Chapter Four

Asymptotic Qualitative Behavior

As in the previous chapter, we will study in this chapter sequences $\{x(k) : k \in \mathbb{N}\}$ given through

$$x(k+1) = A \otimes x(k), \qquad k \in \mathbb{N}, \tag{4.1}$$

with initial vector $x(0) = x_0 \in \mathbb{R}^n_{\max}$ and $A \in \mathbb{R}^{n\times n}_{\max}$. Provided that A is irreducible with unique eigenvalue λ and associated eigenvector v, it follows for $x(0) = v$ and $k \geq 0$ that $x(k) = A^{\otimes k} \otimes x(0) = \lambda^{\otimes k} \otimes v$. In words, the vectors $x(k)$ are proportional to v, and we may therefore say that the *qualitative* asymptotic behavior of $x(k)$ is completely characterized by v. We have already encountered this type of limit in the heap model as described in Section 1.3. The qualitative limiting behavior of $x(k)$ falls into one of two possible scenarios: $x(k)$ reaches the eigenspace of A and behaves according to $x(k+1) = \lambda \otimes x(k)$ for k sufficiently large, or $x(k)$ enters into a periodic regime (to be defined below). Hence, there are two sources of a possible nonuniqueness of the limiting behavior of $x(k)$:

- $x(k)$ enters the eigenspace and this space is of dimension two or higher (there are several nonproportional eigenvectors), or
- $x(k)$ enters a periodic regime.

This chapter is organized as follows. In Section 4.1 the concept of a periodic regime is introduced, and the close relation between periodic regimes, eigenvalues, and eigenvectors is discussed. The eigenspace of irreducible matrices is studied in Section 4.2. A class of matrices with unique qualitative behavior is discussed in Section 4.3. Section 4.4 deals with a proper limit concept for max-plus sequences by means of the projective space. Eventually, in Section 4.5 it is shown that higher-order max-plus recurrence relations can be reduced to first-order ones.

4.1 PERIODIC REGIMES

Let $A \in \mathbb{R}^{n\times n}_{\max}$. A *periodic regime* is a set of vectors $x^1, \ldots, x^d \in \mathbb{R}^n_{\max}$ for some $d \geq 1$ such that a finite number ρ exists that satisfies

$$\rho \otimes x^1 = A \otimes x^d \quad \text{and} \quad x^{i+1} = A \otimes x^i, \qquad i \in \underline{d-1}.$$

If $\overline{x^i} \neq \overline{x^j}$ for $i, j \in \underline{d-1}$ with $i \neq j$, then $x^1, \ldots, x^d$ is said to be of *period* d. A consequence of the above definition is that $x^1, \ldots, x^d$ are eigenvectors of $A^{\otimes d}$ associated with eigenvalue ρ (see exercise 1). If A is irreducible with cyclicity $\sigma(A)$, then A will possess periodic regimes of period $\sigma(A)$ or less. In this context, one may wonder whether the fact that ρ is an eigenvalue of $A^{\otimes d}$ implies that $(1/d) \times \rho$ is an eigenvalue of A. The following theorem gives a positive answer. Moreover, it shows that an eigenvector can be found via a periodic regime.

THEOREM 4.1 *Let $x^1, \ldots, x^d$ be a periodic regime for matrix A with $\rho \otimes x^1 = A \otimes x^d$. Then A has an eigenvalue λ that satisfies $\rho = \lambda^{\otimes d}$, and a corresponding eigenvector v is given by*

$$v = \bigoplus_{j=1}^{d} \lambda^{\otimes (d-j)} \otimes x^j .$$

Proof. We prove the theorem by showing that $A \otimes v = \lambda \otimes v$. Indeed, we have that

$$\begin{aligned} A \otimes v &= A \otimes \left(\bigoplus_{j=1}^{d} \lambda^{\otimes (d-j)} \otimes x^j \right) \\ &= \bigoplus_{j=1}^{d} A \otimes \lambda^{\otimes (d-j)} \otimes x^j . \end{aligned}$$

Noticing that $A \otimes x^j = x^{j+1}$ and $A \otimes x^d = \lambda^{\otimes d} \otimes x^1$ yields

$$\begin{aligned} \bigoplus_{j=1}^{d} A \otimes \lambda^{\otimes (d-j)} \otimes x^j &= \lambda^{\otimes d} \otimes x^1 \ \oplus \bigoplus_{j=1}^{d-1} \lambda^{\otimes (d-j)} \otimes x^{j+1} \\ &= \lambda^{\otimes d} \otimes x^1 \ \oplus \bigoplus_{l=2}^{d} \lambda^{\otimes (d-l+1)} \otimes x^l \\ &= \bigoplus_{l=1}^{d} \lambda^{\otimes (d-l+1)} \otimes x^l \\ &= \lambda \otimes \left(\bigoplus_{l=1}^{d} \lambda^{\otimes (d-l)} \otimes x^l \right) \\ &= \lambda \otimes v, \end{aligned}$$

which proves the claim. □

The above theorem is illustrated with the following example.

