When "Convex Optimization" Meets "Network Flow"

Explore the intersection of two powerful optimization techniques - convex optimization and network flow - and their applications in various domains.

 **by Wai-Shing Luk**

Introduction

This presentation explores the intersection of convex optimization and network flow problems. We will delve into the insights and properties that emerge when these two powerful mathematical frameworks converge.

Overview of Network Flow Problems

Efficient Solutions

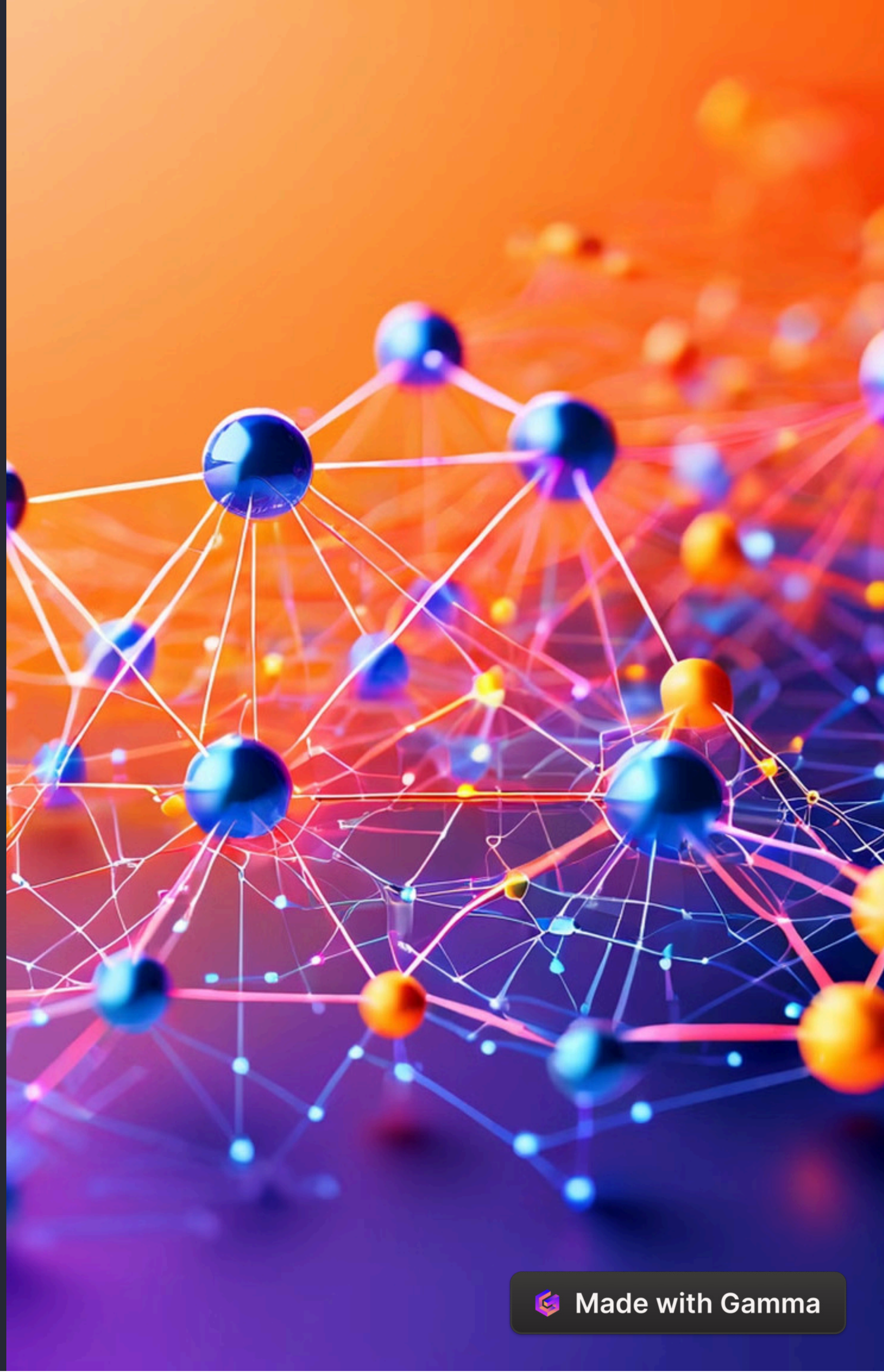
Network flow problems can be solved efficiently and have a wide range of applications.

Additional Constraints

Some problems may have other additional constraints that make them impossible to solve with current network flow techniques.

