Chapter Seven

Petri Nets

In this chapter we will give a brief introduction to Petri nets as a modeling tool. We will show that a subclass of Petri nets, the so-called event graphs, is a suitable modeling aid for the construction of max-plus linear systems (i.e., for the construction of equations like (0.9) or (4.7)). In Section 7.1, the definitions of a Petri net and a timed event graph will be given. The construction of max-plus linear systems, starting from an event graph description of a model, will be treated in Section 7.2 for the autonomous case (i.e., when no external inputs are considered), and in Section 7.3 for the nonautonomous case.

7.1 PETRI NETS AND EVENT GRAPHS

Max-plus algebra allows us to describe the evolution of events on a network subject to synchronization constraints. For a railway network, for instance, the departures of trains are events. An appropriate tool to model events on a certain class of networks is named after C. A. Petri. This class of networks, to be introduced shortly, is therefore called the class of *Petri nets*. A subclass of these Petri nets, called *event graphs*, can be modeled by max-plus linear recurrence relations. Specifically, an event graph description can be transformed into a max-plus model and vice versa.

Petri nets are special directed graphs. The (finite) set of nodes $\mathcal{N}$ can be partitioned into two disjoint subsets $\mathcal{P}$ and $\mathcal{Q}$. The elements of $\mathcal{P}$ are called places and those of $\mathcal{Q}$ are called transitions. Places will be denoted by $p_i,\ i = 1, 2, \ldots, |\mathcal{P}|$, and transitions will be denoted by $q_j,\ j = 1, 2, \ldots, |\mathcal{Q}|$. Arcs from places to transitions exist ($p_i \rightarrow q_j$), as well as from transitions to places ($q_i \rightarrow p_j$), but arcs from places to places or from transitions to transitions do not exist. Hence, the set of arcs, to be denoted by $\mathcal{D}$, satisfies $\mathcal{D} \subset (\mathcal{Q} \times \mathcal{P}) \cup (\mathcal{P} \times \mathcal{Q})$. The set $\mathcal{Q} \times \mathcal{P}$ consists of all elements (q_i, p_j); the set $\mathcal{P} \times \mathcal{Q}$ is likewise defined. Because of this latter property, one says that Petri nets are bipartite directed graphs.

For $(p_i, q_j) \in \mathcal{D}$ we say that node p_i is an upstream place for q_j, and q_j is a downstream transition for p_i. The nodes of $(q_j, p_i) \in \mathcal{D}$ are similarly expressed. In agreement with the notation introduced in Chapter 2, we denote the set of all upstream places of transition q_j by $\pi(q_j)$, i.e., $p_i \in \pi(q_j)$ if and only if $(p_i, q_j) \in \mathcal{D}$. Similarly, the set of all upstream transitions of place p_i is denoted by $\pi(p_i)$; i.e., $q_j \in \pi(p_i)$ if and only if $(q_j, p_i) \in \mathcal{D}$. Downstream relationships are defined analogously by means of the symbol σ instead of π. See also Section 2.1, where these symbols were introduced. A Petri net is called an event graph if all places in the Petri net have exactly one upstream and one downstream transition. Transitions

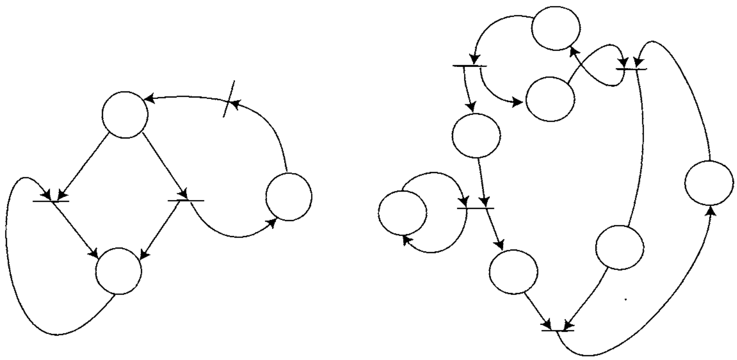

Figure 7.1: A Petri net (left) and an event graph (right). The circles represent places and the bars transitions.

in an event graph can have more than one upstream and/or downstream place. See Figure 7.1 for an example of a Petri net and of an event graph. In such graphical representations, places are always drawn as circles and transitions as bars. These bars can have any orientation (not only horizontal or vertical).

