Chapter Thirteen

Continuous and Synchronized Flows on Networks

So far, we have formulated timed events as discrete flows on networks. In this section, we consider a continuous version of such flows.

One possible way to define, describe, and analyze such continuous flows is by limit arguments in timed event graphs (Chapter 7). In such an approach tokens are split up into mini-tokens (say, one original token consists of N identical mini-tokens); the original corresponding place is replaced by N places in series, with one mini-token in each of them and with transitions in between. The original holding times are divided by N (firing times remain zero). A transition can fire when each of the upstream places contains at least one mini-token. In the limit, when $N \to \infty$, the result is something that is called a continuous flow, which, due to the behavior of transitions, is synchronized.

Instead, we will follow a slightly different route to introduce such flows. Many of the results of Chapter 3 also hold here, just as there are many similarities between recurrence equations and ordinary differential equations.

13.1 DATER AND COUNTER DESCRIPTIONS

Compare the dater and counter descriptions (see Section 0.5) of the same system:

$$x_i(k) = \bigoplus_{j=1}^{n} a_{ij} \otimes x_j(k - b_{ij}), \qquad i \in \underline{n}, \tag{13.1}$$

$$\kappa_i(\chi) = \bigoplus_{j=1}^{n}{}' b_{ij} \otimes \kappa_j(\chi - a_{ij}), \qquad i \in \underline{n}. \tag{13.2}$$

In these descriptions the quantities b_{ij} are natural numbers that refer to the number of (unit) delays in the counting. Quite often $b_{ij} = 1$ (for instance, after having augmented the original state vector, such that the dater equations have become a first-order recurrence equation). The quantities a_{ij}, which are real valued, refer to travel times between the nodes of the network. Quantity $x_i(k)$ refers to the time instant at which the kth event occurs; $\kappa_i(\chi)$ refers to the number of events that have occurred up to (and including) time χ.

In the parlance of Petri nets, b_{ij} is the number of tokens in the place that one passes if traveling from transition q_j to transition q_i; see Chapter 7. The quantities a_{ij} are holding times, and we assume the firing times to be zero. Note that in the Petri net interpretation of (13.1) or (13.2) there is maximally one connection, with one place, between two transitions. In the context of Petri nets, (13.1) and (13.2) can be rewritten as

$$x_i(k) = \bigoplus_{j \in \pi(i)} a_{ij} \otimes x_j(k - b_{ij}), \qquad i \in \underline{n}, \tag{13.3}$$

$$\kappa_i(\chi) = \bigoplus_{j \in \pi(i)}{}' b_{ij} \otimes \kappa_j(\chi - a_{ij}), \qquad i \in \underline{n}, \tag{13.4}$$

where, as before, the set $\pi(i)$ refers to the immediate upstream transitions of q_i. In the latter two equations, the quantities a_{ij} and b_{ij} are assumed to be finitely valued. As a reminder, the notation $\sigma(i)$, to be used later on again, refers to the set of immediate downstream transitions of q_i.

Though (13.3) and (13.4) essentially describe the same phenomena, there is a clear asymmetry between the two models. In (13.3) the delays b_{ij} are integer valued and the coefficients a_{ij} real valued, whereas in (13.4) the delays, here a_{ij}, are real valued and the coefficients b_{ij} integer valued. The extension to be made now is that it does not matter whether one prefers (13.3) or (13.4) for further analysis; both a_{ij} and b_{ij} are assumed to be nonnegative and real valued. Hence, the components of the states x and κ are real valued.

The interpretation of (13.1) and (13.2), with a_{ij} and b_{ij} real valued, is still a (strongly connected) network with n nodes (or transitions in Petri net terminology). These nodes can now fire continuously.

