

# Sampling with Halton Points on $n$ -Sphere

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Abstract

Motivation and Applications

Review of Low Discrepancy Sequence

Our approach

Numerical Experiments

Conclusions



# Abstract



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- ▶ Sampling on  $n$ -sphere ( $S^n$ ) has a wide range of applications, such as:
  - ▶ Spherical coding in MIMO wireless communication
  - ▶ Multivariate empirical mode decomposition
  - ▶ Filter bank design
- ▶ We propose a simple yet effective method which:
  - ▶ Utilizes low-discrepancy sequence
  - ▶ Contains only 10 lines of MATLAB code in our implementation!
  - ▶ Allow incremental generation.
- ▶ Numerical results show that the proposed method outperforms the randomly generated sequences and other proposed methods.



# Motivation and Applications



# Problem Formulation

Desirable properties of samples over  $S^n$

- ▶ Uniform
- ▶ Deterministic
- ▶ Incremental
  - ▶ The uniformity measures are optimized with every new point.
  - ▶ Reason: in some applications, it is unknown how many points are needed to solve the problem in advance



# Motivation

- ▶ The topic has been well studied for sphere in 3D, i.e.  $n = 2$
- ▶ Yet it is still unknown how to generate for  $n > 2$ .
- ▶ Potential applications (for  $n > 2$ ):
  - ▶ Robotic Motion Planning ( $S^3$  and  $SO(3)$ ) (Yershova et al. 2010)
  - ▶ Spherical coding in MIMO wireless communication (Utkovski and Lindner 2006):
    - ▶ Cookbook for Unitary matrices
    - ▶ A code word = a point in  $S^n$
  - ▶ Multivariate empirical mode decomposition (Rehman and Mandic 2010)
  - ▶ Filter bank design (Mandic and others 2011)



# Halton Sequence on $S^n$

- ▶ Halton sequence on  $S^2$  has been well studied (Cui and Freeden 1997) by using cylindrical coordinates.
- ▶ Yet it is still little known for  $S^n$  where  $n > 2$ .
- ▶ Note: The generalization of cylindrical coordinates does NOT work in higher dimensions.





# Review of Low Discrepancy Sequence



# Basic: Van der Corput sequence

- ▶ Generate a low discrepancy sequence over  $[0, 1]$
- ▶ Denote  $\text{vd}(k, b)$  as a Van der Corput sequence of  $k$  points, where  $b$  is the base of a prime number.
- ▶ MATLAB source code is available at [here](#)

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$$[x, y] = [\text{vd}(k, 2), \text{vd}(k, 3)]$$

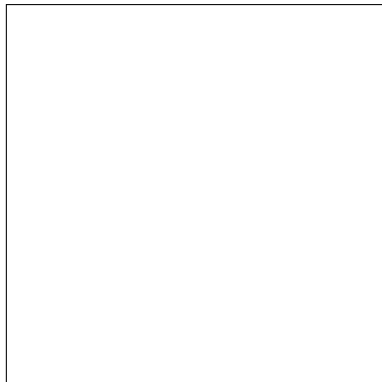


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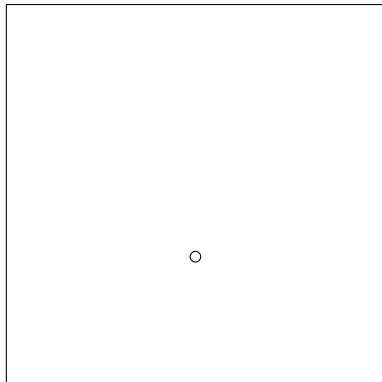


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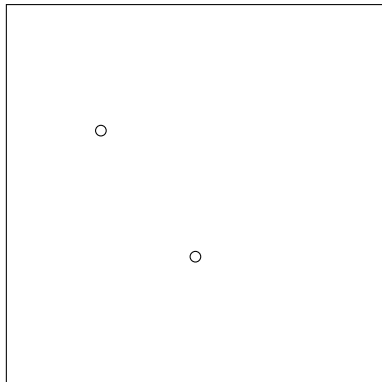


