# Chapter Twelve

# Min-Max-Plus Systems and Beyond

In this chapter min-max-plus systems will be studied. Such systems can be viewed as an extension of max-plus systems in the sense that in addition to the max and plus operators, the min(imization) operator is now also allowed. This gives more flexibility with respect to modeling issues. At the end of this chapter, we will briefly discuss the imbedding of min-max-plus systems in the even more general class of nonexpansive systems.

## 12.1 MIN-MAX-PLUS SYSTEMS

### 12.1.1 Introduction and classification

Min-max-plus systems are described by expressions in which the three operations minimization, maximization, and addition appear. They can be viewed as an extension of max-plus expressions in the sense that minimization has been added as a possible operation. For instance,

$$\min\big(x_1 + 3, \max\big(x_2 - 2, \min(x_1 + 7, x_3)\big), \max(x_3 + 1, x_4 + 2)\big), \qquad (12.1)$$

or, equivalently in the min-max-plus notation,

$$(x_1 \otimes 3) \oplus' \big((x_2 \otimes -2) \oplus ((x_1 \otimes 7) \oplus' x_3)\big) \oplus' \big((x_3 \otimes 1) \oplus (x_4 \otimes 2)\big),$$

is a min-max-plus expression. It will be clear that the class of min-max-plus systems is richer than the class of max-plus systems; that is, one can describe more general phenomena in the former class.

**Example 12.1.1** *Think of the preparation of different meals, each one consisting of various dishes, to be served at the same time in a restaurant. For the preparation, one needs the ingredients at the right time. The preparation will furthermore depend on the labor involved, such as washing the lettuce. The earliest time instant at which all meals can be served is after the last time instant at which all dishes are ready. In the process of preparation, some dishes will probably already be available before the time of serving. Depending on the particulars of these dishes, one should prepare them as late as possible so as not to ruin their taste during the idle time between being ready and being served. Hence, this idle time must be kept to a minimum. Thus, the cook faces a decision process with the maximization operator (the maximum of all time instants at which all dishes are ready), the minimum operator (minimizing the idle times), and the addition (the time needed for washing the ingredients for the salad, boiling the water, etc.).*

Formally, min-max-plus systems can be introduced by means of a recursive definition scheme.

**DEFINITION 12.1** *A min-max-plus expression is an expression that can be thought of as being generated by the following scheme. Variables $x_1, x_2, \ldots, x_n$ taking values in $\mathbb{R}$ are min-max-plus expressions. If $f$ is a min-max-plus expression, then $f \otimes a$ is a min-max-plus expression, where $a \in \mathbb{R}$ is a parameter. If, in addition, $g$ is a min-max-plus expression, then $f \oplus' g$ and $f \oplus g$ are min-max-plus expressions. No other expressions are min-max-plus expressions.*

The most elementary min-max-plus expression is simply a variable, like $x_i$ or $x_j$. One can add constants to these variables $x_i \otimes a$ and take the minimum $x_i \oplus' x_j$ or take the maximum $x_i \oplus x_j$. These latter expressions can be combined once more by means of the $\oplus'$ or $\oplus$ operators to obtain more complex expressions. In this way one can continue. It is easily seen that (12.1) is indeed a min-max-plus expression; however, neither $(x_1 \otimes x_2) \oplus (x_3 \otimes -1)$ nor $x_1 \oplus' 4$ are.

By means of the identities

$$a \oplus (b \oplus c) = a \oplus b \oplus c, \tag{12.2}$$

$$a \oplus' (b \oplus' c) = a \oplus' b \oplus' c, \tag{12.3}$$

$$c \oplus (a \oplus' b) = (c \oplus a) \oplus' (c \oplus b), \tag{12.4}$$

$$c \oplus' (a \oplus b) = (c \oplus' a) \oplus (c \oplus' b), \tag{12.5}$$

each min-max-plus expression $f$ can be transformed into the *conjunctive normal form*; that is, we have

$$f = f_1 \oplus' f_2 \oplus' \cdots \oplus' f_p,$$

for some finite $p \in \mathbb{N}$ and where each $f_i$ is a max-plus expression, i.e.,

$$f_i = (x_1 \otimes a_{i1}) \oplus (x_2 \otimes a_{i2}) \oplus \cdots \oplus (x_n \otimes a_{in}),$$

with $a_{ij} \in \mathbb{R}_{\max}$. The adjective *conjunctive* is related to the logical *and*, which is mathematically often written as $\bigwedge$. The latter symbol refers to the minimum operator. Hence, we have the name *conjunctive normal form*. Each min-max-plus expression $f$ can equally well be transformed into the *disjunctive normal form*; that is, we have

$$f = f_1 \oplus f_2 \oplus \cdots \oplus f_q$$

for some finite $q \in \mathbb{N}$ and where now each $f_i$ is a min-plus expression, i.e.,

$$f_i = (x_1 \otimes b_{i1}) \oplus' (x_2 \otimes b_{i2}) \oplus' \cdots \oplus' (x_n \otimes b_{in}),$$

with $b_{ij} \in \mathbb{R}_{\min}$, defined in Example 1.1.1. The adjective *disjunctive* is related to the logical *or*, which in mathematical expressions becomes the maximum operator, often written as $\bigvee$.

