Metodi Matematici per Equazioni alle Derivate Parziali

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Slides introduttive liberamente adattate da una presentazione del Prof. Claudio Canuto (POLITO)

General Information

- Course Home Page: https://luca-heltai.github.io/nmpde/intro.html
- Elearning: https://elearning.dm.unipi.it/enrol/index.php?id=696
- Course github repository: https://github.com/luca-heltai/nmpde

VARIATIONAL METHODS FOR THE SOLUTION OF PROBLEMS OF EQUILIBRIUM AND VIBRATIONS

R. COURANT

As Henri Poincaré once remarked, "solution of a mathematical problem" is a phrase of indefinite meaning. Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal. Such one sided views seem to reflect human limitations rather than objective values. In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.

The Courant Finite Element (1943)

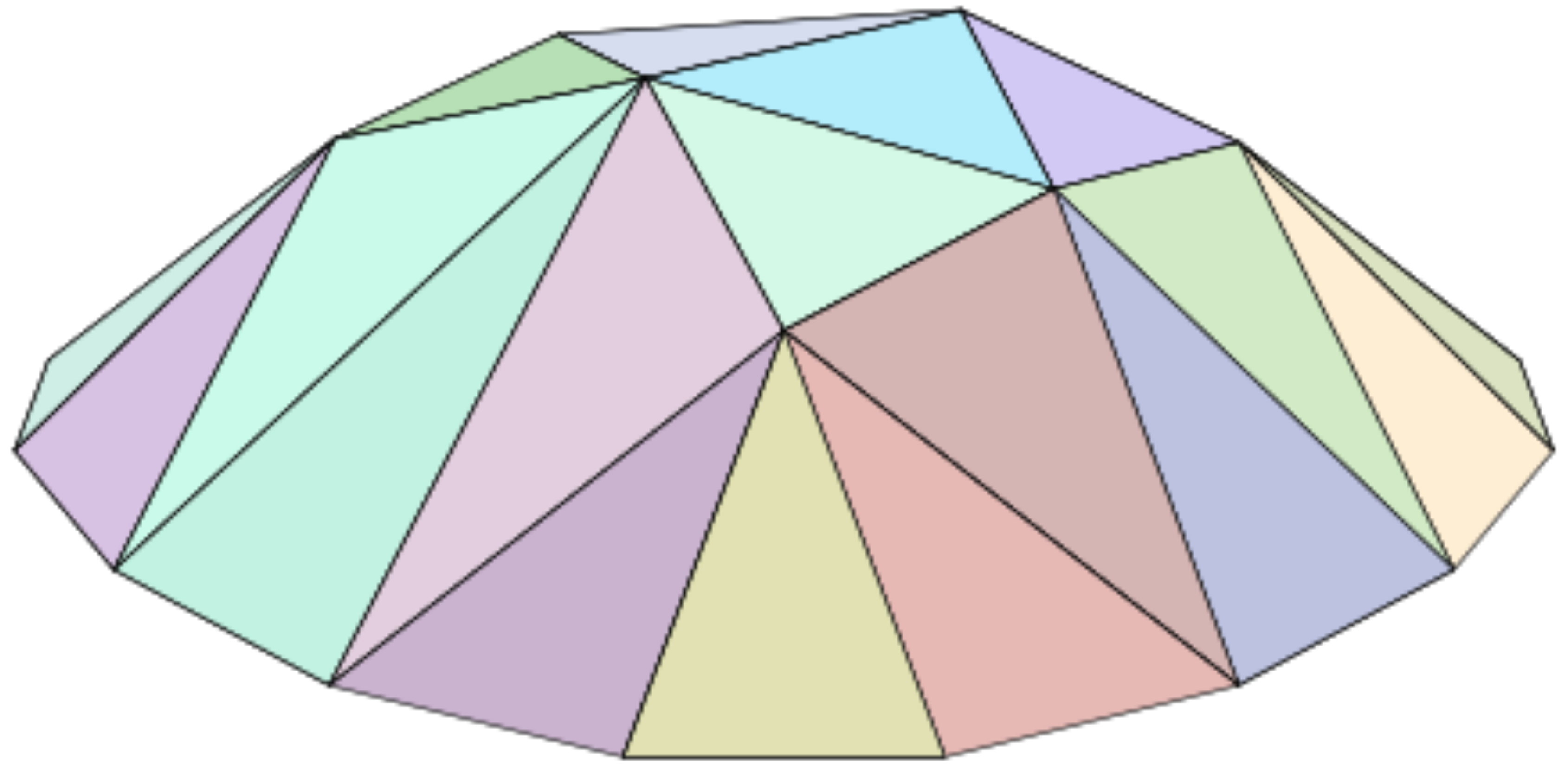
Courant considers minimization problems for convex energies $J : V \to \mathbb{R}$

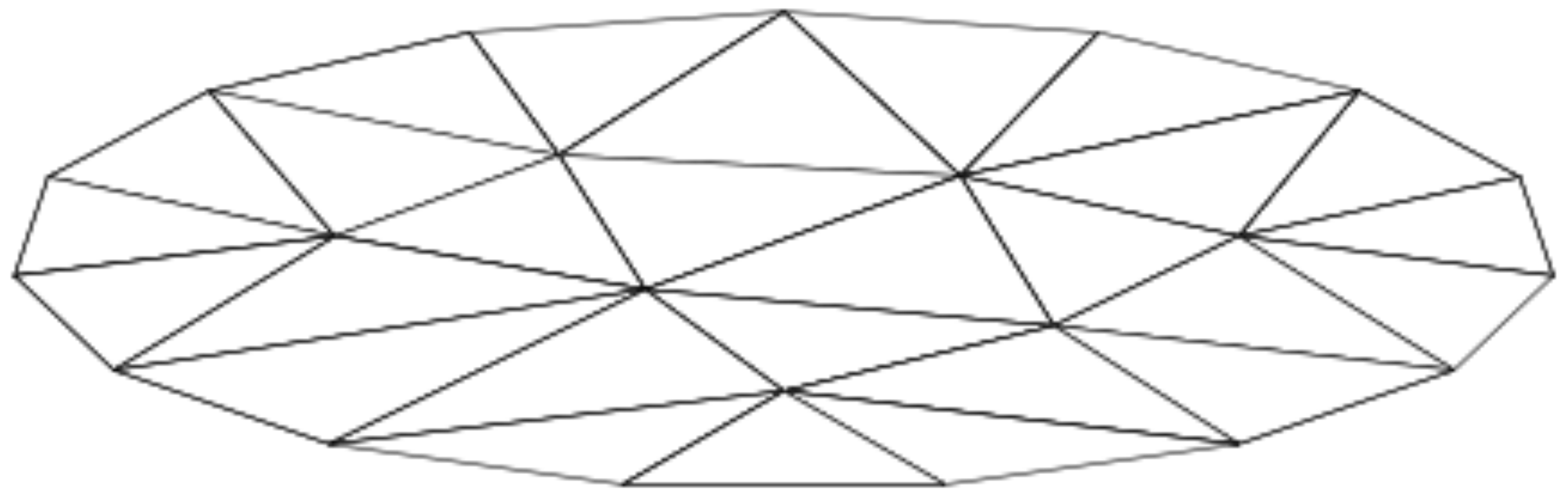
$$J(u) = \min_{v \in V} J(v),$$

approximated by the Rayleigh-Risz method, where V is replaced by

$$V_n = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}.$$

He suggests to choose functions "*linear on a net made of triangles*" as φ_k.





MR7838 (4,200e) 49.0X
Courant, R. [Courant, Richard]
Variational methods for the solution of problems of equilibrium and vibrations.
Bull. Amer. Math. Soc. 49 (1943), 1–23.

The early age of Finite Elements

- Engineers take the lead…

  (Argyris, Clough, Zienkiewicz, …)

- … mathematicians follow them

  (Céa, Aubin, Strang, Nitsche, Bramble, Babuška, ...)

- First books on Finite Elements

  (Zienkiewicz-Cheung 1967, Strang-Fix 1973)

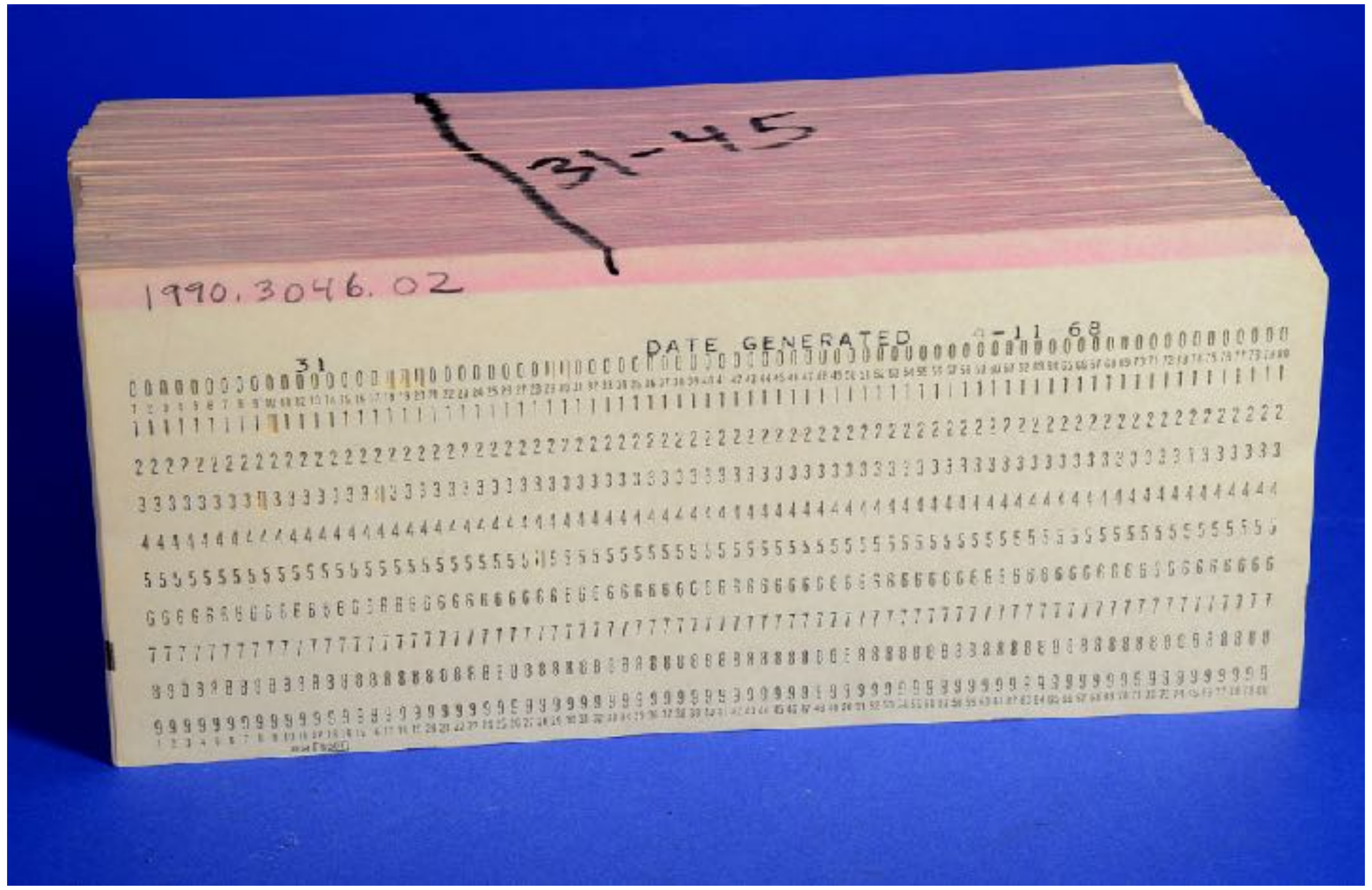
The Three Musketeers: J. Argyris, R. W. Clough and O. C. Zienkiewicz
(Courtesy of DRJ Owen and YT Feng)

