Chapter One

Max-Plus Algebra

In the previous chapter we described max-plus algebra in an informal way. The present chapter contains a more rigorous treatment of max-plus algebra. In Section 1.1 basic concepts are introduced, and algebraic properties of max-plus algebra are studied. Matrices and vectors over max-plus algebra are introduced in Section 1.2, and an important model, called *heap of pieces* or *heap model*, which can be described by means of max-plus algebra, is presented in Section 1.3. Finally, the projective space, a mathematical framework most convenient for studying limits, is introduced in Section 1.4.

1.1 BASIC CONCEPTS AND DEFINITIONS

Define $\varepsilon \stackrel{\text{def}}{=} -\infty$ and $e \stackrel{\text{def}}{=} 0$, and denote by $\mathbb{R}_{\max}$ the set $\mathbb{R} \cup \{\varepsilon\}$, where $\mathbb{R}$ is the set of real numbers. For elements $a, b \in \mathbb{R}_{\max}$, we define operations $\oplus$ and $\otimes$ by

$$a \oplus b \stackrel{\text{def}}{=} \max(a, b) \qquad \text{and} \qquad a \otimes b \stackrel{\text{def}}{=} a + b\,. \tag{1.1}$$

Clearly, $\max(a, -\infty) = \max(-\infty, a) = a$ and $a + (-\infty) = -\infty + a = -\infty$, for any $a \in \mathbb{R}_{\max}$, so that

$$a \oplus \varepsilon = \varepsilon \oplus a = a \qquad \text{and} \qquad a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon\,, \tag{1.2}$$

for any $a \in \mathbb{R}_{\max}$. The above definitions are illustrated with some numerical examples as follows:

$$\begin{aligned}
5 \oplus 3 &= \max(5,3) = 5\,,\\
5 \oplus \varepsilon &= \max(5,-\infty) = 5\,,\\
5 \otimes \varepsilon &= 5 - \infty = -\infty = \varepsilon\,,\\
e \oplus 3 &= \max(0,3) = 3\,,\\
5 \otimes 3 &= 5 + 3 = 8\,.
\end{aligned}$$

The set $\mathbb{R}_{\max}$ together with the operations $\oplus$ and $\otimes$ is called *max-plus algebra* and is denoted by

$$\mathcal{R}_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e)\,.$$

As in conventional algebra, we simplify the notation by letting the operation $\otimes$ have priority over the operation $\oplus$. For example,

$$5 \otimes -9 \oplus 7 \otimes 1$$

has to be understood as

$$(5 \otimes -9) \oplus (7 \otimes 1) .$$

Notice that $(5 \otimes -9) \oplus (7 \otimes 1) = 8$, whereas $5 \otimes (-9 \oplus 7) \otimes 1 = 13$.

The operations $\oplus$ and $\otimes$ defined in (1.1) have some interesting algebraic properties. For example, for $x, y, z \in \mathbb{R}_{\max}$, it holds that

$$\begin{aligned} x \otimes (y \oplus z) &= x + \max(y, z) \\ &= \max(x + y, x + z) \\ &= (x \otimes y) \oplus (x \otimes z) , \end{aligned}$$

which in words means that $\otimes$ distributes over $\oplus$. Below we give a list of algebraic properties of max-plus algebra.

- Associativity:

$$\forall x, y, z \in \mathbb{R}_{\max} : \quad x \oplus \big(y \oplus z\big) = \big(x \oplus y\big) \oplus z$$

  and

$$\forall x, y, z \in \mathbb{R}_{\max} : \quad x \otimes \big(y \otimes z\big) = \big(x \otimes y\big) \otimes z .$$

- Commutativity:

$$\forall x, y \in \mathbb{R}_{\max} : \quad x \oplus y = y \oplus x \qquad \text{and} \qquad x \otimes y = y \otimes x .$$

- Distributivity of $\otimes$ over $\oplus$:

$$\forall x, y, z \in \mathbb{R}_{\max} : \quad x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z) .$$

- Existence of a zero element:

$$\forall x \in \mathbb{R}_{\max} : \quad x \oplus \varepsilon = \varepsilon \oplus x = x .$$

- Existence of a unit element:

$$\forall x \in \mathbb{R}_{\max} : \quad x \otimes e = e \otimes x = x .$$

- The zero is absorbing for $\otimes$:

$$\forall x \in \mathbb{R}_{\max} : \quad x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon .$$

- Idempotency of $\oplus$:

$$\forall x \in \mathbb{R}_{\max} : \quad x \oplus x = x .$$

Powers are introduced in max-plus algebra in the natural way using the associative property. We denote the set of natural numbers including zero by $\mathbb{N}$ and define for $x \in \mathbb{R}_{\max}$

$$x^{\otimes n} \stackrel{\text{def}}{=} \underbrace{x \otimes x \otimes \cdots \otimes x}_{n \text{ times}} \tag{1.3}$$

for all $n \in \mathbb{N}$ with $n \neq 0$, and for $n = 0$ we define $x^{\otimes 0} \stackrel{\text{def}}{=} e\,(= 0)$. Observe that $x^{\otimes n}$, for any $n \in \mathbb{N}$, reads in conventional algebra as

$$x^{\otimes n} = \underbrace{x + x + \cdots + x}_{n \text{ times}} = n \times x\,.$$

For example,

$$5^{\otimes 3} = 3 \times 5 = 15.$$

Inspired by this we similarly introduce negative powers of real numbers, as in

$$8^{\otimes -2} = -2 \times 8 = -16 = 16^{\otimes -1}\,,$$

for example. In the same vein, max-plus roots can be introduced as

$$x^{\otimes \alpha} = \alpha \times x\,,$$

for $\alpha \in \mathbb{R}$. For example,

$$8^{\otimes \frac{1}{2}} = \frac{1}{2} \times 8 = 4$$

and

$$12^{\otimes -\frac{1}{4}} = -\frac{1}{4} \times 12 = -3 = 3^{\otimes -1}\,.$$

Continuing with the algebraic point of view, we show that max-plus algebra is an example of an algebraic structure, called a *semiring*, to be introduced next.

