Sampling with Halton Points on n-Sphere

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October 21, 2018

Abstract

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- Sampling on n-sphere (S^n) has a wide range of applications, such as:
  - Spherical coding in MIMO wireless communication
  - Multivariate empirical mode decomposition
  - Filter bank design
- We propose a simple yet effective method which:
  - Utilizes low-discrepancy sequence
  - Contains only a few lines of Python code in our implementation!
  - Allow incremental generation.
- Numerical results show that the proposed method outperforms the randomly generated sequences and other proposed methods.

Motivation and Applications

Problem Formulation

Desirable properties of samples over S^n

- Uniform
- Deterministic
- Incremental
  - The uniformity measures are optimized with every new point.
  - Reason: in some applications, it is unknown how many points are needed to solve the problem in advance

Motivation

- The topic has been well studied for sphere in 3D, i.e. $n = 2$
- Yet it is still unknown how to generate for $n > 2$.
- Potential applications (for $n > 2$):
  - Robotic Motion Planning (S^3 and SO(3)) (Yershova et al. 2010)
  - Spherical coding in MIMO wireless communication (Utkovski and Lindner 2006):
    - Cookbook for Unitary matrices
    - A code word = a point in S^n
  - Multivariate empirical mode decomposition (Rehman and Mandic 2010)
  - Filter bank design (Mandic and others 2011)

Halton Sequence on S^n

- Halton sequence on S^2 has been well studied (Cui and Freeden 1997) by using cylindrical coordinates.
- Yet it is still little known for S^n where $n > 2$.
- Note: The generalization of cylindrical coordinates does NOT work in higher dimensions.

Review of Low Discrepancy Sequence

Basic: Van der Corput sequence

- Generate a low discrepancy sequence over $[0, 1]$
- Denote $\mathrm{vdc}(k, b)$ as a Van der Corput sequence of k points, where b is the base of a prime number.

Figure 1: Example of Van der Corput sequence

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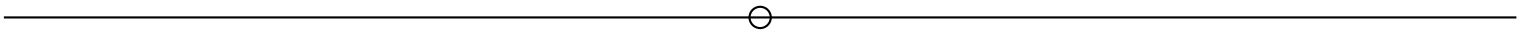

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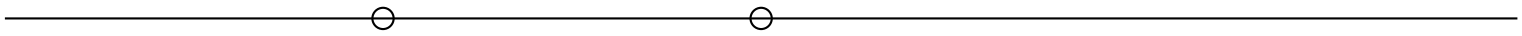

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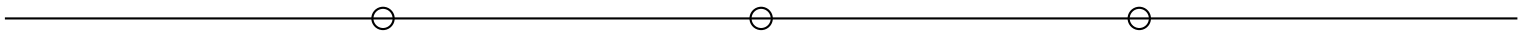

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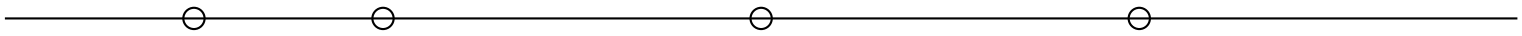

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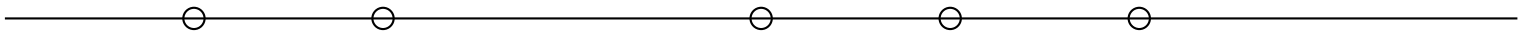

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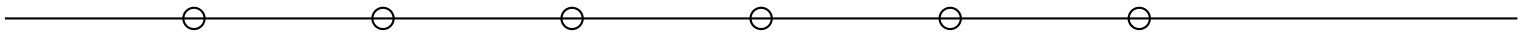

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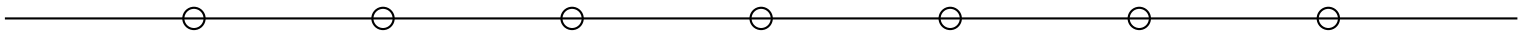

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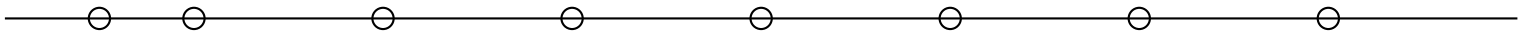

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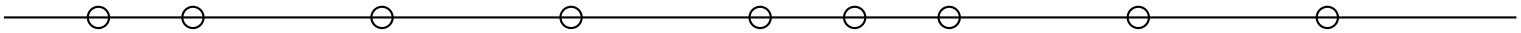

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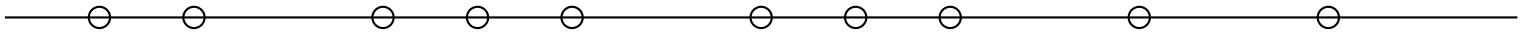