In terms of applications, places often represent conditions and transitions represent events. A condition being fulfilled is indicated by a token allocated to the corresponding place (in the graph this is indicated by a dot in the circle representing the place concerned). The event symbolized by a transition can take place (in Petri net terminology, we say that the transition is *enabled*) if all upstream places contain at least one token. If a transition is enabled, then it can execute the event it symbolizes. If the event takes place, one says that the transition *fires*. If an event happens (i.e., the corresponding transition fires), then a token is taken away from each of the upstream places and one token is added to each of the downstream places. An example is given in Figure 7.2. Note that if the number of upstream places differs from the number of downstream places of a transition, the total number of tokens before and after the firing will be different. The number of tokens in each of the places, called the marking of the Petri net, is indicated by the vector $\mathcal{M} = (m_1, m_2, \ldots, m_{|\mathcal{P}|})^\top$. Each element m_i is a natural number. If a transition fires, the vector $\mathcal{M}$ will in general change. The initial marking is indicated by $\mathcal{M}_0$. Usually the order of firings is not uniquely determined.

A Petri net (or event graph) is called a *timed Petri net* (or *timed event graph*) if a holding time is attached to each place. This holding time associated with place p_i will be indicated by τ_i. It represents the time that a token must spend in the place before it can play its role in the enabling of the downstream transition. (If there is more than one place upstream of this transition, then the tokens in all these places each must have spent their holding times before the transition is enabled.) This firing, or event happening, is supposed to be instantaneous, i.e., it does not take time. We also say that the firing time equals zero. To be very explicit, time durations are only attached to places and not to transitions. The vector whose elements are the holding times will be indicated by $\mathcal{T}$.

Summing up, we get the following definition.

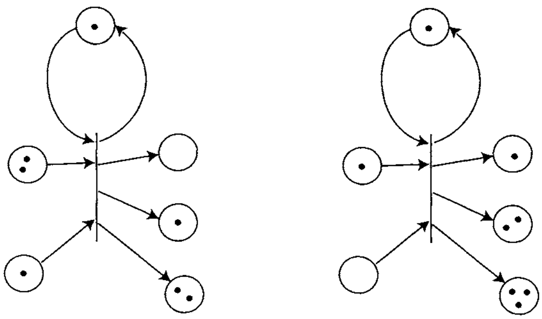

Figure 7.2: The token distribution before (left) and after (right) a firing of a transition. Note that if a place is both up- and downstream, the number of tokens before and after the firing will be the same.

DEFINITION 7.1 *A timed Petri net $\mathcal{G}$ is characterized by $\mathcal{P}, \mathcal{Q}, \mathcal{D}, \mathcal{M}_0$, and $\mathcal{T}$, where $\mathcal{P}$ is the set of places, $\mathcal{Q}$ is the set of transitions, $\mathcal{D}$ is the set of arcs from transitions to places and vice versa, $\mathcal{M}_0$ is the initial marking, and $\mathcal{T}$ is the vector of holding times. If each place has exactly one upstream and one downstream transition, then the (timed) Petri net is called a (timed) event graph.*

From this point on, we will restrict ourselves to the study of event graphs. In event graphs, the location of each place is characterized by its upstream and downstream transitions. If only one place between two transitions exists (keeping track of the directions of the arcs), it is uniquely determined by these transitions. For this reason a place p_l will also be indicated by the notation p_{ji}, where the subscript i refers to the upstream transition q_i and j to the downstream transition q_j. The same convention of notation will be used for holding times and markings. Only in the case where there is more than one place between two transitions are the notations p_{ji}, τ_{ji}, and m_{ji} ambiguous.

