Geometry, Algebra and Computation

Wai-Shing Luk

Fudan University

2016-11-06

Introduction

Geometry and Algebra

- Geometry
  - Points, lines, triangles, circles, conic sections...
  - Collinear, concurrent, parallel, perpendicular...
  - Distances, angles, areas, quadrance, spread, quadrea...
  - Midpoint, bisector, orthocenter, pole/polar, tangent...
- Algebra
  - Addition, multiplication, inverse...
  - Elementry algebra: integer/rational/real/complex... numbers.
  - Abstract algebra: rings, fields...
  - Linear algebra: vector, matrix, determinant, dot/cross product...
- Two subjects are related by coordinates.

Key points

- Our earth is not flat and our universe is non-Euclidean.
- Non-Euclidean geometry is much easier to learn than you might think.
- Our curriculum in school is completely wrong.
- Euclidean geometry is non-symmetric. Three sides determines a triangle, but three angles does not determines a triangle. It might not be true in general geometries. Euclidean geometry is just a special case.
- Yet Euclidean geometry is more computational efficient and is still useful in our small-scale daily life.
- Incidenceship promotes integer arithmetic; non-oriented measurement promotes rational aritmetric; oriented measurement promotes floating-point arithmetric. Don't kill a chicken with cow knife.

Projective Geometry

Projective Plane's Basic Elements

Projective Plane Concept

- Only involve "Points" $\mathbf{P}$ and "Lines" $\mathbf{L}$.
- "Points" (or lines) are assumed to be distinguishable (`Equality-Comparable`).
- Unless mention specifically, objects in different names are assumed to be distinct, i.e. $A \neq B$.

Incidence

- A point either lies on a line, or not.
- If a point A lies on a line a, then $a \cdot A$ is true.
- For convenience, we also define $A \cdot a$ such that $A \cdot a = a \cdot A$
- In C++, define a boolean function `incident(P,L)->bool`.

Projective Point

- Exactly one line passes through two distinct points.
- In C++, define a function `L(P,P) -> L` that returns a line joined by two points.
- Denote join(A,B), or simply AB as a line joined by A and B.
- We have:
  - $AB = BA$
  - $A \cdot AB$ and $B \cdot AB$ are always true.

Projective Line

- Exactly one point met by two distinct lines.
- In C++, define a function `P(L,L) -> P` that returns a point met by two lines.
- Denote meet(a,b), or simply ab as a point met by a and b.
- We have:
  - $ab = ba$
  - $a \cdot ab$ and $b \cdot ab$ are always true.

Relations with other Geometries:

- In Euclidean Geometry, parallel lines are met at points in *infinity*.
- In Hyperbolic Geometry, parallel lines are met at points outside the *null circle*.
- There is no parallel line in Elliptic/Spherical Geometry.

Example 1: Euclidean Geometry

- Point: projection of 3D vector $[x, y, z]$ to 2D plane $z = 1$:

$$(x', y') = (x/z, y/z)$$

- Vector $[x, y, z]$ and $[\lambda x, \lambda y, \lambda z]$ for all $\lambda \neq 0$ are representing the same point (x', y').
- For instance, $[1, 5, 6]$ and $[10, 50, 60]$ are representing the same point $(1/6, 5/6)$
- $[p] = [x, y, 0]$ is a point at *infinity*.
- Line: $a \cdot x' + b \cdot y' + c = 0$, denoted by a vector $[a, b, c]$.
- $[a, b, c]$ and $[\lambda a, \lambda b, \lambda c]$ for all $\lambda \neq 0$ are representing the same line.
- $[l] = [0, 0, 1]$ is the line at *infinity*.
- $[0, 0, 0]$ is not a valid point or line.

Calculation by Vector Operations

- Let $v_1 = [x_1, y_1, z_1]$ and $v_2 = [x_2, y_2, z_2]$.
- The dot product $v_1 \cdot v_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$.
- The cross product $v_1 \times v_2 = [y_1 z_2 - z_1 y_2, -x_1 z_2 + z_1 x_2, x_1 y_2 - y_1 x_2]$.
- A point p lies on a line l if and only if $[p] \cdot [l] = 0$
- join(p_1, p_2) $= [p_1] \times [p_2]$.
- meet(l_1, l_2) $= [l_1] \times [l_2]$.
- Two lines are parallel if and only if $a_1 b_2 = b_1 a_2$
- Exercise: calculate the line equation that joins the points $(5/8, 7/8)$ and $(-5/6, 1/6)$.

Example 2: Spherical/Elliptic Geometry

- It turns out that the vector notations and operators can also represent other geometries such as spherical/Elliptic geometry.
- "Point": projection of 3D vector $[x, y, z]$ to the unit sphere.

$$(x', y', z') = (x/r, y/r, z/r)$$

  where $r^2 = x^2 + y^2 + z^2$.
- Two points on the opposite poles are considered the same point here.
- "Line": $[a, b, c]$ represents the *great circle* intersected by the unit sphere and the plane $a \cdot x + b \cdot y + c \cdot z = 0$.
- $[x, y, z]$ is called *Homogeneous Coordinates*.

Example 3: Poker Card Geometry

- "coordinates" is **not** a necessary requirement of projective geometry.
- A "line" may contain finite number of "points".
- Consider the poker cards in the following arrangement:

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}
A	2	3	4	5	6	7	8	9	10	J	Q	K
2	3	4	5	6	7	8	9	10	J	Q	K	A
4	5	6	7	8	9	10	J	Q	K	A	2	3
10	J	Q	K	A	2	3	4	5	6	7	8	9

- For example, meet(l_2, l_5) $=$ 5, join(J, 4) $= l_8$.
- We call this *Poker Card Geometry* here.

Concept in C++17

```
template <class P>
concept bool ProjectivePlaneH =
    Equality_comparable<P> &&
    requires { typename P::dual; } &&
    requires (P p, P q, typename P::dual l) {
        { incident(p, l) } -> bool;  // incidence
        { (p, q) } -> typename P::dual;  // join or meet
    };

template <class P, class L=typename P::dual>
concept bool ProjectivePlane =
    ProjectivePlaneH<P> && ProjectivePlaneH<L>;
```

Collinear, Concurrent and Coincidence

- Three points are called *collinear* if they all lie on the same line.
- Three lines are called *concurrent* if they all meet at the same point.
- In C++, define a boolean function `coI(P,P,P)->bool`.
- col(A, B, C) is true if and only if $AB \cdot C$ is true.
- Similarly, col(a, b, c) is true if and only if $ab \cdot c$ is true.
- In general, col($\{A_1, A_2, ..., A_n\}$) is true if and only if $A_1A_2 \cdot X$ is true for all X in the rest of points $\{A_3, A_4, ..., A_n\}$.
- Unless mention specifically, $ABCD...$ is assumed to be coincidence.

Pappus Theorem

- Given A, B, C and D, E, F are collinear. Let G=meet(AE, BD), H=meet(AF, CD), I=meet(BF, CE). Then G, H, I are collinear.
- Note: this theorem is only true for real projective geometry.
- Exercise: verify that this theorem holds for the poker card geometry.

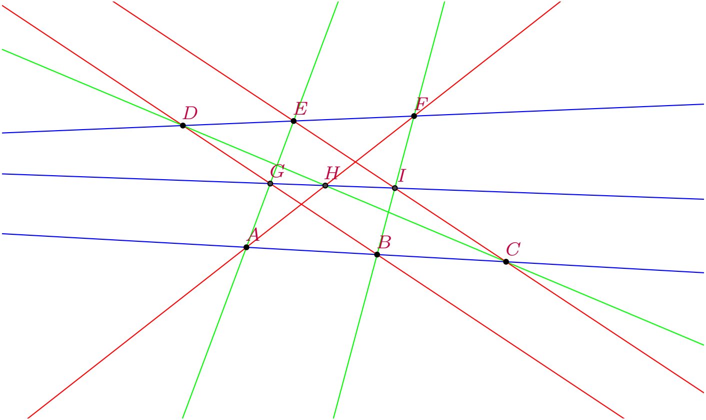

Figure 1: An instance that Pappus' theorem holds

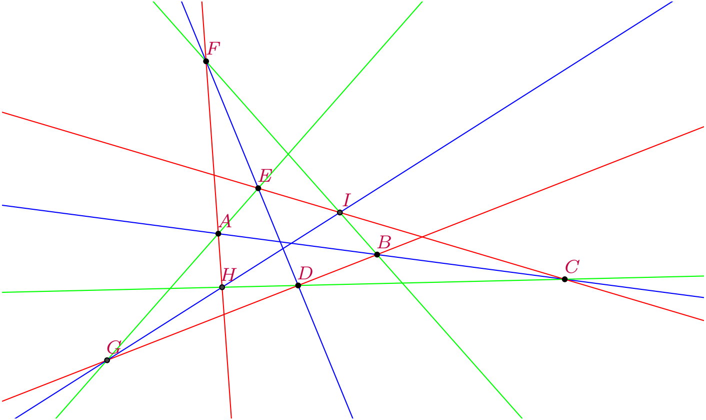

Figure 2: Another instance that Pappus' theorem holds

Triangles and Trilaterals

- If three points A, B, and C are not collinear, they form a triangle, denoted as $\{ABC\}$.
- If three lines a, b, and c are not concurrent, they form a trilateral, denoted as $\{abc\}$.
- Triangle $\{ABC\}$ and trilateral $\{abc\}$ are dual if $A = bc$, $B = ac$ and $C = ab$.

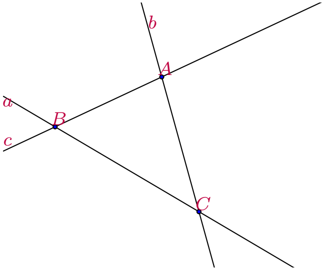

Figure 3: Triangle

Projectivities and Perspectivities

Projectivities

- An ordered set (A, B, C) (either collinear or not) is called a projective of a concurrent set abc if and only if $A \cdot a$, $B \cdot b$ and $C \cdot c$.
- Denote Projectivity as $(A, B, C) \barwedge abc$.
- An ordered set (a, b, c) (either concurrent or not) is called a projective of a collinear set ABC if and only if $A \cdot a$, $B \cdot b$ and $C \cdot c$.
- Denote Projectivity as $(a, b, c) \barwedge ABC$.
- If each ordered set is coincident, we may write:
  - $ABC \barwedge abc \barwedge A'B'C'$
  - Or simply $ABC \barwedge A'B'C'$

Perspectivities

- An ordered set (A, B, C) is called a perspectivity of an ordered set (A', B', C') if and only if $(A, B, C) \barwedge abc$ and $(A', B', C') \barwedge abc$ for some concurrent set abc.
- Denote Perspectivity as $(A, B, C) \doublebarwedge (A', B', C')$.
- An ordered set (a, b, c) is called a perspectivity of an ordered set (a', b', c') if and only if $(a, b, c) \barwedge ABC$ and $(a', b', c') \barwedge ABC$ for some collinear set ABC.
- Denote Perspectivity as $(a, b, c) \doublebarwedge (a', b', c')$.

