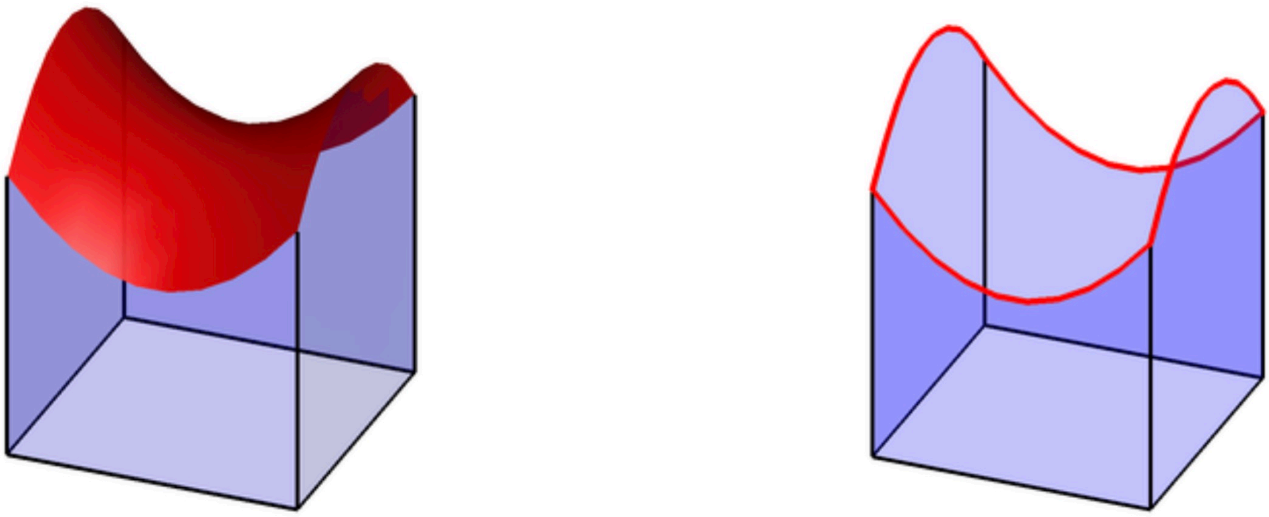

Metodi Matematici per Equazioni alle Derivate Parziali

Nitsche's trick, trace, and inverse estimates

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Can we estimate $u-u_h$ in a weaker norm, say, $\|u-u_h\|_{0,\Omega}$ instead of $\|u-u_h\|_{1,\Omega}$?

For $\|u-\Pi u\|$, yes (Bramble-Hilbert, Denis Lions, scaling argument) but how about $\|u-u_h\|_0$? Remember: u_h is solution of FE —

$\forall f \in V^*$ $V \equiv H^1_0(\Omega)$

$Au = f$ in V^* admits unique sol if A is coercive (α)

$Au_h = f$ in V_h^* $V_h \subset V$

$\Downarrow$

$$\|u-u_h\|_V \lesssim \frac{\|A\|_*}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|$$

$$\| u - u_h \|_1 \lesssim \frac{\|A\|_*}{\alpha} \| u - I_h u \|_{1,\Omega}$$

$$\lesssim \frac{\|A\|_*}{\alpha} h^k \; |u|_{k+1,\Omega}$$

$$h := \max_m \{ h_m \}$$

$\exists \sigma > 0 \,|\, \rho_m \geq \sigma h_m \quad \forall m$ quasi uniform grid

