# Geometry, Algebra and Computation (II)

Wai-Shing Luk

Fudan University

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Cayley-Klein geometry

Basic measurement

# Cayley-Klein geometry

# Key points

- Gravity/electromagnetic force between two objects is inversely proportional to the square of their distance.
- Distance and angle may be powerful for oriented measures. But quadrance and spread are more energy saving for non-oriented measures.
- Euclidean Geometry is a degenerate case.

# Cayley-Klein Geometry

- Projective geometry can further be categorized by a specific polarity.
- Except degenerate cases, $(A^{\perp})^{\perp} = A$ and $(a^{\perp})^{\perp} = a$
- A fundamental cone $\mathcal{F} = (\mathbf{A}, \mathbf{B})$ is defined by a pole/polar pair such that $[A^{\perp}] = \mathbf{A} \cdot [A]$ and $[a^{\perp}] = \mathbf{B} \cdot [a]$.
- To visualize the Cayley-Klein Geometry, we may project the objects to the 2D plane.
- In hyperbolic geometry, the projection of the fundamental conic to the 2D plane is a unit circle, which is called *null circle*. The distance and angle measures could be negative outside the null circle.
- We may consider Euclidean geometry as a hyperbolic geometry where the null circle is expanded toward the infinity.
- In this section, we use the vector notation $p = [A]$ and $l = [a]$.

# Fundamental Cone with a pole and polar

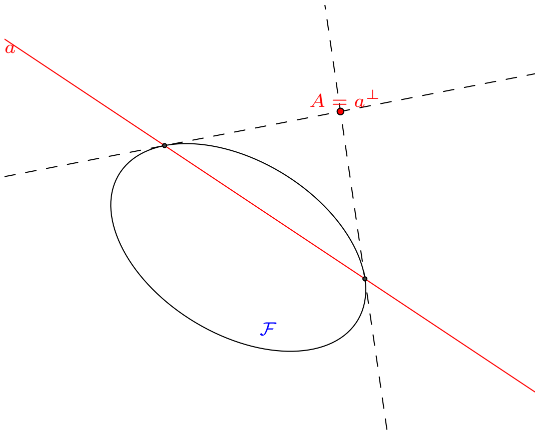


Figure 1: Fudanmental Cone with a pole and polar

# Examples

Let $p = [x, y, z]$ and $l = [a, b, c]$

- Hyperbolic geometry:
  - $\mathbf{A} \cdot p \equiv [x, y, -z]$
  - $\mathbf{B} \cdot l \equiv [a, b, -c]$
- Elliptic geometry:
  - $\mathbf{A} \cdot p \equiv [x, y, z]$
  - $\mathbf{B} \cdot l \equiv [a, b, c]$
- Euclidean geometry (degenerate conic):
  - $\mathbf{A} \cdot p \equiv [0, 0, z]$
  - $\mathbf{B} \cdot l \equiv [a, b, 0]$
- psuedo-Euclidean geometry (degenerate conic):
  - $\mathbf{A} \cdot p \equiv [0, 0, z]$
  - $\mathbf{B} \cdot l \equiv [a, -b, 0]$

# Examples (cont'd)

- Perspective view of Euclidean geometry (degenerate conic):
  - Let $l$ be the line of infinity.
  - Let $p$ and $q$ are two complex conjugate points on $l$. Then
  - $\mathbf{A} \equiv l \cdot l^T$ (outer product)
  - $\mathbf{B} \equiv p \cdot q^T + q \cdot p^T$

# Orthogonality

- A line $l$ is said to be perpendicular to line $m$ if $l^{\perp}$ lies on $m$, i.e., $m^{\mathsf{T}}\mathbf{B}l = 0$.
- To find a perpendicular line of $l$ that passes through $p$, join $p$ to the pole of $l$, i.e., $\text{join}(p, l^{\perp})$. We call this the *altitude* line of $l$.
- For duality, a point $p$ is said to be perpendicular to point $q$ if $q^{\mathsf{T}}\mathbf{A}p = 0$.
- The altitude point can be defined similarly.
- Note that Euclidean geometry does not have the concept of the perpendicular point because every $p^{\perp}$ is the line of infinity.

# Orthocenter of triangle

- Theorem 1 (Orthocenter and ortholine). The altitude lines of a non-dual triangle meet at a unique point $O$, called the *orthocenter* of the triangle.
- Although there is “center” in the name, orthocenter could be outside a triangle.
- Theorem 2. If the orthocenter of triangle $\{ABC\}$ is $O$, then the orthocenter of triangle $\{OBC\}$ is $A$.

