Cayley-Klein geometry

Wai-Shing Luk

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Introduction

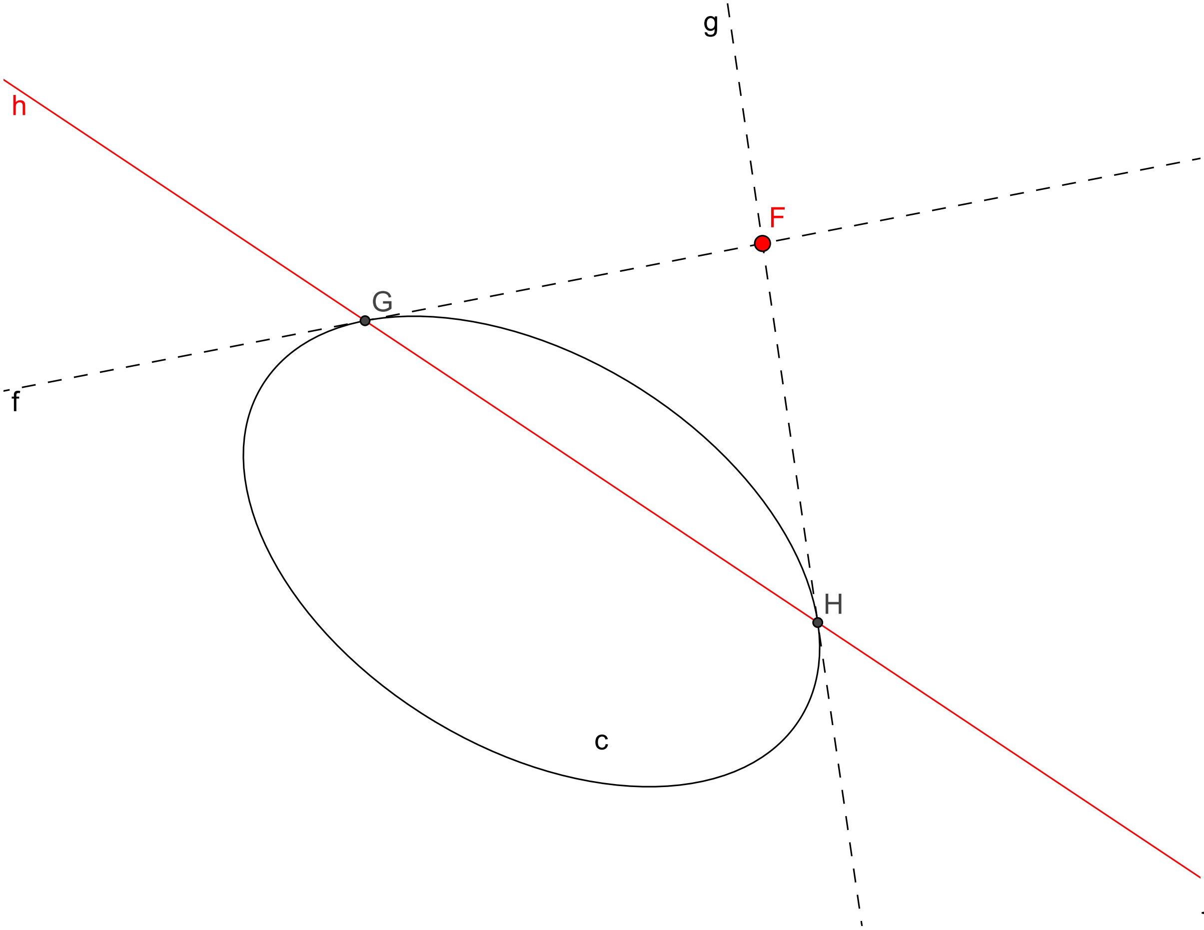
Key points

- Gravity/electromagnetic force between two objects is inversely proportional to the square of their distance.
- Distance and angle may be powerful for oriented measures. But quadrance and spread are more energy saving for non-oriented measures.
- Euclidean Geometry is a degenerate case.