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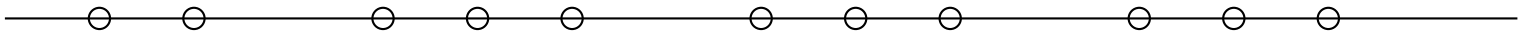

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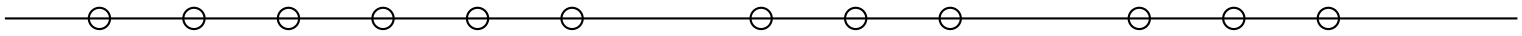

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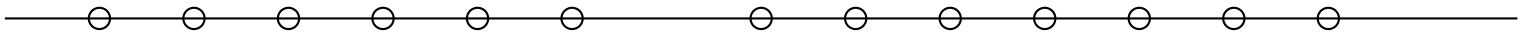

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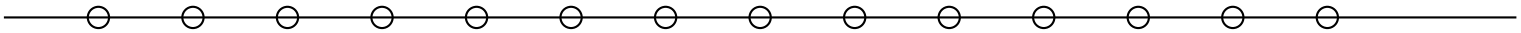

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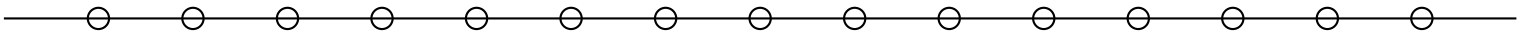

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Python code

```python
def vdc_basic(n, base=2):
    vdc, denom = 0.0, 1.0
    while n:
        denom *= base
        n, remainder = divmod(n, base)
        vdc += remainder / denom
    return vdc

def vdc(n, base=2):
    '''
    n - number of vectors
    base - seeds
    '''
    for i in range(n):
        yield vdc_basic(i, base)
```

Halton sequence on $[0, 1]$

- Halton sequence: using 2 Van der Corput sequences with different bases.
- Example:

$$[x, y] = [\mathrm{vdc}(k, 2), \mathrm{vdc}(k, 3)]$$

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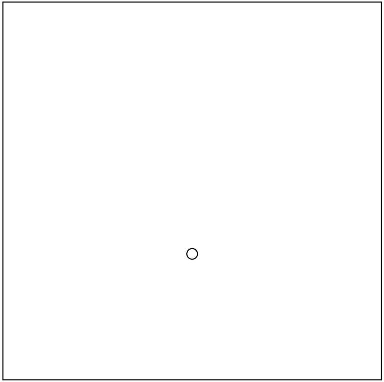

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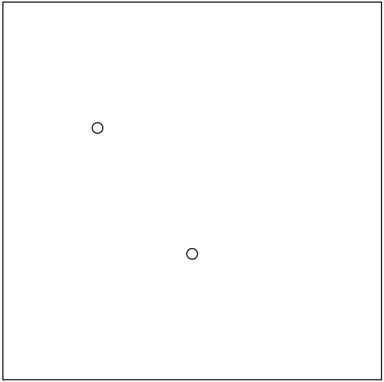

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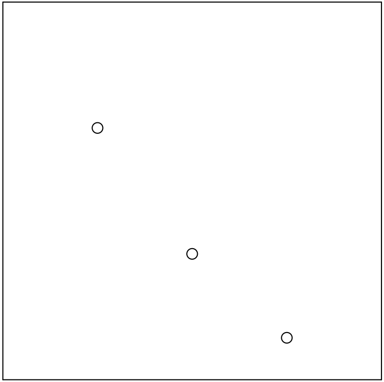

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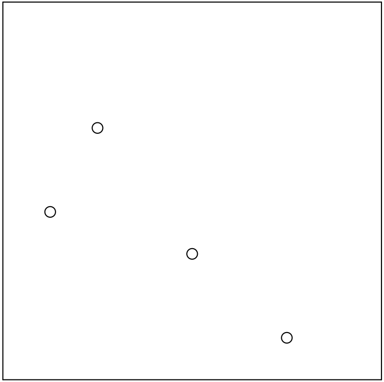

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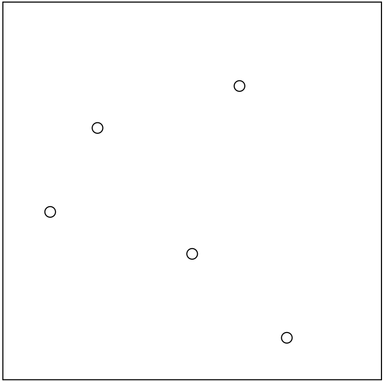

