Discretization of the Helmholtz equation in 2D with finite differences

March 27, 2017

In this notebook we compute the stencils for the finite difference discretization of the Helmholtz boundary value problem with Sommerfeld boundary conditions in $\Omega = (0,1)^2$:

$$
\begin{aligned}
-\Delta u - k^2 u &= f \text{ in } \Omega \\
\partial_n u - iku &= g \text{ on } \partial\Omega
\end{aligned}
$$

Let n_x, n_y be the number of interior points in the x and y directions (including the endpoints), $h_x = 1/(n_x+1), h_y = 1/(n_y+1)$ and $G = \{(x_i, y_j) : 0 \leq i \leq n_x + 1, 1 \leq j \leq n_y + 1\}$ be the corresponding grid. We first compute the equations for the interior points. For the point $x_{ij} = (ih_x, jh_y)$, we obtain the following equation

$$f_{ij} h_x^2 h_y^2 = -h_x^2 \left(u_{(i,j+1)} + u_{(i,j-1)}\right) - h_y^2 \left(u_{(i+1,j)} + u_{(i-1,j)}\right) + u_{(i,j)} \left(-h_x^2 h_y^2 k_{ij}^2 + 2h_x^2 + 2h_y^2\right),$$

equivalent to,

$$f_{ij} = -h_y^{-2} \left(u_{(i,j+1)} + u_{(i,j-1)}\right) - h_x^{-2} \left(u_{(i+1,j)} + u_{(i-1,j)}\right) + u_{(i,j)} \left(-k_{ij}^2 + 2h_y^{-2} + 2h_x^{-2}\right)$$

We set now the linear equation for the non-corner points on the boundary. We begin with points of the form $x_{(0,j)} = (0, jh_y)$ on the west boundary $(x = 0, 1 < y < 1)$. At the point $x_{(0,j)}$ the boundary condition equals $-u_{x(0,j)} - ik_{(0,j)} u_{(0,j)} = g_{(0,j)}$. We approximate the derivative with forward differences and obtain

$$u_{x(0,j)} = h_x^{-1}(u_{(1,j)} - u_{(0,j)}) - (1/2) h_x u_{xx(0,j)} + O(h_x^2)$$

The second derivative $u_{xx(0,j)}$ in the previous expression can be substituted using the equation $-u_{xx} - u_{yy} - k^2 u = f$ (extended by continuity to the boundary point)

$$-f_{(0,j)} - k_{(0,j)}^2 u_{(0,j)} - u_{yy(0,j)}$$

And the second derivative $u_{yy(0\,j)}$ can be approximated using central differences (with order $O(h_y^2)$) to obtain

$$f_{(0,j)} + k_{(0,j)}^2 u_{(0,j)} + \frac{1}{h_y^2}\left(-2u_{(0,j)} + u_{(0,j+1)} + u_{(0,j-1)}\right)$$

We substitute the expression for the second derivative in the expression of the approximation of the first derivative, to obtain

$$-\frac{h_x}{2}\left(-f_{(0,j)} - k_{(0,j)}^2 u_{(0,j)} - \frac{1}{h_y^2}\left(-2u_{(0,j)} + u_{(0,j+1)} + u_{(0,j-1)}\right)\right) + \frac{1}{h_x}\left(-u_{(0,j)} + u_{(1,j)}\right)$$

Finally, the boundary condition $-u_{x(0,j)} - ik_{(0,j)}u_{(0,j)} = g_{(0,j)}$ gives

$$-f_{(0,j)} - k_{(0,j)}^2 u_{(0,j)} + \frac{2u_{(0,j)}}{h_y^2} - \frac{u_{(0,j+1)}}{h_y^2} - \frac{u_{(0,j-1)}}{h_y^2} - \frac{2i}{h_x}k_{(0,j)}u_{(0,j)} + \frac{2u_{(0,j)}}{h_x^2} - \frac{2u_{(1,j)}}{h_x^2}$$