Cayley-Klein Geometry

- Projective geometry can further be categorized by a specific polarity.
- Except degenerate cases, $(A^\perp)^\perp = A$ and $(a^\perp)^\perp = a$
- A fundamental cone $\mathcal{F} = (\mathbf{A}, \mathbf{B})$ is defined by a pole/polar pair such that $[A^\perp] = \mathbf{A} \cdot [A]$ and $[a^\perp] = \mathbf{B} \cdot [a]$.
- To visualize the Cayley-Klein Geometry, we may project the objects to the 2D plane.
- In hyperbolic geometry, the projection of the fundamental conic to the 2D plane is a unit circle, which is called *null circle*. The distance and angle measures could be negative outside the null circle.
- We may consider Euclidean geometry as a hyperbolic geometry where the null circle is expanded toward the infinity.
- In this section, we use the vector notation $p = [A]$ and $l = [a]$.

Fundamental Cone with a pole and polar



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Examples

- Let $p = [x, y, z]$ and $l = [a, b, c]$

- Hyperbolic geometry:
  - $\mathbf{A} \cdot p \equiv [x, y, -z]$
  - $\mathbf{B} \cdot l \equiv [a, b, -c]$

- Elliptic geometry:
  - $\mathbf{A} \cdot p \equiv [x, y, z]$
  - $\mathbf{B} \cdot l \equiv [a, b, c]$

- Euclidean geometry (degenerate conic):
  - $\mathbf{A} \cdot p \equiv [0, 0, z]$
  - $\mathbf{B} \cdot l \equiv [a, b, 0]$

- psuedo-Euclidean geometry (degenerate conic):
  - $\mathbf{A} \cdot p \equiv [0, 0, z]$
  - $\mathbf{B} \cdot l \equiv [a, -b, 0]$

Examples (cont'd)

- Perspective view of Euclidean geometry (degenerate conic):
  - Let l be the line of infinity.
  - Let p and q are two points on l. Then
  - $\mathbf{A} \equiv l \cdot l^T$ (outer product)
  - $\mathbf{B} \equiv p \cdot q^T + q \cdot p^T$