Example 4.1.1 *Consider the matrix given in (0.10):*

$$A = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}.$$

Taking $x(0) = (0,0)^\top$ yields the sequence as given in (0.3):

$$x(1) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \ x(2) = \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \ x(3) = \begin{pmatrix} 13 \\ 11 \end{pmatrix}, \ x(4) = \begin{pmatrix} 16 \\ 16 \end{pmatrix}, \ldots,$$

which is a periodic regime of period 2 *with $\rho = 8$. In particular,*

$$x(k) \in \overline{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}, \qquad k \in \{1, 3, 5, \ldots\},$$

and

$$x(k) \in \overline{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}, \qquad k \in \{0, 2, 4, \ldots\}.$$

By Theorem 4.1, $\lambda = \rho/2 = 4$, *which can be easily checked by inspecting the communication graph of A. Indeed,* $\mathcal{G}(A)$ *consists of the elementary circuits* $(1,1)$, $(2,2)$, *and* $((1,2),(2,1))$. *The circuit* $((1,2),(2,1))$ *is critical with average weight equal to* 4. *Theorem 4.1 also yields an eigenvector of A. Take*

$$x^1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \quad x^2 = \begin{pmatrix} 8 \\ 8 \end{pmatrix};$$

then

$$\lambda^{\otimes 1} \otimes x^1 \oplus \lambda^{\otimes 0} \otimes x^2 = 4 \otimes \begin{pmatrix} 5 \\ 3 \end{pmatrix} \oplus 0 \otimes \begin{pmatrix} 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

yields an eigenvector of A.

4.2 CHARACTERIZATION OF THE EIGENSPACE

Let A have finite eigenvalue λ. The eigenspace $V(A, \lambda)$ of matrix A is the set of all eigenvectors of A corresponding to λ, and $V(A, \lambda)$ obviously is a linear space. The eigenspaces of A and A_λ coincide. Indeed, for $v \in V(A)$, it holds for any j that

$$[\lambda \otimes v]_j = [A \otimes v]_j \Leftrightarrow v_j = [A \otimes v]_j - \lambda \Leftrightarrow e \otimes v_j = [A_\lambda \otimes v]_j . \quad (4.2)$$

In fact, for any β, the eigenspaces of A and A_β coincide. Hence, we may conveniently work with either A or A_λ for the analysis of the eigenspace of A. Recall that we write $[A]_{\cdot k}$ to indicate the kth column of A and that we denote the nodes of the critical graph by $\mathcal{N}^c(A)$, i.e., $\eta \in \mathcal{N}^c(A)$ if η lies on a critical circuit. Given an eigenvalue λ and an associated eigenvector v, the *saturation graph*, denoted by $\mathcal{S}(A, v)$, consists of those arcs (j, i) of $\mathcal{G}(A)$ such that $a_{ij} \otimes v_j = \lambda \otimes v_i$ with $v_i, v_j \neq \varepsilon$. Of course, the saturation graph $\mathcal{S}(A, v)$ also depends on λ in principle, but that will not be reflected in the notation.

LEMMA 4.2 *For* $A \in \mathbb{R}_{\max}^{n \times n}$ *with finite eigenvalue* λ *and associated finite eigenvector* v, *the following hold:*

- *For each node* i *in* $\mathcal{S}(A, v)$, *there exists a circuit in* $\mathcal{S}(A, v)$ *from which node* i *can be reached in a finite number of steps.*
- *The circuits of any saturation graph have average circuit weight* λ.
- *If, in addition,* A *is irreducible, then circuits of any saturation graph belong to the critical graph.*

Proof. Let i be a node of the saturation graph. Then there exists a node j in the saturation graph such that $\lambda \otimes v_i = a_{ij} \otimes v_j$ with $v_i, v_j \neq \varepsilon$. Repeating this argument, we find a node k such that $\lambda \otimes v_j = a_{jk} \otimes v_k$ with $v_j, v_k \neq \varepsilon$. Repeating this argument an arbitrary number of times, say, m, we get a path in $\mathcal{S}(A, v)$ of length m. If $m > n$, the constructed path must contain a circuit.

We turn to the proof of the second part of the lemma. Let

$$\rho = ((i_1, i_2), (i_2, i_3), \ldots, (i_l, i_{l+1} = i_1))$$

be a circuit in $\mathcal{S}(A, v)$. By definition,

$$\lambda \otimes v_{i_{k+1}} = a_{i_{k+1} i_k} \otimes v_{i_k},$$

which implies

$$\lambda^{\otimes l} \otimes v_{i_1} = \bigotimes_{k=1}^{l} a_{i_{k+1} i_k} \otimes v_{i_1}.$$

Hence,

$$\lambda^{\otimes l} = \bigotimes_{k=1}^{l} a_{i_{k+1} i_k},$$

but the expression on the right-hand side of the above formula is the weight of the circuit ρ, which thus has average weight λ.

