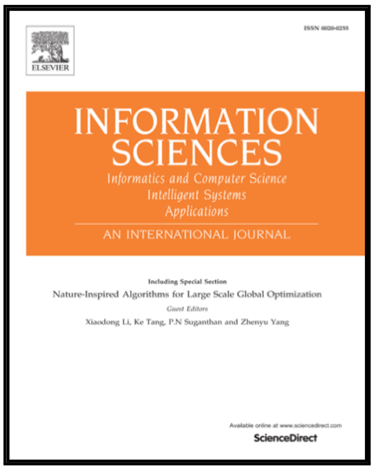

Hybrid Bi-Objective Portfolio Optimization with Pre-Selection Strategy

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Highlights

- A pre-selection strategy based on quadratic programming is proposed.
- The assets that have great contribution to the Pareto frontier can be filtered by applying pre-selection.
- The evolution of the assets combination modes replaces the evolution of the individual.
- Local search and evolutionary algorithm are exployed to exploit the more useful assets combination modes.

Hybrid Bi-Objective Portfolio Optimization with Pre-Selection Strategy

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Hybrid Bi-Objective Portfolio Optimization with Pre-Selection Strategy

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Abstract

Classical Markowitz mean-variance model is widely used for portfolio assets selection and allocation, which aims at simultaneously maximizing the expected return of the portfolio and minimizing portfolio variance. Many numerical approaches and metaheuristic algorithms have been proposed to effectively solve this portfolio optimization problem under an ideal conditon. However, introducing various realistic constraints inadvertently leads to a non-convex search space, which has hindered the application of many classic, exact algorithms such as quadratic programming (QP). The increasing size of available assets and complex constraints has made the effectiveness of metaheuristic algorithms deteriorated. This paper proposes a hybrid bi-objective algorithm combining with the respective advantages of local search algorithm, evolutionary algorithm and QP with a pre-selection strategy. The algorithm first down select the assets that have greater contribution to the Pareto frontier by applying the pre-selection strategy. Then local search and evolutionary algorithm combined with QP are employed to fully exploit the useful assets combination modes to lead the search process toward the frontier direction quickly. The experimental study demonstrates that the proposed hybrid approach can obtain faster and better convergence compared with eight state-of the-art multi-objective evolutionary algorithms. The results also show

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that the proposed method with the pre-selection strategy always displays a closer proximity to the Pareto frontier compared with k-means strategy.

Keywords: Portfolio optimization, assets selection, evolutionary algorithm, bi-objective optimization, quadratic programming, local search.

1. Introduction

The selection and allocation of limited capital to the various financial assets is one of the most challenging problems in financial management. There are many assets that can be selected for investment, such as stocks, securities, golds, real estates, and etc. Investors need to choose multiple assets from lots of available options to form a single portfolio. Mathematically, Harry Markowitz introduced the framework of portfolio selection problem using a mean-variance model. The Markowitz model assumes that investors would like to maximize return under an acceptable risk level or minimize the risk to assure a certain return level. This model is generally expressed as a bi-objective optimization problem of maximizing expected return while minimizing risk. In the construction of a portfolio, investors must select both the types of assets, as well as their allocations (proportion or units). The resulted Pareto frontier can then be obtained by solving the bi-objective optimization problem, which can provide various Pareto-optimal options for the investors with the different risk preferences.

Currently, the majority of the portfolio optimization researches focuses on the advancement of the original Markowitz model incorporating more real-world constraints [29]. Floor-ceiling constraints [8, 25, 26, 36], cardinality constraints [7, 8, 23, 25, 26, 31, 36], round-lot constraints [23, 25, 26, 34, 36], turnover constraints [22], and transaction costs [31] make the original mean-variance Markowitz model [27] more realistic in practice and complexity. These realistic constraints nevertheless impose the decision space to be discontinuous and often lead to a non-convex search space, and make the Pareto frontier to show different distributions [7-8]. As a result, these constraints hinder the application of the usual quadratic programming. The mixed integer quadratic programming methods cannot guarantee to find optimal solutions quickly. If there is not enough time to allow the algorithm to terminate regularly, the best valid portfolio calculated so far can only be used instead [6].

Recently, multi-objective metaheuristic approaches have been extended into portfolio selection and portfolio optimization [15, 16, 23, 29] in the financial and economics applications and etc. Among them, evolutionary Algorithms (EAs) require little domain knowledge to operate and Multi-Objective EAs (MOEAs) have shown ability to simultaneously search for a set of non-dominated solutions, which inspire the researchers with a great interest in addressing the complex problems with specific mathematical features including non-convexities and discontinuities in the objective space [6, 12, 33]. The first application of MOEAs for handling portfolio optimization problems [3] can be traced back to 1993.

However, the representation of a portfolio using EAs will frequently produce infeasible solutions, which often fail to satisfy the basic budget constraint. Encoding scheme is the first important issue to be attended for portfolio selection problem. A real-number encoding scheme [2, 37] and a hybrid representation [8, 35, 36] are used extensively. For a hybrid encoding scheme with binary representation [35, 36], each bit from the binary chromosome determines whether an asset would be chosen or not. As for a hybrid encoding scheme with integer representation [8], the integer vector represents the identity tags of various assets available. In order to meet the basic budget constraint, the weights assigned to assets in a given portfolio are generally normalized to derive the new feasible weight vector [24]. All assets are sorted in the decreasing order according to their weights until the summation of the included weights is greater than 1, then the weight of the last included asset will be reassigned to satisfy the unity constant [37].

In addition to encoding scheme which are used to generate feasible solutions, the designs to handling constraints during the optimization process were also developed. Deb *et al*. [12] proposed a customized hybrid elitist Nondominated Sorting Genetic Algorithm (NSGA-II) procedure to handle the non-smooth conditions. The initialization procedure and the recombination and mutation operators are all customized, which make the method to always create feasible solutions. To handle the cardinality constraints, Fieldsend *et al*. [17] considered the cardinality as an additional objective to be optimized. Then a simple (1+1)-evolution strategy was used to solve the three-objective optimization problem. Vijayalakshmi Pai and Michel [38] exploited *k*-means cluster analysis to eliminate the cardinality constraint and reduced the number of design variables leading to a fast convergence. Lin and Liu developed three possible models for portfolio selection problems with minimum transaction lots, and devised a corresponding genetic algorithm to obtain the solutions accordingly [24]. Lwin *et al*. [25, 26] proposed an efficient learning-guided hybrid multi-objective evolutionary algorithm to solve the portfolio problems with four real-world constraints.

Comparisons of the performance of different multi-objective techniques aiming at solving the practical portfolio problem have been studied. Considering the classical mean-variance model with the exact cardinality constraints, round-lot constraints, and fixed asset weights bounds, Skolpadungket *et al*. [33] compared the performance of six different MOEAs including the original Vector Evaluated Genetic Algorithm (VEGA) [32], two modified versions of VEGA, NSGA-II [11], Multi-Objective Genetic Algorithm (MOGA) [18], and advanced version of Strength Pareto Evolutionary Algorithm (SPEA2) [40]. The computational results showed that SPEA2 was the best method among all six with respect to the two chosen quality indicators of generational distance and solutions dispersion. Anagnostopoulos and Mamanis [1] compared the effectiveness of five different MOEAs (i.e., improved design of Niched Pareto Genetic Algorithm (NPGA2) [14], NSGA-II [11], Pareto Envelope-based Selection Algorithm (PESA) [10], ε-based Multi-Objective Evolution Algorithm (ε-MOEA) [21], and SPEA2 [40]) together with a Single Objective Evolutionary Algorithm (SOEA) on the mean-variance model with cardinality constraints. According to ε-indicator and hypervolume metric, NSGA-II and SPEA2 presented the most promising performance.

These state-of-the-art multi-objective evolutionary algorithms showed the advances in the adaption for solving the portfolio optimization problems. While, the frequency of generating infeasible solutions is high, repairing these solutions exerts a significant running time, which leads to the low efficiency and efficacy.

Many researchers have turned to search for the hybrid methods of metaheuristics and mathematics [6, 13, 20, 28, 30, 31]. Eddelbüttel [13] used a QP solver within a genetic algorithm to solve the index-tracking problem. Branke *et al*. [6] proposed a hybrid algorithm by combining NSGA-II and the critical line search to solve the portfolio problem with an upper limit for each individual asset and for the sum of all "heavyweights." Moral-Escudero *et al*. [28] and Ruiz-Torrubiano *et al*. [30] also made the uses of genetic algorithms and quadratic programming to solve the minimal risk portfolio optimization. Gaspero *et al*. [20] proposed a hybrid metaheuristics by combining a local search approach including steepest descent strategy with a quadratic programming procedure.

The common spirit in these designs is that the metaheuristics determines which assets should be included in portfolio, while the optimal weights are calculated by the QP solver for asset allocation. It is often noted that QP only played a subordinate role to help metaheuristics to search for the global optimum. The main advantage of QP is that it can generate the feasible solutions for portfolio allocation and reduce the time of repairing the infeasible solutions. The key contributor to search for optimal solutions is often left to the metaheuristics. How to construct a metaheuristic to effectively select a subset of assets became an important step in the design of the hybrid approaches. As the scale of available assets increases, the complexity and the volume of the search space grows exponentially. Therefore, how to reduce the size of the assets search space without sacrificing the quality of Pareto-optimal solutions found becomes a top priority.