DEFINITION 1.1 *A semiring is a nonempty set R endowed with two binary operations $\oplus_R$ and $\otimes_R$ such that*

- *$\oplus_R$ is associative and commutative with zero element ε_R;*
- *$\otimes_R$ is associative, distributes over $\oplus_R$, and has unit element e_R;*
- *ε_R is absorbing for $\otimes_R$.*

Such a semiring is denoted by $\mathcal{R} = (R, \oplus_R, \otimes_R, \varepsilon_R, e_R)$. If $\otimes_R$ is commutative, then $\mathcal{R}$ is called commutative, *and if $\oplus_R$ is idempotent, then it is called* idempotent.

Max-plus algebra is an example of a commutative and idempotent semiring. Are there other meaningful semirings? The answer is yes, and a few examples are listed below.

Example 1.1.1

- *Identify $\oplus_R$ with conventional addition, denoted by $+$, and $\otimes_R$ with conventional multiplication, denoted by $\times$. Then the zero and unit element are $\varepsilon_R = 0$ and $e_R = 1$, respectively. The object $\mathcal{R}_{\mathrm{st}} = (\mathbb{R}, +, \times, 0, 1)$ – the subscript* st *refers to "standard" – is an instance of a semiring over the real numbers. Since conventional multiplication is commutative, $\mathcal{R}_{\mathrm{st}}$ is a commutative semiring. Note that $\mathcal{R}_{\mathrm{st}}$ fails to be idempotent. However, as is well known, $\mathcal{R}_{\mathrm{st}}$ is a ring and even a field with respect to the operations $+$ and $\times$. See the notes section for some further remarks on semirings and rings.*

- *Min-plus algebra is defined as* $\mathcal{R}_{\min} = (\mathbb{R}_{\min}, \oplus', \otimes, \varepsilon', e)$, *where* $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$, $\otimes'$ *is the operation defined by* $a \otimes' b \stackrel{\text{def}}{=} \min(a, b)$ *for all* $a, b \in \mathbb{R}_{\min}$, *and* $\varepsilon' \stackrel{\text{def}}{=} +\infty$. *Note that* $\mathcal{R}_{\min}$ *is an idempotent, commutative semiring.*

- *Consider* $\mathcal{R}_{\min,\max} = (\overline{\mathbb{R}}, \oplus', \oplus, \varepsilon', \varepsilon)$, *with* $\overline{\mathbb{R}} = \mathbb{R} \cup \{\varepsilon, \varepsilon'\}$, *and set* $\varepsilon \oplus \varepsilon' = \varepsilon' \oplus \varepsilon = \varepsilon'$. *Then* $\mathcal{R}_{\min,\max}$ *is an idempotent, commutative semiring. In the same vein,* $\mathcal{R}_{\max,\min} = (\overline{\mathbb{R}}, \oplus, \oplus', \varepsilon, \varepsilon')$ *is an idempotent, commutative semiring provided that one defines* $\varepsilon \oplus' \varepsilon' = \varepsilon' \oplus' \varepsilon = \varepsilon$.

- *As a last example of a semiring of a somewhat different nature, let* S *be a nonempty set. Denote the set of all subsets of* S *by* R*; then* $(R, \cup, \cap, \emptyset, S)$, *with* $\emptyset$ *the empty set, and* $\cup$ *and* $\cap$ *the set-theoretic union and intersection, respectively, is a commutative, idempotent semiring. The same applies to* $(R, \cap, \cup, S, \emptyset)$.

The above list of examples explains why we choose an algebraic approach. Any statement that is proved for a semiring will immediately hold in any of the above algebras. Apart from the structural insight this provides into the relationship between the different algebras, the algebraic approach also saves a lot of work.

To illustrate this, consider the following problem. Is it possible to define inverse elements (i.e., inverse with respect to the $\oplus_R$ operation) in an idempotent semiring? For example, consider an idempotent semiring $\mathcal{R} = (R, \oplus_R, \otimes_R, \varepsilon_R, e_R)$, with $\mathbb{R}$ included in R, such as the max-plus or min-plus semiring. For example, is it possible to find a solution of

$$5 \oplus_R x = 3\,? \tag{1.4}$$

As in conventional algebra, it is tempting to subtract 5 on both sides of the above equation in order to obtain

$$x = 3 \oplus_R (-5)$$

as a solution. However, is it possible to give meaning to -5 in the above equation? Take, for example, max-plus algebra. Then, equation (1.4) reads

$$\max(5, x) = 3\,. \tag{1.5}$$

Obviously, there exists no number that makes equation (1.5) true. On the other hand, in min-plus algebra, equation (1.5) reads

$$\min(5, x) = 3$$

and has the solution $x = 3$. Now interchange the numbers 3 and 5 in equation (1.4), yielding $3 \oplus_R x = 5$. This equation has no solution in min-plus algebra and has the obvious solution $x = 5$ in max-plus algebra.

Whether an equation has a solution may depend on the algebra. This raises the question whether a particular semiring (i.e., a particular interpretation of the symbols $\oplus_R$, $\otimes_R$, e_R, and ε_R) exists such that *all* equations of type (1.4) can be solved. The following lemma provides an answer.