THEOREM 7.2 *The number of tokens in any circuit of an event graph is constant.*

Proof. This is straightforward. □

As an immediate consequence of the above theorem, if a circuit contains zero tokens, then the transitions within this circuit will never fire. An event graph is called *live* if each circuit contains at least one token. See exercise 3 at the end of the chapter for liveness of Petri nets.

Example 7.1.1 *In Figure 7.3, Figure 0.1 is repeated in more abstract form (left), and the corresponding event graph is given on the right. The stations (or more properly, the departures of the trains per track) are indicated by transitions. A train on a track is symbolized by a token in the place on this track. Saying that the trains at station S_1 are leaving is equivalent to saying that transition q_1 fires. In order for it to fire, the incoming trains must have arrived or, equivalently, the tokens in the upstream places must have spent their holding time. The holding time is here defined to be the travel time (in which the changeover time has been subsumed).*

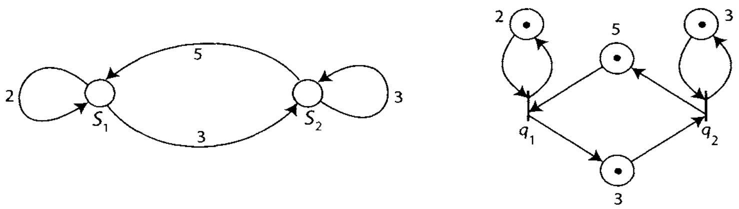

Figure 7.3: Figure 0.1 (left) and the corresponding timed event graph (right).

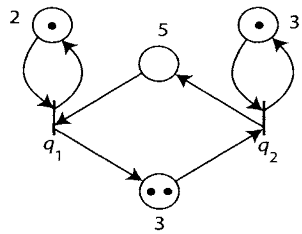

Figure 7.4: Figure 7.3 (right) after one firing.

Once transition q_1 has fired, the token distribution becomes as given in Figure 7.4. It will be obvious that transition q_1 cannot immediately fire a second time, since the place between transition q_2 (station S_2) and transition q_1 no longer contains a token. In order for transition q_1 to fire a second time, this place must have a token, which will be the case after transition q_2 has fired (and the holding time has been spent).

Example 7.1.2 *In Figure 7.5, the event graphs corresponding to (0.13), (0.15), (0.16), and exercise 6 of Chapter 0 are given. Let us consider the construction of the graphs corresponding to (0.13) and (0.16) in some detail. The other two are left as exercises.*

Equation (0.13) represents a two-dimensional system; hence, we have two transitions, one corresponding to each of the two states. Because there are four connections (line segments), the event graph will contain four places. All places contain one token (interpret a token as a train), except for the place between transition q_1 and transition q_2, which has two tokens. This is due to the argument $k - 1$ of $x_1(k - 1)$ at the right-hand side of the second equation of (0.13). The numbers attached to the places in the figure refer to the travel times.

Equation (0.16) represents a three-dimensional system, and hence, there are three transitions. Places exist between those transitions for which the corresponding element in the system matrix in (0.16) is finite. All these places contain exactly one token, which is due to the fact that (0.16) is a first-order system.

Note that the definition of a timed event graph as given does not uniquely determine all (future) firing times. In order to achieve that, it is necessary to add initial

