Chapter Zero

Prolegomenon

In this book we will model, analyze, and optimize phenomena in which the order of events is crucial. The timing of such events, subject to synchronization constraints, forms the core. This zeroth chapter can be viewed as an appetizer for the other chapters to come.

0.1 INTRODUCTORY EXAMPLE

Consider a simple railway network between two cities, each with a station, as indicated in Figure 0.1. These stations are called S_1 and S_2, respectively, and are connected by two tracks. One track runs from S_1 to S_2, and the travel time for a train along this track is assumed to be 3 time units. The other track runs from S_2 to S_1, and a voyage along this track lasts 5 time units. Together, these two tracks form a circuit. Trains coming from S_1 and arriving in S_2 will return to S_1 along the other track, and trains that start at S_2 will, after having visited S_1, come back to S_2. Apart from these two tracks, two other tracks, actually circuits, exist, connecting the suburbs of a city with its main station. A round trip along these tracks lasts 2 units of time for the first city and 3 time units for the second city. Of course, local stations exist in these suburbs, but since they will not play any role in the problem, they are not indicated. We want to design a timetable subject to the following criteria:

- The travel times of the trains along each of the tracks are fixed (and given).
- The frequency of the trains (i.e., the number of departures per unit of time) must be as high as possible.
- The frequency of the trains must be the same along all four tracks, yielding a timetable with regular departure times.
- The trains arriving at a station should wait for each other in order to allow the changeover of passengers.
- The trains at a station depart the station as soon as they are allowed.

We will start with a total number of four trains in the model, one train on each of the outer circuits and two trains on the inner circuit. The departure time of the two trains at station S_1, one in the direction of S_2 and the other one to serve the suburbs, will be indicated by x_1. These two trains depart at the same time because of the requirement of the changeover of passengers and that trains depart as soon

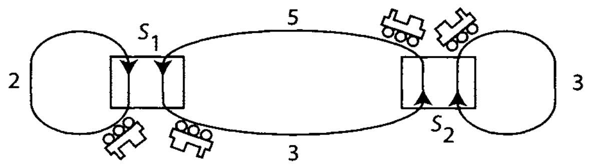

Figure 0.1: The railway network. The numbers along the tracks refer to travel times.

as possible. Similarly, x_2 is the common departure time of the two trains at S_2. Together, the departure times are written as the vector $x \in \mathbb{R}^2$. The first departure times during a day, in the early morning, will be given by $x(0)$. The trains thereafter leave at the time instants given by the two elements of the vector $x(1)$ and so on. The kth departure times are indicated by $x(k-1)$. These departures are called *events* in the model. Because of the rules given, it follows that

$$\begin{aligned} x_1(k+1) &\geq x_1(k) + a_{11} + \delta, \\ x_1(k+1) &\geq x_2(k) + a_{12} + \delta. \end{aligned} \tag{0.1}$$

The quantities a_{ij} denote the travel time from the station indicated by the second subscript (S_j) to the station indicated by the first subscript (S_i), and δ denotes the time reserved for the passengers to change from one train to the other. Without loss of generality, δ can be thought of being part of the travel time. In other words, the travel time can be defined as the actual travel time to which the changeover time, or transfer time, has been added. Hence, from now on it will be assumed that $\delta = 0$. Substituting $a_{11} = 2$ and $a_{12} = 5$, it follows that

$$x_1(k+1) \geq \max(x_1(k) + 2, x_2(k) + 5).$$

Similarly, the departure times at S_2 must satisfy

$$x_2(k+1) \geq \max(x_1(k) + 3, x_2(k) + 3).$$

Since the frequency of the departures must be as high as possible and the trains depart as soon as possible, the inequalities in the latter two expressions will, in fact, have to be equalities, which leads to

$$\begin{aligned} x_1(k+1) &= \max(x_1(k) + 2, x_2(k) + 5), \\ x_2(k+1) &= \max(x_1(k) + 3, x_2(k) + 3). \end{aligned} \tag{0.2}$$

Then, if the initial departure times $x(0)$ are given, all future departure times are uniquely determined. If for instance $x_1(0) = x_2(0) = 0$, then the sequence $x(k)$, for $k = 0, 1, \ldots$, becomes