LEMMA 1.2 *Let $\mathcal{R} = (R, \oplus_R, \otimes_R, \varepsilon_R, e_R)$ be a semiring. Idempotency of $\oplus_R$ implies that inverse elements with respect to $\oplus_R$ do not exist.*

Proof. Suppose that $a \neq \varepsilon_R$ had an inverse element with respect to $\oplus_R$, say, b. In formula, this is

$$a \oplus_R b = \varepsilon_R .$$

Adding a on both sides of the above equation yields

$$a \oplus_R a \oplus_R b = a \oplus_R \varepsilon_R .$$

By idempotency, the left-hand side of the above equation equals $a \oplus_R b$, whereas the right-hand side is equal to a. Hence, we have

$$a \oplus_R b = a ,$$

which contradicts $a \oplus_R b = \varepsilon_R$. □

Lemma 1.2 thus gives a negative answer to the above question, because no idempotent semiring exists for which negative numbers can be defined. Observe that this does not contradict the fact that $\mathcal{R}_{\mathrm{st}}$, defined in Example 1.1.1, is a semiring because $\mathcal{R}_{\mathrm{st}}$ is not idempotent. The fact that we cannot subtract in an idempotent semiring explains why the methods encountered later, when studying max-plus algebra, will differ significantly from those in conventional algebra.

1.2 VECTORS AND MATRICES

In this section matrices over $\mathbb{R}_{\max}$ will be introduced. The set of $n \times m$ matrices with underlying max-plus algebra is denoted by $\mathbb{R}_{\max}^{n\times m}$. For $n \in \mathbb{N}$ with $n \neq 0$, define $\underline{n} \stackrel{\mathrm{def}}{=} \{1, 2, \ldots, n\}$. The element of a matrix $A \in \mathbb{R}_{\max}^{n\times m}$ in row i and column j is denoted by a_{ij}, for $i \in \underline{n}$ and $j \in \underline{m}$. Matrix A can then be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} .$$

Occasionally, the element a_{ij} will also be denoted as

$$[A]_{ij}, \qquad i \in \underline{n}, j \in \underline{m}. \tag{1.6}$$

The sum of matrices $A, B \in \mathbb{R}_{\max}^{n\times m}$, denoted by $A \oplus B$, is defined by

$$\begin{aligned} [A \oplus B]_{ij} &= a_{ij} \oplus b_{ij} \\ &= \max\big(a_{ij}\, ,\, b_{ij}\big), \end{aligned} \tag{1.7}$$

for $i \in \underline{n}$ and $j \in \underline{m}$. For example, let

$$A = \begin{pmatrix} e & \varepsilon \\ 3 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & 11 \\ 1 & \varepsilon \end{pmatrix}; \tag{1.8}$$

then $[A \oplus B]_{11} = e \oplus -1 = \max(0, -1) = 0 = e$. Likewise, it follows that $[A \oplus B]_{12} = \varepsilon \oplus 11 = \max(-\infty, 11) = 11$, $[A \oplus B]_{21} = 3 \oplus 1 = \max(3, 1) = 3$, and $[A \oplus B]_{22} = 2 \oplus \varepsilon = \max(2, -\infty) = 2$. In matrix notation,

$$A \oplus B = \begin{pmatrix} e & 11 \\ 3 & 2 \end{pmatrix}.$$

Note that for $A, B \in \mathbb{R}_{\max}^{n \times m}$ it holds that $A \oplus B = B \oplus A$ (see Exercise 4).

For $A \in \mathbb{R}_{\max}^{n \times m}$ and $\alpha \in \mathbb{R}_{\max}$, the scalar multiple $\alpha \otimes A$ is defined by

$$[\alpha \otimes A]_{ij} = \alpha \otimes a_{ij} \tag{1.9}$$

for $i \in \underline{n}$ and $j \in \underline{m}$. For example, let A be defined as in (1.8) and take $\alpha = 2$; then $[2 \otimes A]_{11} = 2 \otimes e = 2 + 0 = 2$. Likewise, it follows that $[2 \otimes A]_{12} = \varepsilon$, $[2 \otimes A]_{21} = 5$, and $[2 \otimes A]_{22} = 4$, yielding, in matrix notation,

$$2 \otimes A = \begin{pmatrix} 2 & \varepsilon \\ 5 & 4 \end{pmatrix}.$$

For matrices $A \in \mathbb{R}_{\max}^{n \times l}$ and $B \in \mathbb{R}_{\max}^{l \times m}$, the matrix product $A \otimes B$ is defined as

$$\begin{aligned} [A \otimes B]_{ik} &= \bigoplus_{j=1}^{l} a_{ij} \otimes b_{jk} \\ &= \max_{j \in \underline{l}} \left\{ a_{ij} + b_{jk} \right\} \end{aligned} \tag{1.10}$$

for $i \in \underline{n}$ and $k \in \underline{m}$. This is just like in conventional algebra with $+$ replaced by max and $\times$ by $+$. Notice that $A \otimes B \in \mathbb{R}_{\max}^{n \times m}$, i.e., has n rows and m columns. For example, let A and B be defined as in (1.8); then the elements of $A \otimes B$ are given by

$$\begin{aligned} [A \otimes B]_{11} &= e \otimes (-1) \oplus \varepsilon \otimes 1 = \max(0 - 1, -\infty + 1) = -1, \\ [A \otimes B]_{12} &= e \otimes 11 \oplus \varepsilon \otimes \varepsilon = \max(0 + 11, -\infty - \infty) = 11, \\ [A \otimes B]_{21} &= 3 \otimes (-1) \oplus 2 \otimes 1 = \max(3 - 1, 2 + 1) = 3, \end{aligned}$$

and

$$[A \otimes B]_{22} = 3 \otimes 11 \oplus 2 \otimes \varepsilon = \max(3 + 11, 2 - \infty) = 14,$$

yielding, in matrix notation,

$$A \otimes B = \begin{pmatrix} -1 & 11 \\ 3 & 14 \end{pmatrix}.$$

Notice that the matrix product in general fails to be commutative. Indeed, for the above A and B

$$B \otimes A = \begin{pmatrix} 14 & 13 \\ 1 & \varepsilon \end{pmatrix} \neq A \otimes B.$$