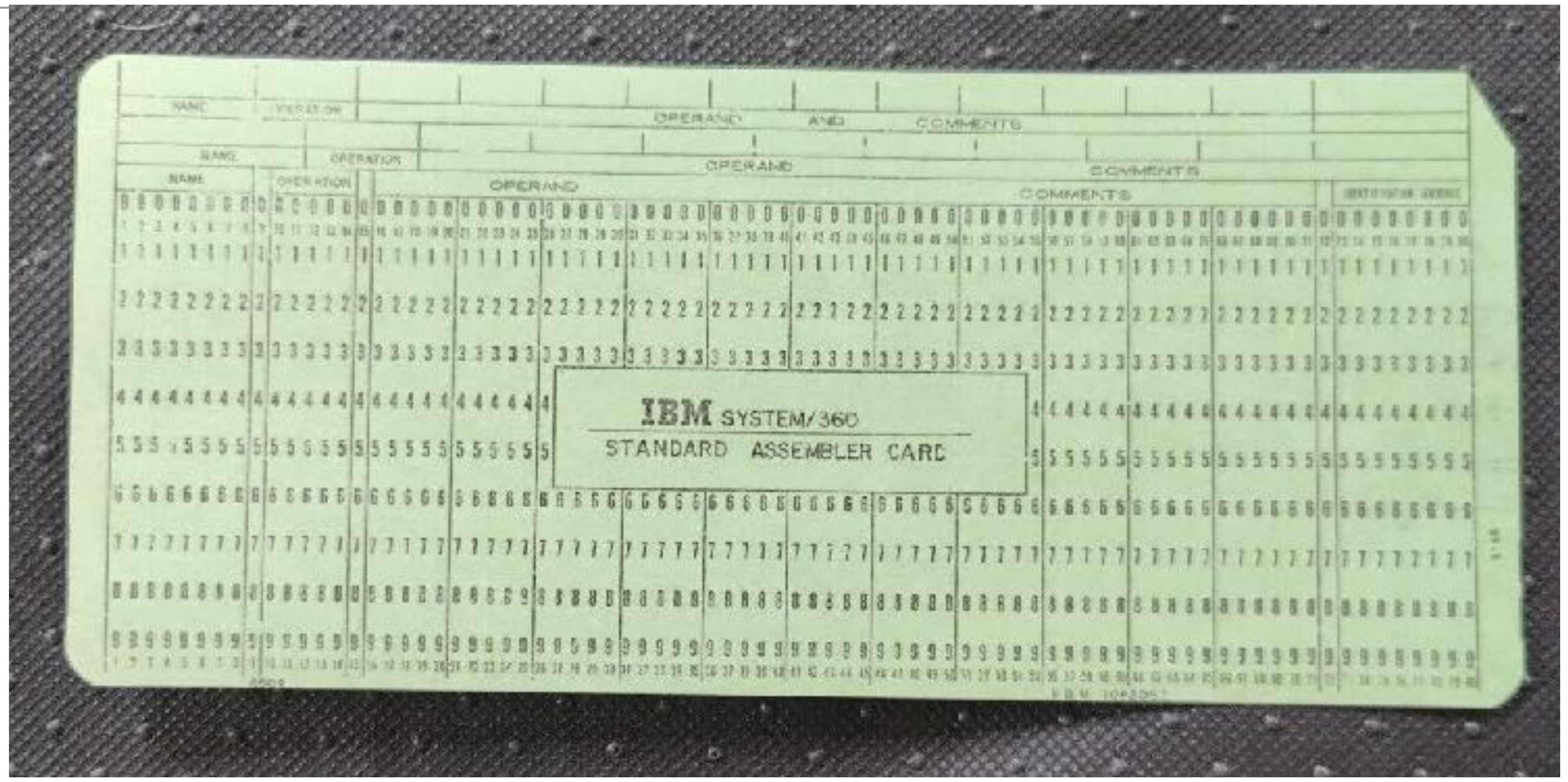
PDP/11

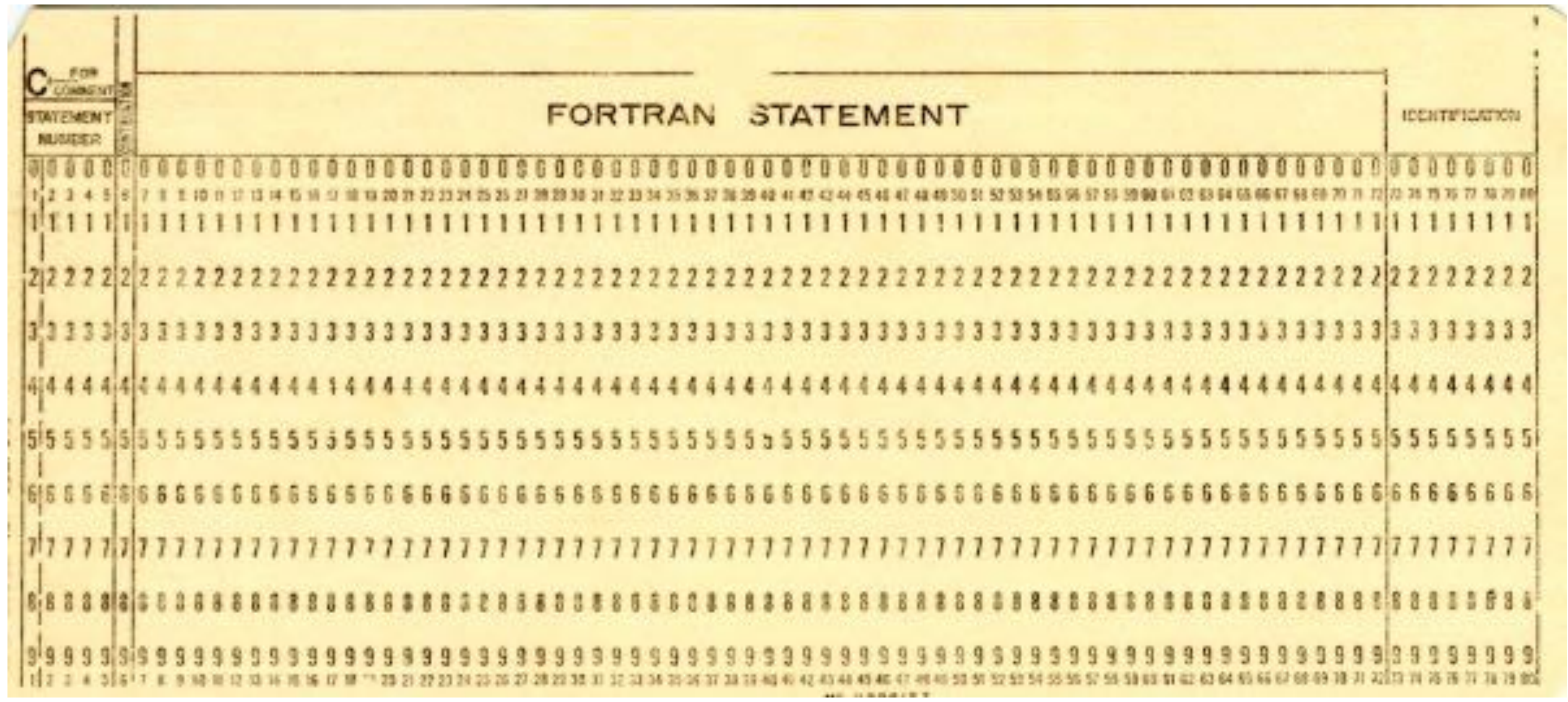
IBM/360

A nightmare for a student….

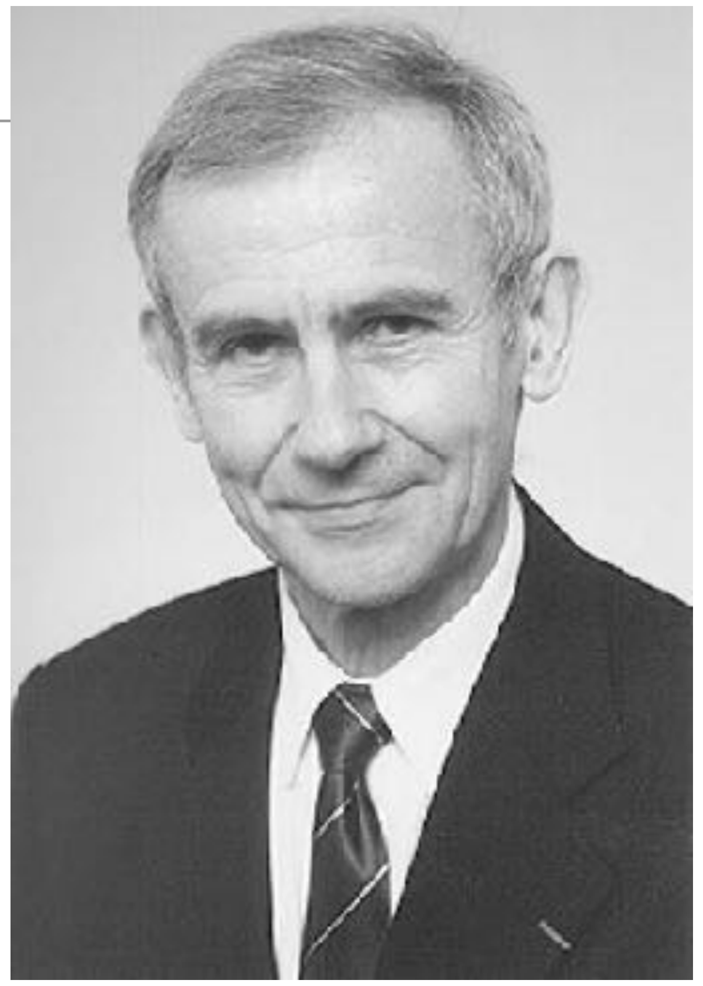
…lot of punched cards







Approximation theory in Sobolev spaces



MR336957 (49 #1730) 65D05 41A05
Ciarlet, P. G. [Ciarlet, Philippe G.]; Raviart, P.-A. [Raviart, Pierre-Arnaud]
General Lagrange and Hermite interpolation in R^n with applications to finite element methods.
Arch. Rational Mech. Anal. 46 (1972), 177–199.

Let $u_{\text{ex}} \in V$ be the exact solution and $u_{\text{fem}} \in V_{\text{fem}} \subset V$ be the finite-element solution of a BVP.

A fundamental results in the error analysis of the Finite Element Method (Céa) says

$$\|u_{\text{ex}} - u_{\text{fem}}\|_V \;\leq\; C \inf_{v_{\text{fem}} \in V_{\text{fem}}} \|u_{\text{ex}} - v_{\text{fem}}\|_V .$$

To bound the r.h.s., one can build an "interpolant" $\mathcal{I}_{\text{fem}} u_{\text{ex}} \in V_{\text{fem}}$ of u_{ex}, for which

$$\|u_{\text{ex}} - \mathcal{I}_{\text{fem}} u_{\text{ex}}\|_V \;\leq\; \bar{C}\, \varepsilon_{\text{fem}} \|u_{\text{ex}}\|_W$$

if $u_{\text{ex}} \in W \Subset V$. Then,

$$\|u_{\text{ex}} - u_{\text{fem}}\|_V \;\leq\; C\bar{C}\, \varepsilon_{\text{fem}} \|u_{\text{ex}}\|_W$$

1974: Brezzi's theorem

MR365287 (51 #1540) 49B30 65K05
Brezzi, F. [Brezzi, Franco]
On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. (French summary)
Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 8 (1974), *no. R-2,* 129–151.

Let $A : V \to V'$ and $B : V \to Q'$ be linear continuous operators between Hilbert spaces. Given $f \in V'$ and $g \in Q'$ consider the constrained minimization problem

$$\text{Find } u \in V \;: \qquad u = \arg\min_{Bv=g} J(v)$$

with $J(v) = \frac{1}{2}\langle Av, v\rangle - \langle f, v\rangle$. This is equivalent to the saddle point problem

$$\text{Find } (u,p) \in V \times Q \;: \qquad \begin{cases} Au + B'p = f & \text{in } V', \\ Bu \qquad\quad = g & \text{in } Q'. \end{cases}$$

This is well-posed iff $A : K \to K'$ is isomorphism (with $K = \ker B$), and $B : V \to Q'$ is surjective. The latter condition is the celebrated (LBB) *inf-sup condition*: there exists $\beta > 0$ such that

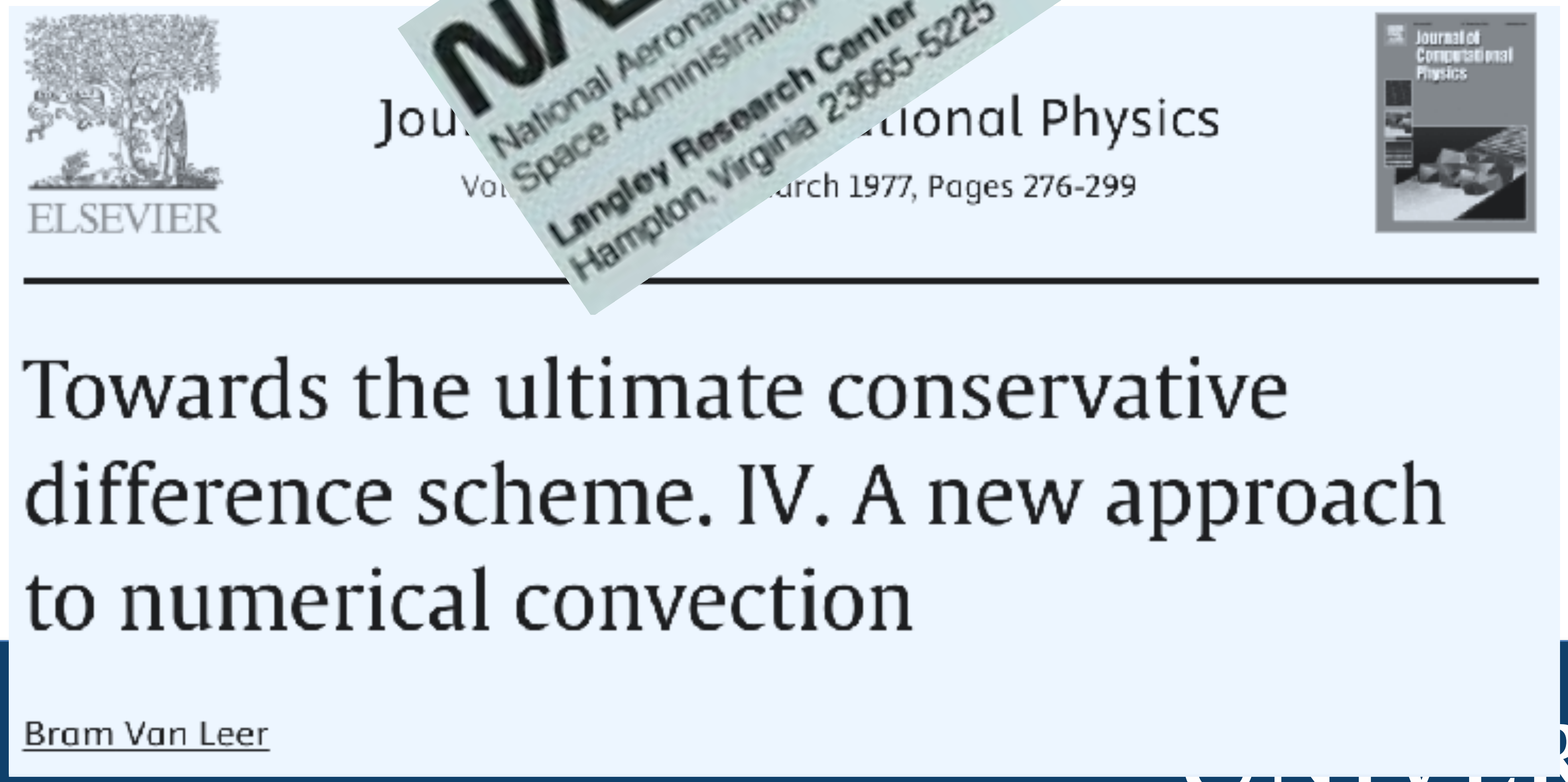
$$\inf_{q\in Q} \frac{\|B'q\|_{V'}}{\|q\|_Q} = \inf_{q\in Q} \sup_{v\in V} \frac{\langle Bv, q\rangle}{\|v\|_V \|q\|_Q} \geq \beta.$$

Brezzi's paper considers the discrete version of this result for finite dimensional $V_\delta \subset V$ and $Q_\delta \subset Q$, and studies the error between (u,p) and (u_δ, p_δ).