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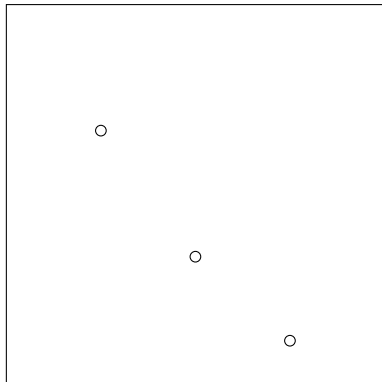


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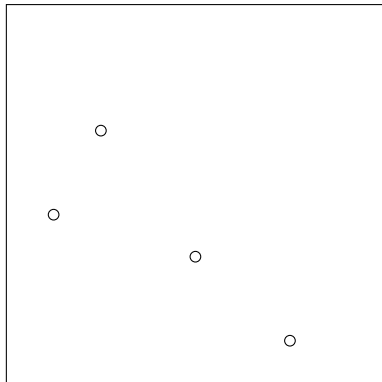


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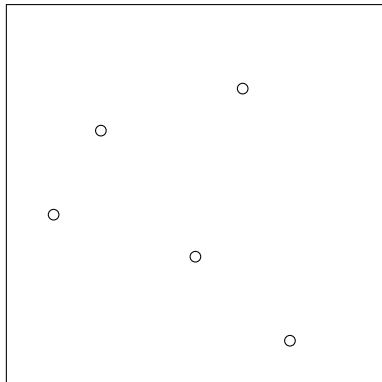


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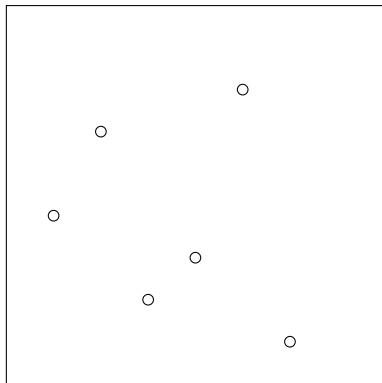


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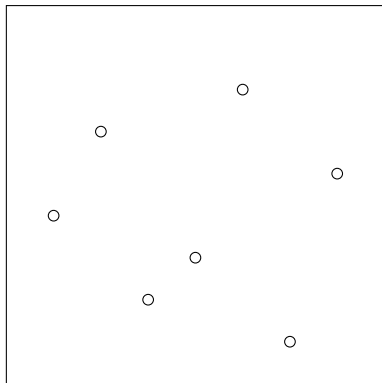


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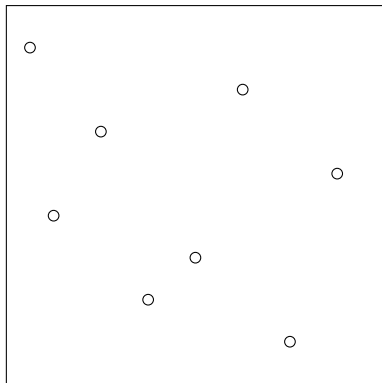


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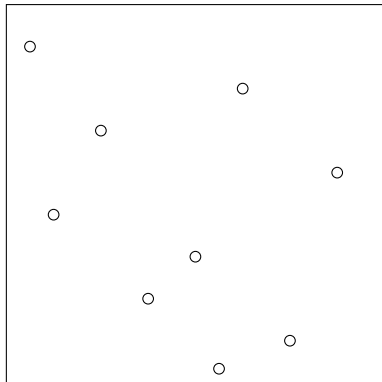


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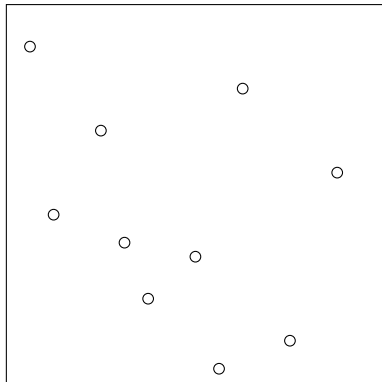


