Geometry, Algebra and Computation

Wai-Shing Luk

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Projective Geometry

Introduction

Geometry and Algebra

- Geometry
  - Points, lines, triangles, circles, conic sections…
  - Collinear, concurrent, parallel, perpendicular…
  - Distances, angles, areas, quadrance, spread, quadrea…
  - Midpoint, bisector, orthocenter, pole/polar, tangent…
- Algebra
  - Addition, multiplication, inverse…
  - Elementary algebra: integer/rational/real/complex… numbers.
  - Abstract Algebra: rings, fields…
  - Linear algebra: vector, matrix, determinant, dot/cross product…
- Two subjects are related by coordinates.

Key points

- Our earth is not flat and our universe is non-Euclidean.
- Non-Euclidean geometry is much easier to learn than you might think.
- Our curriculum in school is completely wrong.
- Euclidean geometry is non-symmetric. Three sides determine a triangle, but three angles do not determine a triangle. It might not be true in general geometries. Euclidean geometry is just a special case.
- Yet Euclidean geometry is more computationally efficient and is still used in our small-scale daily life.
- Incidenceship promotes integer arithmetic; non-oriented measurement promotes rational arithmetic; oriented measurement promotes floating-point arithmetic. Don't use a machine gun to hunt rabbit.

Projective Plane's Basic Elements

Projective Plane Concept

- Only involve "Points" and "Lines".
- "Points" (or "lines") are assumed to be distinguishable.
- Denote A = B as A and B are referred to the same point.
- E.g., $(1/3, 2/3)$ = $(10/30, 20/30)$
- We have the following rules:
  - A = A (reflective)
  - If A = B, then B = A (symmetric)
  - If A = B and B = C, then $A = C$ (transitive)
- Unless mention specifically, objects in different names are assumed to be distinct, i.e. $A \neq B$.
- The idea can be generalized to higher dimensions. However, we restrict to 2D only here.

Incidence

- A point either lies on a line or not.
- If a point A lies on a line l, denote $l \circ A$.
- For convenience, we also denote as $A \circ l$.
- We have $A \circ l = l \circ A$

Projective Point and Line

- Projective Point
  - Exactly one line passes through two distinct points.
  - Denote join(A, B) or simply AB as a line joined by A and B.
  - We have:
    - AB = BA
    - $AB \circ A$ and $AB \circ B$ are always true.
- Projective Line
  - Exactly one point met by two distinct lines.
  - Denote meet(l, m) or simply lm as a point met by l and m.
  - We have:
    - lm = ml
    - $lm \circ l$ and $lm \circ m$ are always true.

- Duality: "Point" and "Line" are interchangable here.

- "Projective geometry is all geometry." (Arthur Cayley)

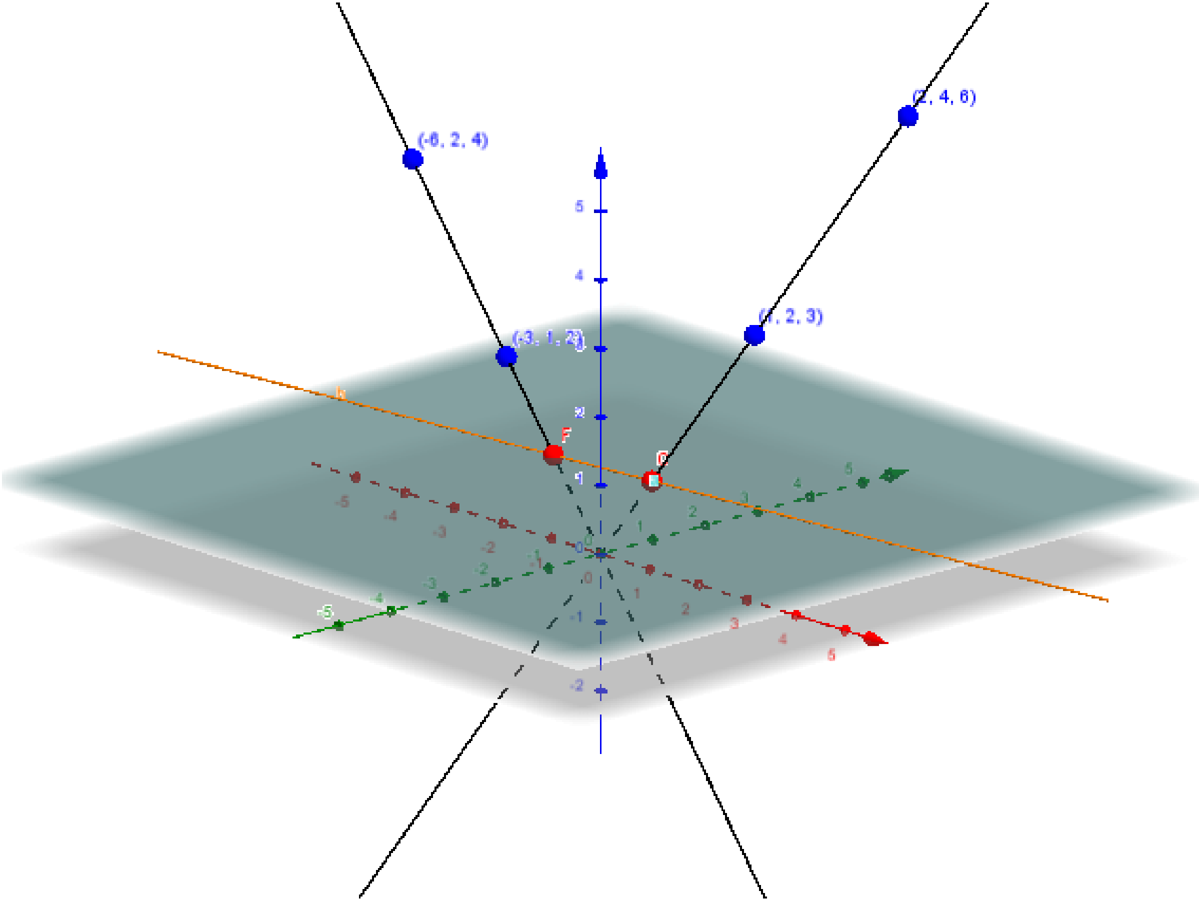
Example 1: Euclidean Geometry

- Point: projection of a 3D vector $p = [x, y, z]$ to 2D plane $z = 1$:

$$(x', y') = (x/z, y/z)$$

- $[x, y, z]$ and $[\alpha x, \alpha y, \alpha z]$ for all $\alpha \neq 0$ are representing the same point.
- For instance, $[1, 5, 6]$ and $[-10, -50, -60]$ are representing the same point $(1/6, 5/6)$
- $p_\infty = [x, y, 0]$ is a point at *infinity*.
- Line: $ax' + by' + c = 0$, denoted by a vector $[a, b, c]$.
- $[a, b, c]$ and $[\alpha a, \alpha b, \alpha c]$ for all $\alpha \neq 0$ are representing the same line.
- $l_\infty = [0, 0, 1]$ is the line at *infinity*.
- $[0, 0, 0]$ is not a valid point or line.