**Example 12.1.2** *Consider expression (12.1):*

$$\begin{aligned}
&(x_1 \otimes 3) \oplus' \big((x_2 \otimes -2) \oplus ((x_1 \otimes 7) \oplus' x_3)\big) \oplus' \big((x_3 \otimes 1) \oplus (x_4 \otimes 2)\big) \\
&\overset{(12.4)}{=} (x_1 \otimes 3) \oplus' \big(((x_2 \otimes -2) \oplus (x_1 \otimes 7)) \oplus' ((x_2 \otimes -2) \oplus x_3)\big) \\
&\qquad\qquad \oplus' \big((x_3 \otimes 1) \oplus (x_4 \otimes 2)\big) \\
&\overset{(12.3)}{=} (x_1 \otimes 3) \oplus' \big((x_2 \otimes -2) \oplus (x_1 \otimes 7)\big) \oplus' \big((x_2 \otimes -2) \oplus (x_3)\big) \\
&\qquad\qquad \oplus' \big((x_3 \otimes 1) \oplus (x_4 \otimes 2)\big).
\end{aligned}$$

*The last expression is in the conjunctive normal form.*

An expression in conjunctive normal form can also be written in disjunctive normal form (or the other way around), as is shown by the following example.

**Example 12.1.3** *We have*

$$\begin{aligned}(c \oplus d) \oplus' (a \oplus b) &\overset{(12.5)}{=} (c \oplus' (a \oplus b)) \oplus (d \oplus' (a \oplus b)) \\ &\overset{(12.5)}{=} ((c \oplus' a) \oplus (c \oplus' b)) \oplus ((d \oplus' a) \oplus (d \oplus' b)) \\ &\overset{(12.2)}{=} (c \oplus' a) \oplus (c \oplus' b) \oplus (d \oplus' a) \oplus (d \oplus' b).\end{aligned}$$

**DEFINITION 12.2** *A min-max-plus function of dimension $n$ is a mapping $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^n$, where the components $\mathcal{M}_i$ of $\mathcal{M}$ are min-max-plus expressions of the $n$ variables $x_1, x_2, \ldots, x_n$.*

Max-plus algebra and min-plus algebra have been already introduced in Section 0.5 and formally defined in Section 1.1. Please note that the min operator is a nonlinear operator in max-plus algebra and that the max operator is nonlinear in min-plus algebra. The following are properties of $\mathcal{M}$:

- $\mathcal{M}$ is monotone; that is, if $x, \bar{x} \in \mathbb{R}^n$ such that $x \leq \bar{x}$, then $\mathcal{M}(x) \leq \mathcal{M}(\bar{x})$, where these inequalities must be interpreted componentwise.
- $\mathcal{M}$ is homogeneous; that is, $\mathcal{M}(\alpha \otimes x) = \alpha \otimes \mathcal{M}(x)$ for any scalar $\alpha \in \mathbb{R}$ and any $x \in \mathbb{R}^n$, where the scalar multiplication in both cases refers to componentwise addition of $\alpha$.
- $\mathcal{M}$ is nonexpansive; that is, $||\mathcal{M}(x) - \mathcal{M}(\bar{x})||_\infty \leq ||x - \bar{x}||_\infty$ for arbitrary $x, \bar{x} \in \mathbb{R}^n$, where $||.||_\infty$ refers to the supremum norm (i.e., the $l^\infty$-norm; see Section 3.2). For a further discussion of nonexpansive mappings, see Section 12.2.2.

In the scientific literature, functions that satisfy the above three properties are called *topical functions*. The class of topical functions is essentially larger than the class of min-max-plus functions. See the notes section of this chapter for some further information and also Example 12.2.1.

If all components $\mathcal{M}_i(x)$ are (re)written in the conjunctive normal form, then we can formally write $\mathcal{M}(x) = \min_{j \in J}(A_j \otimes x)$, where $J$ is a finite set and where all $A_j$ are matrices over $\mathbb{R}_{\max}$ with size $n \times n$. Such a representation is called a *max-representation* of $\mathcal{M}$. If $\mathcal{M}$ is (re)written as $\max_{j \in J'}(B_j \otimes' x)$, with $J'$ being a finite set and the $B_j$ all being $n \times n$ matrices over $\mathbb{R}_{\min}$, then the latter representation of $\mathcal{M}$ is called a *min-representation*. The max-representation (and similarly the min-representation) of a mapping $\mathcal{M}$ is not necessarily unique, as is shown by the next example.

**Example 12.1.4** *If*

$$\begin{aligned}\mathcal{M}_1(x) &= \min\big(\max(x_1 + 1, x_2 + 4), x_2\big), \\ \mathcal{M}_2(x) &= \min\big(\max(x_1 + 3, x_2 + 2), \max(x_1 + 5, x_2 - 2)\big),\end{aligned}$$

*with* $x = (x_1, x_2)^\top$, *then both* $\mathcal{M}(x) = \min_{j=1,2}(A_j \otimes x)$ *and* $\mathcal{M}(x) = \min_{j=3,4}(A_j \otimes x)$ *are max-representations, where*

$$A_1 = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}, A_2 = \begin{pmatrix} \varepsilon & 0 \\ 5 & -2 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 4 \\ 5 & -2 \end{pmatrix}, A_4 = \begin{pmatrix} \varepsilon & 0 \\ 3 & 2 \end{pmatrix}.$$

**DEFINITION 12.3** *A min-max-plus system of dimension $n$ is a system with state $x(k) = (x_1(k), x_2(k), \ldots, x_n(k))^\top$, which evolves according to $x(k+1) = \mathcal{M}(x(k))$, $k \geq 0$, where $\mathcal{M}$ is a min-max-plus function of dimension $n$.*

Subclasses of min-max-plus systems can be defined for which specific properties are known to hold. Two such subclasses, those of separated and of bipartite min-max-plus systems, will be dealt with briefly in the coming sections. The definitions are as follows.