The equation for the point $x_{(0,j)}$ is

$$\frac{2u_{(0,j)}}{h_y^2} + \frac{2u_{(0,j)}}{h_x^2} - k_{(0,j)}^2 u_{(0,j)} - \frac{u_{(0,j+1)}}{h_y^2} - \frac{u_{(0,j-1)}}{h_y^2} - \frac{2u_{(1,j)}}{h_x^2} - \frac{2i}{h_x}k_{(0,j)}u_{(0,j)} = 2\frac{g_{(0,j)}}{h_x} + f_{(0,j)}$$

On the south boundary, we have (non-corner) points of the form $(ih_x, 0)$, where $0 < i < n_x$. At the point $x_{(i,0)}$, the boundary condition is $-u_{y(i,0)} - ik_{(i,0)}u_{(i,0)} = 0$. The equation for this point is

$$\frac{2u_{(i,0)}}{h_x^2} + \frac{2u_{(i,0)}}{h_y^2} - k_{(i,0)}^2 u_{(i,0)} - \frac{u_{(i+1,0)}}{h_x^2} - \frac{u_{(i-1,0)}}{h_x^2} - \frac{2u_{(i,1)}}{h_y^2} - \frac{2i}{h_y}k_{(i,0)}u_{(i,0)} = 2\frac{g_{(i,0)}}{h_y} + f_{(i,0)}$$

Further, we consider the (non-corner) points on the east boundary, of the form $(1, jh_y)$, where $0 < j < n_y$. At the point $x(n_x+1, j) = ((n_x+1)h_x, jh_y) = (1, jh_y)$, the boundary condition is

$$u_{x(n_x+1,j)} - ik_{(n_x+1,j)}u_{(n_x+1,j)} = g_{(n_x+1,j)}.$$

We approximate the derivative $u_{x(n_x+1,j)}$ with backward differences and obtain

$$u_{x(n_x+1,j)} = h_x^{-1}(u_{(n_x+1,j)} - u_{(n_x,j)}) + (1/2)h_x u_{xx(n_x+1,j)} + O(h_x^2)$$

$$\frac{h_x u_{xx(n_x+1,j)}}{2} + \frac{1}{h_x}\left(u_{(n_x+1,j)} - u_{(n_x,j)}\right)$$

The second derivative $u_{xx(n_x+1,j)}$ in the previous expression can be substituted using the equation $-u_{xx} - u_{yy} - k^2 u = f$ (extended by continuity to the boundary point (?))

$$-f_{(n_x+1,j)} - k^2_{(n_x+1,j)} u_{(n_x+1,j)} - u_{yy(n_x+1,j)}$$

And the second derivative $u_{yy(n_x+1\,j)}$ can be approximated using central differences (with order $O(h_y^2)$) to obtain

$$-f_{(n_x+1,j)} - k^2_{(n_x+1,j)} u_{(n_x+1,j)} + \frac{2u_{(n_x+1,j)}}{h_y^2} - \frac{u_{(n_x+1,j+1)}}{h_y^2} - \frac{u_{(n_x+1,j-1)}}{h_y^2}$$

We substitute the expression for the second derivative in the expression of the approximation of the first derivative, to obtain

$$\frac{h_x}{2}\left(-f_{(n_x+1,j)} - k^2_{(n_x+1,j)} u_{(n_x+1,j)} - \frac{1}{h_y^2}\left(-2u_{(n_x+1,j)} + u_{(n_x+1,j+1)} + u_{(n_x+1,j-1)}\right)\right) + \frac{1}{h_x}\left(u_{(n_x+1,j)} - u_{(n_x,j)}\right)$$