Euclidean 2D plane from 3D vector



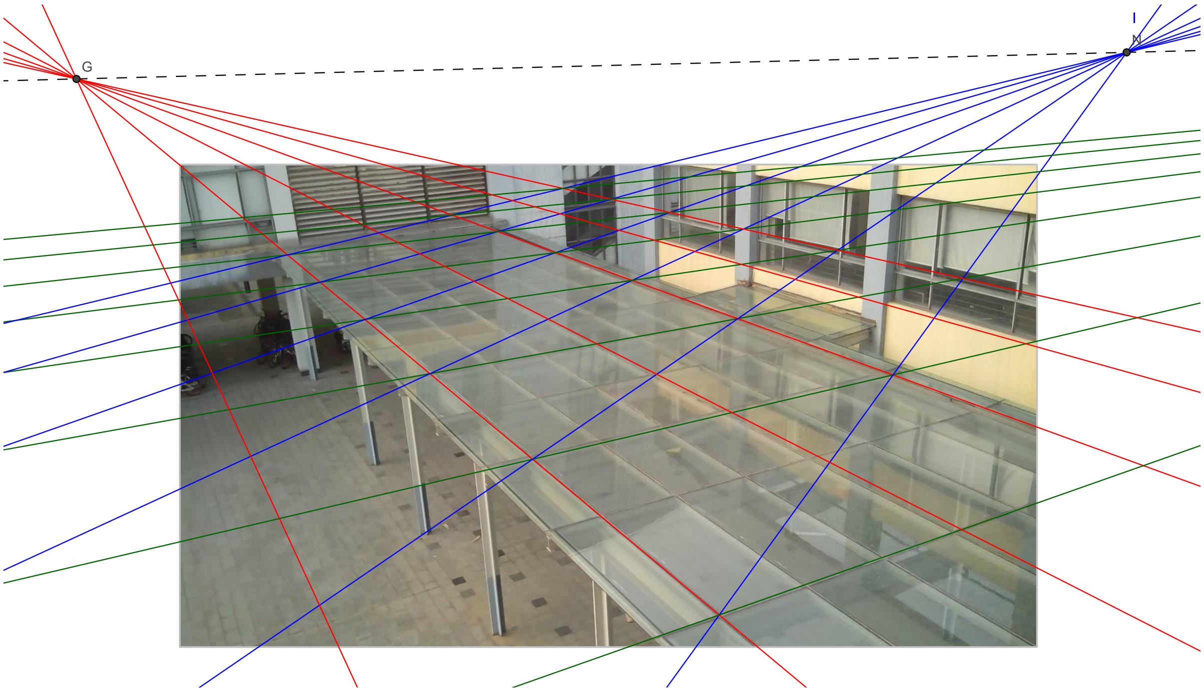
{#fig:euclidean}

Calculation by Vector Operations

- Let $v_1 = [x_1, y_1, z_1]$ and $v_2 = [x_2, y_2, z_2]$.
  - dot product $v_1 \cdot v_2$ = $v_1^T v_2$ = $x_1x_2 + y_1y_2 + z_1z_2$.
  - cross product $v_1 \times v_2$ = $[y_1z_2 - z_1y_2, -x_1z_2 + z_1x_2, x_1y_2 - y_1x_2]$
- Then, we have:
  - $A \circ a$ if and only if $[A] \cdot [a] = 0$
  - Join of two points: $[AB]$ = $[A] \times [B]$
  - Meet of two lines: $[lm]$ = $[l] \times [m]$
  - $A = B$ if and only if $[A] \times [B] = [0, 0, 0]$
- Example: the linear equation that joins the point $(1/2, 3/2)$ and $(4/5, 3/5)$ is $9x + 3y - 9 = 0$, or $3x + y = 3$.
- Exercise: Calculate the line equation that joins the points $(5/8, 7/8)$ and $(-5/6, 1/6)$.

Example 2: Perspective View of Euclidean Geometry

- It turns out that we can choose any line on a plane as the line of infinity.



{#fig:euclidean2}

Example 3: Spherical/Elliptic Geometry

- Surprisingly, the vector notations and operators can also represent other geometries such as spherical/Elliptic geometry.

- "Point": projection of 3D vector $[x, y, z]$ to the unit sphere.

$$(x', y', z') = (x/r, y/r, z/r)$$

  where $r^2 = x^2 + y^2 + z^2$.

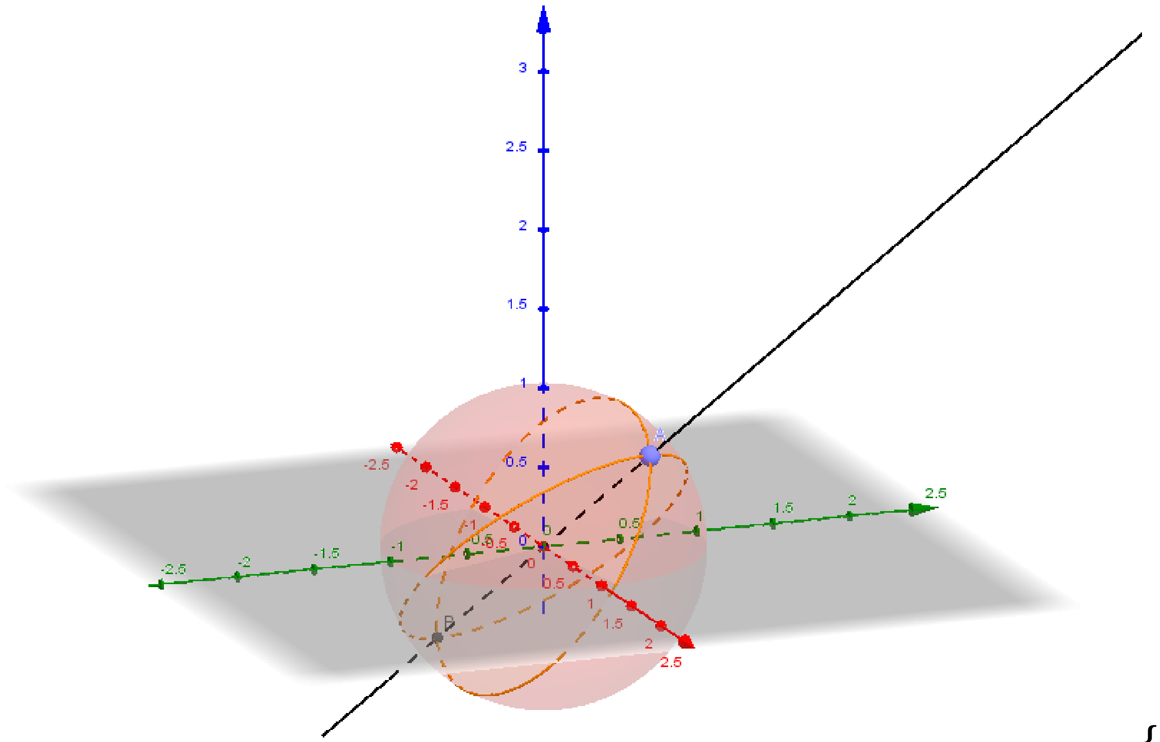
- Two points on the opposite poles are considered the same point here.

- "Line": $[a, b, c]$ represents the *great circle* intersected by the unit sphere and the plane $ax + by + cz = 0$.

- $[x, y, z]$ is called *Homogeneous Coordinates*.

- Here, the coordinates could be in integer numbers, rational numbers (ratio of two integers), real numbers, complex numbers, or finite field numbers, or even polynomial functions.

Spherical Geometry from 3D vector



{#fig:sphere}

Example 4: Hyperbolic Geometry from 3D vector

- A velocity "point": projection of a 3D vector $[p] = [x, y, t]$ to 2D plane $t = 1$:

$$(v_x, v_y) = (x/t, y/t)$$

Counter-examples

- In some quorum systems, two "lines" are allowed to meet at more than one points. Therefore, only the very special case it is a projective geometry.
- In some systems, a line from A to B is not the same as the line from B to A, so they cannot form a projective geometry.
- "Symmetry" is an important keyword in projective geometry.