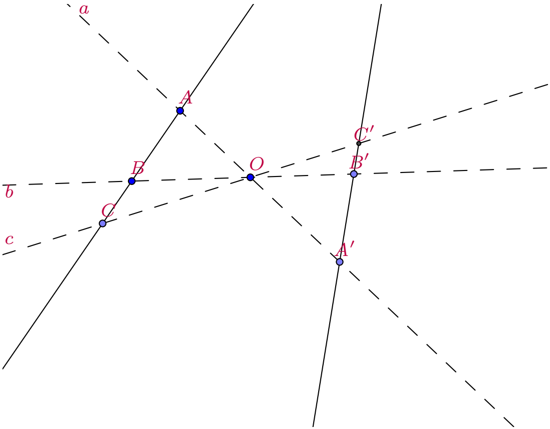

Figure 4: An instance of perspectivity

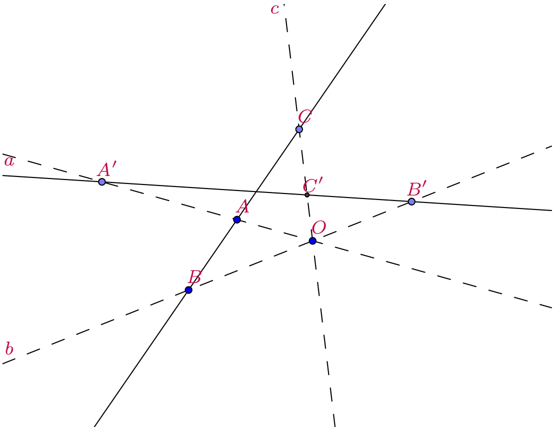

Figure 5: Another instance of perspectivity

Perspectivity

- Similar definition for more than three points:
  - $(A_1, A_2, A_3, ...A_n) \overset{=}{\barwedge} (A'_1, A'_2, A'_3, ...A'_n)$.
- To check perspectivity:
  - First construct a point $O :=$ meet$(A_1A'_1, A_2A'_2)$.
  - For the rest of points, check if X, X', O are collinear.
- Note that $(A, B, C) \overset{=}{\barwedge} (D, E, F)$ and $(D, E, F) \overset{=}{\barwedge} (G, H, I)$ does not imply $(A, B, C) \overset{=}{\barwedge} (G, H, I)$.

Desargues's Theorem

- Let trilateral $\{abc\}$ be the dual of triangle $\{ABC\}$ and trilateral $\{a'b'c'\}$ be the dual of triangle $\{A'B'C'\}$. Then, $\{ABC\} \overset{=}{\wedge} \{A'B'C'\}$ if and only if $\{abc\} \overset{=}{\wedge} \{a'b'c'\}$.
- Note: this theorem is only true for real projective geometry.

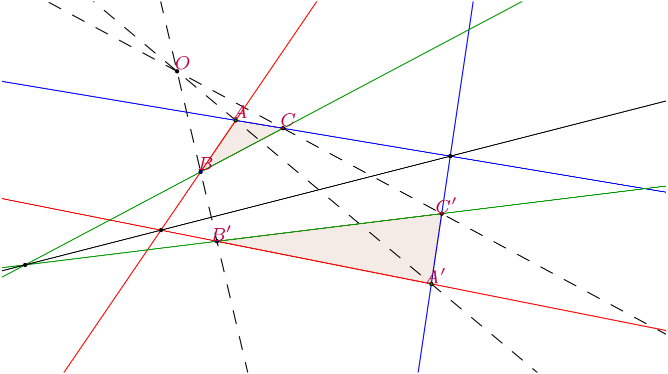

Figure 6: An instance that Desargues theorem holds

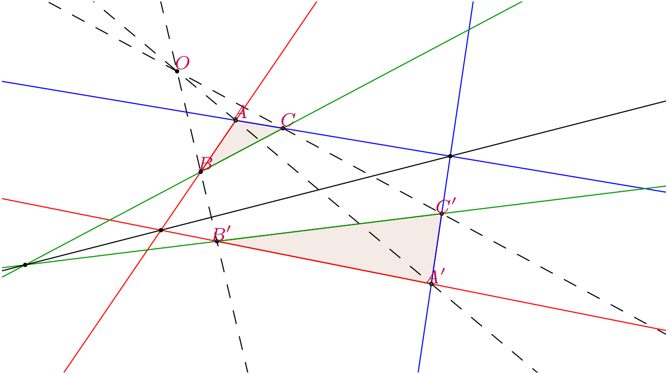

Figure 7: Another instance that Desargues theorem holds

Quadrangles

- If four points P, Q, R and S none of three are collinear, they forms a quadrangle, denoted as $\{PQRS\}$.
- Totally there are six lines formed by $\{PQRS\}$.
- Note that Quadrangle here does not needed to be convex.

Quadrilateral Sets

- Assume they meet another line l at A, B, C, D, E, F, where
  - $A =$ meet(PQ, l), $D =$ meet(RS, l)
  - $B =$ meet(QR, l), $E =$ meet(PS, l)
  - $C =$ mett(PR, l), $F =$ meet(QS, l)
- We call the six points as a quadrilateral set, denoted as $(AD)(BE)(CF)$.

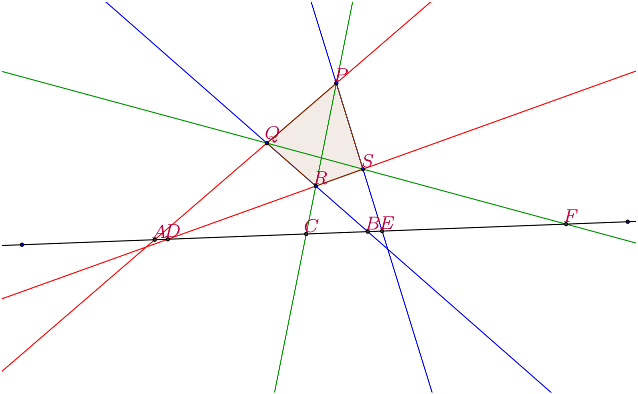

Figure 8: quadrilateral set

Harmonic Sets

- In a quadrilateral set $(AD)(BE)(CF)$, if $A = D$ and $B = E$, then it is called a harmonic set.
- The Harmonic relation is denoted by $H(AB, CF)$.
- Then C and F is called a harmonic conjugate.
- Theorem: If $ABCF \doublebarwedge A'B'C'F'$, then $H(AB, CF) = H(A'B', C'F')$.
- In other words, perspectivity preserves harmonic relation.
- Theorem: If $ABCF \barwedge A'B'C'F'$, then $H(AB, CF) = H(A'B', C'F')$.
- In other words, projectivity preserves harmonic relation.

Polarities

- A *polarity* is a projective correlation of period 2.
- We call a the *polar* of A, and A the pole of a.
- Denote $a = A^{\perp}$ and $A = a^{\perp}$.
- Hence, $A = (A^{\perp})^{\perp}$ and $a = (a^{\perp})^{\perp}$.
- It may happen that A and a are incident, so that each is *self-conjugate*.

The Use of a Self-Polar Triangle

- Any projective correlation that relates the three vertices of one triangle to the respectively opposite sides is a polarity.
- Thus, any triangle $\{ABC\}$, any point P not on a side, and any line p not throughout a vertex, determine a definite polarity $(ABC)(Pp)$.

The Conic

- Historically *ellipse* (including *circle*), *parabola*, and *hyperbola*.
- The locus of self-conjugate points is a *conic*.
- Their polars are its *tangents*.
- Any other line is called a *secant* or a *nonsecant* according as it meets the conic twice or not at all, i.e., according as the involution of conjugate points on it is hyperbolic or elliptic.

Construct the polar of a point

- To construct the polar of a given point C, not on the conic, draw any two secants PQ and RS through C; then the polar joins the two points meet(QR, PS) and meet(RP, QS).

Figure 9: Construct the polar of a point

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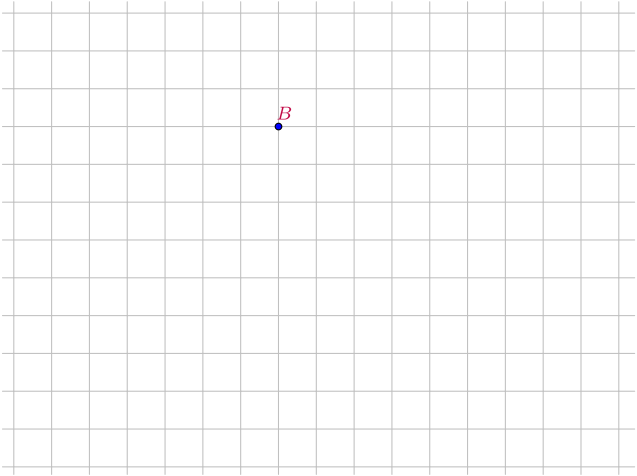

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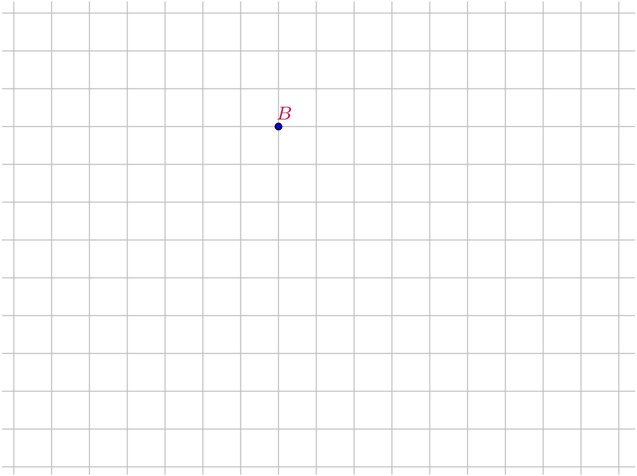

