Chapter Five

Numerical Procedures for Eigenvalues of Irreducible Matrices

In this chapter we discuss two numerical procedures for irreducible matrices over max-plus algebra. The first one, called *Karp's algorithm*, will be presented in Section 5.1 and yields the eigenvalue of an irreducible matrix. The second one, called a *power algorithm*, to be presented in Section 5.2, yields the eigenvalue and a corresponding eigenvector. Notice that we have already encountered an algorithm for computing the eigenvalue in Chapter 2. Indeed, by Theorem 2.9 the eigenvalue of an irreducible matrix A is equal to the maximal average circuit weight of the communication graph of A.

We start in this chapter from a matrix $A \in \mathbb{R}_{\max}^{n \times n}$ and consider the recurrence relation that plays a central role in this book,

$$x(k+1) = A \otimes x(k), \tag{5.1}$$

for all $k \geq 0$. If $\lambda \in \mathbb{R}_{\max}$ and $v \in \mathbb{R}_{\max}$ are an eigenvalue and an eigenvector of A, respectively, then the solution of (5.1) for $x(0) = v$ is given by $x(k) = \lambda^{\otimes k} \otimes v$. It has been shown in Section 2.2 that if matrix A is irreducible, both λ and v exist and are finite (see Lemma 2.8 and Theorem 2.9). Moreover, λ is unique.

5.1 KARP'S ALGORITHM

In this section, we present Karp's algorithm for computing the eigenvalue of an irreducible square matrix. The following theorem, which is due to Karp [55], provides a characterization of the eigenvalue that is different from the one in Theorem 2.9.

THEOREM 5.1 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be irreducible with eigenvalue λ. Then*

$$\lambda = \max_{i=1,\ldots,n} \min_{k=0,\ldots,n-1} \frac{[A^{\otimes n}]_{ij} - [A^{\otimes k}]_{ij}}{n-k}, \tag{5.2}$$

where $j \in \underline{n}$ can be chosen arbitrarily and division has to be understood in conventional algebra.

Proof. Recall from Theorem 2.9 that λ can be interpreted as the maximal average circuit weight. To prove the current theorem, consider first the case where $\lambda = 0$. Then $\mathcal{G}(A)$ contains no circuits with positive weight, and there is at least one circuit with weight zero. Since $\mathcal{G}(A)$ contains no circuits of positive weight, it follows from Lemma 2.2 that

$$A^* = \bigoplus_{k=0}^{n-1} A^{\otimes k},$$

which reads in conventional algebra as $[A^*]_{ij} = \max_{0 \leq k \leq n-1} [A^{\otimes k}]_{ij}$, for all $i, j \in \underline{n}$. This implies

$$\min_{0 \leq k \leq n-1} \left([A^{\otimes n}]_{ij} - [A^{\otimes k}]_{ij} \right) = [A^{\otimes n}]_{ij} - [A^*]_{ij} \leq 0,$$

for all $i, j \in \underline{n}$, which gives

$$\min_{0 \leq k \leq n-1} \frac{[A^{\otimes n}]_{ij} - [A^{\otimes k}]_{ij}}{n-k} = \frac{[A^{\otimes n}]_{ij} - [A^*]_{ij}}{n-k} \leq 0,$$

for all $i, j \in \underline{n}$. Now pick any $j \in \underline{n}$, and to complete the proof for $\lambda = 0$, show that an $i \in \underline{n}$ exists such that