Finally, we deal with the third part of the lemma. Theorem 2.9 implies that λ (the eigenvalue of A) equals the maximal average circuit weight of $\mathcal{G}(A)$. According to the second part of the lemma, circuits in the saturation graph have average circuit weight λ. Hence, the average weight of any circuit in the saturation graph equals the maximal average circuit weight of $\mathcal{G}(A)$, and the circuit thus belongs to the critical graph. □

LEMMA 4.3 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be irreducible and have eigenvalue λ, and let v be an eigenvector associated with λ. Then, A_λ^* has eigenvalue e, and v is an associated eigenvector.*

Proof. Note that

$$(E \oplus A_\lambda) \otimes v = v \quad \text{and} \quad A_\lambda^* = (E \oplus A_\lambda)^{\otimes(n-1)};$$

see exercise 9. Hence,

$$A_\lambda^* \otimes v = (E \oplus A_\lambda)^{\otimes(n-1)} \otimes v = v.$$

□

As the following lemma shows, for an irreducible matrix A, the vectors $[A_\lambda^*]_{\cdot i}$, with $i \in \mathcal{N}^c(A)$, constitute a basis of the eigenspace of A.

LEMMA 4.4 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be irreducible with eigenvalue λ, and let v be an eigenvector associated with λ. Then v can be written as*

$$v = \bigoplus_{i \in \mathcal{N}^c(A)} a_i \otimes [A_\lambda^*]_{\cdot i},$$

with $a_i \in \mathbb{R}_{\max}$.

Proof. By Lemma 4.3, A_λ^* has eigenvalue e, and provided that v is an eigenvector of A associated to λ, v is also an eigenvector of A_λ^* for eigenvalue e.

Consider two nodes i, j in $\mathcal{S}(A_\lambda, v)$ such that there exists a path from i to j, say, $((i_1, i_2), (i_2, i_3), \ldots, (i_l, i_{l+1}))$, with $i = i_1$ and $j = i_{l+1}$. This gives

$$[A_\lambda]_{i_{k+1} i_k} \otimes v_{i_k} = v_{i_{k+1}}, \qquad k \in \underline{l}.$$

Hence, $v_j = a \otimes v_i$ with

$$a = \bigotimes_{k=1}^{l} [A_\lambda]_{i_{k+1} i_k} \leq \left[A_\lambda^{\otimes l}\right]_{ji} \leq [A_\lambda^*]_{ji}. \tag{4.3}$$

Using the fact that $v_j = a \otimes v_i$ yields for an arbitrary node $\eta \in \underline{n}$

$$\begin{aligned} [A_\lambda^*]_{\eta j} \otimes v_j &= [A_\lambda^*]_{\eta j} \otimes a \otimes v_i \\ &\overset{(4.3)}{\leq} [A_\lambda^*]_{\eta j} \otimes [A_\lambda^*]_{ji} \otimes v_i \\ &\leq [A_\lambda^*]_{\eta i} \otimes v_i, \end{aligned} \tag{4.4}$$

where the last inequality follows from $A_\lambda^* \otimes A_\lambda^* = A_\lambda^*$ (see exercise 3). By applying Lemma 4.2 (with the roles of i and j interchanged) for any j in the saturation graph, there exists a node $i = i(j)$ that belongs to a critical circuit. Inequality (4.4) therefore implies

$$\bigoplus_{j \in \mathcal{S}(A_\lambda, v)} \left[A_\lambda^*\right]_{\eta j} \otimes v_j \leq \bigoplus_{i \in \mathcal{N}^c(A_\lambda)} \left[A_\lambda^*\right]_{\eta i} \otimes v_i, \tag{4.5}$$

for $\eta \in \underline{n}$.

The basic equation for the eigenvector v is $v = A_\lambda^* \otimes v$. By definition, the value of v_η is equal to $[A_\lambda^*]_{\eta j} \otimes v_j$ for some j that has to be in the saturation graph. We do not know which specific node j determines v_η, but since the node has to be in the saturation graph, it holds for $\eta \in \underline{n}$ that

$$v_\eta = \bigoplus_{j \in \mathcal{S}(A_\lambda, v)} \left[A_\lambda^*\right]_{\eta j} \otimes v_j \overset{(4.5)}{\leq} \bigoplus_{j \in \mathcal{N}^c(A_\lambda)} \left[A_\lambda^*\right]_{\eta j} \otimes v_j.$$

Conversely, since v is an eigenvector of A_λ^* for eigenvalue e,

$$v_\eta = [A_\lambda^* \otimes v]_\eta = \bigoplus_{j=1}^{n} \left[A_\lambda^*\right]_{\eta j} \otimes v_j \geq \bigoplus_{i \in \mathcal{N}^c(A_\lambda)} \left[A_\lambda^*\right]_{\eta i} \otimes v_i$$

for $\eta \in \underline{n}$, which completes the proof of the lemma. Specifically, we obtain $a_i = v_i$ for $i \in \mathcal{N}^c(A_\lambda)$. In case some of the columns of A_λ^* are colinear, then the a_i's are nonunique and some can be chosen equal to ε. Noticing that $\mathcal{N}^c(A_\lambda) = \mathcal{N}^c(A)$ concludes the proof. □

The following theorem characterizes the eigenspace of an irreducible square matrix.

