To Weight or not to Weight: Where is the Question? *

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Abstract

We investigate the approximability properties of several weighted problems, by comparing them with the respective unweighted problems. For an appropriate (and very general) definition of niceness, we show that if a nice weighted problem is hard to approximate within r, then its polynomially bounded weighted version is hard to approximate within $r - o(1)$. Then we turn our attention to specific problems, and we show that the unweighted versions of MIN VERTEX COVER, MIN SAT, MAX CUT, MAX DIRECTED CUT, MAX 2SAT, *and* MAX EXACT kSAT *are exactly as hard to approximate as their weighted versions. We note in passing that* MIN VERTEX COVER *is exactly as hard to approximate as* MIN SAT. *In order to prove the reductions for* MAX 2SAT, MAX CUT, MAX DIRECTED CUT, *and* MAX E3SAT *we introduce the new notion of "mixing" set and we give an explicit construction of such sets. These reductions give new non-approximability results for these problems.*

1. Introduction

One of the most important results obtained in the last decade in the field of computational complexity has been the so-called *PCP-theorem* [3, 4], that is, the fact that any language in NP admits a probabilistic verifier of membership proof using a logarithmic number of random bits and a constant number of queries. Besides being *per se* interesting, this theorem led to many negative results in the study of the approximability properties of NP-hard optimization problems. One for all, it was possible to prove that, unless P=NP, the MAX CLIQUE problem is not approximable within a factor n^ϵ for a given constant $\epsilon > 0$. Since the first proof of the PCP-theorem, several other refined proofs of the same result appeared in the literature in order to improve the performance of the verifier with respect to different query complexity parameters [7, 16, 8, 5]. The last of these, due to Bellare, Goldreich and Sudan [5], allowed to significantly improve the lower bounds on the approximability of several important optimization problems, such as MIN VERTEX COVER, MAX CUT, and MAX 2SAT. For most of these problems, however, the lower bounds hold for their weighted version only. For instance, in the case of MAX 2SAT, the repetition of clauses is needed in order to show a 1.01 lower bound: clearly, the repetition of clauses is equivalent to considering a polynomially bounded weighted version of the problem.

On the other hand, many approximation algorithms for optimization problems ensure a particular performance ratio only if weights are not allowed. In order to achieve the same performance ratio in the weighted case, algorithm developers had to consider different and often more complicated techniques. For instance, the 2-approximate algorithm for unweighted MIN VERTEX COVER is based on a simple greedy procedure to find a maximal matching in the graph [19]. In order to obtain the first 2-approximate algorithm for the weighted version, instead, a linear programming formulation of the problem has been used [27]. In some cases, it is not known whether the same performance ratio is obtainable. For instance, while the unweighted MAX CLIQUE problem is approximable within $O(n/\log^2 n)$ where n denotes the number of nodes in the graph [9], the best approximation algorithm for the weighted version of this problem reaches a factor of $O((\log\log n)^2 n/\log^2 n)$ [21].

Finally, it is well-known that several NP-hard optimization problems turn out to be tractable whenever a polynomial bound is imposed on the weights that appear in the instance [17]. For example, the MIN PAR-

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TITION problem can be solved in time $O(nb)$ where n denotes the number of elements and b denotes the sum of their weights: clearly, if any upper bound were imposed on the weights in advance, even a polynomial function of n, this algorithm would be a polynomial-time algorithm for the restricted problem.

The goal of this paper is to study the relative complexity of the arbitrarily weighted version, the polynomially bounded weighted version, and the unweighted version of an optimization problem with respect to their approximability properties. The above three considerations may suggest that these three versions have different hardness of approximation. Surprisingly, we will see that for several interesting problems the approximation threshold is exactly the same for all three versions.

Many optimization problems have been studied both in the weighted and in the unweighted version. A list of such results is contained in the compendium of NP optimization problems by Crescenzi and Kann [11]. From a more structural point of view, we recall the following results.

In [28] Papadimitriou and Yannakakis extend their approach based on logical definability to weighted problems and introduce the class of weighted MAX NP problems. They also show that the very same algorithmic technique can be used to approximate both the problems in MAX NP and those in the weighted version of this class. More recently, Zimand [31] studied the approximability properties of logically defined weighted problems, allowing negative weights (but non-negative objective functions). It turns out that the hardness of approximation dramatically increases when negative weights are admitted.

Finally, in [13] Crescenzi and Trevisan proved that any arbitrarily weighted approximable problem is reducible to a polynomially bounded weighted approximable problem by means of an approximation-preserving reduction (this result has been used by Khanna, Motwani, Sudan and Vazirani [25] to prove the APX-completeness of MAX 3SAT).

The first result of this paper states that, for any weighted optimization problem satisfying a particular "niceness" property, *the approximation threshold of the unbounded version and that of a polynomially bounded version are equal.* Informally, an optimization problem is nice if the feasibility of a solution does not depend on its measure. Many important optimization problems are nice: for instance, MIN VERTEX COVER and all problems in MAX NP, such as MAX CUT and MAX SAT, are nice optimization problems.

