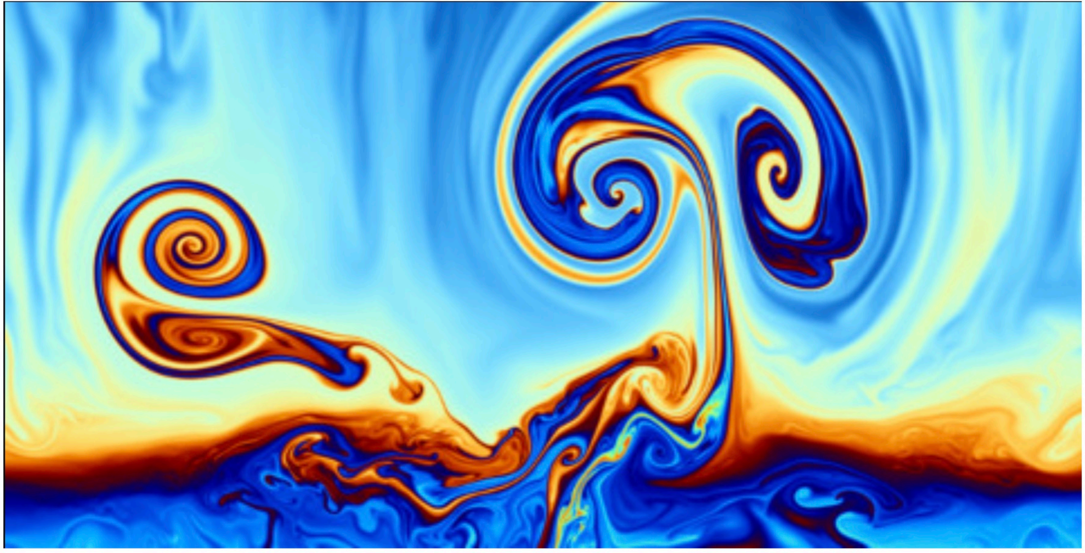


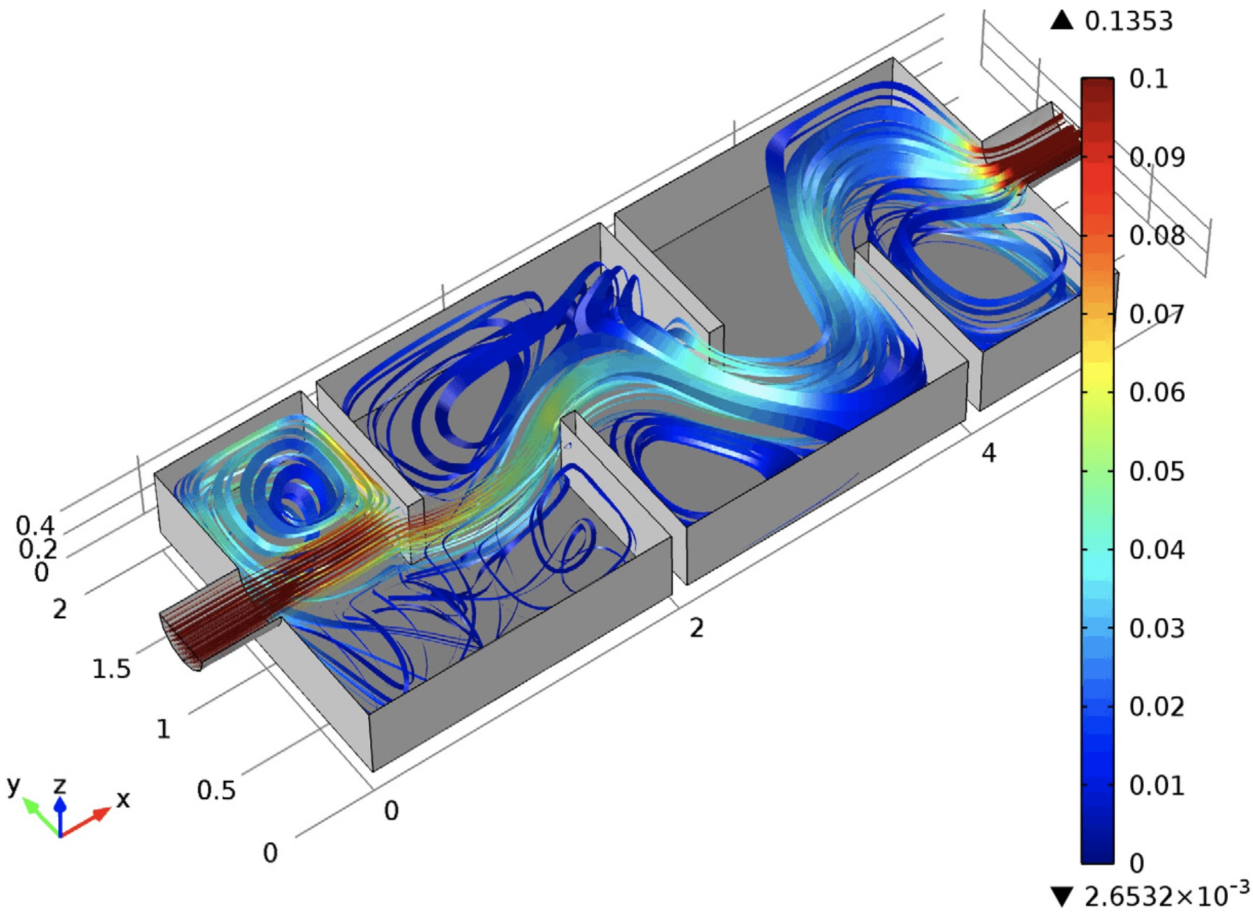
Metodi Matematici per Equazioni alle Derivate Parziali