$$[A^{\otimes n}]_{ij} - [A^*]_{ij} = 0.$$

To that end, consider a critical circuit, say, ζ, and let l be a node on ζ. Next, consider a maximal weight path from j to l, say, ξ. To simplify the notation, set $\gamma = |\zeta|_1$ and $\tau = |\xi|_1$. It then holds that $[A^{\otimes \tau}]_{lj} = [A^*]_{lj}$, where it is assumed that $0 \leq \tau \leq n-1$. Extending ξ by m copies of ζ gives a path from j to l of length $\tau + m \times \gamma$. Denote this path by ξ'. Since the circuit ζ has weight zero, it follows by contradiction that $[A^{\otimes(\tau + m \times \gamma)}]_{lj} = [A^*]_{lj}$ for all integers $m \geq 0$. Now take m such that $\tau + m \times \gamma \geq n$. Then ξ' consists of $\tau + m \times \gamma$ nodes. Let the nth node of ξ' be denoted t. Denote ξ_1 for the subpath of ξ' from node j to node t and ξ_2 for the subpath of ξ' from node t to node l. Clearly, the lengths of ξ_1 and ξ_2 are n and $\tau + m \times \gamma - n$, respectively. Since ξ_1 and ξ_2 are part of a path with maximal weight, they themselves are paths of maximal weight. The weights of ξ_1 and ξ_2 are $[A^{\otimes n}]_{tj}$ and $[A^{\otimes(\tau + m \times \gamma - n)}]_{lt}$, respectively, so that

$$[A^*]_{lj} = [A^{\otimes n}]_{tj} + [A^{\otimes(\tau + m \times \gamma - n)}]_{lt}.$$

For the weight of ξ_1 it holds that $[A^{\otimes n}]_{tj} \leq [A^*]_{tj}$. To prove the equality, assume that $[A^{\otimes n}]_{tj} < [A^*]_{tj}$, and consider a path, say, ξ_0, from j to t of maximal weight $[A^*]_{tj}$. Next, extend ξ_0 by the subpath ξ_2 from t to l. The weight of the path from j to l thus obtained is

$$[A^*]_{tj} + [A^{\otimes(\tau + m \times \gamma - n)}]_{lt},$$

which will be larger than $[A^*]_{lj}$ and which by definition is impossible. Hence, $[A^{\otimes n}]_{ij} = [A^*]_{ij}$ with $i = t$, and (5.2) is correct in the case where the maximal circuit mean is zero.

To conclude the proof, assume that λ is finite but not necessarily zero. Subtracting a constant c from each of the entries of A, it follows that λ, seen as a critical circuit mean, is also reduced by c. Further, $[A^{\otimes k}]_{ij}$ is reduced by $k \times c$, implying that

$$\frac{1}{n-k}\left([A^{\otimes n}]_{ij} - [A^{\otimes k}]_{ij}\right)$$

is reduced by c, for all $i, j \in \underline{n}$ and $k \geq 0$. This means that for any $j \in \underline{n}$

$$\min_{0 \leq k \leq n-1} \frac{[A^{\otimes n}]_{ij} - [A^{\otimes k}]_{ij}}{n-k}$$

is reduced by c. Hence, both sides in (5.2) are reduced by c. Taking c equal to λ, the above case, where the maximum circuit mean was supposed to be zero, can be applied. □

An efficient way of evaluating the eigenvalue of an irreducible matrix with the help of Karp's theorem is the following. Take $x(0) = e_j$ (the jth base vector), and determine $x(k)$ by iterating (5.1). Then $x(k)$ is equal to $[A^{\otimes k}]_{\cdot j}$, for $k \geq 0$. Karp's algorithm can now be stated as follows.

Algorithm 5.1.1 KARP'S ALGORITHM

1. *Choose arbitrary $j \in \underline{n}$, and set $x(0) = e_j$.*

2. *Compute $x(k)$ for $k = 0, \ldots, n$.*

3. *Compute as an eigenvalue*

$$\lambda = \max_{i=1,\ldots,n} \min_{k=0,\ldots,n-1} \frac{x_i(n) - x_i(k)}{n-k}.$$

Karp's algorithm is illustrated with the following examples.