Interpolation error for V_h:

$$I_h : V \longrightarrow V_h \qquad V_h = \operatorname{span}\{v_i\}_{i=1}^N$$

$$u \longrightarrow \tilde{v}^i(u)\, v_i \qquad v^j(v_i) = \delta^j_i$$

$\tilde{v}^i$ is H.B. extension of v^i on V^*

For $u \in H^{k+1}(\Omega)$, $\quad k \geq 1 \quad d = 1,2,3$

$$H^{k+1}(\Omega) \subset C^0(\Omega)$$

more regular space: $\tilde{v}^i \equiv v^i$ $\qquad 0 \leq s \leq \ell \leq k+1$

$$\left(\sum_m \| u - I_h u \|_{s,T_m} \right) \lesssim h^{\ell - s} \; |u|_{\ell,\Omega}$$

$$\| u - u_h \|_1 \lesssim h \; |u|_{2,\Omega}$$

Assume $V \hookrightarrow H$ we identify $H^* \equiv H$

1) $\|u\|_H \lesssim \|u\|_V \quad \forall u \in V$

2) $V = \overline{H}^{\|\cdot\|_V}$

For $f \in H$ we write $\tilde{f}$ Hahn-Banach extension of f to V^*

$$\langle \tilde{f}, v \rangle = (f, v)_H \qquad \forall v \in V$$

$$|\langle \tilde{f}, v\rangle| = |(f,v)|_H \lesssim \|f\|_H \|v\|_H \lesssim \|f\| \, \|v\|_V$$

$\tilde{f}$ is injective

Given $f \in H$ find $u \in V$ s.t.

$$\langle Au, v\rangle = \langle \tilde{f}, v\rangle = (f, v)_H \quad \forall v \in V \qquad Au = \tilde{f}$$

Observation: given $g \in H$ find $\varphi_g \in V$ s.t.

$$\langle Av, \varphi_g \rangle = (g, v)_H \quad \forall v \in V \qquad A^T \varphi_g = g$$

$\forall g \in H, \ \exists! \ \varphi_g \in V$ s.t. $A^T \varphi_g = g$

Theorem (Nitsche's Lemma)

$$\|u - u_h\|_H \lesssim \|u - u_h\|_V \left(\sup_{g \in H} \inf_{\varphi_h \in V_h} \frac{\|\varphi_g - \varphi_h\|_V}{\|g\|_H} \right)$$

We will show that goes to zero

Proof: $\langle A(u-u_h), v_h \rangle = 0 \quad \forall v_h \in V_h$ Cea's lemma

$$\langle A v, \varphi_g \rangle = (g, v)_H \quad \forall v \in V$$

use as test function $e := u - u_h$

$$\langle A(u-u_h), \varphi_g \rangle = (g, u-u_h)_H \quad \forall v \in V$$

we identify H with its dual, we can also define

$$\|w\|_H := \sup_{g \in H} \frac{(g, w)_H}{\|g\|_H}$$

$$(g, u-u_h)_H = \langle A(u-u_h), \varphi_g - \varphi_h \rangle$$

$$\sup \frac{(g, u-u_h)_H}{\|g\|_H} := \|u-u_h\|_H = \sup_{g \in H} \frac{|\langle A(u-u_h), \varphi_g - \varphi_h \rangle|}{\|g\|_H} \quad \forall \varphi_h \in V_h$$

$$\Rightarrow \|u-u_h\|_H \lesssim \|A\|_* \|u-u_h\|_V \left(\sup_{g \in H} \inf_{\varphi_h \in V_h} \frac{\|\varphi_g - \varphi_h\|_V}{\|g\|_H} \right)$$

Assume:

1) k is the sobolev degree of V $\quad \|\cdot\|_V \sim \|\cdot\|_{k,\Omega}$
2) $s (\leq k)$ is the Sobolev degree of H $\quad \|\cdot\|_H \sim \|\cdot\|_{s,\Omega}$
3) A^T is π-regular for ℓ in $[s, s_1]$

That is $\forall g \in H^{\ell} \quad s \leq \ell \leq s_M$

$\exists!$ sol. φ_g to $A^T \varphi_g = g$ and

$$|\varphi_g|_{\ell+r} \lesssim \|g\|_{\ell}$$

Then $$\inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_V \lesssim \|\varphi_g - I_h \varphi_g\|_{k,\Omega} \lesssim h^{-k+\ell+r} |\varphi_g|_{\ell+r}$$

provided that $\ell + r - k \geq 0$ I can use

$$\|\varphi_g - I_h \varphi_g\|_{k,\Omega} \lesssim h^{\ell+r-k} |\varphi_g|_{\ell+r} \lesssim h^{\ell+r-k} \|g\|_{\ell}$$