Banach-Necas-Babuska condition and Petrov Galerkin discretizations

Luca Heltai <luca.heltai@unipi.it>

Dipartimento di Matematica

Università di Pisa



General setting: Banach Spaces

V Banach

W Banach and reflexive

$A \in \mathcal{L}(V, W')$ $\quad a \in \mathcal{L}(V \times W; \mathbb{R})$

$$\boxed{\begin{aligned} A : V &\longrightarrow W' \\ u &\longmapsto Au \in W' \end{aligned}} \iff \begin{aligned} a : V \times W &\longrightarrow \mathbb{R} \\ u, v &\longmapsto a(u,v) \end{aligned}$$

$\boxed{\text{Pb}}$ Given $f \in W'$, find $u \in V$ s.t.

$$Au = f \text{ in } W' \iff a(u,v) = \langle f, v \rangle \quad \forall v \in W$$

$\boxed{\text{Well posedness (Hadamard)}}$

$\exists \alpha$ s.t. $\forall f \in W'$ 1), $\exists!$ u 2) $\in V$ s.t.

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{W'} = \frac{1}{\alpha} \|Au\|_{W'}$$

1) A has to be surjective
2) A has to be injective
3) A has to be bounding

1, 2, 3 are different faces of the same medal

- Closed Range Theorem
- Open Map theorem

$$A : V \longrightarrow W' \qquad A \in \mathcal{L}(V, W')$$

$$A^T : W \longrightarrow V' \qquad a \in \mathcal{L}(V \times W, \mathbb{R})$$

$$K = \ker(A) = \{ v \in V \ \text{s.t.} < Av, w > = 0 \ \forall w \in W \}$$

$$H = \ker(A^T) = \{ w \in W \ \text{s.t.} < v, A^T w > = 0 \ \forall v \in V \}$$

$Z \subset H$ we call "polar of Z" or "annihilator" of Z

$$Z^\circ := \{ f \in H' \mid < f, z > = 0 \ \forall z \in Z \}$$

By construction and continuity of $<\cdot, \cdot>$ we have that

$$\boxed{Z^\circ = \overline{Z^\circ}} \qquad \boxed{\ker(A) = \overline{\ker(A)}}$$

$$\boxed{\begin{aligned} \ker(A) &= \operatorname{Im}(A^T)^\circ \\ \ker(A^T) &= \operatorname{Im}(A)^\circ \end{aligned}}$$

$$0 = < Au, w > = < u, A^T w > \quad \forall v \in K$$

$$0 = < u, A^T w > = < Au, w > \quad \forall w \in H$$

Closed Range Theorem

- $(Z^\circ)^\circ = Z \iff Z = \overline{Z}$
- $\boxed{\ker(A)^\circ = \operatorname{Im}(A^T)} \iff \operatorname{Im}(A^T) = \overline{\operatorname{Im}(A^T)}$
- $\boxed{\ker(A^T)^\circ = \operatorname{Im}(A)} \iff \operatorname{Im}(A) = \overline{\operatorname{Im}(A)}$

Simple and trivial example

$$A : H^1_0(\Omega) \equiv V \longrightarrow L^2(\Omega) \equiv W' \quad (= L^2)$$

$Av = v \qquad \forall v \in H^1_0(\Omega)$

A is not surjective $\quad (\exists q \in L^2$ s.t. $\nabla q \notin L^2$ $($ $)$