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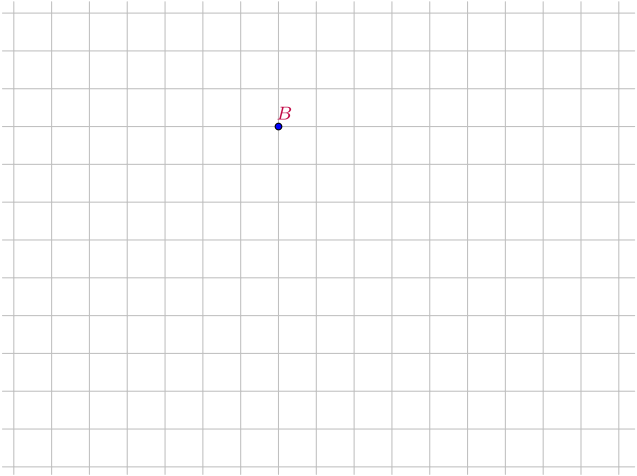

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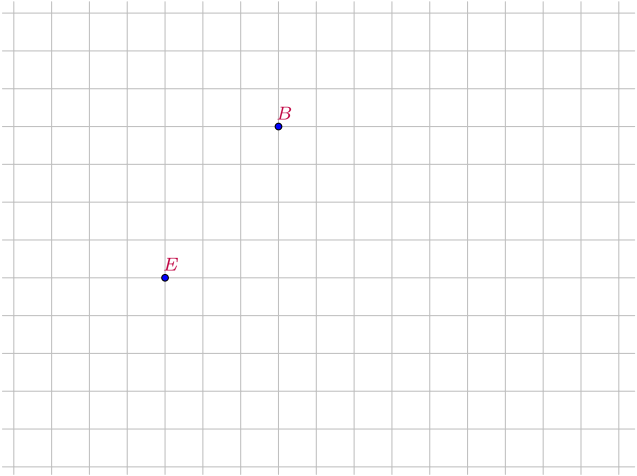

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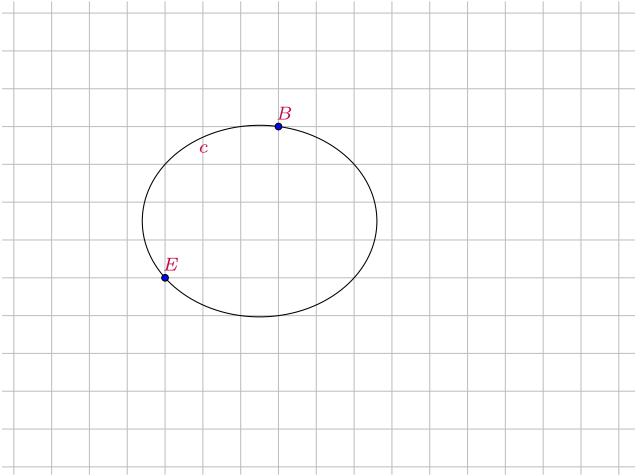

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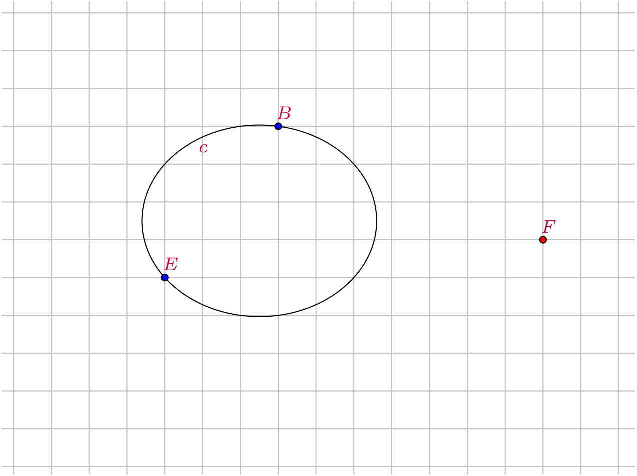

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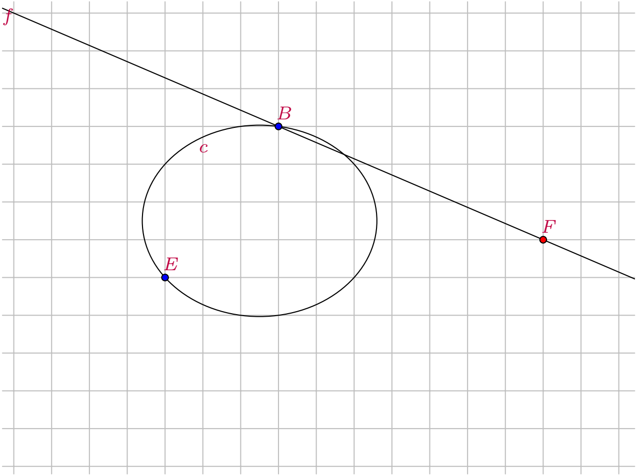

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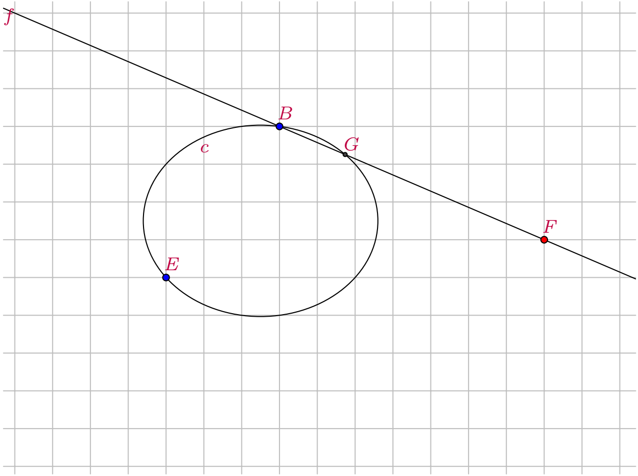

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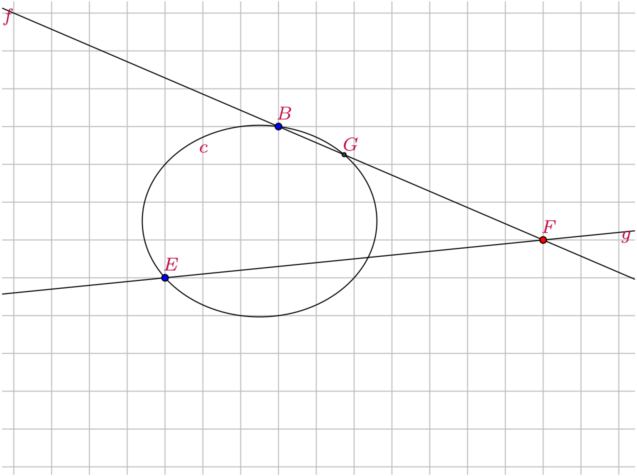

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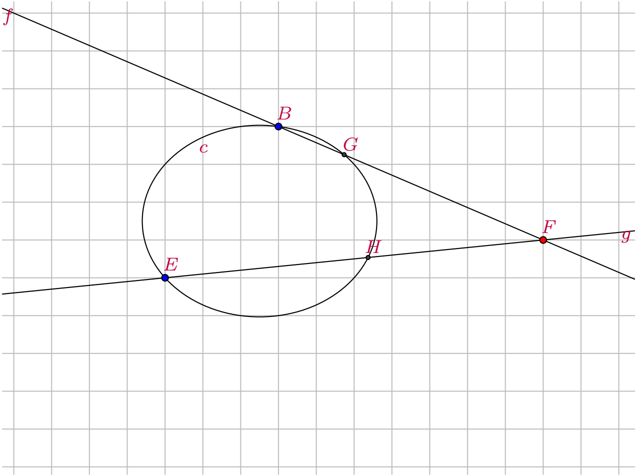

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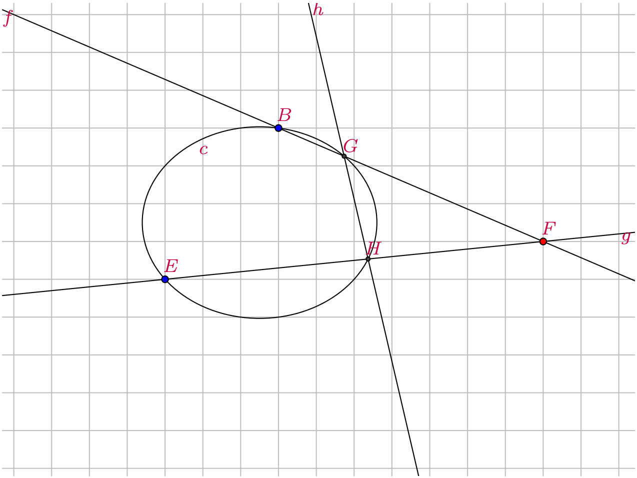

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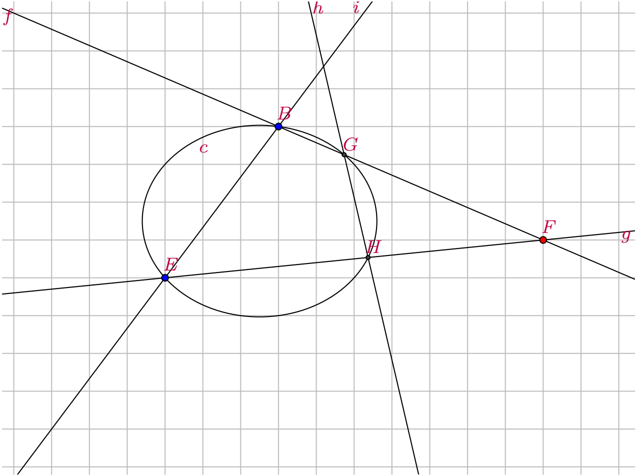