Because of this result, we will restrict our attention on polynomially bounded weighted versions of nice optimization problems. In particular, we will consider the following well-known optimization problems: MIN VERTEX COVER, MIN SAT, MAX CUT, MAX DIRECTED CUT, MAX 2SAT, and MAX EXACT kSAT. For all of them, we will be able to prove that *the approximation threshold of the weighted version is exactly the same as the approximation threshold of the unweighted version.* The first two results will be obtained by showing quite simple approximation preserving reductions between the two versions of the problem (we note in passing that MIN VERTEX COVER is exactly as hard to approximate as MIN SAT). On the other hand, the result for MAX CUT and MAX DIRECTED CUT are based on a more sophisticated reduction technique which is mainly based on the existence of graphs with a particular "mixing" property. The existence of these graphs is a more or less direct consequence of previous results obtained in the field of hash functions. As far as we know, this is the first time that dense expander graphs are used in order to obtain non-approximability results (indeed, in [28] sparse expanders have been used to prove completeness results in the class MAX SNP). In order to extend our technique to MAX EXACT kSAT we introduce the new notion of "mixing" set and we give an explicit construction of such sets (we believe that this notion is *per se* interesting and may be applied to other derandomized constructions).

The above reductions imply improved non-approximability results for the unweighted versions of the MAX CUT and the MAX DIRECTED CUT problems and give the first explicit non-approximability results for the unweighted versions of the MAX E3SAT and the MAX 2SAT problems.

Finally, we will observe that our results can be extended to other maximization problems. In particular, our techniques can be applied to any MAX k-CSP problem [25] whose instances do not contain unary constraints.

An *optimization problem* A consists of three objects: (1) the set I of instances, (2) for any instance $x \in I$, a set $sol(x)$ of solutions, and (3) for any instance $x \in I$ and for any solution $y \in sol(x)$, a measure $m(x, y)$. The goal of an optimization problem is, given an instance x, to find an optimum solution y, that is, a solution whose measure is maximum or minimum depending on whether the problem is a maximization or a minimization one (for a formal definition of NP optimization problems see [10]). In the following opt will denote the function that maps an instance x into the measure of an optimum solution.

Let A be an optimization problem. For any instance x and for any solution $y \in sol(x)$, the *performance ratio* of y with respect to x is defined as

$$R(x,y) = \max\left\{\frac{\mathsf{opt}(x)}{m(x,y)}, \frac{m(x,y)}{\mathsf{opt}(x)}\right\}.$$

Observe that the performance ratio is always a number greater than or equal to 1 and is as close to 1 as the solution is close to an optimum solution.

Let $r : \mathbf{N} \to [1,\infty)$. We say that an algorithm T for an optimization problem A is $r(n)$*-approximate* if, for any instance x of size n, the performance ratio of the feasible solution $T(x)$ with respect to x is at most $r(n)$. If a problem A admits an r-approximate polynomial-time algorithm for some constant $r > 1$, then we will say that A belongs to the *class APX*.

The *approximation threshold* of an optimization problem $A \in$ APX is a real number $r_A \geq 1$ such that, for any $\epsilon > 0$, A admits an $(r_A + \epsilon)$-approximate polynomial-time algorithm but A does not admit an $(r_A - \epsilon)$-approximate polynomial-time algorithm.

Let A and B be two optimization problems and let α be a positive constant. A is said to be α*-AP reducible* to B [12], in symbols $A \leq^{\alpha}_{\mathrm{AP}} B$, if two functions f and g exist such that:

1. For any $x \in I_A$ and for any $r > 1$, $f(x,r) \in I_B$.
2. For any $x \in I_A$, for any $r > 1$, and for any $y \in sol_B(f(x,r))$, $g(x,y,r) \in sol_A(x)$.
3. f and g are computable by two algorithms T_f and T_g, respectively, whose running time is polynomial for any fixed r.
4. For any $x \in I_A$, for any $r > 1$, and for any $y \in sol_B(f(x,r))$, $R_B(f(x,r),y) \leq r$ implies $R_A(x,g(x,y,r)) \leq 1 + \alpha(r-1)$.

Observe that if $A \leq^{\alpha}_{\mathrm{AP}} B$ for any $\alpha > 1$, then the approximation threshold of A is at most equal to the approximation threshold of B. In particular, if $A \equiv^{\alpha}_{\mathrm{AP}} B$ for any $\alpha > 1$, then the two problems have the same approximation threshold.

The unweighted versions of the following problems can be obtained by simply imposing that $\omega(\cdot)$ is a constant function equal to 1.

Min Weighted Vertex Cover

Instance: A pair (G,ω) where $G = (V,E)$ is a graph and $\omega : V \to N$.

Solution: A vertex cover for G, i.e., a subset $V' \subseteq V$ such that, for each edge $(u,v) \in E$, at least one of u and v belongs to V'.

Measure: Sum of the weights of the nodes in the vertex cover.

Min Weighted Sat

Instance: A pair (C,ω) where C is a collection of disjunctive clauses of literals and $\omega : C \to N$.

Solution: A truth assignment for the variables in C.

Measure: Sum of the weights of the clauses satisfied by the truth assignment.

Max Weighted Cut

Instance: A pair (G,ω) where $G = (V,E)$ is a graph and $\omega : E \to N$.

Solution: A partition of V into disjoint sets V_1 and V_2.

Measure: Sum of the weights of the edges with one endpoint in V_1 and one endpoint in V_2.

Max Weighted kSat

Instance: A pair (C,ω) where C is a collection of disjunctive clauses of at most k literals and $\omega : C \to N$.

Solution: A truth assignment for the variables in C.

Measure: Sum of the weights of the clauses satisfied by the truth assignment.

Comment: The Max Weighted EkSat requires that each clause contains exactly k literals.