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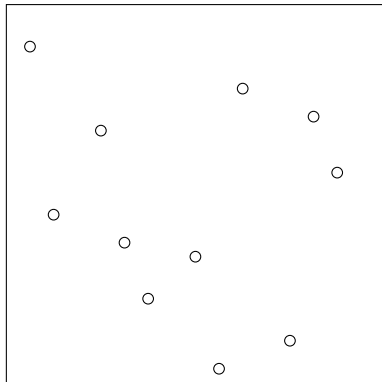


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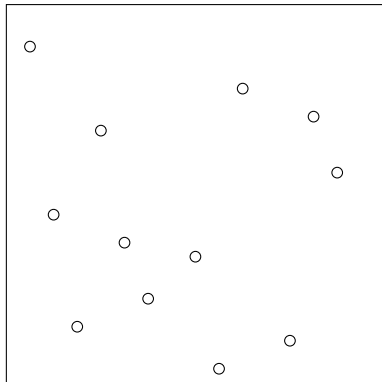


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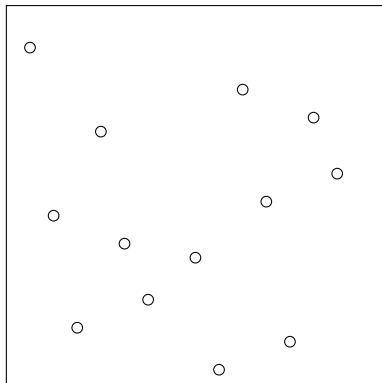


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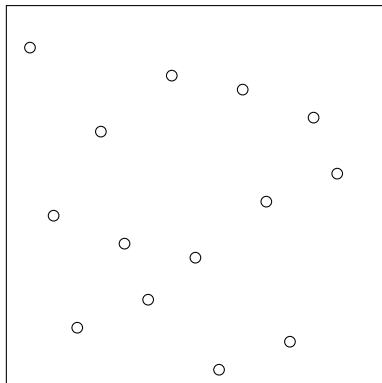


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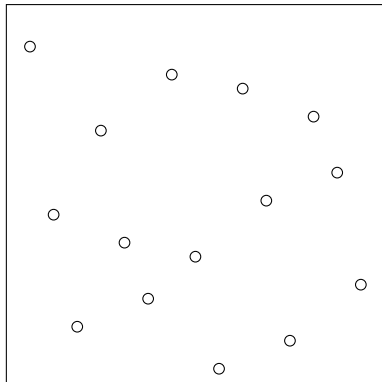


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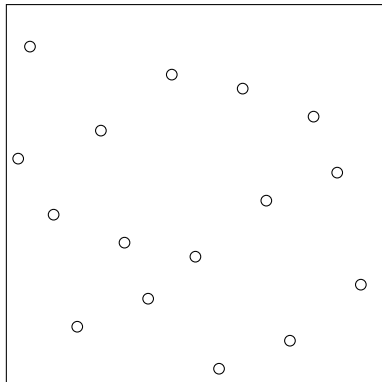


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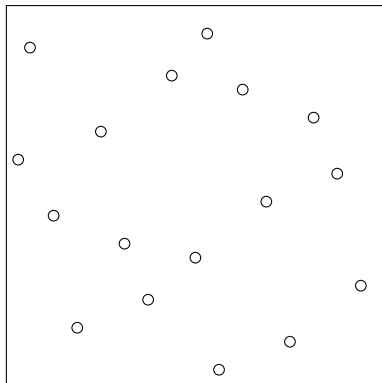


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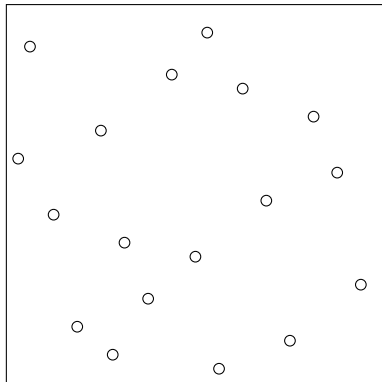


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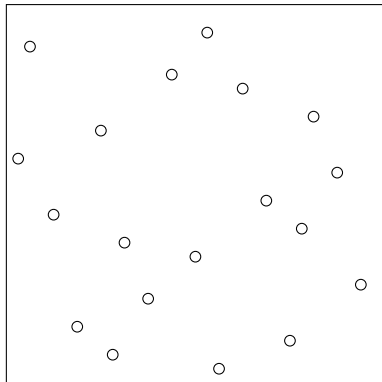