1977

VAX/11-780

VT-100

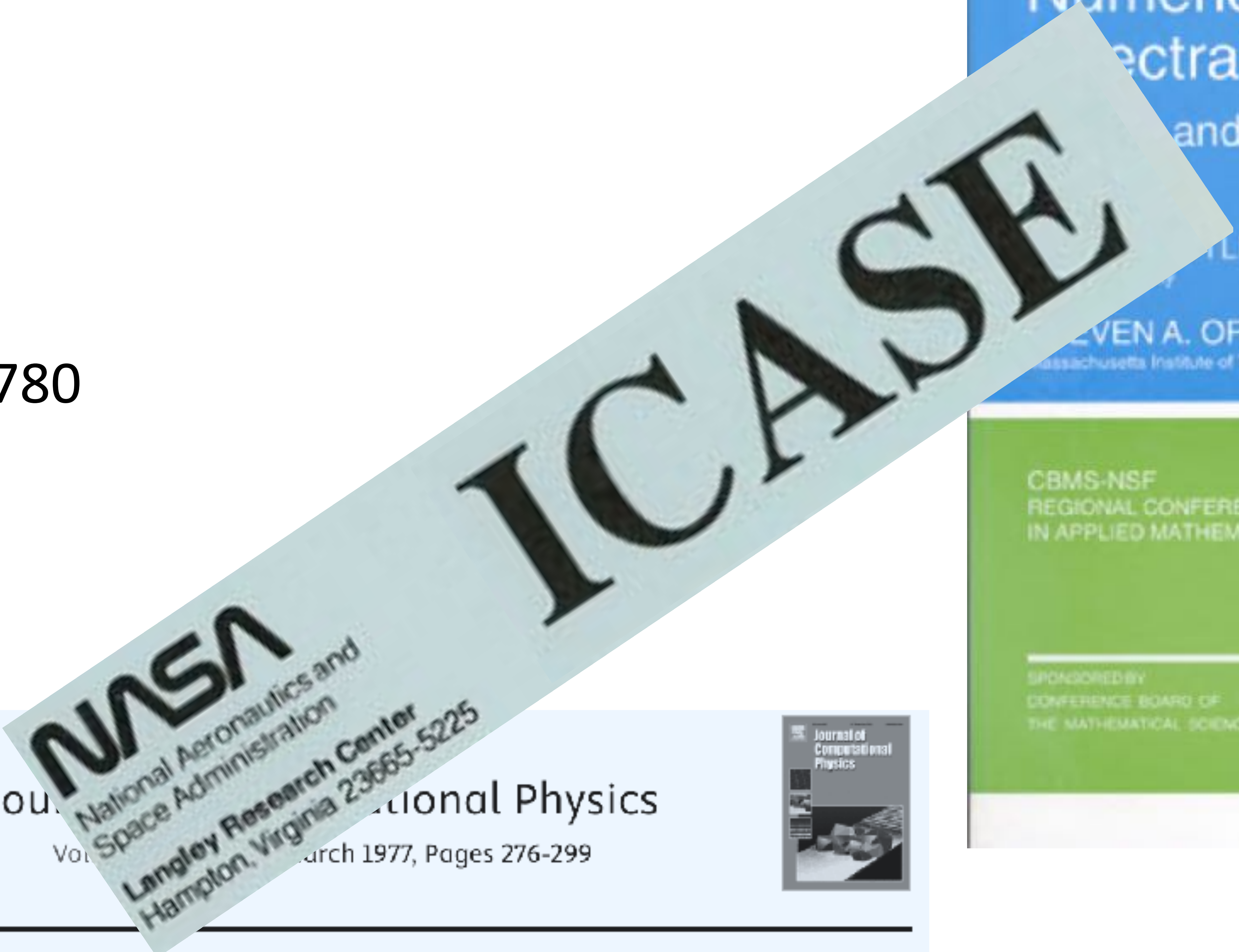

ELSEVIER

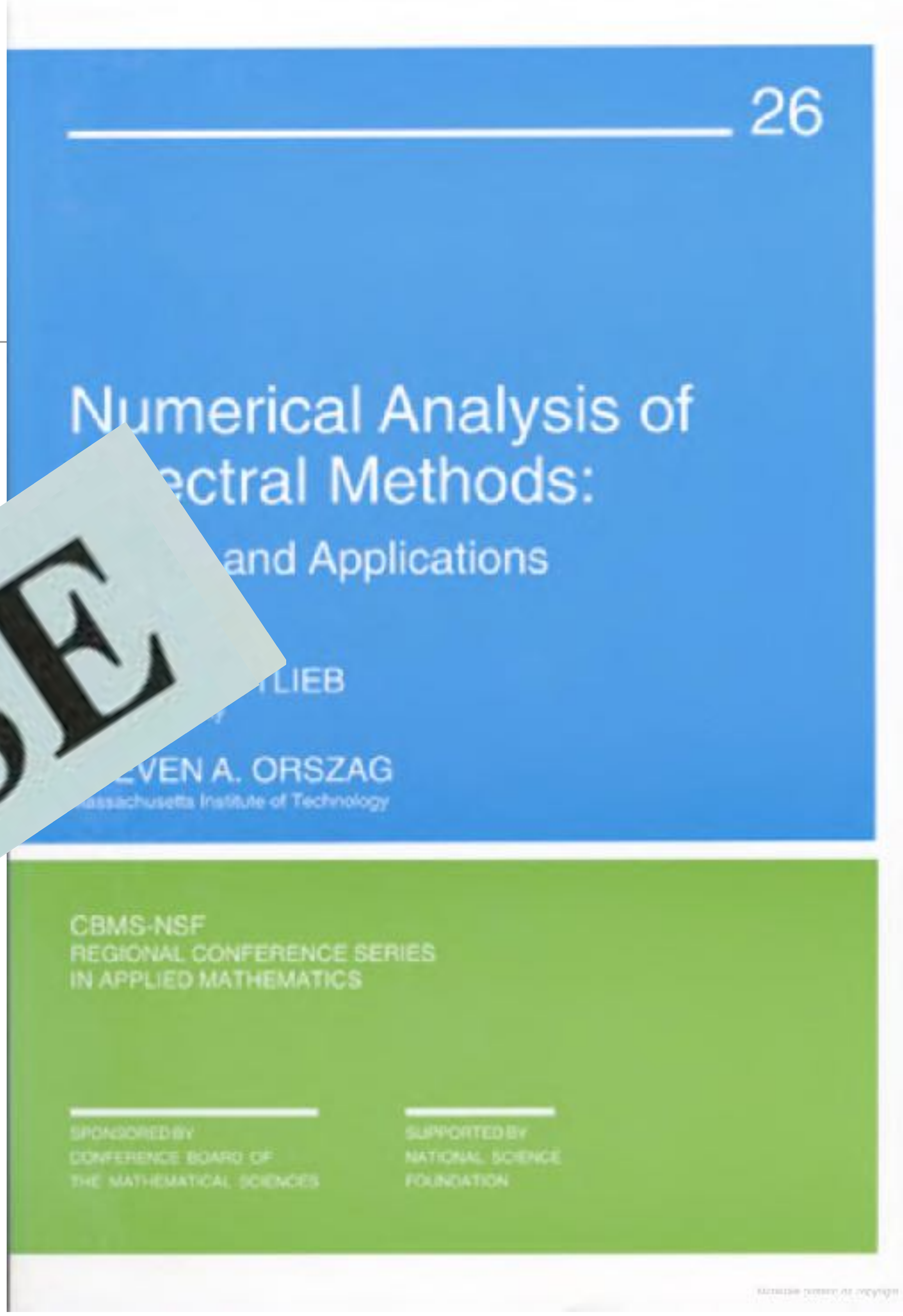
Jou… …ional Physics

Vo… …arch 1977, Pages 276-299

Towards the ultimate conservative difference scheme. IV. A new approach to numerical convection

Bram Van Leer





Numerical conservation laws

- Finite difference / finite volume schemes aim at approximating

$$\partial_t u + \partial_x f(u) = 0$$

  in a conservative, monotone (i.e., oscillation-free near discontinuities), and accurate way.

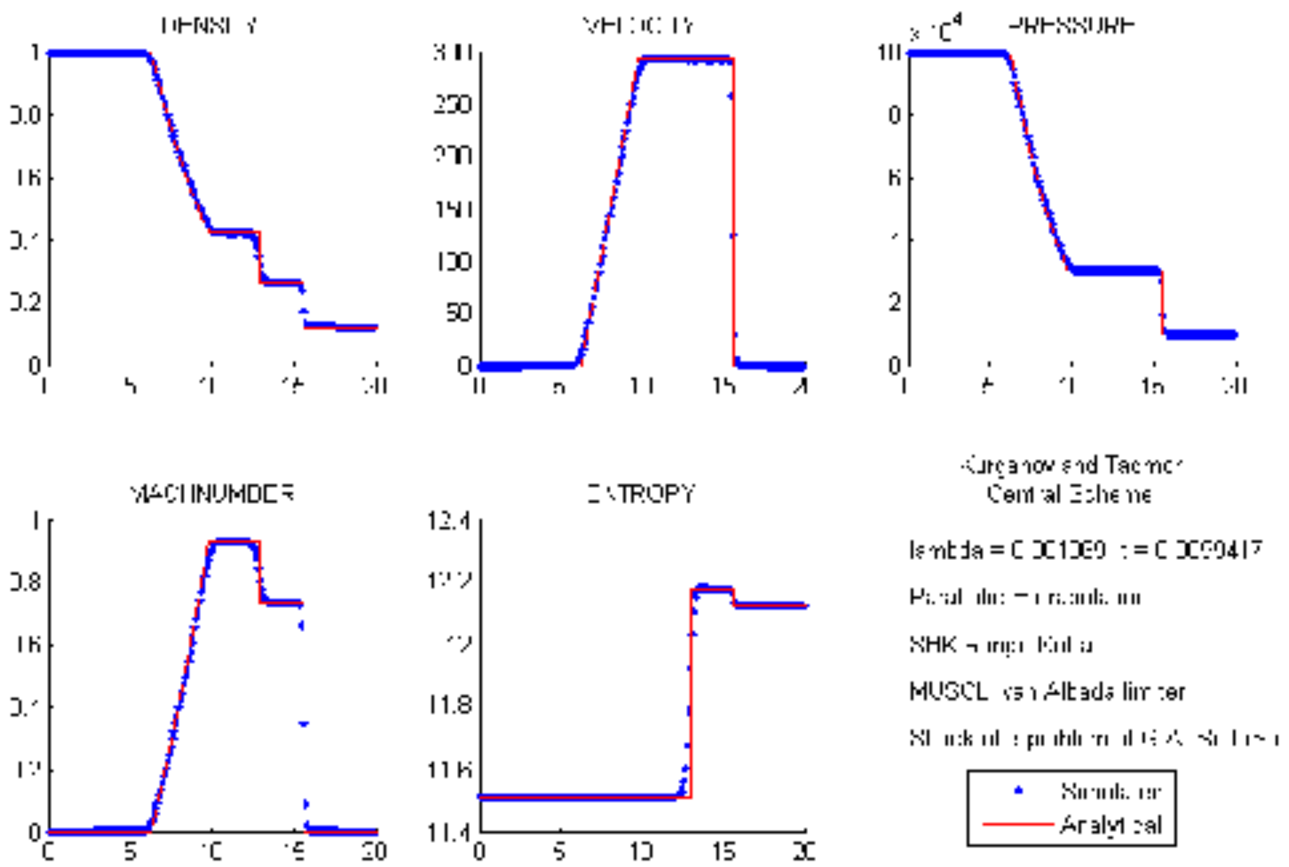
- Godunov barrier: linear and monotone schemes can only be first-order accurate.

- Van Leer introduced nonlinear slope limiters in the flux reconstruction at cell interfaces, thereby granting monotonicity and second-order accuracy.

- His work, based on Godunov's idea of solving local Riemann problems at interfaces, was highly influential on the development of higher-order upwind schemes.

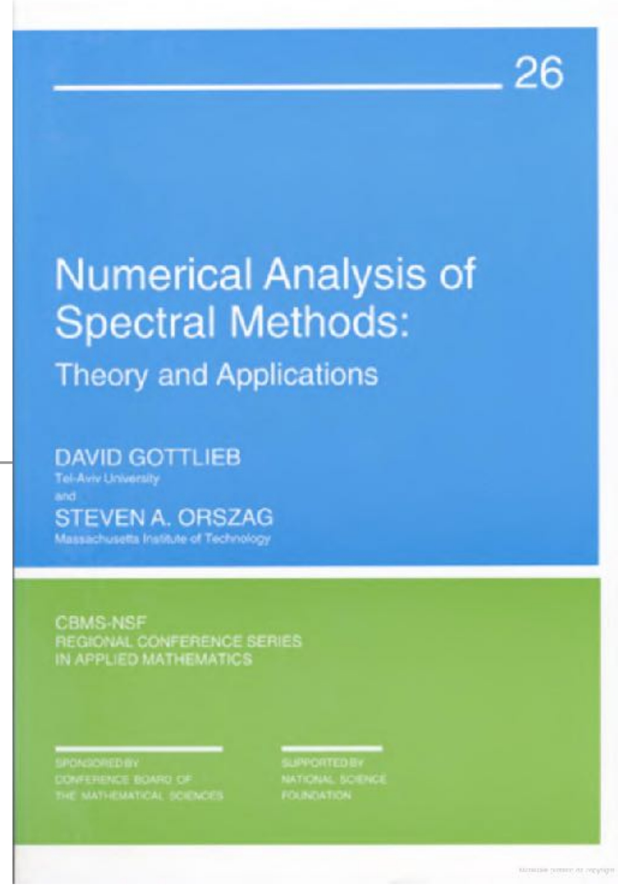
- In the late '80, more sophisticated ENO and WENO schemes were introduced by Harten, Osher, Engquist, Shu, et al

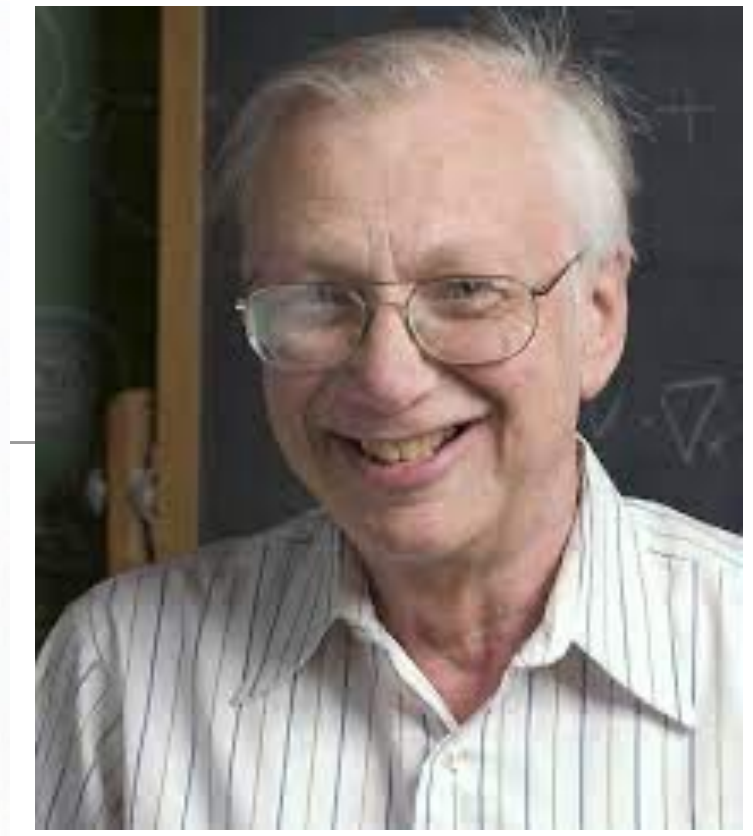
BRAM VAN LEER



Spectral Methods

Steve Orszag pioneered the use of Fourier methods for the direct numerical simulation of periodic turbulent flows, via FFT.





