Chapter Three

Periodic Behavior and the Cycle-Time Vector

This chapter deals with sequences $\{x(k) : k \in \mathbb{N}\}$ generated by

$$x(k+1) = A \otimes x(k),$$

for $k \geq 0$, where $A \in \mathbb{R}_{\max}^{n \times n}$ and $x(0) = x_0 \in \mathbb{R}_{\max}^{n}$ is the initial condition. The sequences are then equivalently described by

$$x(k) = A^{\otimes k} \otimes x_0, \tag{3.1}$$

for all $k \geq 0$.

DEFINITION 3.1 *Let $\{x(k) : k \in \mathbb{N}\}$ be a sequence in $\mathbb{R}_{\max}^{n}$, and assume that for all $j \in \underline{n}$ the quantity η_j, defined by*

$$\lim_{k \to \infty} \frac{x_j(k)}{k},$$

exists. The vector $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^\top$ is called the cycle-time vector *of the sequence $x(k)$. If all η_j's have the same value, this value is also called the* asymptotic growth rate *of the sequence $x(k)$.*

Throughout this chapter the sequences $\{x(k) : k \in \mathbb{N}\}$ will be generated by a recurrence relation as given above, with initial condition x_0. It will be shown that once a cycle-time vector exists, its value is independent of x_0 and is basically determined by matrix A involved; see, for instance, (3.1). For this reason, the vector η will occasionally also be referred to as the *cycle-time vector* of the associated matrix. A similar remark holds with respect to the asymptotic growth rate.

Note that a vector of n identical asymptotic growth rates can be seen as the cycle-time vector in case all the limits $\lim_{k \to \infty} x_j(k)/k$, $j \in \underline{n}$, have the same value. Hence, the notions of cycle-time vector and asymptotic growth rate are closely related. For this reason, the cycle-time vector will occasionally also be referred to as the *asymptotic growth rate*, and vice versa. Hence, the notions cycle-time vector and asymptotic growth rate will be used interchangeably. However, from the context it is always clear which of the two notions, defined in Definition 3.1, is actually meant. Sometimes, even both notions apply.

This chapter deals with the *quantitative* asymptotic behavior of $x(k)$. By quantitative behavior, the cycle-time vector as well as the asymptotic growth rate of $x(k)$ is meant; see Section 1.3 for an application in the heap model. If A is irreducible, with unique eigenvalue λ and associated (finite) eigenvector v, then for $x(0) = v$ it follows that

$$\begin{aligned} x(k) &= A^{\otimes k} \otimes x(0) \\ &= \lambda^{\otimes k} \otimes v \end{aligned}$$

for all $k \geq 0$, which gives for any $j \in \underline{n}$ that

$$\lim_{k\to\infty} \frac{x_j(k)}{k} = \lambda \tag{3.2}$$

and the asymptotic growth rate of $x(k)$ coincides with the eigenvalue of A.

The key question about the limiting behavior is what happens if $x(k)$ is initialized with an *arbitrary finite* vector x_0, not necessarily an eigenvector, or what happens if A is reducible (or not irreducible). This chapter is devoted to answering these questions. Specifically, it will be shown that for regular A the cycle-time vector of $x(k)$ exists and is independent of the initial vector x_0. Further, if A is irreducible, then the asymptotic growth rate of any $x_j(k), j \in \underline{n}$, is equal to the eigenvalue of A.

The chapter is organized as follows. In Section 3.1, a key result is presented characterizing powers of an irreducible matrix by means of its eigenvalue and cyclicity. The cycle-time vector of $x(k)$ for regular matrices is studied in Sections 3.2 and 3.3. In Section 3.4, a special class of reducible matrices is studied that enjoy some nice algebraic properties.

3.1 CYCLICITY AND TRANSIENT TIME

According to Theorem 2.9, any irreducible matrix $A \in \mathbb{R}_{\max}^{n\times n}$ possesses a unique eigenvalue. This section establishes an important statement on the asymptotic behavior of the powers of A in terms of the eigenvalue. Before proving the main result, a number of preliminary technical results will be presented. We start with a fundamental theorem; see, for example, [18].

THEOREM 3.2 *Let $\beta_1, \ldots, \beta_q$ be natural numbers such that their greatest common divisor is one; in symbols,* $\gcd\{\beta_1, \ldots, \beta_q\} = 1$. *Then, there exists a natural number N such that for all $k \geq N$ there are integers $n_1, \ldots, n_q \geq 0$ such that $k = (n_1 \times \beta_1) + \cdots + (n_q \times \beta_q)$.*

The next result is an important one from graph theory. Below, the communication graph $\mathcal{G}(A)$ of an irreducible matrix $A \in \mathbb{R}_{\max}^{n\times n}$ will be considered. The cyclicity of a graph has been introduced in Chapter 2, and for the graph $\mathcal{G}(A)$ it will be denoted by $\sigma_{\mathcal{G}(A)}$. Recall that $\mathcal{G}(A) = (\mathcal{N}(A), \mathcal{D}(A))$, where $\mathcal{N}(A)$ is the set of nodes and $\mathcal{D}(A)$ is the set of directed edges of $\mathcal{G}(A)$. To simplify the notation, in the following we write $\mathcal{N}$, $\mathcal{D}$, and $\sigma_{\mathcal{G}}$ for the set of nodes, the set of directed arcs, and the cyclicity of $\mathcal{G}(A)$, respectively.

LEMMA 3.3 *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be an irreducible matrix, and let the cyclicity of its communication graph be $\sigma_{\mathcal{G}}$. Then, after a suitable relabeling of the nodes of $\mathcal{G}(A)$, the matrix $A^{\otimes\sigma_{\mathcal{G}}}$ corresponds to a block diagonal matrix with $\sigma_{\mathcal{G}}$ blocks on the diagonal. The communication graph of each diagonal block is strongly connected and has cyclicity one. Moreover, the eigenvalues of all diagonal blocks have the same value.*

Proof. Write $\mathcal{G}(A) = (\mathcal{N}, \mathcal{D})$, and consider the relation between nodes $i, j \in \mathcal{N}$ characterized by

$$i\mathcal{K}j \iff \text{the length of every path from } i \text{ to } j \text{ is a multiple of } \sigma_{\mathcal{G}}. \tag{3.3}$$

It can easily be shown that this relation is an equivalence relation on $\mathcal{N}$. Further, let $k_0 \in \mathcal{N}$ be an arbitrarily chosen, but fixed node; then equivalence classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{\sigma_{\mathcal{G}}-1}$ associated with the equivalence relation (3.3) can be introduced as

$$i \in \mathcal{C}_l \iff \text{every path from } k_0 \text{ to } i \text{ has length (mod } \sigma_{\mathcal{G}}) \text{ equal to } l, \tag{3.4}$$

for $l = 0, 1, \ldots, \sigma_{\mathcal{G}} - 1$. It is not difficult to show for any $i, j \in \mathcal{N}$ that $i\mathcal{K}j \iff i, j \in \mathcal{C}_l$ for some $l = 0, 1, \ldots, \sigma_{\mathcal{G}} - 1$.

Assume that there is a path from i to j of length $\sigma_{\mathcal{G}}$. Then it follows that every path from i to j has a length that is a multiple of $\sigma_{\mathcal{G}}$. Indeed, concatenation of the previously mentioned paths with one and the same path from j to i yields circuits whose lengths must be multiples of $\sigma_{\mathcal{G}}$. Hence, every path of length $\sigma_{\mathcal{G}}$ must end in the same class as the class from which it starts. Because $A^{\otimes \sigma_{\mathcal{G}}}$ can be computed by considering all paths of length $\sigma_{\mathcal{G}}$, it follows that $A^{\otimes \sigma_{\mathcal{G}}}$ is block diagonal, possibly after an appropriate relabeling of the nodes according to the classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{\sigma_{\mathcal{G}}-1}$; for instance, by first labeling all nodes in $\mathcal{C}_0$, then all nodes in $\mathcal{C}_1$, and so on.

Further, since for all $i, j \in \mathcal{C}_l$ there is a path from i to j whose length is a multiple of $\sigma_{\mathcal{G}}$, it follows that the block in $A^{\otimes \sigma_{\mathcal{G}}}$ corresponding to class $\mathcal{C}_l$ is irreducible. Indeed, the previous path from i to j can be seen as a concatenation of a number of subpaths, all of length $\sigma_{\mathcal{G}}$ and each going from one node in $\mathcal{C}_l$ to another node in $\mathcal{C}_l$. Now considering all such subpaths of maximal weight, it follows that the communication graph of the block in $A^{\otimes \sigma_{\mathcal{G}}}$ corresponding to class $\mathcal{C}_l$ is strongly connected and that the block itself is irreducible.

Finally, every circuit in $\mathcal{G}(A)$ must go through the equivalence classes $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{\sigma_{\mathcal{G}}-1}$. Indeed, suppose there is a circuit going through just τ of the classes, where $\tau < \sigma_{\mathcal{G}}$. Then there must be a class $\mathcal{C}_l$ and nodes $i, j \in \mathcal{C}_l$ such that there is a path from i to j of length less than or equal to τ. However, this is in contradiction with the fact that any path between nodes of the same class must be a multiple of $\sigma_{\mathcal{G}}$. Hence, it follows that the number of circuits in $\mathcal{G}(A)$ is the same as the number of circuits going though any class $\mathcal{C}_l$. Observe that circuits in $\mathcal{G}(A)$ of length $\kappa \times \sigma_{\mathcal{G}}$ can be associated with circuits in $\mathcal{G}(A^{\otimes \sigma_{\mathcal{G}}})$ of length κ. Since the greatest common divisor of all circuit lengths in $\mathcal{G}(A)$ is $\sigma_{\mathcal{G}}$, it follows that the communication graph of the block in $A^{\otimes \sigma_{\mathcal{G}}}$ corresponding to class $\mathcal{C}_l$ has cyclicity one.