Example 13.1.1 *In a specific country, rosé wine is made by pouring white and red wines together. One tap delivers red wine, the other tap white wine. The separate flows of red and white come together at a transition, which mixes the incoming flows, at equal rates, instantaneously and continuously, into rosé (as long as the incoming streams do not dry up). The outgoing continuous flow of rosé is subsequently led to a bottling machine. The amount of rosé produced in this way, in liters, say, and up to a certain time* χ, *is* $2 \times \min(\kappa_{\text{white}}(\chi), \kappa_{\text{red}}(\chi))$, *where* $\kappa_{\text{white}}(\chi)$ *is the total amount of white wine offered up to time* χ *to the transition and* $\kappa_{\text{red}}(\chi)$ *is likewise defined. If for instance* $\kappa_{\text{white}}(\chi) > \kappa_{\text{red}}(\chi)$, *then part of the white wine must be stored temporarily in a buffer in order to be mixed later on when more red wine becomes available.*

For the solution of (13.3) and (13.4), initial conditions should be given. They are

$$x_j(s) \text{ for } \quad -\max_{l \in \sigma(j)} b_{lj} \le s \le 0,$$

$$\kappa_j(s) \text{ for } \quad -\max_{l \in \sigma(j)} a_{lj} \le s \le 0,$$

respectively, for $j \in \underline{n}$. For (13.3) and (13.4) to be solvable unambiguously, the conditions $a_{ij} > 0$ and $b_{ij} > 0$ are certainly sufficient but not necessary.

13.2 CONTINUOUS FLOWS WITHOUT CAPACITY CONSTRAINTS

The intensity by means of which node q_j fires at time χ is indicated by $v_j(\chi)$. Obviously, $v_j(\chi) \ge 0$. In this section there are no upper bounds on $v_j(\chi)$, i.e., the node can produce at an arbitrarily high rate. Quantity $\kappa_j(\chi)$ denotes the total amount produced by node q_j up to (and including) time χ. In order to have a handy visualization, the outgoing production is assumed to move with unit speed to the downstream nodes. An artificial length of a_{ij} for the connection between nodes q_j and q_i is subsequently assumed, such that the travel time is indeed a_{ij}. Along an arc there is a continuous flow, and its intensity will be denoted by $\phi_j(\chi, l)$, where l is the parameter indicating the exact location along an arc starting at node q_j. Node q_j may have many outgoing arcs, but the flow $\phi_j(\chi, l)$ will be the same along each of them, though it is possible that the range of l will be different for each of these arcs; see Figure 13.1. The beginning of the arc from q_j to q_i coincides with $l = 0$, and $l = a_{ij}$ coincides with its end. As long as the parameters lie in appropriate intervals, it follows that

$$\phi_j(\chi, l) = \phi_j(\chi + s, l + s), \qquad \phi_j(\chi, l) = \phi_j(\chi - l, 0) = v_j(\chi - l).$$

At time χ the total amount of material along the arc from node q_j to node q_i equals

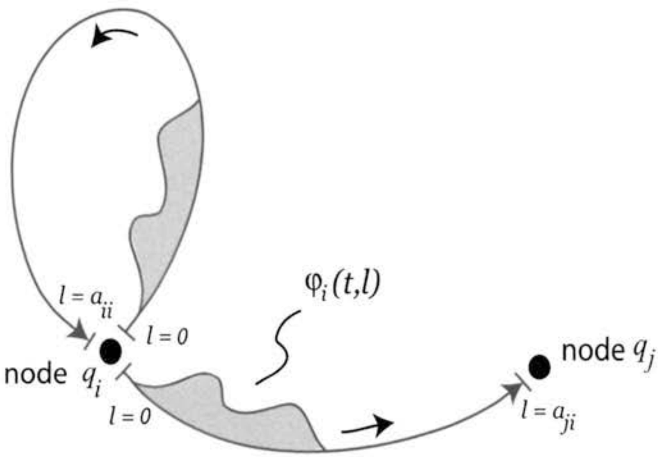

Figure 13.1: Identical continuous flows along the outgoing arcs of a node.

$$\int_0^{a_{ij}} \phi_j(\chi, s)ds. \tag{13.5}$$

The quantity b_{ij} satisfies

$$b_{ij} = \int_0^{a_{ij}} \phi_j(0, s)ds, \tag{13.6}$$

which equals the initial amount of material (a real number!) along the connection between nodes q_j and q_i. The integrands in (13.5) and (13.6) must be considered with care. It is quite possible that these integrands contain Dirac δ-functions. This will particularly happen at the end of an arc, since sometimes material must wait there to be processed by the immediate downstream transition because the other incoming arcs to the same transition have brought in less material thus far. If q_k is a downstream transition to both q_i and q_j and if $\kappa_i(\chi) < \kappa_j(\chi)$, then the flow

ϕ_j will start to build up a δ-function at $l = a_{kj}$, i.e., at the gate of q_k, at time χ. Of course, this δ-function can disappear again later on if $\kappa_i(s) > \kappa_j(s)$ for some s with $s > \chi$. Due to the fact that there are no capacity constraints, it is possible that two incoming δ-functions can be processed together and that on the outgoing arc(s) a δ-function appears which will travel with unit speed to the next transition.