This paper proposes a hybrid bi-objective portfolio optimization with pre-selection strategy. The main contribution of this work is on the pre-selection scheme, which eliminates the potential insignificant assets to reduce the dimensions of the search space to obtain a subset of the original assets under consideration. Meanwhile, a hybrid bi-objective optimization seamlessly integrated with Local Search Algorithm (LSA), EA and QP is applied to realize the evolution of portfolio optimization. In pre-selection stage, these assets that might have potential contributions to the final Pareto frontier will be identified according to the proportion of different assets by solving the portfolio optimization with relaxed constraints based on the QP. Pre-selection strategy also generates different scales of the obtained subset of asset by changing the corresponding constraints to match the need of portfolio problems. Then on the basis of the results of the pre-selection, this paper employs a local search algorithm to fully exploit the useful assets combination mode to lead the search process toward the frontier direction quickly. Finally, Evolutionary Algorithm (EA) based on the global search further improve the quality of assets combination modes and generate optimal frontier. The asset combination mode with binary encoding clearly determines whether an asset will be selected or not in a portfolio.

In the proposed design, QP is repeatedly applied at different stages in portfolio optimization. In pre-selection stage, QP could realize the selection process of good assets with relaxed constraints. In the evolutionary process of hybrid algorithm including LSA and EA, QP is used for solving convex sub-problems and determining optimal weights. The binary encoding of LSA and EA transforms a non-convex problem into a series of convex sub-problems, which provides the appropriate portfolio modes and decides the corresponding assets to be included in a portfolio. All of the non-dominated solutions from these convex sub-problems form the final Pareto frontier of the non-convex problem.

In the remainder of this paper, the classical mean-variance model and its revised model considering practical constraints are first described. Then in Section 3, we describe the motivation and propose the process of pre-selection method in detail. In Section 4, we elaborate on the proposed hybrid method with LSA, EA and QP. After that, in Section 5, we first overview the quality indicators adopted in this paper. Then we provide the performance evaluation at different stages of the proposed method, comparing the performance of the proposed method with chosen state-of-the-art algorithms, and analyzing the experimental results of the pre-selection strategy. The conclusion of this hybrid method is drawn in Section 6.

2. Portfolio optimization

2.1 Bi-objective Optimization Problem I

The mathematical foundation of portfolio optimization was established by Markowitz in 1952 [27], where there were typical two conflicting objectives, maximizing portfolio expected return and minimizing portfolio risk (i.e., the variance of the portfolio's rate of return). This model makes use of the mean and variance of normalized historical asset prices to compute the expected return and risk, respectively. As a result, the Markowitz model is often called as *Markowitz mean-variance model*. The classical Markowitz mean-variance mathematical formulation can be expressed as the following Bi-objective Optimization Problem (named BOP-I in this paper).

$$\text{Minimize} \quad f_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \sigma_{ij} w_j \tag{1}$$

$$\text{Maximize} \quad f_2 = \sum_{i=1}^{N} r_i w_i \tag{2}$$

subject to

$$\text{Budget constraint: } \sum_{i=1}^{N} w_i = 1 \tag{3}$$

$$\text{Box constraint: } 0 \leqslant w_i \leqslant 1. \tag{4}$$

where N is the number of assets available, r_i is the expected value of asset i , which is used for quantifying return. σ_{ij} is the covariance between assets i and j. When $i = j$, σ_{ij} represents the variance of the asset i. The higher the value of the variance, the greater the fluctuations of the price for the corresponding asset will be, so the more portfolio risk is to be expected. w_i is the decision variable (also called weight variable) denoting the proportion of capital to be allocated to asset i. The goal of portfolio optimization is to search for portfolios from the N assets that can simultaneously satisfy the two conflicting objectives, i.e., minimize the total variance in (1) denoting the risk associated with the portfolio, while maximizing its expected return or profit defined in (2). In model BOP-I, the budget constraint ensures that the sum of portfolio equals to the budget available. The box constraint implies that the proportion of the total investment for any asset is limited between 0 and 1, and no short selling is allowed (i.e.,

$\omega_i < 0$).

2.2 Bi-objective Optimization Problem II

In this paper, besides the above two basic constraints, practical constraints imposed by real-world complications including floor-ceiling constraint and cardinality constraint are introduced into the BOP-I model. The Markowitz mean-variance representation with practical constraints can be expressed as the following Bi-objective Optimization Problem-II (named BOP-II).

$$\text{Minimize} \quad f_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i \sigma_{ij} w_j \tag{5}$$

$$\text{Maximize} \quad f_2 = \sum_{i=1}^{N} r_i w_i \tag{6}$$

subject to

$$\text{Budget constraint: } \sum_{i=1}^{N} w_i = 1 \tag{7}$$

$$\text{Floor-ceiling constraint: } w_i = 0 \text{ or } a \leqslant w_i \leqslant b, 0 < a \leqslant b \leqslant 1 \tag{8}$$

$$\text{Cardinality constraint: } Z_L \leqslant Z \leqslant Z_U,\ Z = \sum_{i=1}^{N} z_i,$$

$$z_i = 1 \text{ if } w_i \neq 0; z_i = 0 \text{ if } w_i = 0 \tag{9}$$

The two conflicting objectives (5) and (6) are the same as those defined in (1) and (2). The first constraint is the same as that in the BOP-I, which ensures that the sum of portfolio equals to the budget available. The second constraint implies that an asset is either not invested ($w_i = 0$) or limited between the maximum and minimum non-zero proportion of investment amount ($a_i \leq w_i \leq b_i$, $0 < a_i \leq b_i \leq 1$), and no short selling is allowed. For a practical strategy, very small investment in an asset may not be desired to investors. Higher ratio of the total invested capital to one asset would naturally bring a higher risk. Moreover, investors will not invest in all of the assets in the market. Most likely, they are interested in portfolio of a limited number of assets. So, the cardinality constraint is introduced, which specifies a fixed number of investments or a range in the number of investments. When $Z_L = Z_U$, a fixed number of assets included in the portfolio is required. The above floor-ceiling constraint and cardinality constraint introduce discontinuities in the search space, which lead to the difficulties of the application of standard methods and make the Pareto frontier exhibit different degrees of discontinuity. Relative to BOP-II model, the Pareto frontier of the BOP-I can be solved exactly. So it is usually used to replace the true Pareto frontier of the BOP-II to analyze the performance of the proposed method.

2.3 Portfolio benchmark problems

In order to facilitate the subsequent discussions, we first introduce a set of portfolio optimization problems, which are obtained from the OR-library [4] for comparing study and evaluating performance. These benchmark problems provide the mean return, standard deviation of return, and the correlation matrix for assets in different stock market indices. Their details are summarized in Table 1. With the increasing number of the assets from port 1 to port 5, it becomes more challenging to obtain the respective Pareto frontiers.

Table 1. Descriptions of experimental data sets (see page 44)

3. Pre-selection strategy of a portfolio

The portfolio optimization problem becomes much more complex and computationally expensive, when there are hundreds or even thousands of assets available to select from for investing. In order to reduce the scale of the search space, the assets that might have potential contributions to the final Pareto frontier should be first identified. In this section, we first present the motivation of pre-selection by observing the statistical laws of experimental data, then we demonstrate the detailed process of pre-selection strategy.

3.1 The motivation of pre-selection strategy

Firstly, we observe the composition of the portfolio for the classical Markowitz mean-variance mathematical model (BOP-I) with the budget and box constraints. We can obtain the risk-return tradeoff frontier by solving the ε-constraint problem multiple times via applying QP with different expected returns. For the Port1 problem, we set the size of Pareto frontier generated at 100. In this set of Pareto frontier, each solution represents a portfolio. Different solutions but with the same combinations of assets are considered as the portfolios with the same portfolio mode, and we refer the different combinations of assets as portfolio modes.

Table 2 lists all the 13 portfolio modes of 12 assets (i.e., 2, 5, 9, 13, 15, 16, 17, 26, 28, 29, 30, 31) selected out of 31 assets from port 1, which account for 100 different Pareto frontier solutions, where the grey grid implies that the corresponding assets are selected for this portfolio. The weights of the remaining 19 assets not listed are kept zeros for all the Pareto frontier solutions, which suggest that they have made no contribution to the final Pareto frontier. In Table 2, each row represents a portfolio mode. There are total 13 portfolio modes generated. The size of assets selected in each portfolio mode varies from 1 to 12 (see the penultimate column). The number of Pareto solutions generated by each portfolio mode ranges from 1 to 20 (see the last column). Their accumulated sum is equal to the total number of the solutions generated (1+8+3+8+…+20+9+1=100).

Table 2. Portfolio modes that form a Pareto frontier (see page 44)

In Fig. 1(a), the Pareto frontier and the distribution of 12 assets marked in star (☆) , which form the Pareto frontier for Port1, are shown. The corresponding index of each asset is also given. Their accumulated amounts of weights for each asset in the Pareto population are shown in Fig. 1(b) in ascending order from left to right. Those accumulated amounts of remaining assets marked in plus (+) of Fig. 1(a) are all zeros and they are not shown in Fig. 1(b).

As shown in Fig. 1(b), asset 5 has the largest accumulated investment for the 100 solutions. This manifests that asset 5 is often selected in an portfolio to form the Pareto frontier due to its good features. On the other hand, asset 5 occupies a considerable investing proportion for the most frontier solutions. When asset 5 is not selected, the obtained frontier is obviously worse than otherwise (see Fig. 1 (c)). In general, the larger the accumulated investment is, the more significant impact on final frontier will be. Those assets with zero weights have no contribution to the final Pareto frontier. When those assets with non-zero weights are grouped into the new search space, they would generate the same Pareto frontier as that of the whole space with 31 assets altogether.

Fig. 1. Distribution of assets and their accumulated amount of weights for Port1 (see page 36)

On the basis of discussions above, we continue to observe the composition of the portfolio under the different ceiling constraints. The Port1 problem is repeatedly solved by separately setting the different ceiling constraints via QP. Table 3 lists the results corresponding for different ceiling constraints. The total weights for each asset are ranked in descending order and those assets whose weighted sum equal to zero are not listed. Under the current conditions, the set of assets with non-zero weights constitutes the new search space, which is expected to form the same Pareto frontier compared with the original whole set of assets. When the ceiling constraint is set to 1, 0.8, and 0.5 respectively, the size of assets selected is always equal to 12. The assets selected are also the same. Only the order of their amount of weights slightly changes. When the value of the ceiling constraint is continuously decreased to 0.2, 0.1 and 0.05, the size of assets selected is continuously increased. In the process, the new assets are added in the portfolio under the condition of maintaining the original assets. According to experiment results, we can obtain the same conclusion for the other four portfolio problems similar to what shown in Port1. In order to save space, we give the result of Port5 problem in Table 4.