$H^1_0(\Omega) \equiv \mathrm{Im}(A)$ is dense in $L^2(\Omega)$

But $\mathrm{Im}(A)$ is NOT closed in L^2

$\ker(A) = \{0\}$

$A^T : L^2 \longrightarrow H^{-1}(\Omega) = (H^1_0(\Omega))' = V'$

$\forall q \in L^2 \longrightarrow (A^T q)(u) = \int_\Omega u q = \langle u, A^T q\rangle = \langle u, q\rangle$

$(u,q) = \langle u, q\rangle = 0 \quad \forall q \in L^2 \Rightarrow u = 0$

$\ker(A^T) = \{0\}$

In finite dimensions: $\ker(A) = \{0\}$ $\ker(A^T) = \{0\}$

$\Rightarrow A$ is invertible

In Banach Spaces it is not enough

$\forall w_n \in L^2$ cauchy s.t. $w_n \to w \notin V \equiv H^1_0(\Omega)$

and $w_n \in \mathrm{Im}(A)$, $\Rightarrow \exists v_n \in V$ s.t. $Av_n = w_n$

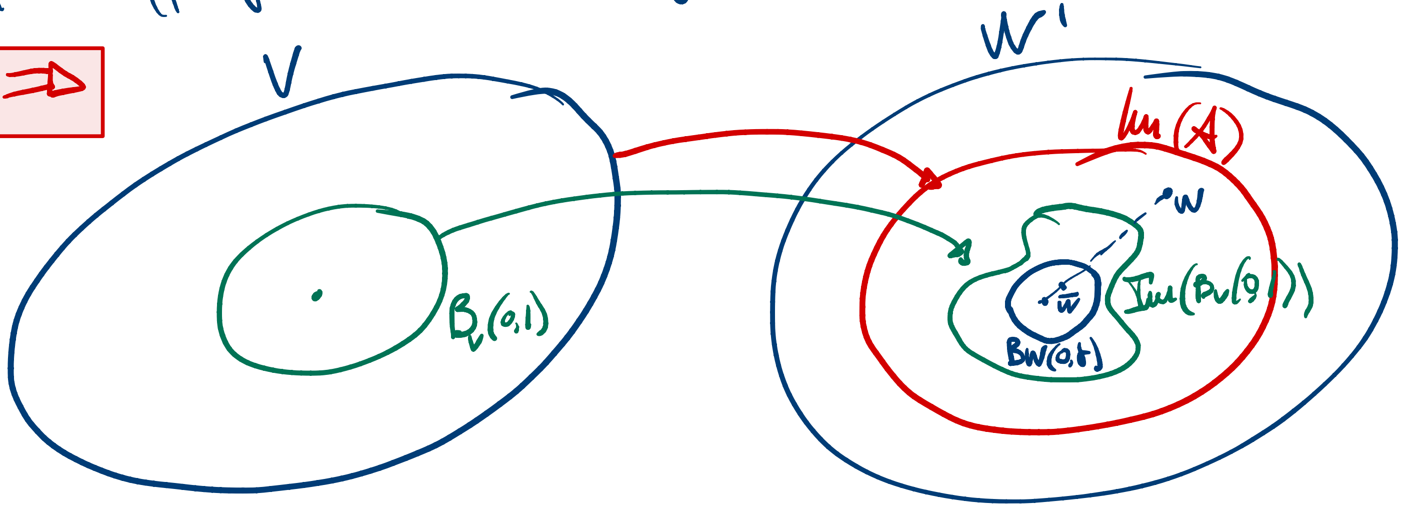
but $v_n \not\to v \in V \qquad \mathrm{Im}(A) \neq \overline{\mathrm{Im}(A)}$

Theo (Open Mapping and Closed Range Theorems)

$\boxed{\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}} \iff \boxed{\exists \alpha \mid \forall w' \in \mathrm{Im}(A) \ \exists u \text{ s.t. } Au = w' \quad \|Au\|_{W'} = \|w'\|_{W'} \geq \alpha \|u\|_V}$

Open Mapping theo: A is surjective, $A(U)$ is open $\forall U$ open

$\boxed{\Rightarrow}$



i) If $\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$ then $\overline{\mathrm{Im}(A)}$ is a linear subspace

$\Rightarrow$ we can apply open map theorem with $\mathrm{Im}(A)$ as target space. A: surjective between