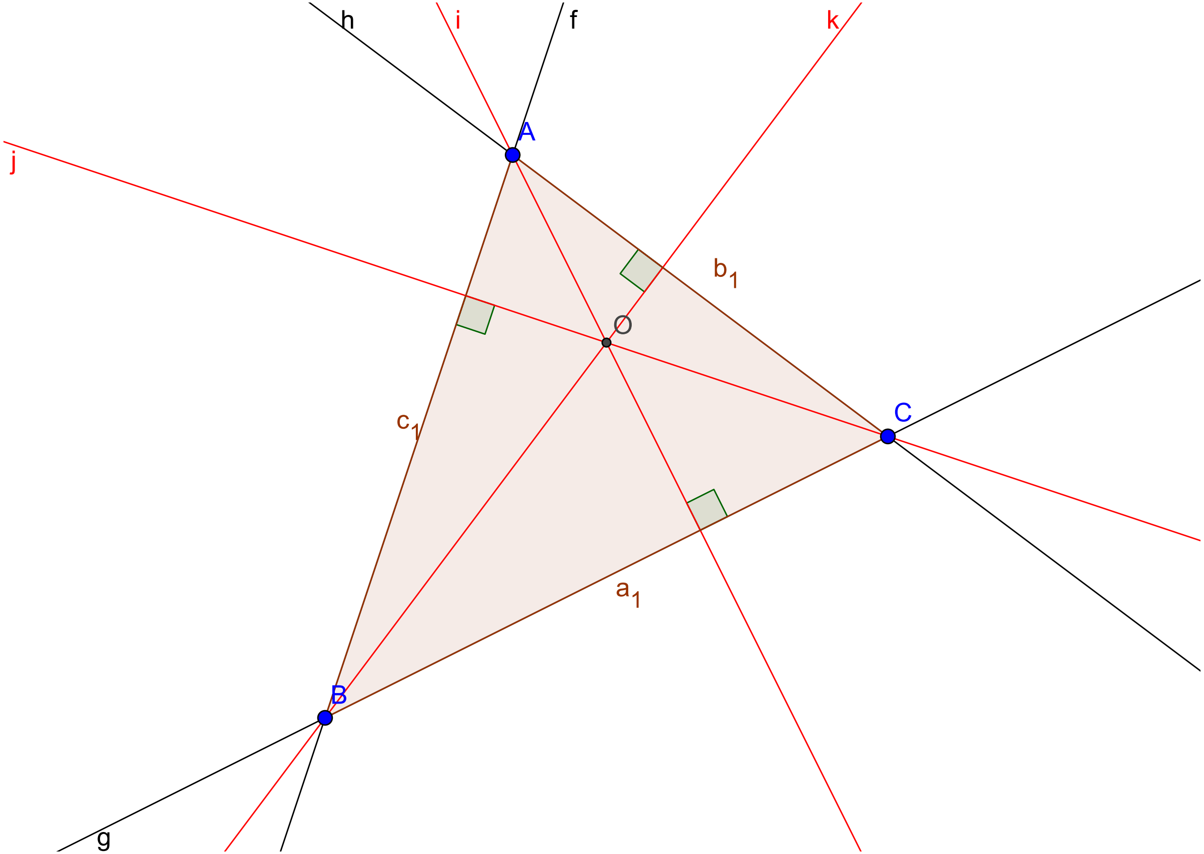
Orthogonality

- A line l is said to be perpendicular to line m if $l^\perp$ lies on m, i.e., $m^T \mathbf{B} l = 0$.
- To find a perpendicular line of l that passes through p, join p to the pole of l, i.e., join($p, l^\perp$). We call this the *altitude* line of l.
- For duality, a point p is said to be perpendicular to point q if $q^T \mathbf{A} p = 0$.
- The altitude point can be defined similarly.
- Note that Euclidean geometry does not have the concept of the perpendicular point because every $p^\perp$ is the line of infinity.

Orthocenter of triangle

- Theorem 1 (Orthocenter and ortholine). The altitude lines of a non-dual triangle meet at a unique point O, called the *orthocenter* of the triangle.
- Although there is "center" in the name, orthocenter could be outside a triangle.
- Theorem 2. If the orthocenter of triangle $\{ABC\}$ is O, then the orthocenter of triangle $\{OBC\}$ is A.

An instance of orthocenter theorem



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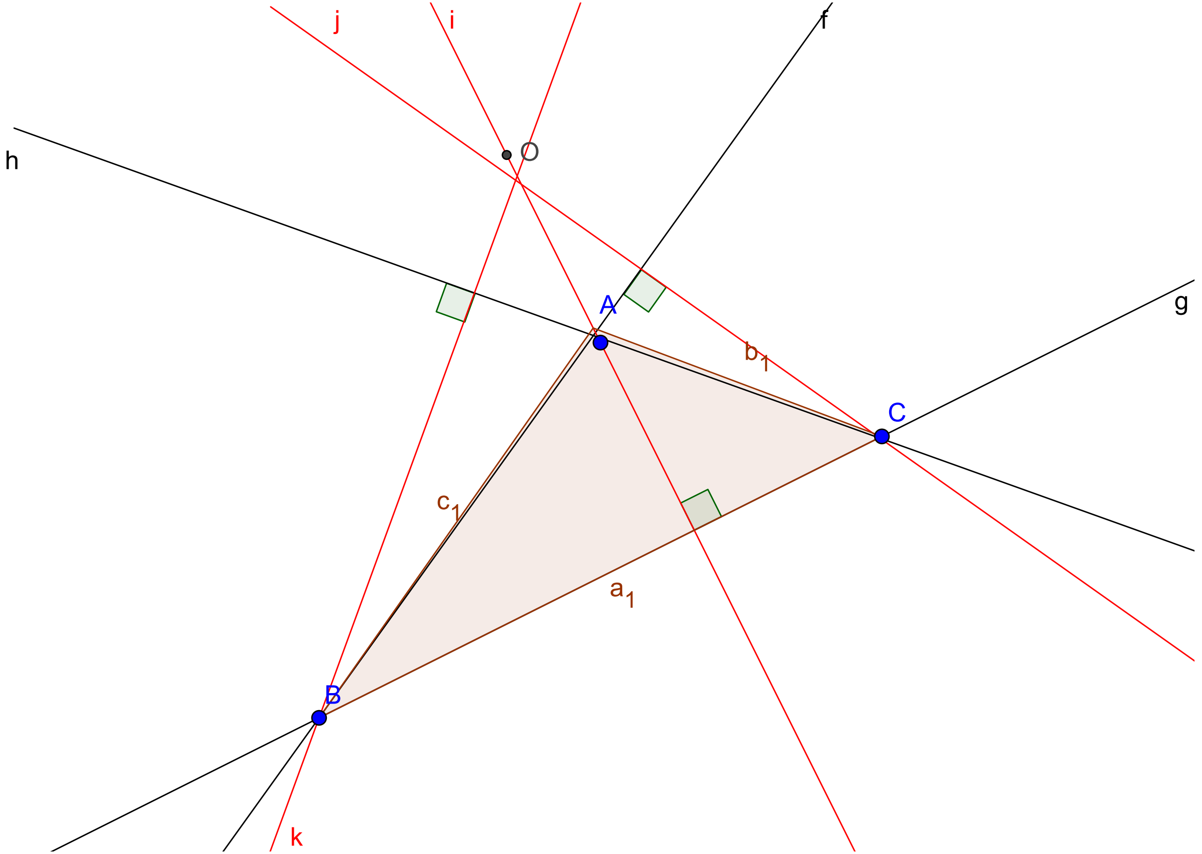
An instance of Theorem 2