Table 3. Assets selected for portfolio with different ceiling constraints of Port1 problem (see page 45)

Table 4. Assets selected for portfolio with different ceiling constraints of Port5 problem (see page 45)

It is noticed that with the decreasing value of the ceiling constraint, these assets are selected in sequence. The sequence is up to their non-dominated features and their relationship although not in an absolute sense. Good assets always appears first even if the scale of the portfolio is small. They are always preserved and not be removed with the increase of the scale.

This regularity provides a motivation for the proposed pre-selection method. We can obtain corresponding pre-selection assets by changing the ceiling constraint. As long as the ceiling constraint and cardinality constraint in BOP-II are within the range specified after pre-selection process (see the first two column in Tables 3 and 4), the original search space can be reduced to the set of assets according to descending order of their sum of weights. The corresponding reduced sets are listed in the last column in Tables 3 and 4. When there are not enough assets in the pre-selection set to satisfy the maximum number of assets of cardinality constraint, the ceiling constraint is decreased until the obtained size of the pre-selection assets meets the need of cardinality constraint. Finally, we can realize the reduction of the whole search space and filter the good assets.

3.2 Pre-selection strategy based on QP

For the mathematical model BOP-I, it is very easy to solved by QP because of its convexity. In this process, assets selection and optimal assets allocation are determined at the same time. Those assets with non-zero weight form the Pareto frontier are selected as equivalent subset of the original assets set.

For the mathematical model BOP-II, the process of pre-selection is similar as that of BOP-I. Considering the non-convexity of BOP-II, we can carry out the pre-selection strategy via QP by solving the relaxed constrained BOP-II. Although more real constraints change the distribution of the Pareto frontier, they cannot change the attributes of which assets are preferably selected, which assets are not selected or selected with a very low percentage.

The constraints of BOP-II are relaxed as the following,

$$0 \leqslant w_i \leqslant b,\ 0 < b \leqslant 1. \tag{10}$$

The floor constraints of BOP-II are relaxed to $0 \leq w_i$. The ceiling constraints are kept without change, $w_i \leq b$. Cardinality constraints are satisfied by the pre-selection strategy. Thus the BOP-II with the relaxed constraint (10) is a convex problem, which is similar as that of the BOP-I (where, $b = 1$). In order to provide enough freedom of selection to generate different combination of assets that satisfy the cardinality constraints, we should guarantee that the scale of the assets after pre-selection must be larger than the value Z_U in BOP-II, and the number exceeded (i.e. the degree of freedom) must reach the specified lower limit d. The more the freedoms of selections is given, the less the possibility of missing Pareto solutions will be, while, the lower the efficiency of the algorithm is. This is clearly not what we hope for. We must balance the quality of Pareto solutions and the efficiency of the algorithm. Meanwhile, we cannot forget the truth that a larger search space often comes with the possibility of been trapped in the local optimal solutions.

In the process of pre-selection, we firstly solve the BOP-II with the relaxed constraints via QP. Then, decide whether the BOP-II with the decreasing ceiling constraint is solved again.

(i) When the size of these assets with non-zero weights is equal to or exceeds the sum $Z_U + d$, the pre-selection process ends. Those assets with non-zero weights are selected to form the new search space.

(ii) When the size of these assets with non-zero weights is less than the sum $Z_U + d$, the value b, i.e., the current ceiling constraint value of each weight, will be further reduced and the pre-selection process will be re-run again until the total pre-selection size is more than the value $Z_U + d$.

Table 5 shows the scale of portfolio after pre-selection for different ceiling constraint value b. Because the size of the pre-selection with different ceiling constraint is uncertain, the degree of selecting freedom will also shows uncertainty. We just specify the lower limit of freedoms, while the upper limit is not concerned in this paper. According to Table 5, after pre-selection, the complexity is significantly reduced compared to that of the original portfolio space.

Table 5. The scale of Portfolio for different ceiling constraints (see page 46)

4. Hybrid bi-objective portfolio optimization

Evolutionary algorithms are population-based and use selection, recombination, and mutation operators to proceed. They have been successfully applied to a wide range of real-world applications. They are very easy to hybridize with other fast local approaches to complement its global search capability.

Observing carefully Table 2 again, we can find that each row represents a convex sub-problem by specifying the weights of those assets in grey grid to be greater than zero while satisfy the floor-ceiling constraints. This convex sub-problem can be easily solved by QP. Thus, the non-convex optimization problem BOP-II can be transformed into a series of convex optimization sub-problems. Here, we use binary encoding to express a portfolio mode as shown in Fig. 2(a). It reflects the existence of the assets in a portfolio. We can continue to derive new modes by evolution of modes. For each portfolio mode, QP is then employed and a group of approximate Pareto solutions are obtained. Each solution in MOEA is no longer represented by a single point but rather by a partial frontier in the mean-variance space determined by QP. From Figs. 2(a) and 2(b), the difference between mode and individual representation can be seen at a glance. Several partial frontiers from different convex sub-problems can be put together and form the non-dominated frontier. The niche-preservation operation of NSGA-II [11] is applied to obtain non-dominated population with the designated size.

Fig. 2. Different representations for portfolio (see page 37)

The obtained solutions in the pre-selection process via QP are close to those which satisfy all of the constraints of BOP-II, so the local search can be first employed and then crossover and mutation are used for later exploitation and exploration to obtain a more accurate and extended frontier region.

4.1 Local search algorithm

Many useful modes can be generated in the pre-selection process. According to these constraints in BOP-II, the portfolio size for each mode is first repaired to the required range of cardinality constraints, where the local repair scheme (RS) is considered as a part of local search. The detailed repair procedure is described in the Algorithm RS (see Fig. 3). Then, those optimal modes that can form the current Pareto frontier are picked out to run local search algorithm.

Fig. 3. The process of algorithm RS (see page 38)

After the repair process, local search process begins. Those modes that can generate more current Pareto solutions than the average are randomly added or deleted or replace an asset until the sum of 1's in the corresponding mode meets the cardinality constraint. The local search process continues until the amount of the modes generated reaches the requirement.

4.2 Multi-objective evolution algorithm of portfolio mode

When the portfolio size and ceiling constraint based on pre-selection with QP are closer to those of the real constraints in BOP-II, the generated solutions via QP bear more information of the final Pareto solutions. On the contrary, the generated solutions via QP are approaching the final Pareto solutions only at a certain extent. It is necessary to employ evolutionary algorithm to search for more new modes for the purpose of exploitation and exploration. The evolutionary algorithm participates in the evolution of portfolio mode instead of that of solution individual. Here, uniform crossover and bit-swap are sequentially executed. Then, the newly generated modes need to be checked and repaired to become feasible by using Steps 3-9 in Algorithm RS (see Fig.3).

In this process of evolution, the new portfolio modes are found continuously, and new non-dominated solutions are generated based on these new modes. As a result, it obtains the final Pareto frontier. Only when the generated modes which have contributed to the current Pareto frontier (marked as optimal modes $M_{optimal}$), these modes would have chances to participate in the evolution of the next generation. The maximal size of the optimal modes set $M_{optimal}$ is the same as that of the Pareto population. The detailed procedure of the multi-objective evolution algorithm of portfolio modes is described in the Algorithm MOEA_PM (see Fig. 4).

For each newly generated portfolio mode, a set of non-dominated solutions is obtained. Thus, the aggregation of these non-dominated solutions may far exceed the designated size of evolutionary population. The niche-preservation operation of NSGA-II will be used to obtain the Pareto solutions with the required size.

Fig. 4. The process of algorithm MOEA_PM (see page 39)

5. Experiments and results analysis

5.1 Quality indicators

In this section, a set of performance metrics including inverse generational distance (IGD) [5], Hypervolume (HV) [41], and C-metric [41] are adopted to measure the performance of the proposed bi-objective portfolio optimization design.

The true Pareto frontier of the BOP-II with the non-convex constraints is difficult to obtain. Instead, we only solve the convex problem, which is an approximate to the BOP-II as close as possible. Here, the obtained solutions by solving the BOP-II with the relaxed constraints (10) are considered as an approximate true Pareto frontier APF_{true}. With the decrease of the ceiling constraint, Pareto frontier APF_{true} moves towards the inferior and the extent of the frontier gradually becomes shorter.

In fact, the non-convex constraints are the subset of the corresponding convex constraints. The Pareto frontier based on non-convex constraints is generally not better than that with the corresponding convex constraints. Thus, these indicators based on approximate Pareto frontier will show worse than that with the true Pareto frontier to a certain extent. So, the relative quality of performance indicators among different algorithms is much more meaningful rather than their absolute measure.

5.2 Performance evaluation of various strategies of the proposed method

In the following, we will measure the performance of the proposed design components individually, including pre-selection strategy, multi-objective evolution, and local search. First, several cases are listed in Table 6, where different cardinality constraints are given. Here, taking into account the space limitation, we just assume that all of the weights satisfy the floor-ceiling conditions with $w_i = 0$, $0.01 \le w_i \le 0.2$.