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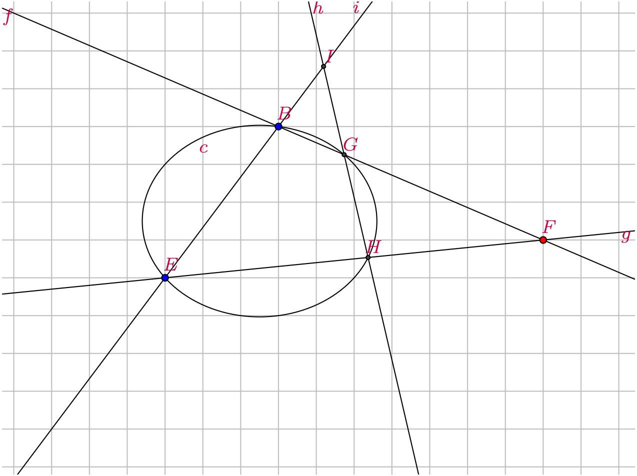

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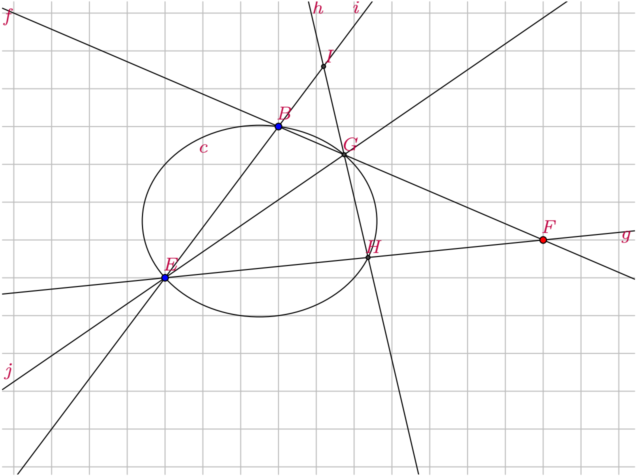

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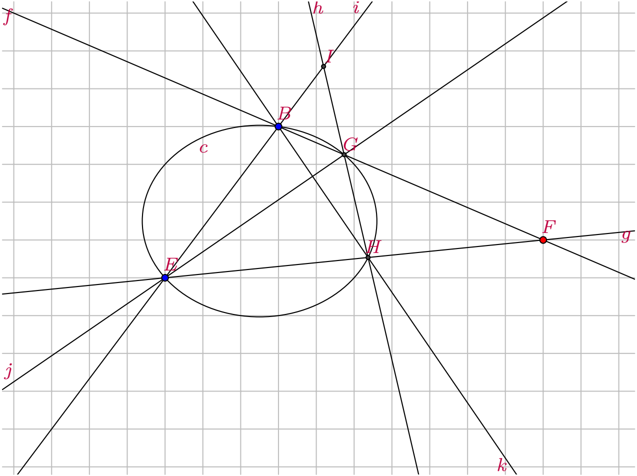

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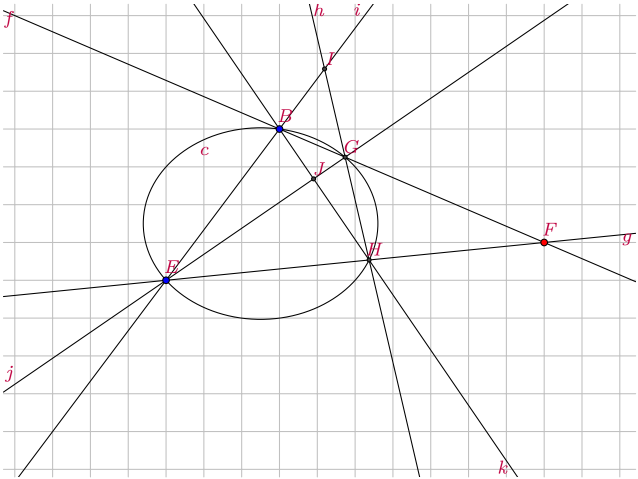

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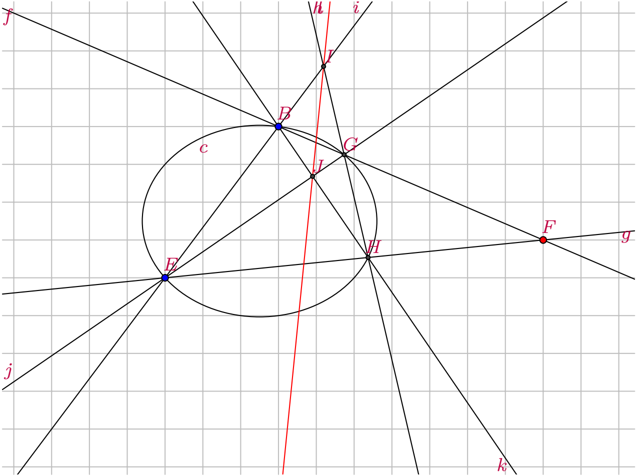

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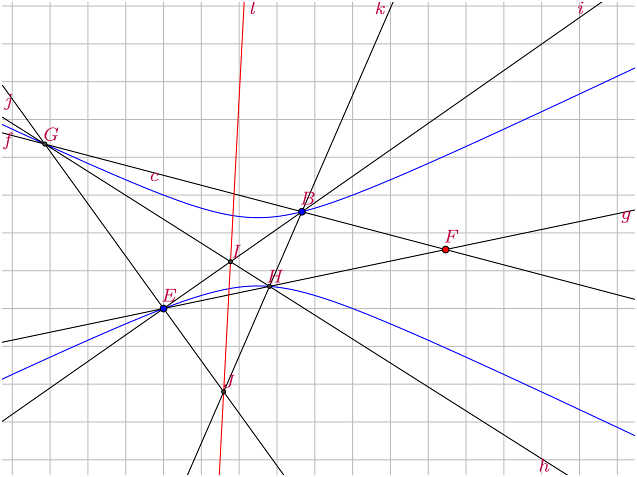

Figure 10: Another example of constructing the polar of a point

Construct the pole from a line

- To construct the pole of a given secant a, draw the polars of any two points on the line; then the common point of two polars is the pole of a.

Figure 11: Constructing the pole of a line

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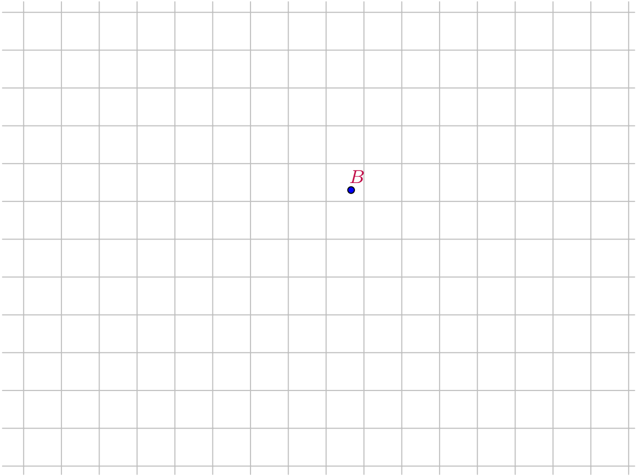

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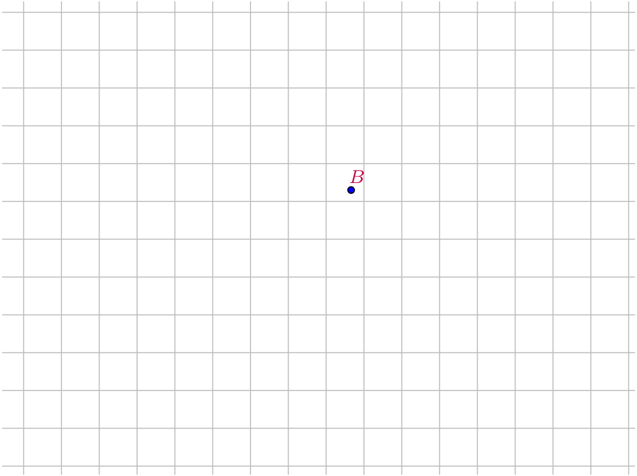

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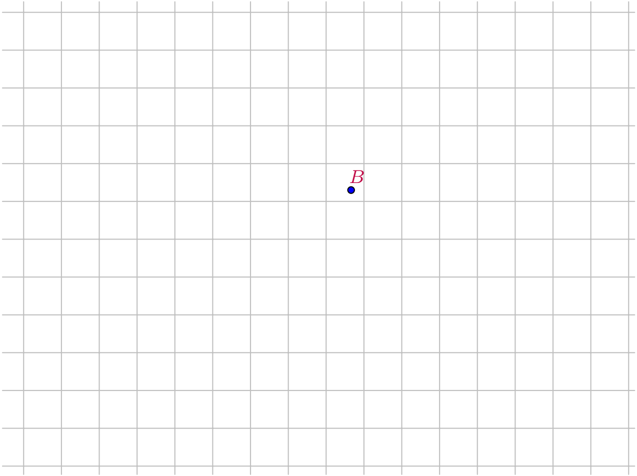

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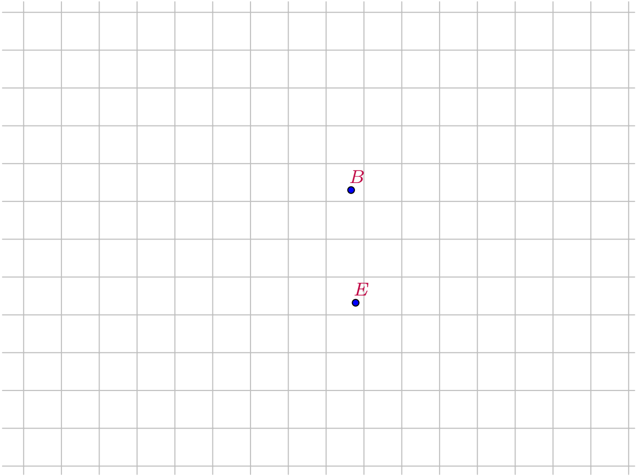