Few years later, the Gottlieb-Orszag book advocates the use of orthogonal polynomials (spectral bases) to treat non-periodic boundary conditions.

The duality between **frequency space** (modes) and **physical space** (grid values) is exploited. Gaussian quadrature nodes are used as collocation points to enforce the equations.

In simple geometries and for smooth solutions, the advantage of using high-order methods in terms of ratio accuracy/unknowns is clearly documented.

(But the Gibbs phenomenon is lurking in the shadow…)

- The numerical analysis was essentially performed *á la Von Neumann*. Something was missing…

The Fab Eighties

IBM PC

OLIVETTI M20

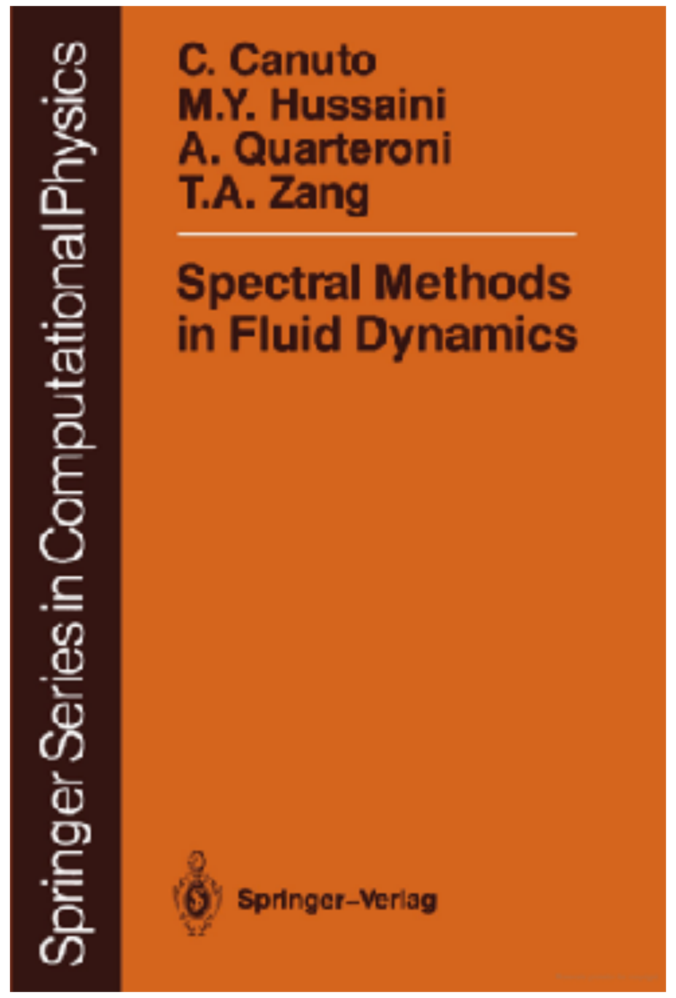
APPLE McIntosh

CRAY XMP

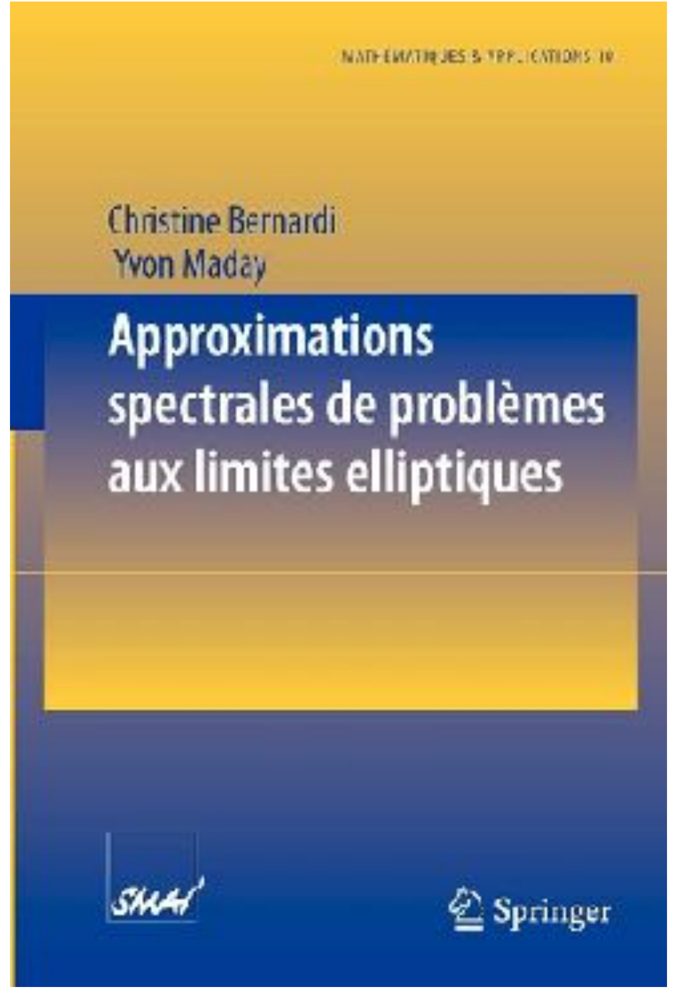
Spectral methods meet Sobolev spaces

- Functional framework for spectral methods
- Use of weak formulations. Interpretation of collocation methods as Galerkin methods with numerical integration.
- Stability and convergence analysis in Sobolev spaces, similar to that of Finite Element Methods.

[Brown University, 2004]



1987

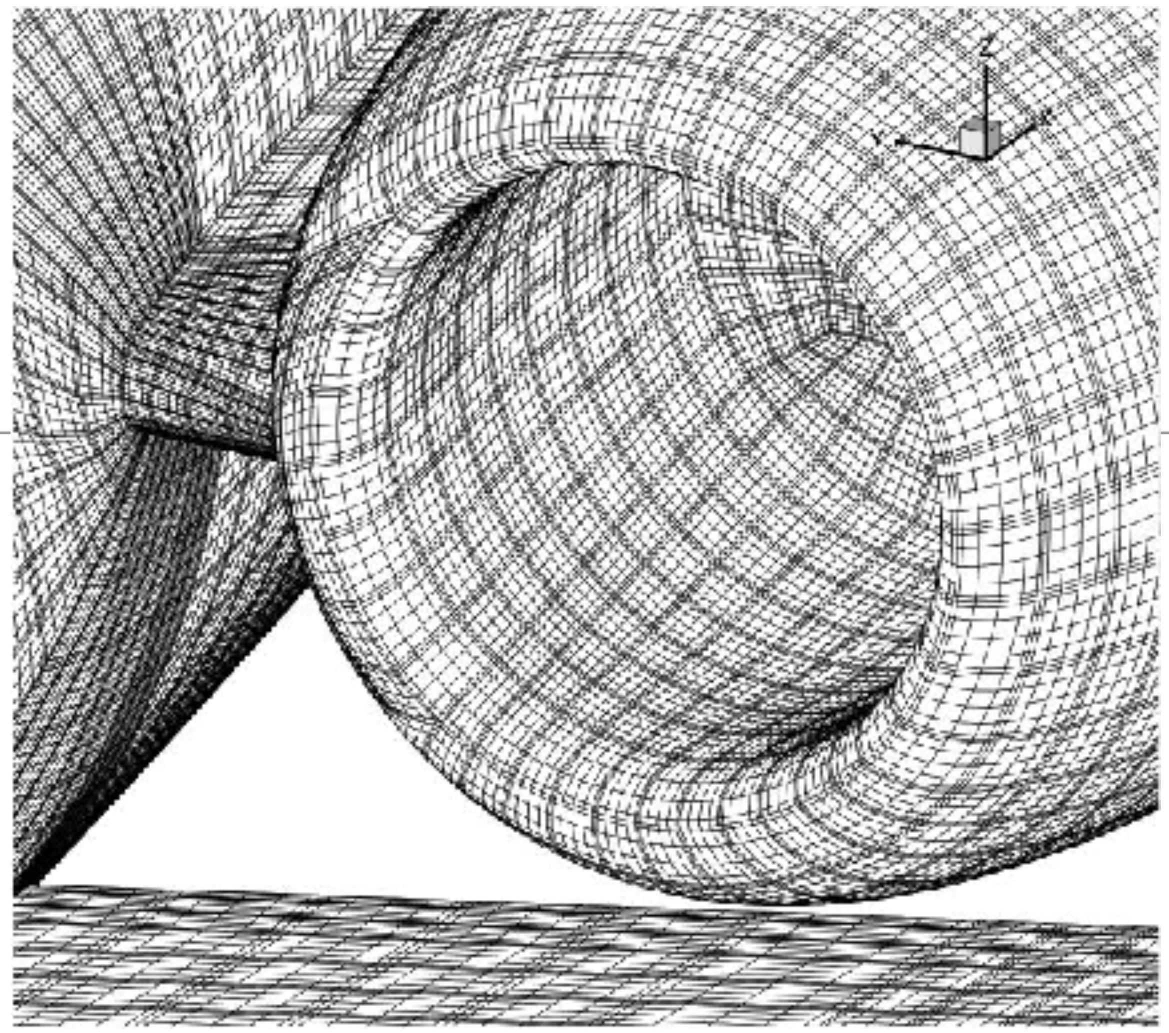


1992

- Early research boosted by French-Italian collaboration (C., Quarteroni, Maday, Bernardi, Funaro, …)

A broader scenario

- Spectral Element Method (SEM) (Patera, 1984)
- Analogies/differences with the p-version (Babuška&Szabo, 1981) and the h-p version of FEM (Gui&Babuška, 1986)

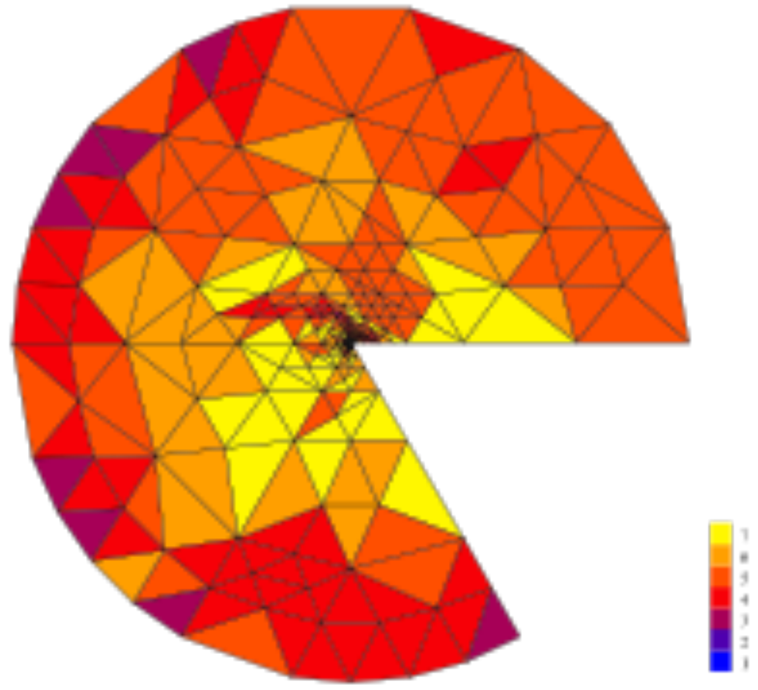


- The first

  International Conference on Spectral and High-Order Methods (ICOSAHOM)

  held in Como in 1989, gathered high-order people from different communities.
- The 14th edition of the Conference was held in 2023 in South Korea.

- One of the major results of SEM or h-p FEM is the **exponential decay** of the error, even in the presence of localized singularities in the solution:
  a suitable combination of *mesh refinement* and *polynomial enrichment* near the singularity does the job.



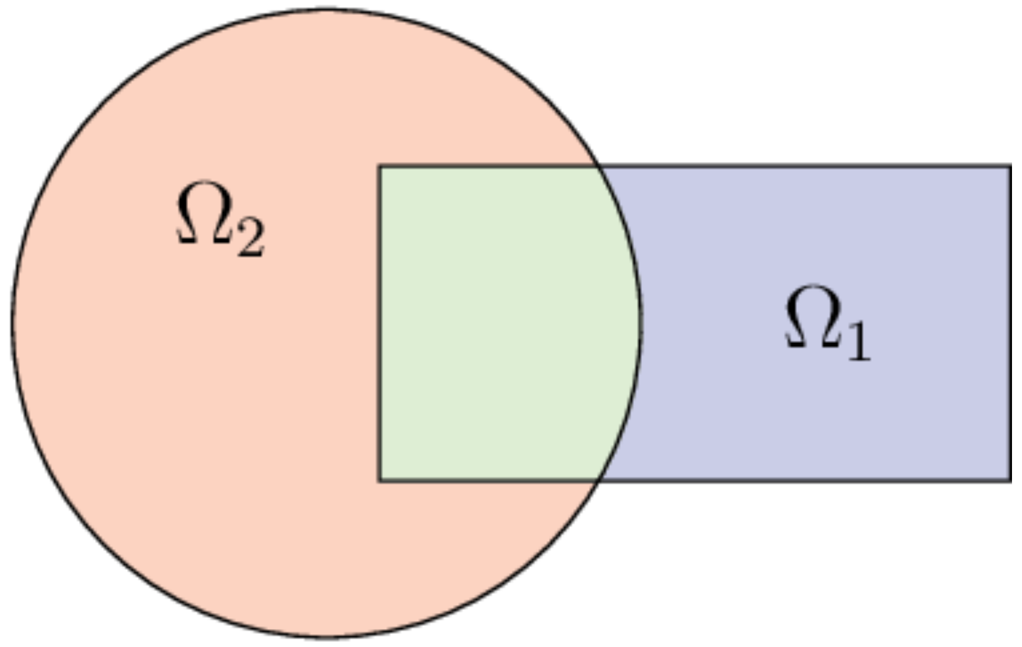
Domain decomposition methods

For Spectral methods:

- handle complex geometries (e.g, Schwarz alternating method (1870), …,
  Mortar Method [Maday, Mavriplis and Patera, 1988], …)

