AMS 206B: Intermediate Bayesian Inference
HOMEWORK #5 (Hypothesis Testing and Model Comparison)

1. Let $X \sim Exp(\theta)$ and assume one wishes to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, with $\theta_1 < \theta_0$.
   (a) Obtain the p-value associated with $X = 3$ when $\theta_0 = 1$.
   (b) Assuming that $\theta_1 = 1/2$, calculate the Bayes factor for $X = 3$ and $\theta_0 = 1$.
   (c) Assuming that $Pr(H_0) = Pr(H_1)$, calculate the posterior probability of H_0.

2. Assume $\theta \sim N(\mu, \tau^2)$ and observe $X \sim N(\theta, \sigma^2)$, with σ^2 known. Suppose one wishes to verify if a quantity θ is smaller than a pre-specified value θ_0.
   (a) Obtain the prior probability of $H_0 : \theta < \theta_0$.
   (b) Obtain the posterior probability of H_0.
   (c) Prove that the probability of H_0 increases after observing $X = x$ if and only if $x < \theta_0(1 - 1/\sqrt{2})$ in the case $\sigma^2 = \tau^2$ and $\mu = 0$.

3. Let $X_i \sim^{i.i.d.} N(\theta, \sigma^2)$. Assume one wishes to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Let $Pr(H_0) = Pr(H_1) = 1/2$, and
$$n_0\sigma_0^2\phi \sim \chi^2_{n_0}$$
   under H_0 and H_1 with $\phi = \sigma^{-2}$. Similarly, let
$$(\theta|\phi, H_1) \sim N(\mu, (c\phi)^{-1})$$
   for some $c > 0$. Compute $BF(H_0; H_1)$.

4. Theory predicts that θ, the melting point of a particular substance under a pressure of 10^6 atmospheres, is 4.01. The procedure for measuring this melting point is fairly inaccurate, due to high pressure. Indeed, it is known that an observation $X \sim N(\theta, 1)$. Five independent experiments give observations of 4.9, 5.6, 5.1, 4.6, and 3.6. The prior probability that $\theta = 4.01$ is 0.5. The remaining values of θ are given the density (0.5) $g_1(\theta)$ where $g_1(\theta)$ is a $N(4.01, 1)$. Formulate and conduct a Bayesian test of the proposed theory.

5. A large shipment of parts is received, out of which 5 are tested for defects. The number of defective parts X, is assumed to have a $Bin(5, \theta)$ distributions. It is also known from past shipments that $\theta \sim Beta(1, 9)$. Assume that $x = 0$ is observed and consider $H_0 : \theta \leq 0.1$ versus $H_1 : \theta > 0.1$. Find the posterior probabilities of the two hypotheses, the posterior odds ratio, and the Bayes factor.