Let $\mathcal{E}(n, m)$ denote the $n \times m$ matrix with all elements equal to ε, and denote by $E(n, m)$ the $n \times m$ matrix defined by

$$[E(n, m)]_{ij} \stackrel{\text{def}}{=} \begin{cases} e & \text{for } i = j, \\ \varepsilon & \text{otherwise.} \end{cases}$$

If $n = m$, then $E(n,n)$ is called the $n \times n$ identity matrix. When their dimensions are clear from the context, $\mathcal{E}(n,m)$ and $E(n,m)$ will also be written as $\mathcal{E}$ and E, respectively. It is easily checked (see exercise 5) that any matrix $A \in \mathbb{R}_{\max}^{n\times m}$ satisfies

$$A \oplus \mathcal{E}(n,m) = A = \mathcal{E}(n,m) \oplus A,$$

$$A \otimes E(m,m) = A = E(n,n) \otimes A.$$

Moreover, for $k \geq 1$ it holds that

$$A \otimes \mathcal{E}(m,k) = \mathcal{E}(n,k) \qquad \text{and} \qquad \mathcal{E}(k,n) \otimes A = \mathcal{E}(k,m).$$

For $\mathbb{R}_{\max}^{n\times m}$, the matrix addition $\oplus$, as defined in (1.7), is associative, commutative, and has zero element $\mathcal{E}(n,m)$. For $\mathbb{R}_{\max}^{n\times n}$ the matrix product $\otimes$, as defined in (1.10), is associative, distributive with respect to $\oplus$, has unit element $E(n,n)$, and $\mathcal{E}(n,n)$ is absorbing for $\otimes$.

The transpose of an element $A \in \mathbb{R}_{\max}^{n\times m}$, denoted by $A^\top$, is defined in the usual way by $[A^\top]_{ij} = a_{ji}$, for $i \in \underline{n}$ and $j \in \underline{m}$. As before, also in matrix addition and multiplication, the operation $\otimes$ has priority over the operation $\oplus$.

The elements of $\mathbb{R}_{\max}^{n} \stackrel{\text{def}}{=} \mathbb{R}_{\max}^{n\times 1}$ are called *vectors*. The jth element of a vector $x \in \mathbb{R}_{\max}^{n}$ is denoted by x_j, which, in the spirit of (1.6), also will be written as $[x]_j$. The vector in $\mathbb{R}_{\max}^{n}$ with all elements equal to e is called the *unit vector* and is denoted by $\mathbf{u}$; in formula, $[\mathbf{u}]_j = e$ for $j \in \underline{n}$. Notice that $\alpha \otimes \mathbf{u}$ denotes a vector with all elements equal to α, for any $\alpha \in \mathbb{R}_{\max}$. For any $j \in \underline{n}$, the jth column of the identity matrix $E(n,n)$ is called the jth *base vector* of $\mathbb{R}_{\max}^{n}$ and is denoted by e_j. Hence, the jth element of e_j has value e, while the other elements of e_j are equal to ε.

Note that for $A \in \mathbb{R}_{\max}^{n\times m}$ and $x \in \mathbb{R}_{\max}^{m}$, the product $A \otimes x$ is defined by (1.10) for $x = B$. Clearly, $A \otimes A$ and higher order powers of A are only defined for $A \in \mathbb{R}_{\max}^{n\times n}$, i.e., for matrices A that are square.

In the following a careful distinction will be made between $\mathbb{R}_{\max}^{n}$ (the set of n-dimensional vectors over $\mathbb{R}_{\max}$), $\mathbb{R}_{\max}^{n\times m}$ (the set of $n \times m$ matrices over $\mathbb{R}_{\max}$), and $\mathbb{R}_{\max}^{n\times n}$ (the set of square $n \times n$ matrices over $\mathbb{R}_{\max}$).

The structure

$$\mathcal{R}_{\max}^{n\times n} = \left(\mathbb{R}_{\max}^{n\times n}, \oplus, \otimes, \mathcal{E}, E\right),$$

with $\oplus$ and $\otimes$ as defined in (1.7) and (1.10), respectively, constitutes a noncommutative, idempotent semiring.

For $A \in \mathbb{R}_{\max}^{n\times n}$, denote the kth power of A by $A^{\otimes k}$ defined by

$$A^{\otimes k} \stackrel{\text{def}}{=} \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}, \tag{1.11}$$

for $k \in \mathbb{N}$ with $k \neq 0$, and set $A^{\otimes 0} \stackrel{\text{def}}{=} E(n,n)$. The above definition is a straightforward extension of (1.3) to matrices. Notice that $[A^{\otimes k}]_{ij}$ has to be carefully distinguished from $(a_{ij})^{\otimes k}$. Indeed, the former is element (i,j) of the kth power of A, whereas the latter is the kth power of element (i,j) of A.

A mapping f from $\mathbb{R}^n_{\max}$ to $\mathbb{R}^n_{\max}$ is called *affine* if $f(x) = A \otimes x \oplus b$ for some $A \in \mathbb{R}^{n\times n}_{\max}$ and $b \in \mathbb{R}^n_{\max}$. If $b = \mathcal{E}$, then f is called *linear*. A recurrence relation $x(k+1) = f(x(k))$, for $k \in \mathbb{N}$, is called *affine* (resp., *linear*) if f is an affine (resp., linear) mapping.

A matrix $A \in \mathbb{R}^{n\times m}_{\max}$ is called *regular* if A contains at least one element different from ε in each row. Regularity is a mere technical condition, for if A fails to be regular, it contains redundant rows, and any system modeled by $x(k+1) = A \otimes x(k)$ can also be modeled by considering a reduced regular version of A in which all redundant rows and related columns are skipped.