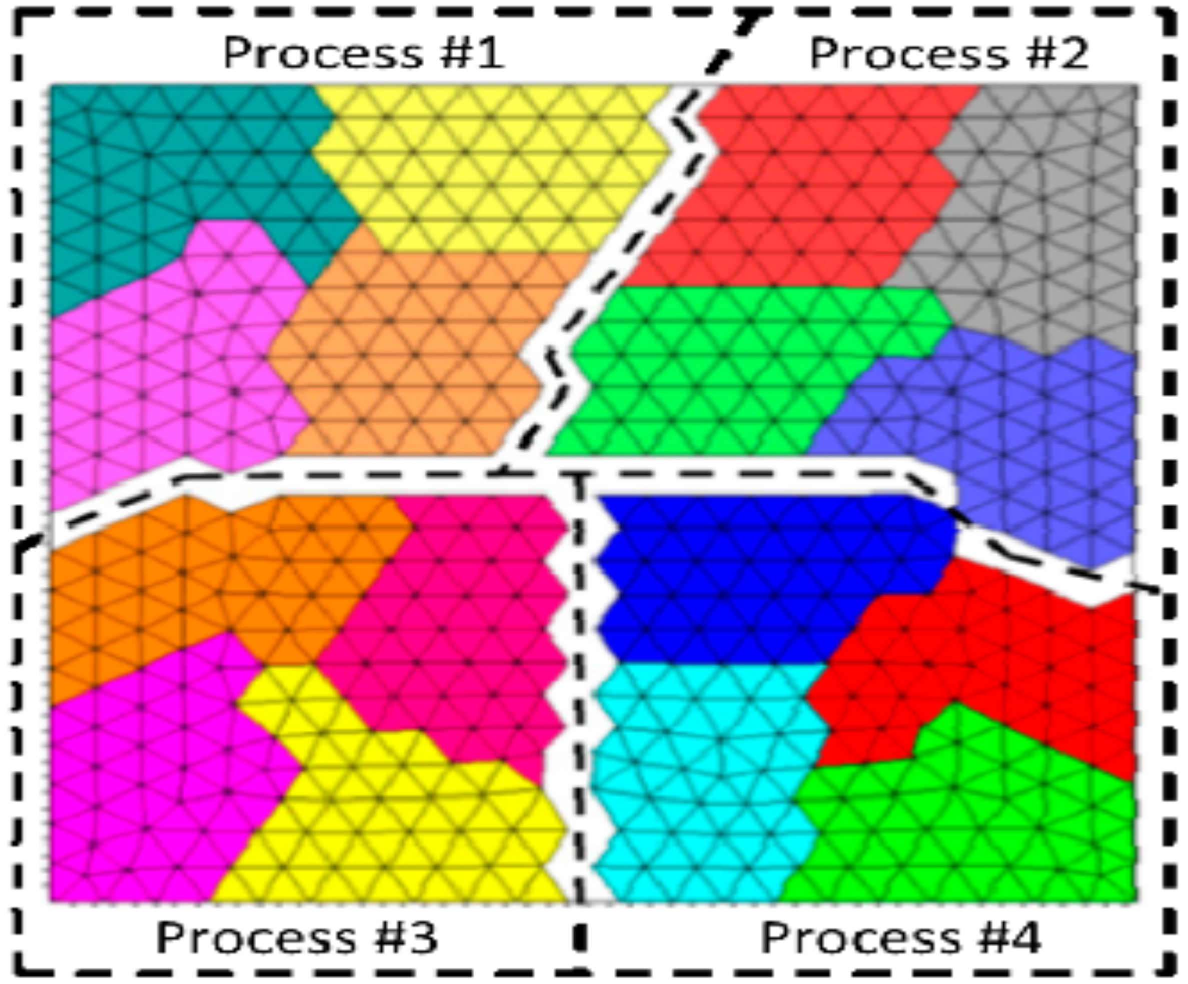
For FEMs, FVMs and others:

- preconditioning, parallelization, enhanced flexibility



DDM bridge several areas:

- Theoretical PDEs
- Numerical PDEs
- Numerical Linear Algebra
- Computer Science



The first

International Conference on Domain Decomposition Methods

was held in Paris in 1987. The 28-th one was held in KAUST, S.A., in February 2024.

VAX STATION (1987)

Discontinuous Galerkin Methods

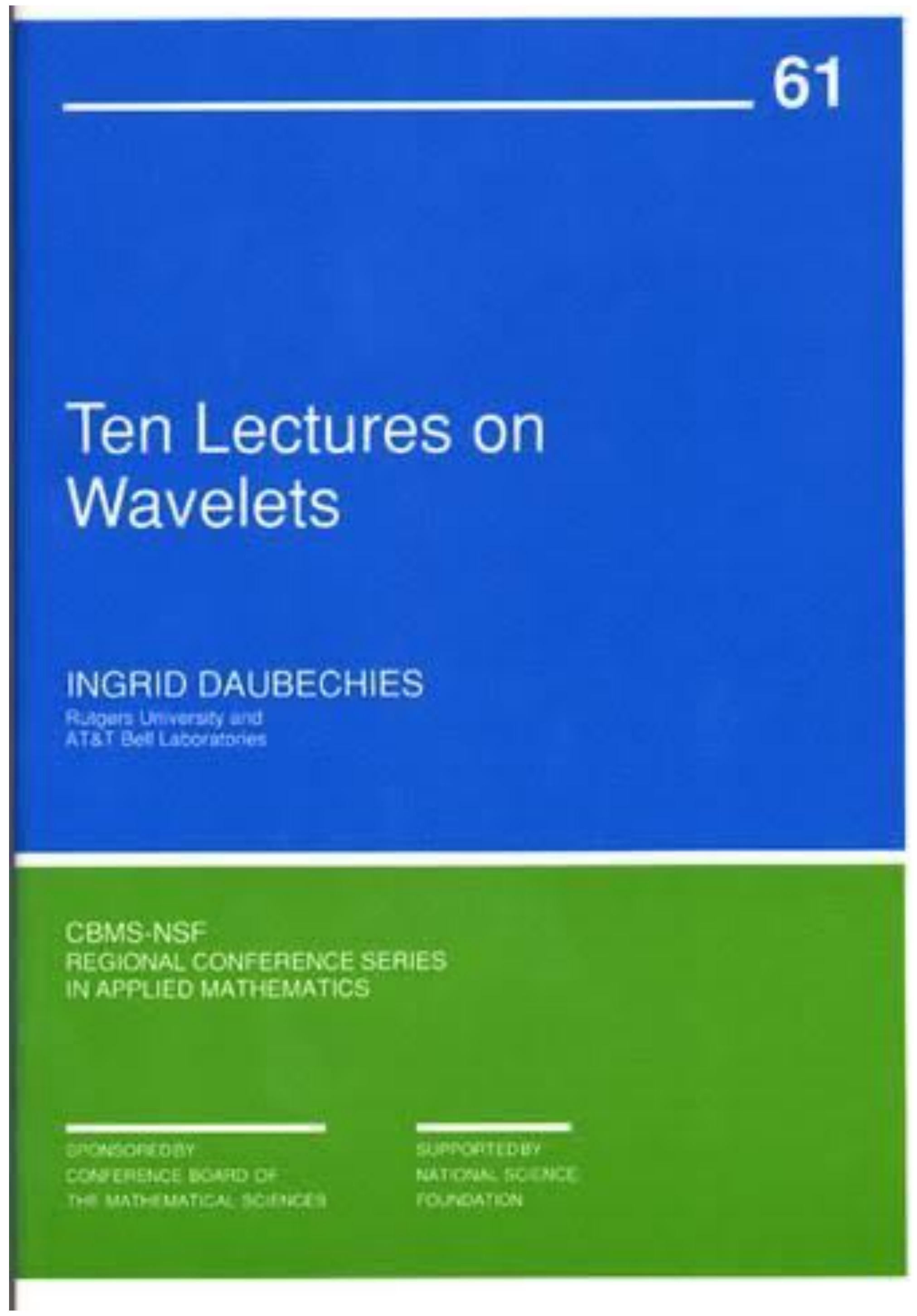
The idea of using piecewise discontinuous functions,
with some ``glue'' between pieces, to approximate the solution of a PDE
dates back to the Seventies:
(Reed&Hill (1973), Douglas&Dupont (1976), Arnold (1979),…)

The DG paradigm helps in handling
- problems with strong variations in the solution or in the coefficients,
- matching of different discretizations

- Influential work by
  - Cockburn&Shu (1989, …) for hyperbolic problems
  - Bassi&Rebay (1997) for elliptic problems
- Bridge between Finite Elements and Finite Volumes (local conservation)

MR1885715 (2002k:65183) 65N30
Arnold, Douglas N. (1-MN-MA); Brezzi, Franco (I-CNR17-IAN);
Cockburn, Bernardo (1-MN-SM);
Marini, L. Donatella [Marini, Luisa Donatella] (I-CNR17-IAN)
Unified analysis of discontinuous Galerkin methods for elliptic problems.
(English summary)
SIAM J. Numer. Anal. 39 (2001/02), *no. 5,* 1749–1779.

A unifying view of DG Methods



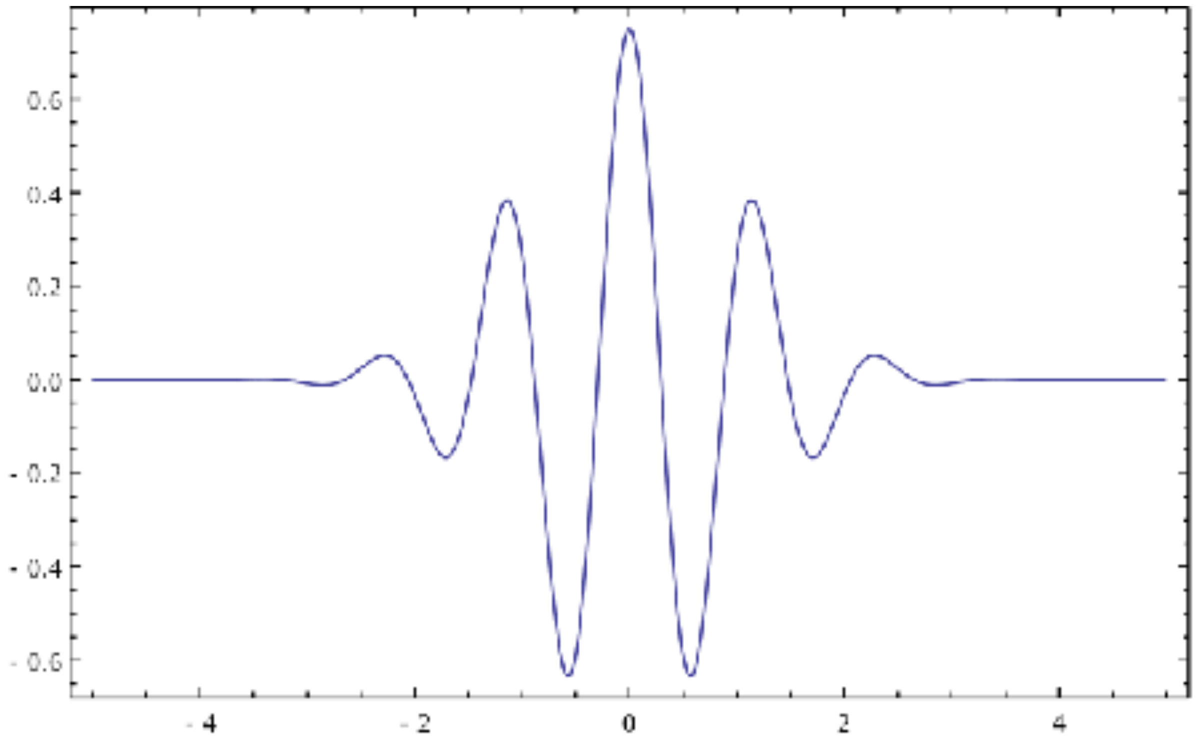
Wavelets ! (1992)

- Orthonormal basis, localized both in physical space and in frequency space

$$v(x) = \sum_{j,k} \hat{v}_{jk}\, \psi_{jk}(x)$$

  - ψ_{jk} is a dilate and translate of a mother ψ
  - Riesz basis property

$$\|v\|_H^2 \simeq \sum_{j,k} |\hat{v}_{jk}|^2$$



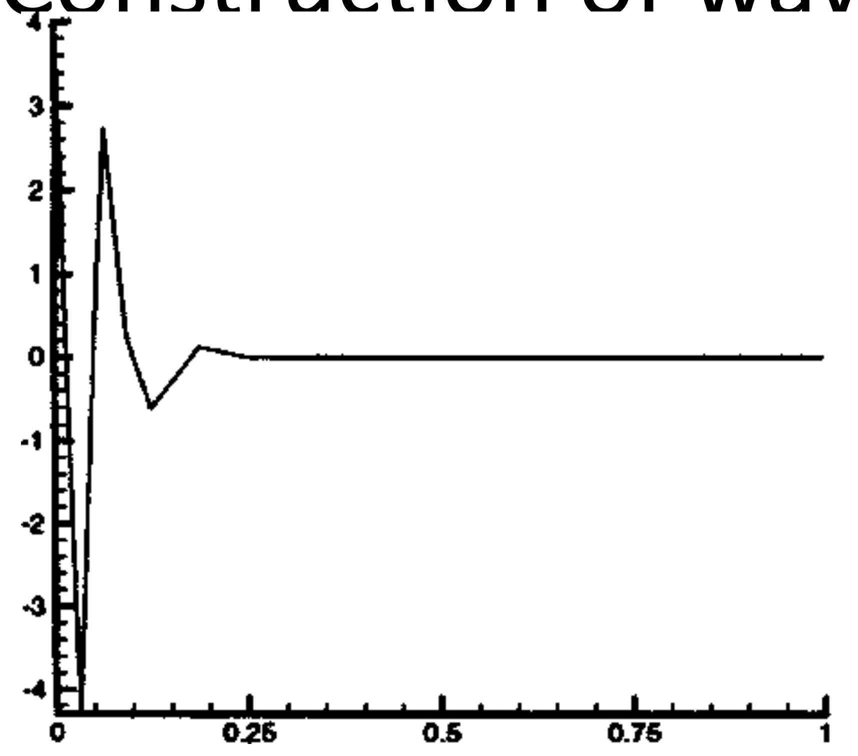
Ingrid DAUBECHIES

- Extension to biorthogonal bases
- Caracterization of function spaces V and their duals V' in terms of decay of wavelet coefficients
- Compression of information, by neglecting the wavelet coefficients with smaller size (image processing, JPEG2000 standard)