Number systems

- Integer number ($\mathbb{Z}$):
  - e.g. $0, 1, 2, 3, \ldots, -1, -2, -3, \ldots$
  - discrete, more computationally efficient.
- Rational number ($\mathbb{Q}[\mathbb{Z}]$):
  - e.g. $0/1, 2/3, -1/3, 1/0$ (i.e. infinity)
  - Multiplication/division is easier than addition/subtraction
- Real number ($\mathbb{R}$):
  - e.g. 0.3, $2^{1/2}$, π
  - May induce round-off errors.
- Finite field, $GF(n)$, where n is a prime number (e.g. $2, 3, 5, 7, 11, 13$) or prime powers (e.g. $4 = 2^2, 8 = 2^3, 9 = 3^2$).
  - Used in Coding Theory

Number systems (cont'd)

- Complex number ($\mathbb{C}$):
  - e.g $1 + \pi i, 1 - 3\pi i$
  - Besides the identity (the only automorphism of the real numbers), there is also the automorphism τ that sends $x + iy$ to $x - iy$ such that $\tau(\tau(x)) = x$.
- Complex number over integer ($\mathbb{C}[\mathbb{Z}]$)
  - e.g. $1 + 2i, 1 - 2i$
  - Also known as Gaussian integer.

- Complex number over Rational ($\mathbb{C}[\mathbb{Q}]$)

- Projective Geometry can work on all these number systems.

- In fact, Projective Geometry can work on any field number. Moreover, the multiplicative inverse is not required.

- "Continuity" is not assumed in Projective Geometry.

Example 4: Poker Card Geometry

- Even "coordinates" is **not** a necessary requirement of projective geometry.
- Consider the poker cards in @tbl:poker_card:
  - For example, meet(l_2, l_5) = `5`, join(`J`, `4`) = l_8.
- We call this *Poker Card Geometry* here.

Table

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}
A	2	3	4	5	6	7	8	9	10	J	Q	K
2	3	4	5	6	7	8	9	10	J	Q	K	A
4	5	6	7	8	9	10	J	Q	K	A	2	3
10	J	Q	K	A	2	3	4	5	6	7	8	9

: Poker Card Geometry {#tbl:poker_card}

Finite projective plane

- Yet we may assign the homogeneous coordinate to a finite projective plane, where the vector operations are in a finite field.
- E.g. The poker card geometry is a finite projective plane of order 3.
- The smallest finite projective plane (order 2) contains only 7 points and 7 lines.
- If the order is a prime number or prime powers, then we can easily construct the finite projective plane via finite field and homogeneous coordinate.
- The non-existence of finite projective plane of order 10 was proved in 1989. The proof took the equivalent of 2000 hours on a Cray 1 supercomputer.
- The existences of many other higher order finite projective planes are still an open question.

Not covered in this work

- Unless mention specifically, we don't discuss finite projective plane further more.
- Although the coordinate system is not a requirement in general projective geometry, practically all examples we are dealing with have homogeneous coordinates. All the proofs of theorems are based on the assumption of homogeneous coordinates.

Basic Properties

Collinear, Concurrent, and Coincidence

- Three points are called *collinear* if they all lie on the same line.
- Three lines are called *concurrent* if they all meet at the same point.
- Denote coincidence relation as coI(A, B, C).
- coI(A, B, C) is true if and only if $AB \circ C$ is true.
- Similarly, coI(a, b, c) is true if and only if $ab \circ c$ is true.
- In general, coI($A_1, A_2, \ldots, A_n$) is true if and only if $A_1 A_2 \circ X$ is true for all X in the rest of points $A_3, A_4, \ldots, A_n$.
- Unless mention specifically, $ABCD \ldots$ is assumed to be coincidence, while $ABCD \ldots$ is assumed none of three are coincident.

Parameterize a line

- The points on the line AB can be parameterized by $\lambda[A] + \mu[B]$ with λ and μ are not both zero.

- For integer coordinates, to show that $\lambda[A] + \mu[B]$ can span all the integer points on the line, we give the exact expression of λ/μ of a point C as follows.

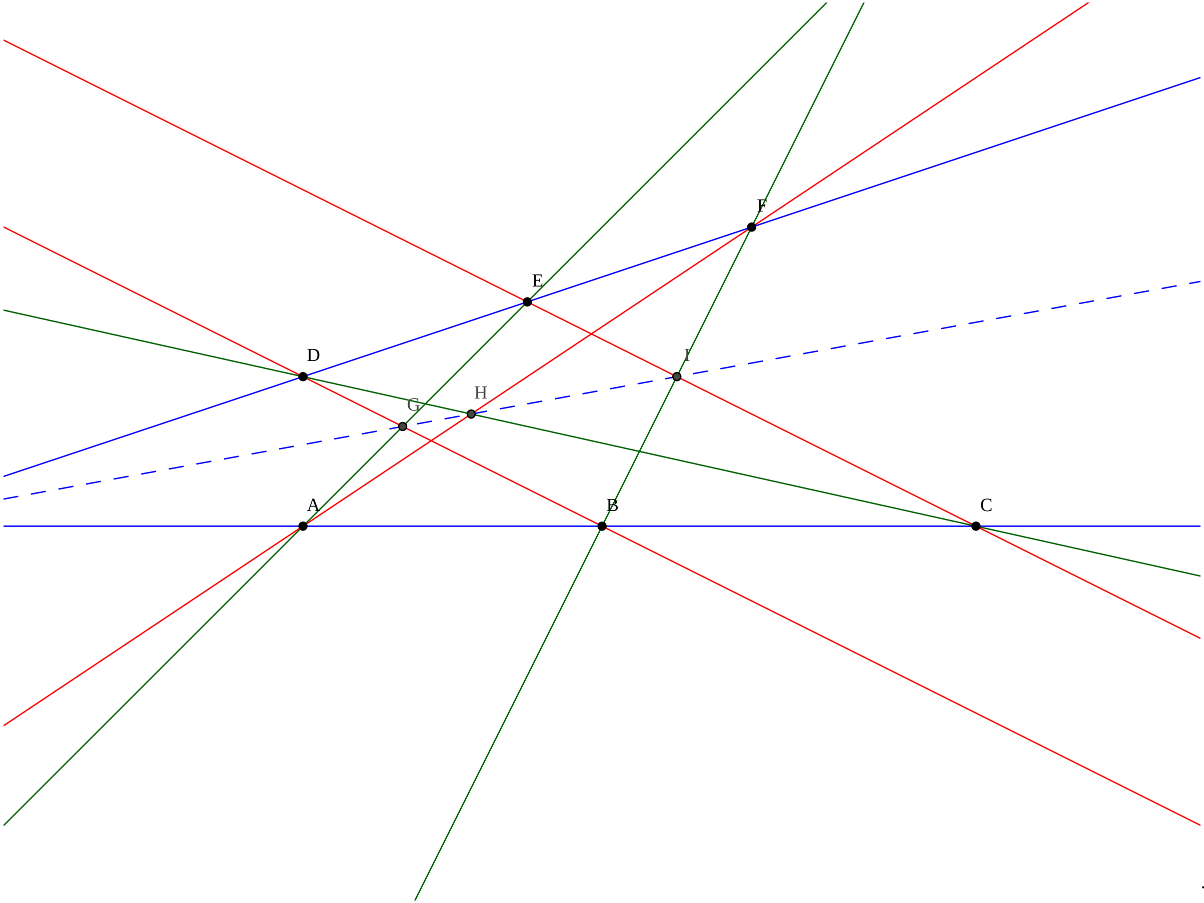
- Let $l = [C] \times ([A] \times [B])$. Then

$$\lambda[A] + \mu[B] = (l^T[B])[A] - (l^T[A])[B]$$

Pappus Theorem

- Theorem (Pappus): Given two lines ABC and DEF. Let G=meet(AE, BD), H=meet(AF, CD), I=meet(BF, CE). Then G, H, I are collinear.
- Sketch of the *proof*:
    - Let $[C] = \lambda_1[A] + \mu_1[B]$.
    - Let $[F] = \lambda_2[D] + \mu_2[E]$.
    - Express $[G], [H], [I]$ in terms of $[A], [B], \lambda_1, \mu_1, \lambda_2, \mu_2$.
    - Simplify the expression $[G] \cdot ([H] \times [I])$ and derive that it is equal to 0. (we may use the Python's symbolic package for the calculation.)
- Exercise: verify that this theorem holds for the poker card geometry with `3`, `6`, `Q` on l_3 and `8`, `9`, `J` on l_8.