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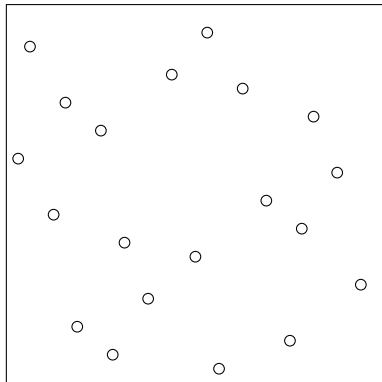


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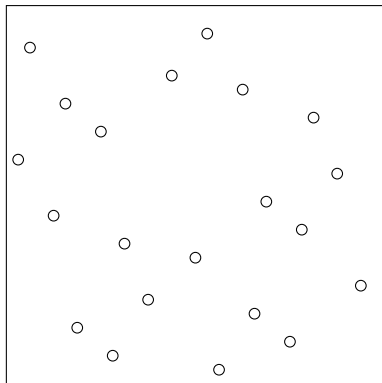


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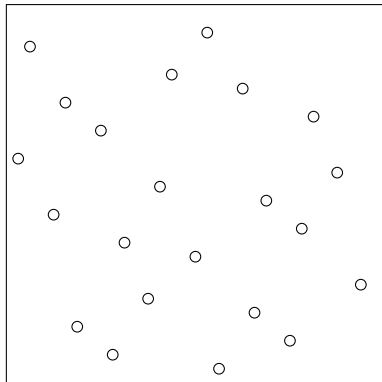


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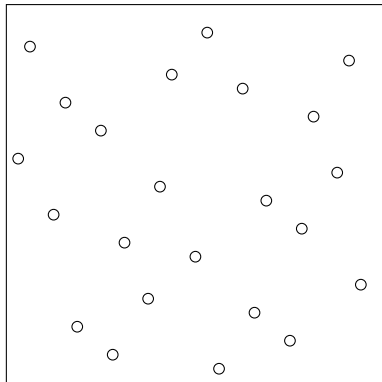


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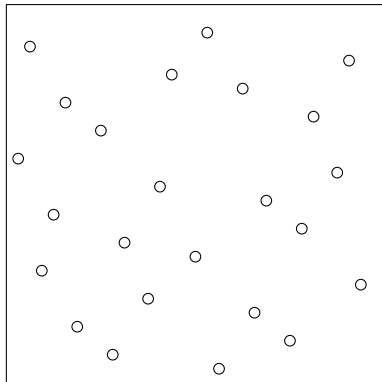


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# Halton sequence on $[0, 1]^n$

- Generally we can generate Halton sequence in a unit hypercube  $[0, 1]^n$ :

$$[x_1, x_2, \dots, x_n] = [\text{vd}(k, b_1), \text{vd}(k, b_2), \dots, \text{vd}(k, b_n)]$$

- A wide range of applications on Quasi-Monte Carlo Methods (QMC).



# Unit Circle $S^1$

Can be generated by mapping the Van der Corput sequence to  $[0, 2\pi]$

- ▶  $\theta = 2\pi \cdot \text{vd}(k, b)$
- ▶  $[x, y] = [\cos \theta, \sin \theta]$

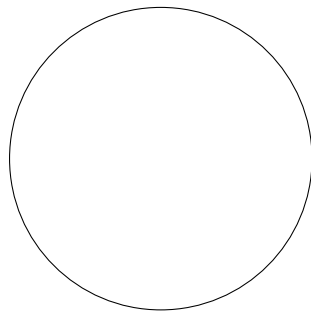


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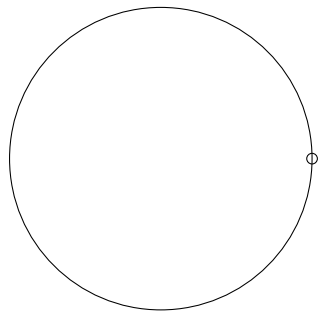