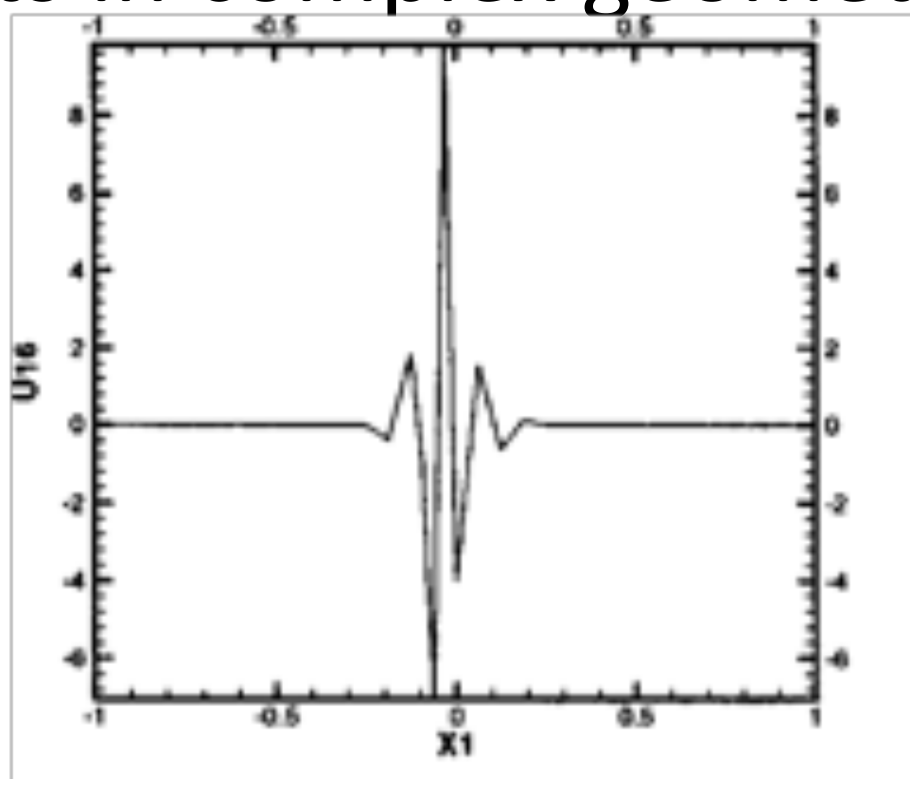
Wavelets and PDEs

- Representation of an operator in wavelet spaces, as an infinite matrix.
- Preconditioning (by diagonal scaling), adaptivity (by adding/removing wavelet functions)

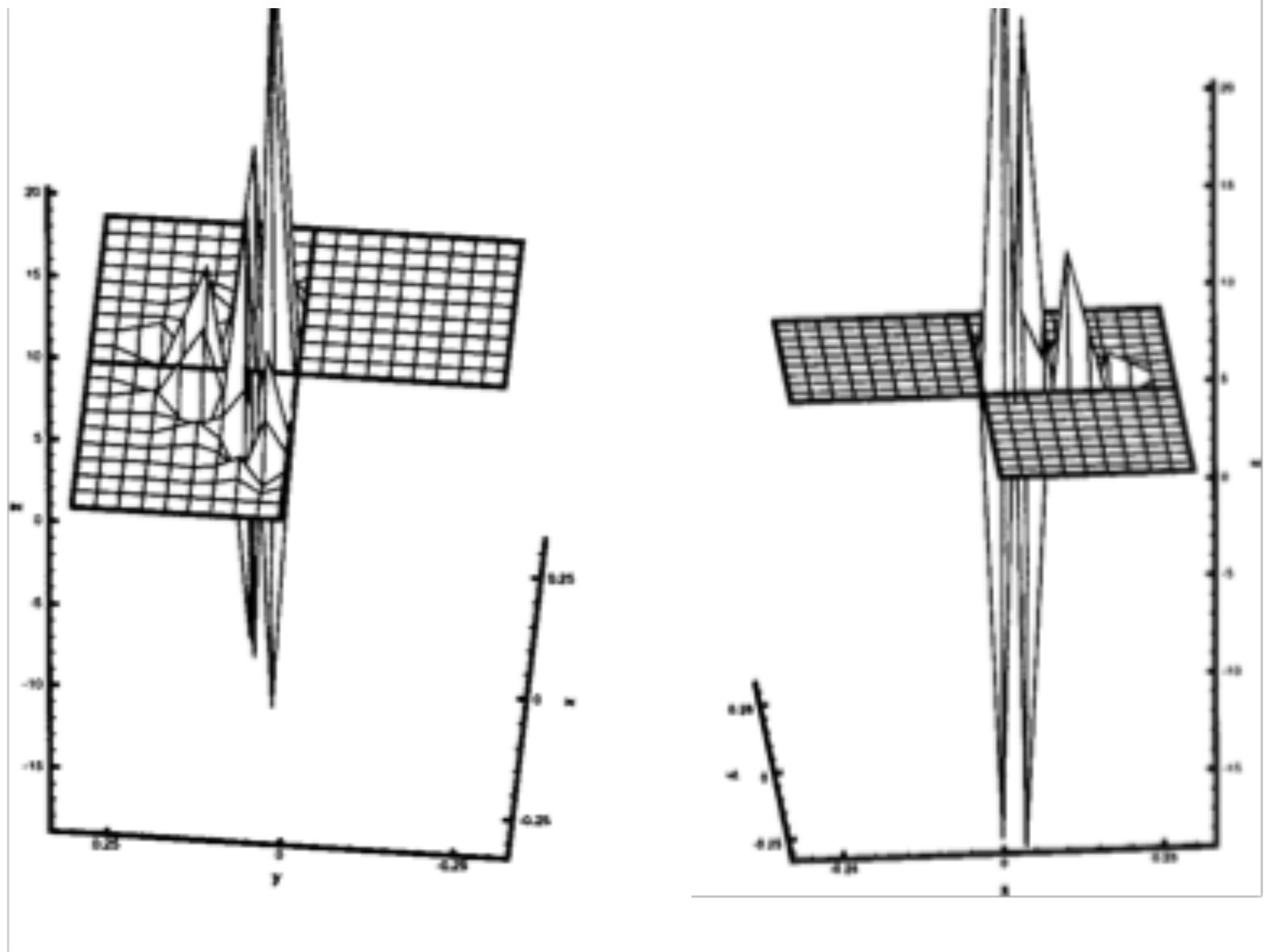
Construction of wavelets in complex geometries:



Boundary conditions



Matching conditions

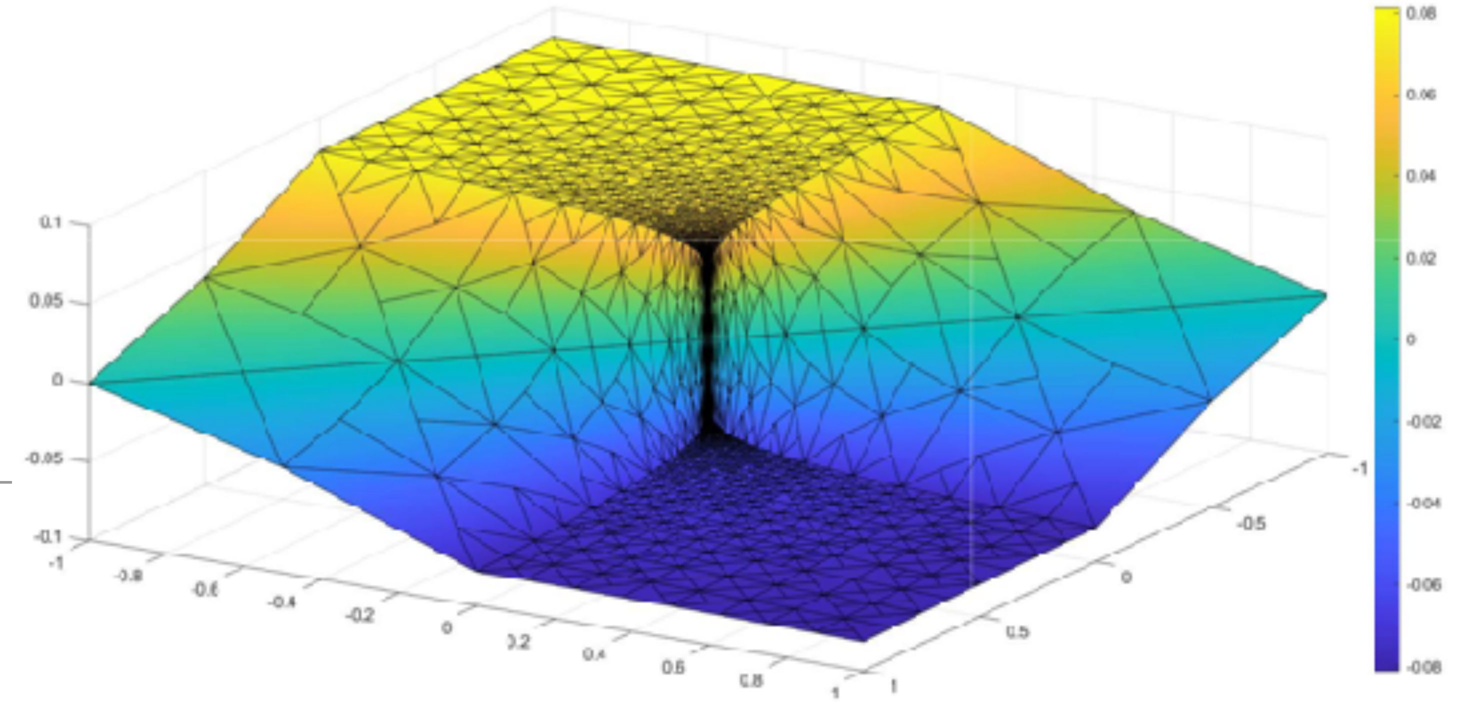


Wavelets in a L-shaped domain

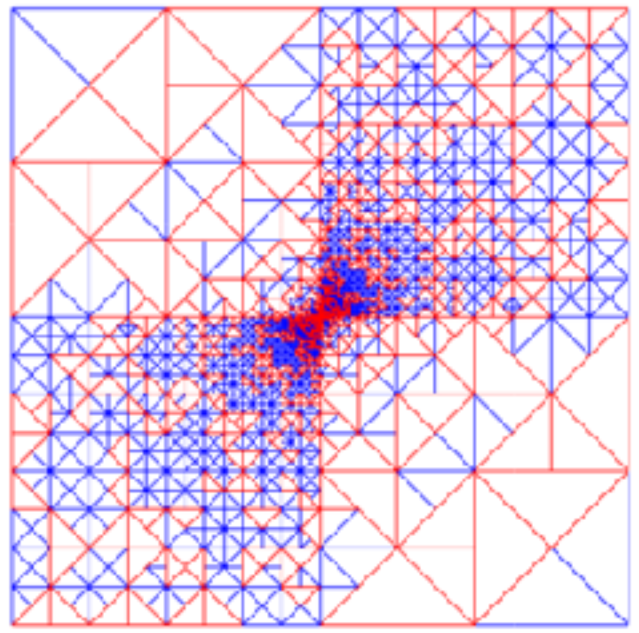
(WEM – Wavelet Element Method)

MR1664902 (99k:42055) 42C15 41A15 65T20
Canuto, Claudio (I-TRNP); Tabacco, Anita (I-TRNP);
Urban, Karsten (D-AACH-G)
The wavelet element method. I. Construction and analysis. (English summary)
Appl. Comput. Harmon. Anal. 6 (1999), *no. 1,* 1–52.

Adaptivity - a nonlinear process



- Problem. Given a budget N (memory, computer time, ...), which is the best investment to approximate the solution of a differential problem?



- Strong impact of wavelets on the theoretical understanding of adaptivity for PDEs:

MR1803124 (2002h:65201) 65N35 65N22 65T60
Cohen, Albert [Cohen, Albert[1]] (F-PARIS6-N);
Dahmen, Wolfgang [Dahmen, Wolfgang A.] (D-AACH-G);
DeVore, Ronald [DeVore, Ronald A.] (1-SC)
Adaptive wavelet methods for elliptic operator equations: convergence rates. (English summary)
Math. Comp. 70 (2001), *no. 233,* 27–75.

- First convergence & optimality result for FEMs:

MR2421046 (2009h:65174) 65N30 41A25
Cascon, J. Manuel [Cascón, José Manuel] (E-SALA);
Kreuzer, Christian (D-AGSB-MI); Nochetto, Ricardo H. (1-MD);
Siebert, Kunibert G. (D-AGSB-MI)
Quasi-optimal convergence rate for an adaptive finite element method. (English summary)
SIAM J. Numer. Anal. 46 (2008), *no. 5,* 2524–2550.

- Representative result: Assume that the best approximation $u_{\text{best},N}$ of u with budget N satisfies for some $s>0$

$$\|u-u_{\text{best},N}\| \simeq N^{-s}, \qquad N \to \infty.$$

Then, a constructive algorithm is given, which computes some u_N with budget N satisfying