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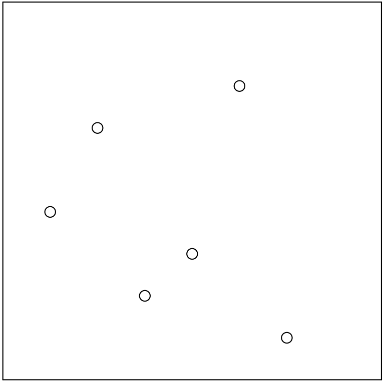

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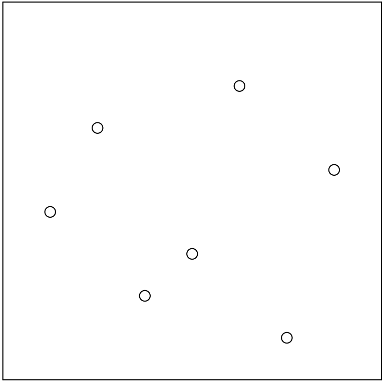

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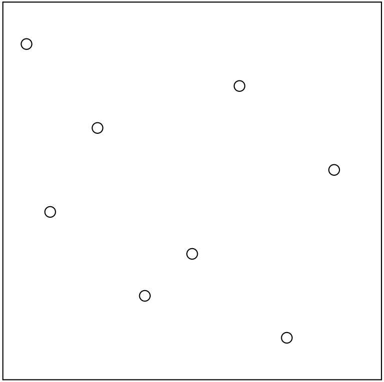

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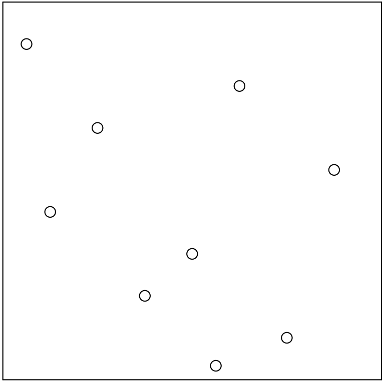

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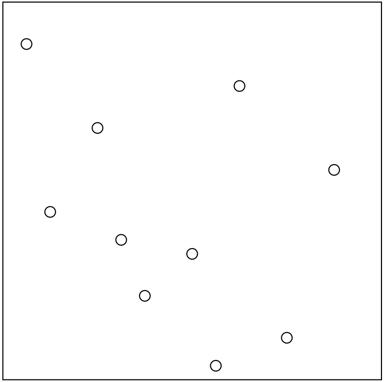

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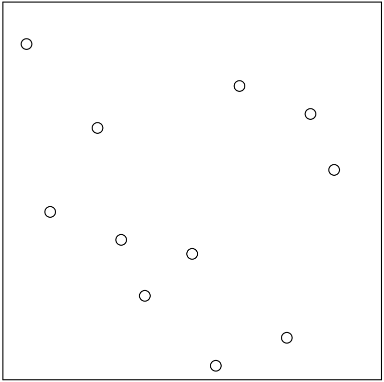

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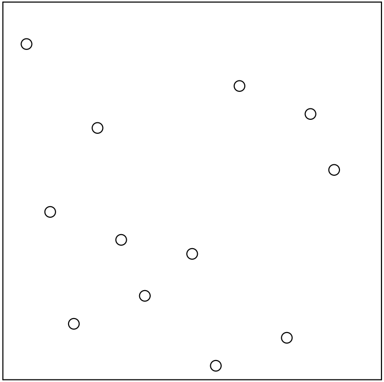

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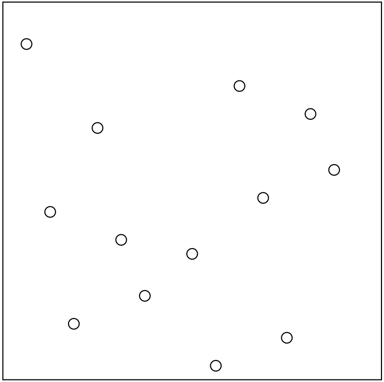

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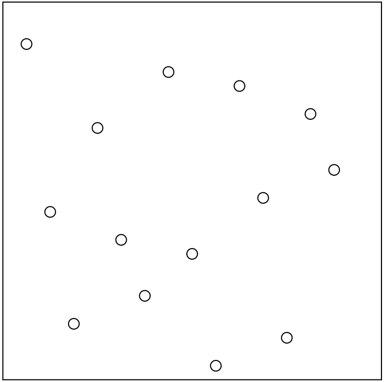

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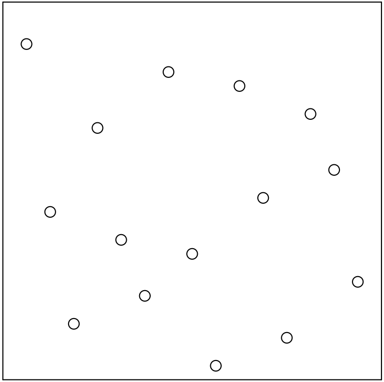