The total amount of material along an arc, as expressed by (13.5), will, in general, be time (i.e., χ) dependent. However, along a circuit the total amount of material is constant.

THEOREM 13.1 *Given a circuit $((q_{i_1}, q_{i_2}), \ldots, (q_{i_k}, q_{i_1}))$, the total amount of material along this circuit, given by*

$$\sum_{l=1}^{k} \int_0^{a_{i_{l+1},i_l}} \phi_{i_l}(\chi, s) ds,$$

where $a_{i_{k+1},i_k} = a_{i_1,i_k}$, is constant (i.e., does not depend on χ).

Proof. A firing transition takes away from every incoming arc exactly as much material as it puts on each of the outgoing arcs. □

Please note that the total amount of material in the network is not necessarily constant.

DEFINITION 13.2 (Cycle mean) *Given a circuit $\zeta = ((q_{i_1}, q_{i_2}), \ldots, (q_{i_k}, q_{i_1}))$, its weight $|\zeta|_{\mathrm{w}}$ and its length $|\zeta|_{\mathrm{l}}$ are defined as*

$$|\zeta|_{\mathrm{w}} = \sum_{l=1}^{k} a_{i_{l+1},i_l}, \qquad |\zeta|_{\mathrm{l}} = \sum_{l=1}^{k} b_{i_{l+1},i_l}.$$

If $|\zeta|_{\mathrm{l}} > 0$, then the cycle mean is defined as $|\zeta|_{\mathrm{w}}/|\zeta|_{\mathrm{l}}$.

This definition of the circuit length is not to be confused with the interpretation of an arc (i) having a length and (ii) along which the material flows with unit speed. The length is the total amount of material in a circuit (which coincides with the total number of tokens for the discrete flows considered in earlier chapters).

Assumption. The following will hold for the remainder of this section:

- The network is strongly connected.
- a_{ij} and b_{ij} are nonnegative along connections between transitions.
- $|\zeta|_{\mathrm{l}} > 0$ and $|\zeta|_{\mathrm{w}} > 0$ for all circuits ζ.

The reason for the assumption $|\zeta|_{\mathrm{l}} > 0$, apart from the fact that it is needed in the definition of a cycle mean, is that if the total amount of material in a circuit would be zero, it will remain zero forever due to Theorem 13.1 and the transitions in this circuit will remain idle forever.

DEFINITION 13.3 *The circuits that have the maximum cycle mean are called* critical. *The corresponding cycle mean is indicated by λ, i.e., $\lambda = \max_\zeta |\zeta|_{\mathrm{w}}/|\zeta|_{\mathrm{l}}$.*

THEOREM 13.4 *For suitably chosen initial conditions, equations (13.2) have a solution*

$$\kappa_i(\chi) = \frac{1}{\lambda} \times \chi + d_i, \tag{13.7}$$

where d_i are constants.

Proof. Suppose that solutions of the form $\kappa_i(\chi) = c_i \times \chi + d_i$ exist (in the remainder of this proof conventional multiplication ($\times$) will not be indicated explicitly anymore). They are then substituted into (13.2), leading to the identities

$$c_i\chi + d_i = \min_{j\in\pi(i)} \{b_{ij} + c_j((\chi - a_{ij}) + d_j)\}\,, \qquad i \in \underline{n}. \tag{13.8}$$

For large values of χ ($\chi \to \infty$), this leads to

$$c_i = \min_{j\in\pi(i)} c_j, \qquad i \in \underline{n}.$$

Due to the assumption of the network being strongly connected, all c_i values must be equal, to be denoted here by c. Now substitute $\chi = 0$ into (13.8), resulting in

$$d_i = \min_{j\in\pi(i)} \{b_{ij} - ca_{ij} + d_j\}\,, \qquad i \in \underline{n},$$