Example 5.1.1 *Let*

$$A = \begin{pmatrix} \varepsilon & 3 & \varepsilon & 1 \\ 2 & \varepsilon & 1 & \varepsilon \\ 1 & 2 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix}.$$

Inspecting the communication graph of A in Figure 5.1, it is easily seen that the graph is strongly connected and, consequently, that matrix A is irreducible. We

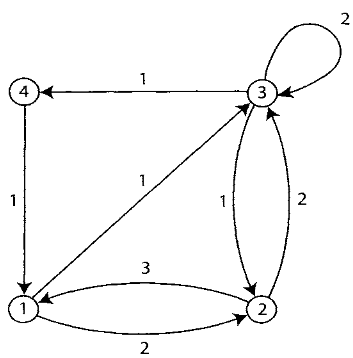

Figure 5.1: Communication graph of Example 5.1.1.

apply Karp's algorithm with $j = 1$, and consider consequently $x(0) = e_1 = (0, \varepsilon, \varepsilon, \varepsilon)^\top$. Note that $n = 4$ and iterating (5.1) four times yields

$$x(1) = \begin{pmatrix} \varepsilon \\ 2 \\ 1 \\ \varepsilon \end{pmatrix}, \quad x(2) = \begin{pmatrix} 5 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 5 \\ 7 \\ 6 \\ 5 \end{pmatrix}, \quad x(4) = \begin{pmatrix} 10 \\ 7 \\ 9 \\ 7 \end{pmatrix}.$$

For $i \in \underline{4}$, the minimization terms $\min_{k=0,\ldots,3} (x_i(4) - x_i(k))/(4-k)$ read

$$\min\left\{\frac{10}{4}, \infty, \frac{5}{2}, \frac{5}{1}\right\} = 2\frac{1}{2}, \qquad \min\left\{\infty, \frac{5}{3}, \frac{5}{2}, 0\right\} = 0,$$

$$\min\left\{\infty, \frac{8}{3}, \frac{5}{2}, \frac{3}{1}\right\} = 2\frac{1}{2}, \qquad \min\left\{\infty, \infty, \frac{5}{2}, \frac{2}{1}\right\} = 2,$$

respectively, and Karp's algorithm yields

$$\lambda = \max\left\{\frac{5}{2}, 0, 2\right\} = 2\frac{1}{2}$$

for the eigenvalue of A.

In the above example, the matrix A was irreducible. In the next two examples, the applicability of Karp's algorithm will be investigated for matrices that are reducible.

Example 5.1.2 *Let*

$$A = \begin{pmatrix} 1 & 2 & \varepsilon & 7 \\ \varepsilon & 3 & 5 & \varepsilon \\ \varepsilon & 4 & \varepsilon & 3 \\ \varepsilon & 2 & 8 & \varepsilon \end{pmatrix}.$$

The communication graph $\mathcal{G}(A)$ is depicted in Figure 5.2. It is clear that the graph is not strongly connected, as there are no paths from node 1 to the other nodes. Hence, the matrix A is reducible. So Theorem 5.1 does not guarantee that Karp's algorithm yields a correct result. Nevertheless, we are going to apply the algorithm and see where it leads. To apply Karp's algorithm we first take $j = 1$, and consequently, as an initial vector (as in the previous example), $x(0) = e_1$. Repeated application of (5.1) yields that

$$x(k) = \begin{pmatrix} k \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}.$$

For $i = 1$ (and $n = 4$) we have

$$\min_{k=0,\ldots,3} \frac{x_i(4) - x_i(k)}{4-k} = \min\{1, 1, 1, 1\} = 1,$$

and for $i \in \{2, 3, 4\}$

$$\min_{k=0,\ldots,3} \frac{x_i(4) - x_i(k)}{4-k} = \min\{0, 0, 0, 0\} = 0,$$

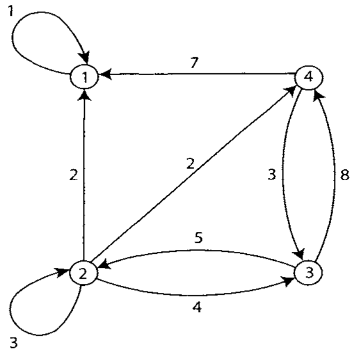

Figure 5.2: Communication graph of Example 5.1.2.

where by convention $\varepsilon - \varepsilon = e$, *i.e.*, $\varepsilon - (-\infty) = 0$; *see (2.7). Taking the maximum for* $i \in \underline{4}$ *results in* $\lambda = 1$ *as the output of Karp's algorithm. Since* $x(1) = A \otimes x(0) = 1 \otimes x(0)$, *it follows that* $\lambda = 1$ *is indeed an eigenvalue of the matrix* A *with* $x(0)$ *as an associated eigenvector.*

