Euclidean Geometry

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Introduction

Rational Trigonometry in Euclidean geometry

Questions?

Introduction

Basic

- Line at infinity $l_\infty = [0, 0, 1]$
- Two special points I and J on l_∞ play an important role in Euclidean Geometry:
  - $\mathrm{I} = [1, -i, 0]$, $\mathrm{J} = [1, i, 0]$
- $\mathbf{A} = l_\infty \cdot l_\infty^\mathsf{T}$
- $\mathbf{B} = \mathrm{I} \cdot \mathrm{J}^\mathsf{T} + \mathrm{J} \cdot \mathrm{I}^\mathsf{T}$
- If we choose another line $l = M \cdot l_\infty$ as line of infinity

Notations

- To distinguish with Euclidean geometry, points are written in capital letters.

Rational Trigonometry in Euclidean geometry

Quadrance and Spread in Euclidean geometry

- The **quadrance** Q between points A_1 and A_2 is:

$$Q = (x_1' - x_2')^2 + (y_1' - y_2')^2$$

- The **spread** s between lines l_1 and l_2 is:

$$s = (a_1 b_2 - a_2 b_1)^2 / (a_1^2 + b_1^2)(a_1^2 + b_1^2)$$

- The **cross** c between lines l_1 and l_2 is:

$$c = 1 - s = (a_1 a_2 + b_1 b_2)^2 / (a_1^2 + b_1^2)(a_1^2 + b_1^2)$$

Triple formulate

- Let A_1, A_2 and A_3 are points with $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$. Let l_1, l_2 and l_3 are lines with $s_1 \equiv s(l_2, l_3)$, $s_2 \equiv s(l_1, l_3)$ and $s_3 \equiv s(l_1, l_2)$.
- Theorem (Triple quad formula): If A_1, A_2 and A_3 are collinear points then

$$(Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2)$$

- Theorem (Triple spread formula): If l_1, l_2 and l_3 are concurrent lines then

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1 s_2 s_3.$$

Spread Law

- Suppose that triangle $\{A_1A_2A_3\}$ form quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$, and it dual trilateral $\{l_1 l_2 l_3\}$ form spreads $s_1 \equiv s(l_2, l_3)$, $s_2 \equiv s(l_1, l_3)$ and $s_3 \equiv s(l_1, l_2)$. Then:
- Theorem (Spread Law)

$$Q_1/s_1 = Q_2/s_2 = Q_3/s_3.$$

- (Compare with the sine law in Euclidean Geometry):

$$d_1/\sin\theta_1 = d_2/\sin\theta_2 = d_3/\sin\theta_3.$$

Cross Law

- Theorem (Cross law)

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3).$$

- (Compare with the Cosine law)

$$d_3^2 = d_1^2 + d_2^2 - 2d_1d_2\cos\theta_3.$$

Right triangles and Pythagoras

- Suppose that $\{A_1A_2A_3\}$ is a right triangle with $s_3 = 1$. Then
- Theorem (Thales)

$$s_1 = Q_1/Q_3 \text{ and } s_2 = Q_2/Q_3.$$

- Theorem (Pythagoras)

$$Q_3 = Q_1 + Q_2.$$

Archimedes' function

- Archimedes' function $A(Q_1, Q_2, Q_3)$

$$Ar(Q_1, Q_2, Q_3) = (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2)$$

- Non-symmetric but more efficient version:

$$Ar(Q_1, Q_2, Q_3) = 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2$$

Theorems

- Theorem (Archimedes' formula): If $Q_1 = d_1^2$, $Q_2 = d_2^2$ and $Q_3 = d_3^2$, then $Ar(Q_1, Q_2, Q_3) =$

$$(d_1 + d_2 + d_3)(d_1 + d_2 - d_3)(d_2 + d_3 - d_1)(d_3 + d_1 - d_2)$$

Theorems (cont'd)

- Theorem: As a quadratic equation in Q_3, the TQF $Ar(Q_1, Q_2, Q_3) = 0$ can be rewritten as:

$$Q_3^2 - 2(Q_1 + Q_2)Q_3 + (Q_1 - Q_2)^2 = 0$$

- Theorem: The quadratic equations

$$(x - p_1)^2 = q_1$$

and

$$(x - p_2)^2 = q_2$$

has a common solutions iff $A(q_1, q_2, (p_1 - p_2)^2) = 0$

Heron's formula (Hero of Alexandria 60BC)

- The area of a triangle with side lengths a, b, c is

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$ is the semi-perimeter.