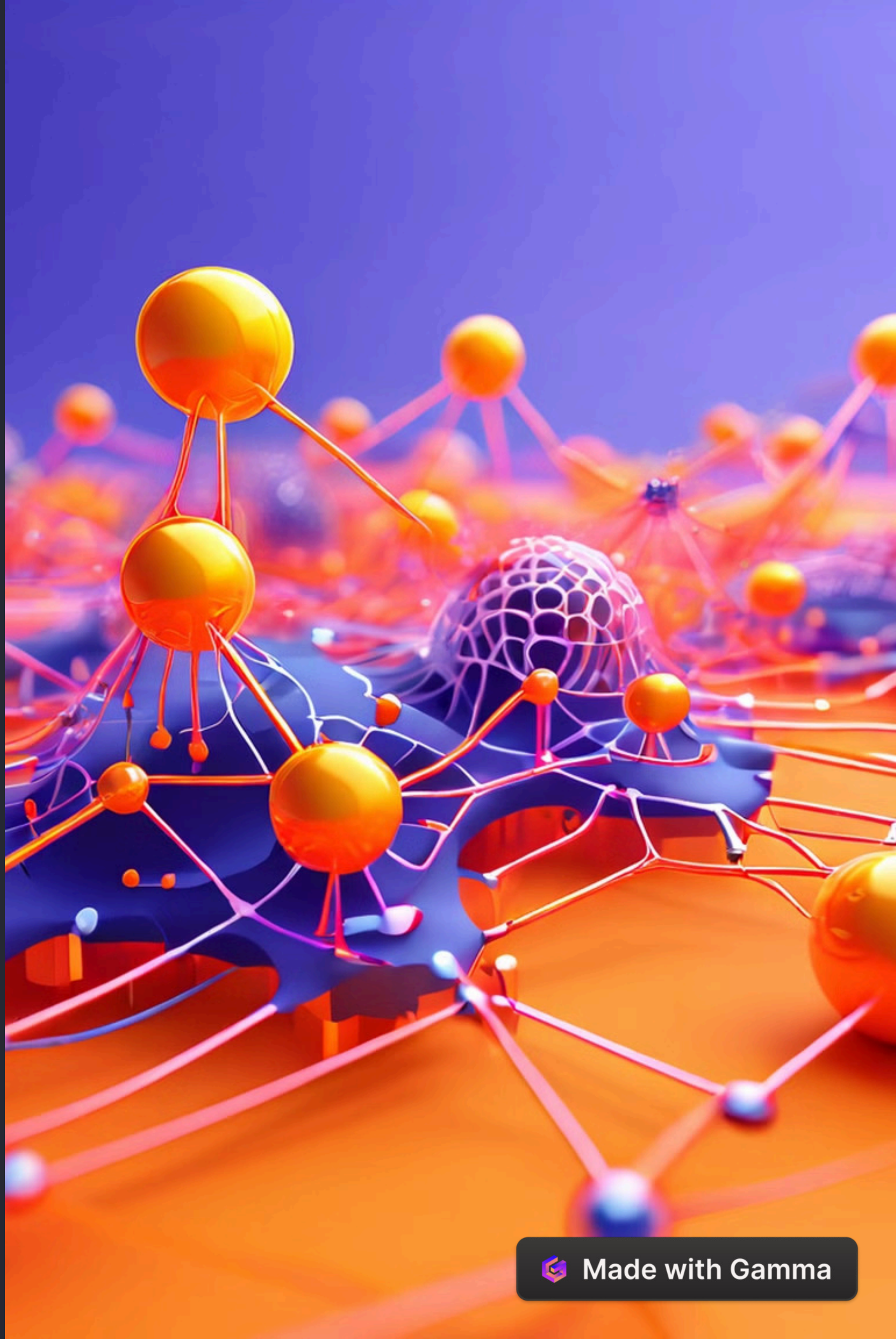
Quasi-Convex Objectives

In some problems, the objective function is quasi-convex rather than convex.



Parametric Potential Problems

Exploring a class of optimization problems where the objective function depends on a parameter. These problems offer insights into network flow and convex optimization techniques.



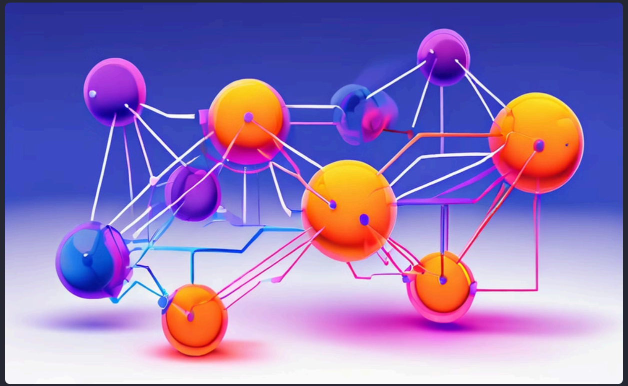
Parametric Potential Problem Definition

Consider the following optimization problem:

