Hyperbolic/Elliptic geometry

Wai-Shing Luk

2016-11-30

Basic

Key points

- In Hyperbolic Geometry,
    - two parallel lines meet outside the null circle.
    - Given a line l, there are more one parallel lines that pass through a point p.
- Notations: To distinguish with Euclidean geometry, lines are written as capital letters.

Quadrance and Spread in Hyperbolic/Elliptic geometry

- For efficiency, quadrance and spread can also be written as follows.
- The **quadrance** $q(a,b)$ between points a and b is:

$$q(a,b) \equiv 1 - (a \cdot b^{\perp})^2/(a \cdot a^{\perp})(b \cdot b^{\perp})$$

- The **spread** $S(L,M)$ between lines L and M is

$$S(L,M) \equiv 1 - (L \cdot M^{\perp})^2/(L \cdot L^{\perp})(M \cdot M^{\perp})$$

- Note: In Hyperbolic Geometry, the quadrance of two points inside the null circle is negative.

Relation with Traditional Distance and Angle

- Hyperbolic:
  - Distance:

$$q(a,b) = \sinh^2(d(a,b))$$

  - Angle:

$$S(L,M) = \sin^2(\theta(L,M))$$

- Elliptic:
  - Distance:

$$q(a,b) = \sin^2(d(a,b))$$

  - Angle:

$$S(L,M) = \sin^2(\theta(L,M))$$

Spread Law

$$S_1/q_1 = S_2/q_2 = S_3/q_3.$$

Triple formulate

- Let a_1, a_2 and a_3 are points with $q_1 \equiv q(a_2, a_3)$, $q_2 \equiv q(a_1, a_3)$ and $q_3 \equiv q(a_1, a_2)$. Let L_1, L_2 and L_3 are lines with $S_1 \equiv S(L_2, L_3)$, $S_2 \equiv S(L_1, L_3)$ and $S_3 \equiv S(L_1, L_2)$.

- Theorem (Triple quad formula): If a_1, a_2 and a_3 are collinear points then

$$(q_1 + q_2 + q_3)^2 = 2(q_1^2 + q_2^2 + q_3^2) + 4q_1q_2q_3$$

- Theorem (Triple spread formula): If L_1, L_2 and L_3 are concurrent lines then

$$(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3.$$

Cross Law

- Theorem (Cross law)

$$(S_1 S_2 q_3 - (S_1 + S_2 + S_3) + 2)^2 = 4(1 - S_1)(1 - S_2)(1 - S_3).$$

- Theorem (Cross dual law)

$$(q_1 q_2 S_3 - (q_1 + q_2 + q_3) + 2)^2 = 4(1 - q_1)(1 - q_2)(1 - q_3).$$

- Note:
  - Given three quadrances, three spreads can be uniquely determined. Same as Euclidean Geometry.
  - Given three spreads, three quadrances can be uniquely determined. Not true in Euclidean Geometry.

Right triangles and Pythagoras

- Theorem (Pythagoras): If L_1 and L_2 are perpendicular lines ($S(L_1, L_2) = 1$) then
$$q_3 = q_1 + q_2 - q_1 q_2.$$
- Theorem (Thales): Suppose that $\{a_1 a_2 a_3\}$ is a right triangle with $S_3 = 1$. Then $S_1 = q_1/q_3$ and $S_2 = q_2/q_3$.

Right parallax

- Theorem (Right parallax): If a right triangle $\{a_1 a_2 a_3\}$ has spreads $S_1 = 0$, $S_2 = S$ and $S_3 = 1$, then it will have only one defined quadrance $q_1 = q$ given by

$$q = (S - 1)/S.$$

- We may restate this result in the form:

$$S = 1/(1 - q).$$

Triangle proportions and barycentric coordinates

Triangle proportions

- Theorem (Triangle proportions): Suppose that d is a point lying on the line $a_1 a_2$. Define the quadrances $r_1 \equiv q(a_1, d)$ and $r_1 \equiv q(a_2, d)$, and the spreads $R_1 \equiv S(a_3 a_1, a_3 d)$ and $R_2 \equiv S(a_3 a_2, a_3 d)$. Then

$$R_1/R_2 = (S_1/S_2)(r_1/r_2) = (q_1/q_2)(r_1/r_2).$$