# An instance of orthocenter theorem

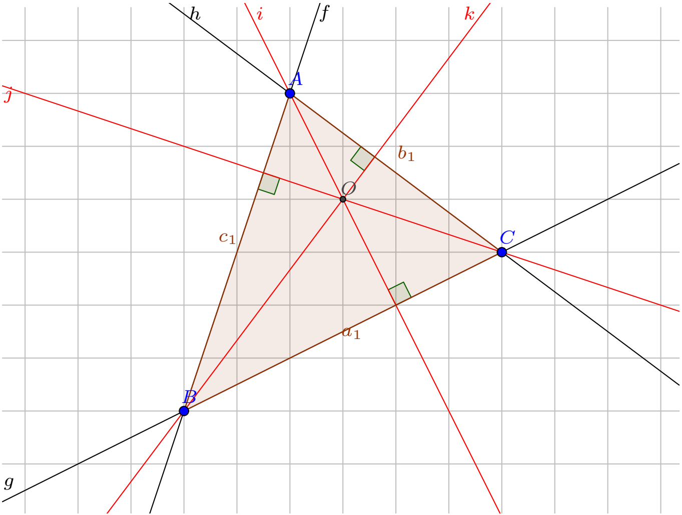


Figure 2: An instance of orthocenter theorem

# An instance of Theorem 2

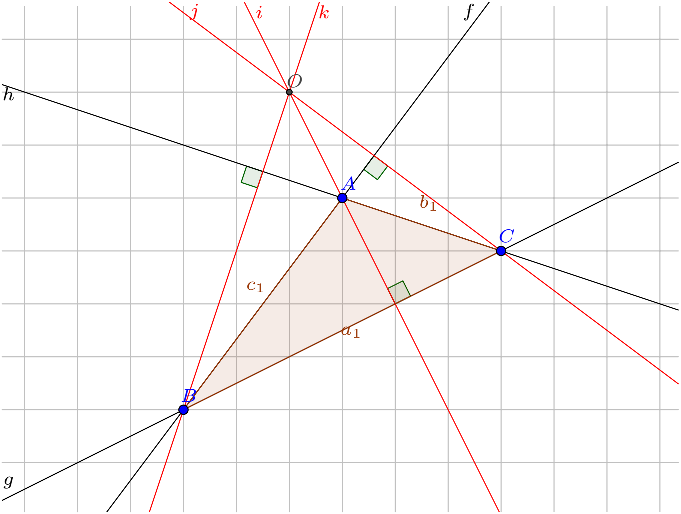


Figure 3: An instance of An instance of Theorem 2

# Involution

- Involutions are closely related to geometric reflections.
- The defining property of an involution $\tau$ is that $\tau(\tau(p)) = p$ for every point $p$.
- Theorem: Let $\tau$ be an involution. Then
  1. there is a line $m$ with $\tau(p) = p$ for every poiny $p$ incident with $m$.
  2. there is a point $o$ with $\tau(l) = l$ for every line $l$ incident with $o$.
- We call the line $m$ a *mirror* and the point $o$ the *center* of the involution.
- If $o$ is at the line of infinity (Euclidean Geometry), then we get an undistorted Euclidean line reflection in $m$.
- If we choose $o = m^{\perp}$, then we keep the fundamental cone invariant.

# Involution (cont'd)

- Theorem: The point transformation matrix $T$ of a projective involution $\tau$ with center $o$ and mirror $m$ is given by

$$(o^\mathsf{T} m)\mathrm{I} - 2om^\mathsf{T}$$

- In other words, $T \cdot p = (o^\mathsf{T} m)p - 2(m^\mathsf{T} p)o$.

