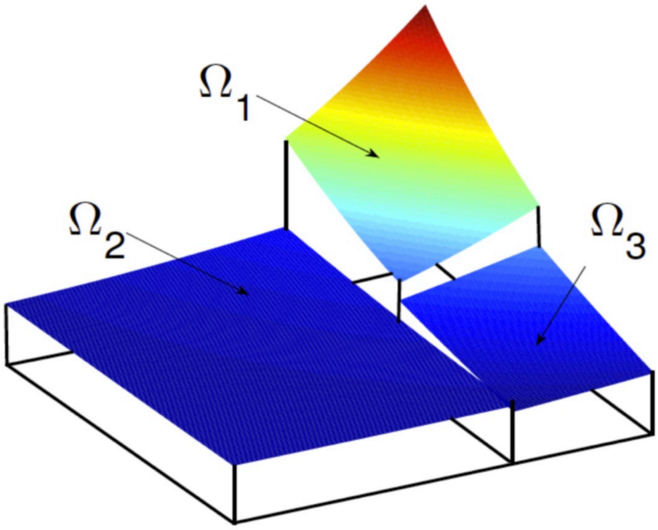

Metodi Matematici per Equazioni alle Derivate Parziali

Discontinuous Galerkin Methods

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Consistency error

Continuous pb: $A : V \longrightarrow V'$, $f \in V'$

find $u \in V$ s.t. $Au = f$ in V'

- $\|Au\|_{V'} \leq \|A\|_* \|u\|_V$

$\|A\|_* := \sup_{v \in V} \frac{\|Av\|_{V'}}{\|v\|}$

$\langle Au, v \rangle \overset{\Leftrightarrow}{\leq} \|A\|_* \|u\|_V \|v\|_V$

$= \sup_{u \in V} \sup_{v \in V} \frac{\langle Au, v\rangle}{\|u\| \|v\|}$

- $\langle Au, u \rangle \geq \alpha \|u\|_V \|u\|_V$

Now we approximate A and f. Take $\boxed{V_h \subset V}$

$A_h : V_h \longrightarrow V_h'$

$f_h \in V_h'$

Family of approximated problems.

Approximated problems: Given $f_h \in V_h'$ find u_h s.t.

$$A_h u_h = f_h \quad \text{in } V_h^*$$

Theorem: if A_h is uniformly α_0 elliptic and Continuous, then

$$\boxed{\|u-u_h\|_V \leq \left(\frac{C}{\alpha_0}+1\right)\inf_{v_h\in V_h}\left(\|u-v_h\|_V + \|Av_h - A_h v_h\|_{V_h'}\right) + \|f-f_h\|_{V_h'}}$$

If $A_h \in V'$ as well (i.e., $a_h : V\times V \xrightarrow{\text{bilinear}} \mathbb{R}$)

$$\boxed{\|u-u_h\|_V \leq \left(1+\frac{C}{\alpha_0}\right)\inf_{v_h\in V_h}\|u-v_h\| + \|A_h u - f_h\|_{V_h'}}$$

* Uniformly α_0-elliptic and continuous: $\exists\, C, \alpha_0$ s.t. $\forall h$

$$\langle A_h u_h, v_h\rangle \leq C\|u_h\|_V \|v_h\|_V$$

$$\langle A_h u_h, u_h\rangle \geq \alpha_0 \|u_h\|_V \|u_h\|_V$$

Note:
$$\boxed{\|f_h\|_{V_h'} := \sup_{v_h\in V_h}\frac{\langle f_h, v_h\rangle}{\|v_h\|_V}}$$

Proof: $\alpha_0\|u_h - v_h\|_V^{\cancel{2}} \leq \langle A_h(u_h-v_h), u_h-v_h\rangle \pm \langle A(u-v_h), u_h-v_h\rangle$

$$\leq \langle f_h, u_h-v_h\rangle - \langle A_h v_h, u_h-v_h\rangle + \langle A(u-v_h), u_h-v_h\rangle$$
$$- \langle f, u_h-v_h\rangle + \langle Av_h, u_h-v_h\rangle$$

$$\leq \|f_h-f\|_{V_h'}\cancel{\|u_h-v_h\|_V} + \|Av_h - A_h v_h\|_{V_h'}\cancel{\|u_h-v_h\|} + \|A\|\,\|u-v_h\|\,\cancel{\|u_h-v_h\|}$$