An instance of Pappus' theorem



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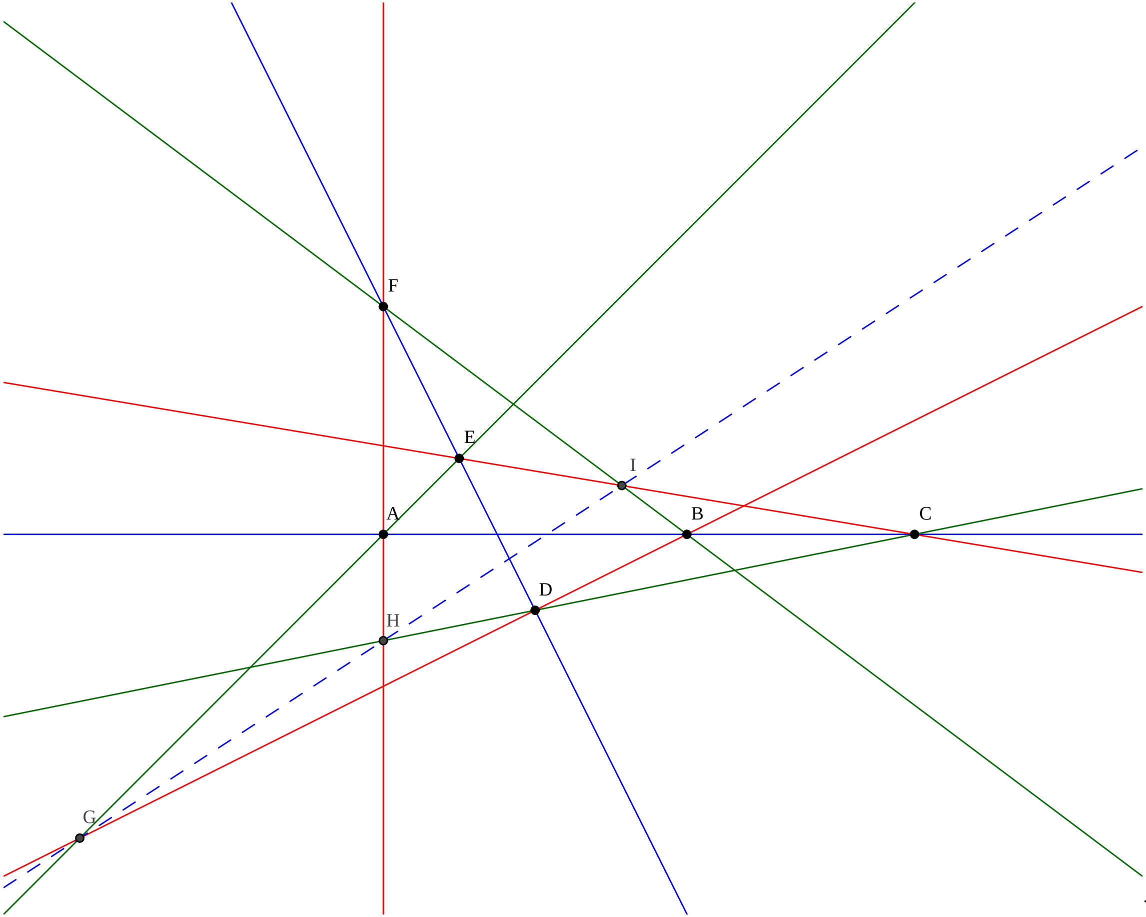
Another instance of Pappus' theorem



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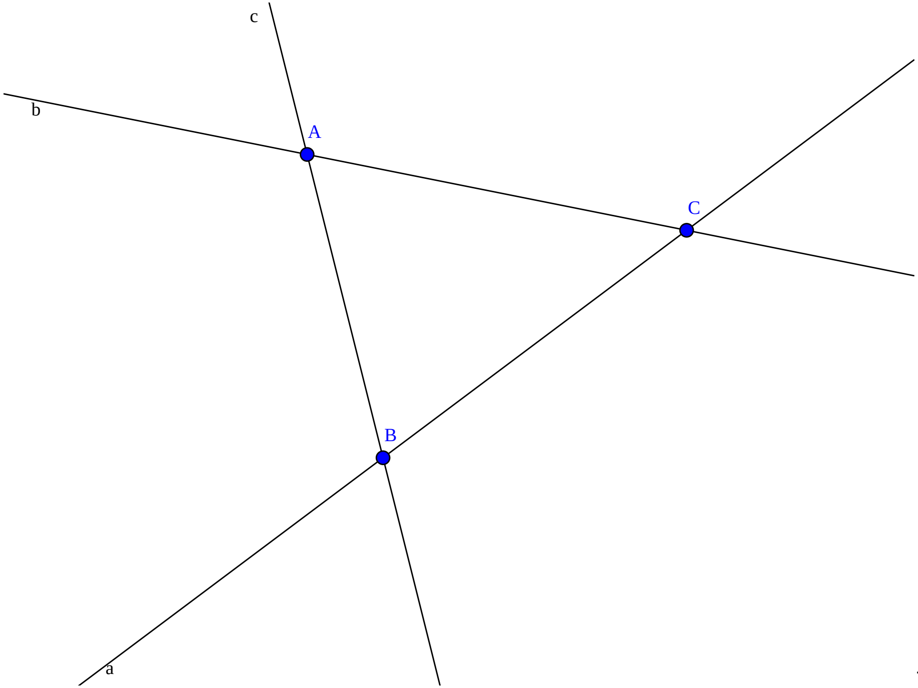
Triangles and Trilaterals

- If three points A, B, and C are not collinear, they form a triangle, denoted as ABC.

- If three lines a, b, and c are not concurrent, they form a trilateral, denoted as abc.

- Triangle ABC and trilateral abc are dual if $A = bc$, $B = ac$ and $C = ab$.