The fact that the eigenvalues of the diagonal blocks are identical follows immediately from the irreducibility of A. □

Example 3.1.1 *Consider the following irreducible matrix and its square,*

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix}, \quad A^{\otimes 2} = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 0 & \varepsilon & 2 & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon & 2 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon \end{pmatrix}.$$

It follows that $\sigma_{\mathcal{G}(A)} = 2$. See Figure 3.1. If we take $k_0 = 1$, then it follows from (3.4) that $\mathcal{C}_0 = \{1, 3, 5\}$ and $\mathcal{C}_1 = \{2, 4\}$. A permutation of the rows and columns of $A^{\otimes 2}$ according to these two classes leads to a block diagonal matrix, being a normal form of $A^{\otimes 2}$,

$$\left(\begin{array}{ccc|cc} 0 & \varepsilon & 1 & \varepsilon & \varepsilon \\ 1 & \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & 0 & 2 \\ \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon \end{array}\right).$$

Also, it is easily seen by the finite diagonal elements in both blocks that the graphs of both blocks have cyclicity one.

COROLLARY 3.4 *Under the conditions of Lemma 3.3, let τ be a multiple of $\sigma_{\mathcal{G}(A)}$. Then, after a relabeling of the nodes of $\mathcal{G}(A)$, the matrix $A^{\otimes \tau}$ corresponds to a block diagonal matrix with $\sigma_{\mathcal{G}(A)}$ blocks on the diagonal. The communication graph of each diagonal block is strongly connected and has cyclicity one.*

Proof. The proof may be obtained along the lines of thought of the proof of Lemma 3.3 as far as the block diagonal structure is concerned. For the strong connectedness and cyclicity of the graph of each block, Theorem 3.2 can play a useful role, similar to its application in the proof of Lemma 3.6. □

DEFINITION 3.5 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be such that its communication graph contains at least one circuit. The cyclicity of A, denoted by $\sigma(A)$, is the cyclicity of the critical graph of A.*

It may be surprising that the cyclicity of matrix A is defined via its critical graph and not via its communication graph. The reason for this will become clear later when we show that the cyclicity of the critical graph determines the periodic behavior of powers of A.

Suppose that the critical graph of an irreducible matrix A has q maximal strongly connected subgraphs (m.s.c.s.), and let m.s.c.s. i have cyclicity σ_i. Then each σ_i, with $i \in \underline{q}$, is a multiple of $\sigma_{\mathcal{G}(A)}$, and consequently, $\sigma(A)$ is a multiple of $\sigma_{\mathcal{G}(A)}$.

Example 3.1.2 *(Continuation of Example 3.1.1.) It follows that $\sigma(A) = 4$. Also it is clear that the critical graph $\mathcal{G}^c(A)$ does not cover all nodes of the strongly connected graph $\mathcal{G}(A)$. See also Figure 3.1.*

With Definition 3.5 we now have two types of cyclicity in relation to matrix A, namely, the cyclicity of the communication graph of A, denoted as $\sigma_{\mathcal{G}(A)}$ and

Figure 3.1: Communication graph of Example 3.1.1 (left) and the corresponding critical graph (right).

occasionally referred to as the graph cyclicity of A, and the cyclicity of matrix A itself, denoted by $\sigma(A)$ and occasionally referred to as the matrix cyclicity of A. From Definition 3.5 it follows that $\sigma(A) = \sigma_{\mathcal{G}^c(A)}$; i.e., the matrix cyclicity of A is equal to the cyclicity of its critical graph.

With the above terminology, Lemma 3.3 now states that if A is an irreducible matrix with graph cyclicity $\sigma_{\mathcal{G}(A)}$, then, after a suitable relabeling of the nodes of $\mathcal{G}(A)$, $A^{\otimes \sigma_{\mathcal{G}(A)}}$ corresponds to a block diagonal matrix with square diagonal blocks that are irreducible and have graph cyclicity one. Clearly, the latter result is completely in terms of graph cyclicity. However, it will turn out that a similar result can be stated completely in terms of matrix cyclicity. To derive this result, we introduce first some preliminary notation and results.

Let A be an irreducible matrix, and let $\mathcal{G}^c(A)$ be its critical graph. Now let A^c be the submatrix of A such that the communication graph of A^c is equal to the critical graph of A, i.e., $\mathcal{G}(A^c) = \mathcal{G}^c(A)$. Matrix A^c can be obtained from matrix A by restricting A to those entries that correspond to arcs in $\mathcal{G}^c(A)$. Occasionally, matrix A^c is referred to as a critical matrix. Since all circuits in a critical graph are critical (see Lemma 2.6), it follows that the critical graph of A^c is the communication graph of A^c itself, i.e., $\mathcal{G}^c(A^c) = \mathcal{G}(A^c)$, implying $\sigma_{\mathcal{G}^c(A^c)} = \sigma_{\mathcal{G}(A^c)}$. From this it follows that the matrix cyclicity of A^c (i.e., $\sigma(A^c)$) is equal to the graph cyclicity of A^c (i.e., $\sigma_{\mathcal{G}(A^c)}$). Hence, for the critical matrix A^c both types of cyclicity coincide and are equal to $\sigma(A)$. From the above it follows further that $\mathcal{G}(A^c) = \mathcal{G}^c(A) = \mathcal{G}^c(A^c)$. However, we can prove more.

LEMMA 3.6 *Let A be an irreducible matrix, and let A^c be its corresponding critical matrix. Then, for all $k \geq 1$,*

$$\mathcal{G}((A^c)^{\otimes k}) = \mathcal{G}^c(A^{\otimes k}) = \mathcal{G}^c((A^c)^{\otimes k}).$$

Proof. Note that A^c is a submatrix of A and $(A^c)^{\otimes k}$ is a submatrix of $A^{\otimes k}$. Further, note that $\mathcal{G}^c(\cdot)$ is a subgraph of $\mathcal{G}(\cdot)$, and denote this as $\mathcal{G}^c(\cdot) \subseteq \mathcal{G}(\cdot)$. Then it follows that $\mathcal{G}^c((A^c)^{\otimes k}) \subseteq \mathcal{G}^c((A)^{\otimes k})$ and $\mathcal{G}^c((A^c)^{\otimes k}) \subseteq \mathcal{G}((A^c)^{\otimes k})$.

To prove the converse inclusions, note that any arc in $\mathcal{G}(A^{\otimes k})$ from node j to node i can be interpreted as a path in $\mathcal{G}(A)$ of length k from node j to node i. Then, if a number of arcs in $\mathcal{G}(A^{\otimes k})$ form a circuit, say, of length l, then the associated paths in $\mathcal{G}(A)$ form a circuit of length $k \times l$. Conversely, consider a circuit in $\mathcal{G}(A)$, choose an arbitrary node on the circuit, and traverse the circuit with steps of length k until the chosen node is reached again. If l such steps are needed, then there exists a circuit in $\mathcal{G}(A^{\otimes k})$ of length l. In the same way, critical circuits in $\mathcal{G}(A^{\otimes k})$ of length l correspond to critical circuits in $\mathcal{G}(A)$ of length $k \times l$ and conversely.

If ψ is a critical circuit of length l in $\mathcal{G}(A^{\otimes k})$, then there is a corresponding critical circuit ψ' of length $k \times l$ in $\mathcal{G}(A)$. Because ψ' is critical, it is also a circuit in $\mathcal{G}^{c}(A)$, in turn implying that ψ is a critical circuit in $\mathcal{G}((A^{c})^{\otimes k})$. Hence, it follows that $\mathcal{G}^{c}((A^{c})^{\otimes k}) \supseteq \mathcal{G}^{c}((A)^{\otimes k})$. The remaining inclusion can be proved in a similar way. □

We now are going to see the consequences of the above for Lemma 3.3 applied to the critical A^{c}. Therefore, we first assume that the matrix is irreducible and has cyclicity (both graph and matrix) σ. Then, according to Lemma 3.3, $(A^{c})^{\otimes\sigma}$, again after a suitable relabeling of nodes, corresponds to a block diagonal matrix with square diagonal blocks that are irreducible and have graph cyclicity one. However, since $\mathcal{G}^{c}((A^{c})^{\otimes\sigma}) = \mathcal{G}((A^{c})^{\otimes\sigma})$ (see Lemma 3.6 with $k = \sigma$), the graph of each of the diagonal blocks of $(A^{c})^{\otimes\sigma}$ coincides with its critical graph. Thus, for each diagonal block both cyclicities coincide, and therefore both are one.

If A^{c} is not irreducible, the above can be done for each of m.s.c.s.'s of $\mathcal{G}^{c}(A)$ with their individual cyclicities (graph and matrix). The least common multiple of these cyclicities equals the matrix cyclicity of A. If this cyclicity is denoted by σ, it follows with Corollary 3.4 (note that σ is a multiple of $\sigma_{\mathcal{G}(A)}$) that each diagonal block of $(A^{c})^{\otimes\sigma}$ corresponds to a block diagonal matrix with square diagonal blocks that are irreducible and have cyclicity (both graph and matrix) one. To make the overall block diagonal matrix of the same size as A it possibly has to be augmented with one square block with entries equal to ε. The latter block arises when $\mathcal{G}^{c}(A)$ does not cover all nodes.

In all cases it follows that each finite diagonal block of the block diagonal matrix corresponding to $(A^{c})^{\otimes\sigma}$ is irreducible and has cyclicity (both graph and matrix) one. Taking the least common multiple of all cyclicities, it follows that the cyclicity of the whole matrix $(A^{c})^{\otimes\sigma}$ is one, i.e., $\sigma_{\mathcal{G}^{c}((A^{c})^{\otimes\sigma})} = 1$.

With these preliminary observations, the next lemma can easily be proved.

LEMMA 3.7 *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be an irreducible matrix with cyclicity $\sigma = \sigma(A)$. Then, the cyclicity of matrix $A^{\otimes\sigma}$ is equal to one.*

Proof. According to Lemma 3.6 with $k = \sigma$, it follows that the graphs $\mathcal{G}^{c}(A^{\otimes\sigma})$ and $\mathcal{G}^{c}((A^{c})^{\otimes\sigma})$ are the same. By the above observations, the latter graph has cyclicity one. Hence, the cyclicity of $A^{\otimes\sigma}$, being the cyclicity of the former graph, is also one. □

Finally, as a last prerequisite for the main result of this section, we now treat a special case.