**DEFINITION 12.4** *Consider a min-max-plus system characterized by the min-max-plus function $\mathcal{M}$. If each component of $\mathcal{M}$ is either a max-plus expression or a min-plus expression only, then the system is called* separated.

If, through a possible reordering of the state components, the first $n$ components of $\mathcal{M}$ are max-plus expressions and the last $m$ components are min-plus expressions (by abuse of notation, the dimension of the system now is $n + m$), then, with a renaming of the state variables, we can write

$$x_i(k+1) = \max(x_1(k) + a_{i1}, \ldots, x_n(k) + a_{in}, y_1(k) + b_{i1}, \ldots, y_m(k) + b_{im}),$$
$$y_j(k+1) = \min(x_1(k) + c_{j1}, \ldots, x_n(k) + c_{jn}, y_1(k) + d_{j1}, \ldots, y_m(k) + d_{jm}),$$

for $i \in \underline{n}$ and $j \in \underline{m}$. More concisely, we can write

$$x(k+1) = (A \otimes x(k)) \oplus (B \otimes y(k)), \tag{12.6}$$
$$y(k+1) = (C \otimes' x(k)) \oplus' (D \otimes' y(k)). \tag{12.7}$$

**DEFINITION 12.5** *Bipartite systems form a subclass of the class of separated systems in the sense that bipartite systems are separated systems with $A = \mathcal{E}$; that is, all elements of $A$ are $-\infty$, and $D = \mathcal{E}'$ (i.e., all elements of $D$ are equal to $+\infty$).*

Bipartite systems, as well as separated systems, can be symbolized by a graph with $n$ maximizing nodes representing $x_1, \ldots, x_n$ and $m$ minimizing nodes representing $y_1, \ldots, y_m$. Finite entries of $B$ represent arcs from the $y$-nodes to the $x$-nodes and finite entries of $C$ represent arcs from the $x$-nodes to the $y$-nodes. For bipartite systems there are no other arcs. The word *bipartite* indicates that there are two distinct sets of nodes with arcs from one to the other set and conversely, and no arcs between nodes of the same set. In contrast to the graph of a bipartite system, the graph of a separated system can contain arcs between $x$-nodes, as well as between $y$-nodes.

### 12.1.2 Eigenvalues and cycle times

Throughout this section it is assumed that $\mathcal{M}$ is an $n$-dimensional min-max-plus function. The following two definitions are straightforward generalizations of the notions of eigenvalue and cycle-time vector as already introduced in Part I. The notation $\mathcal{M}^p$, where $p$ is a positive integer, refers to $\mathcal{M}$ applied $p$ times; that is,

$$\mathcal{M}^p(x) = \underbrace{\mathcal{M}(\cdots(\mathcal{M}}_{p \text{ times}}(x))).$$

**DEFINITION 12.6** *The vector $x \in \mathbb{R}^n$ is called an eigenvector for eigenvalue $\lambda \in \mathbb{R}$ if $\mathcal{M}(x) = \lambda + x$. The vector $x \in \mathbb{R}^n$ is a periodic point of $\mathcal{M}$ with period $p$ if it is an eigenvector of $\mathcal{M}^p$ but not of $\mathcal{M}^k$ for any $1 \leq k < p$.*

Though generalizations are possible, we restrict ourselves here to vectors and eigenvectors with finite elements only. This is in contrast to the definition of eigenvectors of max-plus matrices; see Section 2.2. Compare the notion of period in the above definition with the one of cyclicity as defined in Section 3.1.

**THEOREM 12.7** *If the limit $\lim_{k\to\infty}(\mathcal{M}^k(x)/k)$ exists for some finite vector $x$, then it exists for all finite vectors $x$ and the limit is independent of the initial condition $x$.*

*Proof.* Suppose $\lim_{k\to\infty}(\mathcal{M}^k(x)/k) = a$, and let $\bar{x}$ be another finite vector. Then, nonexpansiveness with respect to the supremum norm implies

$$\lim_{k\to\infty}\left\|a - \frac{\mathcal{M}^k(\bar{x})}{k}\right\|_\infty \leq \lim_{k\to\infty}\left(\left\|a - \frac{\mathcal{M}^k(x)}{k}\right\|_\infty + \left\|\frac{\mathcal{M}^k(x) - \mathcal{M}^k(\bar{x})}{k}\right\|_\infty\right)$$

$$\leq \lim_{k\to\infty}\left\|\frac{x - \bar{x}}{k}\right\|_\infty = 0.$$

□

**DEFINITION 12.8** *The cycle-time vector $\chi(\mathcal{M})$ of the mapping $\mathcal{M}$ is defined as $\lim_{k\to\infty}(\mathcal{M}^k(x)/k)$, whenever this limit exists.*

It will be immediately clear now that if $\mathcal{M}$ has an eigenvalue $\lambda$, then the cycle-time vector exists and equals the vector with all components equal to $\lambda$.