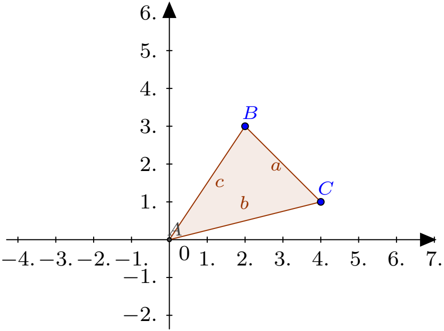


Figure 1: Heron's formula

Archimedes' theorem

- The area of a planar triangle with quadrances Q_1, Q_2, Q_3 is given by
$$16(\text{area})^2 = Ar(Q_1, Q_2, Q_3)$$

- Note: Given Q_1, Q_2. The area is maximum precisely when $Q_3 = Q_1 + Q_2$.

Brahmagupta's formula (convex)

- Brahmagupta's theorem:

$$\text{area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

  where $s = (a+b+c+d)/2$

- Preferred form: $16\text{area}^2 =$

$$(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)$$

Quadratic compatibility theorem

- Two quadratic equations

$$(x - p_1)^2 = q_1, \qquad (x - p_2)^2 = q_2$$

  are compatible iff

$$[(p_1 + p_2)^2 - (q_1 + q_2)]^2 = 4q_1q_2$$

  or

$$Ar(q_1, q_2, (p_1 - p_2)^2) = 0$$

- In this case, if $p_1 \neq p_2$ then there is a unique sol'n:

$$2x = (p_1 + p_2) - (q_1 - q_2)/(p_1 - p_2)$$

Quadruple Quad Formula

- Quadruple Quad Formula $Q(a, b, c, d)$

$$= [(a + b + c + d)^2 - 2(a^2 + b^2 + c^2 + d^2)]^2 - 64abcd$$

- Note that $(a + b + c + d)^2 - 2(a^2 + b^2 + c^2 + d^2)$

  can be computed efficiently as:

$$4(ab + cd) - (a + b - c - d)^2$$

Brahmagupta's formula

- Brahmagupta's formula (convex): $B(a,b,c,d) =$

$$(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)$$

- Robbin's formula (non-convex): $R(a,b,c,d) =$

$$(a+b+c+d)(a+b-c-d)(a-b+c-d)(b+c-a-d)$$

- Brahmagupta's identity

$$Q(a^2,b^2,c^2,d^2) = B(a,b,c,d)\cdot R(a,b,c,d)$$

Cyclic quadrilateral quadrea theorem

$$(\text{Area})^2 - 2m(\text{Area}) + p = 0$$

where

$$\begin{aligned} m &= (Q_{12}+Q_{23}+Q_{34}+Q_{14})^2 - 2(Q_{12}^2+Q_{23}^2+Q_{34}^2+Q_{14}^2) \\ &= 4(ab+cd) - (a+b-c-d)^2 \\ &= 4(Q_{12}Q_{23}+Q_{34}Q_{14}) - (Q_{12}+Q_{23}-Q_{34}-Q_{14})^2) \\ p &= Q(Q_{12}, Q_{23}, Q_{34}, Q_{14}) \end{aligned}$$

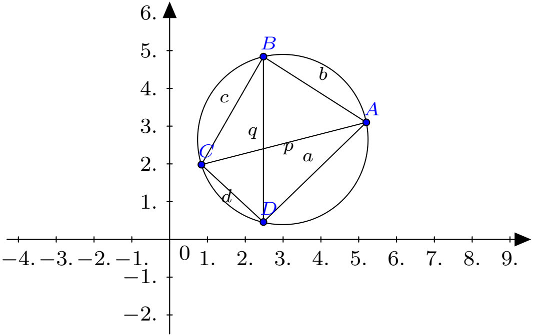
Ptolemy's theorem & generalizations

- **Claudius Ptolemy**: 90-168 A.D. (Alexandria) Astronomer & geographer & mathematician
- **Ptolemy's theorem** If $\{ABCD\}$ is a cyclic quadrilateral with the lengths a, b, c, d and diagonal lengths p, q, then

$$a\,b + c\,d = p\,q$$

[Actually needs convexity!]