LEMMA 3.8 *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be an irreducible matrix with eigenvalue e and cyclicity one. Then there is an N such that $A^{\otimes(k+1)} = A^{\otimes k}$ for all $k \geq N$.*

Proof. Let $\mathcal{G}^{c}(A)$ be the critical graph with node set $\mathcal{N}^{c}$. It will subsequently be shown that there exists an N such that, for all $k \geq N$:

1. $[A^{\otimes k}]_{ii} = [A^{+}]_{ii} = e$ for all $i \in \mathcal{N}^{c}$,

2. $[A^{\otimes k}]_{ij} = [A^+]_{ij}$ for all $i \in \mathcal{N}^c$ and $j \in \underline{n}$,
3. $[A^{\otimes k}]_{ij} = \bigoplus_{l \in \mathcal{N}^c} [A^+]_{il} \otimes [A^+]_{lj}$ for all $i, j \in \underline{n}$.

From these three statements, it follows that $A^{\otimes(k+1)} = A^{\otimes k}$ for all $k \geq N$. Clearly, only statement 3 is required to complete the proof of the lemma. However, statement 2 will be used in proving statement 3, while statement 1 plays a role in the proof of statement 2. The reason for considering the three statements is that they provide a nice structure for the proof of Lemma 3.8. Therefore, to conclude the proof of the lemma, the above three statements will now subsequently be proved.

Statement 1. Consider $i \in \mathcal{N}^c$. Then there is a strongly connected component of $\mathcal{G}^c(A)$, say $\mathcal{G}_1$ with node set $\mathcal{N}_1$, such that $i \in \mathcal{N}_1$. Note that $\mathcal{G}_1$ is a critical (sub)graph, implying that all its circuits are critical. Since the cyclicity of matrix A is one, it further follows that the cyclicity of graph $\mathcal{G}_1$ is equal to one too. Hence, there exist circuits in $\mathcal{G}_1$, say, $\zeta_1, \ldots, \zeta_q$, whose lengths have a greatest common divisor equal to one; in symbols, $\gcd\{|\zeta_1|_1, \ldots, |\zeta_q|_1\} = 1$. Since $\mathcal{G}_1$ is strongly connected, there exists a circuit α in $\mathcal{G}_1$ such that i is a node in α and $\alpha \cap \zeta_j \neq \emptyset$ for $j \in \underline{q}$; that is, α passes through i and through all circuits $\zeta_1, \ldots, \zeta_q$. Then, for any $n_1, \ldots, n_q \in \mathbb{N}$, there is a circuit passing through i, built from circuit α, n_1 copies of circuit ζ_1, n_2 copies of circuit ζ_2 and so on, up to n_q copies of circuit ζ_q. By Theorem 3.2, it follows that there is an N such that for each $k \geq N$, there exist integers $n_1, \ldots, n_q \in \mathbb{N}$, such that

$$k = |\alpha|_1 + n_1 \times |\zeta_1|_1 + \cdots + n_q \times |\zeta_q|_1.$$

For these $n_1, \ldots, n_q$, construct a circuit passing through i, built from circuit α, n_1 copies of circuit ζ_1, n_2 copies of circuit ζ_2 and so on, up to n_q copies of circuit ζ_q. It is clear that the circuit is one in $\mathcal{G}_1$. Therefore, it is itself also a critical circuit with weight e. Since the maximal circuit mean in $\mathcal{G}(A)$ is e, it follows that $[A^{\otimes k}]_{ii} = e$ for all $k \geq N$, by the definition of $[A^+]_{ii}$, also implying that $[A^+]_{ii} = e$.

Statement 2. By the definition of $[A^+]_{ij}$ there exists an l such that $[A^{\otimes l}]_{ij} = [A^+]_{ij}$. In fact, since the eigenvalue of A is e, it follows by Lemma 2.2 even that $l \leq n$. Then it follows that

$$[A^{\otimes(k+l)}]_{ij} \geq [A^{\otimes k}]_{ii} \otimes [A^{\otimes l}]_{ij} = [A^{\otimes l}]_{ij} = [A^+]_{ij} \,,$$

for k large enough, $i \in \mathcal{N}^c$, and $j \in \underline{n}$; see statement 1. Clearly,

$$[A^+]_{ij} = \bigoplus_{m \geq 1} [A^{\otimes m}]_{ij} \geq [A^{\otimes(k+l)}]_{ij} \geq [A^+]_{ij}.$$

If we replace $k + l$ by k, it therefore follows that $[A^{\otimes k}]_{ij} = [A^+]_{ij}$ for all $i \in \mathcal{N}^c, j \in \underline{n}$, with k large enough. Dually, it follows, of course, that $[A^{\otimes m}]_{ij} = [A^+]_{ij}$ for all $i \in \underline{n}, j \in \mathcal{N}^c$, and m large enough.

Statement 3. Take k and m large enough such that $[A^{\otimes k}]_{il} = [A^+]_{il}$ and $[A^{\otimes m}]_{lj} = [A^+]_{lj}$ for all $l \in \mathcal{N}^c$; see statement 2. Then

$$[A^{\otimes(k+m)}]_{ij} \geq [A^{\otimes k}]_{il} \otimes [A^{\otimes m}]_{lj} = [A^+]_{il} \otimes [A^+]_{lj},$$

for all $l \in \mathcal{N}^c$. If we substitute k for $k + m$, then it follows for k large enough that

$$[A^{\otimes k}]_{ij} \geq \bigoplus_{l \in \mathcal{N}^c} [A^+]_{il} \otimes [A^+]_{lj}.$$

Now consider a path from j to i not passing through $\mathcal{N}^c$. Such a path consists of an elementary path and a number of circuits all having negative weight (i.e., a weight less than e). Let the average weight of a noncritical circuit be maximally δ. Then the weight of a path from j to i of length $k+1$ not passing through a node in $\mathcal{N}^c$ can be bounded from above by $[A^+]_{ij} + k \times \delta = [A^+]_{ij} \otimes \delta^{\otimes k}$, where $[A^+]_{ij}$ is a fixed upper bound for the weight of the elementary path and $k \times \delta$ is an upper bound for the total weight of the circuits. Since $\delta < e$ (i.e., $\delta < 0$ in conventional notation), it follows that for k large enough

$$[A^+]_{ij} \otimes \delta^{\otimes k} \leq \bigoplus_{l \in \mathcal{N}^c} [A^+]_{il} \otimes [A^+]_{lj} .$$

Indeed, the right-hand side of the inequality is fixed, while the left-hand side tends to $-\infty$ for k going to $+\infty$. Hence, for k large enough it follows that

$$[A^{\otimes k}]_{ij} = \bigoplus_{l \in \underline{n}} [A^+]_{il} \otimes [A^+]_{lj} = \bigoplus_{l \in \mathcal{N}^c} [A^+]_{il} \otimes [A^+]_{lj} ,$$

for all $i, j \in \underline{n}$. □

We now state the celebrated cyclicity theorem of max-plus algebra.

THEOREM 3.9 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be an irreducible matrix with eigenvalue λ and cyclicity $\sigma = \sigma(A)$. Then there is an N such that*

$$A^{\otimes(k+\sigma)} = \lambda^{\otimes \sigma} \otimes A^{\otimes k}$$

for all $k \geq N$.

Proof. Consider matrix $B = (A_\lambda)^{\otimes \sigma}$. Recall that σ is the cyclicity of the critical graph of A, which is a multiple of the cyclicity of the communication graph of A itself. Then, by Corollary 3.4, after a suitable relabeling of the nodes of $\mathcal{G}(A)$, matrix B is a block diagonal matrix with square diagonal blocks of which the communication graphs are strongly connected and have cyclicity one. By Lemma 3.7 it follows that the matrix cyclicity of B is one, implying that the matrix cyclicity of each of its diagonal blocks is one. Hence, by applying Lemma 3.8 to each diagonal block, it ultimately follows that an M exists such that $B^{\otimes(l+1)} = B^{\otimes l}$, for all $l \geq M$. The latter implies that

$$\left((A_\lambda)^{\otimes \sigma}\right)^{\otimes(l+1)} = \left((A_\lambda)^{\otimes \sigma}\right)^{\otimes l} ,$$

which can be further written as $(A_\lambda)^{\otimes(l \times \sigma + \sigma)} = (A_\lambda)^{\otimes(l \times \sigma)}$ or

$$A^{\otimes(l \times \sigma + \sigma)} = \lambda^{\otimes \sigma} \otimes A^{\otimes(l \times \sigma)},$$

for all $l \geq M$. Finally, note that $A^{\otimes(l \times \sigma + j + \sigma)} = \lambda^{\otimes \sigma} \otimes A^{\otimes(l \times \sigma + j)}$, for any $j, 0 \leq j \leq \sigma - 1$, implying that for all $k \geq N \stackrel{\text{def}}{=} M \times \sigma$ it follows that

$$A^{\otimes(k+\sigma)} = \lambda^{\otimes \sigma} \otimes A^{\otimes k}.$$

□

Consider matrix A as in Theorem 3.9 with eigenvalue λ and cyclicity $\sigma = \sigma(A)$. Note that then also $\sigma = \sigma(A_\lambda)$. The theorem implies that there is a periodic behavior in the sequence of the powers of A_λ with a length σ equal to the cyclicity of A_λ, i.e., $A_\lambda^{\otimes(k+\sigma)} = A_\lambda^{\otimes k}$ for k large enough. With the ideas put forward in Section 3.7 of [5], it can be shown that the cyclicity of A_λ is also the smallest possible length of such a periodic behavior. Hence, the cyclicity of the matrix A can be seen as the minimal length of a periodic behavior in the sequence of the powers of A_λ.

Because of the existence of the integer N, mentioned in Theorem 3.9, it follows that there exists a smallest number $t(A)$ such that

$$A^{\otimes(k+\sigma)} = \lambda^{\otimes\sigma} \otimes A^{\otimes k}$$

for all $k \geq t(A)$. The number $t(A)$ will be called the *transient time* of A.