Apply 1st Strang's lemma to $\|u_h - v_h\| \leq \|f_h - f\|_{V_h'}$ with $\|Av_h - A_h v_h\|_{V_h'} + \|A\| \, \|u - v_h\|_V$

$$\|u - u_h\| \leq \frac{C}{\alpha} \Big[\inf_{v_h \in V_h} \|u - v_h\|_V + |||A_h u - f_h|||_{*,h} \Big] \quad \text{consistency error on } f_h$$

$$\leq \left(\frac{1 + \|A\|}{\alpha_0} \right) \|u - v_h\|_V + \underbrace{\|Av_h - A_h v_h\|_{V_h'}}_{\text{consistency error on } A_h} + \|f_h - f\|_{V_h'}$$

if $A_h \in \mathcal{L}(V, V')$ this simplifies to:

$$\boxed{\|u - u_h\|_h \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V + \|A_h u - f_h\|_{V_h'}}$$

Example: replace $\langle f, v \rangle$ by $\sum_q f(x_q) v(x_q) w_q$

$\Rightarrow$ not in V' (pointwise ev.)

2nd Strang's lemma

$V_h \not\subset V \implies \tilde{V} := V_h + V$

i) Let $|||\cdot|||$ be a norm in $\tilde{V}$ (possibly h-dependent)

ii) $c\|u\| \leq |||u||| \leq C\|u\| \quad \forall u \in V$ equivalent to $\|\cdot\|$ on V

iii) Let A_h be uniformly bounded and coercive in $\tilde{V}$

* $\exists M > 0$ s.t., $\forall h$ $\langle A_h u, v \rangle \leq M |||u||| \, |||v||| \quad \forall v \in \tilde{V}$
* $\exists \alpha > 0$ s.t., $\forall h$ $\langle A_h u, u \rangle \geq \alpha |||u|||^2 \quad \forall u \in \tilde{V}$

Apply 1st Strang's Lemma to A_h, with $|||\cdot|||$ norm:

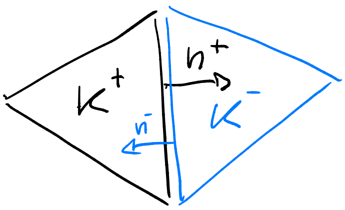
$$|||u - u_h||| \leq \frac{C}{\alpha} \left[\inf_{v_h \in V_h} |||u - v_h||| + |||A_h u - f_h|||_{*,h} \right]$$

Discontinuous Galerkin Methods

$$V_h := \{ v \in L^2(\Omega) \mid v|_K \in \mathbb{P}^e(K) \} \qquad V_h \not\subset H^1(\Omega)$$

Definitions:

- $\mathcal{T} := \bigcup_K \mathring{K} \quad \left(\mathcal{T}_h \setminus \bigcup_K \partial K\right)$ interior of all elements
- $\mathring{\mathcal{E}} := \bigcup_{\substack{K_i \in \mathcal{T}_h \\ K_j \in \mathcal{T}_h}} \overline{K_i} \cap \overline{K_j} =: \underline{\underline{e_{ij}}}$ skeleton of all interior faces
- $\mathcal{E}^{\partial\Omega} := \bigcup_{K_i \in \mathcal{T}_h} \overline{K_i} \cap \partial\Omega$ boundary faces
- $\mathcal{E} := \mathring{\mathcal{E}} \cup \mathcal{E}^{\partial\Omega}$ all faces
- $(u,v)_{\mathcal{T}} := \int_{\mathcal{T}} uv = \sum_K \int_K uv$ "broken" inner product
- $\langle u,v \rangle_{\mathcal{E}} := \int_{\mathcal{E}} uv = \sum_{e \in \mathcal{E}} \int_e uv$ "broken" inner product on faces.