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Involution

- Involutions are closely related to geometric reflections.
- The defining property of an involution τ is that $\tau(\tau(p)) = p$ for every point p.
- Theorem: Let τ be an involution. Then

    1. there is a line m with $\tau(p) = p$ for every poiny p incident with m.
    2. there is a point o with $\tau(l) = l$ for every line l incident with o.

- We call the line m a *mirror* and the point o the *center* of the involution.
- If o is at the line of infinity (Euclidean Geometry), then we get an undistorted Euclidean line reflection in m.
- If we choose $o = m^{\perp}$, then we keep the fundamental cone invariant.

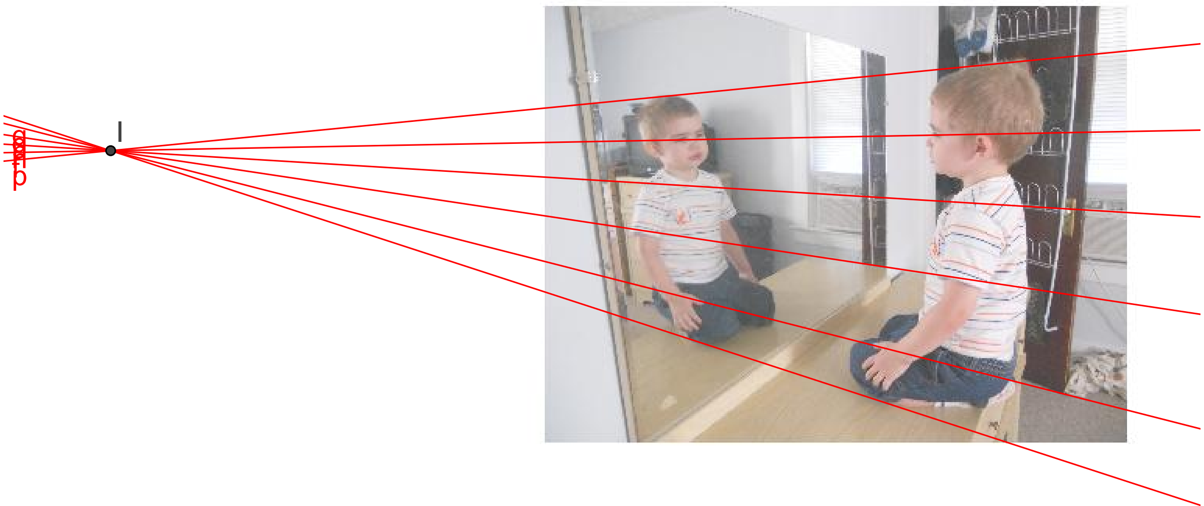
Involution (cont'd)

- Theorem: The point transformation matrix T of a projective involution τ with center o and mirror m is given by

$$(o^T m)\mathrm{I} - 2om^T$$

- In other words, $T \cdot p = (o^T m)p - 2(m^T p)o$.