Theorem 3.9 gives a partial answer to the question about the limiting behavior of $x(k)$ defined in (3.1). Indeed, for any initial condition x_0, the sequence $x(k)$ will show a periodic behavior after at most $t(A)$ transitions as

$$\begin{aligned} x(k+\sigma) &= A^{\otimes(k+\sigma)} \otimes x_0 \\ &= \lambda^{\otimes\sigma} \otimes A^{\otimes k} \otimes x_0 \\ &= \lambda^{\otimes\sigma} \otimes x(k), \end{aligned}$$

for all $k \geq t(A)$. This periodic behavior is characterized through the eigenvalue and the cyclicity of A. If A has cyclicity one, then $x(k+1) = A \otimes x(k) = \lambda \otimes x(k)$ for $k \geq t(A)$ in the above equation. In words, for any initial vector x_0, $x(k)$ becomes an eigenvector of A for $k \geq t(A)$. Put differently, after $t(A)$ steps, $x(k)$ behaves like an eigenvector, and the effect of the initial value x_0 has died out. Hence, we use the name *transient time* for $t(A)$. In the stochastic literature, transient time is called *coupling time*.

Example 3.1.3 *Let*

$$A = \begin{pmatrix} -1 & 11 \\ 1 & \varepsilon \end{pmatrix}.$$

The powers of A are

$$A^{\otimes 2} = \begin{pmatrix} 12 & 10 \\ e & 12 \end{pmatrix}, \quad A^{\otimes 3} = \begin{pmatrix} 11 & 23 \\ 13 & 11 \end{pmatrix},$$

$$A^{\otimes 4} = \begin{pmatrix} 24 & 22 \\ 12 & 24 \end{pmatrix}, \quad A^{\otimes 5} = \begin{pmatrix} 23 & 35 \\ 25 & 23 \end{pmatrix}, \ldots,$$

and we conclude that

$$\begin{aligned} A^{\otimes(k+2)} &= 12 \otimes A^{\otimes k} \\ &= 6^{\otimes 2} \otimes A^{\otimes k}, \qquad k \geq 2\,. \end{aligned}$$

Hence, we obtain $\lambda(A) = 6$, $\sigma(A) = 2$, $\sigma_{\mathcal{G}(A)} = 1$, and $t(A) = 2$. Moreover,

$$A_\lambda = \begin{pmatrix} -7 & 5 \\ -5 & \varepsilon \end{pmatrix},$$

and

$$A_\lambda^{\otimes 2} = \begin{pmatrix} e & -2 \\ -12 & e \end{pmatrix}, \quad A_\lambda^{\otimes 3} = \begin{pmatrix} -7 & 5 \\ -5 & -7 \end{pmatrix},$$

$$A_\lambda^{\otimes 4} = \begin{pmatrix} e & -2 \\ -12 & e \end{pmatrix}, \quad A_\lambda^{\otimes 5} = \begin{pmatrix} -7 & 5 \\ -5 & -7 \end{pmatrix}, \ldots,$$

which gives

$$A_\lambda^* = A_\lambda^+ = \begin{pmatrix} e & 5 \\ -5 & e \end{pmatrix},$$

and, in accordance with Lemma 2.7, both columns of A_λ^* are eigenvectors of A. The fact that the columns are colinear is not a mere coincidence and will be elucidated in Chapter 4.

Even for matrices of small size, the transient time can be arbitrarily large. For example, matrix A with

$$A = \begin{pmatrix} -1 & -N \\ e & e \end{pmatrix},$$

where $N \in \mathbb{N}$ with $N \geq 2$, has transient time $t(A) = N$, while $\lambda(A) = e$ and $\sigma(A) = 1$.

3.2 THE CYCLE-TIME VECTOR: PRELIMINARY RESULTS

This and the following sections deal with the cycle-time vector of sequences $\{x(k) : k \in \mathbb{N}\}$ defined by $x(k+1) = A \otimes x(k)$ for all $k \geq 0$, with $A \in \mathbb{R}_{\max}^{n \times n}$ and $x(0) = x_0 \in \mathbb{R}_{\max}^n$, being the initial condition. In Sections 3.2.2 and 3.2.4 matrix A is assumed to be irreducible. In the other sections matrix A is just supposed to be square and regular.

3.2.1 Uniqueness of the cycle-time vector

As announced, first the dependency of the limit $\lim_{k\to\infty} x(k)/k$ on the initial condition will be investigated, assuming that the limit exists. The actual existence of this limit will be discussed later on. For this purpose, an appropriate norm will be introduced, the so-called l^∞-norm. The l^∞-norm of a vector $v \in \mathbb{R}^n$ is defined as the maximum of the absolute value of all entries of v and will be denoted by $||v||_\infty$. Hence, $||v||_\infty = \max_{i \in \underline{n}} |v_i|$ for every $v \in \mathbb{R}^n$, where $|\,.\,|$ denotes the absolute value. Note that the l^∞-norm of a vector in $\mathbb{R}_{\max}^n$ may be infinite. This happens if at least one of its components is equal to ε. However, when one considers only regular matrices and finite initial conditions, the asymptotic behavior can be expressed entirely in terms of vectors in $\mathbb{R}^n$ (i.e., in terms of finite vectors).

The following property plays a crucial role in proving that the asymptotic behavior is independent of the initial condition.

LEMMA 3.10 *Let $A \in \mathbb{R}_{\max}^{m \times n}$ be a regular (not necessarily square) matrix, then*

$$||(A \otimes u) - (A \otimes v)||_\infty \leq ||u - v||_\infty,$$

for any $u, v \in \mathbb{R}^n$.

Proof. Note that $A \otimes u, A \otimes v \in \mathbb{R}^m$, i.e., both vectors are finite. Set

$$\alpha \stackrel{\text{def}}{=} ||(A \otimes u) - (A \otimes v)||_\infty .$$

Then there is an $i_0 \in \underline{m}$ such that

$$\alpha = \left| \left[(A \otimes u) - (A \otimes v)\right]_{i_0} \right| .$$

Assume that $\alpha = \left[(A \otimes u) - (A \otimes v)\right]_{i_0} \geq 0$; then

$$\alpha = \max_{j \in \underline{n}} \left(a_{i_0 j} + u_j\right) - \max_{l \in \underline{n}} \left(a_{i_0 l} + v_l\right) .$$

Hence, there is a $j_0 \in \underline{n}$ such that

$$\alpha = \left(a_{i_0 j_0} + u_{j_0}\right) - \max_{l \in \underline{n}} \left(a_{i_0 l} + v_l\right) ,$$

which is less than or equal to (take $l = j_0$)

$$\left(a_{i_0 j_0} + u_{j_0}\right) - \left(a_{i_0 j_0} + v_{j_0}\right) = u_{j_0} - v_{j_0} ,$$

implying that

$$\alpha \leq u_{j_0} - v_{j_0} \leq \max_{j \in \underline{n}} \left(u_j - v_j\right) \leq \max_{j \in \underline{n}} |u_j - v_j| = ||u - v||_\infty .$$

Hence, if $\alpha = \left[(A \otimes u) - (A \otimes v)\right]_{i_0} \geq 0$, then $\alpha \leq ||u - v||_\infty$. The same can be shown if $\alpha = \left[(A \otimes u) - (A \otimes v)\right]_{i_0} \leq 0$. Thus, the proof is completed. □

The property $||(A \otimes u) - (A \otimes v)||_\infty \leq ||u - v||_\infty$ for any $u, v \in \mathbb{R}^n$, mentioned in Lemma 3.10, is the so-called nonexpansiveness, in the l^∞-norm, of the mapping $u \in \mathbb{R}^n_{\max} \to A \otimes u \in \mathbb{R}^m_{\max}$. Repeated application of the lemma for a square regular matrix A yields that

$$||(A^{\otimes k} \otimes u) - (A^{\otimes k} \otimes v)||_\infty \leq ||u - v||_\infty , \tag{3.5}$$

for any $u, v \in \mathbb{R}^n$ and all $k \geq 0$. In words, the l^∞-distance between $A^{\otimes k} \otimes u$ and $A^{\otimes k} \otimes v$ is bounded by $\|u - v\|_\infty$. The following theorem shows that nonexpansiveness implies that the cycle-time vector, provided it exists for at least one initial vector, exists for any initial vector and is independent of the specific initial vector. Write $x(k; x_0)$ to express the dependency of $x(k)$ on its initial value, i.e., $x(k; x_0) = A^{\otimes k} \otimes x_0$.

THEOREM 3.11 *Consider the recurrence relation $x(k+1) = A \otimes x(k)$ for $k \geq 0$, with $A \in \mathbb{R}^{n \times n}_{\max}$ a square regular matrix and $x(0) = x_0$ as initial condition. If $x_0 \in \mathbb{R}^n$ is a particular initial condition such that the limit $\lim_{k \to \infty} x(k; x_0)/k$ exists, then this limit exists and has the same value for any initial condition $y_0 \in \mathbb{R}^n$.*

Proof. Assume that $x_0 \in \mathbb{R}^n$ is such that $\lim_{k \to \infty} x(k; x_0)/k = \eta$ with $\eta \in \mathbb{R}^n$. For any $y_0 \in \mathbb{R}^n$, nonexpansiveness implies

$$\begin{aligned} 0 &\leq \left\| \frac{x(k; y_0)}{k} - \frac{x(k; x_0)}{k} \right\|_\infty \\ &\leq \frac{1}{k} || \left(A^{\otimes k} \otimes y_0\right) - \left(A^{\otimes k} \otimes x_0\right) ||_\infty \\ &\leq \frac{1}{k} ||y_0 - x_0||_\infty . \end{aligned}$$

Taking the limit as k tends to ∞ in the above row of inequalities yields

$$\lim_{k\to\infty} \left\| \frac{x(k;y_0)}{k} - \frac{x(k;x_0)}{k} \right\|_\infty = 0 .$$

Hence, as k tends to ∞ the l^∞-distance between $x(k;x_0)/k$ and $x(k;y_0)/k$ tends to zero, which implies that η is the cycle-time vector for any initial value y_0. Note that for the proof it is essential that all elements of x_0 and y_0 are finite. □

3.2.2 Existence of the cycle-time vector for irreducible matrices

The consequence of Theorem 3.11 is that once the cycle-time vector exists, it is independent of the initial condition. Therefore, the next issue to be studied is the actual existence of this vector. In the special case where matrix A is irreducible, the existence of the cycle-time vector, actually the asymptotic growth rate, for a particular initial condition is obvious, as will be shown below.