Next, we take $j = 2$, *and consequently, as an initial vector* $x(0) = e_2 = (\varepsilon, 0, \varepsilon, \varepsilon)^\top$. *Iterating (5.1) we obtain*

$$x(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 2 \end{pmatrix}, \quad x(2) = \begin{pmatrix} 9 \\ 9 \\ 7 \\ 12 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 19 \\ 12 \\ 15 \\ 15 \end{pmatrix}, \quad x(4) = \begin{pmatrix} 22 \\ 20 \\ 18 \\ 23 \end{pmatrix}.$$

For $i \in \underline{4}$, *the expression* $\min_{k=0,\ldots,3} (x_i(4) - x_i(k))/(4-k)$ *gives*

$$\min\left\{\infty, \frac{20}{3}, \frac{13}{2}, \frac{3}{1}\right\} = 3, \qquad \min\left\{\frac{20}{4}, \frac{17}{3}, \frac{11}{2}, \frac{8}{1}\right\} = 5,$$

$$\min\left\{\infty, \frac{14}{3}, \frac{11}{2}, \frac{3}{1}\right\} = 3, \qquad \min\left\{\infty, \frac{21}{3}, \frac{11}{2}, \frac{8}{1}\right\} = 5\frac{1}{2},$$

respectively, and Karp's algorithm yields as an output

$$\lambda = 5\frac{1}{2} = \max\left\{3, 5, \frac{11}{2}\right\}.$$

By trial and error it turns out (see also Section 5.2 for an algorithmic approach) that the vector v, *given by*

$$v = \begin{pmatrix} 24\frac{1}{2} \\ 20 \\ 20\frac{1}{2} \\ 23 \end{pmatrix},$$

satisfies $A \otimes v = \lambda \otimes v$. *Hence,* λ *is indeed an eigenvector of* A *with associated eigenvector* v.

In both cases (i.e., either $j = 1$ or $j = 2, 3, 4$) Karp's algorithm does come up with a correct answer, but the outcome of Karp's algorithm depends on the choice of j.

Example 5.1.3 *Finally, consider a modified version of the previous example in which only the entry a_{11} has been changed:*

$$A = \begin{pmatrix} 6 & 2 & \varepsilon & 7 \\ \varepsilon & 3 & 5 & \varepsilon \\ \varepsilon & 4 & \varepsilon & 3 \\ \varepsilon & 2 & 8 & \varepsilon \end{pmatrix} .$$

As in the previous example, the communication graph of A is not strongly connected. Hence, the matrix A is reducible. Again, in what follows we apply Karp's algorithm even though Theorem 5.1 does not guarantee that the algorithm will yield a correct result.

We first take $j = 1$, and consequently, as an initial vector $x(0) = e_1$. Repeated application of (5.1) yields that

$$x(k) = \begin{pmatrix} 6k \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} .$$

For $i = 1$ we obtain

$$\min_{k=0,\ldots,3} \frac{x_i(4) - x_i(k)}{4 - k} = \min\{6, 6, 6, 6\} = 6 .$$

As in the previous example, for $i = 2, 3, 4$ the expression is equal to zero. Taking the maximum of the above expressions for $i \in \underline{4}$ yields $\lambda = 6$ as the output of Karp's algorithm. Since it is clear that $x(1) = A \otimes x(0) = 6 \otimes x(0)$, $\lambda = 6$ is indeed an eigenvalue of the matrix A with $x(0)$ as an associated eigenvector.