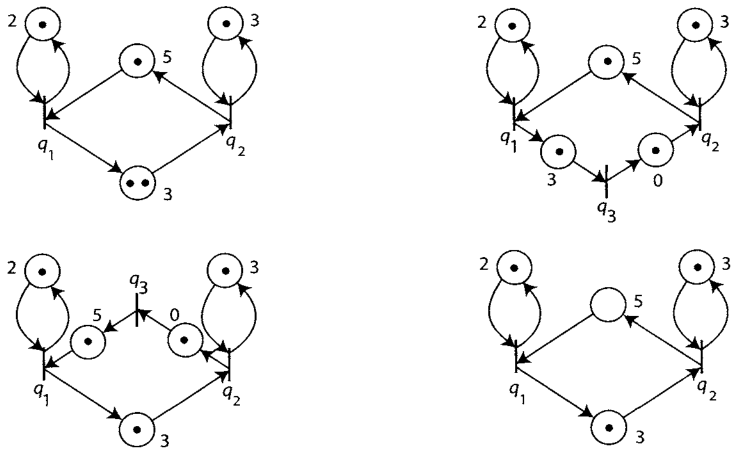

Figure 7.5: The event graphs corresponding to (0.13), (0.15), (0.16), and exercise 8 of Chapter 0.

conditions; to be precise, for every token one should specify in which place it finds itself and how long it has already been there (that is, how much of its holding time it has already consumed).

In the examples considered so far, all transitions had upstream and downstream places. It is quite possible to study event graphs with transitions that do not have such places. A transition without an upstream place is called a *source transition*, or simply a *source*. Such a transition is supposed to be enabled by the outside world. Similarly, a transition that does not have a downstream place is called a *sink transition*) (or a *sink*). Sink transitions deliver tokens to the outside world. If there are no sources in the network, then one talks about an *autonomous* network and calls it *nonautonomous* otherwise. It is assumed that only transitions can be sources or sinks (which is no loss of generality, since one can always add a transition upstream or downstream of a place if necessary). A source transition is an input of the network; a sink transition is an output of the network.

7.2 THE AUTONOMOUS CASE

As before, let τ_{ji} denote the holding time of the place between transition q_i and q_j, tacitly assuming that there is only one such place, and define $a_{ji} \stackrel{\text{def}}{=} \tau_{ji}$. Though generalizations are possible, here the focus is on constant holding times, i.e., they are independent of the counter k. Let $x_j(k)$ denote the time at which transition q_j fires for the kth time. We can define the vector $x(k) = (x_1(k), \ldots, x_{|Q|}(k))^\top$ as the state of the system.

With any event graph, matrices $A_0, \ldots, A_M$ can be associated, all of size $|Q| \times |Q|$. To obtain $[A_m]_{jl}$ consider all places between transitions q_l and q_j having initially m tokens, and take as $[A_m]_{jl}$ the maximum of the corresponding holding times. It is possible that more than one place between two transitions exists. In such a case, $[A_m]_{jl} = a_{jl}$, where a_{jl} is the largest of the holding times with respect to all places between transitions q_l and q_j with m tokens. In other words, when there is only one such a place,

$$[A_m]_{jl} = \begin{cases} a_{jl} & \text{if the number of tokens in place } p_{q_j q_l} \text{ equals } m, \\ \varepsilon & \text{otherwise,} \end{cases} \tag{7.1}$$

for $m = 0, 1, \dots, M$, where M is defined as the maximum number of tokens with respect to all places.

If one considers the state variable $x_i(k)$, which denotes the kth time that transition q_i fires, then the vector $x(k) = (x_1(k), \dots, x_{|\mathcal{Q}|}(k))^\top$ satisfies the max-plus (linear) equation

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus \cdots \oplus A_M \otimes x(k-M), \tag{7.2}$$

for $k \geq 0$.

Remark. The matrices A_m in (7.2) were defined with respect to a token distribution at the initial time (or at another time instant). To start with, one could take a picture of the whole net and count the number of tokens in each of the places in this picture and subsequently construct the A_m matrices. When k increases by one, each transition has fired once, and the token distribution will again be the same. This is not to be confused with the firing of a single transition. Compare, for instance, the token distributions of Figures 7.3 (right) and 7.4. The system descriptions based on these two figures will be different, though the same instants at which tokens move is described. This feature was already encountered in Section 0.4. In conventional algebra one has a somewhat similar phenomenon: a coordinate transformation yields different mathematical representations of the same system.