$V \longrightarrow \mathrm{Im}(A) \equiv \tilde{W}'$

ii) $B_V(0,1)$: unit ball in $V := \{ v \in V \text{ s.t. } \|v\|_V < 1 \}$ (open set)

iii) $A(B_V(0,1)) \supset O_W \Rightarrow \exists \gamma \text{ s.t. } B_W(0,\gamma) \subset \mathrm{Im}(B_V(0,1))$

iv) $\forall 0 \neq w \in \mathrm{Im}(A), \alpha < \gamma \quad \bar{w} = \alpha \dfrac{w}{\|w\|_{W'}} \in B_W(0,\gamma)$

$\Rightarrow \exists z \in B_V(0,1) \text{ s.t. } Az = \bar{w} \quad \Rightarrow \|z\| \leq 1$

v) $v = z \cdot \frac{\|w\|_{W'}}{\alpha} \Rightarrow Av = w$

$$\Rightarrow \|v\| = \|z\| \frac{1}{\alpha} \|w\|_{W'} \lesssim \frac{1}{\alpha} \cdot \|Av\|$$

$$\|Av\| \geq \alpha \|v\| \qquad \underline{\text{QED}}$$

$\boxed{\Leftarrow}$

$\|Av\| \geq \alpha \|v\| \Longrightarrow \operatorname{Im}(A) = \overline{\operatorname{Im}(A)}$

1) $w_n \in \operatorname{Im}(A)$ Cauchy $\Rightarrow \exists w$ s.t. $w_n \to w \in W'$

2) $\forall w_n \in \operatorname{Im}(A)$ $\exists v_n$ s.t. $Av_n = w_n$

$$\|w_n\|_{W'} = \|Av_n\|_{W'} \geq \alpha \|v_n\|_V$$

w_n Cauchy $\Rightarrow$ v_n is also cauchy

$\Rightarrow \exists v \in V$ s.t. $v_n \to v$

A is continuous $\Rightarrow Av_n \longrightarrow Av \in \operatorname{Im}(A)$

$\Rightarrow \operatorname{Im}(A)$ is closed

$\boxed{\text{Equivalent statements}}$ same for A^T

i) A^T is surjective

ii) A is injective and $\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}$

iii) A is bounding $\left(\exists \alpha \mid \|Au\|_{W'} \geq \alpha \|u\|_V \right)$

iv) the inf sup condition is satisfied

$$\exists \alpha \quad \text{s.t.}$$

$$\inf_{v \in V} \sup_{w \in W} \frac{\langle Av, w \rangle}{\|u\|_V \|w\|_W} \geq \alpha$$

A surjective $V \longrightarrow W'$ $\alpha > 0$

P1) $\forall w \in \mathrm{Im}(A)$ $\exists v_w \in V$ s.t.

$$A v_w = w \quad \text{and}$$

$$\|A v_w\|_{W'} \geq \alpha \|w\|_V$$

Implies

P2) $\|A^T w\|_{V'} \geq \alpha \|w\|_W \quad \forall w \in W$

P2) $\Rightarrow$ P1) $\boxed{\text{if } V \text{ is Reflexive}}$

Hilbert Case and $\boxed{\text{BNB}}$

$$A : V \longrightarrow W' \qquad V, W \text{ Hilbert spaces}$$

$$\exists \alpha \mid \forall f \in W' \quad \exists!\, u \in V \quad \text{s.t. } Au = f$$

$$\|u\| \leq \frac{1}{\alpha}\|f\| = \frac{1}{\alpha}\|Au\|$$

$$\Longleftrightarrow$$

BNB for Hilbert spaces

$$\boxed{\begin{array}{l} \exists \alpha > 0 \quad \text{s.t.} \\ \text{i)} \quad \inf\limits_{u \in V} \sup\limits_{w \in W} \dfrac{\langle Au, w\rangle}{\|u\|_V \|w\|_W} = \alpha \\ \\ \text{ii)} \quad \inf\limits_{w \in W} \sup\limits_{u \in V} \dfrac{\langle Au, w\rangle}{\|u\|_V \|w\|_W} = \alpha \end{array}}$$

For Banach, ii) becomes $\ker(A^T) = \{0\}$

LAX MILGRAM : $W \equiv V$,

$\Rightarrow$ BNB (the converse is false)

$$\langle Au, u\rangle \geq \alpha \|u\|^2 \Rightarrow \frac{\langle Au, u\rangle}{\|u\| \, \|u\|} \geq \alpha \quad \forall u \in V$$

Cea's lemma for Petrov-Galerkin.