$$\|u-u_N\| \simeq N^{-s}, \qquad N \to \infty.$$

New challenges…

AMD
FRONTIER

Uncertainty quantification

Reduced-order models

Polytopal meshes

High-dimensional problems

Immersed methods

AMD FRONTIER

Reduced-order models

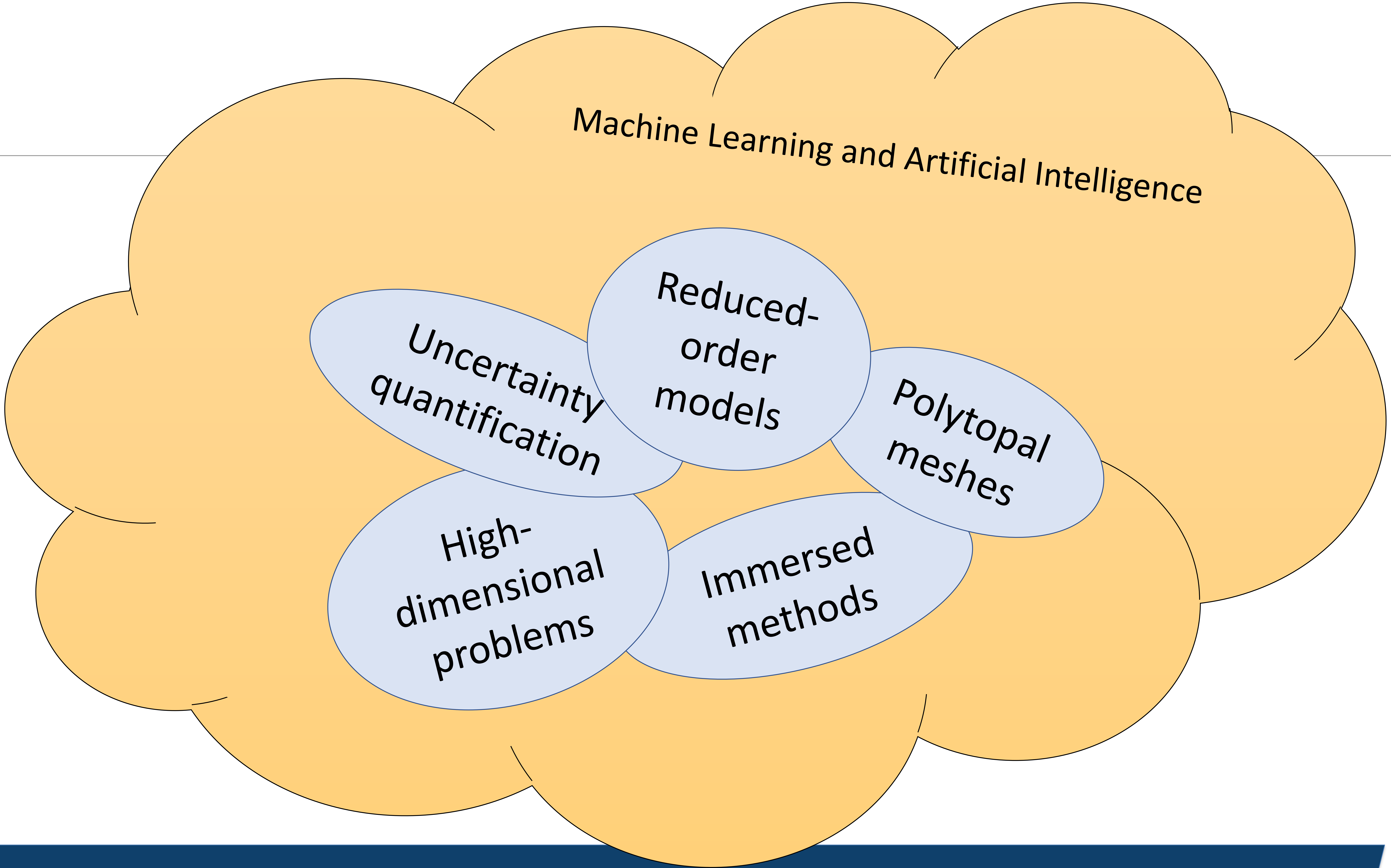
Uncertainty quantification

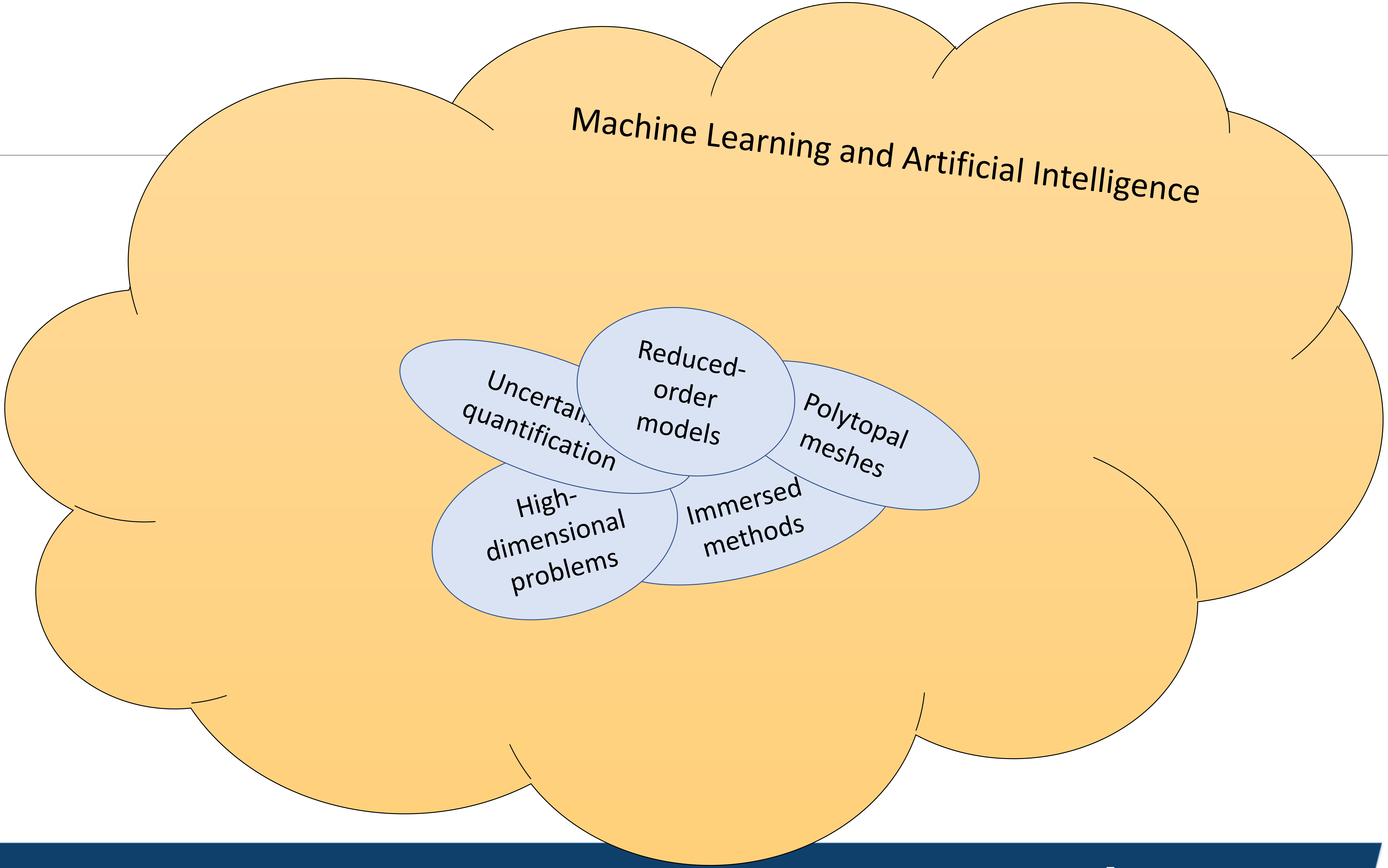
Machine Learning and Artificial Intelligence

Polytopal meshes

High-dimensional problems

Immersed methods


Machine Learning and Artificial Intelligence
Reduced-order models
Uncertainty quantification
Polytopal meshes
High-dimensional problems
Immersed methods