$n^+ :=$ outer normal to K^+

$n^- :=$ outer normal to K^-

- $[u] := \begin{cases} u^+ n^+ + u^- n^- & x \in \mathring{\mathcal{E}} \\ u \cdot n & x \in \mathcal{E}^{\partial\Omega} \end{cases}$

- $\{v\} := \begin{cases} \frac{1}{2} v^+ + \frac{1}{2} v^- & x \in \mathring{\mathcal{E}} \\ v & x \in \mathcal{E}^{\partial\Omega} \end{cases}$

Property ①: $\underline{a^+ \underline{n}^+ \cdot \underline{b}^+ + a^- \underline{n}^- \cdot \underline{b}^- = [a]\{\underline{b}\} + \{a\}[\underline{b}]}$

Jump of a scalar is a vector $\quad [u] = u^+ \underline{n}^+ + u^- \underline{n}^-$

Jump of a vector is a scalar $\quad [\nabla u] = \nabla u^+ \cdot \underline{n}^+ + \nabla u^- \cdot \underline{n}^-$

Model Problem

| $f \in L^2(\Omega)$, Ω is convex and polygonal

$$1) \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega = \Gamma \end{cases}$$

$\Rightarrow u \in H^2(\Omega) \cap H^1_0(\Omega)$

Split on each K, and study the following: ($u \in H^2(\Omega) \cap H^1_0(\Omega)$)

$$\begin{aligned} &i) && -\Delta u - f = 0 && \text{on every } K \\ &ii) && [\![u]\!]|_e = 0 && \text{on } \mathcal{E} \\ &iii)^* && [\![\nabla u]\!]|_e = 0 && \text{on } \mathcal{E}^0 \end{aligned} \quad (2)$$

A sol. to (1) is also a sol to i-iii)

Space $H^2_{\mathcal{E}_h} := \{ v \in L^2(\Omega) \mid v|_K \in H^2(K) \}$

Variational form of (2), as general as possible (Brezzi, Cockburn, Marini, Süli, 2004)

B_0, $\underline{B}_1$, B_2 s.t.

$$(-\Delta u - f, B_0 v)_{\mathcal{T}} + \langle [\![u]\!], \underline{B}_1 v \rangle + \langle [\![\nabla u]\!], B_2 v \rangle = 0$$

$B_0 v = v$

$$\sum_K - \int_{\partial K} \nabla u \cdot n\, v = -\langle [\nabla u], \{v\} \rangle_{\mathcal{E}}$$

$$\underbrace{(\nabla u, \nabla v)_{\mathcal{T}} - (f, v)_{\mathcal{T}} - \langle [\![\nabla u]\!], \{v\} \rangle}_{(-\Delta u - f, B_0 v)} + \langle [\![u]\!], \underline{B}_1 v \rangle + \langle [\![\nabla u]\!], B_2 v \rangle = 0$$

Set $B_2 v = \{v\}$

$$\Rightarrow (\nabla u, \nabla v)_{\mathcal{T}} - (f, v)_{\mathcal{T}} + \langle [\![u]\!], \underline{B}_1 v \rangle = 0$$

$\alpha=1$

Method	$B_0 v$	$\mathbf{B}_1 v$	$B_2 v$
classical C^1-conforming	v	$([\![u]\!] \equiv 0)$	$([\![\boldsymbol{\sigma}(u)]\!] \equiv 0)$
classical C^0-conforming	v	$([\![u]\!] \equiv 0)$	v
IP [20]	v	$([\![u]\!] \equiv 0)$	$v + s_2 [\![\alpha \nabla v]\!]$
B.O. [7]	v	$\{\alpha \nabla v\}$	$\{v\}$
NIPG [26]	v	$\{\alpha \nabla v\} + s_1 [\![v]\!]$	$\{v\}$
IP [4, 28, 1]	v	$-\{\alpha \nabla v\} + s_1 [\![v]\!]$	$\{v\}$
D.S.W. [19]	v	$s_1 [\![v]\!]$	$\{v\}$