Table 6. The problems with different cardinality constraints (see page 46)

For Port1 problem, after the first pre-selection with relaxed constraint b = 0.2 implemented, the size of assets selected is down to 15. It is assumed that the lower limit of selecting freedom is set to d=4. Thus, solutions generated from the first pre-selection not only satisfy the cardinality constraints $5 \le Z \le 10$ (case 1) and $Z = 10$ (case 2), but also reach the requirement of selecting freedom. The pre-selection process ends, the following local search and multi-objective evolutionary algorithms can be applied to these solutions. For case 3 and case 4, while the size of assets of the first pre-selection is less than the upper limit of cardinality constraints $10 \le Z \le 25$ (case 3) and $Z = 25$ (case4), the next pre-selection with less upper level of weight than 0.2 (say b = 0.05) will be carried out until the scale of assets selected exceeds the value Z_U+d. According to Table 5, the size of assets of the second pre-selection with relaxed constraint b = 0.05 is 30. We can obtain solutions that satisfy the requirement of cardinality constraints of cases 3 and 4. In Table 6, the first two cases of each portfolio problem only need to run one time pre-selection, and the last two need to perform the second pre-selection. For clarity, we mark them with grey grid in Table 6.

In the pre-selection process, the size of population is 100 for each run of pre-selection. In the local search and evolutionary process, each portfolio mode will generate 100 non-dominated solutions. The size of the modes set in the multi-objective evolutionary process is set as 20 and is run until 20 generations are reached. The crossover rate is selected as 1, and mutation probability is 0.2. 30 independent runs are executed for each case.

Experiment 1: Performance evaluation of multi-objective evolutionary strategy

In this experiment, we compare the performance of the algorithm with and without multi-objective evolution. The algorithm with evolutionary strategy represents the proposed Hybrid Bi-Objective Portfolio Optimization with Pre-selection Strategy (HBOPO-PS). The algorithm without evolution strategy includes all the strategies of HBOPO-PS except for the multi-objective evolution. The experimental parameters are the same as the proposed HBOPO-PS.

Fig. 5 gives the results of *C*-metric for different cases of five portfolio problems with and without running evolution search. These figures show that there are more dominated solutions generated for the design without evolution than those with evolution except for case1. The conceivable reason is that evolutionary algorithm cannot bring obvious improvement for those portfolio problems with a fairly low complexity. Especially, for the last two cases of each portfolio problem, the improved performance is clearly appreciable. Because the stringent constraints result in the rollback, only limit feasible portfolio modes for the corresponding problem can be generated after local search. In order to obtain the Pareto frontier solutions, the evolutionary algorithm is carried out to further improve the quality of solutions. Fig. 5 can directly show that the Pareto frontiers with evolution search are significantly improved compared with the results without evolution.

Fig. 5. Performance on *C*-metric for all 20 cases (see page 40)

Experiment 2: Performance evaluation of pre-selection strategy

In this experiment, we contrast the performance of the algorithm with and without pre-selection strategy. The algorithm with pre-selection refers to the proposed HBOPO-PS, while the algorithm without pre-selection represents the algorithm that only include multi-objective evolutionary strategy. Local search will also be missed because it is based on the results of pre-selection. For the algorithm without pre-selection, the mode size of each generation is set as 30, and the algorithm will carry on until 80 generations for each run.

There are up to N_1 modes generated for the HBOPO-PS, and N_2 (where $N_2 > N_1$) for the algorithm without pre-selection. For each additional time of pre-selection in HBOPO-PS, 100 modes will be increased. In this paper, the maximum number of pre-selection is set to 2, such as cases 3, 4, 7, 8, 11, 12, 15, 16, 19, 20. N_1 and N_2 are computed as the following:

N_1 = 100$\times$2 (for twice pre-selection) + 100 (for local search) + 20 (number of modes for each generation) $\times$ 20 (number of generations) = 700;

N_2 = 30 (number of modes for each generation) $\times$ 80 (number of generations) = 2400.

The number of the modes generated for the algorithm without pre-selection is much more than those for the HBOPO-PS. However, according to Table 7, the performance indicators including IGD, Hypervolume, and *C*-metric of the results based on HBOPO-PS with a reduced assets space are all better than those based on the algorithm without pre-selection of the original space for all five portfolio problems. These results imply that the algorithm without pre-selection is easily trapped in the local optimal and cannot escape from them facing the large scale of search space.

Experiment 3: Performance evaluation of local search strategy

Although local search strategy alone cannot necessarily obtain high quality of non-dominated solutions, but it still plays an important role in the proposed HBOPO-PS. The algorithm with local search refers to the proposed HBOPO-PS, while the algorithm without local search represents the design that only multi-objective evolutionary strategy is carried on after pre-selection. There are N_3 modes generated for the algorithm without local search. The number of the modes for HBOPO-PS is 700, which is the same as the above part. N_3 is computed as the following:

N_3 = 100$\times$2 (for twice pre-selection) + 30 (number of modes for each generation) $\times$ 30 (number of generation) = 1100.

The number of the modes generated for the algorithm without local search is also more than that with the HBOPO-PS ($N_3 > N_1$, N_1=700). However, Table 7 shows that the IGD, Hypervolume and *C*-metric of the results without local search are much worse than those of HBOPO-PS.

Table 7. The comparing results with different strategies of HBOPO-PS (see page 47)

5.3 Performance comparison with selected approaches

Comparison 1: Comparing with the state-of-the-art evolutionary algorithms

In this subsection, the algorithmic performance of our proposed method, HBOPO-PS, will be compared with eight chosen state-of the-art multi-objective evolutionary algorithms.

Firstly, the proposed method is compared with one of the most regarded designs, Evolutionary Multi-Objective Portfolio Optimization with initialization technique based on Ordered Representation (EMOPO-OR3) [8]. For EMOPO-OR3, we apply the same experimental parameters specified in the original paper [8]. The maximal function evaluations used is 100,000. Here, we only consider Port3 problem given in the original paper [8] as an example to observe their performance. Different box constraints and cardinality constraints are imposed.

As can be seen from Fig. 6, the Pareto frontiers obtained from the two methods both distributed near the true Pareto frontier. When the cardinality constraint is below 30, the frontier obtained corresponds to the higher risk-return region. While for the higher cardinality constraint, the frontier obtained corresponds to the lower risk-return region. When the floor and ceiling constraints become more stringent from $0 \leq w_i \leq 1$ to $w_i = 0$, or $0.01 \leq w_i \leq 0.12$, the whole frontier obtained is moving away from the true Pareto frontier. Under different box constraints and cardinality constraints, HBOPO-PS always shows better performance compare to that of EMOPO-OR3. The frontier of HBOPO-PS is closer to the true Pareto frontier and their distribution is broader towards the upper and lower bounds.

Fig. 6. The results of different methods for Port3 problem (see page 41)

Secondly, our proposed method is compared with nine state-of-the-art multi-objective algorithms including NSGA-II [11], SPEA2 [40], improved design of Niched Pareto Genetic Algorithm (NPGA2) [14], Pareto Envelope-based Selection Algorithm (PESA) [10], PESA-II [9], ε-based Multi-Objective Evolution Algorithm (ε-MOEA) [21], Multiobjective Evolutionary Algorithm based on Decomposition (MOEA/D) [39] and EMOPO-OR3 [8]. The maximum number of assets allowed in the portfolio was fixed to be ten (Z_U =10) for all test problems. For the weights of each asset, the floor and ceiling constraints are 0.01 and 1, respectively. In order to ensure a fair comparison, we have used identical archive sizes (i.e., 250) and the same solutions generated (i.e., 100,000).

30 different optimization runs have been carried out in all test problems considered. The crossover probability of P_c=0.9, and mutation probability of P_m=1/n (where n is the number of decision variables for the above real-coded MOEAs) are used. Apart from these settings, some additional parameters should also be chosen. For real-coded NSGA-II, we use distribution indexes for crossover and mutation operators as $\eta_c = 20$ and $\eta_m = 20$, respectively. The division of the search space for PESA is set equal to 100, the niche radius for NPGA2 to 0.07 and the size of tournament size to 8. For ε-MOEA, we choose the approximate ε value equal to 0.00432, 0.0046, 0.00442, 0.004675, and 0.00458 for test portfolio problems, Port1, Port2, Port3, Port4, and Port5, respectively. T in MOEA/D is set to 10.

In this study, we use the Hypervolume and IGD metrics. In order to allow the objectives to contribute approximately equally to Hypervolume, each objective function value is normalized according to the following equation:

$$f_i^{'} = \frac{f_i - f_i^{\min}}{f_i^{\max} - f_i^{\min}}$$

where $f_i^{\min}$ and $f_i^{\max}$ are the minimum and maximum values in the ith objective. The value of $f_i^{\min}$ and $f_i^{\max}$ are chosen as the 10% difference of the optimal values obtained [1]. Table 8 shows these values for each problem instance. The reference point required to compute the hypervolume indicator is $Z^{ref} = \{1,\ 0\}$ normalized for all problems.

Table 8. Values of objective function for each problem (see page 48)

As can be seen in Table 9 that shows the median and the standard deviation of hypervolume and IGD indicators for each portfolio problem, the results demonstrate the proposed method HBOPO-PS has the absolute superiority compared with other eight algorithms. The results of the proposed method not only show high quality, but also reflect the superior stability.

Table 9. The performance evaluation metrics for different algorithms (see page 49)

Comparison 2: Comparing with k-Means cluster strategy

In this subsection, we compare the results between the proposed pre-selection strategy and that employing *k*-means cluster analysis [38] to eliminate the cardinality constraint, where the *k*-means cluster analysis grouped the available assets into *k* clusters based on the two strategies and marked them as *k*-means-strategy1 and *k*-means-strategy2, respectively. *K* represents the number of the assets in cardinality constraint. *K*-means-strategy1 is based solely on the observation of return and variance to be considered for clustering, while *k*-meas-strategy2 considers not only the return and variance, but also the correlations between the assets. For the *k*-means cluster, the final assets in portfolio optimization are picked from each of the *k* clusters. So, the *k*-means cluster is also considered as one of pre-selection strategies here.