Mirror Image



{#looking-at-
self}

Basic measurement

Basic measure between point and line

- A basic measure between p and l, denoted by $p^T l$ (inner product):
    - $p^T l$ can be positive, negative, and zero.
    - $p^T l = 0$ if and only if p lies on l.

Cross Ratio

- Given a line incident with $ABCD$. Arbitrary choose a point O not on the line.
- The cross ratio is defined as:

$$R(A, B; C, D) = (OA \cdot C)(OB \cdot D)/(OA \cdot D)(OB \cdot C)$$

Quadrance and Spread for general cases

- Let $\Phi(x) = x \cdot x^{\perp}$.
- $\Phi(A) = A \cdot A^{\perp} = [A]^T \mathbf{A} [A]$.
- $\Phi(a) = a \cdot a^{\perp} = [a]^T \mathbf{B} [a]$.
- The **quadrance** $q(A, B)$ between points A and B is:

$$q(A, B) \equiv \Phi(AB)/\Phi(A)\Phi(B)$$

- The **spread** $s(l, m)$ between lines l and m is

$$s(l, m) \equiv \Phi(lm)/\Phi(l)\Phi(m)$$

- Note: they are invariant of any projective transformations.

Relation with Traditional Distance and Angle

- Hyperbolic:
  - $q(A, B) = \sinh^2(d(A, B))$
  - $s(l, m) = \sin^2(\theta(l, m))$
- Elliptic:
  - $q(A, B) = \sin^2(d(A, B))$
  - $s(l, m) = \sin^2(\theta(l, m))$
- Euclidean:
  - $q(A, B) = d^2(A, B)$
  - $s(l, m) = \sin^2(\theta(l, m))$

Spread law and Thales Theorem

- Spread Law

$$q_1/s_1 = q_2/s_2 = q_3/s_3.$$

- (Compare with the sine law in Euclidean Geometry):

$$d_1/\sin\theta_1 = d_2/\sin\theta_2 = d_3/\sin\theta_3.$$

- Theorem (Thales): Suppose that $\{a_1a_2a_3\}$ is a right triangle with $s_3 = 1$. Then

$$s_1 = q_1/q_3 \quad \text{and} \quad s_2 = q_2/q_3$$

- Note: in some geometries, two lines are perpendicular does not imply they have a right angle ($s = 1$).

Triangle proportions

- Theorem (Triangle proportions): Suppose that d is a point lying on the line $a_1 a_2$. Define the quadrances $r_1 \equiv q(a_1, d)$ and $r_1 \equiv q(a_2, d)$, and the spreads $R_1 \equiv s(a_3 a_1, a_3 d)$ and $R_2 \equiv s(a_3 a_2, a_3 d)$. Then

$$R_1/R_2 = (s_1/s_2)(r_1/r_2) = (q_1/q_2)(r_1/r_2).$$

Midpoint and Angle Bisector

- There are two angle bisectors for two lines.
- There are two midpoints for two points also in general geometries.
- Let r be the midpint of p and q.
- Then r = $\sqrt{\Phi(p)}q \pm \sqrt{\Phi(q)}p$.
- Let b be the angle bisector of l and m.
- Then b = $\sqrt{\Phi(m)}l \pm \sqrt{\Phi(l)}m$.
- Note:
  - The midpoint could be irrational in general.
  - The midpoint could even be complex, even the two points are real.
  - Two angle bisectors are perpendicular.
  - In Euclidean geometry, another midpoint is at the line of infinity.