If we follow the train of thought put forward in Section 4.5, then equation (7.2) can be written as

$$x(k) = A_0^* \otimes A_1 \otimes x(k-1) \oplus \cdots \oplus A_0^* \otimes A_M \otimes x(k-M), \tag{7.3}$$

for $k \geq 0$, where

$$A_0^* = \bigoplus_{i=0}^{|\mathcal{Q}|-1} A_0^i, \tag{7.4}$$

provided that the communication graph of A_0 does not contain circuits with positive weight.

As a next step, transform (7.3) into a first-order recurrence relation. In order to do so, we take as new state vector the $(|\mathcal{Q}| \times M)$–dimensional vector

$$\tilde{x}(k) = (x^\top(k), x^\top(k-1), \dots, x^\top(k-M+1))^\top \tag{7.5}$$

and the $(|\mathcal{Q}| \times M) \times (|\mathcal{Q}| \times M)$–dimensional matrix $\tilde{A}$ given by

$$\begin{pmatrix} A_0^* \otimes A_1 & A_0^* \otimes A_2 & \cdots & \cdots & A_0^* \otimes A_M \\ E & \mathcal{E} & \cdots & \cdots & \mathcal{E} \\ \mathcal{E} & E & \mathcal{E} & \cdots & \mathcal{E} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \vdots \\ \mathcal{E} & \cdots & \mathcal{E} & E & \mathcal{E} \end{pmatrix}.$$

Then (7.3) can be written as

$$\tilde{x}(k) = \tilde{A} \otimes \tilde{x}(k-1), \qquad k \geq 0. \tag{7.6}$$

The above equation is called the *standard autonomous equation*. Any live autonomous event graph can be modeled by a standard autonomous equation.

Remark. Equation (7.3) is a recurrence relation (by abuse of language it is often called a *difference equation*). If numerical values for the vectors $x(-1)$, $x(-2), \ldots, x(-M)$ are given, then these values constitute the initial condition, and the future evolution of the state is uniquely determined. In the same way, $\tilde{x}(-1)$ is an initial condition for (7.6). From a mathematical point of view, there are no restrictions on the numerical values for these initial conditions. Given a physical interpretation, however, limitations for these values may exist. An example is the vector of initial conditions of which the components represent both arrival and departure times. The arrival time of a train at a station should occur before the departure time of the same train.

Sometimes the token distribution at a certain time instant is interpreted as an initial condition (compare taking a picture of the whole net in the previous remark). This is not fully correct, however, since one does not know how long a token has already resided in its place with only this information. In other words, one does not know how much of the holding time has already been consumed. One could say that the token distribution only provides partial information for the initial condition.

Often the dimension of the obtained state space is unnecessarily large. Usually quite a few places in the original event graph can be combined into a single place. As an example, two places in a series with a transition in between that has neither upstream nor downstream places can be replaced by a single place. This causes a smaller $|Q|$ and, hence, a smaller dimension. Algorithm 8.3.1 in Chapter 8 describes this reduction of the dimension of the state space in a systematic way.

Example 7.2.1 *Let us consider a circular track with three stations along which two trains run in one direction. The trains run from station S_1 to S_2, from S_2 to S_3, from S_3 to S_1, and so on. For safety reasons it is assumed that a train cannot leave station S_i before the preceding train has left S_{i+1}, $i \in \underline{3}$, with $S_4 = S_1$, or in other words, a train cannot leave a station before the platform at the next station is free. This model is symbolized in the Petri net in Figure 7.6, in which the transitions are denoted by S_i, $i \in \underline{3}$. The trains move counterclockwise, whereas the tokens in the places in the clockwise circuit represent the conditions of the next station being free. If $x_i(k)$ represents the kth departure from station S_i and $a_{i+1,i}$, $i \in \underline{3}$, with $a_{4,3} = a_{1,3}$, is the travel time between S_i and S_{i+1}, then*