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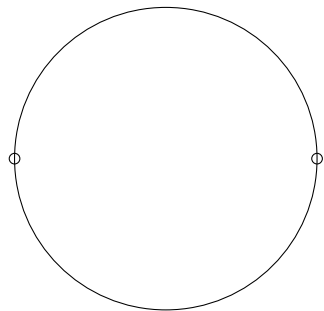


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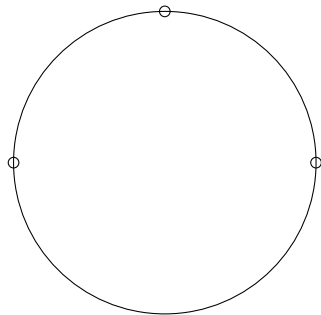


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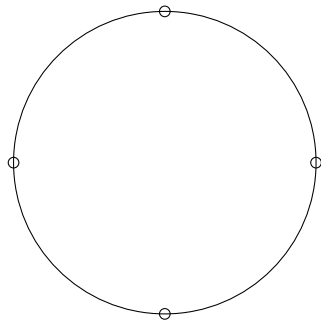


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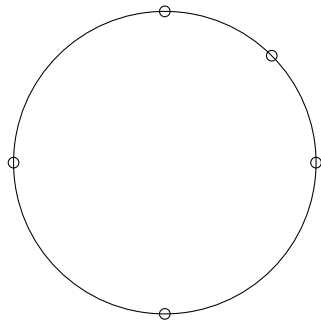


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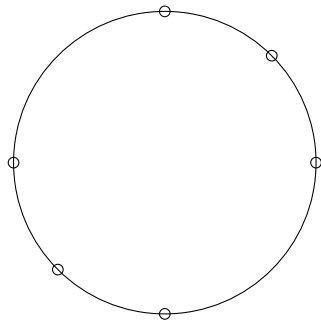


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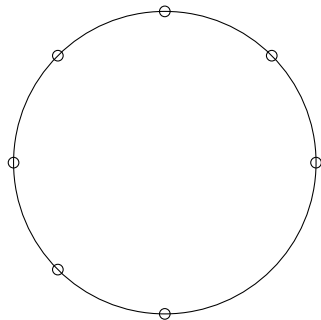


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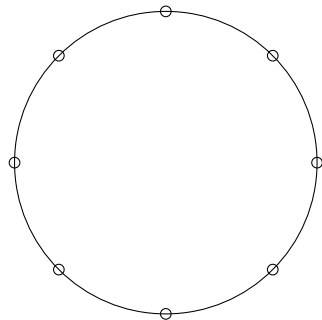


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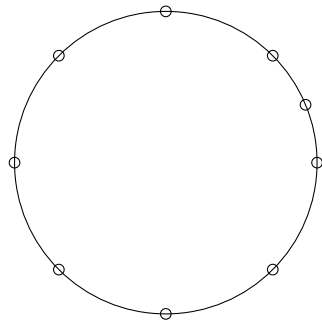


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- ▶  $[x, y] = [\cos \theta, \sin \theta]$

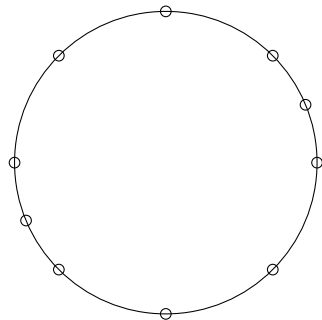


Figure 3: Sequence mapping to a unit circle

# Unit Circle $S^1$

Can be generated by mapping the Van der Corput sequence to  $[0, 2\pi]$

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- ▶  $[x, y] = [\cos \theta, \sin \theta]$

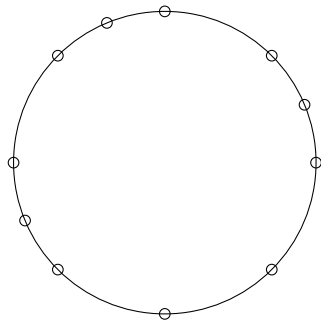


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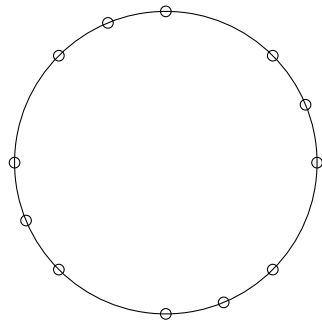