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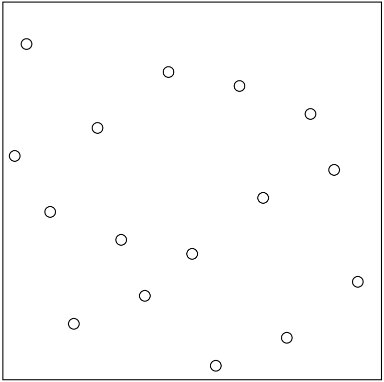

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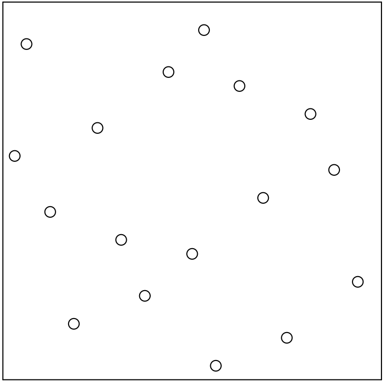

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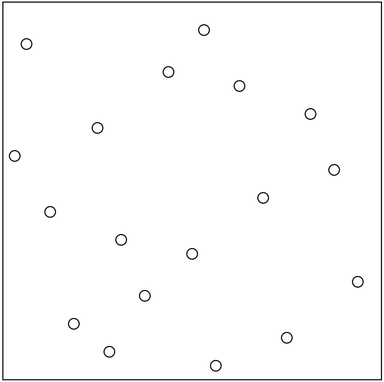

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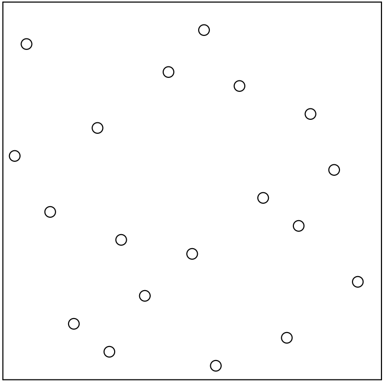

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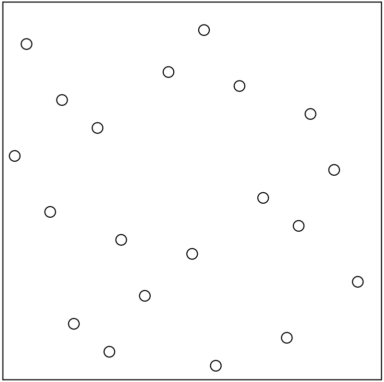

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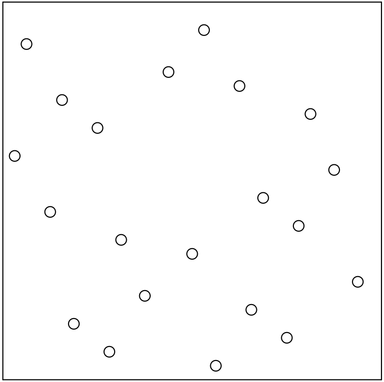

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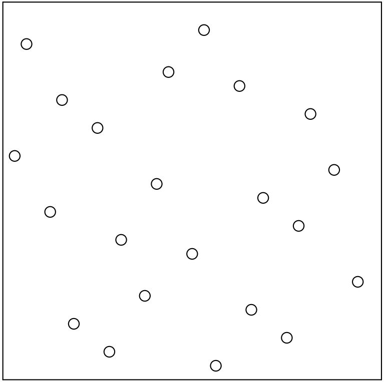

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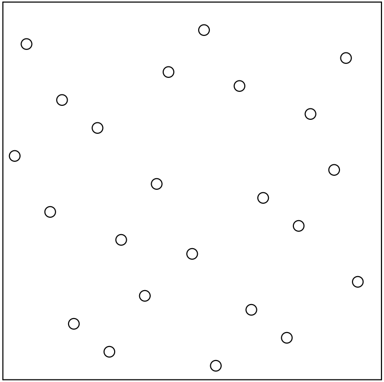

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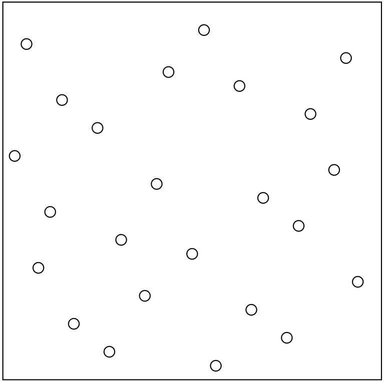

Figure 2: Example of Halton sequnce

Halton sequence on $[0, 1]^n$

- Generally we can generate Halton sequence in a unit hypercube $[0, 1]^n$:

$$[x_1, x_2, \ldots, x_n] = [\mathrm{vdc}(k, b_1), \mathrm{vdc}(k, b_2), \ldots, \mathrm{vdc}(k, b_n)]$$

- A wide range of applications on Quasi-Monte Carlo Methods (QMC).