Ptolemy's theorem



Exercise

- Ex. $A_1 = (1, 0)$, $A_2 = (3/5, 4/5)$, $A_3 = (-12/13, 5/13)$, $A_4 = (15/17, -8/17)$
- Then the quadrances are:

$$Q_{12} = 4/5, Q_{23} = 162/65, Q_{34} = 882/221, Q_{14} = 4/17$$

- The diagonal quadrances are:

$$Q_{24} = 144/85, Q_{13} = 50/13$$

Ptolemy's theorem (rational version)

- Ptolemy's theorem (rational): If $\{A_1 A_2 A_3 A_4\}$ is a cyclic quadrilateral with quadrances $Q_{ij} \equiv Q(A_i, A_j), i, j = 1, 2, 3, 4$ then

$$Ar(Q_{12}Q_{34}, Q_{23}Q_{14}, Q_{13}Q_{24}) = 0$$

- Ex. For $A_1 = (1, 0)$, $A_2 = (3/5, 4/5)$, $A_3 = (-12/13, 5/13)$, $A_4 = (15/17, -8/17)$ with

$$Q_{12} = 4/5, Q_{23} = 162/65, Q_{34} = 882/221,$$

$$Q_{14} = 4/17, Q_{24} = 144/85, Q_{13} = 50/13$$

- we can verify directly that $A(Q_{12}Q_{34}, Q_{23}Q_{14}, Q_{13}Q_{24}) = 0$.
- Note that with the rational form of Ptolemy's theorem, the three quantities appear *symmetrically*: so *convexity* of the cyclic quadrilateral is no longer required!

Proof of Ptolemy's theorem

Sketch of the proof:

- Without loss of generality, we can assume that the circle is a unit circle.
- Recall that a point on a unit circle can be parameterized as:

$$uc = [\lambda^2 - \mu^2, 2\lambda\mu, \lambda^2 + \mu^2],$$

  where λ and μ are not both zero.
- Let $A_i = uc(\lambda_i, \mu_i), i = 1, 2, 3, 4.$
- Express $Q_{ij} \equiv Q(A_i, A_j), i, j = 1, 2, 3, 4$ in terms of λ's and μ's
- Express $Ar(Q_{12}Q_{34}, Q_{23}Q_{14}, Q_{13}Q_{24})$ in terms of λ's and μ's.
- Simplify the expression and derive that it is equal to 0. (we may use the Python's symbolic package for the calculation. It took about 8 minutes on my computer :-)

Python Code

```
from __future__ import print_function
from pprint import pprint
import numpy as np
from fractions import *
from proj_geom import *

def quad1(x1, z1, x2, z2):
    if isinstance(x1, int):
        return (Fraction(x1,z1) - Fraction(x2,z2))**2
    else:
        return (x1/z1 - x2/z2)**2

def quadrance(a1, a2):
    return quad1(a1[0], a1[2], a2[0], a2[2]) + \
           quad1(a1[1], a1[2], a2[1], a2[2])

def uc_point(lambda1, mu1):
    return pg_point([lambda1**2 - mu1**2,
              2*lambda1*mu1, lambda1**2 + mu1**2])

def Ar(a, b, c):
    ''' Archimedes's function '''
    return (4*a*b) - (a + b - c)**2
```

Python Code (II)

```
if __name__ == "__main__":
    import sympy
    sympy.init_printing()

    lambda1, mu1 = sympy.symbols("lambda1 mu1", integer=True)
    lambda2, mu2 = sympy.symbols("lambda2 mu2", integer=True)
    lambda3, mu3 = sympy.symbols("lambda3 mu3", integer=True)
    lambda4, mu4 = sympy.symbols("lambda4 mu4", integer=True)
    a1 = uc_point(lambda1, mu1)
    a2 = uc_point(lambda2, mu2)
    a3 = uc_point(lambda3, mu3)
    a4 = uc_point(lambda4, mu4)
    q12 = quadrance(a1, a2)
    q23 = quadrance(a2, a3)
    q34 = quadrance(a3, a4)
    q14 = quadrance(a1, a4)
    q24 = quadrance(a2, a4)
    q13 = quadrance(a1, a3)
    t = Ar(q12*q34, q23*q14, q13*q24)
    t = sympy.simplify(t)
    print(t) # get 0
```

Backup

```
> pandoc -t latex -F pandoc-crossref -o temp2.svg .\01proj_geom
> pandoc -t beamer -F pandoc-crossref -o temp2.svg .\01proj_geo
```

Questions?