Finally, the boundary condition $u_{x(n_x+1,j)} - ik_{(n_x+1,j)} u_{(n_x+1,j)} = g_{(n_x+1,j)}$ gives

$$-\frac{f_{(n_x+1,j)} h_x}{2} - \frac{h_x u_{(n_x+1,j)}}{2} k^2_{(n_x+1,j)} + \frac{h_x u_{(n_x+1,j)}}{h_y^2} - \frac{h_x u_{(n_x+1,j+1)}}{2h_y^2} - \frac{h_x u_{(n_x+1,j-1)}}{2h_y^2} - ik_{(n_x+1,j)} u_{(n_x+1,j)} + \frac{u_{(n_x+1,j)}}{h_x} - \frac{u_{(n_x,j)}}{h_x}$$

$$-f_{(n_x+1,j)} - k^2_{(n_x+1,j)} u_{(n_x+1,j)} + \frac{2u_{(n_x+1,j)}}{h_y^2} - \frac{u_{(n_x+1,j+1)}}{h_y^2} - \frac{u_{(n_x+1,j-1)}}{h_y^2} - \frac{2i}{h_x} k_{(n_x+1,j)} u_{(n_x+1,j)} + \frac{2u_{(n_x+1,j)}}{h_x^2} - \frac{2u_{(n_x,j)}}{h_x^2}$$

This leads to the following equation for the point $x_{(n_x+1,j)}$:

$$\frac{2u_{(n_x+1,j)}}{h_y^2} + \frac{2u_{(n_x+1,j)}}{h_x^2} - k^2_{(n_x+1,j)} u_{(n_x+1,j)} - \frac{u_{(n_x+1,j+1)}}{h_y^2} - \frac{u_{(n_x+1,j-1)}}{h_y^2} - \frac{2u_{(n_x,j)}}{h_x^2} - \frac{2i}{h_x} k_{(n_x+1,j)} u_{(n_x+1,j)} = \frac{2g_{(n_x+1,j)}}{h_x} + f_{(n_x+1,j)}$$

We finish with the north boundary, where the points have the form $x_{(i,n_y+1)} = (ih_x, 1)$ with $0 < i < n_x$. Similarly to the previous cases, the equation for this point is

$$\frac{2u_{(i,n_y+1)}}{h_x^2} + \frac{2u_{(i,(n_y+1))}}{h_x^2} - k^2_{(i,n_y+1)} u_{(i,n_y+1)} - \frac{u_{(i+1,n_y+1)}}{h_x^2} - \frac{u_{(i-1,n_y+1)}}{h_x^2} - \frac{2u_{(i,n_y)}}{h_y^2} - \frac{2i}{h_y} k_{(i,n_y+1)} u_{(i,n_y+1)} = \frac{2g_{(i,n_y+1)}}{h_y} + f_{(i,n_y+1)}$$

In summary, we have obtained the following equations:

For interior points of the form $x_{(i,j)} = (ih_x, jh_y)$ where $0 < i < n_x + 1$ and $0 < j < n_y + 1$:

$$-\frac{\left(u_{(i,j+1)} + u_{(i,j-1)}\right)}{h_y^2} - \frac{\left(u_{(i+1,j)} + u_{(i-1,j)}\right)}{h_x^2} + \left(\frac{2}{h_x^2} u_{(i,j)} + \frac{2}{h_y^2} - k_{ij}^2\right) u_{(i,j)} = f_{ij}$$

For non-corner points on the west boundary, of the form $x_{(0,j)} = (0, jh_y)$ where $0 < j < n_y + 1$:

$$\frac{2u_{(0,j)}}{h_y^2} + \frac{2u_{(0,j)}}{h_x^2} - k_{(0,j)}^2 u_{(0,j)} - \frac{u_{(0,j+1)}}{h_y^2} - \frac{u_{(0,j-1)}}{h_y^2} - \frac{2u_{(1,j)}}{h_x^2} - \frac{2i}{h_x} k_{(0,j)} u_{(0,j)} = 2\frac{g_{(0,j)}}{h_x} + f_{(0,j)}$$

For non-corner points on the south boundary, of the form $x_{(i,0)} = (ih_x, 0)$ where $0 < i < n_x + 1$:

$$\frac{2u_{(i,0)}}{h_x^2} + \frac{2u_{(i,0)}}{h_y^2} - k_{(i,0)}^2 u_{(i,0)} - \frac{u_{(i+1,0)}}{h_x^2} - \frac{u_{(i-1,0)}}{h_x^2} - \frac{2u_{(i,1)}}{h_y^2} - \frac{2i}{h_y} k_{(i,0)} u_{(i,0)} = 2\frac{g_{(i,0)}}{h_y} + f_{(i,0)}$$