Now we take $j = 2$, and consequently, as an initial vector $x(0) = e_2$. Iterating, we obtain

$$x(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 2 \end{pmatrix}, \quad x(2) = \begin{pmatrix} 9 \\ 9 \\ 7 \\ 12 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 19 \\ 12 \\ 15 \\ 15 \end{pmatrix}, \quad x(4) = \begin{pmatrix} 25 \\ 20 \\ 18 \\ 23 \end{pmatrix} .$$

For $i \in \underline{4}$, the expression $\min_{k=0,\ldots,3} (x_i(4) - x_i(k))/(4 - k)$ is equal to

$$\min\left\{\infty, \frac{23}{3}, \frac{16}{2}, \frac{6}{1}\right\} = 6, \quad \min\left\{\frac{20}{4}, \frac{17}{3}, \frac{11}{2}, \frac{8}{1}\right\} = 5,$$

$$\min\left\{\infty, \frac{14}{3}, \frac{11}{2}, \frac{3}{1}\right\} = 3, \quad \min\left\{\infty, \frac{21}{3}, \frac{11}{2}, \frac{8}{1}\right\} = 5\frac{1}{2},$$

respectively. Taking the maximum of the above expressions for $i \in \underline{4}$ yields again 6 as a candidate for an eigenvalue. It is an eigenvalue since an eigenvector has already been given above.

To conclude this section, we note that Karp's algorithm results in only one value. This value is the eigenvalue of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ if the matrix is irreducible. If A is reducible, Karp's algorithm also yields an eigenvalue depending on the choice of j. However, the associated eigenvector may have elements equal to ε. Notice that Karp's algorithm provides no means for computing an eigenvector. This drawback of Karp's algorithm is overcome by the power algorithm to be presented in the next section, which simultaneously computes the eigenvalue and a corresponding eigenvector.

5.2 THE POWER ALGORITHM

In this section we assume again that the $n \times n$ matrix A is irreducible. By Theorem 3.9, $x(k)$ following (5.1) will eventually enter a periodic regime. If we evoke Theorem 4.1, then the eigenvalue and a corresponding eigenvector can be computed from such a periodic regime. The resulting algorithm, called the *power algorithm*, is given below.

Algorithm 5.2.1 POWER ALGORITHM

1. *Take an arbitrary initial vector* $x(0) = x_0 \neq \mathbf{u}[\varepsilon]$; *that is,* x_0 *has at least one finite element.*
2. *Iterate (5.1) until there are integers* p, q *with* $p > q \geq 0$ *and a real number* c, *such that* $x(p) = x(q) \otimes c$, *i.e., until a periodic regime is reached.*
3. *Compute as the eigenvalue* $\lambda = c/(p-q)$ *(division in conventional sense).*
4. *Compute as an eigenvector* $v = \displaystyle\bigoplus_{j=1}^{p-q} \left(\lambda^{\otimes(p-q-j)} \otimes x(q+j-1)\right)$.

In the following we review the three examples treated in the previous section.

Example 5.2.1 *Reconsider Example 5.1.1. Recall that matrix A is irreducible. Applying the power algorithm, we take as an initial vector $x(0) = e_1$. Iterating (5.1) we obtain (see also Example 5.1.1)*

$$x(1) = \begin{pmatrix} \varepsilon \\ 2 \\ 1 \\ \varepsilon \end{pmatrix}, \quad x(2) = \begin{pmatrix} 5 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 5 \\ 7 \\ 6 \\ 5 \end{pmatrix}, \quad x(4) = \begin{pmatrix} 10 \\ 7 \\ 9 \\ 7 \end{pmatrix}.$$

Note that $x(4) = 5 \otimes x(2)$. In the power algorithm, we therefore have that $p = 4, q = 2$, and $c = 5$, so that consequently $\lambda = 2\frac{1}{2}$. The vector v resulting from the algorithm equals

$$v = (\lambda \otimes x(2)) \oplus x(3) = \begin{pmatrix} 7\frac{1}{2} \\ 4\frac{1}{2} \\ 6\frac{1}{2} \\ 4\frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 7 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 7\frac{1}{2} \\ 7 \\ 6\frac{1}{2} \\ 5 \end{pmatrix}.$$

It is easy to see that indeed

$$A \otimes v = \begin{pmatrix} 10 \\ 9\frac{1}{2} \\ 9 \\ 7\frac{1}{2} \end{pmatrix} = \lambda \otimes v.$$

Hence, v is an eigenvector of matrix A for eigenvalue $\lambda = 2\frac{1}{2}$.