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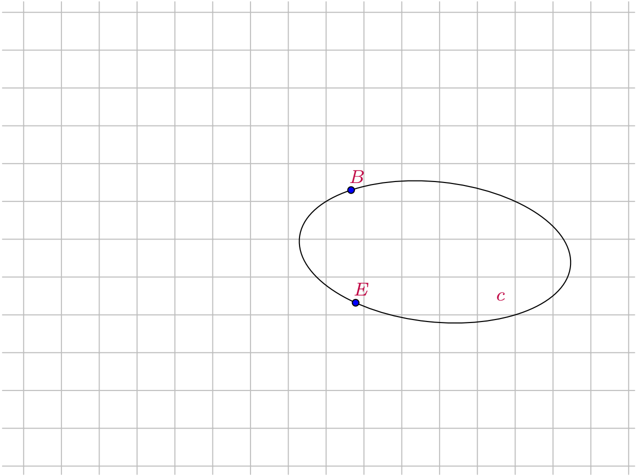

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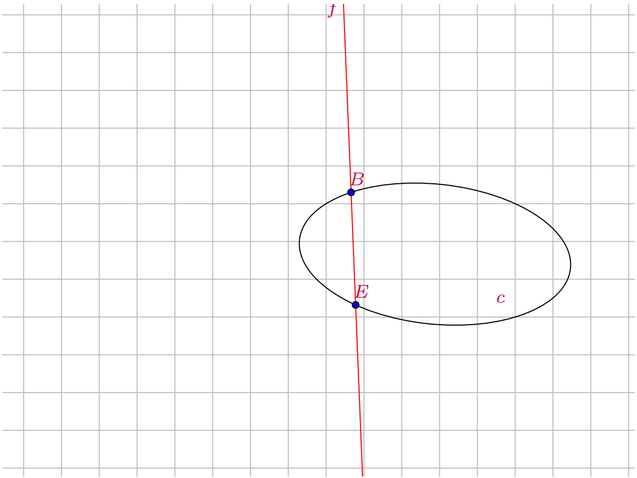

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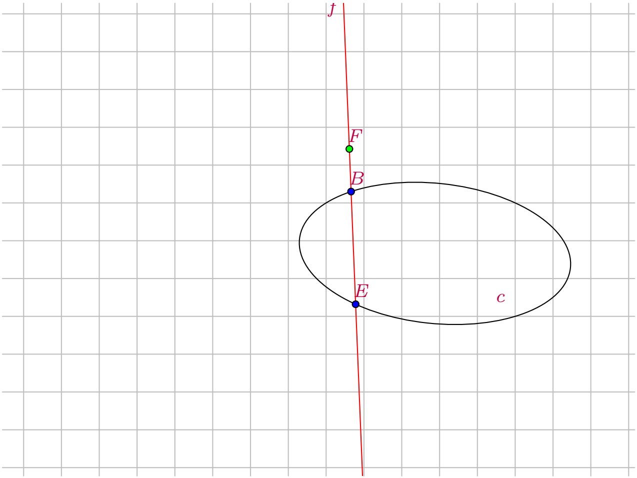

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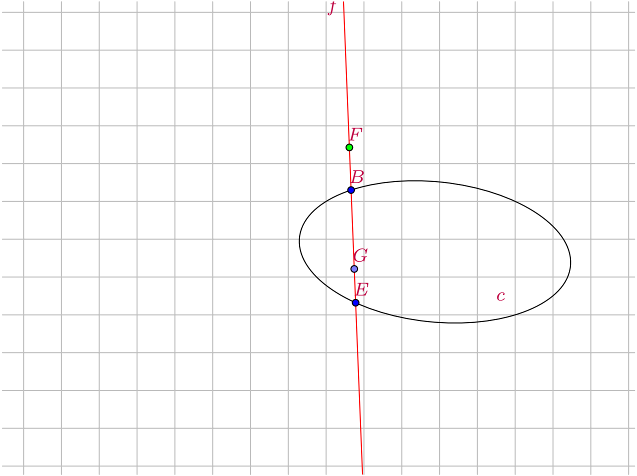

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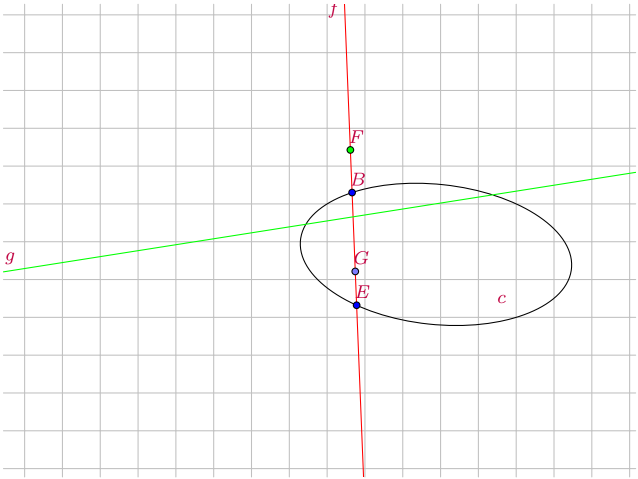

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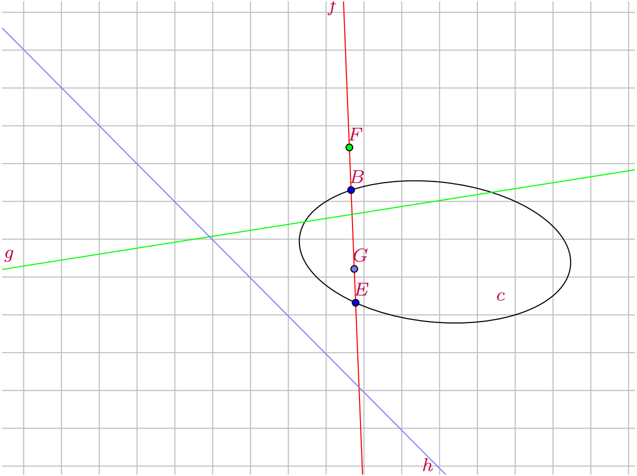

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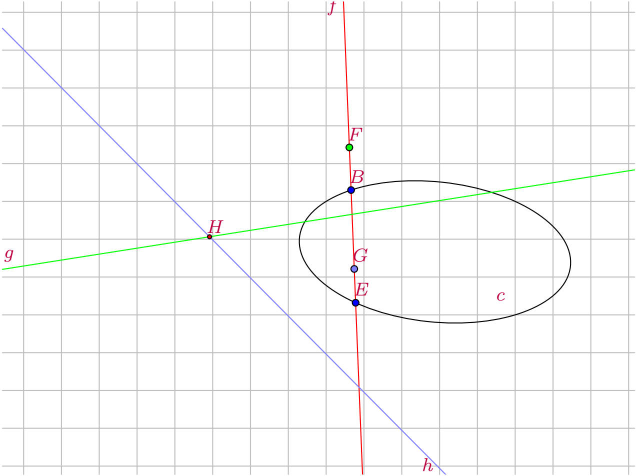


Figure 11: Constructing the pole of a line

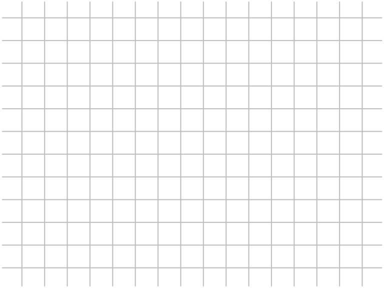


Figure 12: Another example of constructing the pole of a line

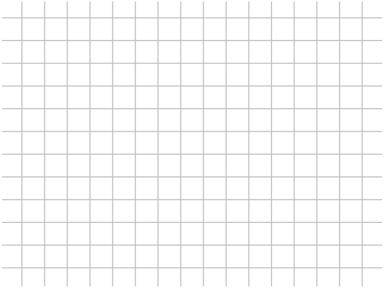


Figure 12: Another example of constructing the pole of a line

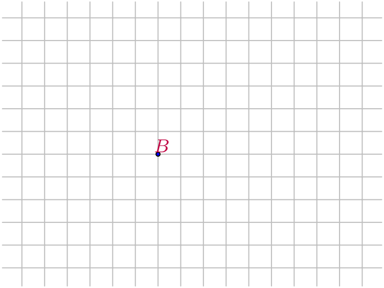


Figure 12: Another example of constructing the pole of a line

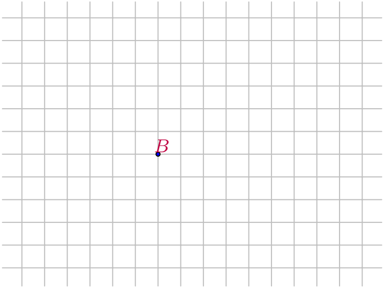
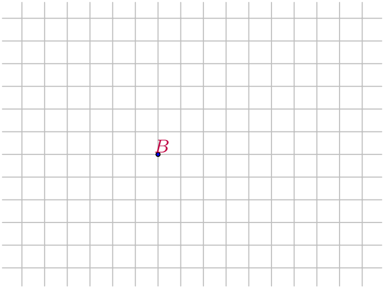


Figure 12: Another example of constructing the pole of a line



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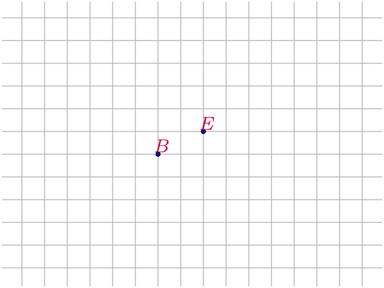

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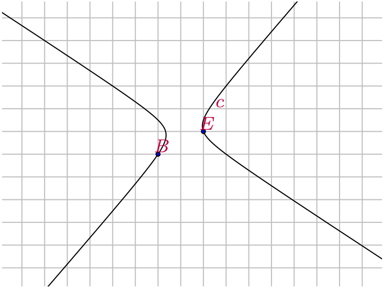


Figure 12: Another example of constructing the pole of a line

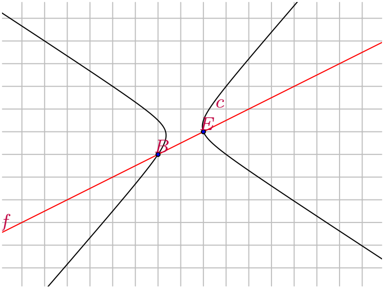


Figure 12: Another example of constructing the pole of a line

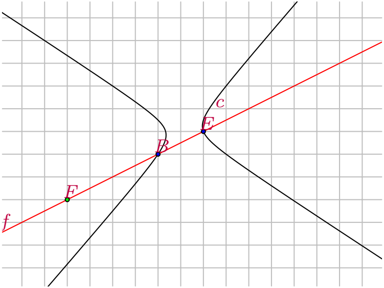
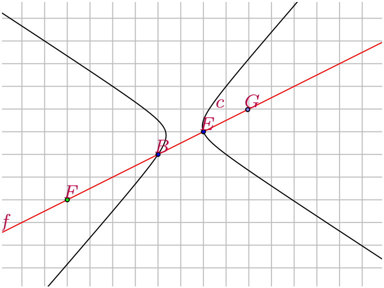