SIPG

(3) $(\nabla u, \nabla v)_{\mathcal{T}} - \langle \{\nabla u\}, [\![v]\!] \rangle_{\mathcal{E}} - \langle \{\nabla v\}, [\![u]\!] \rangle_{\mathcal{E}} + \langle s_1 [\![u]\!], [\![v]\!] \rangle_{\mathcal{E}} = (f, v)_{\mathcal{T}}$

Assume a norm : $|||u|||^2 := |u|^2_{1,\Omega} + \sum_e \frac{1}{|e|} \| [\![u]\!] \|^2_{0,e}$ on $\tilde{V} = V_h + V$

$\mathbb{P}^\ell(\mathcal{T})$ $H^1_0(\Omega)$

Take $u \in H^1_0(\Omega)$, $\langle A_h u, u \rangle = (\nabla u, \nabla u) = |u|^2_{1,\Omega}$

for $u \in V \equiv H^1_0(\Omega)$ A_h is cont and coercive (in V)

for $u \in V_h$ $|||\cdot|||$ is a norm that is equiv. to $|u|_{1,\Omega}$ in V.

and it is indeed a norm on V_h $\Rightarrow$ $|||u||| = 0 \Leftrightarrow u = 0$ in $\mathcal{T}$

$$\left(\sqrt{\varepsilon} a \pm \frac{b}{\sqrt{\varepsilon}}\right)^2 = \varepsilon a^2 + \frac{b^2}{\varepsilon} \pm 2ab \geq 0$$

$$\pm ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon} \quad \forall \varepsilon > 0$$

$u, v \in H^2(\overline{K})$, $u, v \in \mathbb{P}^\ell(K)$

$$\left| \int_e \nabla u \cdot n \, v \right| \leq \|\nabla u\|_{0,e} \|v\|_{0,e} \leq |u|_{1,e} \|v\|_{0,e} \leq C_I h_e^{-\frac{1}{2}} |u|_{\frac{1}{2},e} \|v\|$$

$$\leq C_T C_I h_e^{-\frac{1}{2}} |u|_{1,K} \|v\|_{0,e}$$

$$\langle A_h u, v \rangle \leq |u|_{1,\mathcal{T}} |v|_{1,\mathcal{T}} + C |u|_{1,\mathcal{T}} \left(\sum_{e \in \mathcal{E}} \frac{1}{h_e} \| [\![v]\!] \|_{0,e} \right) + C |v|_{1,\mathcal{T}} \left(\sum_{e \in \mathcal{E}} \frac{1}{h_e} \| [\![u]\!] \|_{0,e} \right)$$

$$+\sum_e s_1 \|[u]\|_{0,e} \|[v]\|_{0,e}$$

$\exists M$ s.t. $\langle A_h u, v\rangle \leq M |||u|||\ |||v|||$

Ellipticity: $\exists \alpha$ s.t.

$$\langle A_h u, u\rangle \geq \alpha |||u|||^2$$

$$\langle A_h u, u\rangle = |u|_{1,\Omega}^2 + \sum_e \left[-2\int_e \{\nabla u\}[u] + s_1 \int_e [u][u]\right]$$

$\textcircled{1}$, $\textcircled{2}$

we have to choose s_1 s.t. $\exists\ c, \alpha$ ind. of h s.t.

$$\langle A_h u, v\rangle \lesssim c\ |||u|||\ |||v|||$$

and s.t.

$$\langle A_h u, u\rangle \gtrsim \alpha |||u|||^2$$

$$\Rightarrow \quad s_1 \geq \frac{c}{h}$$

(this guarantees we control 1 with 2)

$$|||u-u_h||| \leq \inf_{v_h\in V_h}\left[\, |||v-v_h||| + \cancel{|||A_h u - f_h|||_{*,h}}^{\,0} \right]$$

Theorem:

$$|||v - I_h v||| \leq c\, h^{\ell}\, |v|_{\ell+1,\Omega}$$

$v \in H^{\ell+1}$ $\quad \ell \geq 1$

$$|||v - I_h v|||_h^2 = |v - I_h v|_{1,K}^2 + \sum_{e\in\partial K} \frac{1}{h_e} \cancel{\|[v - I_h v]\|_{0,e}^2}^{\,0}$$