Example 5.2.2 *Reconsider matrix A in Example 5.1.2. Recall that matrix A is reducible, which implies that the power algorithm will thus not necessarily find an eigenvalue and an associated eigenvector.*

First, we take as an initial vector $x(0) = e_1$. Applying (5.1) we obtain

$$x(1) = \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} = 1 \otimes \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} = 1 \otimes x(0),$$

and it immediately follows that $A \otimes v = v \otimes \lambda$ with $\lambda = 1$ and $v = x(0)$.

Next, we take as an initial vector $x(0) = e_2$. We iterate (5.1), which gives among others

$$x(3) = \begin{pmatrix} 19 \\ 12 \\ 15 \\ 15 \end{pmatrix}, \quad x(4) = \begin{pmatrix} 22 \\ 20 \\ 18 \\ 23 \end{pmatrix}, \quad x(5) = \begin{pmatrix} 30 \\ 23 \\ 26 \\ 26 \end{pmatrix}.$$

It is clear that $x(5) = 11 \otimes x(3)$. Therefore, in the power algorithm we have that $p = 5, q = 3$, and $c = 11$, so that consequently $\lambda = 5\frac{1}{2}$. The vector v resulting from the algorithm equals

$$v = (\lambda \otimes x(3)) \oplus x(4) = \begin{pmatrix} 24\frac{1}{2} \\ 17\frac{1}{2} \\ 20\frac{1}{2} \\ 20\frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} 22 \\ 20 \\ 18 \\ 23 \end{pmatrix} = \begin{pmatrix} 24\frac{1}{2} \\ 20 \\ 20\frac{1}{2} \\ 23 \end{pmatrix}.$$

It is easy to verify that $A \otimes v = \lambda \otimes v$. Hence, v is an eigenvector of the matrix A for the eigenvalue $\lambda = 5\frac{1}{2}$.

In the above, the power algorithm does come up with an eigenvalue and an eigenvector, although its outcome depends on the choice of $x(0)$. It is not difficult to see that as long as at least one of the components $x_i(0), i \in \{2, 3, 4\}$, of the initial vector $x(0)$ has a finite value, then the power algorithm will yield the eigenvalue $\lambda = 5\frac{1}{2}$. If $x_i(0) = \varepsilon$ for all $i \in \{2, 3, 4\}$ and $x_1(0)$ is finite, then the algorithm will yield the eigenvalue $\lambda = 1$.

Note that matrix A has normal form and the eigenvalues of the two diagonal blocks are 1 and $5\frac{1}{2}$, respectively. If we start with a fully finite initial condition, it follows, according to Theorem 3.15, that the cycle-time vector has value $5\frac{1}{2}$ in each of its components. From the above it further follows that $(\mathbf{u}[5\frac{1}{2}], v)$ is a generalized eigenmode of A.

Example 5.2.3 *Reconsider Example 5.1.3. As in the previous example, the matrix A is reducible, and the power algorithm will thus not necessarily yield an eigenvalue and an associated eigenvector of A. If we take as an initial vector $x(0) = e_1$, then we obtain similarly as in the previous example that $x(1) = 6 \otimes x(0)$. Hence, we immediately are in a periodic regime, and it follows that $A \otimes v = \lambda \otimes v$ with $\lambda = 6$ and $v = x(0)$.*

However, if we take as an initial vector $x(0) = e_2$ and iterate, we obtain

$$\begin{array}{ccccccc} x(0) & & x(1) & & x(2) & & x(3) \\ \begin{pmatrix} \varepsilon \\ 0 \\ \varepsilon \\ \varepsilon \end{pmatrix} & \rightarrow & \begin{pmatrix} 2 \\ 3 \\ 4 \\ 2 \end{pmatrix} & \rightarrow & \begin{pmatrix} 9 \\ 9 \\ 7 \\ 12 \end{pmatrix} & \rightarrow & \begin{pmatrix} 19 \\ 12 \\ 15 \\ 15 \end{pmatrix} & \rightarrow \end{array}$$