Figure 12: Another example of constructing the pole of a line



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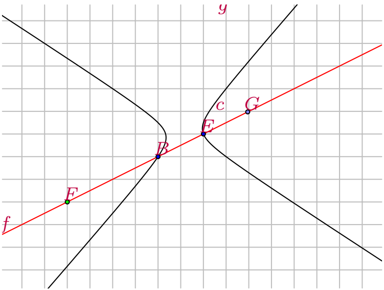

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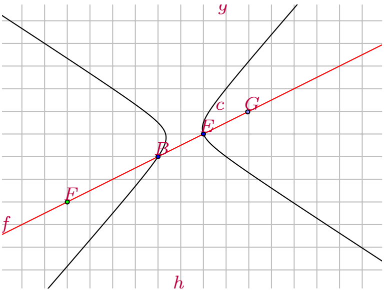

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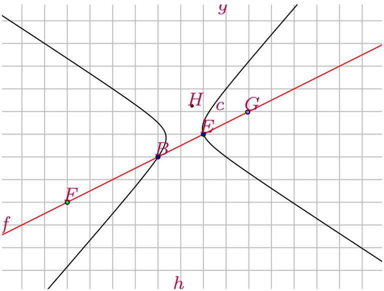


Figure 12: Another example of constructing the pole of a line

Construct the tangent of a point on a conic

- To construct the tangent at a given point P on a conic, join P to the pole of any secant through P.

Figure 13: Construct the tangent of a point on a conic

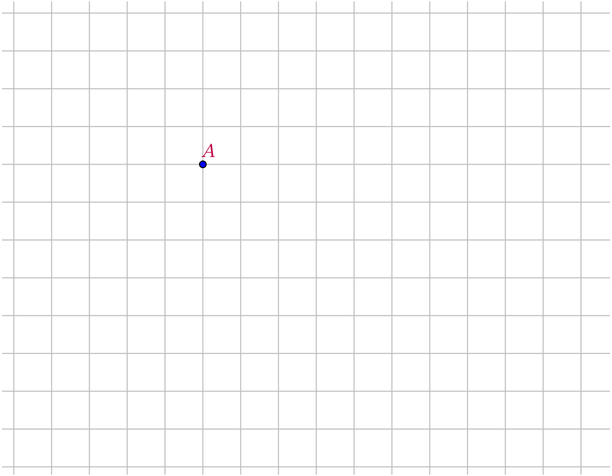


Figure 13: Construct the tangent of a point on a conic

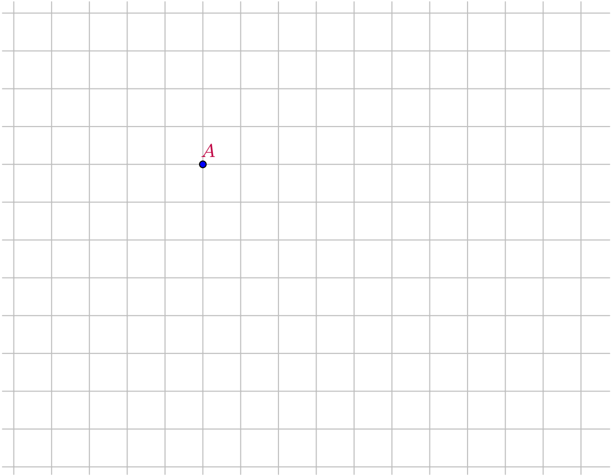
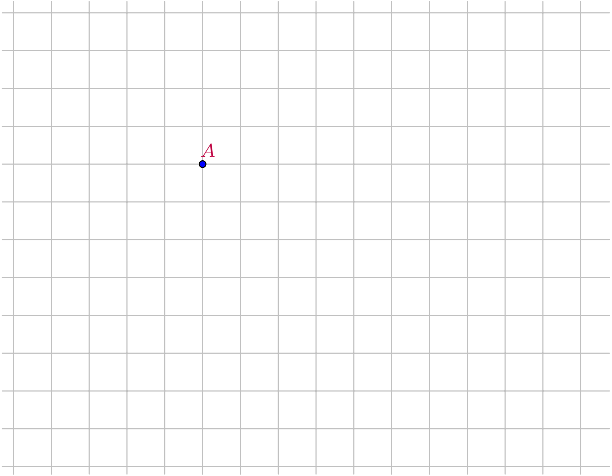


Figure 13: Construct the tangent of a point on a conic



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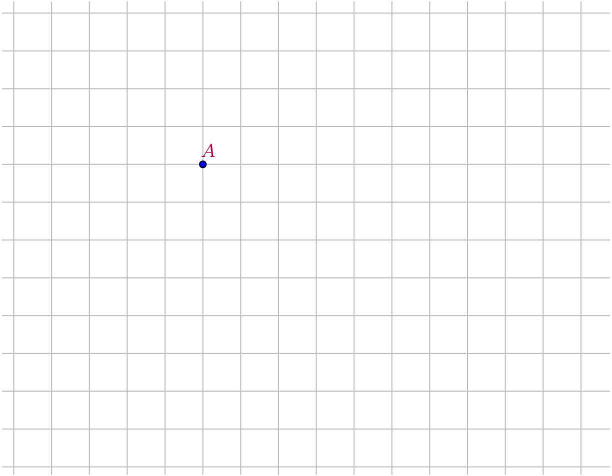

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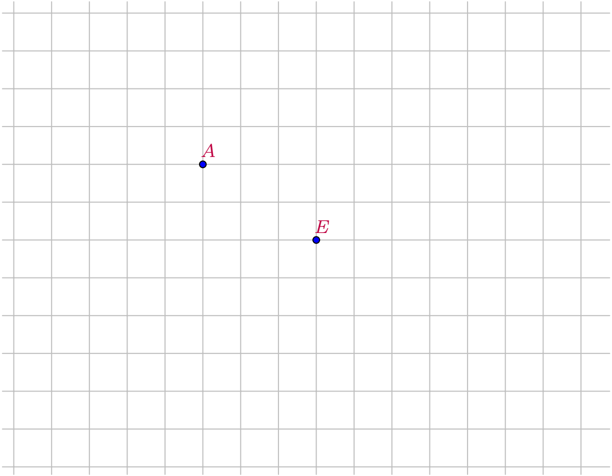


Figure 13: Construct the tangent of a point on a conic

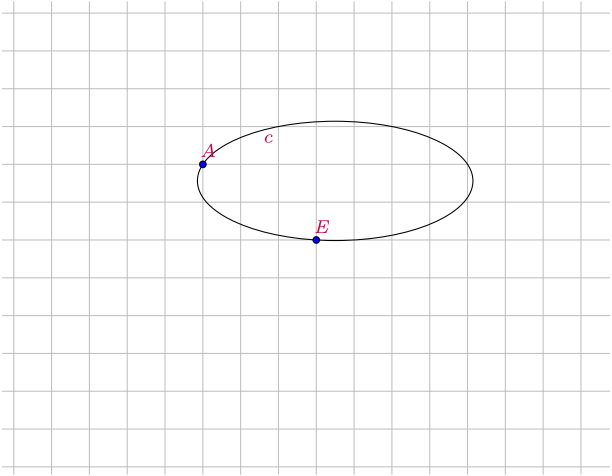


Figure 13: Construct the tangent of a point on a conic

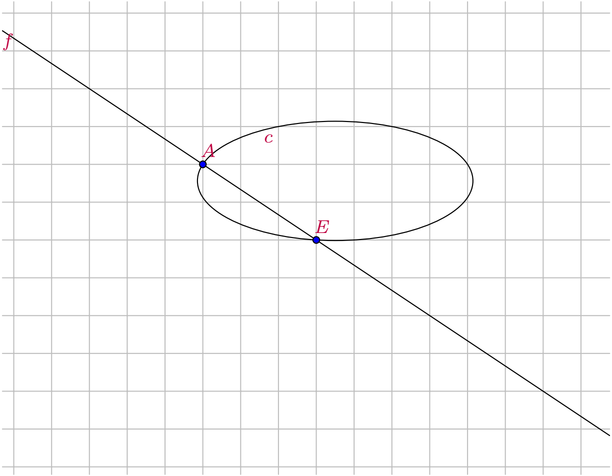


Figure 13: Construct the tangent of a point on a conic

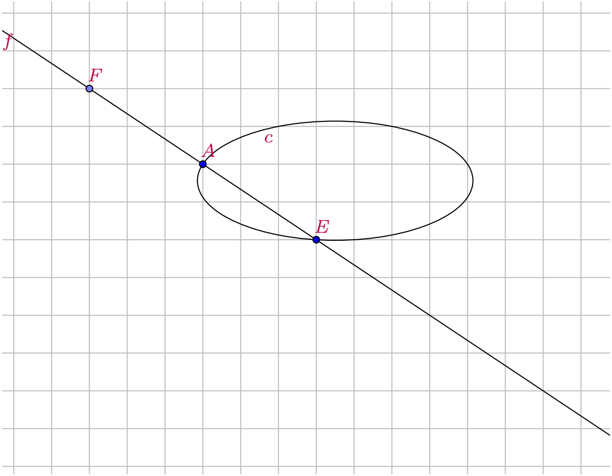


Figure 13: Construct the tangent of a point on a conic

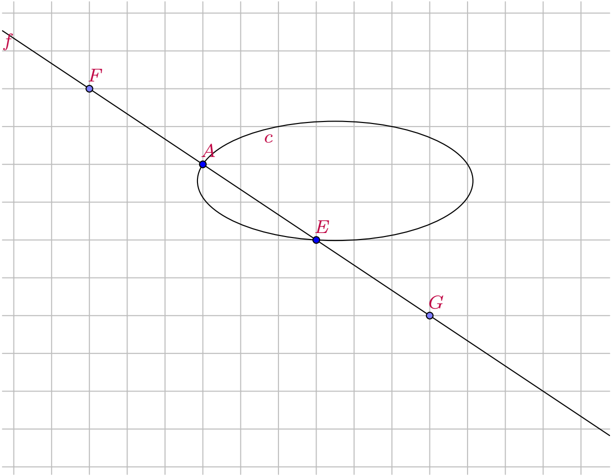
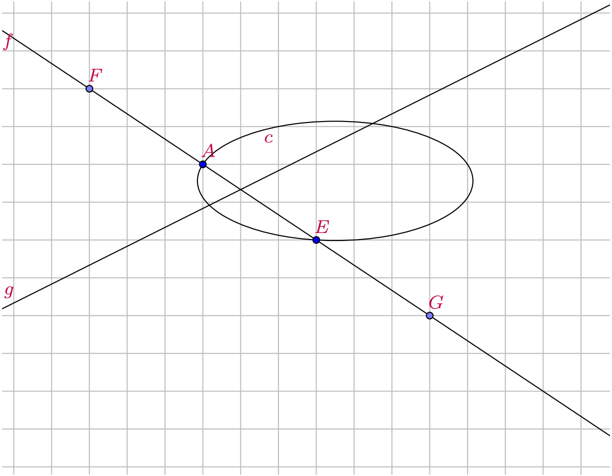


