BASKIN SCHOOL OF ENGINEERING
Department of Applied Mathematics and Statistics

First Year Exam, Take Home Question (Applied Mathematics)

Due by 2PM, Wednesday, June 20, 2012

Instructions:

Please work individually on this problem. Do not share with anyone any information or comments about your findings or the models and methods you use. You are required to write a report using a word processing software (i.e., LaTeX or Microsoft Word). You are required to email your report as ONE PDF file to Hongyun Wang at `hongwang@ams.ucsc.edu`

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Please take care to organize and present the material in the best possible way. Be informative but concise. You should include a summary of your work at the beginning of the report, include and annotate all relevant figures and tables in the body of the report, write your conclusions in a separate section and list the references (if any) that you consider appropriate. Your report should consist of no more than 8 letter-size pages (typeset with 11pt or larger font and margins on all four sides of at least 1 inch), including all figures, tables, and appendices (but excluding the MATLAB codes); answers longer than 8 pages will lose credit for excess length.

Exam Problem:

Consider the forced oscillation of the Van der Pol system described by

$$\begin{cases} y'' + \varepsilon\,(y^2-1)\,y' + y = \alpha\,\varepsilon\cos\left[(1+\beta\varepsilon)t+\gamma\right] \\ y(0)=a\ , \quad y'(0)=0 \end{cases} , \qquad \varepsilon \to 0_+ \tag{1}$$

where ε is the small parameter, and α and β are two other parameters. We use the method of multi-scale expansion to study the forced oscillation.

Let $T_0 = t$ and $T_1 = \varepsilon\, t$. We write $y(t)$ as

$$y(t) = y(T_0, T_1) = a_0(T_0, T_1) + \varepsilon\, a_1(T_0, T_1) + \cdots$$

Recall that in the absence of forcing ($\alpha = 0$), the multi-scale expansion is given by

$$a_0(T_0, T_1) = \frac{2}{\sqrt{1 + \left(\frac{4}{a^2} - 1\right) \exp(-T_1)}} \cos(T_0)$$

Notice that for $\beta \neq 0$, the frequency of forcing is different from the intrinsic frequency of the Van der Pol equation. Here we consider the case of $\alpha > 0$.

Part 1: Show that the leading term of $y(T_0, T_1)$ has the form

$$a_0(T_0, T_1) = A(T_1) \exp(iT_0) + \overline{A(T_1)} \exp(-iT_0) \tag{2}$$

where $A(T_1)$ is a complex function and $\overline{A}$ denotes the complex conjugate of A.

Part 2: We write $A(T_1)$ in the exponential form

$$A(T_1) = R(T_1) \exp\left(i[\theta(T_1) + \beta T_1 + \gamma]\right)$$

where both $R(T_1)$ and $\theta(T_1)$ are real functions. Show that

- $2R(T_1)$ is the amplitude of the oscillation ($a_0(T_0, T_1)$) and
- $\theta(T_1)$ is the difference between the phase angle of oscillation and the phase angle of forcing.

Thus, if $(R(T_1), \theta(T_1))$ attains a steady state, it means that the oscillation ($a_0(T_0, T_1)$) has a constant amplitude and follows the forcing with a constant difference in phase angle.

Part 3: Show that $(R(T_1), \theta(T_1))$ satisfies the ODE system

$$\begin{cases} \dfrac{dR}{dT_1} = -\dfrac{1}{2}(R^3 - R) - \dfrac{\alpha}{4}\sin(\theta) \\[2ex] \dfrac{d\theta}{dT_1} = -\beta - \dfrac{\alpha}{4R}\cos(\theta) \end{cases} \tag{3}$$

and initial conditions

$$R(0) = \frac{a}{2}\,, \qquad \theta(0) = -\gamma$$

Part 4: Using (3), show that if the oscillation ($a_0(T_0, T_1)$) attains a steady state amplitude, then it must follow the forcing with a constant difference in phase angle. In other words, it is impossible for the non-linearity of Van der Pol equation to overcome the forcing frequency and maintain the intrinsic frequency. On the other hand, it is possible for the forcing to overcome the intrinsic frequency and drive the oscillation.

Part 5: Consider a numerical grid for the parameter space:

$$\{(\alpha, \beta)\} = [0.02 : 0.04 : 1.5] \times [0.02 : 0.02 : 0.4]$$

Scan through the numerical grid. At every grid point (α, β) in the parameter space, find numerically ALL steady state solutions of (3). Count ONLY steady states with $R > 0$. Create a figure with α on the horizontal axis and β on the vertical axis. Plot the region of parameter space where (3) has exactly k steady states (with $R > 0$), respectively for $k = 0, 1, 2, 3, ...$ Plot all regions in one figure. Describe clearly which region is which.
(If time permits, you are encouraged to repeat this part on a refined numerical grid.)

Part 6: At every grid point (α, β) in the parameter space, determine numerically the stability of each steady state (with $R > 0$). Plot the region of parameter space where (3) has exactly k stable steady states, respectively for $k = 0, 1, 2, 3, ...$ Plot all regions in one figure. Describe clearly which region is which.
(If time permits, you are encouraged to repeat this part on a refined numerical grid.)

Part 7: For $\beta = 0.1$, scan through the numerical grid for α:

$$\{\alpha\} = [0.02 : 0.01 : 1.5]$$

Create a plot with α on the horizontal axis and R on the vertical axis. At every value of α, plot (α, R) of each steady state as

- a red filled circle if it is stable or
- a blue open circle if it is unstable.

Plot (α, R) of all steady states at all values of α in one figure. Describe the bifurcation(s) and the type(s) of bifurcation(s) you observe in the figure.
(If time permits, you are encouraged to repeat this part on a refined numerical grid.)

Part 8: For $\beta = 0.4$, scan through the numerical grid for α:

$$\{\alpha\} = [0.5 : 0.01 : 2.0]$$

At every value of α, plot (α, R) of each steady state as

- a red filled circle if it is a stable node or
- a blue open circle if it is an unstable node or
- a green filled square if it is a stable focus or
- a black open square if it is an unstable focus or
- a magenta triangle if it is a saddle point.

Plot (α, R) of all steady states at all values of α in one figure. Describe the bifurcation(s) and the type(s) of bifurcation(s) you observe in the figure.
(If time permits, you are encouraged to repeat this part on a refined numerical grid.)

Part 9: Based on your findings in Part 8, try to guess a value α_0 such that (3) has a limit cycle when $(\alpha, \beta) = (\alpha_0, 0.4)$. Solve (3) numerically to confirm the limit cycle. If unsuccessful, try another guess.

Part 10 (optional, finish and write up parts 1-9 before attempting part 10):
For $\alpha > 0$ and $\beta > 0$, prove that (3) has at least one steady state with $R > 0$.

You must include your MATLAB codes for the numerical solution at the end of your report. The MATLAB codes do not count toward the 8-page limit.