For non-corner points on the east boundary, of the form $x_{(n_x,j)} = (1, jh_y)$ where $0 < j < n_y + 1$:

$$\frac{2u_{(n_x+1,j)}}{h_y^2} + \frac{2u_{(n_x+1,j)}}{h_x^2} - k_{(n_x+1,j)}^2 u_{(n_x+1,j)} - \frac{u_{(n_x+1,j+1)}}{h_y^2} - \frac{u_{(n_x+1,j-1)}}{h_y^2} - \frac{2u_{(n_x,j)}}{h_x^2} - \frac{2i}{h_x} k_{(n_x+1,j)} u_{(n_x+1,j)} = \frac{2g_{(n_x+1,j)}}{h_x} + f_{(n_x+1,j)}$$

For non-corner points on the northern boundary, of the form $x_{(i,n_y)} = (ih_x, 1)$ where $0 < i < n_x + 1$

$$\frac{2u_{(i,n_y+1)}}{h_x^2} + \frac{2u_{(i,(n_y+1))}}{h_x^2} - k_{(i,n_y+1)}^2 u_{(i,n_y+1)} - \frac{u_{(i+1,n_y+1)}}{h_x^2} - \frac{u_{(i-1,n_y+1)}}{h_x^2} - \frac{2u_{(i,n_y)}}{h_y^2} - \frac{2i}{h_y} k_{(i,n_y+1)} u_{(i,n_y+1)} = \frac{2g_{(i,n_y+1)}}{h_y} + f_{(i,n_y+1)}$$

We continue with the corner points on the boundary of the domain. At the point $x_{(0,0)} = (0,0)$, the boundary condition $\partial_n u - iku = 0$ in the horizontal direction results in the equation

$$-u_{x(0,0)} - ik_{(0,0)} u_{(0,0)} = g_{(0,0^+)}.$$

Similarly, in the vertical direction the boundary condition is

$$-u_{y(0,0)} - ik_{(0,0)} u_{(0,0)} = g_{(0^+,0)}.$$

Approximating the derivatives $u_{x(0,0)}$ and $u_{y(0,0)}$ by forward differences leads to We multiply the boundary conditions by $2h_y^{-1}$ and $2h_x^{-1}$ respectively

Summing these two equations we obtain

$$u_{xx(0,0)} + u_{yy(0,0)} - \frac{2i}{h_y} k_{(0,0)} u_{(0,0)} + \frac{2u_{(0,0)}}{h_y^2} - \frac{2u_{(0,1)}}{h_y^2} - \frac{2i}{h_x} k_{(0,0)} u_{(0,0)} + \frac{2u_{(0,0)}}{h_x^2} - \frac{2u_{(1,0)}}{h_x^2}$$

Therefore, we have the equation

$$u_{xx(0,0)} + u_{yy(0,0)} - \frac{2i}{h_y} k_{(0,0)} u_{(0,0)} + \frac{2u_{(0,0)}}{h_y^2} - \frac{2u_{(0,1)}}{h_y^2} - \frac{2i}{h_x} k_{(0,0)} u_{(0,0)} + \frac{2u_{(0,0)}}{h_x^2} - \frac{2u_{(1,0)}}{h_x^2} = 2g_{(0^+,0)} h_y^{-1} + 2g_{(0,0^+)} h_x^{-1}$$

We substitute in this expression $u_{xx(0,0)} + u_{yy(0,0)} = -f_{(0,0)} - k^2_{(0,0)} u_{(0,0)}$

$$-f_{(0,0)} - k^2_{(0,0)} u_{(0,0)} - \frac{2i}{h_y} k_{(0,0)} u_{(0,0)} + \frac{2u_{(0,0)}}{h_y^2} - \frac{2u_{(0,1)}}{h_y^2} - \frac{2i}{h_x} k_{(0,0)} u_{(0,0)} + \frac{2u_{(0,0)}}{h_x^2} - \frac{2u_{(1,0)}}{h_x^2}$$