A matrix $A \in \mathbb{R}^{n\times n}_{\max}$ is called *strictly lower triangular* if $a_{ij} = \varepsilon$, for $1 \leq i \leq j \leq n$. If $a_{ij} = \varepsilon$, for $1 \leq i < j \leq n$, then A is called *lower triangular*. Matrix A is said to be *(strictly) upper triangular* if $A^\top$ is (strictly) lower triangular.

For countable sets the max operator has to be understood as a supremum. More formally, let $\{a_i : i \in \mathbb{N}\}$ be a countable set, with $a_i \in \mathbb{R}_{\max}$; then

$$\bigoplus_{i\geq 0} a_i \stackrel{\text{def}}{=} \bigoplus_{i=0}^{\infty} a_i \stackrel{\text{def}}{=} \sup_{i\geq 0} a_i .$$

For max-plus algebra one easily verifies Fubini's rule; namely, that for $\{a_{ij} \in \mathbb{R}_{\max} : i, j \in \mathbb{N}\}$,

$$\bigoplus_{i\geq 0}\bigoplus_{j\geq 0} a_{ij} = \bigoplus_{j\geq 0}\bigoplus_{i\geq 0} a_{ij} . \tag{1.12}$$

Indeed, for any $k, j \geq 0$ it follows that $a_{kj} \leq \bigoplus_{i\geq 0} a_{ij}$, implying that

$$\bigoplus_{j\geq 0} a_{kj} \leq \bigoplus_{j\geq 0}\bigoplus_{i\geq 0} a_{ij} ,$$

for any $k \geq 0$, and consequently that

$$\bigoplus_{k\geq 0}\bigoplus_{j\geq 0} a_{kj} \leq \bigoplus_{j\geq 0}\bigoplus_{i\geq 0} a_{ij} .$$

The inverse inequality follows from similar arguments.

1.3 A FIRST MAX-PLUS MODEL

In this section, we present an important example of a max-plus system, called a *heap model*. In a heap model, solid pieces are piled up according to a mechanism resembling the Tetris game. However, the pieces can only fall downwards vertically and cannot be moved horizontally or rotated. More specifically, consider the pieces labeled a, b, and c, as given in Figures 1.1 to 1.3. The pieces occupy columns out of a finite set of columns. The set of column numbers is given by $\mathcal{R}$, in the example the set $\{1, 2, \ldots, 5\}$. When the pieces are piled up according to a fixed sequence, like a b a c b, for example, this results in the heap shown in Figure 1.4.

Situations like the one pictured in Figure 1.4 typically arise in scheduling problems. Here, pieces represent tasks that compete for a limited number of resources,

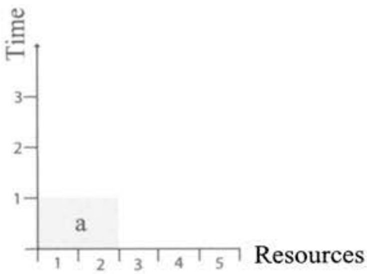

Figure 1.1: Piece a.

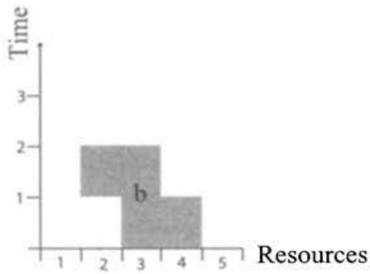

Figure 1.2: Piece b.

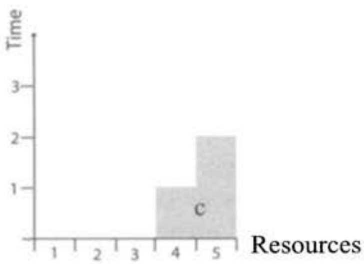

Figure 1.3: Piece c.

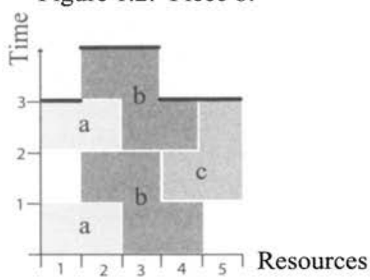

Figure 1.4: The heap w=abacb.

represented by the columns. The covering of a particular column by an individual piece can be interpreted as the amount of time required by the task (represented by the particular piece) of this resource.

Consider, for example, piece b in Figure 1.2. The idea is that piece b represents the time span for which resources have to be allocated in order to complete a certain task. More precisely, if processing the task is initiated at time t, then resource 2 will be occupied from time $t+1$ until time $t+2$, resource 3 will be occupied from time t until time $t+2$, resource 4 will be occupied from time t until time $t+1$, and resources 1 and 5 will not be occupied at all. The depiction of the processing times for this task given by piece b can be translated into mathematical terms by means of so-called contours. The upper and lower contours of a piece describe the covering of a piece lying on ground level. For example, the upper contour of piece b, denoted by $u(\mathrm{b})$, is

$$u(\mathrm{b}) = (\varepsilon, 2, 2, 1, \varepsilon)^\top,$$

and the lower contour, denoted by $l(\mathrm{b})$, reads

$$l(\mathrm{b}) = (\varepsilon, 1, \mathrm{e}, \mathrm{e}, \varepsilon)^\top,$$

where ε in the same location in $u(\mathrm{b})$ and $l(\mathrm{b})$ represents the fact that the piece does not cover the particular resource. The resources covered by piece b are denoted by $\mathcal{R}(\mathrm{b})$, so that $\mathcal{R}(\mathrm{b}) = \{2, 3, 4\}$. For piece a it follows that $\mathcal{R}(\mathrm{a}) = \{1, 2\}$. The upper contour of a is given by

$$u(\mathrm{a}) = (1, 1, \varepsilon, \varepsilon, \varepsilon)^\top,$$

and the lower contour equals

$$l(\mathrm{a}) = (\mathrm{e}, \mathrm{e}, \varepsilon, \varepsilon, \varepsilon)^\top.$$