2. Polynomially Bounded Weights vs. Unbounded Weights

In the following we will consider optimization problems whose goal is to find a set of objects satisfying a given property with the maximum (or minimum) weight sum. More formally, we say that an optimization problem A is a *subset problem* if:

- Any instance x of A is a triple (U, x', ω), where $U = \{u_1, \ldots, u_n\}$ is a set of objects, x' is a (possibly empty) string, and $\omega : U \to N$ assigns a weight $\omega(u)$ to any object $u \in U$.
- With any solution $y \in sol(x)$ is associated a subset $S \subseteq U$, and the measure of y is equal to $\sum_{u \in S} \omega(u)$.

For example, consider the weighted version of the MAX SAT problem. An instance is a triple (C, ϵ, ω) where C is the set of clauses and $\omega : C \to N$ is the function that weights the clauses. A solution is a truth-assignment to the variables in C: with any solution we can then associate the subset $C' \subseteq C$ of satisfied clauses and the measure of the truth-assignment is the sum of the weights of the elements of C'.

A subset problem A is *nice* if, for any instance (U, x', ω), if y is a feasible solution, then, for any function $\omega' : U \to N$, y is a feasible solution for (U, x', ω') as well. Roughly speaking, this property says that the definition of feasible solution in A is independent of the weights. It is easy to see that, for instance, MIN WEIGHTED VERTEX COVER, MAX WEIGHTED CUT and MAX WEIGHTED 3SAT are nice.

Let A be a subset problem and p be a polynomial. We denote by A^p the restriction of A to instances x such that the sum of the weights is at most $p(|x|)$.

The following result exploits the scaling techniques used in [22] to develop a fully polynomial-time approximation scheme for the knapsack problem.

Theorem 1 *Let A be a nice subset problem in* APX *and let $r > 1$. A polynomial p exists such that if A^p is r-approximable, then A is $(r + 1/n)$-approximable.*

Corollary 1 *Let A be a nice subset problem such that $A \in$ APX. Then a polynomial p exists such that A and A^p have the same approximability threshold.*

It is easy to see that the previous theorem can be extended to any nice subset problem that admits an n^c-approximate algorithm where c is a constant and n denotes the number of objects in the instance. For example, it is possible to prove that a polynomial p exists such that if MAX WEIGHTED CLIQUE^p is approximable within $r(n)$ then MAX WEIGHTED CLIQUE is approximable within $r(n) + 1/n$, where n denotes the number of nodes of the input graph.

3. Minimization Problems

The MIN VERTEX COVER and the MIN SAT problems are both nice ones. In this section, we will mainly deal with the unweighted and the polynomially bounded weighted version of these two problems.

Lemma 1 *For any polynomial p,* MIN WEIGHTED VERTEX COVER^p $\leq^1_{\mathrm{AP}}$ MIN VERTEX COVER.

PROOF: Let $G = (V, E)$ be a graph and $\omega : V \to N$ be a polynomially bounded weight function for its nodes. Let us define the graph $G^\omega = (V^\omega, E^\omega)$ as follows. For any node $u \in V$ such that $\omega(u) = w$, V^ω contains w distinct nodes $u^1, \ldots, u^w$. If $(u, v) \in E$ is an edge of G, then E^ω contains all the edges (u^i, v^j) for $i = 1, \ldots, \omega(u)$ and $j = 1, \ldots, \omega(v)$. We note that, given G and ω, G^ω can be constructed in polynomial time.

Let C be a solution for (G, ω) whose measure is c. Then there exists a solution for G^ω of size c. Indeed, we can simply consider the set $C^\omega = \{u^i \in V^\omega : u \in C \wedge i = 1, \ldots, \omega(u)\}$.

On the other hand, let C be a solution for G^ω of size c. From C, we can recover in polynomial time a solution C' for (G, ω) whose measure is at most c. To this aim, assume that a node $u \in V$ and two indices $i, j \in \{1, \ldots, \omega(u)\}$ exist such that $u^i \in C$ and $u^j \notin C$. Then, $C - \{u^i\}$ is still a feasible solution for G^ω. Indeed, u^i and u^j have the same neighborhood, and since $u^j \notin C$ it follows that all the nodes that are adjacent to u^i belong to C and thus there is no need of taking u^i in the cover. By an easy induction argument, it follows that there exists a subset C^ω of C such that for any $u \in U$ either all its copies belong to C^ω, or none does. C^ω can be transformed into a solution C' for (G, ω) of measure $|C^\omega| \leq c$. $\diamond$

The following two lemmas can be proved by using arguments similar to the reduction between MAX CLIQUE and zero-free bit PCP given in [5] and the reduction between zero-free bit PCP and MAX CLIQUE implicitly given in [15], respectively.[1]

Lemma 2 MIN VERTEX COVER$\leq^1_{\mathrm{AP}}$ MIN SAT.

Lemma 3 *For any polynomial p,* MIN WEIGHTED SAT^p *is 1-AP reducible to* MIN WEIGHTED VERTEX COVER^p.

From the above three lemmas, the main result of this section follows.

Theorem 2 *The approximability threshold of the four problems* MIN (WEIGHTED) SAT *and* MIN (WEIGHTED) VERTEX COVER *is the same.*

[1] From the proof of these two lemmas it is also possible to show that MIN SAT is equivalent to its restricted version where each variable occurs exactly twice, once positively and once negatively. Note that a similar result is not known to hold for MAX SAT.

