

# Solutions to Problem 7 & 11

## Homework Assignment 2

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### Problem 7, HW 2

Consider the LINEX loss function defined by

$$L(\theta - \theta(x)) = e^{c(\theta - \theta(x))} - c(\theta - \theta(x)) - 1.$$

- (a) Show that  $L(\theta - \theta(x)) \geq 0$  and plot this as a function of  $(\theta - \theta(x))$  when  $c = .1, .5, 1, 2$ .
- (b) Find  $\hat{\theta}(x)$  the estimator that minimizes the Bayesian expected posterior loss.
- (c) Find  $\hat{\theta}(x)$  when  $x_1, \dots, x_n \mid \theta \sim N(\theta, 1), \theta \sim N(\mu, \tau^2)$ .

### Solution:

(a)

Let  $a = \theta - \theta(x)$ , then  $L(\theta - \theta(x)) = L(a) = e^a - a - 1$ .

$$\begin{aligned}\frac{d}{da}L(a) &:= 0 \\ e^a - 1 &= 0 \\ a &= \ln(1) \\ a &= 0\end{aligned}$$

So,  $L(a)$  reaches a local extrema at  $a = 0 \Rightarrow L(a) = e^0 - 0 - 1 = 0$ . That is, there is a local extrema at  $(0, 0)$ . And since  $L''(a) = e^a > 0$ ,  $(0, 0)$  is a local minimum.

Therefore,  $L(\theta - \theta(x)) \geq 0$ .

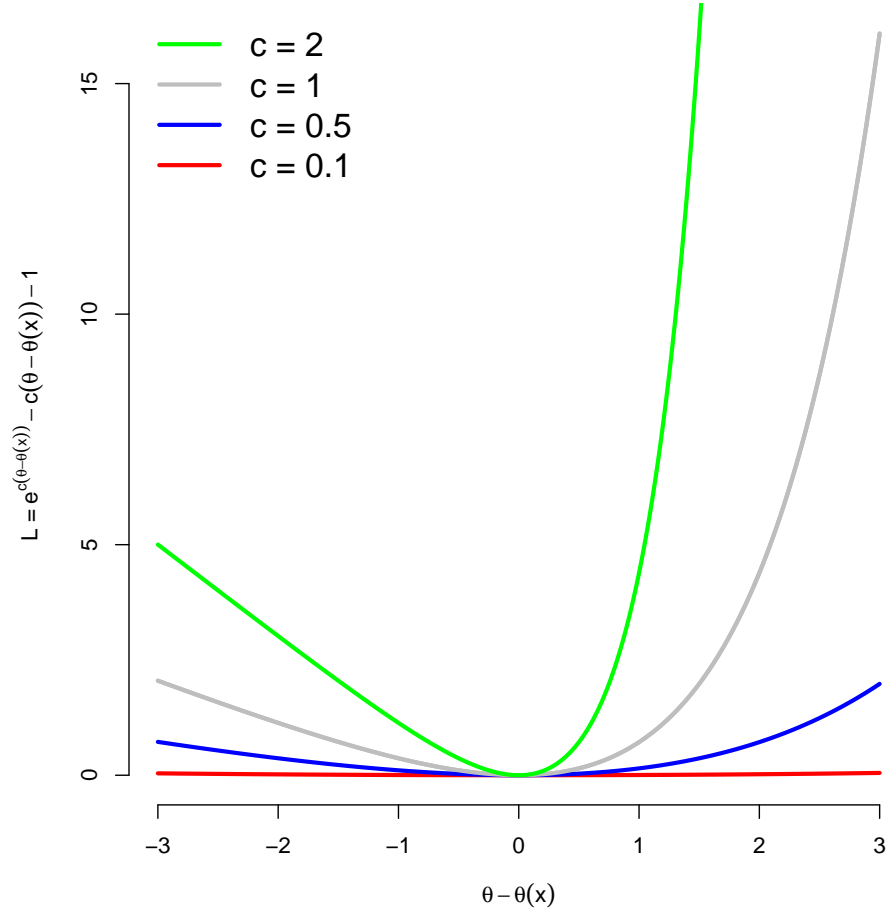


Figure 1: Plot for 7(a). Linex loss as a function of  $\theta - \theta(x)$ , evaluated at various  $c$ .