# Python Code

```python
from proj_geom import *

def is_perpendicular(l, m):
    return m.incident(dual(l))

def altitude(p, l):
    return p * dual(l)

def orthocenter(a1, a2, a3):
    t1 = altitude(a1, a2*a3)
    t2 = altitude(a2, a1*a3)
    return t1*t2

class reflect:
    def __init__(self, m, O):
        self.m = m
        self.O = O
        self.c = dot(m, O)

    def __call__(self, p):
        return pk_point(self.c, p, -2 * dot(self.m, p), self.O)
```

# Basic measurement

# Quadrance and Spread for general cases

- Let $\Omega(x) = x \cdot x^{\perp}$.
- $\Omega(A) = A \cdot A^{\perp} = [A]^{\mathsf{T}}\mathbf{A}[A]$.
- $\Omega(a) = a \cdot a^{\perp} = [a]^{\mathsf{T}}\mathbf{B}[a]$.
- The **quadrance** $q(A, B)$ between points $A$ and $B$ is:

$$q(A, B) \equiv \Omega(AB)/\Omega(A)\Omega(B)$$

- The **spread** $s(l, m)$ between lines $l$ and $m$ is

$$s(l, m) \equiv \Omega(lm)/\Omega(l)\Omega(m)$$

- Note: they are invariant of any projective transformations.

# Python Code

```
import numpy as np
from fractions import *

def omega(l):
    return dot(l, dual(l))

def measure(a1, a2):
    omg = omega(a1*a2)
    if isinstance(omg, int):
        return Fraction(omg, omega(a1) * omega(a2))
    else:
        return omg / (omega(a1) * omega(a2))

def quadrance(a1, a2):
    return measure(a1, a2)

def spread(l1, l2):
    return measure(l1, l2)
```

# Relation with Traditional Distance and Angle

- Hyperbolic:
  - $q(A, B) = \sinh^2(d(A, B))$
  - $s(l, m) = \sin^2(\theta(l, m))$
- Elliptic:
  - $q(A, B) = \sin^2(d(A, B))$
  - $s(l, m) = \sin^2(\theta(l, m))$
- Euclidean:
  - $q(A, B) = d^2(A, B)$
  - $s(l, m) = \sin^2(\theta(l, m))$

# Measure dispersion among points on a unit sphere

Usual way:

```
nsimplex, n = K.shape
maxd = 0
mind = 1000
for k in range(nsimplex):
  p = X[K[k,:],:]
  for i in range(n-1):
    for j in range(i+1, n):
      dot = dot(p[i,:], p[j,:])
*     q = 1.0 - dot*dot
*     d = arcsin(sqrt(q))
      if maxd < d:
        maxd = d
      if mind > d:
        mind = d
*dis = maxd - mind
```

Better way:

```
nsimplex, n = K.shape
maxd = 0
mind = 1000
for k in range(nsimplex):
  p = X[K[k,:],:]
  for i in range(n-1):
    for j in range(i+1, n):
      dot = dot(p[i,:], p[j,:])

*     q = 1.0 - dot*dot
      if maxq < q:
        maxq = q
      if minq > q:
        minq = q
*dis = arcsin(sqrt(maxq)) \
*      - arcsin(sqrt(minq))
```

# Spread law and Thales Theorem

- Spread Law
$$q_1/s_1 = q_2/s_2 = q_3/s_3.$$

- (Compare with the sine law in Euclidean Geometry):
$$d_1/\sin\theta_1 = d_2/\sin\theta_2 = d_3/\sin\theta_3.$$

- Theorem (Thales): Suppose that $\{a_1 a_2 a_3\}$ is a right triangle with $s_3 = 1$. Then
$$s_1 = q_1/q_3 \quad \text{and} \quad s_2 = q_2/q_3$$

- Note: in some geometries, two lines are perpendicular does not imply they have a right angle ($s = 1$).

# Triangle proportions

- Theorem (Triangle proportions): Suppose that $d$ is a point lying on the line $a_1a_2$. Define the quadrances $r_1 \equiv q(a_1, d)$ and $r_1 \equiv q(a_2, d)$, and the spreads $R_1 \equiv s(a_3a_1, a_3d)$ and $R_2 \equiv s(a_3a_2, a_3d)$. Then

$$R_1/R_2 = (s_1/s_2)(r_1/r_2) = (q_1/q_2)(r_1/r_2).$$

# Midpoint and Angle Bisector

- There are two angle bisectors for two lines.
- There are two midpoints for two points also in general geometries.
- Let $r$ be the midpint of $p$ and $q$.
- Then $r = \sqrt{\Phi(p)}q \pm \sqrt{\Phi(q)}p$.
- Let $b$ be the angle bisector of $l$ and $m$.
- Then $b = \sqrt{\Phi(m)}l \pm \sqrt{\Phi(l)}m$.
- Note:
    - The midpoint could be irrational in general.
    - The midpoint could even be complex, even the two points are real.
    - Two angle bisectors are perpendicular.
    - In Euclidean geometry, another midpoint is at the line of infinity.