PROOF: From Corollary 1 it follows that a polynomial p exists such that MIN WEIGHTED SAT^p and MIN SAT have the same approximability threshold and MIN WEIGHTED VERTEX COVER^p and MIN VERTEX COVER have the same approximability threshold. The above three lemmas also imply that MIN WEIGHTED SAT^p $\equiv^1_{\rm AP}$ MIN WEIGHTED VERTEX COVER^p $\equiv^1_{\rm AP}$ MIN VERTEX COVER$\equiv^1_{\rm AP}$ MIN SAT. The theorem thus follows. ◇

4. The MAX WEIGHTED CUT Problem

In this section we prove how to reduce r-approximating the MAX WEIGHTED CUT^p problem to $(r - o(1))$-approximating the simple MAX CUT problem, for any polynomial p. We first give a rough sketch of the reduction, and then we present it formally. Let $(G = (V, E), \omega)$ be a weighted graph; we define a graph $\hat{G} = (\hat{V}, \hat{E})$ in the following way: for each node $u \in V$ there are N nodes $u^1, \ldots, u^N$ in $\hat{V}$, and for each edge $(u, v) \in E$ of weight w, $\hat{E}$ contains the set of edges $\{(u^i, v^j)|(i, j) \in S^w\}$, where S^w is a random set of w elements of $N \times N$. Any cut of cost c in G clearly yields a cut of cost c in $\hat{G}$. Consider now a cut $\hat{C}$ in $\hat{G}$. For any $u \in V$, let p_u be the ratio of nodes $u^1, \ldots, u^N$ in $\hat{C}$, and let $q_u = 1 - p_u$. Since the sets S^w are random, it follows that

$$m(\hat{G}, \hat{C}) \sim \sum_{(u,v)\in E} \omega(u, v)(p_u q_v + p_v q_u).$$

Now define a random cut C in G such that the probability that $u \in V$ is equal to p_u (observe that this is also known as a random rounding). Clearly, the expected measure of this cut satisfies the following equality:

$$\begin{aligned} \mathbf{E}[m(G, \omega, C)] &= \sum_{(u,v)\in E} \omega(u, v)(p_u q_v + p_v q_u) \\ &\sim m(\hat{G}, \hat{C}). \end{aligned}$$

We will now be more formal and we will show how to *derandomize* the reduction. We first define a kind of graphs that will play the role of the random sets S^w.

Definition 1 *A bipartite graph $G = (V_1, V_2, E)$ is said to be (n, w, δ, ϵ)-mixing if $|V_1| = |V_2| = n$, and for any two subsets $A \subseteq V_1$, $B \subseteq V_2$ with at least δn nodes, the following holds*

$$\left| Cut(A, B) - w\frac{|A||B|}{n^2} \right| \leq \epsilon w \frac{|A||B|}{n^2}$$

where $Cut(A, B)$ denotes the number of edges between a node in A and a node in B.

We note in passing that a (n, w, δ, ϵ)-mixing graph has at least $(1 - \epsilon)w$ and at most $(1 + \epsilon)w$ edges.

It is well known that expander graphs have good mixing properties. Indeed, the proof of the following theorem is mainly based on the explicit construction of expanders in order to construct mixing graphs.

Theorem 3 *For any $\epsilon, \delta > 0$, two constants c and n_0 exist such that for any $n > n_0$ and for any $w \geq cn$, an (n, w, ϵ, δ)-mixing graph can be constructed in time polynomial in n.*

We are now ready to state and prove the main result of this section.

Theorem 4 *For any $\alpha > 1$,* MAX WEIGHTED CUT $\leq^\alpha_{\rm AP}$ MAX CUT.

PROOF: Let $(G = (V, E), \omega)$ be an instance of MAX WEIGHTED CUT. Without loss of generality (see Theorem 1) we can assume that a polynomial q exists such that $\omega(e) \leq q(|V|)$ for any edge $e \in E$.

Let ϵ, δ be such that $0 < \epsilon \leq 1/2$, $0 < \delta < 1/2$, and

$$\alpha \geq \frac{(1 + \epsilon)}{(1 - \epsilon)(1 - 12\delta)}.$$

Let c, n_0 be the constants of Theorem 3 relative to ϵ and δ. We assume that there exists an integer $N > n_0$ such that, for any edge e, $cN \leq \omega(e) \leq N^2$ (otherwise we can multiply each weight by $c^2 n_0^2 w_{\max}$, where $w_{\max}$ is the maximum weight, and then set $N = cn_0 w_{\max}$).

In the following, $[N]$ denotes the set $\{1, \ldots, N\}$. We construct an unweighted graph $\hat{G} = (\hat{V}, \hat{E})$ as follows. For any $cN \leq w \leq N^2$ let $M_w = ([N], [N], E_w)$ be the (N, w, ϵ, δ)-mixing graph whose existence is guaranteed by Theorem 3. We let $\hat{V}$ contain N copies of any vertex of V, and, for any edge e of weight $\omega(e)$ of G, we let $\hat{E}$ contain the edges of $M_{\omega(e)}$. More formally,

$$\hat{V} = \{v^i \;:\; v \in V \text{ and } i = 1, \ldots, N\}$$

and

$$\hat{E} = \{(u^i, v^j) \;:\; (u, v) \in E \text{ and } (i, j) \in E_{\omega(u,v)}\}.$$

Claim 1: for any cut C in (G, ω) there exists a cut $\hat{C}$ in $\hat{G}$ such that $m(\hat{G}, \hat{C}) \geq (1 - \epsilon)m(G, \omega, C)$.