An example of triangle and trilateral



{#fig:triangle}

Projectivities and Perspectivities

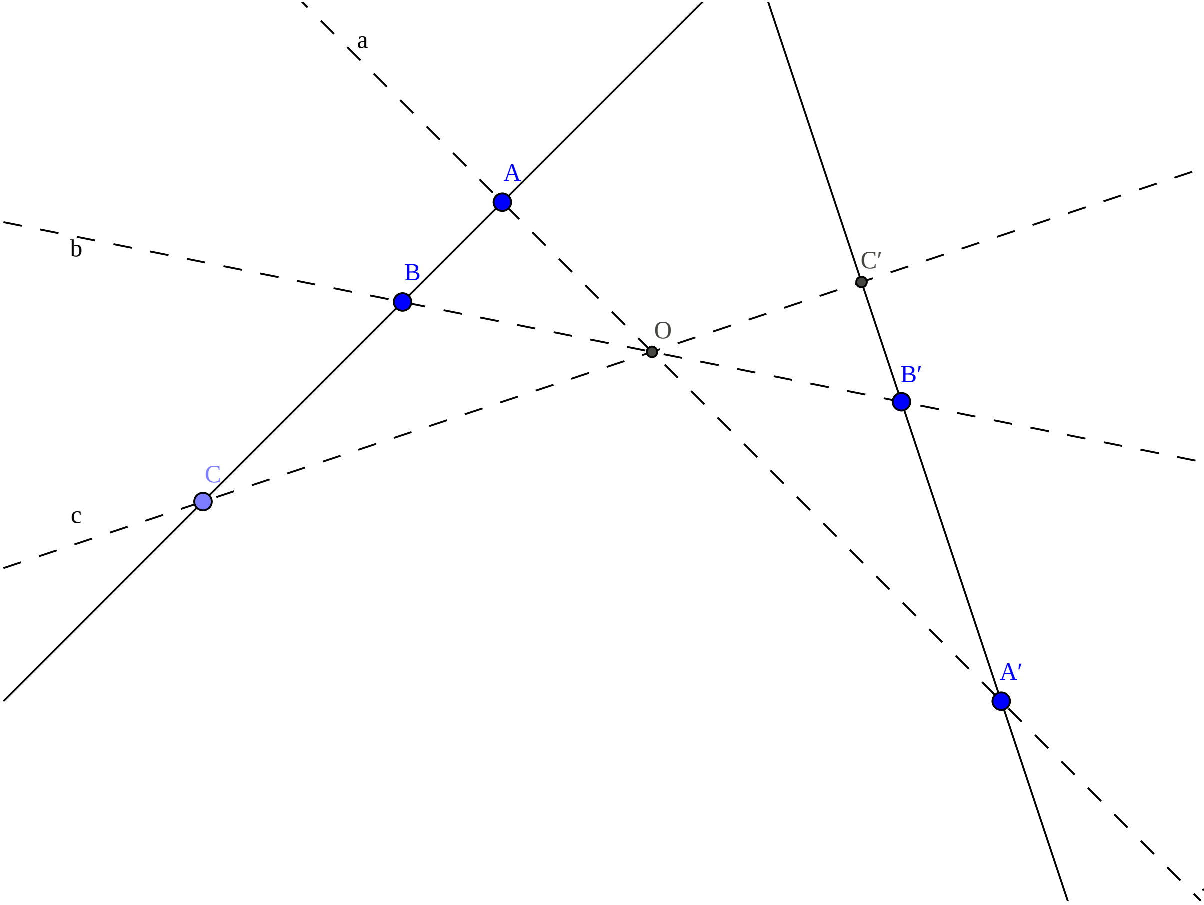
Projectivities

- An ordered set (A, B, C) (either collinear or not) is called a projective of a concurrent set abc if and only if $A \circ a$, $B \circ b$ and $C \circ c$.
- Denote this as $(A, B, C) \barwedge abc$.
- An ordered set (a, b, c) (either concurrent or not) is called a projective of a collinear set ABC if and only if $A \circ a$, $B \circ b$ and $C \circ c$.
- Denote this as $(a, b, c) \barwedge ABC$.
- If each ordered set is coincident, we may write:
    - $ABC \barwedge abc \barwedge A'B'C'$
    - Or simply $ABC \barwedge A'B'C'$

Perspectivities

- An ordered set (A, B, C) is called a perspectivity of an ordered set (A', B', C') if and only if $(A, B, C) \barwedge abc$ and $(A', B', C') \barwedge abc$ for some concurrent set abc.
- Denote this as $(A, B, C) \doublebarwedge (A', B', C')$.
- An ordered set (a, b, c) is called a perspectivity of an ordered set (a', b', c') if and only if $(a, b, c) \barwedge ABC$ and $(a', b', c') \barwedge ABC$ for some collinear set ABC.
- Denote this as $(a, b, c) \doublebarwedge (a', b', c')$.

An instance of perspectivity



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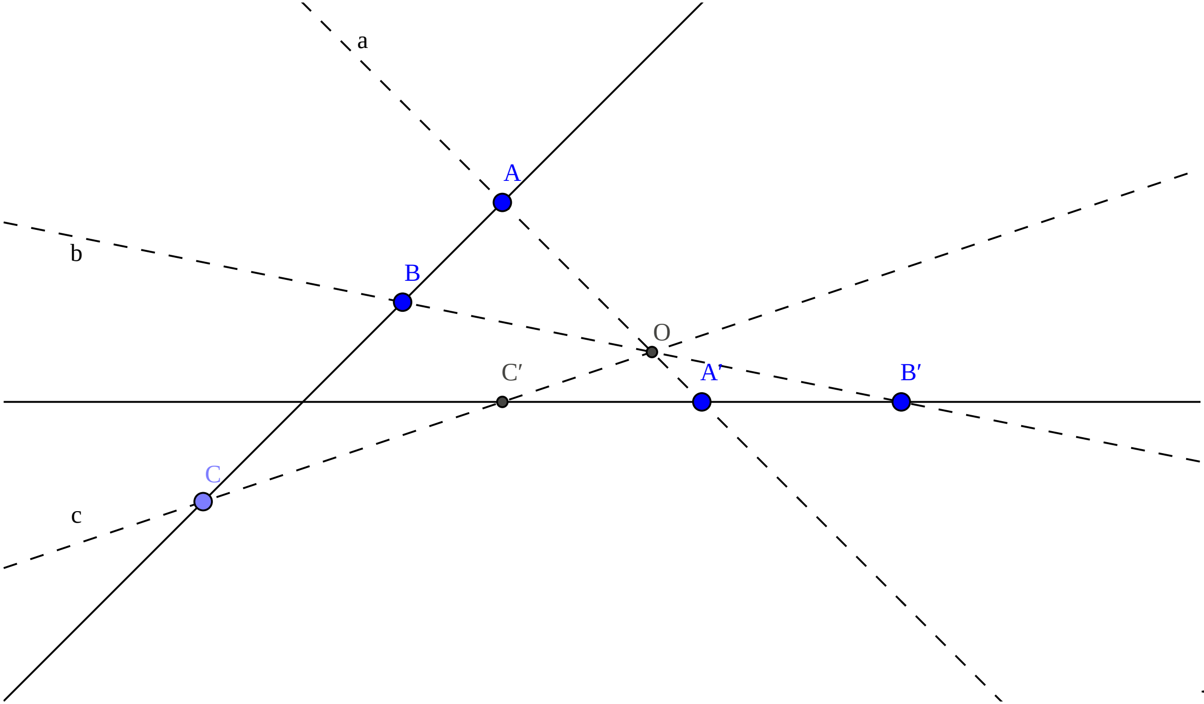
Another instance of perspectivity



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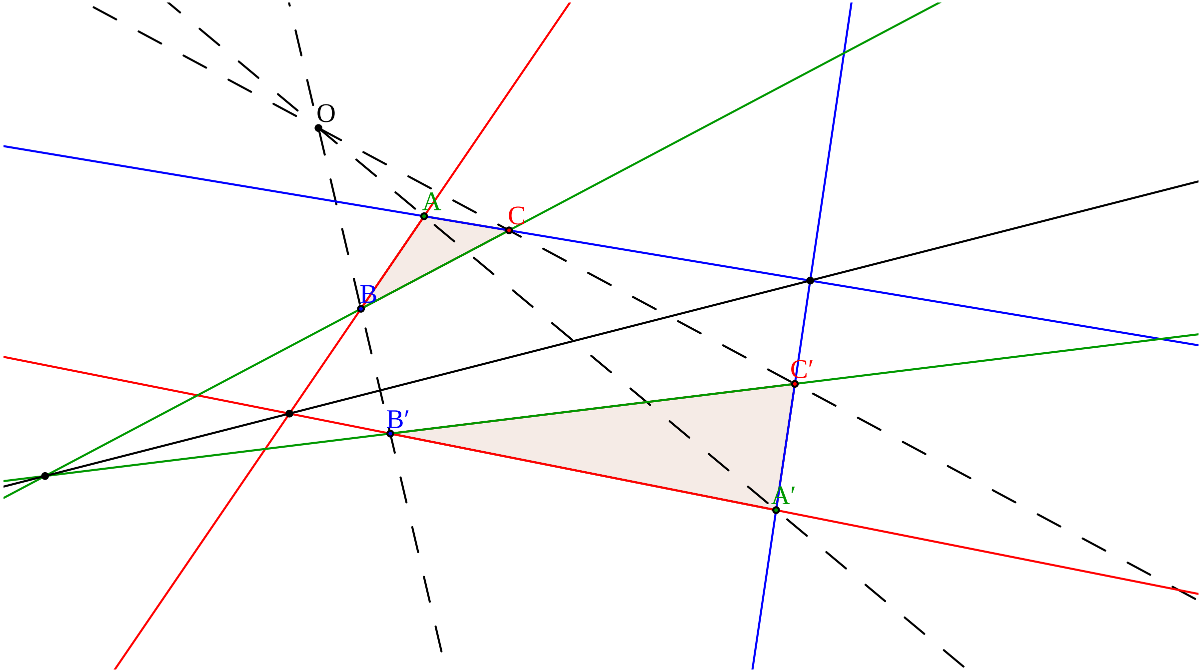
Perspectivity