$$\begin{aligned} x_1(k+1) &= \max\{x_3(k+1) + a_{13}, x_2(k+1)\}, \\ x_2(k+1) &= \max\{x_1(k) + a_{21}, x_3(k+1)\}, \\ x_3(k+1) &= \max\{x_2(k) + a_{32}, x_1(k)\}, \end{aligned}$$

which, like (7.2) and (7.4) can be written as

$$x(k+1) = \begin{pmatrix} \varepsilon & 0 & a_{13} \\ \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \otimes x(k+1) \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ a_{21} & \varepsilon & \varepsilon \\ 0 & a_{32} & \varepsilon \end{pmatrix} \otimes x(k)$$

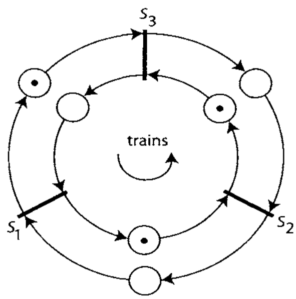

Figure 7.6: A Petri net of a circular track with two trains.

$$= \begin{pmatrix} a_{21} \oplus a_{13} & a_{13} \otimes a_{32} & \varepsilon \\ a_{21} & a_{32} & \varepsilon \\ 0 & a_{32} & \varepsilon \end{pmatrix} \otimes x(k). \tag{7.7}$$

In the derivation of the latter equality in (7.7), it has been tacitly assumed that all a_{ij} are nonnegative. Since the evolution of x_1 and x_2 is not influenced by the evolution of x_3 (the third column of the latter system matrix only contains ε's), the reduced state $x_{\text{red}} \stackrel{\text{def}}{=} (x_1, x_2)^\top$ can be introduced, which satisfies

$$x_{\text{red}}(k+1) = \begin{pmatrix} a_{21} \oplus a_{13} & a_{13} \otimes a_{32} \\ a_{21} & a_{32} \end{pmatrix} \otimes x_{\text{red}}(k). \tag{7.8}$$

7.3 THE NONAUTONOMOUS CASE

Let $\mathcal{Q}_I \subset \mathcal{Q}$ denote the set of input transitions, while $\mathcal{Q}$ remains the set of all transitions. If q_i is such an input transition, then the holding time of a place between q_i and another transition $q_j \in \mathcal{Q} \setminus \mathcal{Q}_I$ is indicated by b_{ji}. If the maximal initial marking of all places downstream of any input transition is denoted by M', then $|\mathcal{Q} \setminus \mathcal{Q}_I| \times |\mathcal{Q}_I|$-dimensional matrices $B_0, \ldots, B_{M'}$ are defined as

$$[B_m]_{jl} = \begin{cases} b_{jl} & \text{if the number of tokens in place } p_{q_j q_l} \text{ equals } m, \\ \varepsilon & \text{otherwise,} \end{cases}$$

for $m = 0, 1, \ldots, M'$. In words, to obtain $[B_m]_{jl}$ we consider all upstream transitions of place j that are input transitions having initially m tokens in their preceding place. If there happens to be more than one such place, we define $[B_m]_{jl}$ to be the maximum of the corresponding holding times. Furthermore, we let $u(k)$ be a $|\mathcal{Q}_I|$-dimensional vector, where $u_i(k)$ denotes the kth firing time of the appropriate input transition.

The vector of the kth firing times satisfies the (linear) equation

$$\begin{aligned} x(k) = {} & A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus \cdots \oplus A_M \otimes x(k-M) \\ & \oplus B_0 \otimes u(k) \oplus B_1 \otimes u(k-1) \oplus \cdots \oplus B_{M'} \otimes u(k-M'), \end{aligned} \tag{7.9}$$

for $k \geq 0$. A possible initial condition is that $x_j(k)$, $j \in \underline{|\mathcal{Q}| \setminus |\mathcal{Q}_I|}$, and $u_j(k)$, $j \in \underline{|\mathcal{Q}_I|}$, equal ε if $k < 0$.