Write $\mathcal{M}$ once more in its conjunctive normal form, $\mathcal{M}(x) = \min_{A_i \in S}(A_i \otimes x)$, where the set $S$ is defined as all possible $n \times n$ $A$ matrices by taking any combination of a max-plus expression in each component of $\mathcal{M}$. (In Example 12.1.4, for instance, $S$ consists of all four matrices given $A_i, i \in \underline{4}$.) For any $A_i \in S$ and any $x$, it follows that

$$\mathcal{M}(x) \leq A_i \otimes x,$$

and hence,

$$\mathcal{M}^2(x) = \mathcal{M}(\mathcal{M}(x)) \leq \mathcal{M}(A_i \otimes x) \leq A_i \otimes (A_i \otimes x) = A_i^{\otimes 2} \otimes x,$$

where the inequalities follow from the monotonicity property. Continuing, we get $\mathcal{M}^k(x) \leq A_i^{\otimes k} \otimes x$, and, for $k \to \infty$, $\chi(\mathcal{M}) \leq \chi(A_i)$. This inequality holds for any $i$, and thus,

$$\chi(\mathcal{M}) \leq \min_{A_i \in S} \chi(A_i). \tag{12.8}$$

This inequality provides an upper bound for $\chi(\mathcal{M})$, whenever it exists.

*Remark.* Inequality (12.8) needs some extra attention since comparing vectors by means of the (scalar) ordering relation $\leq$ only provides a partial ordering. By the definition of $S$ it follows that an $A_{i^*} \in S$ exists such that $\chi(A_{i^*}) \leq \chi(A_i)$, for all $i$, and this latter inequality holds componentwise. The proofs of these statements are left as an exercise (see exercise 2).

We can do the same analysis again, but now starting from the disjunctive normal form, $\mathcal{M}(x) = \max_{B_j \in T}(B_j \otimes' x)$, where the set $T$ is defined as all possible $n \times n$ $B$ matrices by taking any combination of a min-plus expression in each component of $\mathcal{M}$. Since $B_j \otimes' x \leq A_i \otimes x$, for any $i, j$ combination, we obtain

$$\max_{B_j \in T} \chi(B_j) \leq \min_{A_i \in S} \chi(A_i), \tag{12.9}$$

and if $\chi(\mathcal{M})$ exists, it must have a value between these two terms. The *duality conjecture* asserts that the inequality in (12.9) can be replaced by the equality sign. For a proof of this assertion, which thus has become a truth, see, for instance, [12] and [24]. If $\mathcal{M}$ has an eigenvalue, then the proof is simple as shown by the following theorem.

**THEOREM 12.9** *If $\mathcal{M}$ has an eigenvalue, then the duality conjecture holds.*

*Proof.* Call the eigenvalue and corresponding eigenvector $\lambda$ and $v$, respectively, i.e., $\mathcal{M}(v) = \lambda + v$. Hence, $\chi(\mathcal{M})$ exists because $\mathcal{M}^k(v) = \lambda^{\otimes k} \otimes v = k \times \lambda + v$ for $k = 1, 2, \ldots$, implying that each component of $\chi(\mathcal{M})$ equals $\lambda$. For at least one $i$, $\mathcal{M}(v) = A_i \otimes v$ and $\chi(\mathcal{M}) = \chi(A_i)$. In the same way, $\chi(\mathcal{M}) = \chi(B_j)$ for some $j$. The two latter equalities prove the equality sign in (12.9). □

### 12.1.3 Results on separated systems

The notation for a separated system will be the one given in (12.6) and (12.7). Please be reminded of the fact that the state is $(x^\top, y^\top)^\top$, which has size $n + m$.

**THEOREM 12.10** *Assume we are given a separated system characterized by $\mathcal{M}$ and as defined by (12.6) and (12.7), with $A$ and $D$ being irreducible matrices and both $B$ and $C$ having at least one finite element. Then, the mapping $\mathcal{M}$ has an eigenvalue $\lambda$ if and only if $\lambda_{\max} \leq \lambda_{\min}$, where $\lambda_{\max}$ is the eigenvalue of $A$ (in the max-plus algebra sense) and where $\lambda_{\min}$ is the eigenvalue of $D$ (in the min-plus algebra sense). Moreover, if $\lambda_{\max} \leq \lambda_{\min}$, then $\lambda$ is unique and satisfies $\lambda_{\max} \leq \lambda \leq \lambda_{\min}$.*

At least two proofs of this theorem exist in [71] and [24]; both are rather long, and we do not give them here. However, it is easily argued that $\lambda_{\max} \leq \lambda_{\min}$ is a necessary condition for the existence of $\lambda$. If we disregard the matrices $B$ and $C$, then $\mathcal{M}$ consists of two uncoupled systems,

$$x(k+1) = A \otimes x(k), \tag{12.10}$$

$$y(k+1) = D \otimes' y(k), \tag{12.11}$$

with the substate $x$ growing with an average rate of $\lambda_{\max}$ (i.e., on the average $x_i(k+1) = x_i(k) + \lambda_{\max}$ for $i \in \underline{n}$) and the substate $y$ growing with an average rate of $\lambda_{\min}$ (on the average $y_j(k+1) = y_j(k) + \lambda_{\min}$ for $j \in \underline{m}$). Adding the part $B \otimes y(k)$ to the right-hand side of (12.10) such as to obtain (12.6) can only further speed up the rate of $x$ (in the sense that the time instants $x(k+1)$ will occur later), whereas adding the term $C \otimes' x(k)$ to (12.11), so as to obtain (12.7), can only slow down the rate of $y$ (in the sense that the time instants $y(k+1)$ will occur sooner). Hence, if $\lambda_{\max} > \lambda_{\min}$, then the rates of the two subsystems can only grow further apart, and it will therefore be impossible for the two (sub-)states $x$ and $y$ to grow at an identical average rate. Hence, for the existence of a common average rate the inequality $\lambda_{\max} \leq \lambda_{\min}$ is needed.