THEOREM 4.5 *Let A be irreducible, and let A_λ^* be defined as in (2.10).*

(i) *If node i belongs to the critical graph, then $[A_\lambda^*]_{\cdot i}$ is an eigenvector of A.*

(ii) *The eigenspace of A is given by*

$$V(A) = \left\{ v \in \mathbb{R}^n_{\max} \,:\, v = \bigoplus_{i \in \mathcal{N}^c(A)} a_i \otimes [A^*_\lambda]_{\cdot i} \quad \textit{for } a_i \in \mathbb{R}_{\max} \right\}.$$

(iii) *For i, j belonging to the critical graph, there exists $a \in \mathbb{R}$ such that*

$$a \otimes [A^*_\lambda]_{\cdot i} = [A^*_\lambda]_{\cdot j} \tag{4.6}$$

if and only if i and j belong to the same m.s.c.s. of the critical graph.

Proof. Note that irreducibility of A implies regularity of A and therefore part (i) has already been proved in Lemma 2.7.

For the proof of part (ii), notice that the columns $[A^*_\lambda]_{\cdot i}$ for $i \in \mathcal{N}^c(A)$ are eigenvectors of A, which is part (i) of the theorem. Since any linear combination of eigenvectors is again an eigenvector, it follows that

$$\bigoplus_{i \in \mathcal{N}^c(A)} a_i \otimes [A^*_\lambda]_{\cdot i},$$

with $a_i \in \mathbb{R}_{\max}$ and at least one a_i finite, is an eigenvector of A. Conversely, any eigenvector can be written as a linear combination of the columns $[A^*_\lambda]_{\cdot i}$ for $i \in \mathcal{N}^c(A)$, which has been shown in Lemma 4.4. This proves part (ii).

We now turn to the proof of part (iii). We first prove that if i and j belong to the same m.s.c.s., then (4.6) holds. Subsequently it will be shown that if i and j do not belong to the same m.s.c.s., then (4.6) cannot hold.

If i and j belong to the same m.s.c.s. of the critical graph of A_λ, then $[A^*_\lambda]_{ji} \otimes [A^*_\lambda]_{ij} = e$ and, hence,

$$\begin{aligned} [A^*_\lambda]_{li} \otimes [A^*_\lambda]_{ij} &\le [A^*_\lambda]_{lj} \\ &= [A^*\lambda]_{lj} \otimes [A^*_\lambda]_{ji} \otimes [A^*_\lambda]_{ij} \\ &\le [A^*_\lambda]_{li} \otimes [A^*_\lambda]_{ij}, \qquad \forall l \in \underline{n}, \end{aligned}$$

which shows that

$$[A^*_\lambda]_{li} \otimes [A^*_\lambda]_{ij} = [A^*_\lambda]_{lj}, \qquad \forall l \in \underline{n}.$$

Hence, (4.6) has been proved with $a = [A^*_\lambda]_{ij}$.

Now suppose that i and j do not belong to the same m.s.c.s., but that notwithstanding (4.6) holds true. The ith and jth components of (4.6) read

$$a \otimes e = [A^*_\lambda]_{ij} \quad \text{and} \quad a \otimes [A^*_\lambda]_{ji} = e,$$

respectively, from where it follows that

$$[A^*_\lambda]_{ij} \otimes [A^*_\lambda]_{ji} = e.$$

As a consequence, the circuit formed by the arcs (i, j) and (j, i) has average weight e, and therefore, nodes i and j belong to the critical graph of A and, hence, belong to the same m.s.c.s. (of this critical graph). Thus, a contradiction has been obtained. □

The application of the above theorem is illustrated with the following series of examples.

Example 4.2.1 *Consider the matrix*

$$A = \begin{pmatrix} 2 & 5 \\ 3 & 3 \end{pmatrix}.$$

Matrix A is irreducible with unique eigenvalue $\lambda = \lambda(A) = 4$, and the critical graph of A consists of the nodes $\{1, 2\}$. The critical graph has thus one m.s.c.s. and $\sigma(A) = 2$. By computation, it follows that

$$A_\lambda^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By Theorem 4.5, the vectors

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are eigenvectors of A. Clearly, these vectors are max-plus multiples of each other. This also follows from Theorem 4.5 since the associated nodes/indices belong to the same m.s.c.s. of the critical graph. Hence, it follows that the eigenspace of A is given by

$$V(A) = \left\{ v \in \mathbb{R}^2_{\max} : v = a \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \textit{ for } a \in \mathbb{R} \right\}$$

or, more concisely,

$$V(A) = \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}.$$

Compare the above result with Example 4.1.1. There, we computed an eigenvector of A but had no information whether the obtained eigenvector already characterized the complete eigenspace of A. Theorem 4.5 shows that the eigenspace of A is of dimension one. Moreover, it provides an algebraic way of computing all vectors in $V(A)$.