This *k*-means clustering partitions the whole assets set into *k* clusters and minimizes intra-cluster variance while maximizing inter-cluster variability. *K*-means strategy realizes the cardinality constraint from the problem model naturally. This strategy allows the investors to make their choices of one asset from each of the clusters. In order to observe the effects of different pre-selections and reduce the influence caused by the difference of their following respective quadratic programming and evolution strategy, we randomly generate the same designated scale (i.e., 200) of the assets allocation modes after pre-selection or *k*-means cluster and compare their frontier solutions by uniformly applying the quadratic programming.

Figs. 7-9 illustrate the results for the Port4 and Port5 portfolio problems. We notice that the frontiers of the three strategies consistently become shorter and farer away from the true Pareto frontier with the tightened boundary conditions for the same given Z = 60 from the Figs. 7-8. According to Figs. 8 and 9, the frontiers also show the similar trend, when the value of Z (from Z = 40 or Z = 30 to Z = 60) becomes bigger for the same bounding constraints. The frontiers with Z = 60 become shorter and are obviously farer away from the true Pareto frontier compared with Z = 40 for Port4 and Z = 30 for Port5 problems.

From Figs. 7-9, the overall observation is that the proposed pre-selection approach always displays a closer proximity to those Pareto frontiers compared with the two k-means cluster strategies, while the k-means strategy1 performs worse than that of the k-means strategy2 because of ignoring the correlations between assets.

Fig. 7. The results of different pre-selection strategies with $Z = 60$ and $0 \leq w_i \leq 1$ (see page 42)

Fig. 8. The results of different pre-selection strategies with $Z = 60$ and $0.01 \leq w_i \leq 0.2$ (see page 43)

Fig. 9. The results of different pre-selection strategies with $0.01 \leq w_i \leq 0.2$ (see page 43)

6. Conclusion

In this paper, a hybrid method seamlessly integrating multi-objective evolutionary algorithm and quadratic programming with pre-selection for solving the real-world portfolio optimization problem involving basic, bounding and cardinality constraints is proposed. The proposed hybrid method consists of three different stages. Pre-selection stage mainly applies QP to solve the portfolio optimization problem with relaxed constraints and obtains the subset of assets according to their weights appearing in the final solutions, which usually produces the quality Pareto frontier. The results of the pre-selection provide the initial solutions for local search and subsets of assets available for the follow up evolution. Local search stage further acquires useful portfolio modes satisfying the designated constraints by extracting and repairing the initial solutions. After that, the operators of crossover and mutation of evolutionary algorithm stage are employed to exploit more new modes to balance the need of exploitation and exploration. Niche-preservation operation of NSGA-II is applied to the union of the non-dominated solutions generated by portfolio modes and the final Pareto frontier of the portfolio problem is obtained.

Pre-selection strategy based on QP avoids the mistakes of the subjective selection effectively. Its scale of subset of assets after pre-selection can be adjusted by setting the different ceiling constraint. When the ceiling constraint is decreased, the subset of assets will include more assets or remain unchanged, and these assets previously appeared will continue to stay in the new subset. Their main part of the subset also remains relatively stable. Experiments demonstrate that good assets are always selected even if the scale of the portfolio is small, and they are always preserved and do not get eliminated with the increase of the scale.

We test the performance of the proposed hybrid method on the five portfolio benchmark problems of real stock market according to three indicators in IGD, Hypervolume, and *C*-metric. Each stage has made different contributions to the overall performance. Pre-selection effectively reduces the complexity of generating optimal portfolio from the whole assets search space. Local search exploits the useful frontier modes near the optimal solutions provided by the results of pre-selection. Multi-objective evolutionary algorithm provides a complement global search to balance the need of exploitation and exploration.

The performance of HBOPO-PS has been compared with the eight evolutionary multi-objective optimization algorithms. It has been demonstrated that the resulted frontier of HBOPO-PS is closer to the true Pareto frontier and their distribution is broader towards the upper and lower bounds. Furthermore, HBOPO-PS has been tested for its effectiveness of pre-selection strategy. It has also been shown that the method with the pre-selection strategy proposed always displays the closer proximity to the true Pareto frontier compared with the two *k*-means cluster strategies. In summary, this paper has proposed an effective design in addressing the basic, bounding, and cardinality constraints in real-world portfolio optimization problems. We plan to adapt this approach toward solving some other interesting constrained real-world finance and economy problems, such as transaction costs, class and turnover constraints.

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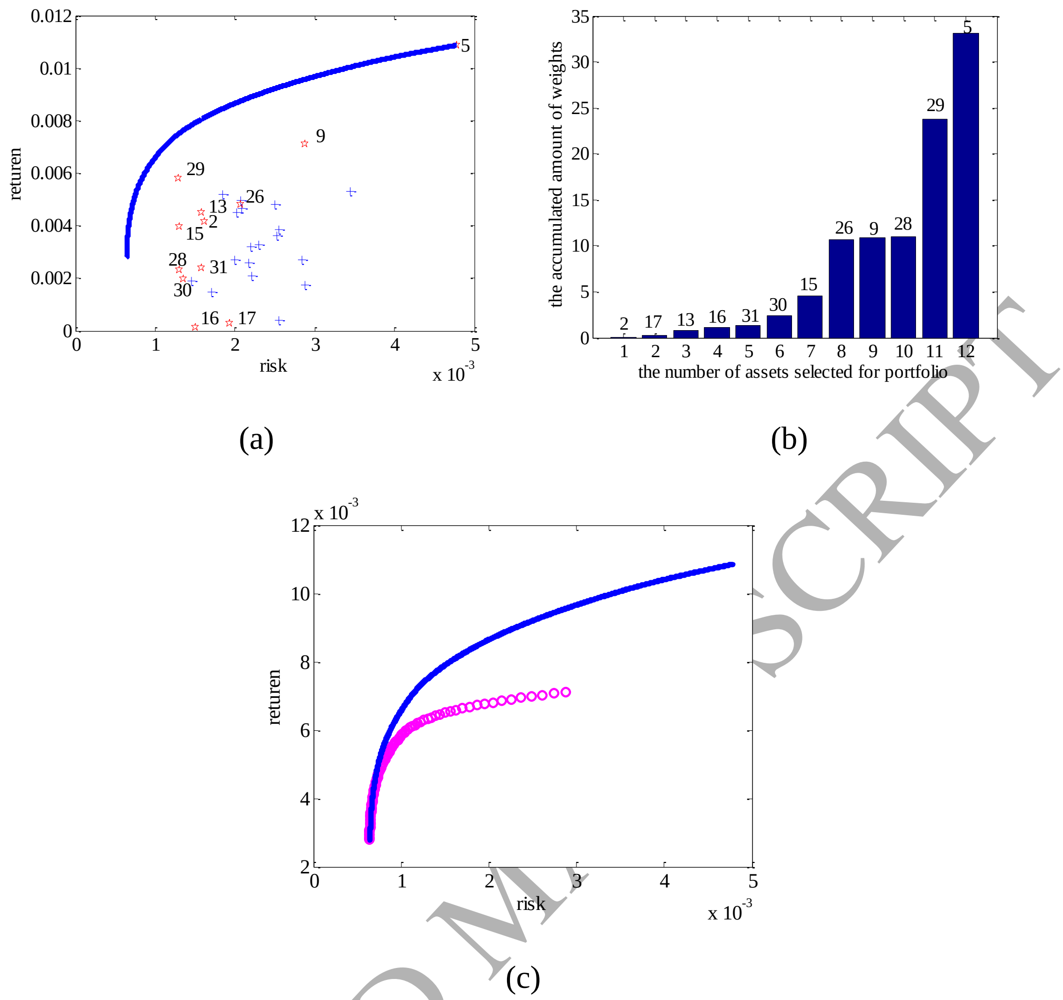


Fig. 1. Distribution of assets and their accumulated amount of weights for Port1 (a) Distribution of assets (b) The accumulated amount of weights (c) Approximate frontier after losing asset 5 (—•—: Pareto frontier, ☆: Assets that can form the Pareto frontier, +: Remaining assets, o : Approximate frontier)

1	2	3	4	5	… …	N-1	N	Assets space
0	0	1	1	1		0	1	3,4,5,… …N
0	1	1	0	0		1	0	2,3,… …, N-1

(a)

1	2	3	4	5	… …	N-1	N
0	0	0.1	0.3	0.2		0	0.4
0	0.2	0.05	0	0		0.3	0

(b)

Fig. 2. Different representations for portfolio (a) Mode representation for portfolio (b) Corresponding individual representation for portfolio

Algorithm RS

Input:

P_{OP} : The approximate Pareto solutions population generated by QP in the pre-selection process

Output:

P_1 : The current Pareto solutions set

M_{optmal} : The optimal modes set, which contributes to the current Pareto frontier

M_{all} : All the obtained modes set until now

1 Extract modes M_P from the population P_{OP};

2 **for** $i = 1$; $i \le N_M$; i++ **do** /* N_M represents the size of modes M_P */

/* Repair each mode M_{Pi} */

3 **if** $Z_L \le | M_{Pi} | \le Z_U$ **then**

4 Keep this mode M_{Pi} unchanged; /*$| M_{Pi} |$ represents the size of assets in mode M_{Pi} */

5 **else if** $| M_{Pi} | > Z_U$ **then**

6 Randomly generate the number of deleted assets N_{delete}, which satisfies the designated range,

$| M_{Pi} | - Z_U \le N_{delete} \le | M_{Pi} | - Z_L$, and randomly remove N_{delete} assets included in the current mode;

7 **else if** $| M_{Pi} | < Z_L$ **then**

8 Randomly generate the number of added assets N_{add}, which satisfies the designated range,

$Z_L - | M_{Pi} | \le N_{add} \le Z_U - | M_{Pi} |$, and randomly add N_{add} assets excluded in the current mode;