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 11 \end{pmatrix}, \begin{pmatrix} 16 \\ 16 \end{pmatrix}, \ldots . \tag{0.3}$$

Compare this sequence with the following one, obtained if the initial departure times are $x_1(0) = 1$ and $x_2(0) = 0$ (i.e., the first trains at S_2, one in each direction, still leave at time 0, but the first trains at S_1 now leave at time 1),

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 12 \end{pmatrix}, \begin{pmatrix} 17 \\ 16 \end{pmatrix}, \ldots . \tag{0.4}$$

With the interdeparture time being the time duration between two subsequent departures along the same track, both sequences have the same *average* interdeparture time equal to 4, but the second sequence has exactly this interdeparture time, whereas the first sequence has it only on average (the average of the interdeparture times 3 and 5). If these sequences were real timetable departures, then most people would prefer the second timetable since it is regular.

A question that might arise is whether it would be possible to have a "faster" timetable (i.e., a timetable with a smaller average interdeparture time) by choosing appropriate initial departure times? The answer is no. The reason is that the time duration for a train to go around on the inner circuit is equal to 8, and there are two trains on this circuit. Hence, the average interdeparture time can never be smaller than $8/2 = 4$.

As a direct generalization of (0.2), one can study

$$x_i(k+1) = \max(x_1(k) + a_{i1}, x_2(k) + a_{i2}, \ldots, x_n(k) + a_{in}), \tag{0.5}$$

for $i = 1, 2, \ldots, n$, and in fact, the study of these equations will be the central theme of this book. The x_i's could, for instance, be the departure times in a more general railway network, with the explicit possibility that some of the terms in the right-hand side are not present. In the terminology of train networks this would mean, as in reality, that there are no direct tracks between some of the stations. This absence of terms is solved by allowing the value $-\infty$ for the corresponding a_{ij}-quantities. The value $-\infty$ will never contribute to the max operation, and thus there is no need to change the notation in (0.5). In the parlance of train networks, if a direct connection between stations S_j and S_i does not exist, then simply define its corresponding travel time to be equal to $-\infty$, i.e., $a_{ij} = -\infty$. This is done for mathematical convenience. At first sight one might be tempted to set $a_{ij} = +\infty$ for a nonexisting track. Setting $a_{ij} = +\infty$, however, refers to an existing track with an extremely high travel time, and a term with such an a_{ij}-quantity in a max-plus expression will dominate all other terms.

We can draw a directed graph based on (0.5). Such a graph has n nodes, one node corresponding to each x_i, and a number of directed arcs (i.e., arrows), one arc from node j to node i for each $a_{ij} \neq -\infty$. More details follow in Chapter 2.

0.2 ON THE NOTATION

Equation (0.5) can be written more compactly as

$$x_i(k+1) = \max_{j=1,2,\ldots,n} (a_{ij} + x_j(k)), \qquad i = 1, 2, \ldots, n. \tag{0.6}$$

Many readers will be familiar with linear recurrence relations of the form

$$z_i(k+1) = \sum_{j=1}^{n} c_{ij} z_j(k), \qquad i = 1, 2, \ldots, n.$$

For conceptual reasons the above equation will also be written as

$$z_i(k+1) = \sum_{j=1}^{n} c_{ij} \times z_j(k), \qquad i = 1, 2, \ldots, n, \tag{0.7}$$

and the reader will now immediately notice the resemblance between (0.6) and (0.7). The "only" difference between the two expressions is that the maximization and addition in the first equation are replaced by the addition and multiplication, respectively, in the second equation. In order to make this resemblance more clear we will change the notation of the operations in (0.6). For (0.6) we will henceforth write

$$x_i(k+1) = \bigoplus_{j=1}^{n} a_{ij} \otimes x_j(k), \qquad i = 1, 2, \ldots, n. \tag{0.8}$$

For the pronunciation of the symbol $\bigoplus$ and related ones, which will be introduced later, the reader is referred to Table 0.1.

symbol	pronunciation	symbol	pronunciation
$\bigoplus$	big o-plus	$\oplus$	o-plus
$\bigotimes$	big o-times	$\otimes$	o-times
$\bigoplus'$	big o-plus prime	$\otimes'$	o-times prime

Table 0.1: The pronunciation of some symbols.