The resulting boundary condition equals

$$\frac{2u_{(0,0)}}{h_y^2} + \frac{2u_{(0,0)}}{h_x^2} - k^2_{(0,0)} u_{(0,0)} - \frac{2u_{(0,1)}}{h_y^2} - \frac{2u_{(1,0)}}{h_x^2} - \frac{2i}{h_y} k_{(0,0)} u_{(0,0)} - \frac{2i}{h_x} k_{(0,0)} u_{(0,0)} = f_{(0,0)} + 2g(0^+, 0) h_y^{-1} + 2g(0, 0^+) h_x^{-1}$$

We continue with the southeast corner point $x_{(n_x+1,0)} = (1, 0)$. At this point the boundary conditions on the vertical and horizontal directions are $-u_{y(n_x+1,0)} - ik_{(n_x+1,0)} u_{(n_x+1,0)} = g(1^-, 0)$ and $u_{x(n_x+1,0)} - ik_{(n_x+1,0)} u_{(n_x+1,0)} = g(1, 0^+)$.

The resulting boundary condition is

$$\frac{2u_{(n_x+1,0)}}{h_x^2} + \frac{2u_{(n_x+1,0)}}{h_y^2} - k^2_{(n_x+1,0)} u_{(n_x+1,0)} - \frac{2u_{(n_x,0)}}{h_x^2} - \frac{2u_{(n_x+1,1)}}{h_y^2} - \frac{2i}{h_y} k_{(n_x+1,0)} u_{(n_x+1,0)} - \frac{2i}{h_x} k_{(n_x+1,0)} u_{(n_x+1,0)} = f_{(n_x+1,0)} + 2g(1^-, 0) h_y^{-1} + 2g(1, 0^+) h_x^{-1}$$

At the northwest corner point $x_{(0,n_y+1)} = (0, 1)$ the equation is

$$\frac{2u_{(0,n_y+1)}}{h_x^2} + \frac{2u_{(0,n_y+1)}}{h_y^2} - k^2_{(0,n_y+1)} u_{(0,n_y+1)} - \frac{2u_{(1,n_y+1)}}{h_x^2} - \frac{2u_{(0,n_y)}}{h_y^2} - \frac{2i}{h_y} k_{(0,n_y+1)} u_{(0,n_y+1)} - \frac{2i}{h_x} k_{(0,n_y+1)} u_{(0,n_y+1)} = f_{(0,n_y+1)} + 2g(0^+, 1) h_y^{-1} + 2g(0, 1^-) h_x^{-1}$$

At the northeast corner point $x_{(n_x+1,n_y+1)} = (1, 1)$ the equation is

$$\frac{2u_{(n_x+1,n_y+1)}}{h_x^2} + \frac{2u_{(n_x+1,n_y+1)}}{h_y^2} - k^2_{(n_x+1,n_y+1)} u_{(n_x+1,n_y+1)} - \frac{2u_{(n_x,n_y+1)}}{h_x^2} - \frac{2u_{(n_x+1,n_y)}}{h_y^2} - \frac{2i}{h_y} u_{(n_x+1,n_y+1)} - \frac{2i}{h_x} u_{(n_x+1,n_y+1)} = f_{(n_x+1,n_y+1)} + 2g(1^-, 1) h_y^{-1} + 2g(1, 1^-) h_x^{-1}$$

END OF REVISED VERSION The complete set of equations is the following:

For interior points of the form $x_{(i,j)} = (ih_x, jh_y)$ where $1 < i < n_x$ and $1 < j < n_y$:

$$-h_x^2 \left(u_{(i,j+1)} + u_{(i,j-1)}\right) - h_y^2 \left(u_{(i+1,j)} + u_{(i-1,j)}\right) + u_{(i,j)} \left(-h_x^2 h_y^2 k_{ij}^2 + 2h_x^2 + 2h_y^2\right) = f_{ij} h_x^2 h_y^2$$

