Projective Geometry in 1D

Wai-Shing Luk

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Introduction

Key points

- A simplified version of the projective plane.
- Möbius transformation can be viewed as a projective transform of a complex projective point.

Projective Line's Basic Elements

Projective Line Concept

- Only involve "Points".
- "Points" is assumed to be distinguishable.
- Denote A = B as A and B are referred to the same point.
- E.g., $(1/3)$ = $(10/30)$
- We have the following rules:
  - A = A (reflective)
  - If A = B, then B = A (symmetric)
  - If A = B and B = C, then $A = C$ (transitive)
- Unless mention specifically, objects in different names are assumed to be distinct, i.e. $A \neq B$.

Homogenous Coordinates

- Let $v_1 = [x_1, y_1]$ and $v_2 = [x_2, y_2]$.
  - dot product $v_1 \cdot v_2$ = $v_1^T v_2$ = $x_1 x_2 + y_1 y_2$.
  - cross product $v_1 \times v_2$ = $x_1 y_2 - y_1 x_2$
- Then, we have:
  - $A = B$ if and only if $[A] \times [B] = 0$
- Example: the point $(5/10)$ and $(3/6)$ is the same because $5 \cdot 6 - 3 \cdot 10 = 0$
- The cross product is also used as a basic measure between two points.
- The cross ratio of four points $R_1(a, b; c, d)$ is given by:

$$R_1(a, b; c, d) = (a \times c)(b \times d)/(a \times d)(b \times c)$$

Example 1: Euclidean Geometry

- Point: projection of a 2D vector $p = [x, y]$ to 1D line $y = 1$:

$$(x') = (x/y)$$

- $p_\infty = [x, 0]$ is a point at *infinity*.
- $[0, 0]$ is not a valid point.

Example 1: Euclidean Geometry (measurement)

- The **quadrance** Q between points A_1 and A_2 is:

$$Q = (x'_1 - x'_2)^2 = (x_1/y_1 - x_2/y_2)^2$$

- Let A_1, A_2 and A_3 are points with $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$.
- TQF (Triple quad formula):

$$(Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2)$$

- TQF (non-symetric form):

$$(Q_1 + Q_2 - Q_3)^2 = 4(Q_1 Q_2)$$

Euclidean 1D plane from 2D vector

Example 2: Elliptic Geometry

- "Point": projection of 2D vector $[x, y]$ to the unit circle.

$$(x', y') = (x/r, y/r)$$

  where $r^2 = x^2 + y^2$.

- Two points on the opposite poles are considered the same point here.

Example 2: Elliptic Geometry (measurement)

- The measure of two points is the "spread" of the point.
- The **spread** S between points A_1 and A_2 is:

$$s(A_1, A_2) = 1 - (x_1x_2 + y_1y_2)^2/(x_1^2 + y_1^2)(x_2^2 + y_2^2)$$

- Let A_1, A_2 and A_3 are points with $S_1 \equiv S(A_2, A_3)$, $S_2 \equiv S(A_1, A_3)$ and $S_3 \equiv S(A_1, A_2)$.
- TSF (Triple spread formula):

$$(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3.$$

Example 4: Hyperbolic Geometry

- A velocity "point": projection of a 2D vector $[p] = [x, t]$ to 1D line $t = 1$:

$$(v) = (x/t)$$

- The measure of two velocity points is the relative speed of two points.

$$\begin{aligned} \mathrm{Speed}(p, q) &= (x_p t_q - t_p x_q)^2/(x_p^2 - t_p^2)(x_q^2 - t_q^2) \\ &= (v_p - v_q)^2/(v_p^2 - 1)(v_q^2 - 1) \end{aligned}$$

- Assume that the speed of light is normalized as 1. Then Speed(p, q) can never exceed 1 when $|v_p| \leq 1$ and $|v_q| \leq 1$.

Projective Transformation

- Given a nonsingular matrix T = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The transformation

$$[x', y'] = \tau([x, y]) = [ax + by, cx + dy]$$

- Let $z = x/y$, the formula becomes:

$$z' = (az + b)/(cz + d)$$

- This is exactly the Möbius transformation, where z is a complex number.
- Möbius transformation plays an important role in the electromagetic theory.
- There are two fixed points in this transformation, considering infinity as also a fixed point.