$$V_h \subset V \quad , \quad Q_h \subset Q$$

$$\Pi_h : V_h \to V \qquad P_h : W_h \longrightarrow W$$

$$\boxed{\begin{aligned} &\langle \Pi_h v, v_h \rangle = \langle v, v_h \rangle \quad \forall\, v_h \in V_h \\ &\langle P_h p, q_h \rangle = \langle p, q_h \rangle \quad \forall\, q_h \in Q_h \end{aligned}}$$

$$\Pi_h^T : V' \longrightarrow V_h'$$

$$P_h^T : W' \longrightarrow W_h'$$

Given $f \in W'$, find $u \in V$ s.t.

$$\underline{a(u,v) = \langle f, v \rangle \qquad \langle Au, v \rangle = \langle f, v \rangle} \qquad \forall v \in W$$

Given $f \in W'$, find $u \in V_h$ s.t.

$$a(u_h, v_h) = \langle f, v_h \rangle \qquad \forall\, v_h \in W_h$$

$$a(u - u_h, v_h) = 0 \qquad \forall\, v_h \in W_h$$

Discrete inf sup: $\exists\, \alpha_h$ (independent on h)

(1)
$$\inf_{u_h \in V_h} \sup_{v_h \in W_h} \frac{a(u_h, v_h)}{\|u_h\|\,\|v_h\|} \geq \alpha_h$$

(2)
$$\inf_{v_h \in W_h} \sup_{u_h \in V_h} \frac{a(u_h, v_h)}{\|u_h\|\,\|v_h\|} \geq \alpha_h$$

$$\alpha_h \leq \alpha$$

$$\|u - u_h\| \leq \|u - v_h\| + \|v_h - u_h\|$$

$$\leq \|u - v_h\| + \frac{1}{\alpha}\|A(v_h - \underline{\underline{u_h}})\|_{*,W_h}$$

$$= \|u - v_h\| + \frac{1}{\alpha}\|A(v_h - \underline{\underline{u}})\|_{*,W_h}$$

$$\|u - u_h\| \leq \left(\underset{=}{1} + \frac{\|A\|}{\alpha}\right)\|u - v_h\| \qquad \forall\, v_h \in V_h$$

$$\Longleftrightarrow$$

$$\|u - u_h\| \leq \left(1 + \frac{\|A\|}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|$$

$$\frac{1}{\alpha_h} = \|A_h^{-1}\| \quad \Rightarrow \quad \|u - u_h\| \leq \left(1 + \|A_h^{-1}\|\,\|A\|\right) \inf_{v_h \in V_h} \|u - v_h\|$$

$$\|A^{-1} f\| = \|u\| \qquad \|u\| \leq \|A^{-1}\|\,\|f\| \qquad = \|A^{-1}\|$$

$$\Uparrow$$

$$\|f\| \leq \alpha \|u\| \quad \Rightarrow \quad \|u\| \leq \frac{1}{\alpha}\|f\|$$

Who is A_h?

$$\langle A_h u_h, v_h \rangle = \langle A u_h, v_h \rangle \quad \forall u_h \in V_h, \ \forall v_h \in W_h$$

$$\langle A_h \Pi_h u_h, P_h v_h \rangle = \langle A \Pi_h u_h, P_h v_h \rangle$$

$$\Rightarrow \quad A_h = P_h^T A \Pi_h$$

$$A_h : V \longrightarrow W'$$

$$\underset{V}{u} \xrightarrow{\Pi_h} \underset{V_h}{\Pi_h u} \xrightarrow{A} \underset{W'}{A\Pi_h u} \xrightarrow{P_h^T} \underset{W_h'}{P_h^T A \Pi_h u}$$

$$< P_h^T A \Pi_h u, w > \;=\; < A\Pi_h u, P_h w >$$

$$\forall v_h \in V_h \qquad \Pi_h v_h = v_h \quad \Rightarrow$$

$$\underbrace{P_h^T A \Pi_h}_{A_h} v_h = P_h^T A v_h \qquad \forall v_h \in V_h$$

$$v_h = A_h^{-1} P_h^T A v_h \qquad \forall v_h \in V_h$$

$$\|u - u_h\| = \|u - v_h + v_h - u_h\|$$

$$= \|u - v_h + A_h^{-1} P_h^T A (v_h - u_h)\|$$

By construction $P_h^T A u = P_h^T f$ $\qquad P_h^T A u = P_h^T A_h u_h$

$$P_h^T A_h u_h = P_h^T f \qquad = P_h^T A u_h$$

$$= \|u - v_h + A_h^{-1} P_h^T A (v_h - u)\|$$

$$\|u - u_h\| \leq \Big(1 + \underbrace{\|A_h\|^{-1}}_{= \frac{1}{\alpha_h}} \|A\|\Big) \|v_h - u\| \qquad \forall v_h \in V_h$$