# Midpoint in Euclidean geometry

- Let $l$ be the line of infinity.
- $\mathbf{A} \equiv l \cdot l^T$
- $\Phi(p) = p^\mathsf{T} \mathbf{A} p = (p^\mathsf{T} l)^2$.
- Then, the midpoint $r = (q^\mathsf{T} l)p \pm (p^\mathsf{T} l)q$.
- One midpoint $(q^\mathsf{T} l)p - (p^\mathsf{T} l)q$ in fact lies on $l$.

# Constructing angle bisectors using a conic

1. For each line construct the two tangents $(t_f^1, t_f^2)$ and $(t_g^1, t_g^2)$ of its intersection points with the fundamental conic to that conic.
2. The following lines are the two angle bisectors:
   - join(meet($t_f^1, t_g^1$), meet($t_f^2, t_g^2$))
   - join(meet($t_f^1, t_g^2$), meet($t_f^2, t_g^1$))

   *Remark: the tangents in elliptic geometry have complex coordinates. However, the angle bisectors are real objects again.*

# Constructing a pair of angle bisectors

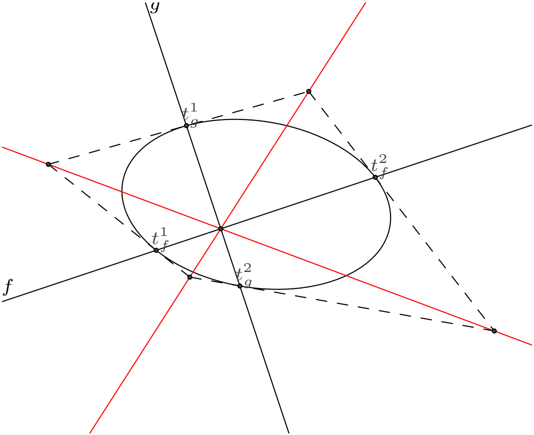


Figure 4: Constructing a pair of angle bisectors

# Angle Bisector Theorem

- Let $a$, $b$, $c$ be three lines such that none of them tangents to the fundamental conic.
- Then one set of angle bisector $m^1_{ab}, m^1_{bc}, m^1_{ac}$ are concurrent.
- Furthermore, the points $\text{meet}(m^2_{ab}, c)$, $\text{meet}(m^2_{bc}, a)$, $\text{meet}(m^2_{ac}, b)$ are collinear.

# An instance of complete angle bisector theorem

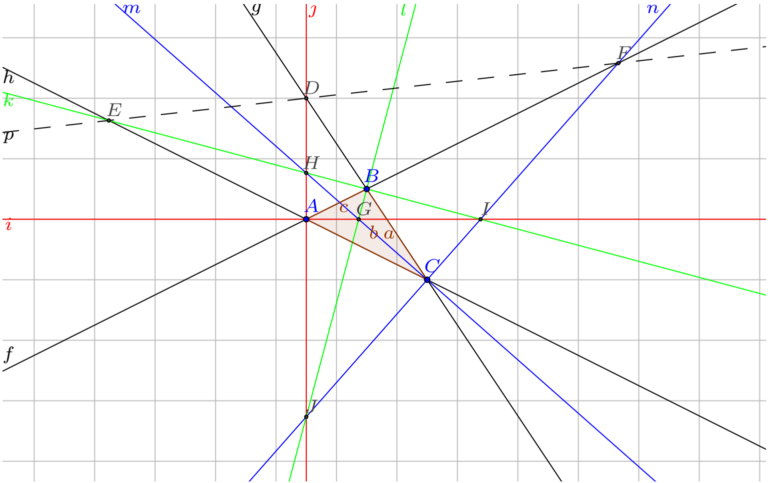


Figure 5: An instance of complete angle bisector theorem

# Midpoint theorem

- Let $p$, $q$, $r$ be three points such that none of them lies on the fundamental conic.
- Then one set of midpoints $m^1_{pq}$, $m^1_{qr}$, $m^1_{pr}$ are collinear.
- Furthermore, the lines join($m^2_{pq}, r), join(m^2_{qr}, p$), join($m^2_{pr}, q$) meet at a point.

# backup

> http://melpon.org/wandbox/permlink/Rsn3c3AW7Ud8E1qX