Proof of Claim 1: let $\hat{C} = \{u^i \;:\; u \in C \text{ and } i \in [N]\}$. Then,

$$\begin{aligned} m(\hat{G}, \hat{C}) &= \sum_{(u,v)\in E, |\{u,v\}\cap C|=1} |E_{\omega(u,v)}| \\ &\geq \sum_{(u,v)\in E, |\{u,v\}\cap C|=1} (1 - \epsilon)\omega(u, v) \\ &= (1 - \epsilon)m(G, \omega, C). \end{aligned}$$

Claim 2: assume that, for any node u in G, we are given two rationals $p_u, q_u \geq 0$ such that $p_u + q_u = 1$, then we can find in polynomial time a cut C such that

$$m(G,\omega,C) \geq \sum_{(u,v)\in E} \omega(u,v)(p_u q_v + q_u p_v).$$

Proof of Claim 2: this is a folklore result. For the sake of completeness we sketch the proof: consider the *random* cut where each vertex u is chosen independently and with probability p_u. The expected measure of such a cut is clearly equal to $\sum_{(u,v)\in E} \omega(u,v)(p_u q_v + q_u p_v)$. Using the method of conditional probabilities (see e.g. [2]), a cut with measure not smaller than this expectation is constructible in polynomial time.

Claim 3: given a cut $\hat{C}$ in $\hat{G}$, we can construct in polynomial time a cut C for G such that $m(G,\omega,C) \geq m(\hat{G},\hat{C})(1-12\delta)/(1+\epsilon)$.

Proof of Claim 3: we assume that $\hat{C}$ is a local optimum, and thus

$$m(\hat{G},\hat{C}) \geq 1/2 \sum_{e\in E}(1-\epsilon)\omega(e) \geq 1/4 \sum_{e\in E}\omega(e).$$

We first transform $\hat{C}$ into a new cut $\hat{C}'$ such that, for any $u \in V$, at least δN copies of u belong to $\hat{C}'$ and at least δN copies belong to $\hat{V} - \hat{C}'$. Clearly $m(\hat{G},\hat{C}')$ can be smaller than $m(\hat{G},\hat{C})$. However, from the mixing property of each graph $M_{\omega(u,v)}$, it follows that

$$\begin{aligned} m(\hat{G},\hat{C}') &\geq m(\hat{G},\hat{C}) - 2\sum_{e\in E}(1+\epsilon)\delta\omega(e) \\ &\geq m(\hat{G},\hat{C})(1-12\delta). \end{aligned}$$

Let $p_u = |\{u^i : u^i \in \hat{C}'\}|/N$. Using the preceding claim we can construct a cut C such that

$$m(G,\omega,C) \geq \sum_{(u,v)\in E} \omega(u,v)(p_u(1-p_v) + p_v(1-p_u)).$$

On the other hand, consider the measure of $\hat{C}'$: for any edge $(u,v) \in E$ we have to consider the edges connecting the Np_u copies of u in $\hat{C}'$ with the $N - Np_v$ copies of v not in $\hat{C}'$, plus the edges connecting the $N - Np_u$ copies of u not in $\hat{C}'$ with the Np_v copies of v in $\hat{C}'$. Recalling that such edges are chosen from a mixing graph, we have that

$$\begin{aligned} m(\hat{G},\hat{C}') &\leq \sum_{(u,v)\in E}\Big[\frac{(1+\epsilon)\omega(u,v)}{N^2}(Np_u(N-Np_v)) \\ &\qquad + \frac{(1+\epsilon)\omega(u,v)}{N^2}(Np_v(N-Np_u))\Big] \\ &\leq (1+\epsilon)m(G,\omega,C). \end{aligned}$$

From the above claims it follows that given a solution $\hat{C}$ for $\hat{G}$ whose performance ratio is at most r, we can construct in polynomial time a solution C whose performance ratio is

$$\begin{aligned} \frac{\mathsf{opt}(G,\omega)}{m(G,\omega,C)} &\leq \frac{\mathsf{opt}(\hat{G})/(1-\epsilon)}{m(\hat{G},\hat{C})(1-12\delta)/(1+\epsilon)} \\ &\leq \frac{1+\epsilon}{(1-12\delta)(1-\epsilon)} r \leq \alpha r. \end{aligned}$$

We have thus shown that MAX WEIGHTED CUT $\leq^{\alpha}_{\mathrm{AP}}$ MAX CUT. ◇

Remark 1 *A similar argument can be applied to* MAX DIRECTED CUT *thus showing that* MAX WEIGHTED DIRECTED CUT $\leq^{\alpha}_{\mathrm{AP}}$ MAX DIRECTED CUT, *for any* $\alpha > 1$.

From the results of Bellare, Goldreich and Sudan [6] and from the results of Sorkin, Sudan, Trevisan and Williamson [29] we thus obtain the following.

Corollary 2 MAX CUT *is not* 54/53-*approximable and* MAX DIRECTED CUT *is not* 43/42-*approximable unless* P = NP.

This improves over the non-approximability result by Bellare, Goldreich and Sudan (see [6], page 57).

5. The MAX SAT Problem

In order to deal with various versions of the MAX SAT problem we have to introduce the notion of *mixing set* of tuples.