LEMMA 3.12 *Consider the recurrence relation $x(k+1) = A \otimes x(k)$ for $k \geq 0$, with $A \in \mathbb{R}_{\max}^{n\times n}$ an irreducible matrix having eigenvalue $\lambda \in \mathbb{R}$. Then, for all $j \in \underline{n}$*

$$\lim_{k\to\infty} \frac{x_j(k;x_0)}{k} = \lambda$$

for any initial condition $x(0) = x_0 \in \mathbb{R}^n$.

Proof. Let v be an eigenvector of A. Initializing the recurrence relation with $x_0 = v$ gives

$$\lim_{k\to\infty} \frac{1}{k} x_j(k;x_0) = \lambda,$$

for all $j \in \underline{n}$; see (3.2). Since by Theorem 3.11 once the asymptotic growth rate exists, it is independent of x_0, and the proof is completed. □

3.2.3 The generalized eigenmode for general matrices

In Lemma 3.12 the asymptotic growth rate for the recurrence relation $x(k+1) = A \otimes x(k)$ is characterized in the case where matrix A is irreducible. To give a similar characterization in the case where matrix A is not necessarily irreducible, the concept of a *generalized eigenmode* is introduced. In the following definition, the symbols $+$ and $\times$ stand for vector addition and scalar multiplication, respectively, in the conventional sense.

DEFINITION 3.13 *A pair of vectors $(\eta, v) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a* generalized eigenmode *of the regular matrix A if for all $k \geq 0$*

$$A \otimes (k \times \eta + v) = (k+1) \times \eta + v .$$

The vector η in a generalized eigenmode will be shown to coincide with the cycle-time vector. We illustrate the above definition with the following example.

Example 3.2.1 *Consider*

$$A = \begin{pmatrix} a & b \\ \varepsilon & c \end{pmatrix},$$

with $a, c \in \mathbb{R}$ and $b \in \mathbb{R}_{\max}$. First, let $b = \varepsilon$. Then it is straightforward that

$$\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} e \\ e \end{pmatrix} \right)$$

is a generalized eigenmode of A.

Now assume that $b \neq \varepsilon$. For $a \geq c$, it follows easily that

$$\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b - c \\ e \end{pmatrix} \right)$$

is a generalized eigenmode, and for $a \leq c$, a generalized eigenmode is given by

$$\left(\begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} b - c \\ e \end{pmatrix} \right).$$

The notion of a generalized eigenmode can be seen as an extension of an eigenvalue/eigenvector pair, as will be illustrated in the following. For $\mu \in \mathbb{R}_{\max}$, let

$$\mathbf{u}[\mu] \stackrel{\text{def}}{=} \mu \otimes \mathbf{u}$$

denote the vector having value μ in each of its entries. Thus, if $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$ are such that $A \otimes v = \lambda \otimes v$, then $A \otimes \lambda^{\otimes k} \otimes v = \lambda^{\otimes(k+1)} \otimes v$ for all $k \geq 0$, implying that for all $k \geq 0$

$$A \otimes (k \times \mathbf{u}[\lambda] + v) = (k+1) \times \mathbf{u}[\lambda] + v.$$

Hence, the cycle-time vector η can be seen as an extension of the notion of eigenvalue, whereas the vector v remains to play its role as eigenvector.

It follows that if a generalized eigenmode of a regular matrix exists, then the cycle-time vector exists and is unique. Indeed, assume that (η, v) is a generalized eigenmode of the regular matrix A and consider the recurrence relation $x(k+1) = A \otimes x(k)$ with $x_0 = v$. Then, by induction, it follows from the definition of a generalized eigenmode that $x(k) = k \times \eta + v$, so that the cycle-time vector satisfies $\lim_{k\to\infty} x(k)/k = \eta$, where it is crucial that the vector v is completely finite. From Theorem 3.11 it is known that this limit is independent of the initial condition, implying the uniqueness of the cycle-time vector.

Unlike the cycle-time vector, the second vector in a generalized eigenmode is not uniquely determined. Indeed, if (η, v) constitutes a generalized eigenmode of the regular matrix A, so does the pair $(\eta, \nu \otimes v)(= (\eta, \mathbf{u}[\nu] + v))$ for any $\nu \in \mathbb{R}$.

3.2.4 Preliminary results on inhomogeneous recurrence relations

In order to have a better understanding of the limiting behavior of reducible matrices, we will study a natural extension of the recurrence relation $x(k+1) = A \otimes x(k)$ in which A is an irreducible matrix. To that end, consider the recurrence relation

$$x(k+1) = A \otimes x(k) \oplus \bigoplus_{j=1}^{m} B_j \otimes u_j(k), \tag{3.6}$$

where A is an $n \times n$ matrix over $\mathbb{R}_{\max}$ and $B_1, \ldots, B_m$ are matrices over $\mathbb{R}_{\max}$ of suitable sizes; that is, for each $j \in \underline{m}$, matrix B_j is an $n \times m_j$ matrix over $\mathbb{R}_{\max}$ for some appropriate $m_j \geq 1$, and $u_j(k)$ denotes a vector (!) with m_j elements. The latter is in contrast with the notation so far, where $u_j(k)$ is used to indicate the jth element of $u(k)$. The reader should be warned that for the time being $u_j(k)$ will denote a vector of suitable size for any $j \in \underline{m}$.

Next assume the following:

- A is irreducible, so that the eigenvalue of A, denoted by $\lambda = \lambda(A)$, exists.
- Each of the matrices $B_1, \ldots, B_m$ contains at least one finite element, i.e., $B_j \neq \mathcal{E}$ for all $j \in \underline{m}$.
- For $j \in \underline{m}$, each of the sequences $u_j(k), k \geq 0$, is of the form
$$u_j(k) = w_j \otimes \tau_j^{\otimes k}, \qquad k \geq 0,$$
for some vector $w_j \in \mathbb{R}^{m_j}$ and scalar $\tau_j \in \mathbb{R}$.

Denote $\tau = \bigoplus_{j \in \underline{m}} \tau_j$, i.e., $\tau = \max\{\tau_1, \ldots, \tau_m\}$. Now it is claimed that there exists an integer $K \geq 0$ and a vector $v \in \mathbb{R}^n$ such that the sequence defined by

$$x(k) = v \otimes \mu^{\otimes k}, \quad \text{with} \quad \mu = \lambda \oplus \tau,$$

satisfies the recurrence relation (3.6) for all $k \geq K$.

To prove the claim, two cases will be distinguished, namely, $\lambda > \tau$ and $\lambda \leq \tau$.

Case $\lambda > \tau$. Take v to be an eigenvector of matrix A corresponding to eigenvalue λ, and recall that v is finite by the irreducibility of A, i.e., $v \in \mathbb{R}^n$; see Lemma 2.8. Further, choose v such that $v \otimes \lambda > \bigoplus_{j=1}^m B_j \otimes w_j$. The latter inequality can always be satisfied in combination with $A \otimes v = \lambda \otimes v$. Indeed, if $A \otimes v = \lambda \otimes v$, but not $\lambda \otimes v > \bigoplus_{j=1}^m B_j \otimes w_j$, then replace v by $v \otimes \rho$ with

$$\rho > \left(\bigoplus_{j=1}^m B_j \otimes w_j)^{\mathrm{top}} + (-(\lambda \otimes v) \right)^{\mathrm{top}} .$$

In the latter, the notation γ^{top} denotes the maximal element of vector γ. Then, with $\mu = \lambda > \tau_j$ for all $j \in \underline{m}$, it follows that for all $k \geq 0$

$$\mu \otimes v \otimes \mu^{\otimes k} = A \otimes v \otimes \mu^{\otimes k} > \bigoplus_{j=1}^m B_j \otimes w_j \otimes \mu^{\otimes k} \geq \bigoplus_{j=1}^m B_j \otimes w_j \otimes \tau_j^{\otimes k}.$$

Hence,

$$v \otimes \mu^{\otimes (k+1)} = A \otimes v \otimes \mu^{\otimes k} > \bigoplus_{j=1}^m B_j \otimes w_j \otimes \tau_j^{\otimes k}$$

for all $k \geq 0$, so that recurrence relation (3.6) is fulfilled for all $k \geq 0$, with $x(k) = v \otimes \mu^{\otimes k}$ and $u_j(k) = w_j \otimes \tau_j^{\otimes k}$ for $j \in \underline{m}$.

Case $\lambda \leq \tau$. Recall that $\tau = \max\{\tau_1, \ldots, \tau_m\}$. Assume without loss of generality that $\tau = \tau_j$ for $j = 1, \ldots, r$ and that $\tau > \tau_j$ for $j = r+1, \ldots, m$, with $1 < r \leq m$; that is, the maximum in $\max\{\tau_1, \ldots, \tau_m\}$ is attained by the first

r τ's. The latter can always be accomplished by a renumbering of the sequences $u_j(k),\ j \in \underline{m}$. Now take vector v to be a solution of

$$v = A_\tau \otimes v \oplus \bigoplus_{j=1}^{r} \left(B_j\right)_\tau \otimes w_j, \tag{3.7}$$

where A_τ denotes the matrix obtained from matrix A by subtracting τ from all of its finite elements and similarly for $\left(B_j\right)_\tau$, for $j \in \underline{m}$. Because $\lambda \leq \tau$, the communication graph of A_τ only contains circuits with a nonpositive weight. Therefore, it follows from Theorem 2.10 that a solution v of (3.7) is given by

$$v = \left(A_\tau\right)^* \otimes \left(\bigoplus_{j=1}^{r} \left(B_j\right)_\tau \otimes w_j \right).$$

Because A (and thus A_τ) is irreducible, matrix $\left(A_\tau\right)^*$ is completely finite. Further, since $\bigoplus_{j=1}^{r} \left(B_j\right)_\tau \otimes w_j$ contains at least one finite element, it follows that v is finite (i.e., $v \in \mathbb{R}^n$). Note that by adding τ to both sides of (3.7), it follows that v satisfies

$$v \otimes \tau = A \otimes v \oplus \bigoplus_{j=1}^{r} B_j \otimes w_j.$$

Then, with $\mu = \tau = \tau_j$ for $j = 1, \ldots, r$, it follows for all $k \geq 0$ that

$$v \otimes \mu^{\otimes(k+1)} = A \otimes v \otimes \mu^{\otimes k} \oplus \bigoplus_{j=1}^{r} B_j \otimes w_j \otimes \tau_j^{\otimes k},$$

leading to the inequality

$$v \otimes \mu^{\otimes(k+1)} \leq A \otimes v \otimes \mu^{\otimes k} \oplus \bigoplus_{j=1}^{m} B_j \otimes w_j \otimes \tau_j^{\otimes k}. \tag{3.8}$$

However, since $\mu > \tau_j$ for $j = r + 1, \ldots, m$, there exists an integer $K \geq 0$ such that for all $k \geq K$

$$v \otimes \mu^{\otimes(k+1)} \geq \bigoplus_{j=r+1}^{m} B_j \otimes w_j \otimes \tau_j^{\otimes k}.$$

Hence, for all $k \geq K$ the inequality in (3.8) in fact is an equality, so that the recurrence relation (3.6) is fulfilled for all $k \geq K$, with $x(k) = v \otimes \mu^{\otimes k}$ and $u_j(k) = w_j \otimes \tau_j^{\otimes k}$, for $j \in \underline{m}$.