9 **end**

10 **end** /* These feasible modes are grouped into M_{all}*/

11 Repeat running QP for each M_{Pi}, $i = 1, \ldots N_M$ until all modes are executed, obtain solutions set Q by non-dominated sorting method and the current modes set M_{all} ;

12 Apply the niche-preservation operation of NSGA-II to solutions set Q to obtain Pareto population P_1 with the designated size;

13 Determine optimal modes set M_{optmal}, which can generate solutions existed in the current Pareto population P_1.

Fig. 3. The process of algorithm RS

Algorithm MOEA_PM

Input:

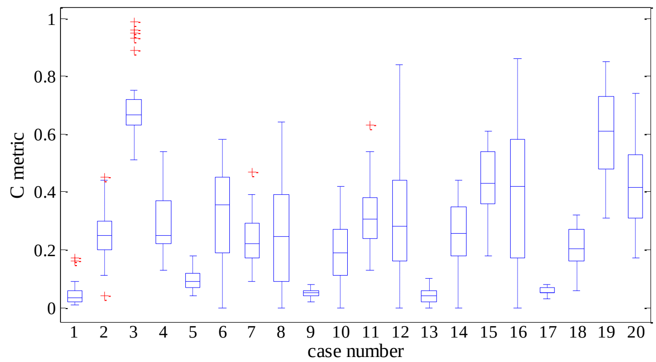
P_1 : The current Pareto population
$M_{optimal}$: The optimal modes set, which contributes to the current Pareto frontier
M_{all} : All the obtained modes set until now
Gen : Generation
N_P : The size of Pareto population
N_M : The size of optimal modes

Output:

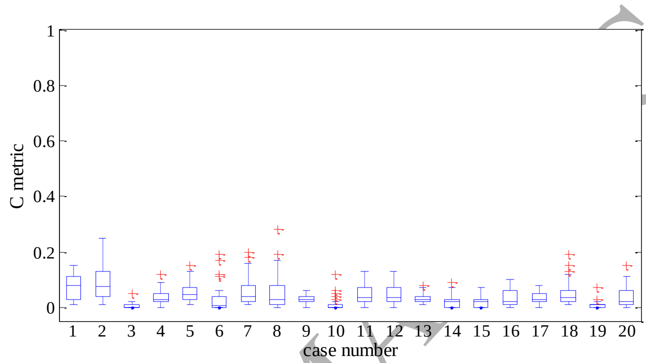
P_{t+1} : The Pareto population of $t + 1$
$M_{optimal}$: The updated optimal modes set
M_{all} : The updated modes set until now

for $t = 1$; $t \le Gen$; t++ **do**
  for $i = 1$; $i \le N_M$; i++ **do**
    Randomly select M_1, $M_2 \in M_{optimal}$;
    Run uniform crossover operator and bit-swap;
    Generate the new mode M_{newi} ;
    Repair the mode M_{newi} (see the steps 3-9 in Algorithm RS), and reserve those different from the original modes M_{all}, $M_{all} = M_{newi} \cup M_{all}$;
    Run QP for mode M_{new} and obtain the non-dominated solutions set Q_{ti}, $Q_t = Q_t \cup Q_{ti}$;
  end
  $P_t = P_t \cup Q_t$;
  Apply the niche-preservation operation of NSGA-II to obtain current Pareto population P_{t+1};
  Extract modes $M_{optimal}$ from the population P_{t+1};
end

Fig. 4. The process of algorithm MOEA_PM

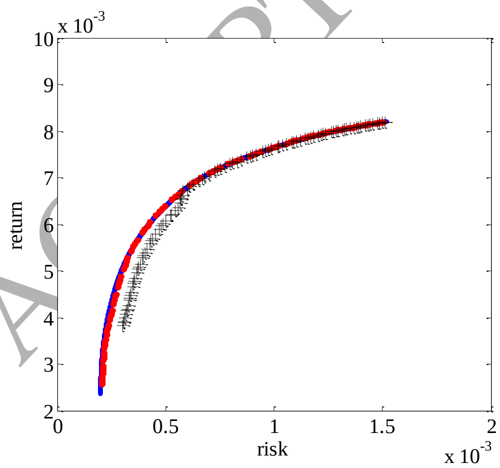


(a)

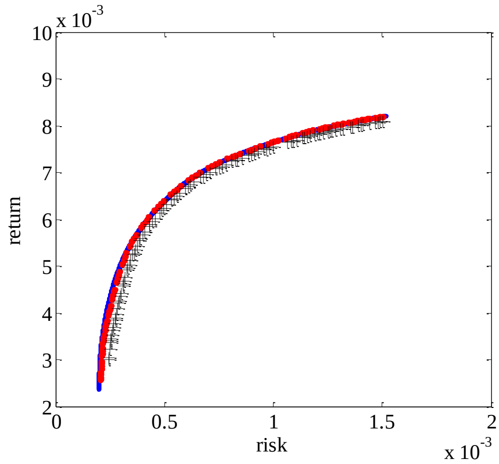


(b)

Fig. 5. Performance on *C*-metric for all 20 cases (a) HBOPO-PS vs. No evolution (b) No evolution vs. HBOPO-PS



(a)



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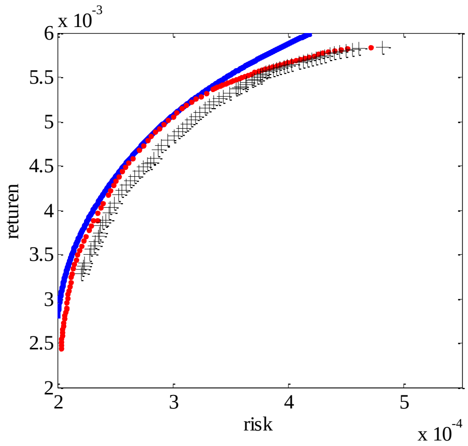
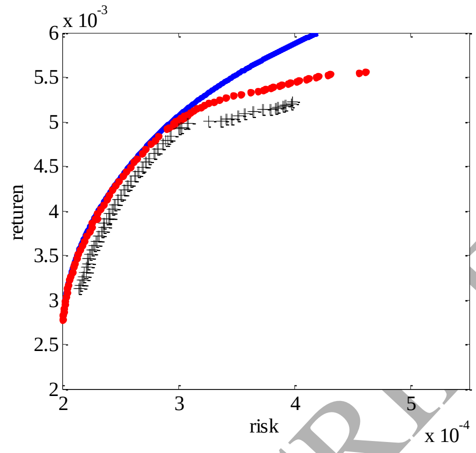
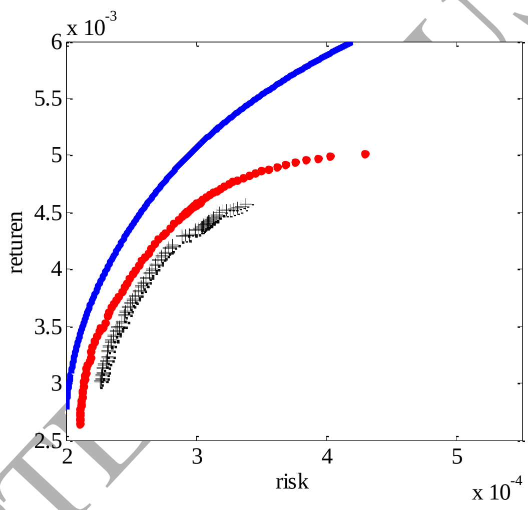


(c)



(d)



(e)

Fig. 6. The results of different methods for Port3 problem (a) $2 \leq Z \leq 3$ (b) $1 \leq Z \leq 4$ (c) $15 \leq Z \leq 20$ (d) $25 \leq Z \leq 30$ (e) $50 \leq Z \leq 55$ (where $0 \leq w_i \leq 1$ for the case (a) and (b), $w_i = 0$, or $0.01 \leq w_i \leq 0.12$ for the case (c), (d) and (e). —•—: the true Pareto frontier, —•—: the Pareto frontier with HBOPO-PS, + : the Pareto frontier with EMOPO-OR3)