which can be written in min-plus notation as

$$d = R \otimes' d,$$

where $d = (d_1, \ldots, d_n)^\top$ and where element (i, j) of the matrix R equals $b_{ij} - ca_{ij}$, provided that $j \in \pi(i)$; otherwise, this element equals $+\infty$. The vector d is an eigenvector of R, corresponding to the eigenvalue e (in the min-plus algebra sense). The remaining question is whether c can be chosen such that R has an eigenvalue e. Due to strong connectedness again, R is irreducible, and hence, it has only one eigenvalue that equals the minimum cycle mean (recall that we are currently working in min-plus algebra; the definition of a minimum cycle mean will be obvious)

$$\min_{\zeta} \frac{\sum_{(j,i)\in\zeta}(b_{ij} - ca_{ij})}{|\zeta|_1} = \min_{\zeta} \frac{\sum_{(j,i)\in\zeta} b_{ij} - c\sum_{(j,i)\in\zeta} a_{ij}}{|\zeta|_1}. \tag{13.9}$$

If we choose $c = (\sum_{(j,i)\in\zeta^*} b_{ij})/(\sum_{(j,i)\in\zeta^*} a_{ij})$, where ζ^* is a circuit for which the minimum in (13.9) is attained, then $c = \lambda^{-1}$ and the minimum cycle mean in (13.9) becomes zero as required. Now given that (13.7) is a solution of (13.2), the initial conditions as mentioned in the statement of Theorem 13.4 are determined by this solution. This concludes the proof. □

Remark. With all quantities specified as in Theorem 13.4, note that the density functions $\phi_i(t, s)$ are constant; that is, $\phi_i(t, s) = c$ for all appropriate t, s, with a possible exception of the endpoints of the corresponding arcs, where a δ-function may be present (of which the magnitude is constant with time again).

The solution given in the proof of Theorem 13.4 is, in the sense to be given, the best one. Let us concentrate on a critical circuit. During an interval of $\sum a_{ij}$ time units, where the summation is over all arcs of this circuit, any transition within this critical circuit can never produce more than an amount of $\sum b_{ij}$, the summation being again over all arcs of the circuit. In the linear solution in the statement of Theorem 13.4, any transition on the critical circuit produces exactly an amount of material equal to $\sum b_{ij}$ in each of the outgoing arcs during $\sum a_{ij}$ time units.

Example 13.2.1 *Solutions other than (13.7) may exist to (13.2). By means of this example, it will be shown that such solutions indeed exist. The solutions to be presented fluctuate in a periodic way around the linear solution obtained in Theorem 13.4. Exactly the same phenomenon has been observed for linear systems in conventional discrete max-plus (or min-plus) algebra. If one starts with an eigenvector as an initial condition, then the state behaves linearly with respect to time (one could say that this solution has period one). Other solutions are generally possible with a period larger than one and which fluctuate around the linear behavior just mentioned.*

We are given a network with $n = 2$. It is assumed that all four arcs (from transition q_i to q_j, $i, j \in \underline{2}$) exist and that

$$\lambda \stackrel{\text{def}}{=} \frac{a_{12} + a_{21}}{b_{12} + b_{21}} > \frac{a_{ii}}{b_{ii}}, \qquad i \in \underline{2}. \tag{13.10}$$

Try a solution of the form

$$\kappa_i(\chi) = \lambda^{-1}\chi + \alpha_i \sin(\beta(\chi - r_i)) + s_i, \qquad i \in \underline{2},$$

with $r_2 = s_2 = 0$ and $\beta > 0$, and

$$|\alpha_i|\beta < \lambda^{-1}, \qquad i \in \underline{2}. \tag{13.11}$$

The latter two conditions ensure the solutions, if they exist, to be nondecreasing. This solution is substituted into (13.2), leading to the identities

$$\begin{aligned} s_1 + \alpha_1 \sin(\beta(\chi - r_1)) &= \min\{-\lambda^{-1}a_{11} + \bar{b}_{11} + \alpha_1 \sin(\beta(\chi - a_{11} - r_1)), \\ &\qquad -\lambda^{-1}a_{12} + b_{12} + \alpha_2 \sin\beta(\chi - a_{12})\}, \\ \alpha_2 \sin(\beta\chi) &= \min\{-\lambda^{-1}a_{21} + \bar{b}_{21} + \alpha_1 \sin(\beta(\chi - a_{21} - r_1)), \\ &\qquad -\lambda^{-1}a_{22} + b_{22} + \alpha_2 \sin\beta(\chi - a_{22})\}, \end{aligned} \tag{13.12}$$