Example 4.2.2 *Consider the matrix*

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

Matrix A is irreducible with a unique eigenvalue $\lambda = \lambda(A) = 0$, and the critical graph of A consists of the nodes $\{1, 2\}$ and circuits (or self-loops) $(1, 1)$ and $(2, 2)$. Thus, the critical graph has two m.s.c.s.'s, namely, the circuits $(1, 1)$ and $(2, 2)$, and $\sigma(A) = 1$. For A it holds that

$$A = A^{\otimes k} = A_\lambda = A^+ = A^* = A_\lambda^*, \qquad k \in \mathbb{N}.$$

By Theorem 4.5, the vectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

are eigenvectors of A. Obviously, these vectors are not max-plus multiples of each other. This fact also follows from Theorem 4.5 since the associated nodes/indices belong to different m.s.c.s.'s of the critical graph. By Theorem 4.5 it further follows that the eigenspace of A is given by

$$V(A) = \left\{ v \in \mathbb{R}^2_{\max} : v = a_1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus a_2 \otimes \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ for } a_1, a_2 \in \mathbb{R} \right\}.$$

Example 4.2.3 *Consider A in Example 3.1.3. The critical graph of A has one m.s.c.s. consisting of the circuit $((1,2),(2,1))$. It follows that $\lambda = \lambda(A) = 6$. Application of Theorem 4.5 yields*

$$A^*_\lambda = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix} \quad \text{and} \quad V(A) = \overline{\begin{pmatrix} 5 \\ 0 \end{pmatrix}}.$$

4.3 PRIMITIVE MATRICES

From a system-theoretical point of view, we are interested in the limiting behavior of $x(k)$. More precisely, we are interested in the behavior of $\overline{x(k)}$ for k large. Let A in (4.1) be irreducible. Theorem 3.9 and the text thereafter imply that for any initial vector x_0 it holds that $x(k)$ enters after at most $t(A)$ iterations a periodic regime with period $\sigma(A)$ or smaller. If A has cyclicity one, then $x(k)$ enters after at most $t(A)$ iterations the eigenspace of A; in formula, we write

$$x(k) \in V(A) \quad \text{for} \quad k \geq t(A).$$

Generally speaking, there are two possible scenarios: $x(k)$ enters the eigenspace, or $x(k)$ enters an orbit whose period is larger than one and maximally $\sigma(A)$. We call the set of all initial conditions x_0 such that $A^{\otimes k} \otimes x_0$ eventually reaches the set $\overline{v}$ (resp., the periodic regime $x^1, \ldots, x^d$) for some eigenvector v the *domain of attraction* of v (resp., $x^1, \ldots, x^d$), with $d > 1$. For example, for the matrix given in Example 4.1.1, the vector $x = (0,0)^\top$ lies in the domain of attraction of the periodic regime $(5,3)^\top, (8,8)^\top$.

Let A be irreducible, and suppose that the critical graph of A has a single m.s.c.s. of cyclicity one. Such matrices are called *primitive*. Primitive matrices form an important class of square matrices because for any primitive matrix A the sequence $\{A^{\otimes k} \otimes x_0 : \; k \in \mathbb{N}\}$ has, independent of the initial vector $x_0 \in \mathbb{R}^n$, a unique limit behavior. A way of expressing this is by saying that $V(A) = \overline{v}$ for some $v \in \mathbb{R}^n$ and that the domain of attraction of v is equal to $\mathbb{R}^n$. More explicitly, let A be primitive and let $x(k) = A^{\otimes k} \otimes x_0$ for $k \geq 0$, with $x_0 \in \mathbb{R}^n$; then, independently of the initial vector x_0, the following hold:

- The asymptotic growth rate is equal to the eigenvalue of A:

$$\lim_{k \to \infty} \frac{x_j(k)}{k} = \lambda, \qquad j \in \underline{n},$$

  where λ is the eigenvalue of A (see Lemma 3.12).

- There exists a unique $\overline{v} \in \mathbb{PR}^n$ such that

$$\forall k \geq t(A) : \qquad \overline{v} = \overline{x(k)}$$

(see exercise 2).

A way of phrasing the latter point is by saying that $x(k)$ reaches after at most $t(A)$ iterations its asymptotic regime; any influence of the initial vector has died out and the growth rate of $x(k)$ coincides with the eigenvalue of A.

Example 4.3.1 *Let $A \in \mathbb{R}_{\max}^{n \times n}$ model a railway network with n stations. Suppose that A is irreducible with eigenvalue λ. Then any eigenvector v of A provides a regular timetable; that is, initializing x_0 to v, the kth departure of a train from station j happens at time $\lambda^{\otimes k} \otimes v_j = k \times \lambda + v_j$. If A is irreducible and its critical graph has more than one m.s.c.s., several nonproportional regular timetables exist; that is, $v, w \in V(A)$ exist such that $\overline{v} \neq \overline{w}$, and initializing x_0 to either v or w again results in a regular behavior for the departure time of the kth train from station j, namely, $\lambda^{\otimes k} \otimes v_j = k \times \lambda + v_j$ or $\lambda^{\otimes k} \otimes w_j = k \times \lambda + w_j$, respectively.*

Suppose that we intend to operate the network with timetable $v \in V(A)$, but unfortunately—due to some external influence—the first trains can only start at time vector $z \geq v$ (with at least one component being a strict inequality). If A is of cyclicity 2 or higher, then $x(k+1) = A \otimes x(k)$, with $x_0 = z$, may enter a periodic regime of period $\sigma(A)$. Formally, z may lie in the domain of attraction of a periodic regime with period d, where $1 < d \leq \sigma(A)$. In words, the network will not completely recover from the initial delay, and no regular timetable is reached. If, however, A is irreducible and primitive, then there is a unique timetable (up to scalar multiplication), and any initial delay will eventually die out and the network will return to its unique regular timetable (apart from possibly the same constant shift in all departure times). This kind of phenomenon will be discussed in detail in Section 9.1.