- Similar definition for more than three points:
  - $(A_1, A_2, A_3, \ldots, A_n) \overset{=}{\wedge} (A'_1, A'_2, A'_3, \ldots, A'_n)$.
- To check perspectivity:
  - First construct a point O := meet($A_1 A'_1$, $A_2 A'_2$).
  - For the rest of points, check if X, X', O are collinear.
- Note that $(A, B, C) \overset{=}{\wedge} (D, E, F)$ and $(D, E, F) \overset{=}{\wedge} (G, H, I)$ does not imply $(A, B, C) \overset{=}{\wedge} (G, H, I)$.

Desargues's Theorem

- Theorem (Desargues): Let trilateral abc be the dual of triangle ABC and trilateral $a'b'c'$ be the dual of triangle $A'B'C'$. Then, $ABC \overline{\overline{\wedge}} A'B'C'$ if and only if $abc \overline{\overline{\wedge}} a'b'c'$.
- Sketch of the *proof*:
  - Let O be the perspective point.
  - Let $[A'] = \lambda_1[A] + \mu_1[O]$.
  - Let $[B'] = \lambda_2[B] + \mu_2[O]$.
  - Let $[C'] = \lambda_3[C] + \mu_3[O]$.
  - Let $[G]$ = meet(join($[A]$, $[B]$),join($[A']$, $[B']$))
  - Let $[H]$ = meet(join($[B]$, $[C]$),join($[B']$, $[C']$))
  - Let $[I]$ = meet(join($[A]$, $[C]$),join($[A']$, $[C']$))
  - Express $[G], [H], [I]$ in terms of $[A], [B], [C], [O], \lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \mu_3$.
  - Simplify the expression $[G] \cdot ([H] \times [I])$ and find that it is equal to 0. (we may use the Python's symbolic package for the calculation.)
  - Dual to the duality, the only-if part can be proved using the same technique.

An instance of Desargues's theorem



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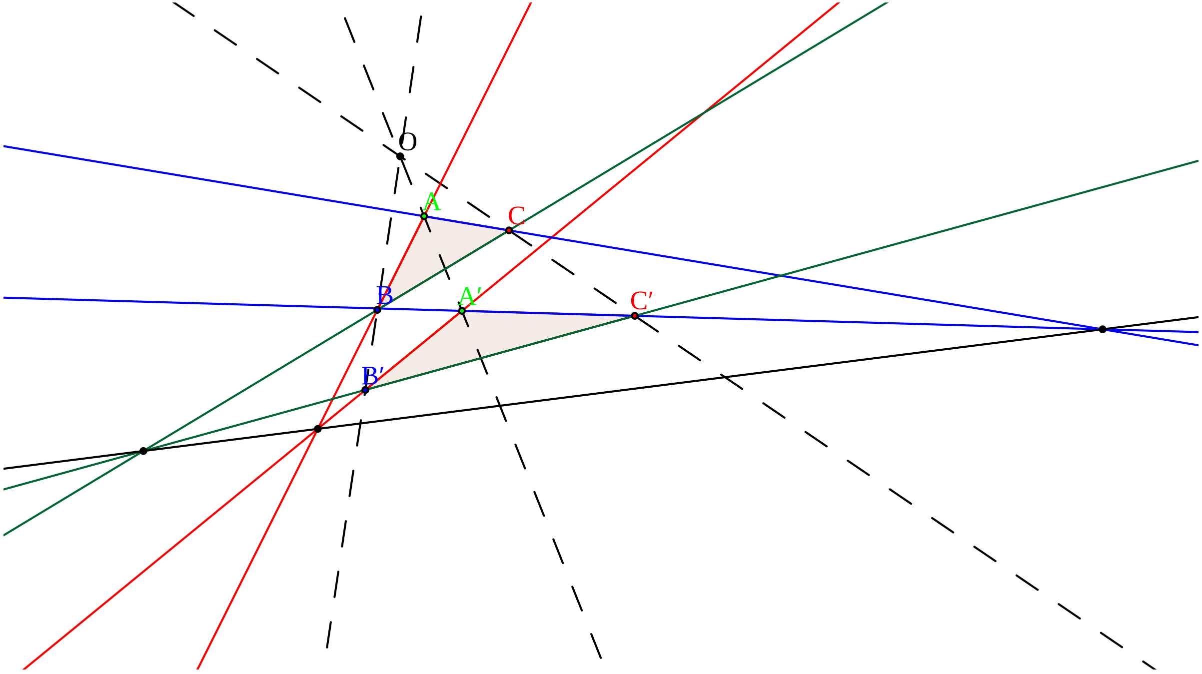
Another instance of Desargues's theorem



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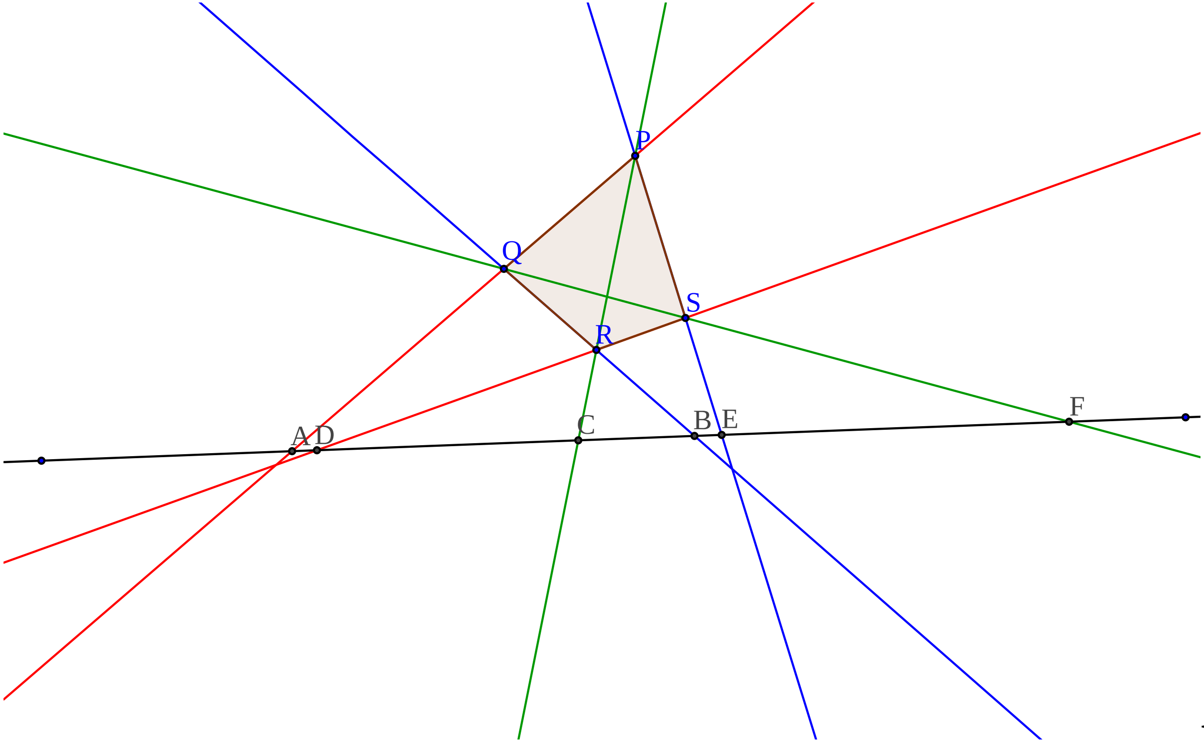
Projective Transformation