$$\ell = s \Rightarrow \quad \|\varphi_g - I_h \varphi_g\| \lesssim h^{\ell+r-k} \|g\|_s$$

$$\|u - u_h\|_H \lesssim \|u - u_h\|_V \, h^{s+r-k}$$

Example: Poisson: A on Ω smooth $\Rightarrow r = 2$

$\|\cdot\|_V = \|\cdot\|_{1,\Omega} \qquad \|\cdot\|_H = \|\cdot\|_{0,\Omega}$

$$\|u - u_h\|_{1,\Omega} \lesssim h^{k} |u|_{k,\Omega}$$

$$\|u - u_h\|_{0,\Omega} \lesssim \|u - u_h\|_{1,\Omega} \, h^{\overset{0}{s} + \overset{2}{r} - \overset{1}{k}}$$ $s + r - k = 1$

$$\boxed{\begin{aligned}\|u-u_h\|_{1,\Omega} &\lesssim h^k\,|u|_{k+1,\Omega}\\ \|u-u_h\|_{0,\Omega} &\lesssim h^{k+1}\,|u|_{k+1,\Omega}\end{aligned}}$$

Inverse Estimates

It's trivial to say that $\|u\|_k \lesssim \|u\|_s$ $s\geq k$ $\forall u \in H^s$

the inverse is possible on $\mathcal{T}_h$ for elements of V_h.

We'll show: Local inverse Estimates

FE: $\hat{T}, \hat{P}, \hat{\Sigma}$, 1) $\ell \geq 0$ s.t. $\hat{P} \subset W^{\ell,\infty}(\hat{T})$

$\exists\, \sigma$ s.t.

$\forall m,\ \rho_m \geq \sigma h_m$

2) $h \leq 1$

3) $T_m = F_m(\hat{T})$, $P_m = \#_{F_m}\hat{P}$, $\Sigma_m = \#_{F_m}\hat{\Sigma}$

$FE_m = T_m, P_m, \Sigma_m$

F_m is affine

$0 \leq k \leq \ell$

$1 \leq p \leq +\infty$

$$\boxed{\|v\|_{\ell,p,T_m} \lesssim h_m^{k-\ell}\,\|v\|_{k,p,T_m}}$$

Proof 1) all finite dim. norm are equivalent.

$\forall\, \hat{v} \in \hat{P}$ $\quad \|\hat{v}\|_{\ell,p,\hat{T}} \lesssim \|\hat{v}\|_{0,p,\hat{T}}$ $\quad$ (1)

$\forall\ v = \hat{v} \circ F_m^{-1} \in P_m$

(1)

(2) $$|v|_{\ell,p,T_m} \lesssim h_m^{-\ell} J_m^{\frac{1}{p}} |\hat{v}|_{\ell,p,\hat{T}} \lesssim h_m^{-\ell} J_m^{\frac{1}{p}} \|\hat{v}\|_{0,p,\hat{T}}$$

$$\lesssim h_m^{-\ell} \|v\|_{0,p,\hat{T}}$$

$h \leq 1 \quad \Rightarrow \quad \|v\|^p_{\ell,p,T_m} = \sum_{|m|\leq \ell} |v|^p_{|m|,p,T_m}$

insert (2) here, explain $h \leq 1$

$$\boxed{\|v\|_{\ell,p,T_m} \lesssim h_m^{-\ell} \|v\|_{0,p,\hat{T}}}$$

$0 \leq k \leq \ell \quad , \quad 0 \leq |\alpha| \leq \ell \quad \alpha$ multiindex