If the circles around $\bigoplus$ and $\otimes$ in (0.8) were omitted, we would get equations of type (0.7), be it that the summations still are different in notation. To be very clear, the evolutions with respect to the parameter k of the processes characterized by (0.6) and (0.8) will be the same and will generally be different from the evolution resulting from (0.7). In conventional linear algebra the scalar equations (0.7) can be written in vector form as

$$z(k+1) = Cz(k).$$

In the same way, we will write (0.5) in vector form as

$$x(k+1) = A \otimes x(k), \tag{0.9}$$

with $\otimes$ to indicate that the underlying process is not described in terms of the conventional linear algebra (upon which (0.7) is based). The algebra underlying equations of the form (0.5) and (0.9) is called *max-plus algebra*. The relation between $\bigoplus$ in (0.8) and $\oplus$ in Table 0.1 is clarified by

$$\bigoplus_{j=1}^{n} (a_{ij} \otimes x_j) = (a_{i1} \otimes x_1) \oplus (a_{i2} \otimes x_2) \oplus \cdots \oplus (a_{in} \otimes x_n)$$

for any $i = 1, 2, \ldots, n$.

The reason for wanting to emphasize the resemblance between the notation in conventional algebra and that of max-plus algebra is partly historical. More important, many well-known concepts in conventional linear algebra can be carried over to max-plus algebra, which can be emphasized with a similar notation. The next section on eigenvectors will make this particularly clear, where the evolution of the state of model (0.9) will be considered in more detail.

For $k = 1$ we get $x(1) = A \otimes x(0)$, and for $k = 2$

$$\begin{aligned} x(2) &= A \otimes x(1) \\ &= A \otimes (A \otimes x(0)) \\ &= (A \otimes A) \otimes x(0) \\ &= A^{\otimes 2} \otimes x(0). \end{aligned}$$

The associative property, together with others, will be discussed in Chapter 1. Instead of $A \otimes A$, we have simply written $A^{\otimes 2}$, where the $\otimes$ symbol in the exponent indicates a matrix power in max-plus algebra. Continuing along these lines, we get

$$\begin{aligned} x(3) &= A \otimes x(2) \\ &= A \otimes (A^{\otimes 2} \otimes x(0)) \\ &= (A \otimes A^{\otimes 2}) \otimes x(0) \\ &= A^{\otimes 3} \otimes x(0), \end{aligned}$$

and in general,

$$\begin{aligned} x(k) &= A \otimes x(k-1) \\ &= A \otimes (A^{\otimes(k-1)} \otimes x(0)) \\ &= (A \otimes (A^{\otimes(k-1)})) \otimes x(0) \\ &= \underbrace{(A \otimes A \otimes \cdots \otimes A)}_{k \text{ times}} \otimes x(0) \\ &= A^{\otimes k} \otimes x(0). \end{aligned}$$

The matrices $A^{\otimes 2}, A^{\otimes 3}, \ldots$ can be calculated directly.

As an example, let us consider the example in Section 0.1 once more. The equations governing the departure times were given by (0.2) or, equivalently, by (0.9) with

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}. \tag{0.10}$$

Then we get (details on vector and matrix multiplication follow in Chapter 1)

$$A^{\otimes 2} \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix} \otimes \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} (2 \otimes 2) \oplus (5 \otimes 3) & (2 \otimes 5) \oplus (5 \otimes 3) \\ (3 \otimes 2) \oplus (3 \otimes 3) & (3 \otimes 5) \oplus (3 \otimes 3) \end{pmatrix}$$

$$= \begin{pmatrix} \max(2+2, 5+3) & \max(2+5, 5+3) \\ \max(3+2, 3+3) & \max(3+5, 3+3) \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 6 & 8 \end{pmatrix}.$$

If the initial departure times are $x_1(0) = 1$ and $x_2(0) = 0$, then we can directly calculate that

$$x(2) = A^{\otimes 2} \otimes x(0) = \begin{pmatrix} 8 & 8 \\ 6 & 8 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \max(8+1, 8+0) \\ \max(6+1, 8+0) \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \tag{0.11}$$

which is in complete agreement with the corresponding result in (0.4).