- Given a function τ that transforms a point A to another point $\tau(A)$.
- If A, B, and C are collinear and we always have $\tau(A)$, $\tau(B)$, and $\tau(C)$ collinear. Then the function τ is called a projective transformation.
- In Homogeneous coordinate, a projective transformation is any non-singular matrix times a vector.

Quadrangles and Quadrilateral Sets

- If four points P, Q, R and S none of three are collinear, they form a quadrangle, denoted as $PQRS$.
- Note that Quadrangle here could be convex or self-intersecting.
- Totally there are six lines formed by $PQRS$.
- Assume they meet another line l at A, B, C, D, E, F, where
  - A = meet(PQ, l), D = meet(RS, l)
  - B = meet(QR, l), E = meet(PS, l)
  - C = meet(PR, l), F = meet(QS, l)
- We call the six points as a quadrilateral set, denoted as $(AD)(BE)(CF)$.

Quadrilateral set



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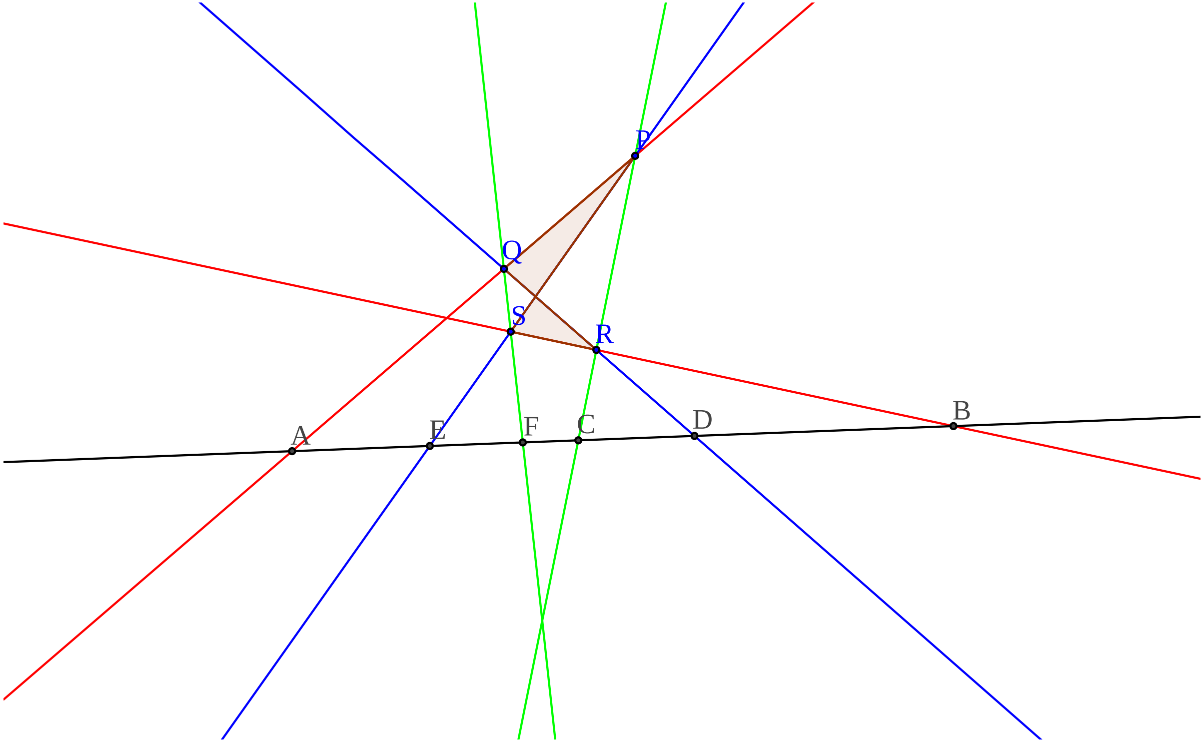
Another quadrilateral set



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Harmonic Sets

- In a quadrilateral set $(AD)(BE)(CF)$, if $A = D$ and $B = E$, then it is called a harmonic set.
- The Harmonic relation is denoted by $H(AB, CF)$.
- Then C and F is called a harmonic conjugate.
- Theorem: If $ABCF \barwedge abcd$, then $H(AB, CF) = H(ab, cf)$.
- In other words, projectivity preserves harmonic relation.
- Theorem: If $ABCF \doublebarwedge A'B'C'F'$, then $H(AB, CF) = H(A'B', C'F')$.
- In other words, perspectivity preserves harmonic relation.

Polarities

- A *polarity* is a projective correlation of period 2.
- We call a the *polar* of A, and A the pole of a.
- Denote $a = A^{\perp}$ and $A = a^{\perp}$.
- Except degenerate cases, $A = (A^{\perp})^{\perp}$ and $a = (a^{\perp})^{\perp}$.
- It may happen that A is incident with a so that each is *self-conjugate*.
- The locus of self-conjugate points defines a *conic*. However, the polarity is a more general concept than a conic, because some polarities may not have self-conjugate points (or their self-conjugate points are complex).

The Use of a Self-Polar Triangle

- Any projective correlation that relates the three vertices of one triangle to the respectively opposite sides is a polarity.
- Thus, any triangle ABC, any point P not on a side, and any line p not throughout a vertex, determine a definite polarity $(ABC)(Pp)$.

The Conic

- Historically *ellipse* (including *circle*), *parabola*, and *hyperbola*.
- The locus of self-conjugate points is a *conic*.
- Their polars are its *tangents*.
- Any other line is called a *secant* or a *nonsecant* according to as it meets the conic twice or not at all, i.e., according to as the involution of conjugate points on it is hyperbolic or elliptic.
- Note: Intersecting a conic with a line may result of an irrational intersection point.

Construct the polar of a point using a conic

- To construct the polar of a given point C, not on the conic, draw any two secants PQ and RS through C; then the polar joins the two points meet(QR, PS) and meet(RP, QS).

Example of constructing the polar of a point

\begin{figure}[hp] \centering \input{pole2polar.tikz} \caption{Example of constructing the polar of a point} \end{figure}

Another example of constructing the polar of a point

\begin{figure}[hp] \centering \input{pole2polar2.tikz} \caption{Another example of constructing the polar of a point} \end{figure}

Construct the pole from a line

- To construct the pole of a given secant a, draw the polars of any two points on the line; then the common point of two polars is the pole of a.

Constructing the pole of a line

\begin{figure}[hp] \centering \input{polar2pole.tikz} \caption{Constructing the pole of a line} \end{figure}

Construct the tangent of a point on a conic

- To construct the tangent at a given point P on a conic, join P to the pole of any secant through P.