Definition 2 *Let $n > 0$, $k > 0$ be integers; a set $S \subseteq [n]^k$ is k-ary (n,w,ϵ,δ)-mixing if for any k sets $A_1,\ldots,A_k \subseteq [n]$, each with at least δn elements, the following holds:*

$$\left| |S \cap A_1 \times \cdots \times A_k| - w\frac{|A_1|\cdots|A_k|}{n^k} \right| \leq \epsilon w \frac{|A_1|\cdots|A_k|}{n^k}.$$

For any fixed k and for sufficiently large values of n and w, k-ary mixing sets are efficiently constructible.

Theorem 5 *For any integer $k \geq 2$ and for any two rationals $\epsilon, \delta > 0$, two constants n_0 and c exist such that, for any $n \geq n_0$ and for any w with $cn \leq w \leq n^k$, a k-ary (n,w,ϵ,δ)-mixing set is constructible in time polynomial in n.*

PROOF: We first prove the theorem assuming that $k = 2^h$ is a power of two. We proceed by induction on h. For $h = 1$ the theorem follows from Theorem 3. Assume that the theorem holds for $k = 2^h$ and consider the case $2k = 2^{h+1}$. Let $\epsilon, \delta > 0$ be arbitrary positive constants, let $\bar{\epsilon}, \bar{\delta}$ be greater than zero and such that

$$\frac{(1+\bar{\epsilon})^3}{(1-\bar{\epsilon})^2} - 1 \leq \epsilon \ , \ 1 - \frac{(1-\bar{\epsilon})^3}{(1+\bar{\epsilon})^2} \leq \epsilon \text{ and } \bar{\delta} \leq \frac{(1-\bar{\epsilon})}{1+\bar{\epsilon}}\delta^k.$$

The inductive hypothesis ensures the existence of constants c, n_0 such that for any $n, m > n_0$ and any $w > cm$, a 2^h-ary $(n, cn, \bar{\epsilon}, \delta)$-mixing set and a binary $(m, w, \bar{\epsilon}, \bar{\delta})$-mixing set exist. Let $n \geq n_0$ and $w \geq c^2(1+\bar{\epsilon})n$ be integers, let S be a k-ary $(n, cn, \bar{\epsilon}, \delta)$-mixing set with m elements and let B be a binary $(m, w, \bar{\epsilon}, \bar{\delta})$-mixing set. Note that B exists since

$$w \geq c^2(1+\bar{\epsilon})n \geq cm.$$

We can fix an arbitrary (say, lexicographic) order among the tuples of S, that is a bijection between S and $[m]$. Using this bijection, we can see the pairs of B as $2k$-tuples over $[n]$. Under this mapping, we are going to show that B is a $2k$-ary (n, w, ϵ, δ)-mixing set.

Let $A_1, \ldots, A_{2k} \subseteq [n]$ be any family of $2k$ sets such that, for any $i = 1, \ldots, 2k$ $|A_i| \geq \delta n$. We define two sets T and U such that

$$T = S \cap (A_1 \times \cdots \times A_k) \text{ and } U = S \cap (A_{k+1} \times \cdots \times A_{2k}).$$

We first note that, from the mixing property of S, it follows that

$$(1-\bar{\epsilon})cn \leq m \leq (1+\bar{\epsilon})cn,$$

that

$$(1-\bar{\epsilon})cn\frac{|A_1|\cdots|A_k|}{n^k} \leq |T| \leq (1+\bar{\epsilon})cn\frac{|A_1|\cdots|A_k|}{n^k},$$

and that

$$(1-\bar{\epsilon})cn\frac{|A_{k+1}|\cdots|A_{2k}|}{n^k} \leq |U| \leq (1+\bar{\epsilon})cn\frac{|A_{k+1}|\cdots|A_{2k}|}{n^k}.$$

We observe that $|T|, |U| \geq \bar{\delta}m$ and thus, from the mixing property of B, it follows that

$$(1-\bar{\epsilon})w\frac{|T||U|}{m^2} \leq |B \cap (T \times U)| \leq (1+\bar{\epsilon})w\frac{|T||U|}{m^2}.$$

Putting all the pieces together we get

$$\begin{aligned}\frac{(1-\bar{\epsilon})^3}{(1+\bar{\epsilon})^2}w\frac{|A_1|\cdots|A_{2k}|}{n^{2k}} &\leq |B \cap (A_1 \times \cdots \times A_{2k})| \\ &\leq \frac{(1+\bar{\epsilon})^3}{(1-\bar{\epsilon})^2}w\frac{|A_1|\cdots|A_{2k}|}{n^{2k}}.\end{aligned}$$

To conclude the proof, we note that from a k-ary (n, w, ϵ, δ)-mixing set S (where $w \leq n^{k-1}$) we can obtain a $(k-1)$-ary (n, w, ϵ, δ)-mixing set S' as follows

$$S' = \{(a_1, \ldots, a_{k-1}) : \exists a_{k+1}.(a_1, \ldots, a_{k-1}, a_k) \in S\}.$$

◇

Theorem 6 *For any $\alpha > 1$,* MAX WEIGHTED E3SAT $\leq_{\mathrm{AP}}^{\alpha}$ MAX E3SAT.