**Example 12.1.5** *A separated system is given by means of its matrices*

$$A = \begin{pmatrix} -\infty & 1 & -\infty \\ -\infty & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 3 & -\infty \\ 3 & -\infty & -\infty \\ -\infty & -\infty & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} +\infty & +\infty & 3 \\ +\infty & 3 & +\infty \\ +\infty & +\infty & 3 \end{pmatrix}, \quad D = \begin{pmatrix} +\infty & 4 & 3 \\ 6 & +\infty & +\infty \\ +\infty & 9 & 6 \end{pmatrix}.$$

*The corresponding graph is shown in Figure 12.1. It is easily verified that $A$ is irreducible and that $\lambda_{\max} = 4/3$. Similarly, $D$ is irreducible and $\lambda_{\min} = 5$. Hence, $\lambda$ must exist. One way to find $\lambda$ is to use the duality conjecture, i.e., (12.9) with the equality sign. Other (numerical) approaches are mentioned in Section 12.1.5. Whatever method is used, the eigenvalue is $\lambda = 14/5$, with corresponding eigenvector $v = \frac{1}{5}(30, 28, 26, 27, 29, 27)^\top$.*

The communication graph of the system (12.6) and (12.7) can be given. It consists of $n$ maximizing nodes and $m$ minimizing nodes. The finite elements of the matrices $A$, $B$, $C$, and $D$ represent directed arcs. Now, the critical graph can be defined in the usual way. Toward this end one considers (12.6) and (12.7), in which an eigenvector is substituted. In each of the $n+m$ components of (12.6) and (12.7) those terms on the right-hand side that take care of the equality sign characterize a critical arc. All these critical arcs together (there are at least $n+m$) form at least one circuit, called a *critical circuit*.

For the example above, for instance, $n = 3$, $m = 3$, and the critical circuit is formed by the nodes $x_1, x_3, y_1, x_2, y_2, x_1$, visited in this order. Indeed, this circuit has average weight $14/5$. The critical graph thus constructed depends on the eigenvector chosen. The order is here first to compute an eigenvector and eigenvalue pair and then to determine the critical graph.

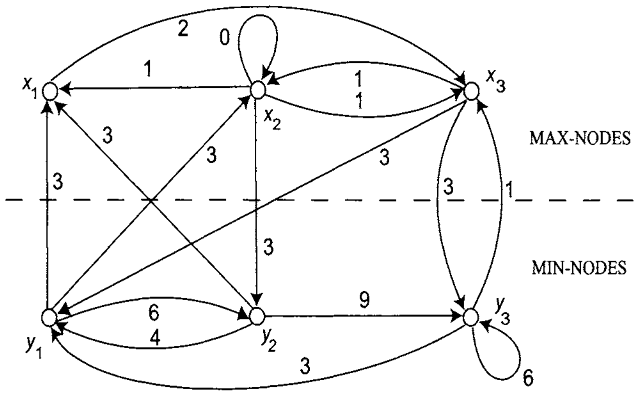


Figure 12.1: Graph corresponding to Example 12.1.5.

One could have realized beforehand that the eigenvalue should be equal to the average weight of a circuit. Since there is a finite number of circuits, one must choose out of a finite number of possibilities for $\lambda$. In general, the critical circuit is neither the slowest one (in the average sense) nor the fastest one (in the average sense). In the above example, the circuit $((y_3, x_3), (x_3, x_2), (x_2, y_2), (y_2, y_3))$ has average weight $14/4$, which is larger than $\lambda = 14/5$. Similarly, the circuit $((y_1, x_2), (x_2, x_3), (x_3, y_3), (y_3, y_1))$ has average weight $10/4$ which is smaller than $\lambda$.

*Remark.* Nothing has been said about the possibility of non-colinear eigenvectors. In principle, it is therefore possible that the critical circuit depends on the eigenvector chosen.

### 12.1.4 Results on bipartite systems

The notation for a bipartite system will be the one given in (12.6) and (12.7) with $A$ and $D$ nonexisting, i.e., $A = \mathcal{E}$ and $D = \mathcal{E}'$. Hence, a bipartite system is characterized by two matrices $B$ and $C$, such that

$$x(k+1) = B \otimes y(k), \qquad y(k+1) = C \otimes' x(k). \tag{12.12}$$

It will be assumed in this section that each row of $B$ and each row of $C$ contain at least one finite entry. Systems that satisfy this assumption are called *regular*, just as for max-plus algebra.

**THEOREM 12.11** *Consider the regular bipartite system (12.12). If the matrix pair $(B, C)$ is irreducible, then an eigenvalue (with corresponding eigenvector) exists.*

A matrix pair being irreducible is an extension of a single matrix being irreducible. The definition is as follows. If $\sigma$ denotes a permutation of $\underline{n}$ and $\tau$ a permutation of $\underline{m}$, then the $n \times m$ matrix $W(\sigma, \tau)$ is obtained from the $n \times m$ matrix $W$ by permuting the rows and columns of $W$ according to $\sigma$ and $\tau$, respectively.

**DEFINITION 12.12** *The matrix pair $(B, C)$ is irreducible if no permutations $\sigma$ of $\underline{n}$ and $\tau$ of $\underline{m}$ exist such that*

$$B(\sigma,\tau) = \begin{pmatrix} B_{11} & B_{12} \\ \mathcal{E} & B_{22} \end{pmatrix}, \qquad C(\tau,\sigma) = \begin{pmatrix} C_{11} & \mathcal{E}' \\ C_{21} & C_{22} \end{pmatrix},$$

*where*

- *the sizes of $B_{ij}$ and $C_{ji}^{\top}$, $i, j \in \underline{2}$, are identical (the submatrices $B_{ii}$ and $C_{jj}$ are not necessarily square), and*
- *$B_{11}$ and $C_{22}$ are regular.*

*Otherwise, the pair $(B, C)$ is called* reducible.