In general, the entry of the matrix $A^{\otimes 2}$ in row i and column j is given by

$$[A^{\otimes 2}]_{ij} = \bigoplus_{l=1}^{n} a_{il} \otimes a_{lj} = \max_{l=1,\dots,n} (a_{il} + a_{lj}).$$

The quantity $[A^{\otimes 2}]_{ij}$ can be interpreted as the maximum (with respect to l) over all connections (think of the directed graph representing A) from node j to node i via node l. In terms of the train example, this maximum refers to the maximum of all travel times from j to i via l. More generally, $[A^{\otimes k}]_{ij}$ will denote the maximum travel time from j to i via $k-1$ intermediate nodes. In graph-theoretical terms one speaks about paths of length k, instead of connections via $k-1$ intermediate nodes, starting at node j and ending at node i.

0.3 ON EIGENVALUES AND EIGENVECTORS

We consider the notion of eigenvalue and eigenvector in max-plus algebra. Given a square matrix A of size $n \times n$, assume that

$$A \otimes v = \lambda \otimes v, \tag{0.12}$$

where λ is a scalar and v is an n-vector such that not all of its components are equal to $-\infty$. The notation $\lambda \otimes v$ refers to an n-vector whose ith element equals $\lambda \otimes v_i$, i.e., is equal to $\lambda + v_i$. If λ and v are as described in (0.12), then λ is called an *eigenvalue* of matrix A and v a corresponding *eigenvector*.

If $x(0)$ is an eigenvector of A corresponding to the eigenvalue λ, then the solution of (0.9) can be written as

$$x(1) = A \otimes x(0) = \lambda \otimes x(0),$$

$$x(2) = A \otimes x(1) = A \otimes (\lambda \otimes x(0)) = \lambda^{\otimes 2} \otimes x(0),$$

and in general,

$$x(k) = \lambda^{\otimes k} \otimes x(0), \qquad k = 0, 1, 2, \dots .$$

Note that the numerical evaluation of $\lambda^{\otimes k}$ in max-plus algebra is equal to $k \times \lambda$ in conventional algebra. It is easily seen that the eigenvector is not unique. If the same constant is added to all elements of $x(0)$, then the resulting vector will again be an eigenvector. This is reminiscent of the situation in conventional algebra in which the eigenvectors are determined up to a multiplicative factor.

As an example, observe that

$$\begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix} \otimes \begin{pmatrix} 1+h \\ h \end{pmatrix} = 4 \otimes \begin{pmatrix} 1+h \\ h \end{pmatrix}$$

for arbitrary h. Thus, it is seen that the matrix from the example in Section 0.1 has eigenvalue 4. Also compare the sequence resulting for $h = 0$ with (0.4). Moreover, if the initial condition of the corresponding system (0.10) happens to be an eigenvector, then the evolution of the states according to (0.10) leads directly to a regular timetable. Equation (0.11) can be written as

$$x(2) = \begin{pmatrix} 8 & 8 \\ 6 & 8 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix} = \lambda^{\otimes 2} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The existence of eigenvalues and eigenvectors will be studied in more depth in Chapter 2. In Chapters 5 and 6 we will give computational schemes for their calculation.

0.4 SOME MODELING ISSUES

Suppose that the management of the railway company in Section 0.1 decides to buy an extra train in order to possibly speed up the network's behavior (i.e., to obtain a timetable with an average interdeparture time less than $\lambda = 4$). On which circuit should this extra train run? Suppose that the extra train is placed on the track from S_1 to S_2, just outside station S_1 and at the moment that train number k has already left in the direction of S_2. Hence, train number k is in front of the newly added train. If this train number k is renumbered as the $(k-1)$st train and the newly added train gets number k, then the model that yields the smallest possible departure times is given by

$$\begin{aligned} x_1(k+1) &= \max(x_1(k)+2, x_2(k)+5), \\ x_2(k+1) &= \max(x_1(k-1)+3, x_2(k)+3), \end{aligned} \tag{0.13}$$

which can be rewritten as a first-order recurrence relation by introducing an auxiliary variable x_3 with $x_3(k+1) \stackrel{\text{def}}{=} x_1(k)$ as follows:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 2 & 5 & -\infty \\ -\infty & 3 & 3 \\ 0 & -\infty & -\infty \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}. \tag{0.14}$$

In order to interpret this in a different way, one can think of the auxiliary variable x_3 as the departure time at an auxiliary station S_3 situated on the track from S_1 to S_2, just outside S_1, such that the travel time between S_1 and S_3 equals 0 and the travel time from S_3 to S_2 is 3.