where $\bar{b}_{i1} = s_1 + b_{i1}$. Each minimization operation has two arguments. Assume for the moment that these arguments satisfy

$$\begin{aligned} -\lambda^{-1}a_{11} + s_1 + b_{11} + \alpha_1 \sin(\beta(\chi - a_{11} - r_1)) &\geq \\ -\lambda^{-1}a_{12} + b_{12} + \alpha_2 \sin\beta(\chi - a_{12}), &\\ -\lambda^{-1}a_{21} + s_1 + b_{21} + \alpha_1 \sin(\beta(\chi - a_{21} - r_1)) &\leq \\ -\lambda^{-1}a_{22} + b_{22} + \alpha_2 \sin\beta(\chi - a_{22}). & \end{aligned} \tag{13.13}$$

Then, the identities (13.12) become

$$\begin{aligned} s_1 + \alpha_1 \sin(\beta(\chi - r_1)) &= -\lambda^{-1}a_{12} + b_{12} + \alpha_2 \sin\beta(\chi - a_{12}), \\ \alpha_2 \sin(\beta\chi) &= -\lambda^{-1}a_{21} + s_1 + b_{21} + \alpha_1 \sin(\beta(\chi - a_{21} - r_1)). \end{aligned}$$

These equations are indeed identities if

$$\begin{aligned} r_1 &= a_{12}, \\ \beta(a_{12} + a_{21}) &= 2k\pi, \qquad k = 1, 2, \ldots, \qquad (13.14) \\ \alpha_1 &= \alpha_2, \\ s_1 &= -\lambda^{-1}a_{12} + b_{12}, \qquad (13.15) \\ -\lambda^{-1}a_{21} + s_1 + b_{21} &= 0. \qquad (13.16) \end{aligned}$$

The quantities s_1 and λ can be uniquely solved from (13.15) and (13.16). The value of λ fortunately coincides with its value given in (13.10). The value β is determined by (13.14). Thus, it is shown that a periodic solution exists, provided that (13.11) and (13.13) are true. A simple analysis shows that this is the case for $|\alpha_1|(=|\alpha_2|)$ sufficiently small. For $k=1$ in (13.14) we get that β equals the length of the critical circuit divided by 2π. For larger values of k we get higher harmonics. If the results of various k values are combined, then the solution becomes a Fourier series (with period $a_{12}+a_{21}$) added to the linear part.

Depending on the parameters a_{ij} and b_{ij}, one can construct a Fourier series in such a way that the ultimate solution $\kappa_i(\chi)$ becomes piecewise constant (and nondecreasing), thus leading to a real discrete flow. Points at which $\kappa_i(\chi)$ jumps refer to discrete events in the traditional sense.

CONJECTURE 13.5 *If there is a unique critical circuit, then each solution of (13.2), starting from arbitrary initial conditions, converges in finite time either to the linear solution or to a periodic solution as described in the example just given.*

A proof of this conjecture could resemble the train of thought of the proof of the discrete analogue of this theorem, which can be found in Chapter 3. A sketch of such a proof, if possible, would be as follows. Given the value $\kappa_i(\chi)$ for some χ sufficiently large, a critical path along the nodes, backward in time, is constructed according to (13.2), leading back all the way to the initial conditions. This critical path will contain a number of encirclements of the critical circuit, denoted ζ_{crit}. If now the same is done with respect to $\kappa_i(\chi+|\zeta_{\text{crit}}|_{\text{w}})$, then one gets the same critical path, except for the fact that ζ_{crit} will be encircled once more than for the critical path corresponding to $\kappa_i(\chi)$. Therefore, $\kappa_i(\chi+|\zeta_{\text{crit}}|_{\text{w}})=\kappa_i(\chi)+|\zeta_{\text{crit}}|_{1}$.

If the critical circuit is nonunique, then remarks related to the periodicity similar to those in Part I can be given for the continuous case.

