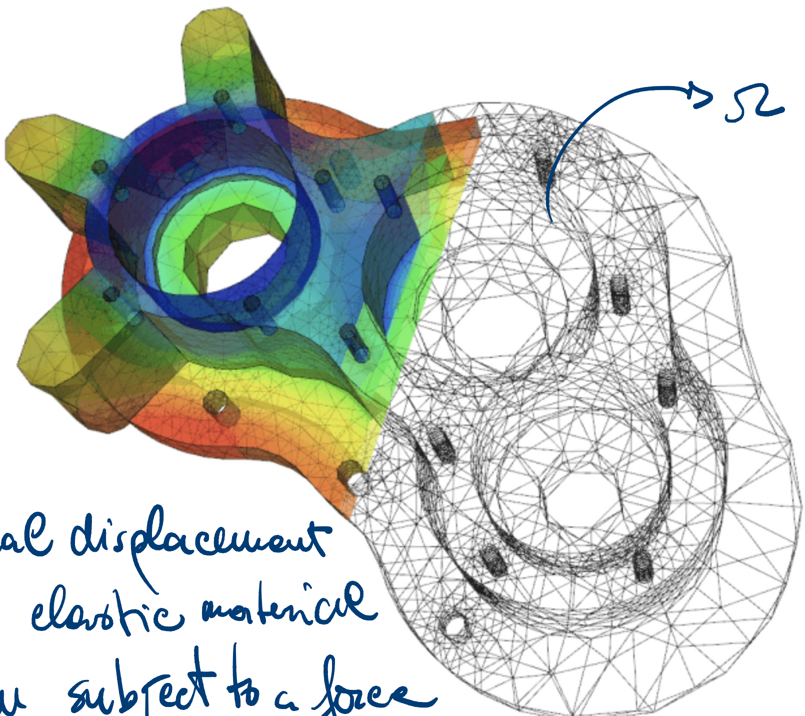
$\Omega \subseteq \mathbb{R}^d$

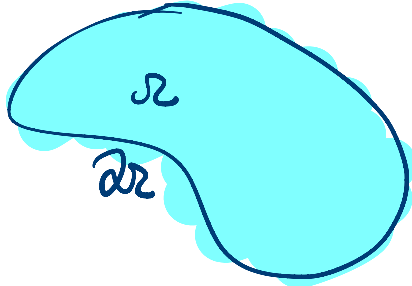


u : vertical displacement of a elastic material when subject to a force

f : data

Prototypical problem :

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



Ω is Lipschitz, bounded subset of $\mathbb{R}^d$

$d = 1, 2, 3$ $(, 4, 5, 6)$

Is formulated in **Weak form**

"$-\Delta$" operator is seen as a weak derivative operator on a Hilbert space

Classical: $-\Delta u := -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$

"$-\Delta$" is interpreted weakly :

"$-\Delta$" : $A : \begin{array}{ccc} V & \longrightarrow & V' \\ u & \longrightarrow & Au \end{array}$

V is Hilbert

$Au = F$ in V'

$\iff$ same thing as

$\langle Au, v \rangle = \langle F, v \rangle \quad \forall v \in V$

$\langle Au, v \rangle := \int_\Omega Au \, v$

$\underline{\underline{Au}}(v) \in \mathbb{R}$

$\Downarrow$

Au is linear functional from $V \longrightarrow \mathbb{R}$

$$\langle Au, v\rangle := \int_\Omega \nabla u \nabla v \, d\Omega$$

$$-\Delta u = -\mathrm{div}(\nabla u)$$

$$\int_\Omega -\Delta u, v \, d\Omega = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\Omega$$

Typical procedure in FEM:

- Replace the space V (infinite dimensional, Hilbert) with V_h, finite dimensional, and $V_h \subset V$

$$Au = F \quad \text{in } V'$$

$$\boxed{Au_h = F \quad \text{in } V_h'}$$

$$\Updownarrow$$

$$\langle Au_h, v_h\rangle = \langle F, v_h\rangle \quad \forall v_h \in V_h$$

$$\langle A v_J u^J, v_i\rangle = \langle F, v_i\rangle \quad i = 1 \dots n$$

$$V_h = \mathrm{span}\{v_i\}_{i=1}^n$$

$$\forall u_h \quad \exists! \{u^i\}_{i=1}^n \in \mathbb{R}^n$$

s.t. $u_h(x) = \underline{u^i v_i(x)}$

sum is implied when repeating indices in "opposite" positions!

$$A \underline{u} = \mathbb{F}$$

$$\boxed{A_{ij} u^j = \mathbb{F}_i}$$

linear system of equations.

$$A_{ij} := \langle A v_j, v_i \rangle$$

$$\mathbb{F}_i := \langle F, v_i \rangle$$

LAX-MILGRAM LEMMA

Assume V is Hilbert with norm $\|\cdot\|$ and assume that a is a bilinear operator on V : $a : V \times V \longrightarrow \mathbb{R}$

$$a(u,v) := \langle Au, v \rangle$$

and assume that there exist $\alpha \in \mathbb{R}$, $\alpha > 0$ s.t.