PROOF: We only give the definition of the reduction: its correctness similarly to the proof of Theorem 4. Let $\phi = \{\mathsf{c}_1, \ldots, \mathsf{c}_m\}$ be a set of clauses, each over three distinct literals, let $\omega(\mathsf{c}_i)$ be the weight of clause c_i, let $X = \{x_1, \ldots, x_n\}$ be the set of variables of ϕ. Let $\epsilon, \delta \in (0, 1)$ be such that

$$\alpha \geq \frac{(1+\epsilon)(1+2\delta)}{1-\epsilon}.$$

Let c, n_0 be such that for any $N \geq n_0$ and $w \geq cN$ a ternary (N, w, ϵ, δ)-mixing set exists. Assume that a $N \geq n_0$ exists such that for any clause c of ϕ, $cN \leq \omega(\mathsf{c}) \leq N^3$ (again, this can be done with no loss of generality). For any $cN \leq w \leq N^3$ let S^w be a ternary (N, w, ϵ, δ)-mixing set. For any clause c of ϕ let $Var(\mathsf{c})$ be the set of variables occurring in c. Finally, let $\mathsf{c}[y/x]$ be the clause obtained by substituting y for x in c. We define a new set of clauses $\hat{\phi}$ over the set of variables $\hat{X} = \{x^i \ : \ x \in X, i \in [N]\}$ as follows.

$$\begin{aligned}\hat{\phi} = \{\mathsf{c}[x^i/x][y^j/y][z^h/z] \quad : \quad & \mathsf{c} \in \phi, (i,j,h) \in S^{\omega(\mathsf{c})}, \\ & Var(\mathsf{c}) = \{x, y, z\}\}.\end{aligned}$$

◇

The above theorem can be easily generalized to MAX WEIGHTED EkSAT for any $k \geq 1$. As a consequence of the theorem, the results of Bellare, Goldreich and Sudan [5, 6], and the 1-AP reduction given by Yannakakis [30] from MAX WEIGHTED 2SAT to MAX WEIGHTED E2SAT, new non-approximability results follow.

Corollary 3 MAX 2SAT *is not 74/73-approximable;* MAX E3SAT *and* MAX 3SAT *are not 27/26-approximable.*

6. Conclusions

We studied the approximability properties of several weighted problems, by comparing them with the respective unweighted versions. For a very general and natural class of weighted problems, we showed that if a problem in the class is hard to approximate within

r, then its polynomially bounded weighted version is hard to approximate within $r - o(1)$. Then we considered specific problems, and we showed that the unweighted versions of Min Vertex Cover, Min Sat, Max Cut, Max 2Sat, and Max EkSat are exactly as hard to approximate as their weighted versions. The reductions for Max Cut implied an improved non-approximability result for the problem, and the reductions for Max E3Sat and Max 2Sat gave the first explicit non-approximability results for these problems.

Our results can be extended to any Max kCSP problem whose instances do not contain unary constraints, such as the d-Regular Hypergraph Trasversal studied by Alimonti and by Kann, Lagergren and Panconesi [1, 24] and the Max k Cut (also known as Max k-Colorable Subgraph) studied by Papadimitriou and Yannakakis [28] and by Kann, Khanna, Lagergren and Panconesi [23].

We believe that our reduction can be extended to the general Max Sat problem. This should require the construction of k-ary (n, w, ϵ, δ)-mixing sets with $k = n^{\Omega(1)}$ and a 1-AP reduction from the general Max Sat problem to the restriction of Max Sat to formulas without unary constraints. The latter step seems quite difficult.

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Appendix

We assume that A is a maximization problem (the proof for minimization problems is similar). Let A be r_0-approximable, let T be an r_0-approximate polynomial-time algorithm for A, and let $t(x) = m(x,T(x))$. For any instance x, we have that

$$t(x) \leq \mathsf{opt}(x) \leq r_0 t(x).$$

Let $x = (U, x', \omega)$ be an instance of size n. Note that $|U| \leq n$. We define a new *scaled down* weight function $\tilde{\omega}$, such that, for any $u \in U$,

$$\tilde{\omega}(u) = \left\lfloor \frac{\omega(u)n^2}{t(x)} \right\rfloor.$$

Note that, since A is nice, the set of feasible solutions for (U, x', ω) is equal to that of $(U, x', \tilde{\omega})$. Moreover, we may assume without loss of generality that $\omega(u) \leq r_0 t(x)$ for any $u \in U$. It follows that $\tilde{\omega}(u) \leq r_0 n^2$ for any $u \in U$. Thus, $\tilde{x} = (U, x', \tilde{\omega})$ is an instance of $A^{r_0 n^3}$.

From the definition of $\tilde{\omega}$ it follows that, for any feasible solution y,

$$m(x,y) \geq m(\tilde{x},y)t(x)/n^2.$$

Moreover,

$$\begin{aligned} m(x,y) &= \sum_{u \in S} \omega(u) \leq \sum_{u \in S} \frac{t(x)}{n^2} \left\lceil \frac{\omega(u)n^2}{t(x)} \right\rceil \\ &\leq (t(x)/n^2) \sum_{u \in S} \left(\left\lfloor \frac{\omega(u)n^2}{t(x)} \right\rfloor + 1 \right) \\ &= (t(x)/n^2)(m(\tilde{x},y) + |S|) \\ &\leq (t(x)/n^2)(m(\tilde{x},y) + n). \end{aligned}$$

On the other hand, we may assume that $m(x,y) \geq t(x)$, otherwise we replace y with the solution computed by the approximation algorithm T. Then,

$$\begin{aligned} \frac{\mathsf{opt}(x)}{m(x,y)} &\leq \frac{(t(x)/n^2)(\mathsf{opt}(\tilde{x}) + n)}{m(x,y)} \\ &\leq \frac{(t(x)/n^2)\mathsf{opt}(\tilde{x})}{(t(x)/n^2)m(\tilde{x},y)} + \frac{t(x)}{n \cdot m(x,y)} \\ &\leq \frac{\mathsf{opt}(\tilde{x})}{m(\tilde{x},y)} + \frac{1}{n}. \end{aligned}$$

Thus if y is an r-approximate solution for $\tilde{x}$, then y is an $(r + \frac{1}{n})$-approximate solution for x. The theorem follows.