(b)

Let  $f(\theta|\mathbf{x})$  be the posterior density for  $\theta$ . Then, the expected posterior loss is

$$\begin{aligned}
 E[L(\theta - \theta(x))] &= \int_{\Theta} \{e^{c(\theta - \theta(x))} - c(\theta - \theta(x)) - 1\} f(\theta|x) d\theta \\
 &= e^{-c\theta(x)} \int_{\Theta} e^{c\theta} f(\theta|x) d\theta - c \int_{\Theta} \theta f(\theta|x) d\theta + c\theta(x) \int_{\Theta} f(\theta|x) d\theta - \int_{\Theta} f(\theta|x) d\theta \\
 &= e^{-c\theta(x)} E_{\theta|x}[e^{c\theta}|x] - cE_{\theta|x}[\theta|x] + c\theta(x) - 1
 \end{aligned}$$

To minimize the expected posterior loss,

$$\begin{aligned}
\frac{d}{d\theta(x)} E[L(\theta - \theta(x))] &:= 0 \\
-ce^{-c\hat{\theta}} E_{\theta|x}[e^{c\theta}|x] + c &= 0 \\
e^{c\hat{\theta}} &= E_{\theta|x}[e^{c\theta}|x] \\
\Rightarrow \hat{\theta} &= \frac{\ln(E_{\theta|x}[e^{c\theta}|x])}{c}
\end{aligned}$$

This is simply the log of the posterior moment generating function divided by  $c$ .

(c)

$\hat{\theta}(x)$  is simply the log of posterior mgf divided by  $c$ . So, all we need to do is compute the posterior of  $\theta$ .

$$\begin{aligned}
f_{\theta|\mathbf{x}}(\theta|\mathbf{x}) &\propto \exp\left\{-\frac{\sum_{i=1}^n (x_i - \theta)^2}{2}\right\} \exp\left\{-\frac{(\theta - \mu)^2}{2\tau^2}\right\} \\
&\propto \exp\left\{\frac{-n\theta^2\tau^2 + 2n\bar{x}\theta\tau^2 - \theta^2 + 2\theta\mu}{2\tau^2}\right\} \\
&\propto \exp\left\{\frac{-\theta^2(n\tau^2 + 1) + 2\theta(n\bar{x}\tau^2 + \mu)}{2\tau^2}\right\} \\
&\propto \exp\left\{\frac{-\theta^2 + 2\theta(\frac{n\bar{x}\tau^2 + \mu}{n\tau^2 + 1})}{2\frac{\tau^2}{n\tau^2 + 1}}\right\}
\end{aligned}$$

$$\Rightarrow \theta|\mathbf{x} \sim N(\tilde{m}, \tilde{v}),$$

where  $\tilde{m} = \frac{n\bar{x}\tau^2 + \mu}{n\tau^2 + 1}$  and  $\tilde{v} = \frac{\tau^2}{n\tau^2 + 1}$ . Finally, using the MGF of the Normal distribution,

$$\begin{aligned}
\hat{\theta}(x) &= \frac{\ln(\exp\{\tilde{m}c + \tilde{v}c^2/2\})}{c} \\
&= \frac{\tilde{m}c + \tilde{v}c^2/2}{c} \\
&= \tilde{m} + \frac{\tilde{v}c}{2}
\end{aligned}$$

$$\Rightarrow \hat{\theta}(x) = \frac{(n\bar{x} + c/2)\tau^2 + \mu}{n\tau^2 + 1}$$

## Problem 11, HW 2

Given that  $X_1, X_2$  are two independent observations from

$$P(X = \theta - 1|\theta) = P(X = \theta + 1|\theta) = \frac{1}{2}$$

where  $\theta$  is an integer.

We are provided with the 0 -1 Loss Function

$$L(\theta, \theta(X_1, X_2)) = \begin{cases} 1 & \text{if } \theta(X_1, X_2) \neq \theta \\ 0 & \text{o.w.} \end{cases}$$

(a)

(i) We proceed to find the risk of the estimator  $\theta_0(X_1, X_2) = \frac{X_1 + X_2}{2}$ .

Let  $\mathbf{X} = (X_1, X_2)'$

$$\begin{aligned}
R(\theta, \theta_0(\mathbf{X})) &= \sum_{x_1 \in \{\theta-1, \theta+1\}} \sum_{x_2 \in \{\theta-1, \theta+1\}} L(\theta, \theta(x_1, x_2)) P(X_1 = x_1, X_2 = x_2) \\
&= 1 \cdot P(X_1 = \theta - 1, X_2 = \theta - 1) + 1 \cdot P(X_1 = \theta + 1, X_2 = \theta + 1) \\
&+ 0 \cdot P(X_1 = \theta - 1, X_2 = \theta + 1) + 0 \cdot P(X_1 = \theta + 1, X_2 = \theta - 1) \\
&= P(X_1 = \theta - 1)P(X_2 = \theta - 1) + P(X_1 = \theta + 1)P(X_2 = \theta + 1) \\
&= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