→ bounded $\quad a(u,v) \leq \|A\| \, \|u\| \, \|v\| \quad \forall u,v \in V$

→ coercive $\quad a(u,u) \geq \alpha \|u\|^2$

Then $\forall F \in V'$, $\exists!\, u \in V$ s.t.

$$\langle Au, v \rangle = a(u,v) = \langle F, v \rangle = F(v) \quad \forall v \in V$$

$$\|u\| \leq \frac{1}{\alpha} \|F\|_*$$

$$\|F\|_* := \sup_{0\neq v\in V} \frac{|F(v)|}{\|v\|} = \sup_{0\neq v\in V} \frac{|\langle F, v\rangle|}{\|v\|}$$

Intermediate result: Rietz representation theorem

$\forall \; F \in V' \quad \exists!\; f \in V$ s.t.

$$(f, v)_V = \langle F, v\rangle \quad \forall v \in V$$

$$\|f\| = \|F\|_*$$

We call $\tau : V' \longrightarrow V$ the Rietz operator that maps Elements of V' to their representation in V.

Useful to define a fixed point problem in V.

$T_\rho(u) = u \iff u$ is solution to

$T_\rho : V \longrightarrow V$

$Au = F$ in V'

$\langle Au, v\rangle = \langle F, v\rangle \quad \forall v \in V$

$\in V' \qquad \in V$

$$\underbrace{-\rho\tau\big(Au - F\big) + u}_{T_\rho(u) := u - \rho\tau(Au - F)} = u$$

By construction if u is s.t. $T_\rho(u) = u$ then $\underline{Au = F}$

Search for ρ s.t. T_ρ is a contraction $(\exists L < 1)$

s.t.

$$\| T_\rho(u_1) - T_\rho(u_2) \| \leq L \| u_1 - u_2 \|$$

$\Rightarrow$ start from u_0 arbitrary, and define

$$u_{k+1} = T_\rho(u_k)$$

$$\| u_{k+1} - u_k \| = \| T_\rho(u_k) - T_\rho(u_{k-1}) \| \leq L \| u_k - u_{k-1} \|$$

$$\| u_{k+1} - u_k \| \leq L^{k-1} \| T_\rho(u_0) - u_0 \|$$

then $\exists\, u$ s.t. $T_\rho(u) = u$

It is unique. If not then

$$\| \tilde{u} - u \| \leq \| T(\tilde{u}) - T(u) \| \leq L \| \tilde{u} - u \|$$

contradiction

$$\| \tilde{u} - u \| < \| \tilde{u} - u \|$$

$\Rightarrow$ u is unique

$$\|T_\rho(u_1) - T_\rho(u_2)\|^2 = \|(u_1-u_2) - \rho\tau A(u_1-u_2)\|^2$$

$$\|u_1-u_2\|^2 + \rho^2\|\tau A(u_1-u_2)\|^2 - 2\rho(\tau A(u_1-u_2),(u_1-u_2))$$

$$\leq L^2\|u_1-u_2\|^2 \qquad \text{with } L<1$$

$$-2\rho(\tau A(u_1-u_2),(u_1-u_2)) = -2\rho\underbrace{\langle A(u_1-u_2),(u_1-u_2)\rangle}_{\geq\alpha\|u_1-u_2\|^2}$$

$$\leq \|u_1-u_2\|^2 + \rho^2\underbrace{\|A\|_*\,\|(u_1-u_2)\|^2}_{\text{boundedness}} - \underbrace{2\rho\alpha\,\|u_1-u_2\|^2}_{\text{coercivity}}$$

$$L^2$$

$$\|T_\rho(u_1) - T_\rho(u_2)\|^2 \leq \boxed{\left(1+\rho^2\|A\|_* - 2\rho\alpha\right)}\|u_1-u_2\|^2$$

$$L<1 \Rightarrow \quad -1 \leq 1+\rho^2\|A\|_* - 2\rho\alpha \leq 1$$

$\rightarrow$ if $\rho < \dfrac{2\alpha}{\|A\|_*}$ then $L<1 \Rightarrow \exists!$ sol. to fixed point.

$$\alpha\|u\|^2 \leq a(u,u) = \langle F, u\rangle \leq \|F\|_*\,\|u\|$$

$$\Rightarrow \boxed{\|u\| \leq \frac{1}{\alpha}\|F\|_*}$$

$\boxed{\text{Galerkin Method}}$ Given $\underline{V_h \subset V}$

Find u_h s.t.