Example of construct the tangent of a point on a conic

\begin{figure}[hp] \centering \input{tangent.tikz} \caption{Construct the tangent of a point on a conic} \end{figure}

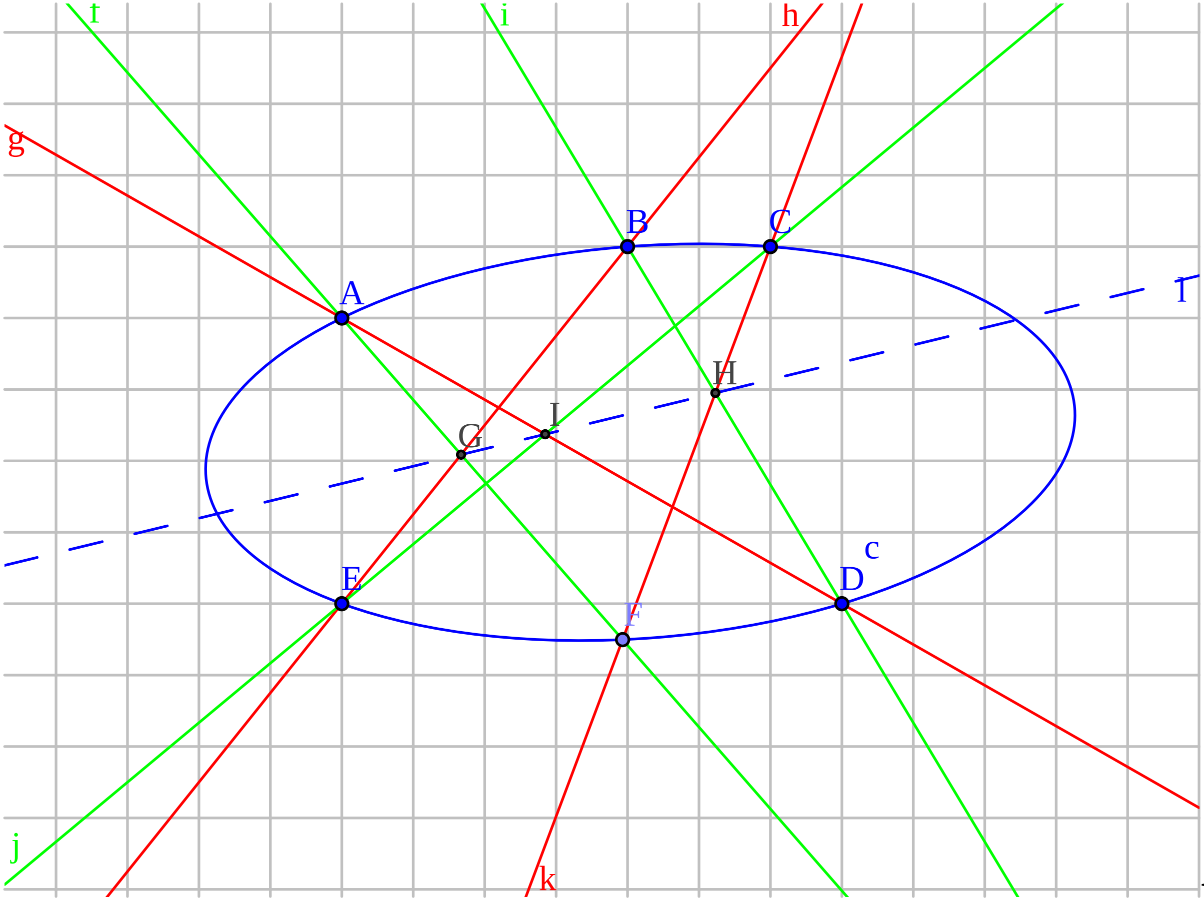
Another example of constructing the tangent of a point on a conic

\begin{figure}[hp] \centering \input{tangent2.tikz} \caption{Another example of constructing the tangent of a point on a conic} \end{figure}

Pascal's Theorem

- If a hexagon is inscribed in a conic, the three pairs of opposite sides meet in collinear points.

An instance of Pascal' theorem



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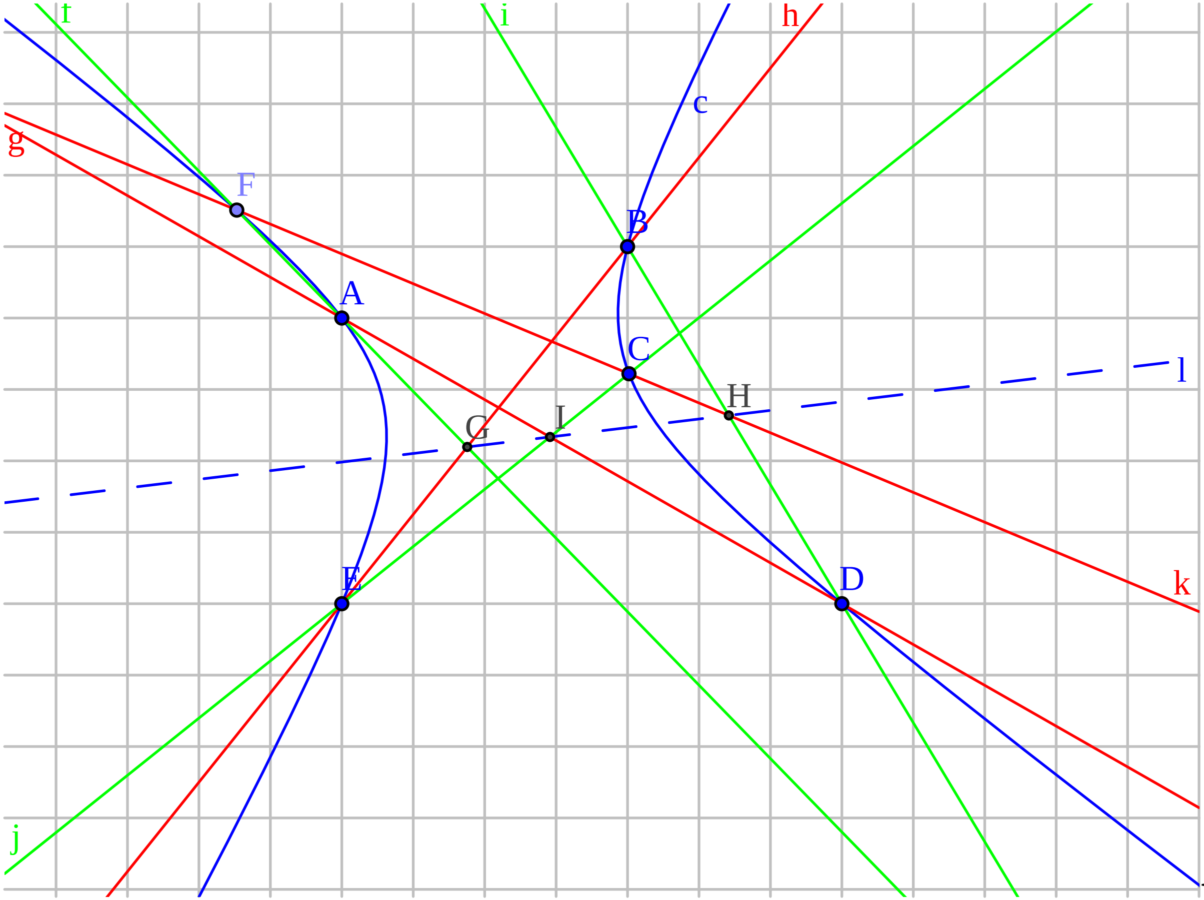
Another instance of Pascal' theorem



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Backup

melpon.org

```
>  pandoc -s --mathjax -t revealjs -V theme=default -o proj_geom.html
>  pandoc -t beamer -o proj_geom.svg proj_geom.md beamer.yaml
>  pandoc -o proj_geom.docx proj_geom.md
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