Given a set of integers, S, return all possible subsets.

Notation and terminology

We say that $S = \langle x_1, x_2, \ldots, x_n \rangle$ is a *set*, in which all x_i are integers and all element placed in non-decending order, that is, $x_1 \leq x_2 \leq \cdots \leq x_n$.

We say that S' is a *subset* of S, when $\forall x \in S', x \in S$. In another word, $S' = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_m} \rangle$, in which, $i_1 < i_2 < \cdots < i_m \leq n$.

We defines a new set with all possible sets containing only the first k elements of S.

SUBSETS$(S, k) = \{\sigma | \sigma = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_m} \rangle$, in which $i_1 < i_2 < \cdots < i_m \leq k \leq n\}$

Then our recursive method relies on how to figure out how to calculate the $difference$ of SUBSETS(S, k), that is,

ΔSUBSETS(S, k) = SUBSETS$(S, k+1)$ - SUBSETS(S, k)

1 S Without Duplicate Elements

When S is a set without duplication elements, all elements are in ascending order:

$$x_1 < x_2 < \cdots < x_n$$

Lemma 1. Δ SUBSETS*(S, k)* $= \{\sigma' | \sigma' = \sigma \cup \{x_{k+1}\}, \sigma \in$ SUBSETS*(S, k)*$\}$

Proof. For $\forall \gamma \in$ SUBSETS$(S, k+1)$, $\gamma = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_m} \rangle$, in which $i_1 < i_2 < \cdots < i_m \leq k+1 \leq n$. γ can fall into the following two categories.

1. If $i_m \neq k+1$, then $x_{i_m} \neq x_{k+1}$. Because $x_k \neq x_{k+1}$, $\gamma \in$ SUBSETS(S, k).
2. If $i_m = k+1$, then $x_{i_m} = x_{k+1}$. Because $x_{i_{m-1}} \neq x_{k+1}$, then $\langle x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}} \rangle \in$ SUBSETS(S, k).

$\square$

From the Lemma 1, we now have the following algorithm:

SUBSETS-WITHOUT-DUPLICATION(S)

```
SORT(S)
n ← length(S)
e ← empty array //SUBSETS(S, 0)
subsets = {e}
for i ← 1 to n
      do subsets ← subsets + DIFFERENCE(subsets, S[i])
return subsets
```

DIFFERENCE($subsets, new\_elem$)

```
n ← length(subsets)
subsets' ← {}
for i ← 1 to n
      do subset' ← subsets[i] + {new_elem}
         subsets' ← subsets' + {subset'}
return subsets'
```

2 S With Duplicate Elements

When S is a set without duplication elements, all elements are in non-descending order:

$$x_1 \leq x_2 \leq \cdots \leq x_n$$

Unlike Lemma 1, from current S, $(i_m < k \leq n) \nRightarrow (x_{i_m} \neq x_k)$.

Lemma 2. *With Duplication,*

$$\Delta\text{SUBSETS}(S,k) = \begin{cases} \{\sigma' | x_k \neq x_{k+1}, \text{then } \sigma' = \sigma \cup \{x_{k+1}\}, \sigma \in \text{SUBSETS}(S,k)\}; \\ \{\sigma' | x_k = x_{k+1}, \text{then } \sigma' = \sigma \cup \{x_{k+1}\}, \sigma \in \Delta\text{SUBSETS}(S,k-1)\}. \end{cases}$$

Proof. For $\forall \gamma \in \text{SUBSETS}(S, k+1)$, $\gamma = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_m} \rangle$, in which $i_1 < i_2 < \cdots < i_m \leq k+1 \leq n$. When $x_k \neq x_{k+1}$, the discussion of lemma 1 remains valid, so as the 1st part of lemma 2.

When $x_k = x_{k+1}$, let's say x_j is the last element less than x_{k+1}, then $\text{SUBSETS}(S,j)$ contains all subsets without any x_{k+1}. $\Delta\text{SUBSETS}(S,j)$ should be all subsets with exactly 1 x_{k+1}. $\Delta\text{SUBSETS}(S,j+1)$ should be all subsets with exactly 2 x_{k+1}.

For $\alpha \in \Delta\text{SUBSETS}(S,j+1)$, let α' be the subset that deduced by getting rid of last element from α, then $\alpha' \in \Delta\text{SUBSETS}(S,j)$. Backwards, $\forall \alpha' \in \Delta\text{SUBSETS}(S,j), \alpha' + \{x_{j+1}\} \in \Delta\text{SUBSETS}(S,j+1)$. From inducation, this applies to the rest element equal to x_{k+1}, hence we prove the 2nd part. $\square$

From the Lemma 2, we now have the following algorithm:

SUBSETS-WITH-DUPLICATION(S)

```
SORT(S)
n ← length(S)
e ← empty array //SUBSETS(S, 0)
subsets ← {e}
diffs ←
for i ← 1 to n
      do
         if i = 1 or S[i] ≠ S[i − 1]
            then diffs[i] ← DIFFERENCE(subsets, S[i])
            else  diffs[i] ← DIFFERENCE(diffs[i − 1], S[i])
         subsets ← subsets + diffs[i]
return subsets
```