For non-corner points on the west boundary, of the form $x_{(0,j)} = (0, jh_y)$ where $1 < j < n_y$:

$$-2h_y^2 u_{(1,j)} - h_x^2 u_{(0,j+1)} - h_x^2 u_{(0,j-1)} + u_{(0,j)} \left(-h_x^2 h_y^2 k^2_{(0,j)} - 2ih_x h_y^2 k_{(0,j)} + 2h_x^2 + 2h_y^2\right) = f_{(0,j)} h_x^2 h_y^2$$

For non-corner points on the south boundary, of the form $x_{(i,0)} = (ih_x, 0)$ where $1 < i < n_x$:

$$-2h_x^2 u_{(i,1)} - h_y^2 u_{(0,j+1)} - h_y^2 u_{(0,j-1)} + u_{(i,0)} \left(-h_x^2 h_y^2 k_{(i,0)}^2 - 2ih_x^2 h_y k_{(i,0)} + 2h_x^2 + 2h_y^2 \right) = f_{(i,0)} h_x^2 h_y^2$$

For non-corner points on the east boundary, of the form $x_{(n_x,j)} = (1, jh_y)$ where $1 < j < n_y$:

$$-2h_y^2 u_{(nx-1,j)} - h_x^2 u_{(nx,j+1)} - h_x^2 u_{(nx,j-1)} + u_{(nx,j)} \left(-h_x^2 h_y^2 k_{(nx,j)}^2 - 2ih_x h_y^2 k_{(nx,j)} + +2h_x^2 + 2h_y^2 \right) = f_{(nx,j)} h_x^2 h_y^2$$

For non-corner points on the northern boundary, of the form $x_{(i,n_y)} = (ih_x, 1)$ where $1 < i < n_x$

$$-2h_x^2 u_{(i,ny-1)} - h_y^2 u_{(i+1,ny)} - h_y^2 u_{(i-1,ny)} + u_{(i,ny)} \left(-h_x^2 h_y^2 k_{(i,ny)}^2 - 2ih_x^2 h_y k_{(i,ny)} + 2h_x^2 + 2h_y^2 \right) = f_{(i,ny)} h_x^2 h_y^2$$

For the corner point $x_{(0,0)} = (0, 0)$

$$-h_x^2 u_{(0,1)} - h_y^2 u_{(1,0)} + u_{(0,0)} \left(-\frac{h_x^2 k_{(0,0)}}{2} h_y^2 - ih_x^2 h_y k_{(0,0)} + h_x^2 - ih_x h_y^2 k_{(0,0)} + h_y^2 \right) = \frac{f_{(0,0)} h_x^2 h_y^2}{2}$$

For the corner point $x_{(n,0)} = (1, 0)$

$$-h_x^2 u_{(n,1)} - h_y^2 u_{(n-1,0)} + u_{(n,0)} \left(-\frac{h_x^2 h_y^2}{2} k_{(n,0)}^2 - ih_x^2 h_y k_{(n,0)} + h_x^2 - ih_x h_y^2 k_{(n,0)} + h_y^2 \right) = \frac{f_{(n,0)} h_x^2 h_y^2}{2}$$

For the corner point $x_{(0,n)} = (0, 1)$

$$-h_x^2 u_{(0,n-1)} - h_y^2 u_{(1,n)} + u_{(0,n)} \left(\frac{h_x^2 h_y^2}{2} k_{(0,n)}^2 - ih_x^2 h_y k_{(0,n)} + h_x^2 - ih_x h_y^2 k_{(0,n)} + h_y^2 \right) = \frac{f_{(0,n)} h_x^2 h_y^2}{2}$$

For the corner point $x_{(n,n)} = (1, 1)$

$$-h_x^2 u_{(n,n-1)} - h_y^2 u_{(n-1,n)} + u_{(n,n)} \left(-\frac{h_x^2 h_y^2}{2} k_{(n,n)}^2 - ih_x^2 h_y k_{(n,n)} + h_x^2 - ih_x h_y^2 k_{(n,n)} + h_y^2 \right) = \frac{f_{(n,n)} h_x^2 h_y^2}{2}$$