$$\begin{array}{ccccccc} x(4) & & x(5) & & x(6) & & x(7) \\ \begin{pmatrix} 25 \\ 20 \\ 18 \\ 23 \end{pmatrix} & \rightarrow & \begin{pmatrix} 31 \\ 23 \\ 26 \\ 26 \end{pmatrix} & \rightarrow & \begin{pmatrix} 37 \\ 31 \\ 29 \\ 34 \end{pmatrix} & \rightarrow & \begin{pmatrix} 43 \\ 34 \\ 37 \\ 37 \end{pmatrix} & \rightarrow \end{array}$$

$$\begin{array}{ccccccccc} x(8) & & x(9) & & x(10) & & x(11) & & \ldots \\ \begin{pmatrix} 49 \\ 42 \\ 40 \\ 45 \end{pmatrix} & \rightarrow & \begin{pmatrix} 55 \\ 45 \\ 48 \\ 48 \end{pmatrix} & \rightarrow & \begin{pmatrix} 61 \\ 53 \\ 51 \\ 56 \end{pmatrix} & \rightarrow & \begin{pmatrix} 67 \\ 56 \\ 59 \\ 59 \end{pmatrix} & \rightarrow & \cdots . \end{array}$$

In the above it is seen that in the long run the first component of $x(k)$ increases each iteration step by 6, while the other components increase on average each iteration step by $5\frac{1}{2}$. Hence, for the chosen initial vector there never can be an overall periodic regime. It turns out that this is always the case for initial vectors $x(0)$ of which at least one of the components $x_i(0), i = 2, 3, 4$, has a finite value. For those initial vectors an overall periodic regime does not exist, and the power algorithm cannot be applied. Again, observe that the above is in correspondence with Theorem 3.15. Indeed, one finds that

$$\eta = \begin{pmatrix} 6 \\ 5\frac{1}{2} \\ 5\frac{1}{2} \\ 5\frac{1}{2} \end{pmatrix}$$

is the cycle-time vector when starting with a fully finite initial condition.

To conclude this section we note that if monitoring the iteration process shows that a periodic regime will not be reached, the power algorithm will not terminate. Note that for termination of the algorithm the matrix A does not necessarily have to be irreducible.

5.3 EXERCISES

1. Consider the matrix
$$A = \begin{pmatrix} 5 & 1 \\ 0 & 6 \end{pmatrix}.$$
Check that matrix A is irreducible, and apply Karp's algorithm to determine the eigenvalue of A.

2. Consider matrix A of exercise 1, and apply the power algorithm to determine the eigenvalue of A and a corresponding eigenvector. Do this starting from
$$\text{(a)} \quad x(0) = \begin{pmatrix} e \\ \varepsilon \end{pmatrix}, \qquad \text{(b)} \quad x(0) = \begin{pmatrix} \varepsilon \\ e \end{pmatrix}.$$

3. Combine Karp's algorithm and the power algorithm to obtain an algorithm that in as few as possible iterations of (5.1) results in the eigenvalue of an irreducible matrix A.

4. Explain why in Example 5.2.2 the power algorithm comes up with the eigenvalue $5\frac{1}{2}$ if at least one of the components $x_i(0), i = 2, 3, 4$, has a finite value.

5. Eplain why in Example 5.2.3 the power algorithm does not work if at least one of the components $x_i(0), i = 2, 3, 4$, has a finite value.

5.4 NOTES

Example 5.1.1 is due to [17], and Examples 5.1.2 and 5.1.3 are based upon [23]. Section 5.2 is based on [83]. Exercise 2 is inspired by [81].

The complexity of Karp's algorithm is of order n^3, while the complexity of the power algorithm is less clear. In particular, the length of the transient behavior (i.e., the number of steps to reach the periodic regime) can be large depending on the value of the entries of the matrix. Only very conservative upper bounds exist for the length of the transition behavior. See, for instance, [50] or [81].

Numerical methods, other than presented in this chapter and the next, have been proposed in the literature, based, for example, on linear programming techniques or on finding the root of the characteristic polynomial in max-plus algebra; see [5] and [75]. The ones presented in the current book turned out to be the most powerful.