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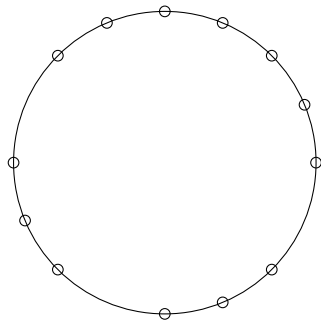


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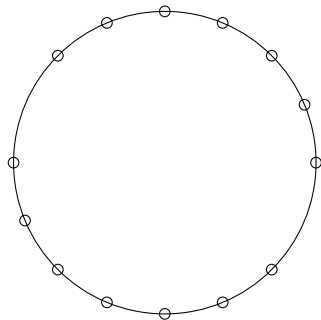


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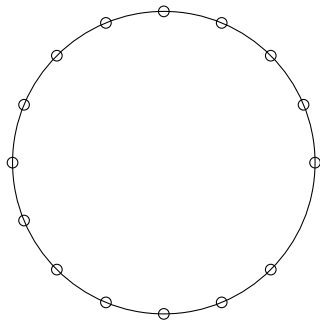


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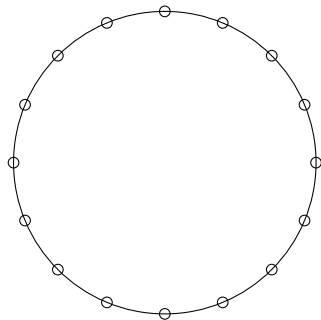


Figure 3: Sequence mapping to a unit circle

# Unit Sphere $S^2$

Has been applied for computer graphic applications (Wong, Luk, and Heng 1997)

- ▶  $[z, x, y]$   
     $= [\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi]$   
     $= [z, \sqrt{1 - z^2} \cos \varphi, \sqrt{1 - z^2} \sin \varphi]$
- ▶  $\varphi = 2\pi \cdot \text{vd}(k, b_1) \text{ \% map to } [0, 2\pi]$
- ▶  $z = 2 \cdot \text{vd}(k, b_2) - 1 \text{ \% map to } [-1, 1]$

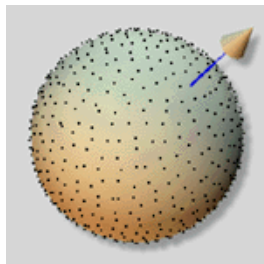


Figure 4: image

# Sphere $S^n$ and $\mathrm{SO}(3)$

- ▶ Deterministic point sets
  - ▶ Optimal grid point sets for  $S^3$ ,  $\mathrm{SO}(3)$  [Lubotzky, Phillips, Sarnak 86] [Mitchell 07]
- ▶ No Halton sequences so far to the best of our knowledge



## Our approach



# $SO(3)$ or $S^3$ Hopf Coordinates

- ▶ Hopf coordinates (cf. (Yershova et al. 2010))
  - ▶  $x_1 = \cos(\theta/2) \cos(\psi/2)$
  - ▶  $x_2 = \cos(\theta/2) \sin(\psi/2)$
  - ▶  $x_3 = \sin(\theta/2) \cos(\varphi + \psi/2)$
  - ▶  $x_4 = \sin(\theta/2) \sin(\varphi + \psi/2)$
- ▶  $S^3$  is a principal circle bundle over the  $S^2$



Figure 5: image



# Hopf Coordinates for $\text{SO}(3)$ or $S^3$

Similar to the Halton sequence generation on  $S^2$ , we perform the mapping:

- ▶  $\varphi = 2\pi \cdot \text{vd}(k, b_1) \% \text{ map to } [0, 2\pi]$
- ▶  $\psi = 2\pi \cdot \text{vd}(k, b_2) \% \text{ map to } [0, 2\pi] \text{ for } \text{SO}(3), \text{ or}$
- ▶  $\psi = 4\pi \cdot \text{vd}(k, b_2) \% \text{ map to } [0, 4\pi] \text{ for } S^3$
- ▶  $z = 2 \cdot \text{vd}(k, b_3) - 1 \% \text{ map to } [-1, 1]$
- ▶  $\theta = \cos^{-1} z$