The reader is referred to the exercises in Section 12.3 in order to show that this definition can be viewed as an extension of the definition of irreducibility for single matrices. A bipartite system characterized by a nonirreducible (i.e., reducible) matrix pair can be written as, after a possible reordering of the components of the state vector,

$$\begin{aligned} x_1(k+1) &= B_{11} \otimes y_1(k) \oplus B_{12} \otimes y_2(k), && (12.13)\\ x_2(k+1) &= \phantom{B_{11} \otimes y_1(k) \oplus{}} B_{22} \otimes y_2(k), \\ y_1(k+1) &= C_{11} \otimes' x_1(k), \\ y_2(k+1) &= C_{21} \otimes' x_1(k) \oplus' C_{22} \otimes' x_2(k), && (12.14) \end{aligned}$$

where the vector $x$ has been split up into two subvectors $x_i$, $i \in \underline{2}$, of appropriate size and similarly for $y$. If the original system is regular, then the individual subsystems

$$\begin{pmatrix} x_1(k+1) \\ y_1(k+1) \end{pmatrix} = \begin{pmatrix} B_{11} \otimes y_1(k) \\ C_{11} \otimes' x_1(k) \end{pmatrix}, \begin{pmatrix} x_2(k+1) \\ y_2(k+1) \end{pmatrix} = \begin{pmatrix} B_{22} \otimes y_2(k) \\ C_{22} \otimes' x_2(k) \end{pmatrix}$$

are both regular bipartite systems. Suppose that both subsystems have an eigenvalue, say, $\lambda_1$ and $\lambda_2$, respectively, with $\lambda_1 > \lambda_2$. Intuitively, the average behavior of the events characterized by the time instants $x_1(k)$ and $y_1(k)$ and parameterized by $k$ in the original model can only become slower; that is, the time instants $x_1(k+1)$ and $y_1(k+1)$ will occur later, due to the term $B_{12} \otimes y_2(k)$ in (12.13). Similarly, the average behavior of the events characterized by $x_2(k)$ and $y_2(k)$ in the original model can only become faster; that is, the time instants $x_2(k+1)$ and $y_2(k+1)$ will occur sooner, due to the term $C_{21} \otimes' x_1(k)$ in (12.14). This is a plausible argument to support the statement that the average rate of growth of $x_1, y_1$ on the one side and of $x_2, y_2$ on the other will never become equal and hence, the eigenvalue for the original nonirreducible system cannot exist. Therefore, for the eigenvalue of the overall system to exist, it must be true that $\lambda_1 \leq \lambda_2$.

The existence of the eigenvalue in Theorem 12.11 depends purely on qualitative properties of the matrix pair $(B, C)$ and not on the numerical values of the finite elements of these matrices (as long as they remain finite). Thus, one talks about the *structural* existence of an eigenvalue. By abuse of language, the expression *structural eigenvalue* is also used. A system characterized by a nonirreducible matrix pair $(B, C)$ may or may not have an eigenvalue. The existence of the latter depends on the numerical values of the elements of the matrices $B$ and $C$. In that case one speaks of a *nonstructural* eigenvalue (provided that it exists). Note that the eigenvalue of a separated system is always nonstructural.

### 12.1.5 Some remarks on algorithmic issues

For general min-max-plus systems, the cycle-time vector can in principle be calculated by employing the duality conjecture, i.e., $\chi(\mathcal{M}) = \min_{A_i \in S} \chi(A_i)$; see (12.8) and (12.9). The set $S$, however, though finite, may be very large. In [24] ideas of the policy algorithm are presented in order to speed up this approach. Recently, the policy algorithm extended to bipartite systems has been shown to work well; see [81].

For irreducible bipartite systems, the following power algorithm yields the eigenvalue and an eigenvector; see [82].

**Algorithm 12.1.1**

1. *Start with an arbitrary vector* $x(0)$.
2. *Iterate* $x(k+1) = \mathcal{M}(x(k))$, $k = 0, 1, \ldots,$ *until there are integers* $p, q$, *with* $p > q \geq 0$ *and a finite real number* $c$ *such that* $x(p) = c \otimes x(q)$.
3. *Define as eigenvalue* $\lambda = c/(p-q)$ *and as candidate eigenvector*
$$v = \bigoplus_{i=1}^{p-q} \lambda^{\otimes(p-q-i)} \otimes x(q+i-1).$$
*Alternatively, one can take the candidate eigenvector*
$$v = {\bigoplus_{i=1}^{p-q}}' \lambda^{\otimes(p-q-i)} \otimes x(q+i-1).$$
4. *If* $\mathcal{M}(v) = \lambda \otimes v$, *then* $v$ *is a correct eigenvector; stop. Otherwise, start again at step 2, with* $x(0) = v$ *as the new initial state vector. Thus, the newly obtained quantities* $\lambda$ *and* $v$ *in step 3 do satisfy* $\mathcal{M}(v) = \lambda \otimes v$.