Unit Circle S^1

Can be generated by mapping the Van der Corput sequence to $[0, 2\pi]$

- $\theta = 2\pi \cdot \mathrm{vdc}(k, b)$
- $[x, y] = [\cos\theta, \sin\theta]$

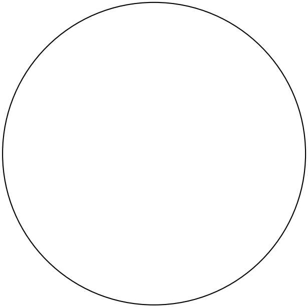

Figure 3: Sequnce mapping to a unit circle

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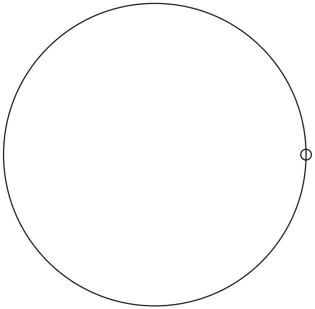

Figure 3: Sequnce mapping to a unit circle

Unit Circle S^1

Can be generated by mapping the Van der Corput sequence to $[0, 2\pi]$

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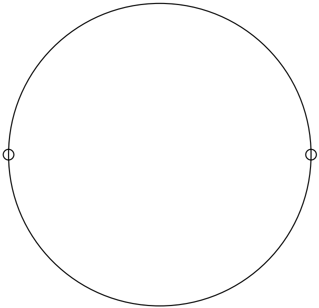

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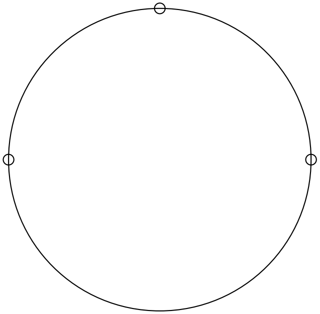

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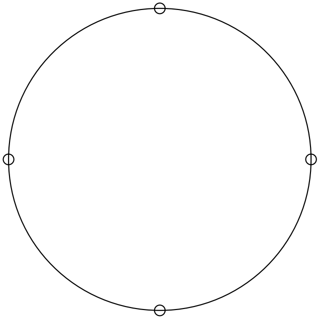

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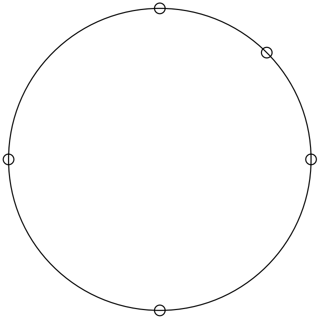

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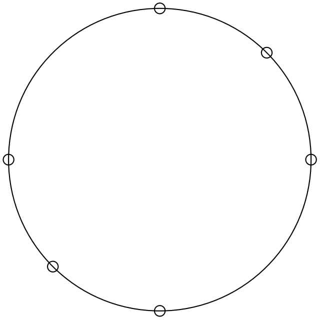

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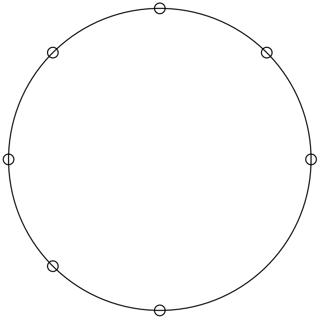

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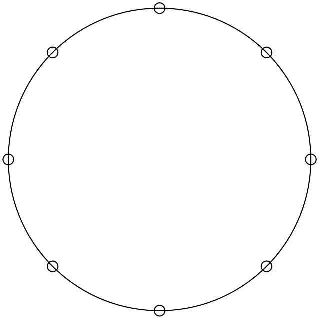

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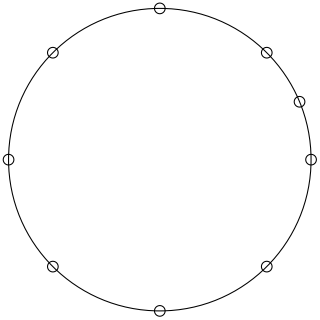

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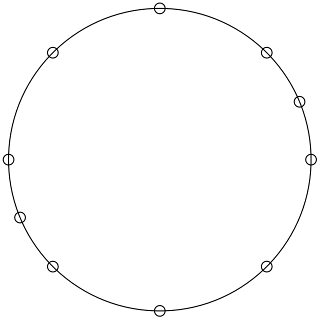