There are, of course, other places where the auxiliary station could be situated, for example, somewhere on the inner circuit or on one of the two outer circuits. If, instead of having S_3 neighboring S_1 as above, one could situate S_3 just before S_2, still on the track from S_1 to S_2, then the equations become

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 2 & 5 & -\infty \\ -\infty & 3 & 0 \\ 3 & -\infty & -\infty \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}. \tag{0.15}$$

Or, with S_3 just after S_2 on the track towards S_1,

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 2 & -\infty & 5 \\ 3 & 3 & -\infty \\ -\infty & 0 & -\infty \end{pmatrix} \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}. \tag{0.16}$$

Each of the three models (0.14), (0.15), and (0.16) essentially describes the same speedup of the network's behavior. It will come as no surprise that the eigenvalues of the three corresponding system matrices are identical. A little exercise shows that these eigenvalues all equal 3. That the eigenvalues, or average interdeparture times, cannot be smaller than 3 is easy to understand since the outer circuit at S_2 has one train and the travel time equals 3. On the inner circuit the average interdeparture time cannot be smaller than $8/3$ (i.e., the total travel time on this circuit divided by the number of trains on it). Apparently, the outer circuit at S_2 has become the bottleneck now. A small calculation will show that eigenvectors corresponding to models (0.14), (0.15), and (0.16) are

$$\begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix},$$

respectively. Since eigenvectors are determined up to the addition of a constant, the eigenvectors given above are scaled in such a way that the first element is equal to zero. At first sight, one may be surprised about the fact that the departure times at S_1 and S_2 differ in the latter two cases. For the first (and second) model, the departure times at S_1 are $0, 3, 6, 9, \ldots$, and for S_2 they are $-2, 1, 4, 7, \ldots$. For the third model these sequences are $0, 3, 6, 9, \ldots$ and $1, 4, 7, 10, \ldots$, respectively. Thus, one notices that the kth departure time at S_2 of model (0.15) coincides with the $(k-1)$st departure time at S_2 of model (0.16). Apparently the geographical shift of station S_3 on the inner circuit, from just before S_2 to just after it, causes a shift in the counting of the departures and their times.

Models (0.14) and (0.15) can be obtained from one another by means of a coordinate transformation in the max-plus algebra sense. With $A_{\text{eqn } (0.14)}$ being the matrix from (0.14) and similarly for $A_{\text{eqn } (0.15)}$, it is a straightforward calculation to show that

$$T^{\otimes -1} \otimes A_{\text{eqn } (0.14)} \otimes T = A_{\text{eqn } (0.15)}, \qquad T^{\otimes -1} \otimes T = E,$$

where E denotes the identity matrix in max-plus algebra (i.e., zeros on the diagonal and $-\infty$'s elsewhere) and

$$T = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & -3 \end{pmatrix}, \qquad T^{\otimes -1} = \begin{pmatrix} 0 & -\infty & -\infty \\ -\infty & 0 & -\infty \\ -\infty & -\infty & 3 \end{pmatrix}.$$

A similar coordinate transformation does not exist between models (0.15) and (0.16). This is left as an exercise. The reader should perhaps at this point already be warned that, in contrast to the transformation just mentioned, the inverse of a matrix does in general not exist in max-plus algebra.

0.5 COUNTER AND DATER DESCRIPTIONS

Traditionally, the introduction of a max-plus system is as given in this chapter. Other approaches to describe the same kind of phenomena exist. Define $\kappa_i(\chi)$ as the number of trains in a certain direction that have left station S_i up to and including time χ. Be aware of the fact that here the argument χ refers to time and κ to a counter. In the notation $x_i(k)$ it is the other way around, since there x_i is the time at which an event takes place and k is the counter. The model of the example in Section 0.1 can be written in terms of the κ-variables as

$$\begin{aligned}\kappa_1(\chi) &= \min(\kappa_1(\chi-2)+1, \kappa_2(\chi-5)+1),\\ \kappa_2(\chi) &= \min(\kappa_1(\chi-3)+1, \kappa_2(\chi-3)+1).\end{aligned} \tag{0.17}$$

At station S_1, for instance, the number of departures up to time χ cannot be more than one plus the number of departures from the station S_1 2 time units ago, due to the outer loop, and also not more than one plus the number of departures from station S_2 5 time units ago, due to the inner loop. Recall that a departure here means a departure in both directions (one on the inner loop, one on the outer loop) simultaneously.