Figure 13: Construct the tangent of a point on a conic



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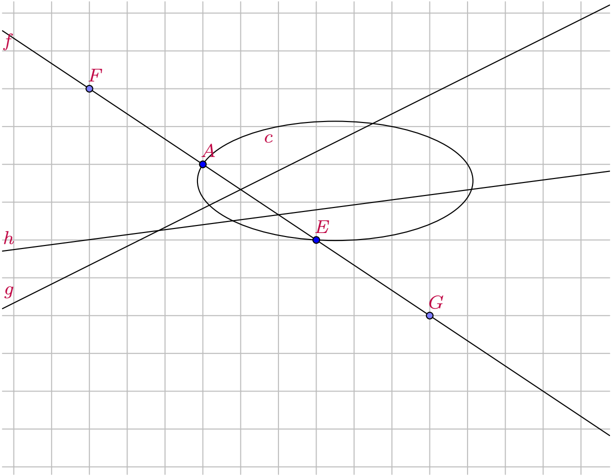


Figure 13: Construct the tangent of a point on a conic

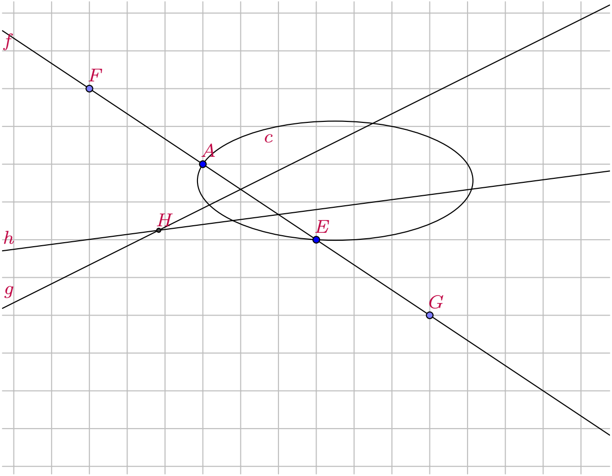


Figure 13: Construct the tangent of a point on a conic

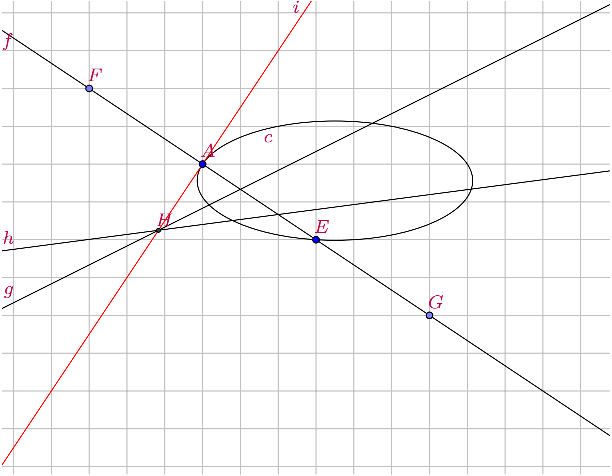


Figure 13: Construct the tangent of a point on a conic

Figure 14: Another example of constructing the tangent of a point on a conic

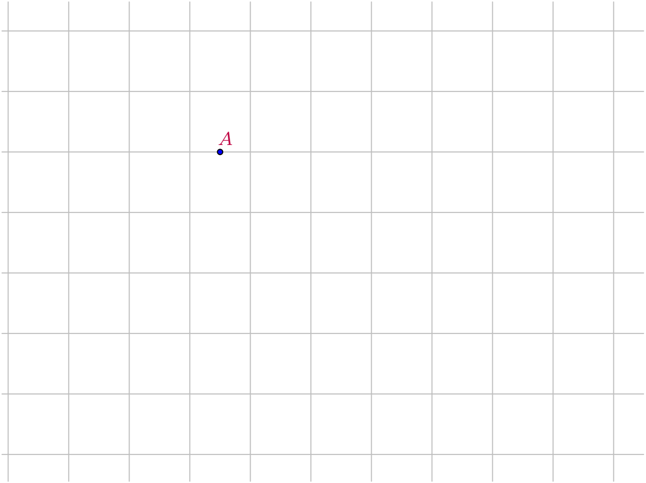
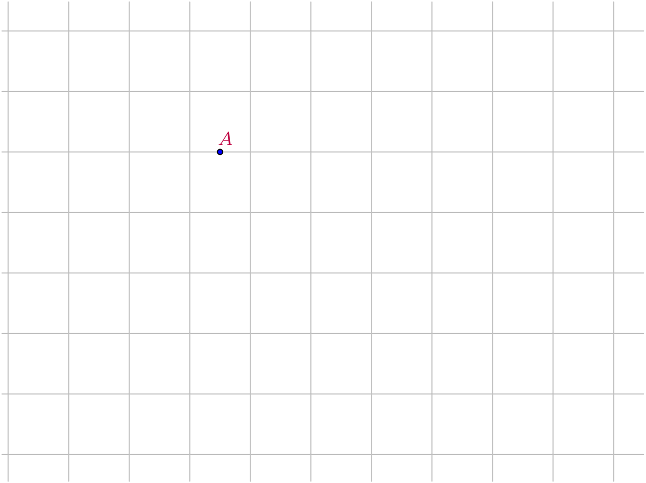


Figure 14: Another example of constructing the tangent of a point on a conic



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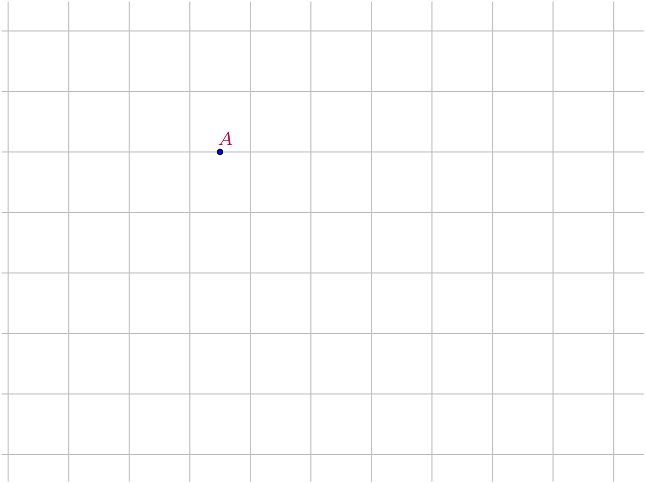

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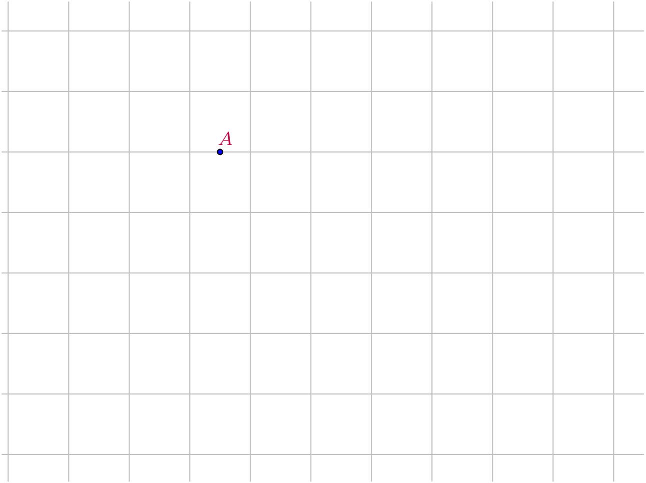

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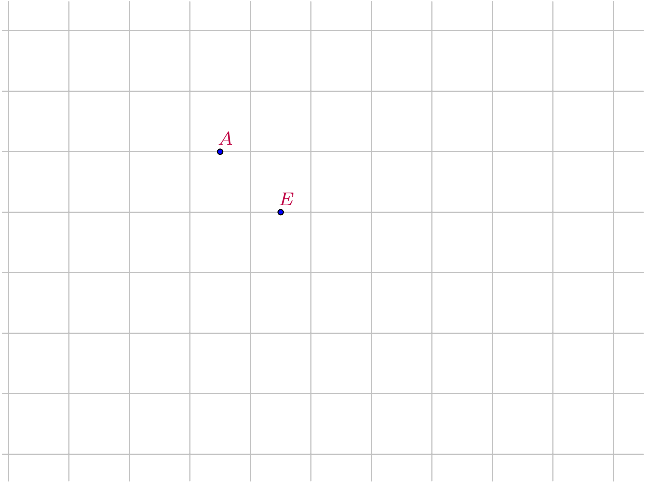

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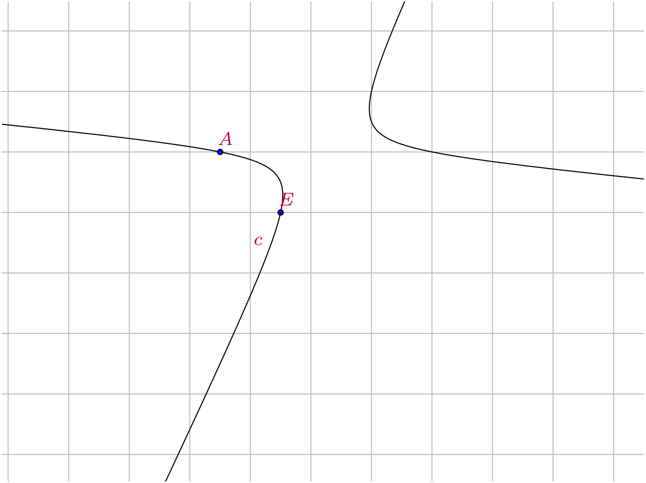