$$\langle A u_h, v_h \rangle = \langle F, v_h \rangle \quad \forall v_h \in V_h$$

$$\langle A u, v_h \rangle = \langle F, v_h \rangle \quad \forall v_h \in V_h \subset V$$

$$\boxed{\langle \underbrace{A(u-u_h)}, v_h \rangle = 0 \qquad \forall v_h \in V_h}$$

the error is A-orthogonal to V_h

$\boxed{\text{Cea's lemma}}$

$\langle A(u-u_h), u_h \rangle = 0$

$$\alpha \| u - u_h \|^{\cancel{2}} \leq \langle A(u-u_h), (u-\underbrace{u_h}_{=0}) \rangle$$

$$\leq \langle A(u-u_h), (u - \underbrace{v_h}_{=0}) \rangle$$

$$\leq \|A\|_* \cancel{\|u-u_h\|} \, \|u - v_h\| \qquad \forall v_h \in V_h$$

$$\|u - u_h\| \leq \frac{\|A\|_*}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$$

Main idea and goal of the course:

- construct V_h s.t. → A-priori (Worst case)

→ $\text{dist}(V, V_h) := \sup_{u \in V} \inf_{v_h \in V_h} \|u - v_h\|$ is "small" (typically of order h^k)

→ ◦ measure $\text{dist}(V, V_h)$, estimate $\|u - u_h\|$

→ ◦ given F, u, find the best possible V_h → A-posteriori p.b.

---- How to do this in 1D ----

$\Omega = (0,1)$, $\partial\Omega := \{0, 1\}$

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Weak form ⇝ $\int_0^1 u'v' = \int_0^1 fv \quad \forall v \in H_0^1([0,1])$

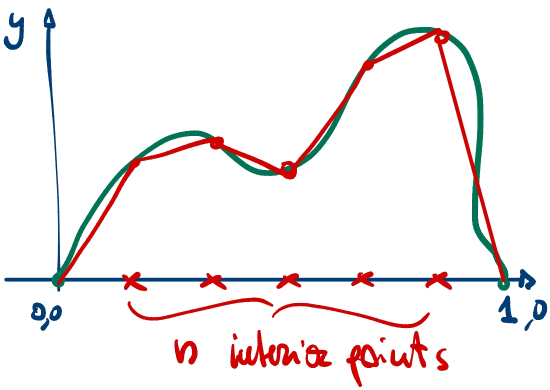
$$H_0^1([0,1]) := \{ v \in L^2([0,1]) \text{ s.t. } v' \in L^2([0,1]), \; v(0) = v(1) = 0 \}$$

$H^s \subset C^0 \quad s > \frac{d}{2}$

$$H_0^1([0,1]) \subset C^0([0,1])$$

How to define V_h ? $\qquad h = \left(\frac{b-a}{n+1}\right) = \frac{1}{n+1}$

$x_i := ih \qquad i = 1, \dots, n$



u_h , u

$$u_h = \sum_{i=1}^{n} u^i v_i$$

$$V_h \subset V \equiv H^1_{\underline{\underline{0}}}([0,1])$$

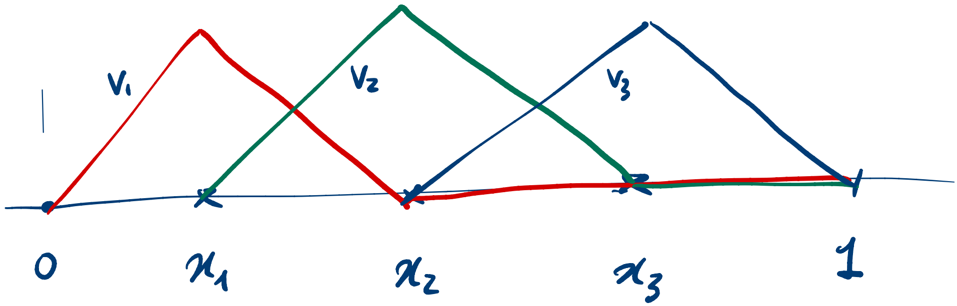
$$V_h = \{ v \in C^0([0,1]) \, , \; v|_{(x_i, x_{i+1})} \in \mathbb{P}^1(x_i, x_{i+1}) \text{ for } i = 0, \dots, n \, , \; v(0) = v(1) = 0 \}$$

We know the dimension of V_h $(\equiv n)$.

- Choose n functionals in V_h', called nodal functionals, s.t. they are linearly independent. (choose a base for V_h'), call them $v^i \in V_h'$
- construct v_i such that $v_i \in V_h$, $v^j(v_i) = \langle v^j, v_i \rangle = \delta^j_{\,i} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- Simplest choice: $v^i(x) := \delta(x - x_i) \quad i = 1, \dots, n$

$$v^i(u) = u(x_i) = \int_0^1 \delta(x - x_i)\, u(x)\, dx$$

- $v_i(x)$ are hat functions:



$$I : V \longrightarrow V_h$$

$$u \longrightarrow v^i(u)\, v_i = \sum_{i=1}^{n} u(x_i)\, v_i$$