We first recall the following folklore result about the mixing properties of expander graphs.

Lemma 4 ([20]) *Let $G = (V, E)$ be a d-regular graph, let λ be an upper bound on the absolute values of all the eigenvalues, save the biggest one, of the adjacency matrix of G, let A and B be any two disjoint sets of nodes, then*

$$\left| Cut(A,B) - \frac{d}{|V|}|A||B| \right| \leq \lambda\sqrt{|A||B|}.$$

As is well known, the largest eigenvalue of the adjacency matrix of a d-regular graph is equal to d, thus we need graphs with second largest eigenvalue much smaller than d. Ramanujan graphs meet our requirement.

Theorem 7 ([26], Theorem 4.4) *Let p and q be primes such that $p \equiv q \equiv 1 \pmod 4$ and $q > p$. Then there exists a graph $Y^{p,q}$ that is $p+1$-regular, has $q+1$ nodes, and such that the second largest eigenvalue of its adjacency matrix is at most $2\sqrt{p}$.*

Dirichlet's stronger version of the Prime Number Theorem helps understanding the density of primes of the form $4k+1$. Let $\pi_{b,c}(n)$ be the number of primes $p \leq n$ such that $p \equiv c \pmod b$, and let $\phi(n)$ be the Euler function, then the following theorem holds.

Theorem 8 *If b and c are co-prime, then*

$$\lim_{n\to\infty} \frac{\pi_{b,c}(n)}{n/\ln n} = \frac{1}{\phi(b)}.$$

The above theorem is due to Dirichlet, Hadamard, and de la Vallèe Poussin, and a proof can be found in [14]. Let now $l_{b,c}(n)$ be the smallest prime $p \geq n$ such that $p \equiv c \pmod b$. From the above Theorem, it is an easy exercise to show that

$$\lim_{n\to\infty} \frac{l_{4,1}(n)}{n} = 1.$$

Said another way, for any ϵ a constant n_0 exists such that for any $n \geq n_0$ a prime $n \leq p \leq n(1+\epsilon)$ exists such that $p \equiv 1 \pmod 4$.

We are now able to prove Theorem 3. Let n_0 be such that for any $m \geq 2n_0 - 1$ a prime $p \equiv 1 \pmod 4$ exists such that $m \leq p \leq m(1+\epsilon/2)$. Let $c = \max\{n_0, 32/(\epsilon^2\delta^2)\}$.

Consider any $n \geq n_0$ and any w $cn \leq w \leq n^2$. If $w > n^2/(1+\epsilon/2)$ then the bipartite graph $\mathrm{K}_{n,n}$ is clearly (n, w, ϵ, δ)-mixing. Otherwise, the above conditions imply the existence of two primes $p \equiv q \equiv 1 \pmod 4$ such that

$$2n-1 \leq q \leq (2n-1)(1+\epsilon/2) \leq 2n(1+\epsilon/2)-1,$$

$$2w/n-1 \leq p \leq (2w/n-1)(1+\epsilon/2) \leq 2w/n(1+\epsilon/2)-1.$$

That is,

$$\frac{w}{n^2(1+\epsilon/2)} \leq \frac{p+1}{q+1} \leq \frac{w(1+\epsilon/2)}{n^2}$$

and thus

$$\left|\frac{p+1}{q+1} - \frac{w}{n^2}\right| \leq \frac{\epsilon}{2}\frac{w}{n^2}. \tag{1}$$

Consider now the graph $Y^{p,q}$, let V_1, V_2 be any two disjoint sets of n vertices of $Y^{p,q}$. Let E be the set of edges of $Y^{p,q}$ that have an endpoint in V_1 and an endpoint in V_2, we shall prove that the bipartite graph $G = (V_1, V_2, E)$ is (n, w, ϵ, δ)-mixing. Let $A \subseteq V_1$ and $B \subseteq V_2$ be any two sets of at least δn nodes, let $a = |A|$, $b = |B|$. Applying Lemma 4 we have that

$$\left|Cut(A,B) - \frac{p+1}{q+1}ab\right| \leq 2\sqrt{pab}$$

and, combining with (1),

$$\begin{aligned}
\left|Cut(A,B) - \frac{w}{n^2}ab\right| &\leq \frac{\epsilon}{2}\frac{w}{n^2}ab + 2\sqrt{pab} \\
&\leq \frac{\epsilon}{2}\frac{w}{n^2}ab + 2\sqrt{\frac{2w}{n}ab} \\
&\leq \frac{w}{n^2}ab\left(\frac{\epsilon}{2} + 2\sqrt{\frac{2n^3}{wab}}\right) \\
&\leq \frac{w}{n^2}ab\left(\frac{\epsilon}{2} + 2\sqrt{\frac{2n^3}{cn\delta^2 n^2}}\right) \\
&\leq \frac{w}{n^2}ab\left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right).
\end{aligned}$$