From now on we restrict ourselves to nonautonomous event graphs for which the communication graph of A_0 does not contain any circuit with positive weight. Then (7.9) is equivalent to

$$\begin{aligned} x(k) = A_0^* \otimes A_1 \otimes x(k-1) \oplus \cdots \oplus A_0^* \otimes A_M \otimes x(k-M) \\ \oplus A_0^* \otimes B_0 \otimes u(k) \oplus \cdots \oplus A_0^* \otimes B_{M'} \otimes u(k-M'), \end{aligned} \qquad (7.10)$$

for all $k \geq 0$. Compare this recurrence relation with (7.3) for the autonomous case, and note that now terms with the input firings have been added.

Define the $((M'+1) \times |\mathcal{Q}_I|)$–dimensional vector

$$\tilde{u}(k) = (u^\top(k), u^\top(k-1), \ldots, u^\top(k-M'))^\top$$

and the $(|\mathcal{Q} \setminus \mathcal{Q}_I| \times M) \times (|\mathcal{Q}_I| \times (M'+1))$ matrix $\tilde{B}$ by

$$\begin{pmatrix} A_0^* \otimes B_0 & A_0^* \otimes B_1 & \cdots & \cdots & A_0^* \otimes B_{M'} \\ \mathcal{E} & \mathcal{E} & \cdots & \cdots & \mathcal{E} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \cdots & \mathcal{E} \end{pmatrix}.$$

Then (7.10) can be written as

$$\tilde{x}(k) = \tilde{A} \otimes \tilde{x}(k-1) \oplus \tilde{B} \otimes \tilde{u}(k), \qquad k \geq 0,$$

where $\tilde{x}$ and $\tilde{A}$ are defined as in the previous section. We call the above equation the *standard nonautonomous equation*. Any nonautonomous event graph (with the communication graph of A_0 not having circuits with positive weight) can be modeled by a standard nonautonomous equation.

Example 7.3.1 *Consider a railway system with external arrivals modeling a single long-distance track with two stations. Let station S_0 represent an external source generating arrivals of trains, and denote the interarrival time of trains by a_0. Assume that the distances between stations S_1 and S_2 and between the source stations S_0 and S_1 are long, so that more than one (actually, arbitrarily many) trains can be present on the intermediate tracks. The travel times are a_{10} and a_{21}, respectively. Each train that enters the system has to pass through stations S_1 and S_2, each of which can handle only one train at a time, and then leaves the line. This is modeled by $a_{11} = a_{22} = e$. We assume that the system starts empty. This description is symbolized as a Petri net model in Figure 7.7. Let $x_j(k)$ denote the*

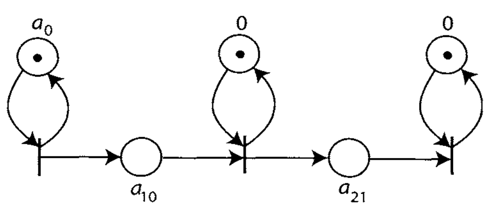

Figure 7.7: A Petri net model of Example 7.3.1.

kth departure time of a train from station S_j, $j \in \underline{2}$. Let $x_0(k)$ denote the time of the kth arrival of a train on the line. The time evolution of the system can then be described by a three-dimensional vector $x(k) = (x_0(k), x_1(k), x_2(k))^\top$ following the homogeneous equation

$$x(k+1) = A \otimes x(k),$$

with $x(0) = \mathbf{u}$ and where

$$A = \begin{pmatrix} a_0 & \varepsilon & \varepsilon \\ a_0 \otimes a_{10} & e & \varepsilon \\ a_0 \otimes a_{10} \otimes a_{21} & a_{21} & e \end{pmatrix}.$$

Observe that A is reducible. Alternatively, we could describe the system via a two-dimensional vector $x_{\mathrm{red}}(k)$ following the inhomogeneous equation

$$x_{\mathrm{red}}(k+1) = \begin{pmatrix} e & \varepsilon \\ a_{21} & e \end{pmatrix} \otimes x_{\mathrm{red}}(k) \oplus \begin{pmatrix} a_{10} \\ a_{10} \otimes a_{21} \end{pmatrix} \otimes u(k+1),$$

for $k \geq 0$, where $u(k) = a_0^{\otimes k} = k \times a_0$ denotes the time of the kth arrival of a train on the line.