4.4 LIMITS IN THE PROJECTIVE SPACE

On $\mathbb{R}^n$, we define

$$||x||_{\mathbb{P}} \stackrel{\text{def}}{=} \bigoplus_{i=1}^{n} x_i \otimes \bigoplus_{i=1}^{n} (-x_i) = \max_{i \in \underline{n}} x_i - \min_{i \in \underline{n}} x_i .$$

It is easy to check that $||x||_{\mathbb{P}}$ is the same for all vectors that belong to $\overline{x}$ (see exercise 4), and we define $||\overline{x}||_{\mathbb{P}} = ||x||_{\mathbb{P}}$. Then $||\overline{x}||_{\mathbb{P}} \geq 0$ for any $\overline{x} \in \mathbb{PR}^n$, and

$$||\overline{x}||_{\mathbb{P}} = 0 \quad \text{if and only if } \; \overline{x} = \overline{\mathbf{u}};,$$

that is, $||\overline{x}||_{\mathbb{P}} = 0$ if and only if for any $x \in \overline{x}$ it holds that all components are equal. For $\alpha \in \mathbb{R}$, let $\alpha \times x$ be defined as the componentwise conventional multiplication of x by α. Thus, $\alpha \times \overline{x} = \overline{\alpha \times x}$, which implies

$$||\alpha \times \overline{x}||_{\mathbb{P}} = |\alpha| \times ||\overline{x}||_{\mathbb{P}}$$

for $\alpha \in \mathbb{R}$ and $\overline{x} \in \mathbb{PR}^n$, where $|\alpha|$ denotes the absolute value of λ. Expression $||\cdot||_{\mathbb{P}}$ also satisfies the triangular inequality. To see this, let $\overline{x}, \overline{y} \in \mathbb{PR}^n$; then, for any $x \in \overline{x}$ and $y \in \overline{y}$,

$$\begin{aligned} ||\overline{x}+\overline{y}||_{\mathbb{P}} &= \max_i(x_i+y_i) \;-\; \min_j(x_j+y_j) \\ &\leq \max_i(\max_k(x_k)+y_i) \;-\; \min_j(\min_l(x_l)+y_j) \\ &= \max_k x_k \;-\; \min_l x_l \;+\; \max_i y_i \;-\; \min_j y_j \\ &= ||\overline{x}||_{\mathbb{P}} \;+\; ||\overline{y}||_{\mathbb{P}}. \end{aligned}$$

Hence, $||\cdot||_{\mathbb{P}}$ is a norm on $\mathbb{PR}^n$. Using the convention in (2.7), we extend the definition of $||\cdot||_{\mathbb{P}}$ to $\mathbb{PR}^n_{\max}$. However, according to the strict use in conventional algebra, $||\cdot||_{\mathbb{P}}$ fails to be a norm on $\mathbb{PR}^n_{\max}$. For any $x \in \mathbb{R}^n_{\max}$ with at least one finite element and at least one element equal to ε, it holds that $||\overline{x}||_{\mathbb{P}} = \infty$, whereas a norm is by definition a mapping to $\mathbb{R}$.

On $\mathbb{PR}^n$, let $x - y$ be defined as the componentwise conventional difference of x and y. With this definition, we obtain a metric $d_{\mathbb{P}}(\cdot,\cdot)$ on $\mathbb{PR}^n$ in the natural way. For $x \in \overline{x}$ and $y \in \overline{y}$, set

$$d_{\mathbb{P}}(\overline{x},\overline{y}) \;=\; ||\overline{x}-\overline{y}||_{\mathbb{P}}$$

or, more explicitly,

$$\begin{aligned} d_{\mathbb{P}}(\overline{x},\overline{y}) &= \bigoplus_{i=1}^{n}(x_i-y_i) \otimes \bigoplus_{j=1}^{n}(y_j-x_j) \\ &= \max_{i\in\underline{n}}(x_i-y_i) \;-\; \min_{j\in\underline{n}}(x_j-y_j) \\ &= \max_{j\in\underline{n}}(y_j-x_j) \;-\; \min_{i\in\underline{n}}(y_i-x_i), \end{aligned}$$

where for the last equality we have used that $\max_i(x_i-y_i) \;=\; -\min_i(y_i-x_i)$. The metric $d_{\mathbb{P}}(\cdot,\cdot)$ is called the *projective metric*. We extend the definition of $d_{\mathbb{P}}(\cdot,\cdot)$ to $\mathbb{PR}^n_{\max}$ by adopting the convention that $\varepsilon - x = \varepsilon$ for $x \neq \varepsilon$. Note that $d_{\mathbb{P}}(\cdot,\cdot)$ fails to be a metric on $\mathbb{PR}^n_{\max}$. To see this, let $\overline{y}$ be such that all components of y are equal to ε. Then, for any $\overline{x} \in \mathbb{PR}^n$, it follows that $d_{\mathbb{P}}(\overline{x},\overline{y}) = 0$, whereas $\overline{x} \neq \overline{y}$.