As both cases are treated, the following theorem has been proved. In fact, if the above is repeated for $A = \varepsilon$, it follows easily that the following theorem holds in the case where $A = \varepsilon$.

THEOREM 3.14 *Consider the recurrence relation given by (3.6) and assume the following:*

- *A is irreducible with eigenvalue $\lambda = \lambda(A)$, or $A = \varepsilon$ with $\lambda = \varepsilon$;*

- *$B_j \neq \mathcal{E}$ for all $j \in \underline{m}$;*
- *$u_j(k) = w_j \otimes \tau_j^{\otimes k}$, $k \geq 0$, with $\tau_j \in \mathbb{R}$ and w_j a finite vector of suitable size for all $j \in \underline{m}$.*

Denote $\tau = \bigoplus_{j \in \underline{m}} \tau_j$. Then there exists an integer $K \geq 0$ and a vector $v \in \mathbb{R}^n$ such that the sequence defined by

$$x(k) = v \otimes \mu^{\otimes k}, \qquad \textit{with} \qquad \mu = \lambda \oplus \tau,$$

satisfies (3.6) for all $k \geq K$.

Note that in Theorem 3.14 recurrence relation (3.6) is satisfied for k larger than or equal to some integer $K \geq 0$. However, in the case where it is possible to reinitialize the sequences $u_j(k) = w_j \otimes \tau_j^{\otimes k}$, $k \geq 0$, by redefining the vectors w_j for $j \in \underline{m}$, it is possible that recurrence relation (3.6) is fulfilled for all $k \geq 0$, i.e., that K is actually zero. Indeed, in such a case, redefine

$$v := v \otimes \mu^{\otimes K}, \qquad w_j := w_j \otimes \tau_j^{\otimes K}, \qquad j \in \underline{m},$$

where v, μ, and K come from Theorem 3.14. Then the sequences

$$x(k) = v \otimes \mu^{\otimes k}, \qquad u_j(k) = w_j \otimes \tau_j^{\otimes k}, \qquad j \in \underline{m},$$

satisfy recurrence relation (3.6) for $k \geq 0$.

3.3 THE CYCLE-TIME VECTOR: GENERAL RESULTS

In this section the existence of a generalized eigenmode for a square reducible matrix $A \in \mathbb{R}_{\max}^{n \times n}$ will be proved. In doing so, also the existence of the cycle-time vector of a square reducible matrix is proved. Further, we will derive explicit expressions for the value of the elements of the cycle-time vector.

3.3.1 Existence of the cycle-time vector for reducible matrices

Consider the recurrence relation

$$x(k+1) = A \otimes x(k), \tag{3.9}$$

with A reducible. Recall that by renumbering the nodes in the communication graph $\mathcal{G}(A)$, matrix A can be brought into a block upper triangular form, called a *normal form* of A, given by

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1q} \\ \mathcal{E} & A_{22} & \cdots & \cdots & A_{2q} \\ \mathcal{E} & \mathcal{E} & A_{33} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{qq} \end{pmatrix},$$

with, the conditions in which for $i \in \underline{q}$,

- either A_{ii} is an irreducible matrix, so that $\lambda_i = \lambda(A_{ii})$ exists, or

- $A_{ii} = \varepsilon$, in which case $\lambda_i = \varepsilon$.

Let the vector $x(k)$ be partitioned according to the above normal form of A as

$$\begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_q(k) \end{pmatrix},$$

where $x_i(k)$ for $i \in \underline{q}$ denotes a vector (!) of suitable size. Again this is in contrast with the notation so far, where $x_i(k)$ is used to indicate the ith element of $x(k)$. Hence, the reader should be aware that for the time being $x_i(k)$ for $i \in \underline{q}$ in principle is a vector.

It follows now that recurrence relation (3.9) can be written as

$$x_i(k+1) = A_{ii} \otimes x_i(k) \oplus \bigoplus_{j=i+1}^{q} A_{ij} \otimes x_j(k), \quad i \in \underline{q}. \tag{3.10}$$

In particular, it follows for $i = q$ that

$$x_q(k+1) = A_{qq} \otimes x_q(k).$$

Since it is assumed throughout that matrix A is regular (i.e., contains at least one finite element in each row), it follows that also A_{qq} is regular. So $A_{qq} \neq \mathcal{E}$, implying that the corresponding m.s.c.s. has a nonempty arc set and, consequently, A_{qq} is irreducible. Hence, there exist a finite vector v_q and a scalar $\xi_q \in \mathbb{R}$ such that the sequence

$$x_q(k) = v_q \otimes \xi_q^{\otimes k}$$

satisfies $x_q(k+1) = A_{qq} \otimes x_q(k)$ for all $k \geq 0$. Indeed, simply take for v_q an eigenvector of A_{qq} corresponding to the eigenvalue $\lambda_q = \lambda(A_{qq})$, and set $\xi_q = \lambda_q$.

The case for general $i \in \underline{q}$ is treated in the following theorem, which will be proved by means of induction with respect to i, going from q back to 1.

THEOREM 3.15 *Consider the recurrence relations given in (3.10). Assume that A_{qq} is irreducible and that for $i \in \underline{q-1}$ either A_{ii} is an irreducible matrix or $A_{ii} = \varepsilon$. Then there exist finite vectors $v_1, v_2, \ldots, v_q$ of suitable sizes and scalars $\xi_1, \xi_2, \ldots, \xi_q \in \mathbb{R}$ such that the sequences*

$$x_i(k) = v_i \otimes \xi_i^{\otimes k}, \qquad i \in \underline{q},$$

satisfy (3.10) for all $k \geq 0$. The scalars $\xi_1, \xi_2, \ldots, \xi_q$ are determined by

$$\xi_i = \bigoplus_{j \in \mathcal{H}_i} \xi_j \oplus \lambda_i,$$

where $\mathcal{H}_i = \{j \in \underline{q} : j > i, A_{ij} \neq \mathcal{E}\}$.

Proof. For $i = q$, the result is immediate. Next, assume that the result is true for some $l+1$, with $1 < l+1 \le q$. Hence, there are finite vectors $v_{l+1}, \ldots, v_q$ of suitable sizes and scalars $\xi_{l+1}, \ldots, \xi_q \in \mathbb{R}$ such that the sequences

$$x_i(k) = v_i \otimes \xi_i^{\otimes k}, \qquad l+1 \le i \le q,$$

satisfy

$$x_i(k+1) = A_{ii} \otimes x_i(k) \oplus \bigoplus_{j=i+1}^{q} A_{ij} \otimes x_j(k), \qquad l+1 \le i \le q,$$

for all $k \ge 0$. Next, consider

$$x_l(k+1) = A_{ll} \otimes x_l(k) \oplus \bigoplus_{j=l+1}^{q} A_{lj} \otimes x_j(k). \tag{3.11}$$

Recall that either A_{ll} is a square irreducible matrix of suitable size or A_{ll} is a 1×1 matrix (i.e., a scalar) equal to ε. Further, note that

$$\bigoplus_{j=l+1}^{q} A_{lj} \otimes x_j(k) = \bigoplus_{j \in \mathcal{H}_l} A_{lj} \otimes x_j(k),$$

with $\mathcal{H}_l = \{j \in \underline{q} : j > l, A_{lj} \ne \mathcal{E}\}$. Hence, (3.11) can be written as

$$x_l(k+1) = A_{ll} \otimes x_l(k) \oplus \bigoplus_{j \in \mathcal{H}_l} A_{lj} \otimes x_j(k). \tag{3.12}$$

By this all the assumptions in Theorem 3.14 with respect to the recurrence relation in (3.12) are satisfied. It therefore follows from Theorem 3.14 that there exists an integer $K_l \ge 0$ and a finite vector/scalar v_l such that the sequence

$$x_l(k) = v_l \otimes \xi_l^{\otimes k}$$

satisfies the recurrence relation (3.12) or, equivalently, (3.11) for all $k \ge K_l$, where

$$\xi_l = \bigoplus_{j \in \mathcal{H}_l} \xi_j \oplus \lambda_l \, .$$

All the sequences $x_i(k), l \le i \le q$, can be reinitialized by redefining

$$v_i := v_i \otimes \xi_i^{\otimes K_l}, \qquad l \le i \le q.$$

It then follows that the (new) sequences

$$x_i(k) = v_i \otimes \xi_i^{\otimes k}, \qquad l \le i \le q,$$

satisfy

$$x_i(k+1) = A_{ii} \otimes x_i(k) \oplus \bigoplus_{j=i+1}^{q} A_{ij} \otimes x_j(k), \qquad l \le i \le q,$$

for all $k \ge 0$. Hence, the result is also true for l, with $1 \le l < q$. This concludes the induction step and also the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.15.