$|\alpha| \leq k \qquad \|D^\alpha v\|_{0,p,T_m} \leq \|v\|_{\ell-k,p,T_m} \lesssim h_m^{k-\ell} \|v\|_{0,p,T_m}$

$$\lesssim h_m^{k-\ell} \|v\|_{k,p,T_m}$$

$\ell - k \leq |\alpha| \leq \ell \qquad \exists\ \beta, \gamma$ s.t. $\alpha = \beta + \gamma \qquad |\beta| = \ell - k$, $|\gamma| \lesssim k$

$$\|D^\alpha\|_{0,p,T_m} = \|D^\beta (D^\gamma)\|_{0,p,T_m} \lesssim \|D^\gamma\|_{\ell-k,p,T_m}$$

$$\lesssim h^{k-\ell} \|D^\gamma v\|_{0,p,T_m} \lesssim h^{k-\ell} \|v\|_{k,p,T_m}$$

If $\mathcal{T}_h$ is quasi uniform, then $\forall v \in V_h$

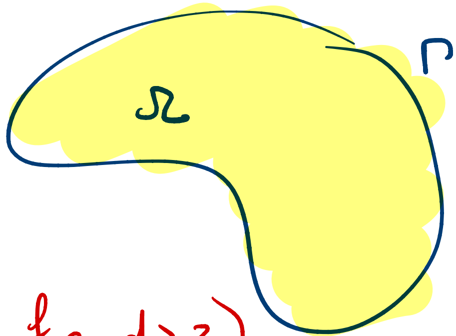
$$\|v\|_{\ell,\Omega} \lesssim h^{r-\ell} \|v\|_{r,\Omega}$$

is global for $0 \leq r \leq \ell \leq k$

When $V_h \subset H^k(\Omega)$

Trace Spaces and Trace inequality

$u \in C^0(\overline{\Omega})$ then $u|_\Gamma$ makes sense point-wise.



What if $u \in H^k \not\subset C^0(\Omega)$

$(H^1(\Omega)$ for $d \geq 2)$

Trace theorem, for $s \in (\frac{1}{2}, 1]$, Ω Lip, $\exists!$ linear bounded mapping $\gamma : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma)$ s.t.

① $$\boxed{\|\gamma v\|_{s-\frac{1}{2},\Gamma} \lesssim \|v\|_{s,\Omega}}$$

and such that $\forall u \in C^0(\overline{\Omega})$ $\gamma u = u|_\Gamma$ pointwise restriction

② γ admits a bounded right inverse E

$$E : H^{s-\frac{1}{2}}(\Gamma) \longrightarrow H^s(\Omega)$$

$$\boxed{\gamma E v = v} \qquad \forall v \in H^{s-\frac{1}{2}}(\Gamma)$$

Fractional Spaces

1) $\Omega \equiv \mathbb{R}^d$ Fourier Transform:

$$(\mathcal{F} v)(\xi) := \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp(-i\,\xi \cdot x)\, v(x)\, dx$$

$$\| v \|_{s,\mathbb{R}^d} := \left\| \left|(1+|\cdot|)^2\right|^{s/2} \hat{v}(\cdot) \right\|_{0,\mathbb{R}^d}$$

$$H^s(\mathbb{R}^d) = \overline{C_c^\infty}^{\|\cdot\|_{s,\mathbb{R}^d}}$$

2) Ω Lip. $s = m + \lambda$ $m \in \mathbb{N}_0$, $\lambda \in (0,1)$

$$\| u \|^2_{s,\Omega} := \| u \|^2_{m,\Omega} + | u |^2_{\lambda,\Omega}$$

$$| u |^2_{\lambda,\Omega} := \sum_{|\alpha| = m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2+2\lambda}}\, dx\, dy$$

$$H_0^s(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{s,\Omega}}$$

Inverse plus Trace inequalities

$$u \in P_k(T) \subset H^s(k) \qquad s > \frac{1}{2}$$

$$\|u\|_{s-\frac{1}{2},\partial T} \lesssim \|u\|_{s,T} \lesssim h^{r-s} \|u\|_{r,T}$$

For example

$$\|\nabla u \cdot n\|_{0,\partial T} \lesssim \|\nabla u\|_{0,\partial T} \lesssim \|\nabla u\|_{\frac{1}{2},T} \lesssim h^{-\frac{1}{2}} \|\nabla u\|_{0,T}$$

$$\lesssim h^{-\frac{1}{2}} |u|_{1,T}$$