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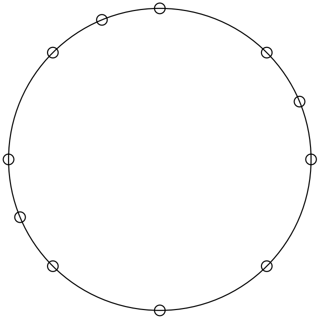

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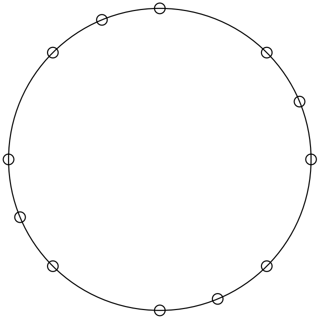

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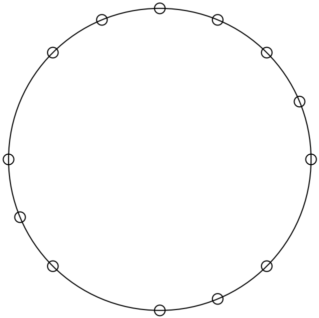

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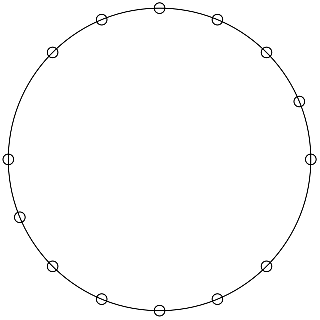

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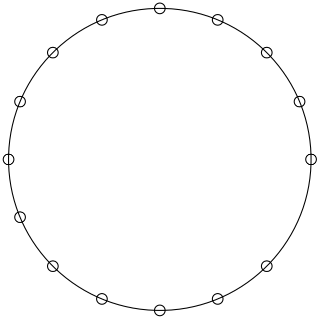

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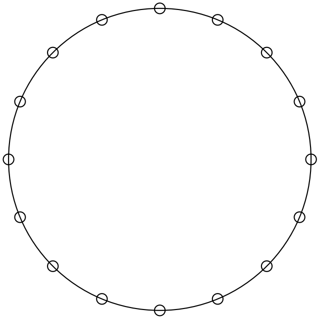

Figure 3: Sequnce mapping to a unit circle

Unit Sphere S^2

Has been applied for computer graphic applications (Wong, Luk, and Heng 1997)

- Use cylindrical mapping.
- $[z, x, y]$
  $= [\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi]$
  $= [z, \sqrt{1-z^2}\cos\varphi, \sqrt{1-z^2}\sin\varphi]$
- $\varphi = 2\pi \cdot \mathrm{vdc}(k, b_1)$ % map to $[0, 2\pi]$
- $z = 2 \cdot \mathrm{vdc}(k, b_2) - 1$ % map to $[-1, 1]$

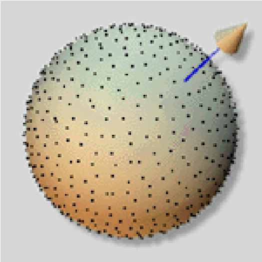

Figure 4: image

Sphere S^n and SO(3)

- Deterministic point sets
  - Optimal grid point sets for S^3, SO(3) (Mitchell 2008; Yershova et al. 2010)
- No Halton sequences so far to the best of our knowledge.
- Note that cylindrical mapping method cannot be extended to higher dimensions.

SO(3) or S^3 Hopf Coordinates

- Hopf coordinates (cf. (Yershova et al. 2010))
  - $x_1 = \cos(\theta/2)\cos(\psi/2)$
  - $x_2 = \cos(\theta/2)\sin(\psi/2)$
  - $x_3 = \sin(\theta/2)\cos(\varphi + \psi/2)$
  - $x_4 = \sin(\theta/2)\sin(\varphi + \psi/2)$
- S^3 is a principal circle bundle over the S^2

Figure 5: image

Hopf Coordinates for SO(3) or S^3

Similar to the Halton sequence generation on S^2, we perform the mapping:

- $\varphi = 2\pi \cdot \mathrm{vdc}(k, b_1)$ % map to $[0, 2\pi]$
- $\psi = 2\pi \cdot \mathrm{vdc}(k, b_2)$ % map to $[0, 2\pi]$ for SO(3), or
- $\psi = 4\pi \cdot \mathrm{vdc}(k, b_2)$ % map to $[0, 4\pi]$ for S^3
- $z = 2 \cdot \mathrm{vdc}(k, b_3) - 1$ % map to $[-1, 1]$
- $\theta = \cos^{-1} z$