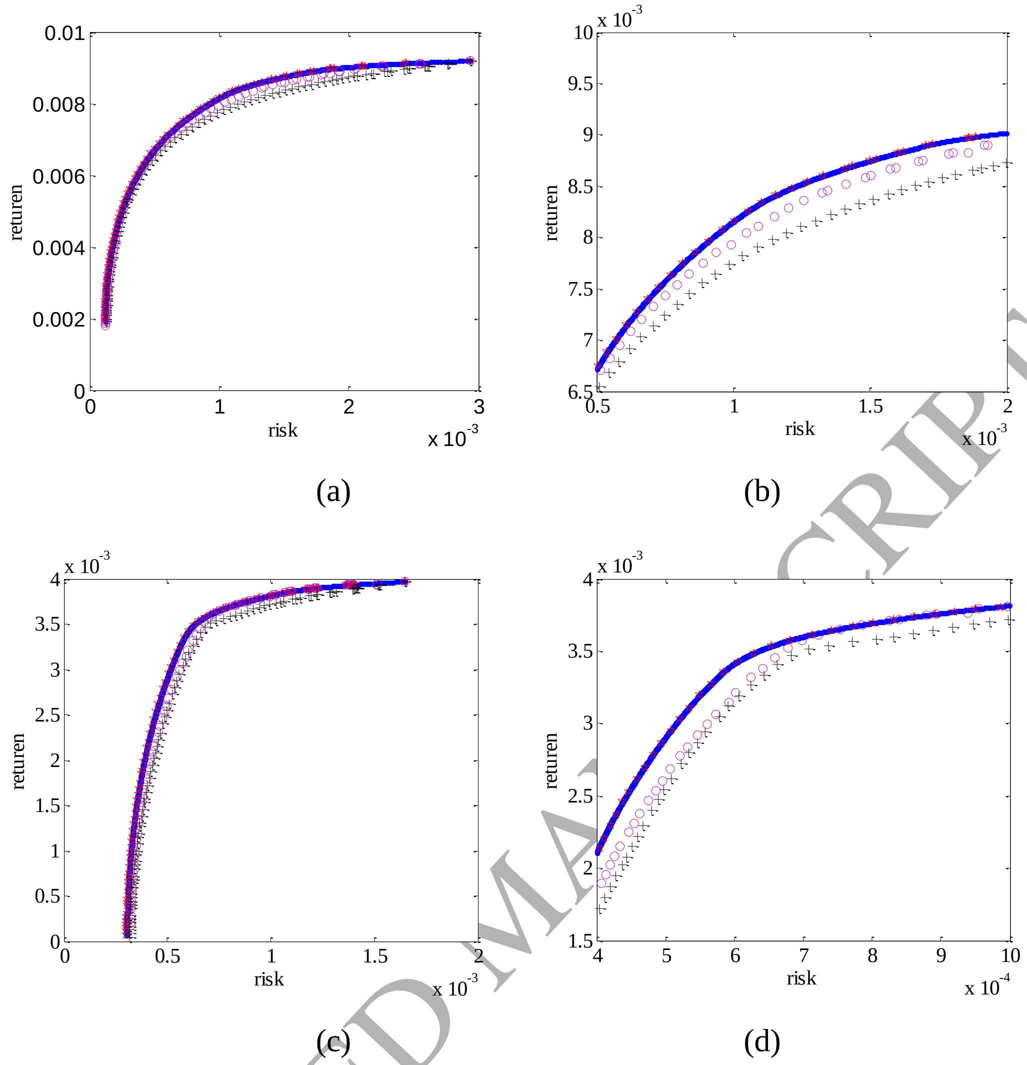
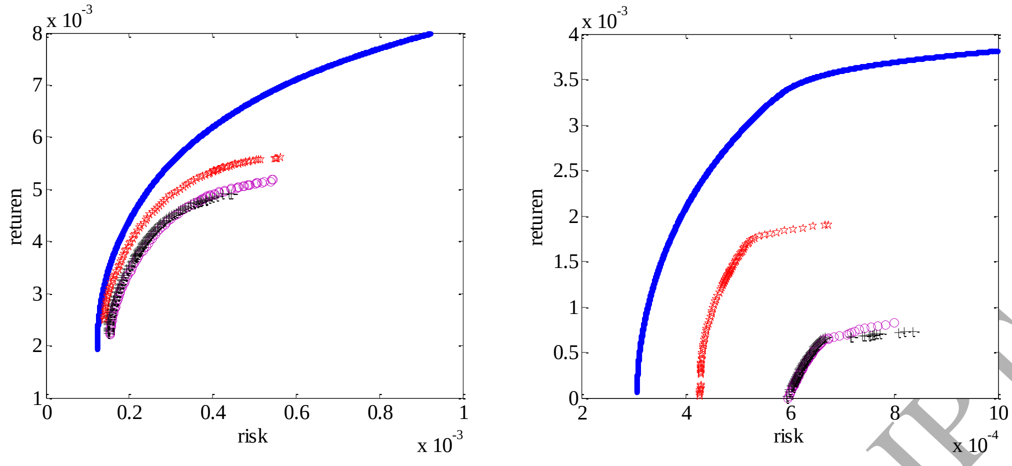
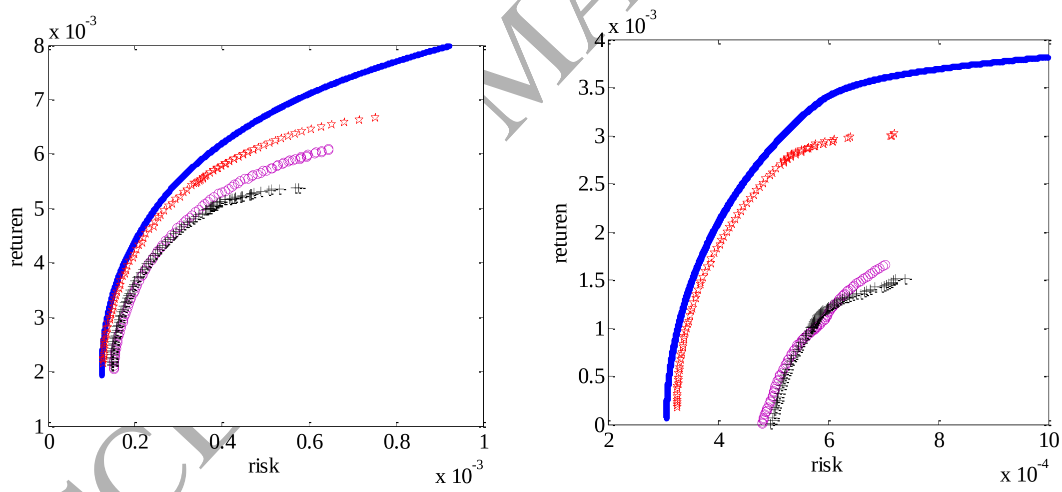


Fig. 7. The results of different pre-selection strategies with $Z = 60$ and $0 \le w_i \le 1$ (a) Port4 problem (b) zoom in graph of Port4 problem (c) Port5 problem (d) zoom in graph of Port5 problem (—•—: Pareto frontier with the basic constraints, ☆: the Pareto frontier with our pre-selection strategy, + : the Pareto frontier with *k*-means strategy 1, o : the Pareto frontier with *k*-means strategy 2)



(a) (b)

Fig. 8. The results of different pre-selection strategies with $Z = 60$ and $0.01 \le w_i \le 0.2$ (a) Port4 problem (b) Port5 problem (—•—: Pareto frontier with the basic constraints, ☆: the Pareto frontier with our pre-selection strategy, +: the Pareto frontier with *k*-means strategy 1, o: the Pareto frontier with *k*-means strategy 2)



(a) (b)

Fig. 9. The results of different pre-selection strategies with $0.01 \le w_i \le 0.2$ (a) Port4 problem with $Z = 40$ (b) Port5 problem with $Z = 30$ (—•—: Pareto frontier with the basic constraints, ☆: the Pareto frontier with our pre-selection strategy, +: the Pareto frontier with *k*-means strategy 1, o: the Pareto frontier with *k*-means strategy 2)

Table 1

Descriptions of experimental data sets

Problem index	Data source	Number of assets
port 1	Hong Kong Hang Seng	31
port 2	German DAX 100	85
port 3	British FTSE 100	89
port 4	US S & P 100	98
port 5	Japan Nikkei 225	225

Table 2

Portfolio modes that form a Pareto frontier

The index of modes	The index of assets for Port1												The size of selected assets	The number of Pareto solutions
	2	5	9	13	15	16	17	26	28	29	30	31		
1													10	1
2													12	8
3													11	3
4													10	8
5													9	6
6													8	2
7													7	3
8													6	17
9													5	4
10													4	18
11													3	20
12													2	9
13													1	1

Table 3

Assets selected for portfolio with different ceiling constraints of Port1 problem

Ceiling constraint value *b*	Assets size selected	The sequence of assets selected according to descending order of their sum of weights
1	12	5,29,28,9,26,15,30,31,16,13,17,2
0.8	12	5,29,28,26,9,15,30,31,16,13,17,2
0.5	12	29,5,28,26,9,15,30,31,16,13,17,2
0.2	15=12+**3**	26,29,28,15,5,9,30,13,31,16,**12**,17,**19**,2,**8**
0.1	20=15+**5**	29,26,13,15,28,31,9,30,5,2,16,12,17,8,**20**,19,**23,22,4,1**
0.05	30=20+**10**	9,13,26,29,15,2,12,5,8,28,4,31,30,22,23,16,19,**11**,17,20,1,**14,10,3,21,24, 27,18,25,7**

Table 4

Assets selected for portfolio with different ceiling constraints of Port5 problem

Ceiling constraint value *b*	Assets size selected	The sequence of assets selected according to descending order of their sum of weights
1	18	62,9,60,40,43,129,214,196,215,97,225,171,98,11,105,115,85,114
0.8	18	62,9,60,40,43,129,196,215,214,97,225,171,98,11,105,115,85,114
0.5	18	62,9,60,40,43,129,196,215,97,225,171,214,98,11,105,115,85,114
0.2	19=18+**1**	62,60,40,9,43,129,215,196,97,171,225,98,11,214,105,115,**165**,85,114
0.1	26=19+**7**	62,40,97,215,43,60,9,171,196,129,165,225,98,11,214,115,105,114,**132, 188,79,2,199**,85,**42,162**
0.05	38=26+**12**	97,40,215,62,196,199,60,9,171,43,165,129,114,162,98,115,79,225,132, 11,214,**158**,105,188,42,**144,201**,2,85,**104,109,186,212,8,89,137,193,28**
0.02	81	… …

Table 5

The scale of portfolio for different ceiling constraints

Problem	Ceiling constraint value b						
	0.02	0.05	0.1	0.2	0.5	0.8	1
Port1-31	/	30	20	15	12	12	12
Port2-85	73	49	36	34	33	33	33
Port3-89	77	50	42	42	42	42	42
Port4-98	87	66	59	57	56	56	56
Port5-225	81	38	26	19	18	18	18