This algorithm even seems to work for a wider class of systems, such as, for example, separated systems as shown now by its application to Example 12.1.5. There one finds, if one starts with the zero-vector, that the subsequent states are

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 2 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ 5 \\ 5 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 8 \\ 8 \\ 9 \\ 8 \end{pmatrix}, \begin{pmatrix} 12 \\ 11 \\ 11 \\ 11 \\ 11 \\ 11 \end{pmatrix}, \begin{pmatrix} 14 \\ 14 \\ 14 \\ 14 \\ 14 \\ 14 \end{pmatrix},$$

and a periodic behavior is obtained after five steps. So, $p = 5$, $q = 0$, and $c = 14$. Generally, there is a transient behavior (i.e., the phase from $x(0)$ to $x(q)$), but little is known about its maximum length; see, however, [50] and [81]. The algorithm now gives that $\lambda = 14/5$ is the eigenvalue and that $v = \frac{1}{5}(60, 58, 56, 57, 59, 57)^\top$ is an eigenvector. These claims for the eigenvalue and eigenvector are easily shown to be correct by substitution into $\mathcal{M}(v) = \lambda + v$. Note that the vector $v$ and the eigenvector in Example 12.1.5 are colinear.

For separated systems another algorithm is given in [71]. Essentially, one studies $||c||_{\mathbb{P}} = \max_i c_i - \min_i c_i$, where $c_i = \mathcal{M}_i(x) - x_i$, $i \in \underline{n}$, as a function of the state $x$. The mapping $\mathcal{M}$ refers to the separated system under consideration. One continuously adapts the vector $x$ in such a way that with these changes the quantity $||c||_{\mathbb{P}}$ decreases and ultimately becomes zero. The $x$ vector for which $||c||_{\mathbb{P}} = 0$ is an eigenvector, and $c_1 = \cdots = c_n$ is the eigenvalue.

## 12.2 LINKS TO OTHER MATHEMATICAL AREAS

### 12.2.1 Link with the theory of nonnegative matrices

Consider once more the specific eigenvalue problem introduced in Chapter 0; that is,

$$\max(2 + v_1, 5 + v_2) = \lambda + v_1,$$
$$\max(3 + v_1, 3 + v_2) = \lambda + v_2.$$

By means of

$$\max(a, b) = \lim_{s \to \infty} \frac{1}{s} \ln(e^{sa} + e^{sb}), \qquad a + b = \lim_{s \to \infty} \frac{1}{s} \ln(e^{sa} e^{sb}), \tag{12.15}$$

the two scalar equations for the eigenvector can be approximated by

$$\frac{1}{s} \ln(e^{s(2+v_1)} + e^{s(5+v_2)}) = \frac{1}{s} \ln(e^{s(\lambda+v_1)}),$$
$$\frac{1}{s} \ln(e^{s(3+v_2)} + e^{s(3+v_2)}) = \frac{1}{s} \ln(e^{s(\lambda+v_2)})$$

or, equivalently, by

$$\begin{array}{rcl} e^{2s}e^{sv_1} + e^{5s}e^{sv_2} & = & e^{s\lambda}e^{sv_1}, \\ e^{3s}e^{sv_2} + e^{3s}e^{sv_2} & = & e^{s\lambda}e^{sv_2}. \end{array} \tag{12.16}$$

The reader will have realized that $e$ here stands for exp (of exponential). For $s \to \infty$ the approximation becomes exact in the appropriate sense. Now note that (12.16) is the eigenvalue equation for the matrix

$$A = \begin{pmatrix} e^{2s} & e^{5s} \\ e^{3s} & e^{3s} \end{pmatrix} \tag{12.17}$$

in conventional algebra, where now the eigenvalue is indicated by $e^{s\lambda}$ and the components of the eigenvector by $e^{sv_i}$, $i \in \underline{2}$. Since the elements of this matrix are positive, the Perron-Frobenius theorem [11] teaches us that a real and positive eigenvalue exists with a corresponding eigenvector of which the elements are real and positive also. Hence, the fact that the eigenvalue and the components of the eigenvector in (12.16), which can actually be seen as a definition of these quantities, are restricted to be positive is not a restriction.

For the sake of completeness, let us solve the eigenvalue for the matrix (12.17) in conventional algebra. Then, the eigenvalue $e^{s\lambda}$ must satisfy $\det(A - e^{s\lambda}I) = 0$, where $I$ denotes the identity matrix in conventional linear algebra. It follows that

$$(e^{s\lambda})^2 - (e^{2s} + e^{3s})e^{s\lambda} - e^{8s} = 0,$$

which has as solutions

$$e^{s\lambda} = \frac{(e^{2s} + e^{3s}) \pm \sqrt{(e^{2s} + e^{3s})^2 + 4e^{8s}}}{2}.$$

For $s \to \infty$ one obtains for the positive eigenvalue $e^{s\lambda} = e^{4s}$ in the appropriate sense, and thus, $\lambda$ of the original problem in max-plus algebra equals 4, which is in complete agreement with the results obtained in Chapter 0.

Actually, what we did above can be interpreted as the calculation of the eigenvalue for a matrix in max-plus algebra via a detour in conventional algebra. The same detour has been used to prove other properties in the theory of max-plus algebra.

### 12.2.2 Imbedding in nonexpansive maps

In Section 12.1.1 we encountered three properties of a min-max-plus function, namely, monotonicity, homogeneity and nonexpansiveness with respect to the $l^\infty$-norm. These properties will briefly be indicated by the symbols M, H and N, respectively. In the literature results are given in the case where nonexpansiveness is defined by means of a different norm (specifically, the $l^1$-norm); however, we will confine ourselves in this section to the $l^\infty$-norm.

**THEOREM 12.13** *If the function $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies* H, *then* M *is equivalent to* N.

In the statement of Theorem 12.13, $\mathcal{M}$ is not necessarily restricted to be a min-max-plus function. It is simply a mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying the above-mentioned properties. Such mappings (satisfying H and M, and equivalently, satisfying H and N) are called *topical*. That the min-max-plus functions form an actual subset of the set of topical functions is shown in the following example of a topical function, which is not min-max-plus; see also exercise 8.