Python Code

```python
def sphere3_hopf(k, b):
    vd = zip(vdc(k, b[0]), vdc(k, b[1]), vdc(k, b[2]))
    for vd0, vd1, vd2 in vd:
        phi = 2*math.pi*vd0   # map to [0, 2*math.pi]
        psy = 4*math.pi*vd1   # map to [0, 4*math.pi]
        z = 2*vd2 - 1         # map to [-1., 1.]
        theta = math.acos(z)
        cos_eta = math.cos(theta/2)
        sin_eta = math.sin(theta/2)
        s = [cos_eta * math.cos(psy/2),
             cos_eta * math.sin(psy/2),
             sin_eta * math.cos(phi + psy/2),
             sin_eta * math.sin(phi + psy/2)]
        yield s
```

Our approach

3-sphere

- Polar coordinates:
  - $x_0 = \cos\theta_3$
  - $x_1 = \sin\theta_3 \cos\theta_2$
  - $x_2 = \sin\theta_3 \sin\theta_2 \cos\theta_1$
  - $x_3 = \sin\theta_3 \sin\theta_2 \sin\theta_1$
- Spherical surface element:

$$dA = \sin^2(\theta_3)\sin(\theta_2)\,d\theta_1\,d\theta_2 d\theta_3$$

n-sphere

- Polar coordinates:
  - $x_0 = \cos\theta_n$
  - $x_1 = \sin\theta_n \cos\theta_{n-1}$
  - $x_2 = \sin\theta_n \sin\theta_{n-1} \cos\theta_{n-2}$
  - $x_3 = \sin\theta_n \sin\theta_{n-1} \sin\theta_{n-2} \cos\theta_{n-3}$
  - $\dots$
  - $x_{n-1} = \sin\theta_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \cos\theta_1$
  - $x_n = \sin\theta_n \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_1$
- Spherical surface element:

$$d^n A = \sin^{n-2}(\theta_{n-1}) \sin^{n-1}(\theta_{n-2}) \cdots \sin(\theta_2)\, d\theta_1\, d\theta_2 \cdots d\theta_{n-1}$$

How to Generate the Point Set

- $p_0 = [\cos\theta_1, \sin\theta_1]$ where $\theta_1 = 2\pi \cdot \mathrm{vdc}(k, b_1)$
- Let $f_j(\theta) = \int \sin^j\theta \mathrm{d}\theta$, where $\theta \in (0, \pi)$.
  - Note 1: $f_j(\theta)$ can be defined recursively as:

$$f_j(\theta) = \begin{cases} \theta & \text{if } j = 0, \\ -\cos\theta & \text{if } j = 1, \\ (1/n)(-\cos\theta\sin^{j-1}\theta + (n-1)\int\sin^{j-2}\theta \mathrm{d}\theta) & \text{otherwise.} \end{cases}$$

  - Note 2: $f_j(\theta)$ is a monotonic increasing function in $(0, \pi)$
- Map $\mathrm{vdc}(k, b_j)$ uniformly to $f_j(\theta)$:
  $t_j = f_j(0) + (f_j(\pi) - f_j(0))\mathrm{vdc}(k, b_j)$
- Let $\theta_j = f_j^{-1}(t_j)$
- Define p_n recursively as:
  $p_n = [\cos\theta_n, \sin\theta_n \cdot p_{n-1}]$

Numerical Experiments

Testing the Correctness

- Compare the dispersion with the random point-set
  - Construct the convex hull for each point-set
  - Dispersion roughly measured by the difference of the maximum distance and the minimum distance between every two neighbour points:

$$\max_{a \in \mathcal{N}(b)} \{D(a,b)\} - \min_{a \in \mathcal{N}(b)} \{D(a,b)\}$$

where $D(a,b) = \sqrt{1 - a^\top b}$

Random sequences

- To generate random points on S^n, spherical symmetry of the multidimensional Gaussian density function can be exploited.
- Then the normalized vector $(x_i/\|x_i\|)$ is uniformly distributed over the hypersphere S^n. (Fishman, G. F. (1996))

Convex Hull with 600 points

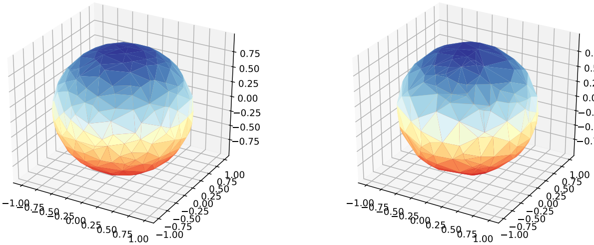

Figure 6: image

Left: our, right: random

Result for S^3

Compared with Hopf coordinate method.