We can now give a precise definition for the limit in (1.14) in Section 1.4. We say that $\overline{x(k)}$ converges to $\overline{x}$ as k tends to ∞ if for any $\delta > 0$ a number $K \in \mathbb{N}$ exists such that for all $k \geq K$ it holds that

$$d_{\mathbb{P}}(\overline{x}(k),\overline{x}) \;<\; \delta.$$

Example 4.4.1 *Consider the sequence*

$$x(k) \;=\; \begin{pmatrix} k+\frac{k}{2+k} \\ k \end{pmatrix}, \qquad k \geq 1.$$

Then, in conventional analysis

$$\lim_{k\to\infty} x(k) \;=\; \begin{pmatrix} \infty \\ \infty \end{pmatrix}.$$

Notice that the above limit fails to capture the fact that $x_1(k) - x_2(k)$ *tends to* 1 *as* k *tends to* ∞.

Consider now the above limit in the projective space; that is, study the (projective) limit of the sequence

$$\overline{x(k)} = \overline{\begin{pmatrix} \frac{k}{2+k} \\ 0 \end{pmatrix}}, \qquad k \in \mathbb{N},$$

as k *tends to* ∞. *Since*

$$\overline{\begin{pmatrix} k+1 \\ k \end{pmatrix}} = \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \qquad k \in \mathbb{N},$$

it follows that

$$\begin{aligned} d_{\mathbb{P}}\left(\overline{x}(k), \overline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}\right) &= d_{\mathbb{P}}\left(\overline{\begin{pmatrix} k + \frac{k}{2+k} \\ k \end{pmatrix}}, \overline{\begin{pmatrix} k+1 \\ k \end{pmatrix}}\right) \\ &= \max\left(\frac{-2}{2+k}, 0\right) - \min\left(\frac{-2}{2+k}, 0\right) \\ &= \frac{2}{2+k}, \end{aligned}$$

which becomes arbitrarily small. Hence, $\overline{x(k)}$ *converges to* $\overline{(1,0)^\top}$ *as* k *tends to* ∞. *The limit in the projective space keeps track of the finite differences* within *the vector* $x(k)$. *Even though* $x(k)$ *diverges (in the conventional sense), convergence of* $x(k)$ *in the projective space to a finite element happens.*

4.5 HIGHER-ORDER RECURRENCE RELATIONS

For $M \geq 0$, let $A_m \in \mathbb{R}_{\max}^{n \times n}$ for $0 \leq m \leq M$ and $x(m) \in \mathbb{R}_{\max}^n$ for $-M \leq m \leq -1$. Then, the (implicit) recurrence relation

$$x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m), \qquad k \geq 0, \tag{4.7}$$

is defined. The above recurrence relation is called an *Mth-order recurrence relation*. So far we have restricted our analysis to first-order recurrence relations with $A_0 = \mathcal{E}$. However, in applications one frequently encounters systems whose dynamics follow a recurrence relation of order two or higher and/or for which $A_0 \neq \mathcal{E}$. As we will show in this section, the Mth-order recurrence relation (4.7) can be transformed into a first-order recurrence relation of the type

$$x(k+1) = A \otimes x(k), \qquad k \geq 0,$$

provided that A_0 in (4.7) has circuit weights less than or equal to zero or has no circuits at all. The starting point is Lemma 2.2, which implies that if A_0 has circuit weights less than or equal to zero, then

$$A_0^* = \bigoplus_{i=0}^{n-1} A_0^{\otimes i}. \tag{4.8}$$

We now turn to the algebraic manipulation of (4.7). Set

$$b(k) \;=\; \bigoplus_{m=1}^{M} A_m \otimes x(k-m).$$

Then (4.7) reduces to

$$x(k) \;=\; A_0 \otimes x(k) \,\oplus\, b(k). \tag{4.9}$$

By Theorem 2.10, (4.9) can be written as

$$x(k) \;=\; A_0^* \otimes b(k)$$

or, more explicitly,

$$x(k) \;=\; A_0^* \otimes A_1 \otimes x(k-1) \,\oplus\, \cdots \,\oplus\, A_0^* \otimes A_M \otimes x(k-M)\,. \tag{4.10}$$

The difference between (4.7) and (4.10) is that in the latter $x(k)$ occurs only in the left-hand side of the equation.

As a next step, we transform (4.10) into a first-order recurrence relation. In order to do so, we set

$$\tilde{x}(k) = (\, x^\top(k-1), x^\top(k-2), \ldots, x^\top(k-M)\,)^\top$$

and

$$\tilde{A} \;=\; \begin{pmatrix} A_0^* \otimes A_1 & A_0^* \otimes A_2 & \ldots & \ldots & A_0^* \otimes A_M \\ E & \mathcal{E} & \ldots & \ldots & \mathcal{E} \\ \mathcal{E} & E & \ddots & & \mathcal{E} \\ \vdots & & \ddots & & \vdots \\ \mathcal{E} & \mathcal{E} & \ldots & E & \mathcal{E} \end{pmatrix}.$$

Then, (4.7) can be written as

$$\tilde{x}(k+1) \;=\; \tilde{A}(k) \otimes \tilde{x}(k), \qquad k \geq 0. \tag{4.11}$$