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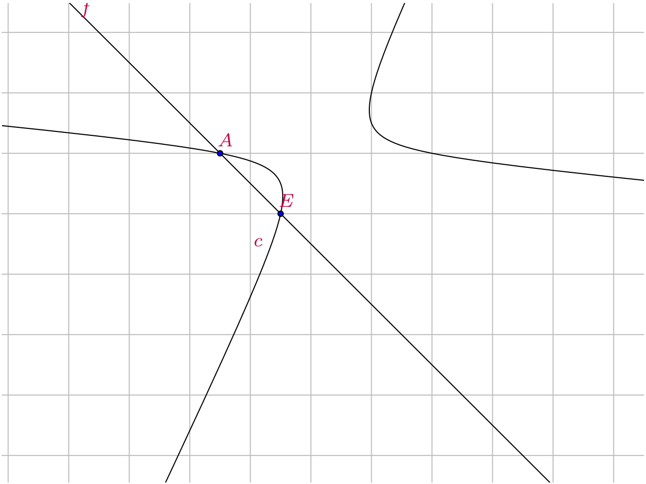

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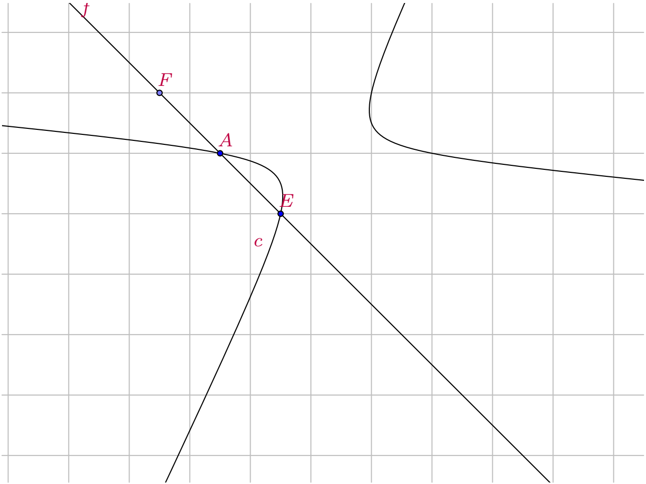

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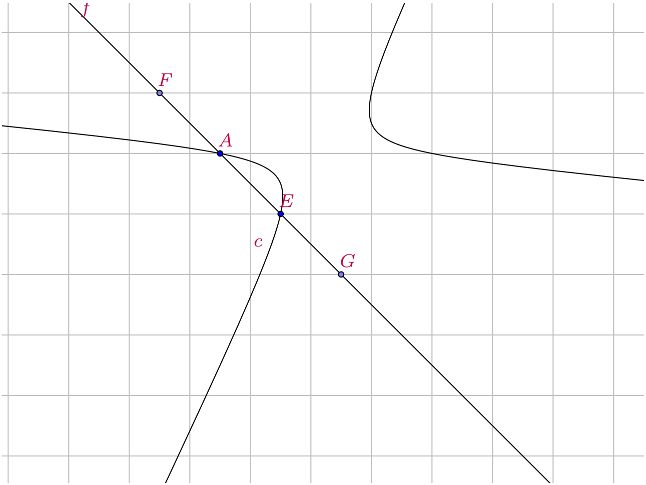

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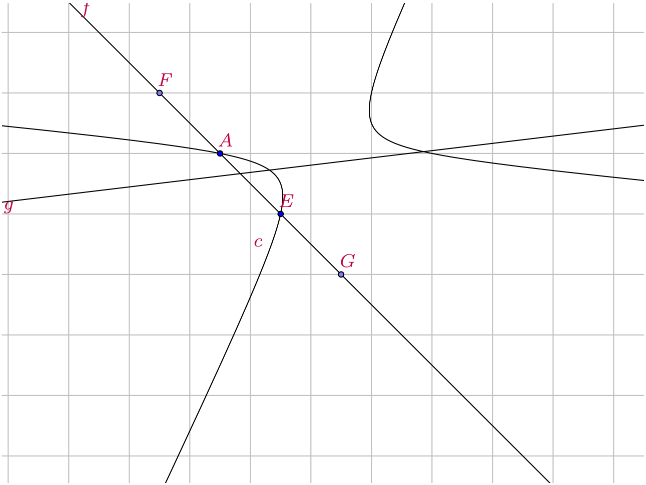

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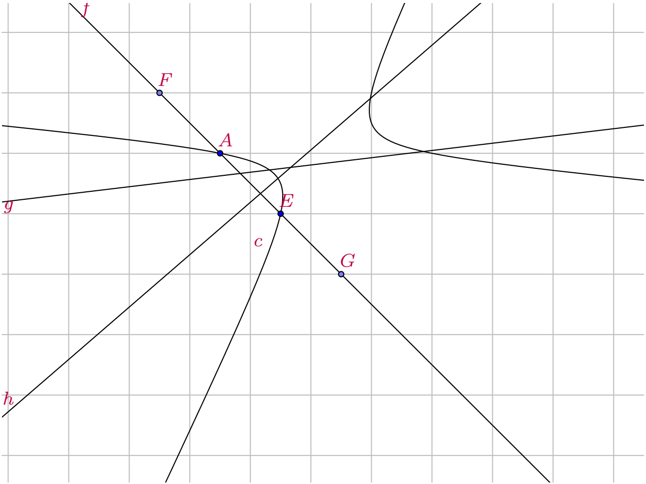

Figure 14: Another example of constructing the tangent of a point on a conic

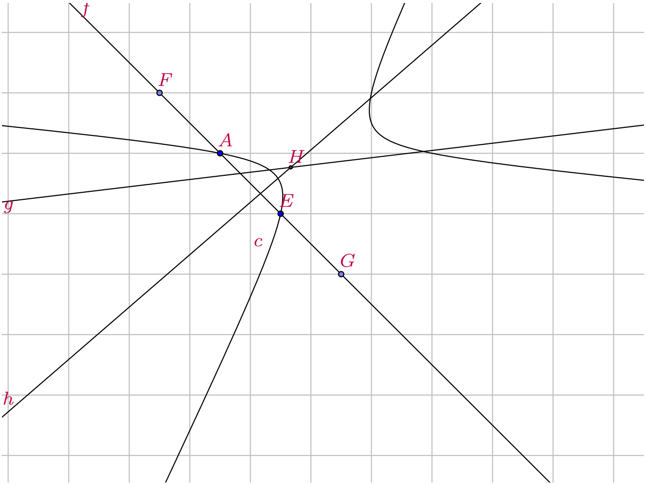

Figure 14: Another example of constructing the tangent of a point on a conic

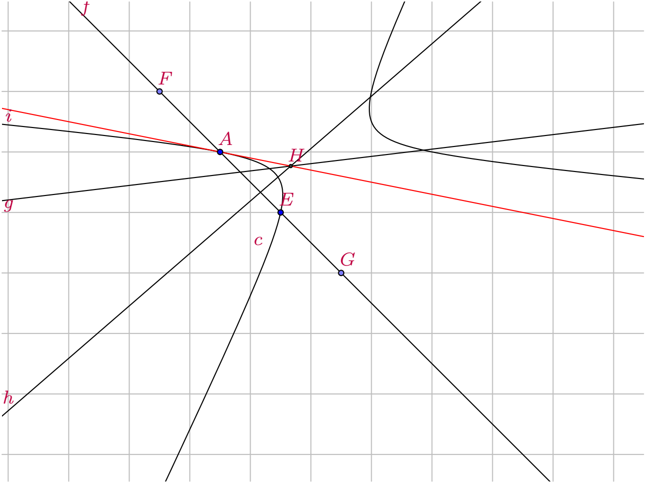

Figure 14: Another example of constructing the tangent of a point on a conic

Pascal's Theorem

- If a hexagon is inscribed in a conic, the three pairs of opposite sides meet in collinear points.

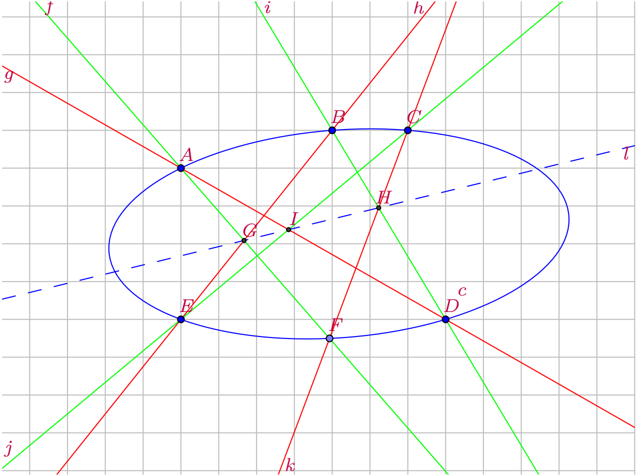

Figure 15: Pascal's theorem

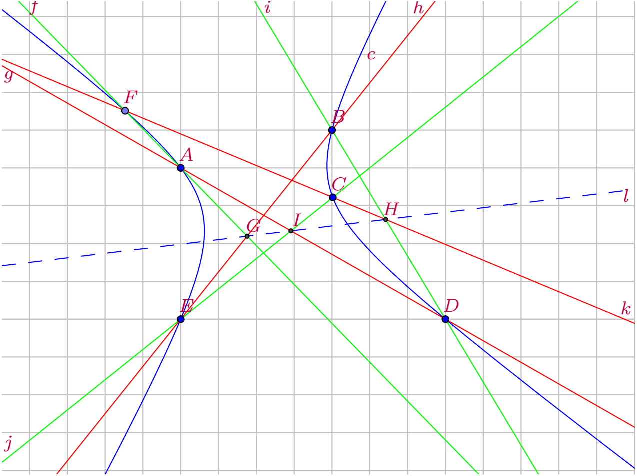

Figure 16: Another instance that Pascal's theorem holds

Backup

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