Equations (0.17) are equations in so-called min-plus algebra, due to the fact that the minimization and addition play a role in the evolution. By augmenting the state space (i.e., introducing auxiliary κ-variables), (0.17) can be written as a set of first-order recurrence relations, which symbolically can be written as

$$\kappa(\chi) = B \otimes' \kappa(\chi - 1), \tag{0.18}$$

where the matrix B is a square matrix. The symbol $\otimes'$ (see Table 0.1 for its pronunciation) indicates that we are working in min-plus algebra. While in max-plus algebra the "number" $-\infty$ was introduced to characterize nonexisting connections, in min-plus algebra this role is taken over by $+\infty$. Indeed, $\min(a, +\infty) = a$ for any finite a. Equations in min-plus algebra are referred to as counter equations and equations in max-plus algebra as dater equations.

0.6 EXERCISES

1. Define the max-plus product $A \otimes B$ of two matrices A and B of size $n \times n$, as (see Chapter 1 for the general definition)

$$[A \otimes B]_{ij} = \bigoplus_{l=1}^{n} a_{il} \otimes b_{lj} = \max_{1 \le l \le n} (a_{il} + b_{lj}),$$

and compute $A^{\otimes 3}$ as $A \otimes A^{\otimes 2}$ for matrix A given in (0.10). In the same spirit compute $A^{\otimes 4}$ using $A^{\otimes 3}$. Check your result by squaring $A^{\otimes 2}$.

2. Compute $A^{\otimes 2}$ for

$$A = \begin{pmatrix} 2 & 5 & -\infty \\ -\infty & 3 & 3 \\ 0 & -\infty & -\infty \end{pmatrix}.$$

3. In Section 0.4 the fifth train was added to the inner circuit. What would be the result if this fifth train were added to the outer circuit at S_2 instead? Write down the equations for that possibility and show that in that case the eigenvalue of the matrix concerned would still be equal to 4.

4. If you were on the board of the railway company, on which circuit would you add yet another train (the sixth one), whereby the fifth one remains on the inner circuit? The resulting eigenvalue (and interdeparture time) should be equal to $8/3$. Can you relate this number to the now-existing bottleneck in the system?
5. Show that a coordinate transformation (in max-plus algebra, of course) between models (0.15) and (0.16) does not exist.
6. Check the derivation of equation (0.17), starting from the example in Section 0.1, and determine matrix B in (0.18). The latter matrix, of size 8×8, will contain quite a few elements $+\infty$.
7. Formulate the notion of eigenvalue and eigenvector for matrices in min-plus algebra. Without computation, what do you expect for the eigenvalue in min-algebra of matrix B of the above exercise?
8. Suppose that in the railway network of Figure 0.1 only three trains are available, one on each circuit. A possible interpretation of the inner circuit now is that it is a single track and that the direction in which the trains (actually the only train) use this track alternates. Derive a model of the form (0.9) for this layout and trains.
9. What is the minimal interdeparture time in the above exercise?

0.7 NOTES

The seminal work in max-plus algebra is [31]. Earlier traces, such as [30], exist. It was revitalized, in a system-theoretical setting, in [28] and subsequent papers of the "French school." The first railway application appeared in [15]. The book [5] describes the state of the art at the beginning of the 1990s. A more recent book with contributions by many authors is [48]. For more background information the reader is referred to the website `www.maxplus.org`.

The name *counter equation* is logical for equations of the form (0.18), since it refers to the question of how often an event has taken place up to a certain time. The generally accepted name *dater equation* for equations of the form (0.9) has French roots, due to the French school, which was seminal in the development of the theory. The word *date* in French refers to the calendar date, as it does in English, but in English there are other connotations. In historical perspective, *timer equation* might have been better to express the fact that the elements of the state x represent time instants.