# Python Code

```
def sphere3_hopf(k, b):  
    vd = zip(vdc(k, b[0]), vdc(k, b[1]), vd(k, b[2]))  
    for vd0, vd1, vd2 in vd:  
        phi = 2*math.pi*vd0    # map to [0, 2*math.pi]  
        psy = 4*math.pi*vd1    # map to [0, 4*math.pi]  
        z = 2*vd2 - 1          # map to [-1., 1.]  
        theta = math.acos(z)  
        cos_eta = math.cos(theta/2)  
        sin_eta = math.sin(theta/2)  
        s = [cos_eta * math.cos(psy/2),  
             cos_eta * math.sin(psy/2),  
             sin_eta * math.cos(phi + psy/2),  
             sin_eta * math.sin(phi + psy/2)]  
    yield s
```



# 3-sphere

- ▶ Polar coordinates:
  - ▶  $x_0 = \cos \theta_3$
  - ▶  $x_1 = \sin \theta_3 \cos \theta_2$
  - ▶  $x_2 = \sin \theta_3 \sin \theta_2 \cos \theta_1$
  - ▶  $x_3 = \sin \theta_3 \sin \theta_2 \sin \theta_1$



# n-sphere

- ▶ Polar coordinates:

- ▶  $x_0 = \cos \theta_n$

- ▶  $x_1 = \sin \theta_n \cos \theta_{n-1}$

- ▶  $x_2 = \sin \theta_n \sin \theta_{n-1} \cos \theta_{n-2}$

- ▶  $x_3 = \sin \theta_n \sin \theta_{n-1} \sin \theta_{n-2} \cos \theta_{n-3}$

- ▶  $\dots$

- ▶  $x_{n-1} = \sin \theta_n \sin \theta_{n-1} \sin \theta_{n-2} \cdots \cos \theta_1$

- ▶  $x_n = \sin \theta_n \sin \theta_{n-1} \sin \theta_{n-2} \cdots \sin \theta_1$

# How to Generate the Point Set

- ▶  $p_0 = [\cos \theta_1, \sin \theta_1]$  where  $\theta_1 = 2\pi \cdot \text{vd}(k, b_1)$
- ▶ Let  $f_j(\theta) = \int \sin^j \theta d\theta$ , where  $\theta \in (0, \pi)$ .  
Note:  $f_j(\theta)$  is a monotonic increasing function in  $(0, \pi)$
- ▶ Map  $\text{vd}(k, b_j)$  uniformly to  $f_j(\theta)$ :  
$$t_j = f_j(0) + (f_j(\pi) - f_j(0))\text{vd}(k, b_j)$$
- ▶ Let  $\theta_j = f_j^{-1}(t_j)$
- ▶ Define  $p_n$  recursively as:  
$$p_n = [\cos \theta_n, \sin \theta_n \cdot p_{n-1}]$$



$S^3$  projected on four different spheres

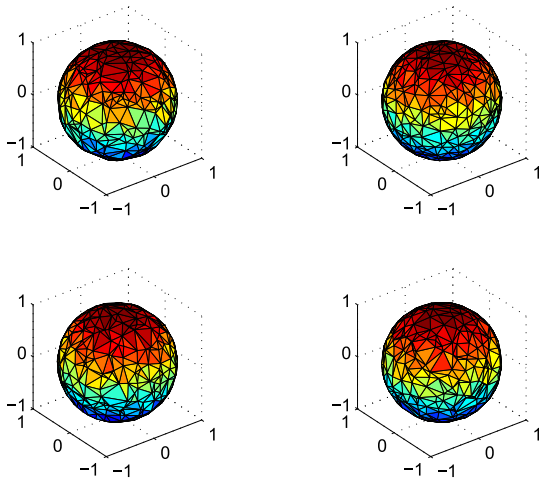


Figure 6: image



# Numerical Experiments



# Testing the Correctness

- ▶ Compare the dispersion with the random point-set
  - ▶ Construct the convex hull for each point-set
  - ▶ Dispersion roughly measured by the difference of the maximum distance and the minimum distance between every two neighbour points:

$$\max_{a \in \mathcal{N}(b)} \{D(a, b)\} - \min_{a \in \mathcal{N}(b)} \{D(a, b)\}$$

where  $D(a, b) = \sqrt{1 - a^T b}$





# Random sequences

- ▶ To generate random points on  $S^n$ , spherical symmetry of the multidimensional Gaussian density function can be exploited.
- ▶ Then the normalized vector  $(x_i/\|x_i\|)$  is uniformly distributed over the hypersphere  $S^n$ . [Fishman, G. F. (1996)]