**Example 12.2.1** *Consider the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ symbolized by $\ln(A\exp(\cdot))$, where $A$ is a positive matrix and both $\ln(\cdot)$ and $\exp(\cdot)$ are defined componentwise, i.e., $(\ln(x))_i = \ln(x_i)$ and $(\exp(x))_i = \exp(x_i)$. As a specific example (with $n = 2$ and $A$ being the matrix of (0.10)),*

$$f_1(x) = \ln(2e^{x_1} + 5e^{x_2}),$$
$$f_2(x) = \ln(3e^{x_1} + 3e^{x_2}).$$

It can be shown that any topical function can be represented as

$$\bigwedge_{i\in I} f_i, \qquad \text{or as} \qquad \bigvee_{j\in J} g_j,$$

where $I$ and $J$ are possibly uncountably infinite and where the components of $f_i$ and $g_j$ are max-plus and min-plus expressions, respectively. Note that if $I$ and $J$ are finite, then these representations are the disjunctive and conjunctive normal forms as already introduced.

One may now wonder whether the theory of eigenvalues and cycle times, as developed for min-max-plus systems in Section 12.1.2, can be carried over to topical functions. Definitions 12.6 and 12.8, as well as Theorem 12.7, are valid in the current context of nonexpansive mappings.

**THEOREM 12.14** *If $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive and if $p$ is the period (i.e., $\mathcal{M}^{p+1}(\cdot) = \mathcal{M}(\cdot)$), then $p \leq (2n)^n$.*

For max-plus systems (or min-plus systems), it is easy to show that a tighter upper bound can be given. In contrast to max-plus systems, the cycle-time vector does not always exist for nonexpansive mappings. In [49] a counterexample to this extent has been given with $n = 3$. For $n = 1, 2$, the cycle-time vector always exists.

## 12.3 EXERCISES

1. Consider the two-dimensional system

$$x(k+1) = \min(\max(x(k)+1, y(k)+7), \max(x(k)+6, y(k)+4)),$$
$$y(k+1) = \min(\max(x(k)+8, y(k)+2), \max(x(k)+3, y(k)+5)).$$

   Calculate the eigenvalue (answer: $\lambda = 5$) and show that the duality conjecture holds for this system. Rewrite the system in its disjunctive normal form, recalculate the eigenvalue, and show the correctness of the duality conjecture now starting from this representation.

2. Prove the statements made in the remark in Section 12.1.2.

3. Prove that the irreducibility of a square matrix $A$ is equivalent to the irreducibility of the matrix pair $(A, B)$, where $B$ is the identity matrix in min-plus algebra.

4. Calculate, by means of the power algorithm, the eigenvalue and an eigenvector of the bipartite system characterized by

$$B = \begin{pmatrix} 2 & -3 & 6 & 2 & -11 \\ 13 & 12 & 19 & -6 & 21 \\ -10 & 8 & 14 & -5 & -16 \end{pmatrix}, \quad C = \begin{pmatrix} 18 & 8 & 4 \\ -11 & 10 & 14 \\ -8 & -4 & 4 \\ 13 & -1 & -7 \\ 4 & 7 & 0 \end{pmatrix}.$$

   (Answer: $\lambda = 3,\ v = (14, 29, 14, 15, 0, 3, 4, 11)^\top$.)

5. Consider the bipartite system characterized by

$$B = \begin{pmatrix} 1 & -\infty \\ -\infty & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 100 \\ 100 & 0 \end{pmatrix}.$$

   Show that this system has at least two independent eigenvectors ($(1, 2, 1, 1)^\top$ and $(2, 2, 2, 1)^\top$). The critical circuit as defined in Section 12.1.3 depends in principle on the eigenvector. However, show that this is not true in the current example.

6. Suppose you are asked to design a timetable of trains on a network of tracks with a real crossing, say, a crossing of an east–west line with a north–south line. On this crossing the order of events is such that the first arriving train, which could arrive from either the east or north, should pass first (and the second arriving train must possibly wait for this first train to have passed), and afterwards the trains of the east–west direction and the north–south direction must alternate. Can you come up with a min-max-plus model to allow for the inclusion of the freedom to order the trains in this way?

7. In Section 12.2.1 a detour via conventional algebra has been defined to prove some results in max-plus algebra. Is a similar detour possible to prove results in the min-max-plus algebra by considering $\min(a, b) = \lim_{s \to \infty} \frac{1}{s} \ln(e^{-sa} + e^{-sb})$ in addition to (12.15)?

8. Show that the mapping $f$ in Example 12.2.1 is topical.

## 12.4 NOTES

Most of the material of Section 12.1 has been taken from [24], [71], and [85]. It can be shown that min-max-plus functions are dense in the class of topical functions in an appropriate setting; see [24]. The definitions of eigenvectors in some papers are more general than the one given here, in the sense that some elements (but not all) may be $-\infty$ or $+\infty$. This may lead to slightly different conditions on uniqueness issues, for instance. In [24], applications of min-max-plus algebra to the area of circuit theory, specifically for the clock schedule verification problem, are claimed. The notion of irreducibility of matrix pairs already shows up in [66]. No efficient algorithms are known to the authors to check the (ir)reducibility of matrix pairs of a large size. A recent paper with an algorithm to calculate the cycle time of min-max-plus systems is [22]. Survey paper [68] deals with the existence of cycle-time vectors (albeit in a different context). Paper [69] is a recent contribution toward the theory of nonexpansive maps.