COROLLARY 3.16 *Consider the recurrence relation given in (3.9), and assume that A is regular. Then there exist finite vectors $\eta, v \in \mathbb{R}^n$ such that the sequence*

$$x(k) = k \times \eta + v$$

satisfies (3.9) for all $k \geq 0$.

Proof. First, by permuting/renumbering, transform the recurrence relation (3.9) into the recurrence relations (3.10). Note that due to the regularity of A the diagonal matrices in the normal form of A satisfy the conditions of Theorem 3.15. Observe that the sequences $x_i(k) = v_i \otimes \xi_i^{\otimes k}$, resulting from Theorem 3.15, also can be written as

$$x_i(k) = k \times \mathbf{u}[\xi_i] + v_i \, .$$

Then stacking the $\mathbf{u}[\xi_i]$'s on top of each other, stacking the v_i's on top of each other, and permuting/renumbering backwards, two vectors remain, say, η and v, such that the sequence

$$x(k) = k \times \eta + v$$

can easily be shown to satisfy (3.9) for all $k \geq 0$. □

Corollary 3.16 shows that any regular matrix $A \in \mathbb{R}_{\max}^{n \times n}$ has a generalized eigenmode. Consequently, the cycle-time vector exists for any regular matrix $A \in \mathbb{R}_{\max}^{n \times n}$.

Note that the cycle-time vector of matrix A in the normal form, as used above, is given by

$$\eta = \left(\mathbf{u}^\top[\xi_1], \mathbf{u}^\top[\xi_2], \ldots, \mathbf{u}^\top[\xi_q]\right)^\top .$$

For matrices A not in normal form, the cycle-time vector is a reshuffled version of the cycle-time vector corresponding to a normal form of A.

3.3.2 Expressions for the elements of the cycle-time vector

Consider again recurrence relation (3.9), and write it in the form of recurrence relations (3.10) by bringing A into normal form.

Assume, as in Section 2.1, that matrix A_{ss} in (3.10) corresponds to the restriction of matrix A to the m.s.c.s. denoted by $[i_s]$. Also note that $A_{sr} \neq \mathcal{E}$ in (3.10) if there exists an arc in $\mathcal{G}(A)$ from a node in m.s.c.s. $[i_r]$ to a node in m.s.c.s. $[i_s]$.

According to Theorem 3.15 the cycle-time vector corresponding to the m.s.c.s. $[i_s]$ is given by $\mathbf{u}[\xi_s]$. This follows simply from the solution for $x_s(k)$, given in Theorem 3.15, written as $x_s(k) = k \times \mathbf{u}[\xi_s] + v_s$. Hence, asymptotically speaking, all components of $x(k)$ corresponding to the nodes in the m.s.c.s. $[i_s]$ grow with the same speed, namely, ξ_s, where these scalars are determined by

$$\xi_s = \bigoplus_{r \in \mathcal{H}_s} \xi_r \oplus \lambda_s \, , \tag{3.13}$$

with $\mathcal{H}_s = \{r \in \underline{q} : r > s, A_{sr} \neq \mathcal{E}\}$. By the above equations the scalars ξ_s, for $s \in \underline{q}$, are determined implicitly. To have explicit expressions for these scalars, consider the following $q \times q$ matrix $\widetilde{A}$ defined by

$$[\widetilde{A}\,]_{sr} = \begin{cases} 0 & r > s, A_{sr} \neq \mathcal{E} \text{ (i.e., } r \in \mathcal{H}_s), \\ \varepsilon & \text{otherwise.} \end{cases}$$

The equation in (3.13) can then be written as

$$\xi = \widetilde{A} \otimes \xi \oplus \lambda, \tag{3.14}$$

where

$$\xi = \left(\xi_1, \xi_2, \ldots, \xi_q\right)^\top, \qquad \lambda = \left(\lambda_1, \lambda_2, \ldots, \lambda_q\right)^\top.$$

In the communication graph $\mathcal{G}(\widetilde{A})$ of matrix $\widetilde{A}$, identify each node with the m.s.c.s. it represents in the communication graph $\mathcal{G}(A)$ of matrix A. Hence, $\mathcal{G}(\widetilde{A})$ has node set $\{[i_1], [i_2], \ldots, [i_q]\}$. Further, note that $\mathcal{G}(\widetilde{A})$ has arcs with weight zero and that $\mathcal{G}(\widetilde{A})$ coincides with the reduced graph of A, introduced in Section 2.1.

Because matrix $\widetilde{A}$ is strictly upper triangular, its communication graph $\mathcal{G}(\widetilde{A})$ does not contain any circuits. Therefore, according to Theorem 2.10, the (unique) solution of (3.14) can be written as

$$\xi = \left(\widetilde{A}\right)^* \otimes \lambda.$$

Recall that $[(\widetilde{A})^*]_{sr}$ is the maximal weight of a path in $\mathcal{G}(\widetilde{A})$ from node $[i_r]$ to node $[i_s]$ and that $[(\widetilde{A})^*]_{sr} = \varepsilon$ if no such path exists. Since all arcs have weight zero, every path has weight zero. Hence, it follows from $\xi = \left(\widetilde{A}\right)^* \otimes \lambda$ that for all $s \in \underline{q}$ component ξ_s of vector ξ satisfies

$$\xi_s = \max\{\lambda_r | \text{there is a path in } \mathcal{G}(\widetilde{A}) \text{ from node } [i_r] \text{ to node } [i_s]\}.$$

If there is a path in $\mathcal{G}(\widetilde{A})$ from node $[i_r]$ to node $[i_s]$, then there is a path in $\mathcal{G}(A)$ from any node $i \in [i_r]$ to any node $j \in [i_s]$ and conversely. From Theorem 3.15 it follows that all nodes in one m.s.c.s. have the same asymptotic growth rate. Hence, their asymptotic growth rate can be identified with the asymptotic growth rate of the m.s.c.s. they belong to. Recall that $[j]$ stands for the m.s.c.s. that node j belongs to. Hence, $\xi_{[j]}$ denotes the asymptotic growth rate corresponding to m.s.c.s. $[j]$. Similarly, $\lambda_{[i]}$ denotes the eigenvalue corresponding to m.s.c.s. $[i]$, i.e., the eigenvalue of $A_{[i][i]}$. Finally, recall that $\pi^*(j)$ stands for the set of nodes i from which there is a path in $\mathcal{G}(A)$ to node j, including node j itself. Then the next theorem follows from the previous discussion.

THEOREM 3.17 *Consider the recurrence relation $x(k+1) = A \otimes x(k)$ for $k \geq 0$, with a square regular matrix $A \in \mathbb{R}^{n \times n}_{\max}$ and an initial condition $x(0) = x_0$. Let $\xi = \lim_{k \to \infty} x(k; x_0)/k$ be the cycle-time vector of A.*

1. *For all $j \in \underline{n}$,*

$$\xi_{[j]} = \bigoplus_{i \in \pi^*(j)} \lambda_{[i]}.$$

2. *For all $j \in \underline{n}$ and any $x_0 \in \mathbb{R}^n$,*

$$\lim_{k \to \infty} \frac{1}{k} x_j(k; x_0) = \bigoplus_{i \in \pi^*(j)} \lambda_{[i]}.$$

Example 3.3.1 *Consider A as given in Example 2.1.3. The communication graph of A has two nontrivial m.s.c.s.'s, given by the node sets $\{1,2,3,4\}$ and $\{5,6,7\}$. The associated arc sets follow directly from the graph in Figure 2.2. The other m.s.c.s.'s in the graph each consist of a single node with no arcs attached. Inspecting the graph in Figure 2.2, it immediately follows that*

$$\lambda_{[1]} = \lambda_{[2]} = \lambda_{[3]} = \lambda_{[4]} = \frac{1}{2}, \; \lambda_{[5]} = \lambda_{[6]} = \lambda_{[7]} = \frac{4}{3}, \; \lambda_{[8]} = \lambda_{[9]} = \lambda_{[10]} = \varepsilon.$$

Applying Theorem 3.17 then gives (see also Figure 2.3)

$$\lim_{k\to\infty} \tfrac{x_j(k)}{k} = \lambda_{[2]} = \tfrac{1}{2}, \qquad j \in \{1,2,3,4\},$$

$$\lim_{k\to\infty} \tfrac{x_j(k)}{k} = \max\left(\lambda_{[2]}, \lambda_{[5]}\right) = \tfrac{4}{3}, \qquad j \in \{5,6,7\},$$

$$\lim_{k\to\infty} \tfrac{x_8(k)}{k} = \max\left(\lambda_{[8]}, \lambda_{[2]}, \lambda_{[5]}\right) = \tfrac{4}{3},$$

$$\lim_{k\to\infty} \tfrac{x_9(k)}{k} = \max\left(\lambda_{[9]}, \lambda_{[2]}, \lambda_{[5]}\right) = \tfrac{4}{3},$$

$$\lim_{k\to\infty} \tfrac{x_{10}(k)}{k} = \max\left(\lambda_{[10]}, \lambda_{[2]}\right) = \tfrac{1}{2},$$

for any finite initial value x_0.

3.4 A SUNFLOWER BOUQUET

In this last section a special class of matrices is considered that will play an important role in Chapter 6.

DEFINITION 3.18 *A matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is called a* sunflower matrix *if*

- *its communication graph $\mathcal{G}(A)$ consists of precisely one circuit and possibly a number of paths and*
- *each node of $\mathcal{G}(A)$, being part of the circuit or of a possible path, has one incoming arc.*

The communication graph of a sunflower matrix will be called a sunflower graph.

Another way of defining such a sunflower graph is to say that a node not already belonging to the unique circuit is connected via a unique path (in backward direction) to this circuit.