maximize $g(\beta)$, **subject to** $y \leq d(\beta)$, $A u = y$

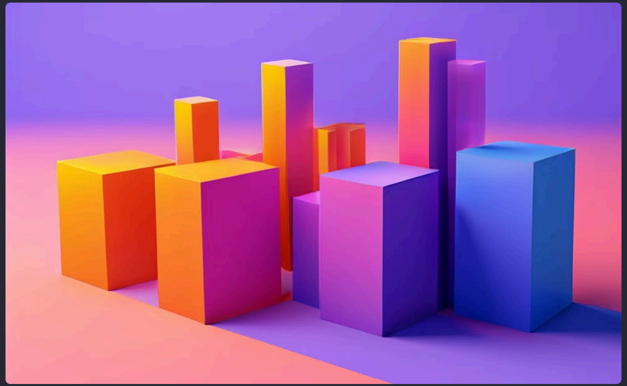
where $g(\beta)$ and $d(\beta)$ are concave functions. This type of problem is known as a **parametric potential problem**. Similar formulations can be used to define **parametric flow problems**.

Network Flow Insights



Negative Cycle Detection

Network flow problems can be solved efficiently by detecting negative cycles using Bellman-Ford-like methods.

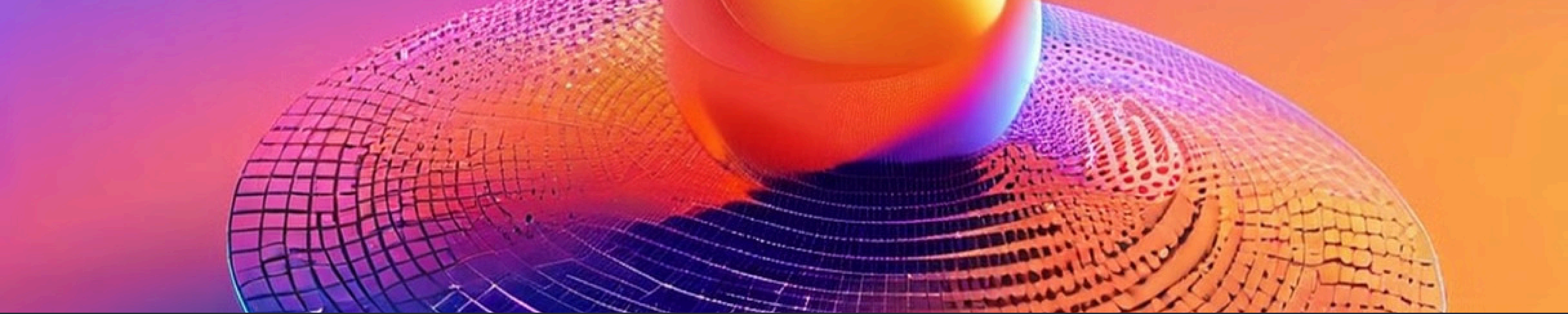


Negative Cycle Condition

If a negative cycle is found, the sum of the edge weights in that cycle will be less than 0.

Feasibility Condition

For a fixed parameter β, the network flow problem is feasible precisely when there exists no negative cycle.



Convex Optimization Insights

Bisection Method

If the sub-gradients of the functions are known, the bisection method can be used to efficiently solve the optimization problem.

Ellipsoid Method

For multi-parameter problems, the ellipsoid method can be employed to find the optimal solution.

Quasi-convex Minimization

Consider the optimization problem where $f(\beta)$ is quasi-convex and $d(\beta)$ are concave:

Maximize	$f(\beta)$
Subject to	$y \leq d(β)$
	$A u = y$

Examples of Quasi-Convex Functions

Square Root of Absolute Value

$\sqrt{|y|}$ is quasi-convex on $\mathbb{R}$.

Logarithm

$\log(y)$ is quasi-linear on $\mathbb{R}_{++}$.

Product Function

$f(x, y) = x y$ is quasi-concave on $\mathbb{R}_{++}^2$.

Linear-Fractional Function

$f(x) = (a^{\mathsf{T}} x + b)/(c^{\mathsf{T}} x + d)$ with domain ${x \,|\, c^{\mathsf{T}} x + d > 0 }$.

Properties of Quasi-Convex Functions

Convex Sublevel Sets

If **f** is quasi-convex, there exists a family of convex functions **ϕt** such that the **t**-sublevel set of **f** is the **0**-sublevel set of **ϕt**.

Monotonicity

ϕt is non-increasing with respect to **t** for fixed **β**, meaning the sublevel sets of **f** are nested.

Fractional Functions

An example of a quasi-convex function is **f(β) = p(β)/q(β)**, where **p** is convex, **q** is concave, **p(β) ≥ 0**, and **q(β) > 0** on the domain of **f**.