## Convex Hull with $\approx 400$ points

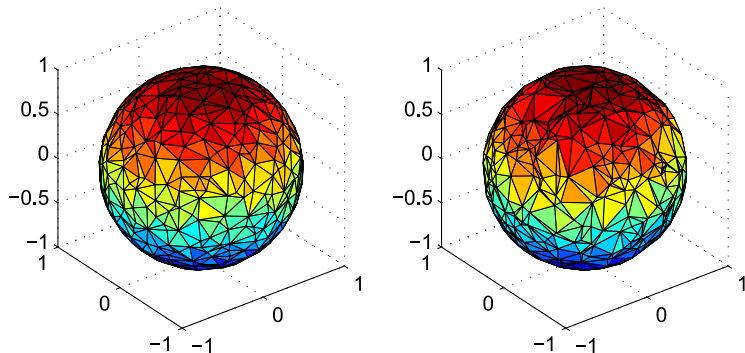


Figure 7: image

Left: our, right: random

# Result for $S^3$

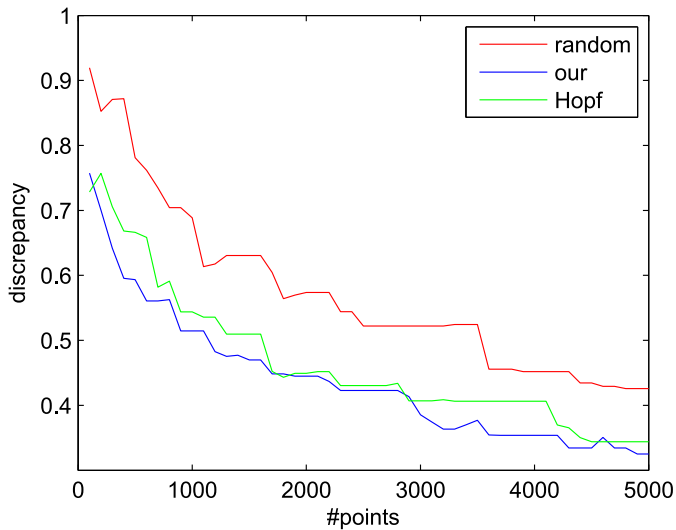


Figure 8: image



# Result for $S^4$

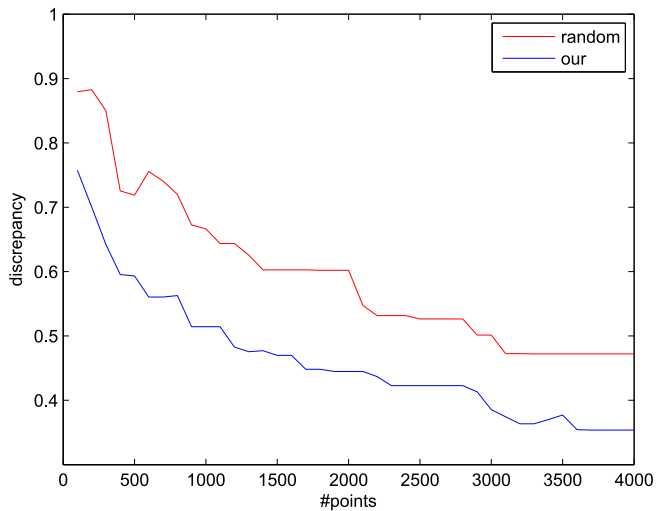


Figure 9: image



# Conclusions



# Conclusions

- ▶ Proposed method generates low-discrepancy point-set in nearly linear time
- ▶ The result outperforms the corresponding random point-set, especially when the number of points is small
- ▶ The MATLAB source code is available in public (or upon request)



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