Machine Learning and Artificial Intelligence
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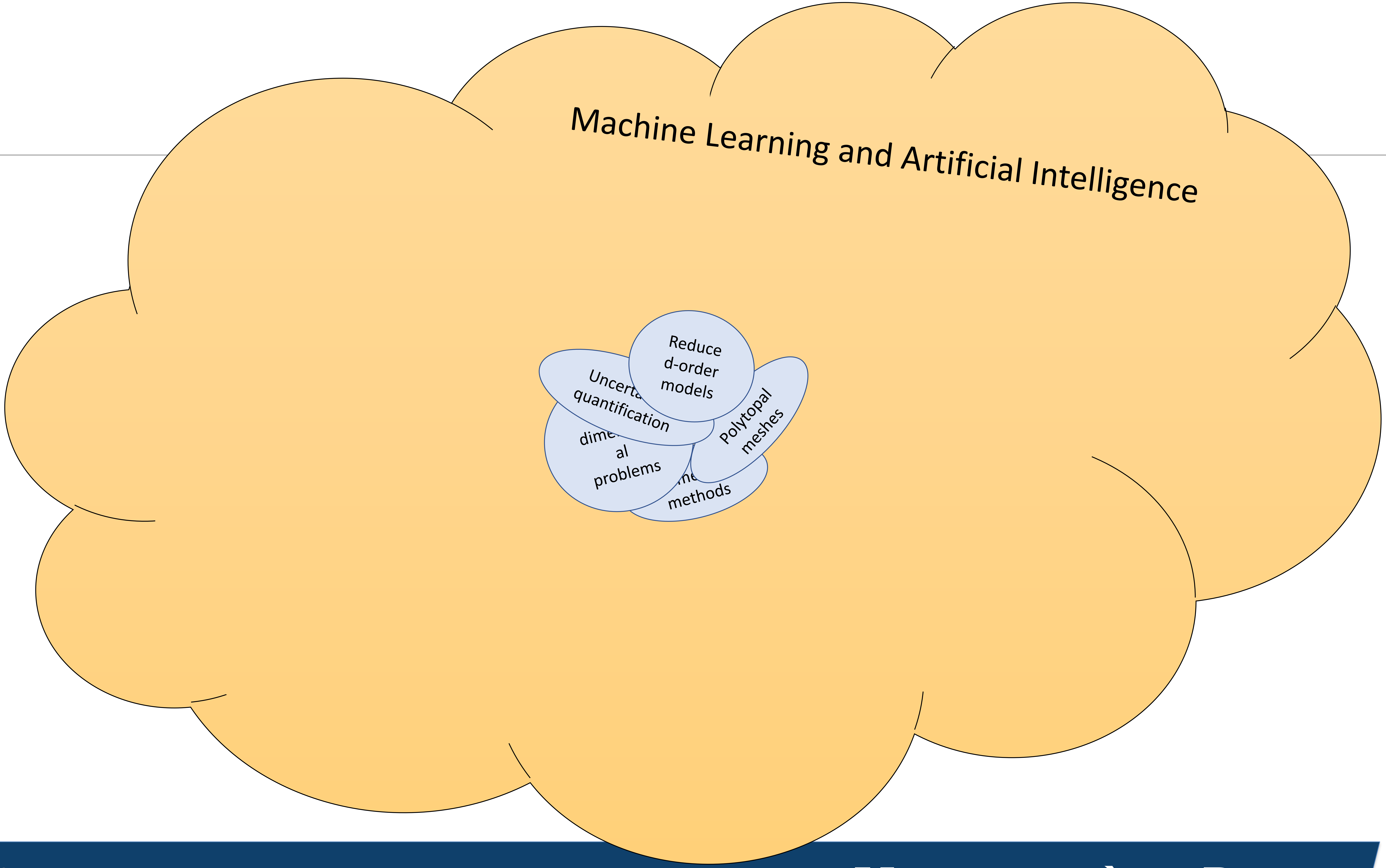
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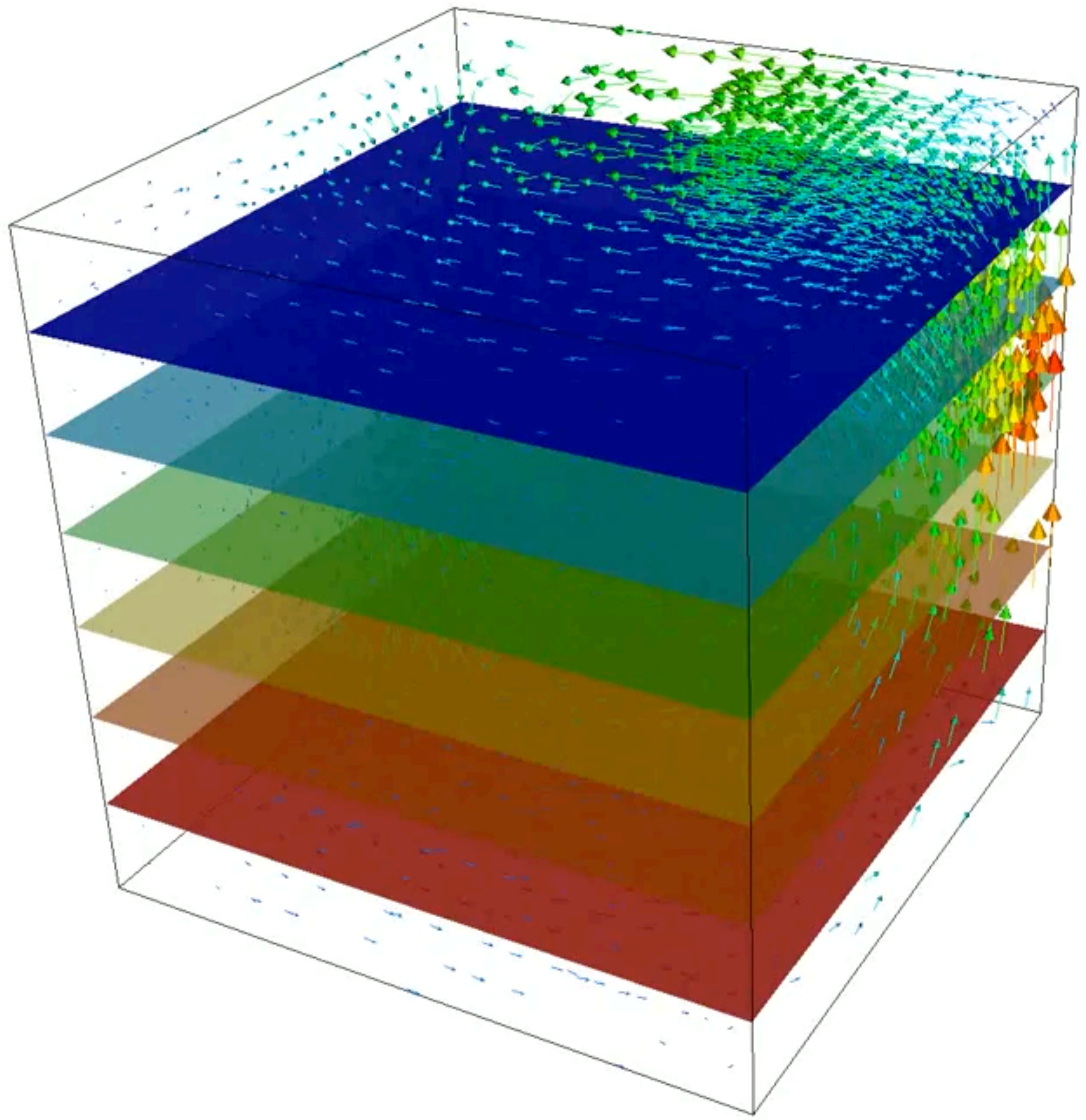
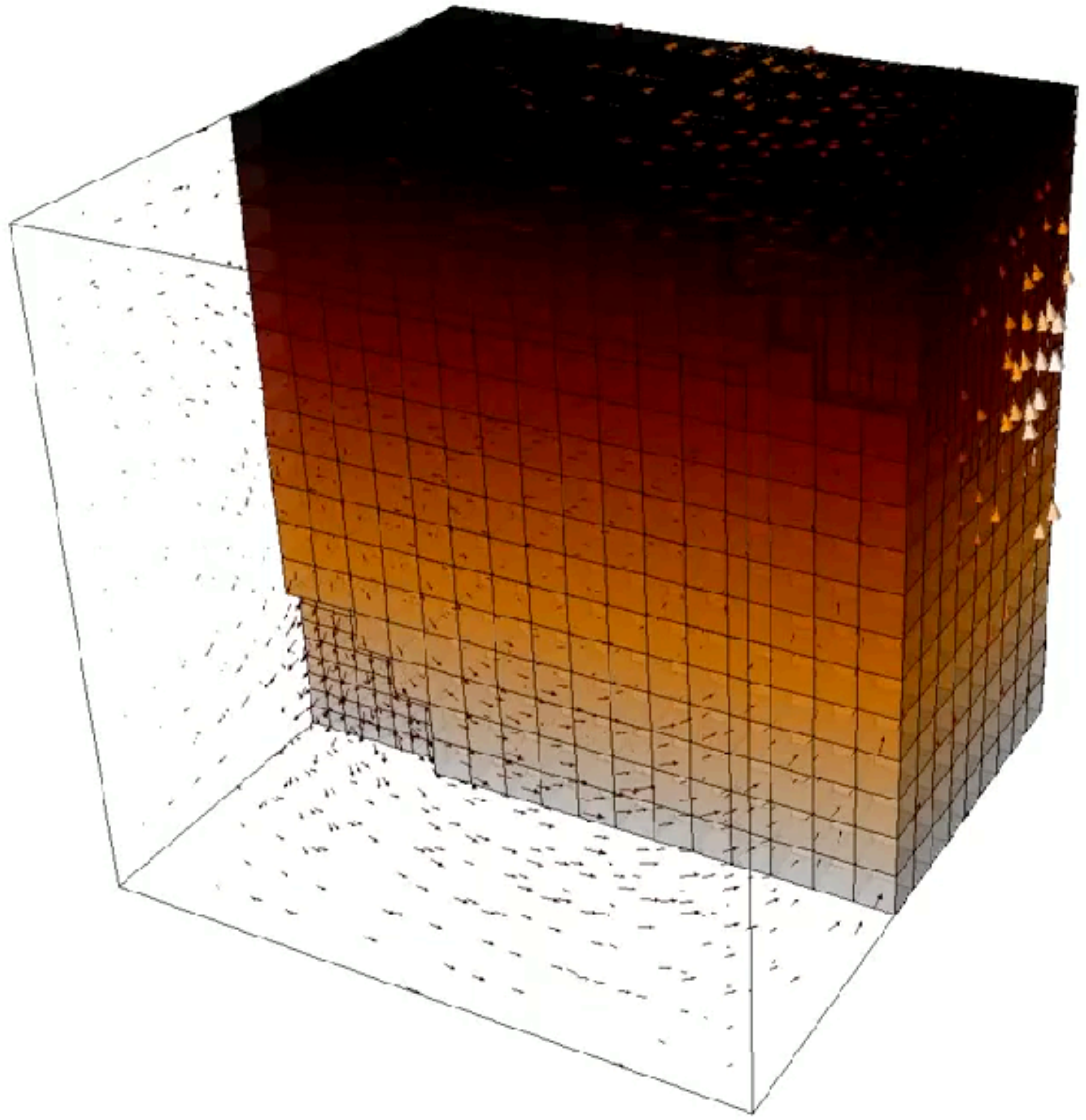
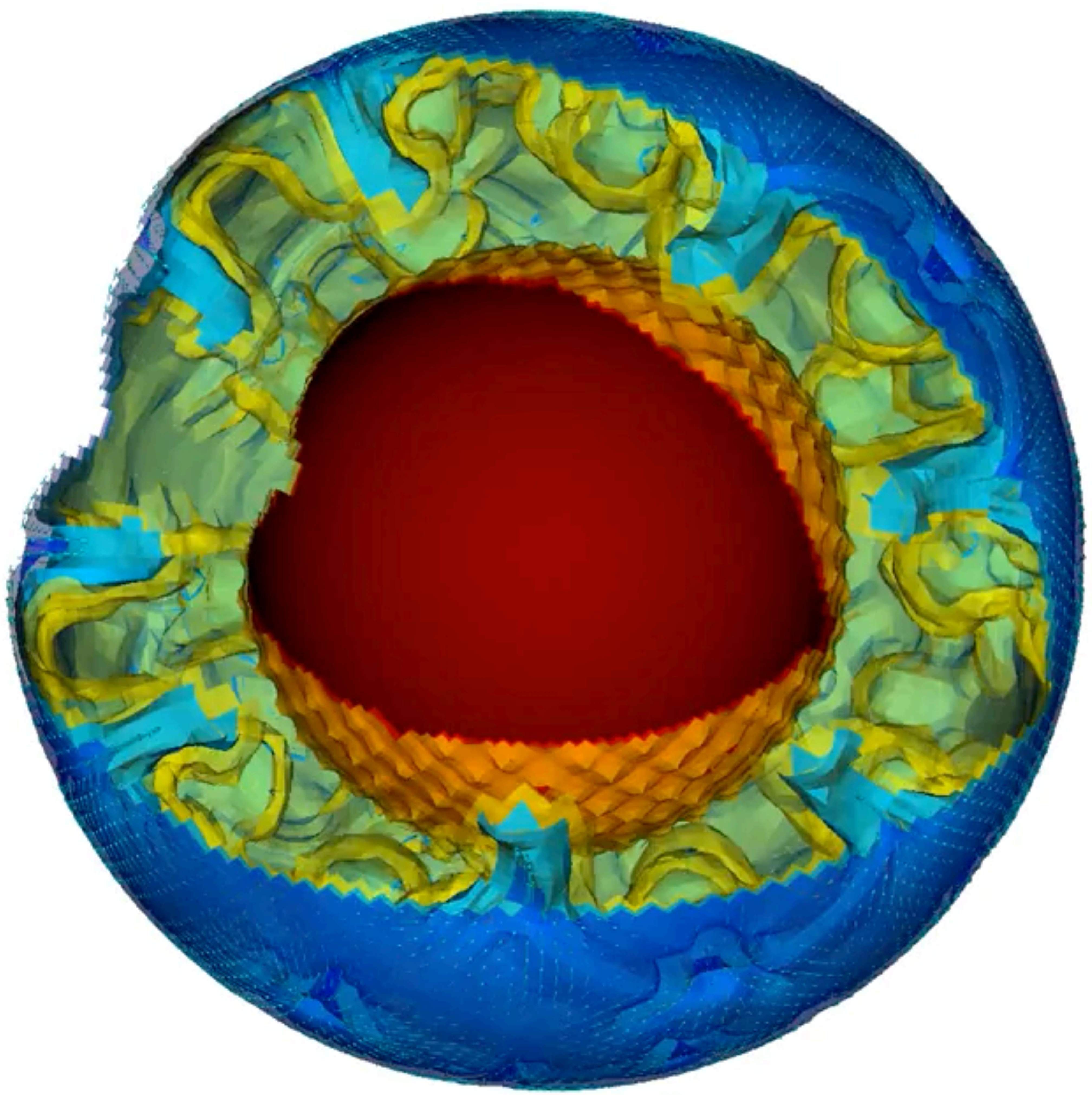
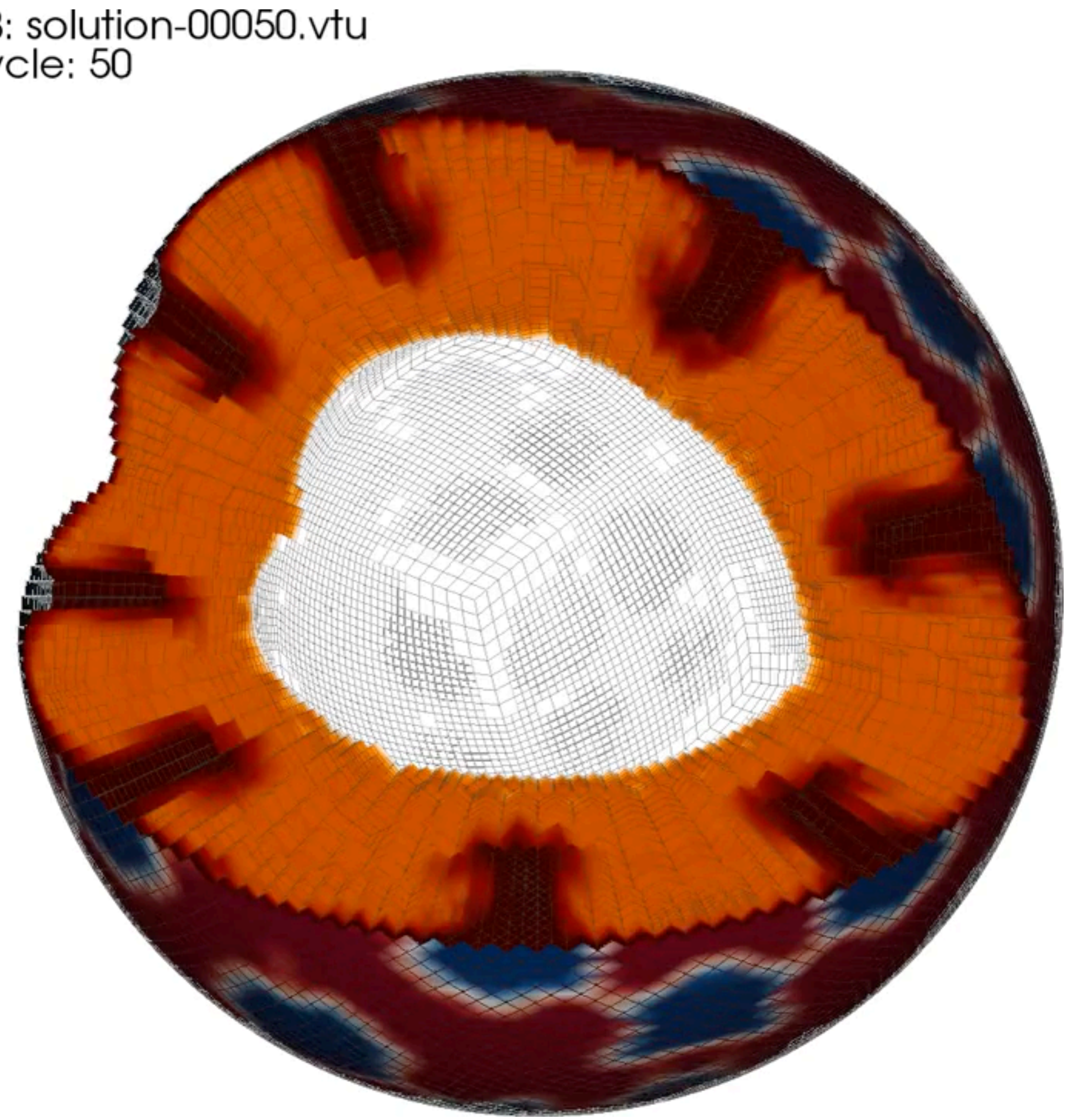
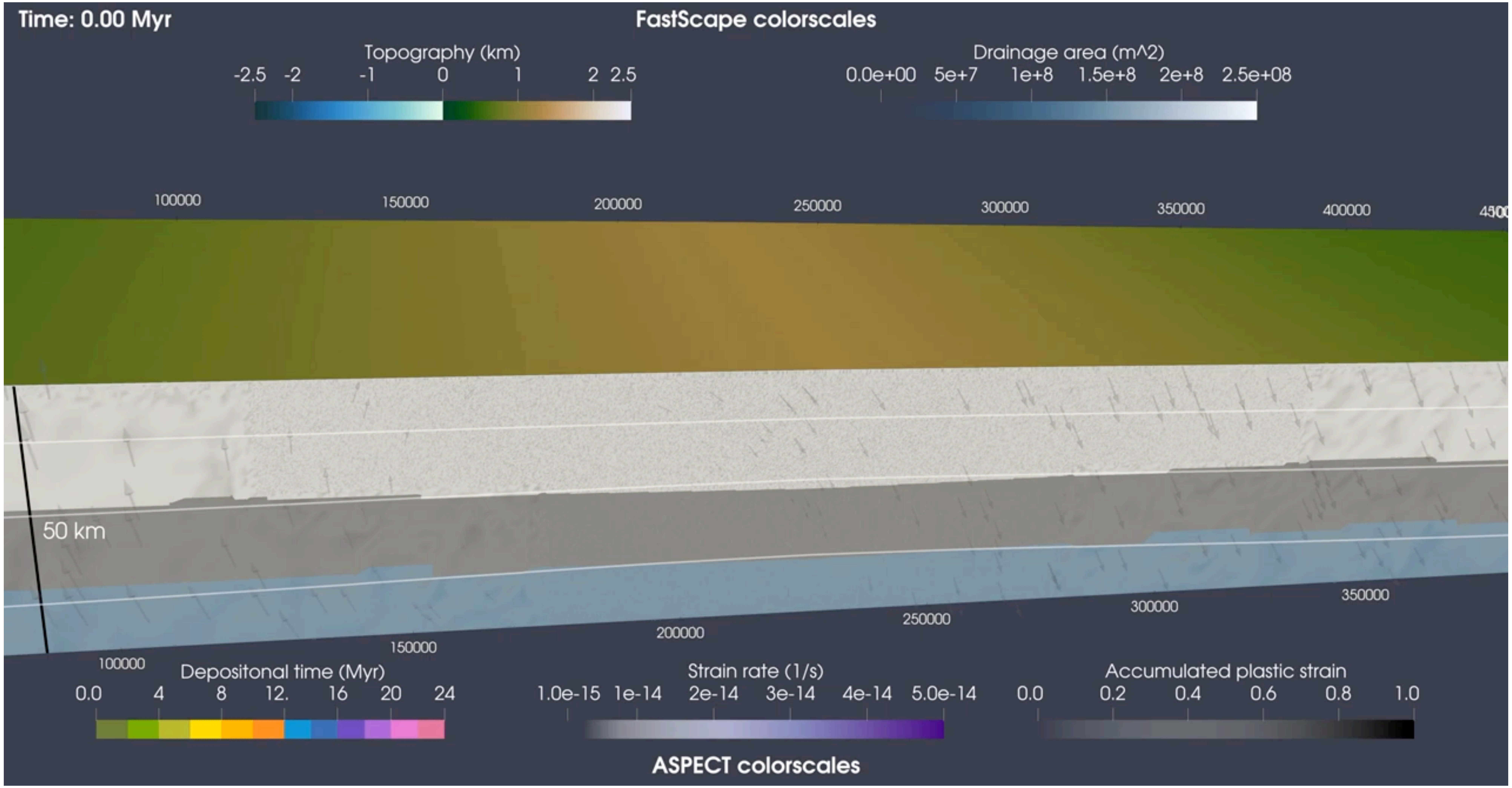
What you will learn








DB: solution-00050.vtu
Cycle: 50



Evolution of Rift Systems and Their Fault Networks in Response to Surface Processes.

Derek Neuharth, Sascha Brune, Thilo Wrona, Anne Glerum, Jean Braun, Xiaoping Yuan. Tectonics. Volume 41, Issue3, 2022.

Exploiting high-contrast Stokes preconditioners to efficiently solve incompressible fluid-structure interaction problems,
M. Wichrowski, S.Stupkiewicz, P.Krzyzanowski, **LH,** 2023.