(ii) Now considering the second estimator  $\theta_1(X_1, X_2) = X_1 + 1$ .

$$\begin{aligned}
R(\theta, \theta_1(\mathbf{X})) &= \sum_{x_1 \in \{\theta-1, \theta+1\}} \sum_{x_2 \in \{\theta-1, \theta+1\}} L(\theta, \theta(x_1, x_2)) P(X_1 = x_1, X_2 = x_2) \\
&= 1.P(X_1 = \theta + 1, X_2 = \theta + 1) + 1.P(X_1 = \theta + 1, X_2 = \theta - 1) \\
&+ 0.P(X_1 = \theta - 1, X_2 = \theta + 1) + 0.P(X_1 = \theta - 1, X_2 = \theta - 1) \\
&= P(X_1 = \theta + 1)P(X_2 = \theta + 1) + P(X_1 = \theta + 1)P(X_2 = \theta - 1) \\
&= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

The two estimators are the same as far as “Frequentist Risk” is concerned.

(b) Need to find an estimator  $\hat{\theta}(X_1, X_2)$  that minimizes Bayesian Expected Loss.

$$\text{Bayesian Expected Loss} = E_{\theta|\mathbf{X}}(L(\theta, \theta(\mathbf{X})))$$

$$\begin{aligned}
E_{\theta|\mathbf{X}}(L(\theta, \theta(\mathbf{X}))) &= 1.P_{\theta|\mathbf{X}}(\theta \neq \theta(\mathbf{X})) + 0.P_{\theta|\mathbf{X}}(\theta = \theta(\mathbf{X})) \\
&= P_{\theta|\mathbf{X}}(\theta \neq \theta(\mathbf{X})) = 1 - P_{\theta|\mathbf{X}}(\theta = \theta(\mathbf{X})).
\end{aligned}$$

In order to minimize the Bayesian Expected Loss, we need to maximize  $P_{\theta|\mathbf{X}}(\theta = \theta(\mathbf{X}))$ .

This implies  $\hat{\theta}(\mathbf{X}) = \arg \max_{\theta} P_{\theta|\mathbf{X}}$ .

Let the prior distribution of  $\theta$  be  $\mathbf{p}(\cdot)$ . Consider the following cases:

*Case 1:*  $X_1 \neq X_2$

$$P(\theta = \frac{X_1 + X_2}{2} | X_1, X_2) = 1.$$

So, the mode of the posterior distribution is  $\frac{X_1 + X_2}{2}$ .

Therefore, when  $X_1 \neq X_2$ ,  $\hat{\theta}(\mathbf{X}) = \frac{X_1 + X_2}{2}$ .

*Case 2:*  $X_1 = X_2$

The posterior distribution of  $\theta$  has mass on  $X_1 + 1$  and  $X_1 - 1$  and the posterior p.m.f is given by

$$P(\theta = X_1 + 1 | \mathbf{X}) = \frac{\mathbf{p}(\theta = X_1 + 1)}{\mathbf{p}(\theta = X_1 + 1) + \mathbf{p}(\theta = X_1 - 1)}$$

and

$$P(\theta = X_1 - 1 | \mathbf{X}) = \frac{\mathbf{p}(\theta = X_1 - 1)}{\mathbf{p}(\theta = X_1 + 1) + \mathbf{p}(\theta = X_1 - 1)}.$$

Thus the mode of  $\theta|\mathbf{X}$  is  $(X_1 + 1)$  if  $\mathbf{p}(\theta = X_1 + 1) \geq \mathbf{p}(\theta = X_1 - 1)$ . Also, the mode of  $\theta|\mathbf{X}$  is  $(X_1 - 1)$  if  $\mathbf{p}(\theta = X_1 + 1) < \mathbf{p}(\theta = X_1 - 1)$ .  
Hence

$$\hat{\theta}(X_1, X_2) = \begin{cases} \frac{X_1+X_2}{2}, & \text{If } X_1 \neq X_2 \\ X_1 + 1, & \text{If } X_1 = X_2 \text{ and } \mathbf{p}(\theta = X_1 + 1) \geq \mathbf{p}(\theta = X_1 - 1) \\ X_1 - 1, & \text{If } X_1 = X_2 \text{ and } \mathbf{p}(\theta = X_1 + 1) < \mathbf{p}(\theta = X_1 - 1) \end{cases}$$