As for piece c, it follows that $\mathcal{R}(\mathrm{c}) = \{4, 5\}$,

$$u(\mathrm{c}) = (\varepsilon, \varepsilon, \varepsilon, 1, 2)^\top,$$

and the lower contour equals

$$l(\mathrm{c}) = (\varepsilon, \varepsilon, \varepsilon, \mathrm{e}, \mathrm{e})^\top.$$

Before continuing, we will introduce some notation. Let $\mathcal{P}$ denote the finite set of pieces in the example $\mathcal{P} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. As already seen above a piece $\eta \in \mathcal{P}$ is characterized by its lower contour, denoted by $l(\eta)$, and its upper contour, denoted by $u(\eta)$. Moreover, the set of resources required by η is denoted by $\mathcal{R}(\eta)$. Let there be $n \in \mathbb{N}$, with $n \neq 0$, resources available. In our example we have $n = 5$. The upper and lower contours of a piece η are vectors over $\mathbb{R}^n_{\max}$, in formula $l(\eta), u(\eta) \in \mathbb{R}^n_{\max}$, such that

$$0 \leq l_r(\eta) \leq u_r(\eta) < \infty,$$

for $r \in \mathcal{R}(\eta)$, and

$$l_r(\eta) = u_r(\eta) = \varepsilon,$$

for $r \notin \mathcal{R}(\eta)$. Associate a matrix $M(\eta)$ with piece η through

$$[M(\eta)]_{rs} = \begin{cases} u_r(\eta) - l_s(\eta) & \text{for } r, s \in \mathcal{R}(\eta), \\ e & \text{for } s = r,\ r \notin \mathcal{R}(\eta), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Elaborating on the upper and lower contours of pieces a, b, and c, respectively, the following matrices are obtained:

$$M(\mathrm{a}) = \begin{pmatrix} 1 & 1 & \varepsilon & \varepsilon & \varepsilon \\ 1 & 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix}, \quad M(\mathrm{b}) = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 1 & 2 & 2 & \varepsilon \\ \varepsilon & 1 & 2 & 2 & \varepsilon \\ \varepsilon & e & 1 & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & e \end{pmatrix},$$

and

$$M(\mathrm{c}) = \begin{pmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & e & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 & 1 \\ \varepsilon & \varepsilon & \varepsilon & 2 & 2 \end{pmatrix}.$$

A sequence of pieces out of $\mathcal{P}$ is called a *heap*. For example, $w = \mathrm{a\,b\,a\,c\,b}$ is a heap; see Figure 1.4. Denote the *upper contour* of heap w by a vector $x_\mathcal{H}(w) \in \mathbb{R}^n_{\max}$, where $(x_\mathcal{H}(w))_r$ is the height of the heap on column r; for example, $x_\mathcal{H}(\mathrm{a\,b\,a\,c\,b}) = (3, 4, 4, 3, 3)^\top$, when starting from ground level. The upper contour of the heap a b a c b is indicated by the boldfaced line in Figure 1.4. For heap w and piece $\eta \in \mathcal{P}$, write $w\,\eta$ for the heap resulting from piling piece η on heap w. Note that the order in which the pieces fall is of importance. The upper contour follows the recurrence relation

$$[x_\mathcal{H}(w\,\eta)]_r = \max\Big\{[M(\eta)]_{rs} + [x_\mathcal{H}(w)]_s \,:\, s \in \mathcal{R}\Big\}, \qquad r \in \mathcal{R}, \qquad (1.13)$$

with initial upper contour $x_{\mathcal{H}}(\emptyset) = \mathbf{u}$, where $\emptyset$ denotes the empty heap. Elaborating on the notational power of the max-plus semiring, we can rewrite the above recurrence relation as

$$[x_{\mathcal{H}}(w\,\eta)]_r = \bigoplus_{s\in\mathcal{R}} [M(\eta)]_{rs} \otimes [x_{\mathcal{H}}(w)]_s, \qquad r \in \mathcal{R},$$

or, in a more concise way,

$$x_{\mathcal{H}}(w\,\eta) = M(\eta) \otimes x_{\mathcal{H}}(w).$$

In words, the upper contour of a heap of pieces follows a max-plus recurrence relation.

For a given sequence $\eta_1, \eta_2, \ldots, \eta_k$ of pieces, set, for notational convenience, $x_{\mathcal{H}}(k) = x_{\mathcal{H}}(\eta_1\,\eta_2\,\cdots\,\eta_k)$ and $M(k) = M(\eta_k)$. Then the upper contour follows the recurrence relation

$$x_{\mathcal{H}}(k+1) = M(k+1) \otimes x_{\mathcal{H}}(k), \qquad k \geq 1,$$

where $x_{\mathcal{H}}(0) = \mathbf{u}$.

In this context two kinds of limits are of interest, the first addressing the asymptotic growth rate of the heap and the second addressing the shape of the upper contour of the heap.

For a given sequence η_k, $k \in \mathbb{N}$, the *asymptotic growth rate* of the heap model is given by

$$\lim_{k\to\infty} \frac{1}{k} x_{\mathcal{H}}(k)\,,$$

provided that the limit exists. For example, if η_k, $k \in \mathbb{N}$, represents a particular schedule, like $\eta_1 = \mathrm{a}, \eta_2 = \mathrm{b}, \eta_3 = \mathrm{c}, \eta_4 = \mathrm{a}, \eta_5 = \mathrm{b}, \eta_6 = \mathrm{c}$, and so forth, then the above limit measures the efficiency of schedule $\mathrm{a\,b\,c}$.