Table 6

The problems with different cardinality constraints

Problem	Pre-selection strategy	Cardinality constraint	Case
Port1	The first time $b = 0.2$	$5 \leq Z \leq 10$	Case 1
		$Z = 10$	Case 2
	The second time $b = 0.05$	$10 \leq Z \leq 25$	Case 3
		$Z = 25$	Case 4
Port2	The first time $b = 0.2$	$5 \leq Z \leq 30$	Case 5
		$Z = 10$	Case 6
	The second time $b = 0.05$	$10 \leq Z \leq 40$	Case 7
		$Z = 40$	Case 8
Port3	The first time $b = 0.2$	$5 \leq Z \leq 30$	Case 9
		$Z = 10$	Case 10
	The second time $b = 0.05$	$10 \leq Z \leq 45$	Case 11
		$Z = 45$	Case 12
Port4	The first time $b = 0.2$	$5 \leq Z \leq 50$	Case 13
		$Z = 10$	Case 14
	The second time $b = 0.05$	$10 \leq Z \leq 60$	Case 15
		$Z = 60$	Case 16
Port5	The first time $b = 0.2$	$5 \leq Z \leq 15$	Case 17
		$Z = 10$	Case 18
	The second time $b = 0.05$	$10 \leq Z \leq 30$	Case 19
		$Z = 30$	Case 20

Table 7

The comparing results with different strategies of HBOPO-PS

Case	IGD (e-5)			Hypervolume (e-6)			*C-metric*			
	HBOPO-PS	No pre-selection	No local search	HBOPO-PS	No pre-selection	No local search	*C*(HBOPO-PS, No pre-selection)	*C*(No pre-selection, HBOPO-PS)	*C*(HBOPO-PS, No local search)	*C*(No local search, HBOPO-PS)
1	**1.9189** 0.4820	13.843 5.1532	2.9507 1.0926	**5.2651** 0.0072	4.6628 0.2282	5.1680 0.0452	**1** 0	0 0	**0.8236** 0.1788	0.0096 0.0175
2	**1.6358** 0.0771	8.0035 4.3740	1.9101 0.2776	**5.2540** 0.0064	4.9271 0.1527	5.2312 0.0144	**0.9880** 0.0514	0 0	**0.5253** 0.2009	0.0600 0.0823
3	**3.9238** 0.1544	5.1533 1.4151	5.2212 0.9673	**5.0604** 0.0083	4.9944 0.0599	4.9884 0.0622	**0.7066** 0.2816	0.1870 0.2395	**0.7060** 0.3042	0.1806 0.2737
4	**7.7948** 0.0680	8.4961 0.1726	8.5481 0.2255	**4.8323** 0.0026	4.7882 0.0113	4.7869 0.0127	**0.8273** 0.1169	0.0080 0.0274	**0.8413** 0.1618	0.0043 0.0140
5	**1.6342** 0.0114	21.491 10.002	6.1653 1.4335	**2.1542** 0.0006	1.6435 0.1168	2.0159 0.0269	**1** 0	0 0	**1** 0	0 0
6	**4.8974** 2.7588	38.802 19.730	9.3665 5.3082	**2.0089** 0.0187	1.3487 0.1457	1.8235 0.0749	**1** 0	0 0	**0.9450** 0.0984	0.0296 0.0705
7	**2.9492** 0.1406	19.325 8.1117	8.4301 3.1435	**2.1121** 0.0014	1.6902 0.0856	1.9436 0.0433	**1** 0	0 0	**0.9993** 0.0036	0.0006 0.0036
8	**14.190** 0.4597	23.887 1.8970	16.069 0.5795	**1.9377** 0.0058	1.7036 0.0266	1.8990 0.0076	**1** 0	0 0	**0.9710** 0.0541	0.0023 0.0081
9	**1.3796** 0.0032	14.542 3.6050	7.4275 2.2769	**2.7006** 0.0001	2.2687 0.0763	2.4947 0.0401	**1** 0	0 0	**1** 0	0 0
10	**5.4349** 2.7336	21.970 7.7242	12.092 4.0459	**2.5657** 0.0151	2.0929 0.1077	2.3339 0.0613	**1** 0	0 0	**0.9356** 0.0875	0.0366 0.0662
11	**1.9695** 0.0449	11.884 3.0968	7.1513 1.9587	**2.6691** 0.0015	2.3445 0.0479	2.5019 0.0343	**1** 0	0 0	**1** 0	0 0
12	**17.904** 0.3735	22.092 1.1098	18.478 0.4484	**2.3522** 0.0055	2.2091 0.0153	2.3374 0.0064	**0.9930** 0.0383	0.0030 0.0164	**0.6863** 0.2448	0.0886 0.1354
13	**2.0466** 0.0115	21.249 6.1785	13.792 4.4185	**7.2400** 0.0011	6.2798 0.1747	6.6115 0.1471	**1** 0	0 0	**1** 0	0 0
14	**4.4338** 1.1228	25.563 8.9846	20.710 7.2976	**7.0211** 0.0183	6.0452 0.2340	6.2546 0.2148	**1** 0	0 0	**0.9686** 0.0485	0.0136 0.0305
15	**3.0979** 0.0816	17.667 4.7078	12.631 3.0747	**7.1255** 0.0053	6.3854 0.1388	6.6213 0.1103	**1** 0	0 0	**1** 0	0 0
16	**53.925** 0.4957	57.285 2.0048	55.218 0.9154	**5.5916** 0.0103	5.4515 0.0437	5.5567 0.0185	**0.9980** 0.0061	0.0003 0.0018	**0.8050** 0.2116	0.0453 0.1049
17	**1.1977** 0.0019	55.339 19.775	2.0343 0.3265	**1.5172** 0.0001	0.5625 0.1485	1.4664 0.0138	**1** 0	0 0	**0.9026** 0.1002	0 0
18	**1.2913** 0.0407	51.988 15.515	1.9496 0.2266	**1.5049** 0.0018	0.6075 0.1530	1.4657 0.0088	**1** 0	0 0	**0.9210** 0.0770	0.0060 0.0188
19	**2.0752** 0.0523	13.843 5.1532	4.1125 0.7529	**1.4656** 0.0015	0.7090 0.0991	1.3896 0.0197	**1** 0	0 0	**0.9860** 0.0359	0 0
20	**6.8016** 0.1280	8.0035 4.3740	7.8833 0.3910	**1.3311** 0.0024	0.7041 0.0063	1.3054 0.0084	**1** 0	0 0	**0.8990** 0.1188	0.0073 0.0285

Table 8

Values of objective function for each problem

problem	$f_1^{\min}$	$f_1^{\max}$	$f_2^{\min}$	$f_2^{\min}$
Port1	0.000578	0.005253	0.00234	0.011950
Port2	0.000130	0.003120	0.00140	0.010800
Port3	0.000185	0.001668	0.00211	0.009030
Port4	0.000120	0.003233	0.00156	0.010000
Port5	0.000270	0.001800	-0.00034	0.004370

Table 9

The performance evaluation metrics for different algorithms

Indicator	Methods	Port1	Port2	Port3	Port4	Port5
Hypervolume	HBOPO-PS	**0.7050** 5.60e-6	**0.8098** 7.61e-7	**0.7197** 1.00e-8	**0.7911** 1.00e-8	**0.8064** 3.30e-5
	NSGA-II	0.7045 1.10e-4	0.8091 2.50e-4	0.7182 3.50e-4	0.7907 1.00e-4	0.8060 1.80e-4
	SPEA2	0.7048 5.00e-5	0.8095 2.0e-4	0.7187 2.00e-4	0.7910 1.40e-4	0.8062 2.00e-4
	NPGA2	0.7040 1.00e-4	0.8088 1.50e-4	0.7176 7.50e-4	0.7900 2.50e-4	0.8055 2.20e-4
	PESA	0.7040 1.00e-4	0.8084 4.00e-4	0.7172 1.00e-3	0.7897 2.60e-4	0.8055 4.00e-4
	PESA-II	0.7042 1.01e-4	0.8087 3.23e-4	0.7179 2.40e-4	0.7900 2.10e-4	0.8057 2.82e-4
	ε-MOEA	0.7049 1.00e-4	0.8092 5.20e-4	0.7180 7.50e-4	0.7906 2.80e-4	0.8058 6.00e-4
	MOEA/D	0.7049 1.15e-5	0.8095 2.43e-5	0.7192 3.20e-5	0.7908 1.31e-5	0.8062 1.70e-5
	EMOPO-OR3	0.7047 2.23e-4	0.8093 4.12e-5	0.7191 2.14e-4	0.7905 2.20e-4	0.8060 2.10e-4
IGD	HBOPO-PS	**3.54e-6** 3.34e-10	**3.85e-6** 4.21e-10	**3.38e-6** 1.84e-10	**3.70e-6** 5.28e-10	**3.83e-6** 2.91e-10
	NSGA-II	3.95e-6 2.80e-10	4.12e-6 3.40e-10	3.72e-6 3.14e-10	4.03e-6 3.80e-10	4.08e-6 2.33e-10
	SPEA2	3.86e-6 3.20e-10	3.93e-6 3.36e-10	3.74e-6 2.73e-10	3.89e-6 4.20e-10	3.95e-6 2.26e-10
	NPGA2	6.30e-6 4.28e-10	6.84e-6 4.71e-10	6.38e-6 5.10e-10	6.72e-6 4.32e-10	6.54e-6 4.17e-10
	PESA	7.24e-6 5.82e-10	7.64e-6 5.43e-10	8.15e-6 5.92e-10	7.41e-6 5.27e-10	8.73e-6 5.60e-10
	PESA-II	6.75e-6 4.32e-10	6.85e-6 4.70e-10	7.32e-6 4.81e-10	7.25e-6 4.24e-10	7.38e-6 4.67e-10
	ε-MOEA	3.85e-6 2.60e-10	4.10e-6 3.15e-10	3.79e-6 3.26e-10	4.07e-6 4.11e-10	4.15e-6 3.40e-10
	MOEA/D	3.82e-6 1.61e-10	3.90e-6 3.38e-10	3.54e-6 2.10e-10	3.84e-6 3.17e-10	3.90e-6 2.26e-10
	EMOPO-OR3	4.37e-6 2.18e-10	4.15e-6 3.74e-10	3.82e-6 3.26e-10	4.57e-6 4.02e-10	4.24e-6 2.73e-10

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