Constructing midpoints using the fundamental conic

Midpoint in Euclidean geometry

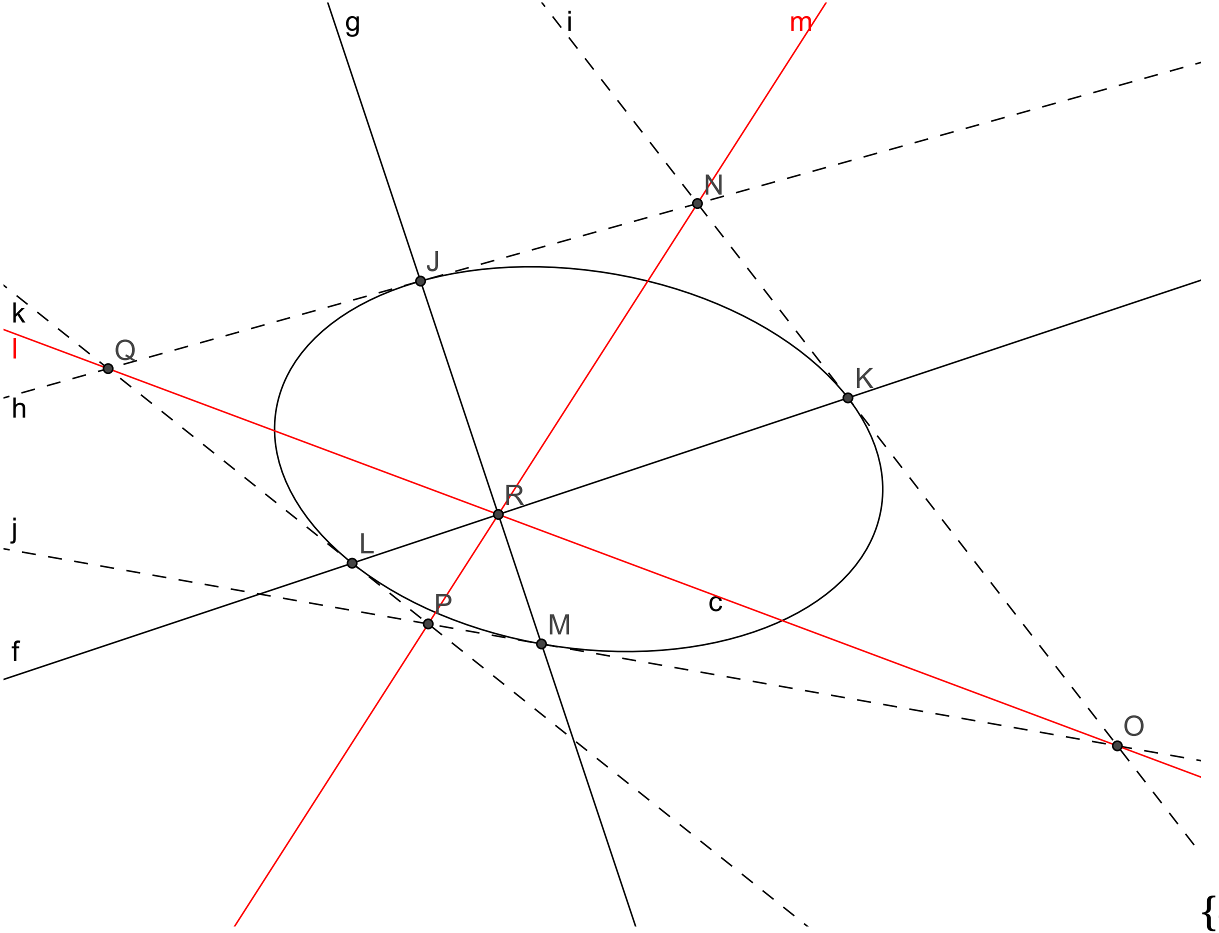
- Let l be the line of infinity.
- $\mathbf{A} \equiv l \cdot l^T$
- $\Phi(p) = p^T \mathbf{A} p = (p^T l)^2$.
- Then, the midpoint $r = (q^T l)p \pm (p^T l)q$.
- One midpoint $(q^T l)p - (p^T l)q$ in fact lies on l.