For a given sequence η_k, $k \in \mathbb{N}$, the *asymptotic form of* $x_{\mathcal{H}}(k)$ can be studied, where *form* means the relative differences of the components of $x_{\mathcal{H}}(k)$. More precisely, in studying the shape of the upper contour the actual height of the heap is disregarded. To that end, the vector of relative differences in $x_{\mathcal{H}}(w)$, called the *shape vector*, is denoted by $s(w)$. For example, the shape of heap $w = \mathrm{a\,b\,a\,c\,b}$ in Figure 1.4 is obtained by letting the boldfaced line (the upper contour) sink to the ground level, yielding the vector $s(w) = (0,1,1,0,0)^\top$. More formally, the shape vector is defined as

$$s_r(w) = (x_{\mathcal{H}}(w))_r - \min\left\{ (x_{\mathcal{H}}(w))_p \,:\, p \in \mathcal{R} \right\}, \qquad r \in \mathcal{R}.$$

Suppose that the sequence in which the pieces appear cannot be controlled (their arrivals may be triggered by an external source). For instance, η_k, $k \in \mathbb{N}$, is a random sequence such that piece a, b, and c appear with equal probability. Set $s(k) \stackrel{\text{def}}{=} s(\eta_1, \eta_2, \ldots, \eta_k)$. Since pieces fall in random order, $s(k)$ is a random variable. Using probabilistic arguments, one can identify sufficiency conditions such that the probability distribution of $s(k)$ converges to a limiting distribution, say, F. Hence, the asymptotic shape of the heap is given by the probability distribution F. By means of F, for example, the probability can be determined that the completion time of tasks typically differs more than t time units over the resources, yielding an indication on how well balanced the schedule η_k, $k \in \mathbb{N}$, is. See the notes section for references.

The asymptotic growth rate will be addressed in Section 3.2, and (the deterministic variant of) the limit of the shape vector will be addressed in Section 4.4.

1.4 THE PROJECTIVE SPACE

To sketch the idea of this section, let A be an $n \times n$ matrix with *positive* elements and let $x(k) \in \mathbb{R}^n$ be defined through

$$x(k+1) = A \otimes x(k), \qquad k \geq 0,$$

with $x(0) = x_0 \in \mathbb{R}^n$. Then, $x(k)$ is monotonically increasing, meaning that each of its components $x_i(k)$, $i \in \underline{n}$, is monotonically increasing. Taking the limit of $x(k)$ as k tends to ∞ will result in $(+\infty, \ldots, +\infty)^\top$ as the limiting vector. Indeed, revisit, for example, recurrence relation (0.10). The matrix describing the travel times has positive entries, and $x(k) = 4^{\otimes k} \otimes x_0$ for $x_0 = (1\ ,\ 0)^\top$; see also Section 0.3. Notice that even though $x(k)$ diverges, the relative differences of $x(k)$ have a limit.

In this section, the modeling of differences within a vector will be explored more closely. Therefore, an equivalence relation on $\mathbb{R}^n_{\max}$ is introduced, denoted by $\cdot \parallel \cdot$, that is defined as

$$\forall y, z \in \mathbb{R}^n_{\max} : \quad y \parallel z \;\Leftrightarrow\; \exists \alpha \in \mathbb{R} : \;\; y = \alpha \otimes z,$$

where the equation on the right-hand side should be read as $y_i = \alpha + z_i$ for all $i \in \underline{n}$. Two vectors $y, z \in \mathbb{R}^n_{\max}$ are said to be *colinear* (resp., *proportional*) if $y \parallel z$.

For $z \in \mathbb{R}^n_{\max}$, write $\overline{z}$ for the equivalence class $\{y \in \mathbb{R}^n_{\max} : y \parallel z\}$. Let $\mathbb{PR}^n_{\max}$ denote the *projective space*; that is, $\mathbb{PR}^n_{\max}$ is the quotient space of $\mathbb{R}^n_{\max}$ by the above equivalence relation. More formally,

$$\mathbb{PR}^n_{\max} = \{\overline{z} : z \in \mathbb{R}^n_{\max}\}.$$

The bar operator is the canonical projection of $\mathbb{R}^n_{\max}$ onto $\mathbb{PR}^n_{\max}$. In the same vein, denote by $\mathbb{PR}^n$ the quotient space of $\mathbb{R}^n$ by the above equivalence relation. With this terminology, the limit of the relative differences in the upper contour as k tends to ∞ now reads

$$\lim_{k \to \infty} \overline{x(k)}, \tag{1.14}$$

provided that the limit exists. For example, let v be an eigenvector of A and let λ be the corresponding eigenvalue. See, for instance, Section 0.3 for an introduction of these notions. Then, because all elements of A are positive, it can be shown (see Chapter 2) that v contains only finite elements and that $\lambda > 0$. Hence, $v \in \mathbb{R}^n$ and $\lambda > 0$ are such that $A \otimes v = \lambda \otimes v$. For $x(0) = v$, it then follows that $x(k) = \lambda^{\otimes k} \otimes v$ and $\overline{x(k)} = \overline{v}$. Hence, the shape is equal to $\overline{v}$ for any k. The projective space turns out to be a convenient mathematical space for speaking about limits of sequences $\{x(k) : k \in \mathbb{N}\}$ stemming from max-plus recurrence relations.

Notice that it actually has not been explained what it means when the above limit is said to exist. A precise definition will be provided in Section 4.4.

1.5 EXERCISES

1. Show that the algebraic structures defined in Example 1.1.1 are indeed semirings.
2. Compute the following:

   (a) $-8 \otimes \varepsilon$

   (b) $(-1)^{\otimes \frac{1}{2}}\ (\stackrel{\mathrm{def}}{=} \sqrt[\otimes]{-1})$

   (c) the product of the next two matrices

$$\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & e \\ 2 & 1 \\ \varepsilon & \varepsilon \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & \varepsilon & 8 & e \\ -1 & \varepsilon & \varepsilon & 7 & 1 \end{pmatrix}$$

3. Show that for any $n \in \mathbb{N}$ numbers $x, y, z \in \mathbb{R}_{\max}$ exist such that

$$z^{\otimes n} = x^{\otimes n} \oplus y^{\otimes n},$$

   i.e., Fermat's theorem is not true over $\mathbb{R}_{\max}$.