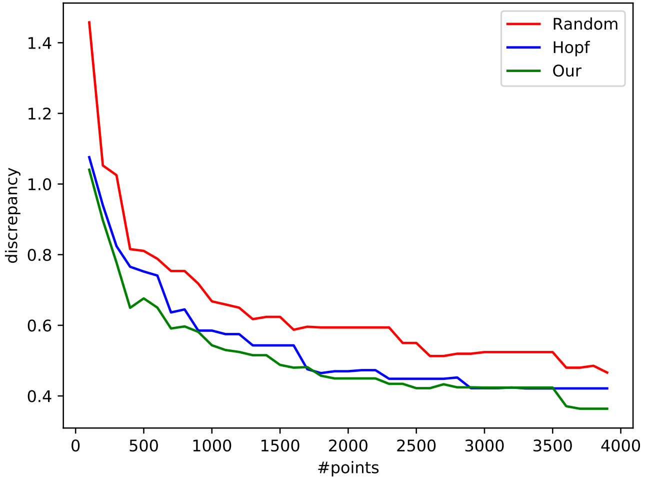

Figure 7: image

Result for S^3 (II)

Compared with cylindrical mapping method.

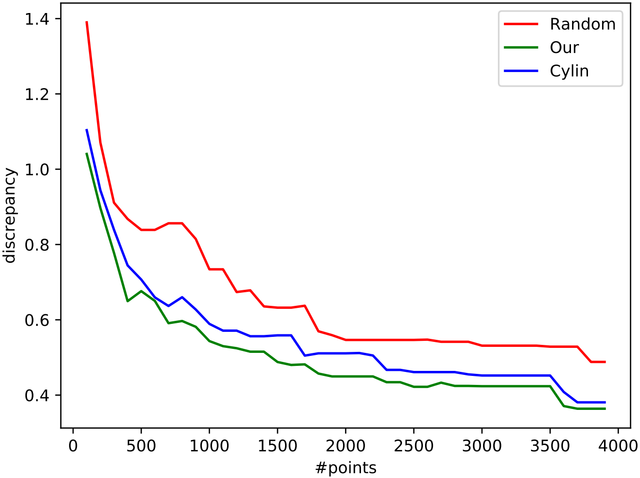

Figure 8: image

Result for S^4

Compared with cylindrical mapping method

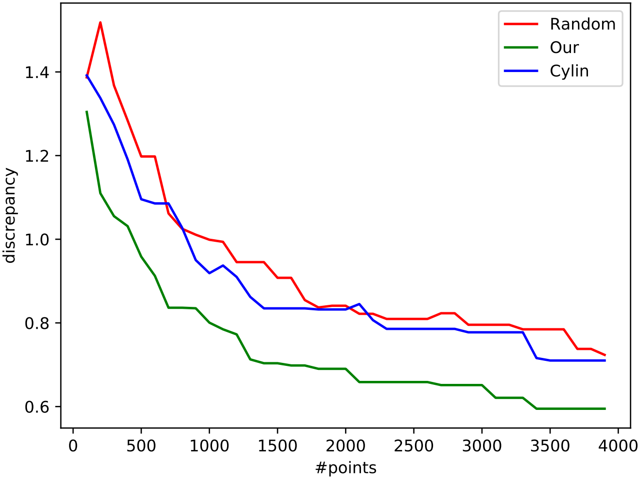

Figure 9: image

Conclusions

Conclusions

- Proposed method generates low-discrepancy point-set in nearly linear time
- The result outperforms the corresponding random point-set, especially when the number of points is small
- Python code is available at here

References I

Cui, Jianjun, and Willi Freeden. 1997. "Equidistribution on the Sphere." *SIAM Journal on Scientific Computing* 18 (2). SIAM:595–609.

Mandic, DP, and others. 2011. "Filter Bank Property of Multivariate Empirical Mode Decomposition." *Signal Processing, IEEE Transactions on* 59 (5). IEEE:2421–6.

Mitchell, Julie C. 2008. "Sampling Rotation Groups by Successive Orthogonal Images." *SIAM Journal on Scientific Computing* 30 (1). SIAM:525–47.

Rehman, Naveed, and Danilo P Mandic. 2010. "Multivariate Empirical Mode Decomposition." *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* 466 (2117). The Royal Society:1291–1302.

Utkovski, Zoran, and Juergen Lindner. 2006. "On the Construction of Non-Coherent Space Time Codes from High-Dimensional Spherical Codes." In *Spread Spectrum Techniques and Applications, 2006 Ieee Ninth International Symposium on*, 327–31. IEEE.

References II

Wong, Tien-Tsin, Wai-Shing Luk, and Pheng-Ann Heng. 1997. "Sampling with Hammersley and Halton Points." *Journal of Graphics Tools* 2 (2). Taylor & Francis:9–24.

Yershova, Anna, Swati Jain, Steven M LaValle, and Julie C Mitchell. 2010. "Generating Uniform Incremental Grids on so (3) Using the Hopf Fibration." *The International Journal of Robotics Research* 29 (7). SAGE Publications:801–12.