Constructing angle bisectors using a conic

1. For each line construct the two tangents (t_f^1, t_f^2) and (t_g^1, t_g^2) of its intersection points with the fundamental conic to that conic.
2. The following lines are the two angle bisectors:
   - join(meet(t_f^1, t_g^1), meet(t_f^2, t_g^2))
   - join(meet(t_f^1, t_g^2), meet(t_f^2, t_g^1))

Remark: the tangents in elliptic geometry have complex coordinates. However, the angle bisectors are real objects again.

Constructing a pair of angle bisectors



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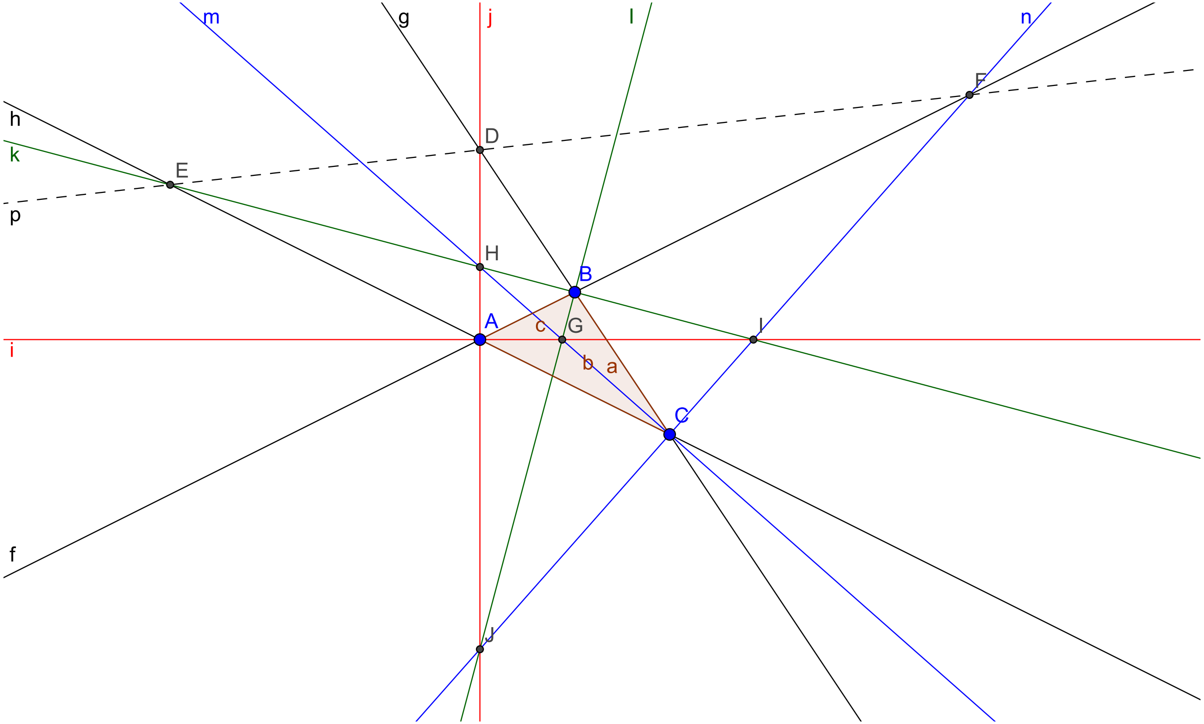
Angle Bisector Theorem

- Let a, b, c be three lines such that none of them tangents to the fundamental conic.
- Then one set of angle bisector $m_{ab}^1, m_{bc}^1, m_{ac}^1$ are concurrent.
- Furthermore, the points meet(m_{ab}^2, c), meet(m_{bc}^2, a), meet(m_{ac}^2, b) are collinear.

An instance of complete angle bisector theorem



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Midpoint theorem

- Let p, q, r be three points such that none of them lies on the fundamental conic.
- Then one set of midpoints $m^1_{pq}, m^1_{qr}, m^1_{pr}$ are collinear.
- Furthermore, the lines join(m^2_{pq}, r), join(m^2_{qr}, p), join(m^2_{pr}, q) meet at a point.

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