BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

Student number: ____________________

First Year Exam: June 15th 2009

INSTRUCTIONS

You are to work individually on this problem. Do not share with anyone any information or comments about your findings or the models and methods you use. You must turn in your solution by 5pm on Wednesday 17th June. You can either print it and take it to Nic Brummell's office (BE125), or send a PDF file to `brummell@ams.ucsc.edu` . Please take care to organize and present the material in the best possible way; be informative but concise. Your paper should consist of no more than 12 letter-size pages (with 11pt or larger type and margins on all four sides of at least 1 inch), including tables and figures. The grading will based on both technical correctness and the presentation of the report. All questions should be answered in an accurate, complete and logically-argued manner.

PROBLEM

The motion of the satellite under gravity can be approximated by the two-body dynamics

$$\begin{aligned} \frac{dx}{dt} &= v_x \\ \frac{dy}{dt} &= v_y \\ \frac{dv_x}{dt} &= -\frac{\mu}{r^3}x \\ \frac{dv_y}{dt} &= -\frac{\mu}{r^3}y \end{aligned} \tag{1}$$

(x, y) is the position of the satellite in an Earth centered inertial frame; and (v_x, v_y) denotes the velocity in x and y directions. $\mu = 398600.44\ km^3/s^2$ is the Earth gravitational constant; and $r = \sqrt{x^2 + y^2}$ is the distance between the satellite and the center of the Earth. The units used are kilometer for position and kilometer per second for velocity.

1. The total mechanical energy and the angular momentum of the satellite are given by

$$E(t) = \frac{m}{2}(v_x^2(t) + v_y^2(t)) - \frac{m\mu}{\sqrt{x^2(t) + y^2(t)}} \qquad \text{(energy)} \tag{2}$$

$$L(t) = m[x(t)v_y(t) - y(t)v_x(t)] \qquad \text{(momentum)} \tag{3}$$

where constant m is the mass of the satellite. Prove that $E(t)$ and $L(t)$ remain constant along any solution of Eqn. (1).

2. Suppose the initial condition $(x(0), y(0), v_x(0), v_y(0))$ satisfies

$$v_x^2(0) + v_y^2(0) - \frac{2\mu}{\sqrt{x^2(0) + y^2(0)}} = -\frac{\mu}{a} \tag{4}$$

where $a > 0$ is a constant. Prove that, from such initial conditions, the trajectory $(x(t), y(t))$ is bounded for all t.

Now consider the numerical solution of Eqn. (1) from the following initial condition

$$\begin{bmatrix} x(0) \\ y(0) \\ v_x(0) \\ v_y(0) \end{bmatrix} = \begin{bmatrix} 12672.328683202\ km \\ 0\ km \\ 0\ km/s \\ 7.31248745491471\ km/s \end{bmatrix} \tag{5}$$

It can be proved that the solution from the given initial condition (5) is periodic with period $T = 86400$ seconds (you don't need to prove this). Without analytic solution, we will use the properties developed before to analysis the accuracy of the numerical solutions.

3. Write a MATLAB program to implement the following Runge-Kutta scheme for the initial value problem of a system of first order ODEs, $dz/dt = f(z)$, $z \in R^n$.

$$\begin{aligned} s_1 &= z_n \\ s_2 &= z_n + \frac{h}{2} f(s_1) \\ s_3 &= z_n + \frac{h}{2} f(s_2) \\ s_4 &= z_n + h f(s_3) \\ z_{n+1} &= z_n + \frac{h}{6}[f(s_1) + 2f(s_2) + 2f(s_3) + f(s_4)] \end{aligned}$$

4. Use above RK scheme to generate the solution from given initial condition (5) for $t \in [0, 10T]$, where $T = 86400$ seconds (1 day) is the period of the solution. Suppose the requirement on the numerical solution is that the changes of the energy and angular momentum in $[0, 10T]$ are less than 1%. Let the step size in above RK scheme be $h = 1200$ seconds. Plot out your numerical solutions and analysis if the given accuracy is achieved. If not, find appropriate step size that generates a satisfactory solution.

5. Since the solution from the given initial condition (5) is periodic with period $T = 86400$ seconds, we can also use the distance between $(x(T), y(T))$ and $(x(0), y(0))$ to evaluate the accuracy of the numerical solution. Find appropriate step size, h, so that the distance between $(x(T), y(T))$ and $(x(0), y(0))$ is less than 50 km.

6. Introduce the polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad v_r = v_x\cos\theta + v_y\sin\theta, \qquad v_t = -v_x\sin\theta + v_y\cos\theta,$$

where v_r and v_t are velocities in radial and tangential direction. Write down the dynamical equation in the polar coordinates. Furthermore, let $k = 0.5(v_r^2 + v_t^2)$ and $p = 1/r$ (k and p are normalized kinetic and potential energy) Show that the dynamical equation in (k, p, v_r, θ) coordinates is

$$\begin{aligned} \frac{dk}{dt} &= -\mu v_r p^2 \\ \frac{dp}{dt} &= -v_r p^2 \\ \frac{dv_r}{dt} &= -\mu p^2 + p(2k - v_r^2) \\ \frac{d\theta}{dt} &= p\sqrt{2k - v_r^2} \end{aligned} \tag{6}$$

7. Prove that, when aforementioned Runge-Kutta scheme is applied to Eqn. (6), the energy is automatically conserved, i.e., $k_{n+1} - \mu p_{n+1} = k_n - \mu p_n$.

8. Apply the given RK scheme on Eqn. (6) and Eqn. (1) with initial condition (5). The time span is $t \in [0, 10T]$, where $T = 86400$. Use the same step size $h = 1200$ seconds in both cases. Compare your results and show which system of equations provide more accurate solutions.