7.4 EXERCISES

1. Give the Petri net representation of (0.14).

2. Give a proof of Theorem 7.2.

3. A Petri net is said to be *live* (for the initial marking $\mathcal{M}_0$) if for any $\mathcal{M}$, obtained after an arbitrary series of firings starting from $\mathcal{M}_0$, and for each transition q there exists another marking $\mathcal{N}$, which can be obtained after a suitable series of firings starting from $\mathcal{M}$, such that q is enabled in $\mathcal{N}$. Prove the following:

    - For a live Petri net, or a live event graph, any transition can be fired an infinite number of times.
    - An event graph is live if and only if each circuit contains at least one token.
    - An event graph of an autonomous system is live if and only if the communication graph of A_0 does not have circuits.

4. Calculate the matrices A_m as given in equation (7.2) for both Figures 7.3 (right) and 7.4. Rewrite the model for Figure 7.4 in the format of the standard autonomous equation. Thus, one has two first-order descriptions of the same event graph. Compare the behaviors of these two descriptions.

5. If the travel times in Example 7.2.1 of the kth departure depend on k, i.e., $a_{ij}(k)$, then prove that (7.8) becomes

$$x_{\mathrm{red}}(k+1) = \begin{pmatrix} a_{21}(k) \oplus a_{13}(k+1) & a_{13}(k+1) \otimes a_{32}(k) \\ a_{21}(k) & a_{32}(k) \end{pmatrix} \otimes x_{\mathrm{red}}(k).$$

7.5 NOTES

Petri nets were the subject of the PhD thesis [78] of C. A. Petri. An excellent overview of the theory can be found in [67]. The chapter in this book on Petri nets is only a minimal introduction to the subject. Many books on the subject have been written. A recent one is [32].

Event graphs are sometimes also referred to as *marked graphs* or *decision-free Petri nets*. Typical examples of applications are the G/G/1 queues, networks of (finite) queues in tandem, kanban systems, flexible manufacturing systems, fork/join queues, or any parallel and/or series composition made by these elements.

More abstract system descriptions, with two-domain formal power series, can be found in [26]. As an example, this description for the Petri net of Figure 7.4 is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \delta^2\gamma & \delta^5 \\ \delta^3\gamma^2 & \delta^3\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where the $\otimes$ symbol in the exponents is suppressed for ease of notation. The exponents of δ refer to the holding times, and the exponents of γ refer to the number of tokens in the corresponding places; for a formal definition see Section 6.3. The operator γ is sometimes called the *shift operator*. The states x_i are the formal power series in γ and δ. With the rules

$$t\gamma^k \oplus \tau\gamma^k = \max(t,\tau)\gamma^k, \qquad k\delta^t \oplus k\delta^\tau = k\delta^{\max(t,\tau)},$$

it is easy to see that the series

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \bigoplus_{k=-\infty}^{+\infty} (\gamma\delta^4)^k \begin{pmatrix} \delta \\ \gamma \end{pmatrix}$$

is a solution to the (implicit) system description given above.

In Chapter 1, a reference was made to system-theoretical concepts. In systems theory one considers input/output relations. In terms of Petri nets, inputs coincide with source transitions and outputs with sink transitions. Many relationships with (conventional) systems theory exist. Well-known concepts in this theory, on stability, feedback, model reference control, and so on, have found corresponding notions within a max-plus setting; see, [29] for such an example, and `www.istia.univ-angers.fr/~hardouin/outils.html` for free downloadable software.