A sunflower matrix is a special kind of regular matrix because it contains in every row precisely one finite entry. Indeed, in the communication graph of a sunflower matrix every node has precisely one predecessor. Further, note that in the graph of a sunflower matrix the circuit is the only strongly connected part and that every node can be reached from this circuit. Hence, if the (maximum) circuit mean is λ, then according to Theorem 3.17 it follows that the asymptotic growth rate of each component equals λ. The cycle-time vector of the matrix is thus given by $\mathbf{u}[\lambda]$, where λ is the circuit mean of the only circuit. To compute the second vector v of a generalized eigenmode of a sunflower matrix A, let the matrix be specified by its n finite entries $a_{i\pi(i)}$ for $i \in \underline{n}$. Then $v \in \mathbb{R}^n$, together with $\mathbf{u}[\lambda]$, should satisfy

$$A \otimes (k \times \mathbf{u}[\lambda] + v) = (k+1) \times \mathbf{u}[\lambda] + v,$$

for all $k \geq 0$ or in more detail, $a_{i\pi(i)} + k \times \lambda + v_{\pi(i)} = (k+1) \times \lambda + v_i$, for all $i \in \underline{n}$ and for all $k \geq 0$. This can be simplified to

$$a_{i\pi(i)} - \lambda + v_{\pi(i)} = v_i,$$

for all $i \in \underline{n}$. Now let i_0 be a node of the circuit and set $v_{i_0} = 0$. Then the components of v corresponding to the other nodes of the circuit can be obtained by going backward around the circuit using $v_{\pi(i)} = v_i - a_{i\pi(i)} + \lambda$. The components of v corresponding to any of the paths can be obtained by going forward along the paths using $v_i = a_{i\pi(i)} - \lambda + v_{\pi(i)}$. Clearly, the latter computations can be done very efficiently and require a number of operations that is linear in n. The above discussion is summarized in the following lemma.

LEMMA 3.19 *Let A be a sunflower matrix. Then $(\mathbf{u}[\lambda], v)$ is a generalized eigenmode $(\mathbf{u}[\lambda], v)$ of A, where λ is the circuit mean of the unique circuit in $\mathcal{G}(A)$ and v is recursively given through*

$$a_{i\pi(i)} - \lambda + v_{\pi(i)} = v_i,$$

for all $i \in \underline{n}$, with $v_{i_0} = 0$ for some initial node i_0 on the circuit.

If the graph of a sunflower matrix A contains nodes that do not belong to the circuit, then A is reducible. Lemma 3.19 can thus be phrased by saying that a sunflower matrix possesses a unique eigenvalue, which is an extension of the result in Theorem 2.9 to a subclass of reducible matrices.

A *bouquet matrix* is a regular matrix whose communication graph consists of a number of disjoint sunflower graphs. It can be seen easily that a bouquet matrix has precisely one finite entry in each row, implying that each node of its communication graph has precisely one predecessor. See Figure 3.2 for an example of the communication graph of a bouquet matrix. Since the communication graph of a bouquet matrix A consists of $r \geq 1$ disjoint graphs, matrix A can (after possible relabeling of the nodes in $\mathcal{G}(A)$) be written in the following block diagonal form:

$$A = \begin{pmatrix} A_1 & \mathcal{E} & \cdots & & \mathcal{E} \\ \mathcal{E} & A_2 & \mathcal{E} & & \vdots \\ \vdots & \mathcal{E} & \ddots & \ddots & \\ & & \ddots & A_{r-1} & \mathcal{E} \\ \mathcal{E} & \cdots & & \mathcal{E} & A_r \end{pmatrix},$$

where each of the block matrices A_i, $i \in \underline{r}$, is a sunflower matrix. Note that the above form should not be confused with the normal form of matrix A, in which

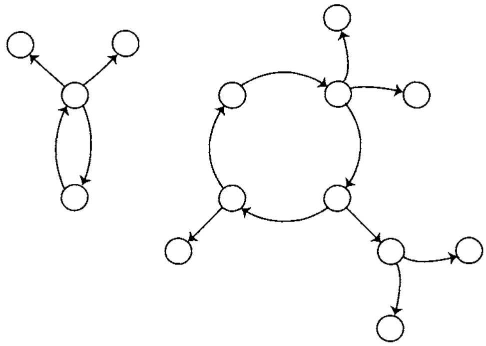

Figure 3.2: Communication graph of a bouquet matrix (consisting of two sunflowers).

each of the diagonal block matrices is either irreducible or equal to ε. In general, sunflower matrices are neither irreducible nor equal to ε. Clearly, Lemma 3.19 can be applied to the individual block matrices A_i, $i \in \underline{r}$, yielding generalized eigenmodes $(\mathbf{u}[\lambda_i], v_i)$, $i \in \underline{r}$, respectively. Let

$$v = (v_1^\top, v_2^\top, \ldots, v_r^\top)^\top \tag{3.15}$$

and

$$\eta = (\mathbf{u}^\top[\lambda_1], \mathbf{u}^\top[\lambda_2], \ldots, \mathbf{u}^\top[\lambda_r])^\top. \tag{3.16}$$

Then, it is easily seen that (η, v) is a generalized eigenmode of bouquet matrix A.

With the above method the generalized eigenmode of a bouquet matrix can in principle be computed. In Section 6.1.1 the latter computations will be presented in a more algorithmic form.

3.5 EXERCISES

1. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a regular square matrix. Recall that $||\,.\,||_\infty$ denotes the l^∞-norm of a vector. Show the following:

   (a) $||A \otimes u - A \otimes v||_\infty \leq ||u - v||_\infty$ for all $u, v \in \mathbb{R}^n$;

   (b) $||A^{\otimes k} \otimes u - u||_\infty \leq k \times ||A \otimes u - u||_\infty$ for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^n$;

   (c) if η denotes the cycle-time vector of A, then $||\eta||_\infty \leq ||A \otimes u - u||_\infty$ for all $u \in \mathbb{R}^n$, implying that $||\eta||_\infty \leq \min_{u \in \mathbb{R}^n} ||A \otimes u - u||_\infty$.

2. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be irreducible, and let e_j be the jth base vector $j \in \underline{n}$. Then for all $i \in \underline{n}$, there exists an $l \in \underline{n}$ such that $[A^{\otimes l} \otimes e_j]_i \neq \varepsilon$. This implies that for all $i \in \underline{n}$ there exists an $l \in \underline{n}$, such that $[A^{\otimes l} \otimes v]_i \neq \varepsilon$ for any for $v \neq \mathbf{u}[\varepsilon]$.

3. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be an irreducible matrix having eigenvalue λ. Then $\lim_{k \to \infty} A^{\otimes k}/k$ is an $n \times n$ matrix with λ in each of its entries.

4. Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $B \in \mathbb{R}_{\max}^{n\times m}$ be given matrices. Consider the inhomogeneous recurrence relation $x(k+1) = A \otimes x(k) \oplus B \otimes u(k)$. Let $x(0)$ and $u(k)$ for all $k \in \mathbb{N}$ be given. Then prove that for all $k \in \mathbb{N}$

$$x(k) = A^{\otimes k} \otimes x(0) \oplus \bigoplus_{l=0}^{k-1} A^{\otimes l} \otimes B \otimes u(k-1-l).$$

5. Show that the relation defined in (3.3) on the node set $\mathcal{N}(A)$ of a communication graph $\mathcal{G}(A)$ with cyclicity $\sigma_{\mathcal{G}(A)}$ is an equivalence relation with equivalence classes defined in (3.4).

6. Consider the matrix

$$A = \begin{pmatrix} \varepsilon & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \\ 0 & 4 & \varepsilon & 0 \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix},$$

and sketch its communication graph $\mathcal{G}(A)$ and its critical graph $\mathcal{G}^c(A)$. From the graphs, determine $\sigma_{\mathcal{G}(A)}, \sigma(A)$, and $\lambda(A)$. Compute $t(A)$ by straightforward (but tedious) multiplication. (Answer: $\sigma_{\mathcal{G}(A)} = 1, \sigma(A) = 3, \lambda(A) = 2, t(A) = 6.$)

7. By looking at the matrix

$$A = \begin{pmatrix} 1 & -6 \\ 0 & 2 \end{pmatrix}$$

and some of its first powers, determine $\lambda(A)$, and $\sigma(A)$ and estimate $t(A)$. (Answer: $\lambda(A) = 2, \sigma(A) = 1, t(A) = 10.$)

8. Determine a generalized eigenmode of the matrices

$$A = \begin{pmatrix} \varepsilon & 3 & \varepsilon \\ 1 & \varepsilon & 1 \\ \varepsilon & \varepsilon & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 1 & \varepsilon \\ \varepsilon & \varepsilon & 3 \\ \varepsilon & 1 & \varepsilon \end{pmatrix}.$$

(Answer: $(\eta, v) = ((4,4,4)^\top, (0,1,4)^\top)$, $(\eta, v) = ((4,2,2)^\top, (1,1,0)^\top)$.)

9. Check that the following matrix is a sunflower matrix, and compute a generalized eigenmode as explained in Section 3.4:

$$A = \begin{pmatrix} \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 7 & \varepsilon \end{pmatrix}.$$

(Answer: $(\eta, v) = ((4,4,4,4,4,4,4,4)^\top, (0,2,-1,-1,0,0,4,7)^\top)$.)

3.6 NOTES

The approach in Section 3.1 for proving Theorem 3.9 is inspired by [5] and [28]. In the latter reference a proof of Theorem 3.9 is given for a special case. Results on the general case, as studied in the present chapter, are also given in [27], which is the report version of [28]. See also the lecture notes [37]. Sections 3.2 and 3.3 are inspired by [23], in which the notion of a generalized eigenmode is introduced. Theorem 3.15 is related to the results obtained in [34]. The explicit expressions and proofs in these chapters follow directly from the standard theory but have not appeared elsewhere. The expressions can be combined with well-known methods to obtain a normal form of a square regular matrix, and they can be used with techniques from previous chapters for computing the eigenvalue and eigenvector of each of the diagonal matrices. Then, using the reduced graph, the generalized eigenmode can be obtained as described in the proofs of Theorem 3.15 and Corollary 3.16. In Chapter 6 an algorithmic approach will be presented based on an approach using bouquet matrices.

The notions of sunflower and bouquet matrices are new. They appear for the first time in this book. Though there are flowers of similar shape, we chose the name *sunflower* because of the famous painting by Vincent van Gogh. In botanic terms, the unique circuit of a sunflower graph corresponds to the disc flower, the part of the sunflower with all the seeds. In the same way, the paths leaving from this circuit correspond to the petals (which together form the corolla).