13.3 CONTINUOUS FLOWS WITH CAPACITY CONSTRAINTS

The basic formulas in this section are

$$v_i(\chi)=\begin{cases} c_i & \text{if } m_{ij}(\chi)>0 \text{ for all } j\in\pi(i),\\ c_i\oplus'\bigoplus'_{j\in\pi(i)}v_j(\chi-a_{ij}) & \text{otherwise,}\end{cases} \tag{13.17}$$

for $i\in\underline{n}$, and

$$\dot{m}_{ij}(\chi)=v_j(\chi-a_{ij})-v_i(\chi), \tag{13.18}$$

for $i,j\in\underline{n}$. Quantity m_{ij} refers to the material sent from node j to node i that has already traveled along the whole arc (j,i) and is piled up at the "entrance" of node i, waiting to be processed. This material cannot be processed immediately, due to the constraint on the firing intensity indicated by c_i. The time derivative of m_{ij}, $\dot{m}_{ij}$, denotes the change of the size of this pile. The term $-v_i(\chi)$ refers to the processing speed, and the term $v_j(\chi-a_{ij})$ denotes the speed of new material to this pile. Quantity v_i denotes the production flow (i.e., the production per time unit) of transition q_i. Quantity $c_i > 0$ denotes the maximum firing intensity (equivalently, the capacity) of node i. The relation with the total amount produced is

$$\kappa_i(\chi) = \int_0^\chi v_i(s)\, ds.$$

In terms of the flow along the arc from q_j to q_i, the quantity $m_{ij}(\chi)$ represents the magnitude of the δ-function at the end of this arc, as described in the previous section. In the current section, traveling δ-functions along the arcs cannot arise; this is in contrast to the previous section. It is easily checked that $m_{ij}(\chi) \geq 0$ for $\chi \geq 0$, provided that the initial condition $m_{ij}(0)$ is nonnegative. It may look somewhat surprising that the quantities b_{ij} have disappeared from (13.17) and (13.18). They are, however, implicitly present in the initial conditions; that is, in order to calculate $v_i(\chi)$ for $\chi > 0$, one needs "old" values of $v_i(\chi)$ (i.e., with $\chi < 0$), and the latter functions with $\chi < 0$ are related to the quantities b_{ij}.

The equivalent expressions of (13.17) and (13.18) in the dater sense are

$$\nu_i(k) = \begin{cases} \dfrac{1}{c_i} & \text{if } \mu_{ij}(k) > 0 \text{ for all } j \in \pi(i), \\ \dfrac{1}{c_i} \oplus \bigoplus_{j \in \pi(i)} \nu_j(k - b_{ij}) & \text{otherwise}, \end{cases} \tag{13.19}$$

for $i \in \underline{n}$, and

$$\dot{\mu}_{ij}(k) = \nu_j(k - b_{ij}) - \nu_i(k), \tag{13.20}$$

for $i, j \in \underline{n}$. The reader is invited to give a meaning to the quantities μ_{ij}.

THEOREM 13.6 *Along a circuit the total amount of material is constant.*

The proof is identical to that of Theorem 13.1.

In the following theorem, the quantity λ appears again. As in Section 13.2, it is equal to the maximum cycle mean. The definition of the cycle mean, however, must now be slightly adapted.

DEFINITION 13.7 *Given a circuit* $\zeta = ((q_{i_1}, q_{i_2}), \ldots, (q_{i_k}, q_{i_1}))$*, the weight* $|\zeta|_{\mathrm{w}}$ *and the length* $|\zeta|_{\mathrm{l}}$ *are defined as*

$$|\zeta|_{\mathrm{w}} = \sum_{l=1}^{k} a_{i_{l+1}, i_l}, \qquad |\zeta|_{\mathrm{l}} = \sum_{l=1}^{k} b_{i_{l+1}, i_l} + m_{i_{l+1}, i_l}(0).$$

If $|\zeta|_{\mathrm{l}} > 0$*, then the cycle mean of* ζ *is defined as* $|\zeta|_{\mathrm{w}}/|\zeta|_{\mathrm{l}}$.

It is understood here that the function $\phi(0, s)$ is a real one, that is, it no longer contains any δ-function and the b_{ij}'s are defined as in (13.6). Here also the functions $\phi_j(0, s)$ and the quantities b_{ij} are related as given in (13.6), except for possible concentrations of material at the end of the arc as just explained. Because of the capacity constraints, $\phi(\chi, s)$ will also be a real function for $\chi > 0$.