4.6 EXERCISES

1. Let $x^1, \ldots, x^d \in \mathbb{R}^n$ be a periodic regime of period d of $A \in \mathbb{R}_{\max}^{n\times n}$ such that $\mu \otimes x^1 = A \otimes x^d$. Show that $x^1, \ldots, x^d$ are eigenvectors of $A^{\otimes d}$ associated with eigenvalue μ.
2. For $A \in \mathbb{R}_{\max}^{n\times n}$ we say that the *eigenvector of A is unique* if for any two eigenvectors v, w it holds that $v = \alpha \otimes w$ for $\alpha \in \mathbb{R}$. Show that if $A \in \mathbb{R}_{\max}^{n\times n}$ is irreducible and if the critical graph of A consists of a single strongly connected subgraph, then the eigenvector of A is unique.
3. Show that $A_\lambda^* \otimes A_\lambda^* = A_\lambda^*$.
4. Show that $||\overline{x}||_{\mathbb{P}}$ is independent of the representative x; that is, for $y \in \overline{x}$ it holds that $||\overline{y}||_{\mathbb{P}} = ||\overline{x}||_{\mathbb{P}}$.
5. Show that $d_{\mathbb{P}}(\cdot,\cdot)$ is a metric on $\mathbb{PR}^n$; that is, show that
   - $d_{\mathbb{P}}(\overline{x},\overline{y}) = d_{\mathbb{P}}(\overline{y},\overline{x})$ for $\overline{x},\overline{y} \in \mathbb{PR}^n$;

- $d_{\mathbb{P}}(\overline{x}, \overline{y}) = 0$ if and only if $\overline{x} = \overline{y}$;
- $d_{\mathbb{P}}(\overline{x}, \overline{y}) + d_{\mathbb{P}}(\overline{y}, \overline{z}) \geq d_{\mathbb{P}}(\overline{x}, \overline{z})$ for $\overline{x}, \overline{y}, \overline{z} \in \mathbb{PR}^n$.

6. Show that max-plus algebra is nonexpansive in the $d_{\mathbb{P}}$-norm; that is, show that for any regular $A \in \mathbb{R}_{\max}^{n \times n}$ and any $x, y \in \mathbb{R}^n$ it holds that

$$d_{\mathbb{P}}(A \otimes x, A \otimes y) \leq d_{\mathbb{P}}(x, y) .$$

7. Let

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}.$$

Compute a basis of $V(A)$.

8. Let

$$x(k) = \begin{pmatrix} \varepsilon & 1 \\ -4 & \varepsilon \end{pmatrix} \otimes x(k) \oplus \begin{pmatrix} 2 & 4 \\ -1 & \varepsilon \end{pmatrix} \otimes x(k-2), \qquad k \geq 2,$$

with given $x(1), x(0) \in \mathbb{R}^2$. Transfer the above recurrence relation into a recurrence relation of the form $\tilde{x}(k+1) = \tilde{A} \otimes \tilde{x}(k)$.

9. Let A be irreducible. Show that $A_\lambda^* = (E \oplus A_\lambda)^{\otimes(n-1)}$, where $\lambda = \lambda(A)$ and n denotes the number of rows and columns of A.

10. If λ is an eigenvalue of matrix A with some appropriate eigenvector, then $\lambda^{\otimes k}$ is an eigenvalue of $A^{\otimes k}$ for the same eigenvector. Conversely, if $\mu = \lambda^{\otimes k}$ is an eigenvalue of $A^{\otimes k}$ with eigenvector w, then

$$v = \bigoplus_{i=1}^{k} A^{\otimes(k-i)} \otimes w \otimes \lambda^{\otimes i}$$

is an eigenvector of A corresponding to eigenvalue λ. Prove this.

4.7 NOTES

Primitive matrices are called *scs1-cyc1 matrices* in the literature. The somewhat awkward expression *scs1-cyc1* stems from the general terminology of calling a matrix whose communication graph has m m.s.c.s.'s and that is of cyclity k a *scs-m-cyc-k matrix*. We refer to [61] for a variety of examples of scs-m-cyc-k matrices with $m, k > 1$.

Mairesse provides a nice graphical representation of the domain of attraction in the projective space for dimension three; see [62] and the extended version [61]. In particular, the eigenvector (resp., periodic regime) in whose domain of attraction an initial value x_0 lies can be deduced from a graphical representation of the eigenspace of A in the projective space.

The set $\mathbb{R}_{\max}$ can be equipped with a metric through an exponential lifting. Set $\exp(-\infty) = \exp(\varepsilon) = 0$, then $|x - y|_{\exp} \overset{\text{def}}{=} |\exp(x) - \exp(y)|$ yields a metric on $\mathbb{R}_{\max}$, and convergence in $\mathbb{R}_{\max}$ can be defined through $|\cdot|_{\exp}$.

Making use of the nonexpansiveness of max-algebra in the $d_{\mathbb{P}}$-norm, one can give an alternative proof of Lemma 3.12 via arguments in the projective space.

The statement of part (iii) of Theorem 4.5 can be generalized: $[A_\lambda^*]_{\cdot i}$ cannot be expressed as a linear combination of other $[A_\lambda^*]_{\cdot j}$'s with the j's belonging to other m.s.c.s.'s than the one to which i belongs. The proof, beyond the scope of this book, can be found in [5].