4. Show that for $A, B, C \in \mathbb{R}_{\max}^{n \times n}$ the following properties are true:

   (a) Associativity: $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ and $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

   (b) Commutativity: $A \oplus B = B \oplus A$

   (c) Distributivity of $\otimes$ over $\oplus$: $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$

5. Let $A \in \mathbb{R}_{\max}^{n \times m}$. Show that

$$A \oplus \mathcal{E}(n, m) = A = \mathcal{E}(n, m) \oplus A\,,$$

$$A \otimes \mathcal{E}(m, k) = \mathcal{E}(n, k)\,,$$

   and

$$\mathcal{E}(k, n) \otimes A = \mathcal{E}(k, m)\,,$$

   for $k \geq 1$. Moreover, show

$$A \otimes E(m, m) = A = E(n, n) \otimes A\,.$$

6. (a) Show that for $\mathcal{R}_{\min,\max}$ to be a semiring, one needs to define $\max(+\infty, -\infty) = \max(-\infty, +\infty) = +\infty$.

   (b) Show that for $\mathcal{R}_{\max,\min}$ to be a semiring, one needs to define $\min(+\infty, -\infty) = \min(-\infty, +\infty) = -\infty$.

   (c) Show that an expression in terms of $\otimes_R$ and $\oplus_R$, in general, will attain different numerical values when evaluated in $\mathcal{R}_{\min,\max}$ or in $\mathcal{R}_{\max,\min}$.

7. Show that (1.13) is indeed the correct recurrence relation.
8. Let $x, y \in \mathbb{R}_{\max}^n$ be such that $\alpha \otimes x = y$ for some $\alpha \in \mathbb{R}$. Show that $\overline{x} = \overline{y}$.
9. Show that for $\overline{x}, \overline{y} \in \mathbb{PR}_{\max}^n$ it generally does not hold that $\overline{x} \oplus \overline{y} = \overline{x \oplus y}$.

10. A semiring $\mathcal{R}$ is said to have zero-divisors if elements $x, y \neq \varepsilon_R$ exist such that $x \otimes_R y = \varepsilon_R$. Show that $\mathcal{R}_{\max}$ is zero-divisor free and that, for $n > 1$, $\mathcal{R}_{\max}^{n \times n}$ possesses zero-divisors. (Hint: Use matrices

$$A = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & e \end{pmatrix}, \quad B = \begin{pmatrix} e & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix},$$

and show that $A \otimes B = \mathcal{E}$.)

11. Let $\mathcal{B} = \{\varepsilon, e\}$. Then $(\mathcal{B}, \oplus, \otimes, \varepsilon, e)$ is called Boolean algebra. Show that Boolean algebra is a semiring.

1.6 NOTES

For an extensive discussion of max-plus algebra and similar structures we refer to [5]. An early reference is [31]. A historical overview of the beginnings of max-plus theory can be found in [36]. This article also contains many more examples of semirings. For more details on idempotency, see [48]. In [21] the solvability of sets of equations over $\mathcal{R}_{\min,\max}$ and $\mathcal{R}_{\max,\min}$ is treated.

From the semiring theory point of view, it seems more natural to use the symbols $+$ and $\times$ instead of $\oplus$ and $\otimes$ and, for consistency, 0 and 1 for the zero and the unit. However, in applications, often hybrid formulas are encountered containing conventional addition and multiplication as well as addition and multiplication in a semiring. For this reason, the notation for semirings will be carefully distinguished from that for operations in conventional algebra.

The term *semiring* originates from the fact that $(R, \oplus_R, \varepsilon_R)$ in the definition of a semiring is a semigroup. Indeed, since that inverse elements with respect to $\oplus_R$ do not exist, it follows that $(R, \oplus_R, \varepsilon_R)$ is not a group but a semigroup (and even a monoid). Consequently, $(R, \oplus_R, \otimes_R, \varepsilon_R, e_R)$, with all the properties stated in Definition 1.1, is not a ring but is just a semiring. In literature, idempotent semirings are also called *dioids*; see [5]. Observe that $\mathcal{R}_{\max}$ is by no means an algebra in the classical sense. The name *max-plus algebra* is only historically justified, and the correct name for $\mathcal{R}_{\max}$ would be *idempotent semiring* or *dioid* (which might explain why the name *max-plus algebra* is still predominant in the literature). The book [45] discusses general aspects of idempotent structures, also in the infinite-dimensional case (in connection with the Hamilton-Jacobi equation). A reference book on general algebraic structures is [44]. The books [64] and [57] focus on applications in physics.

Heap models were introduced in [40] and further studied in [42]; see also the references therein for more details. For applications of heap models to scheduling we refer to [19], [41], and [42]. A variant of the heap model is to consider colored pieces. The basic idea is to normalize the heap, consisting of differently colored pieces, to a certain fixed height. When piling up pieces, the overall height of the heap does not change but its average color does. For example, having only two colors, say, red and blue, the heap will in the limit attain a certain shade of purple representing the limit regime of the schedule.

In discrete-time optimal control or, in Markovian decision theory one encounters the equation

$$V(k, x) = \max_u \big(V(k+1, f(x, u)) + g(x, u)\big),$$

which is a consequence of Bellman's principle of optimality. The underlying model is $x(k+1) = f(x(k), u(k))$ and the costs during time step k are $g(x(k), u(k))$. The function V is the value function. This equation, with the operations addition and maximization can be interpreted and analyzed in the sense of max-plus algebra; see [1] and [65].