THEOREM 13.8 *For appropriately chosen initial conditions, equations (13.17) have a solution* $\kappa_i(\chi) = \frac{1}{\overline{\lambda}}\chi + \overline{d}_i$*, where* $\overline{\lambda} = \max(c_1^{-1}, \ldots, c_n^{-1}, \lambda)$ *and where* $\overline{d}_i$ *are constants.*

Proof. If $\lambda > \max_i c_i^{-1}$ (equivalently, $\lambda^{-1} < \min_i c_i$), then the proof is identical to the one of Theorem 13.4 because none of the constraints is active. If $\lambda^{-1} \geq \min_i c_i$, then the assertion of the theorem follows from direct substitution of the proposed solution into (13.17). □

13.4 EXERCISES

1. Show that equations (13.17) are identical to (13.2) as c_i tends to ∞.

2. Consider a network with three nodes. The transportation times are $a_{21} = 1$, $a_{12} = 1$, $a_{13} = 1$, $a_{31} = 4$, $a_{32} = 3$, and $a_{33} = 2$. The time durations that have not been mentioned refer to nonexisting arcs. The capacity constraints are $c_1 = 2$, $c_2 = 3$, and $c_3 = 4$. The initial values of the flows, $\phi_i(0, s)$, $0 \leq s \leq a_{ji}$, for $i = 1, 2, 3$ and the appropriate downstream nodes q_j, are piecewise constant;

$$
\begin{array}{llll}
\phi_1(s) = 1, & \text{for } 0 \leq s \leq 1; & \phi_1(s) = 2, & \text{for } 1 < s \leq 2; \\
\phi_1(s) = 3, & \text{for } 2 < s \leq 3; & \phi_1(s) = 1, & \text{for } 3 < s \leq 4; \\
\phi_2(s) = 1, & \text{for } 0 \leq s \leq 1; & \phi_2(s) = 2, & \text{for } 1 < s \leq 2; \\
\phi_2(s) = 2, & \text{for } 2 < s \leq 3; & \phi_3(s) = 1, & \text{for } 0 \leq s \leq 1; \\
\phi_3(s) = 1, & \text{for } 1 < s \leq 2. & &
\end{array}
$$

   These functions only have to be considered up to the end of the appropriate arc. At the end of the arcs there are δ-functions with magnitudes

$$
\begin{array}{lll}
m_{21}(0) = 4, & m_{12}(0) = 1, & m_{31}(0) = 1, \\
m_{13}(0) = 2, & m_{32}(0) = 0, & m_{33}(0) = 3.
\end{array}
$$

   The functions ϕ together with these δ-functions can be viewed as a picture taken of the network at time zero. Evaluate the solution $\kappa(\chi)$ of this network for $\chi \geq 0$. Show the following:

   - One of the capacity constraints determines the overall speed.
   - Only the linear solution exists (in other words, there are no periodic solutions).
   - The transient behavior lasts 7 time units.

3. Rephrase and prove Theorem 13.4 in terms of $x_i(k)$, i.e., in the max-plus algebra setting.

4. Suppose that it would be possible to have real numbers of trains, such as $1\frac{2}{3}$ or $\sqrt{7}$ trains. Since under this new rule trains can be arbitrarily small, we may remodel train models as continuous flow models. Given the track layout of Figure 0.1, with the total number of trains being four, how would you split up these trains as a continuous flow on the network so as to get the smallest possible eigenvalue? Your answer should be a constant train density of $4/13$, i.e., $4/13$ train per unit of travel time along each track, leading to $\lambda = 13/4$. Note that in more general networks the λ thus obtained is a lower bound for the similar quantity to be obtained when dealing with integer numbers of trains.

5. Give a meaning to (13.19) and (13.20). Why have the quantities a_{ij} (seemingly) disappeared from this formulation?

13.5 NOTES

This chapter closely follows [72]. Other approaches, with different modeling features to continuous Petri nets, exist. The reader is referred to [2] and [32].

One of the authors often used Example 13.1.1 to jokingly explain about continuous flows, especially in France. Great was his surprise when a colleague from Germany told him that one particular German rosé wine, with the name Schiller, is indeed produced by pouring red and white wines together. In addition, he was told that the name did not derive from the famous poet F. von Schiller, but from the *schillernden Farben* (sparkling color) of the wine. Recently, a French colleague admitted that the French Champagne Rosé can also be produced by adding red wine to classic Champagne, according to strict rules, of course.