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On The Construction of Non-coherent Space Time Codes from High-dimensional Spherical Codes

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***Abstract*—This paper analyzes the performance of non-coherent space time codes obtained from a new class of spherical codes by mapping the surface of a higher-dimensional half-sphere to the Grassmann manifold. The deployed spherical code contains a structure which can be exploited by the receiver, leading to a reasonable decoding complexity. Therefore, high-dimensional coding spaces (Grassmann manifolds) can be used as a basis for the construction. This corresponds to codes with larger block lengths, as opposite to the usual construction of space time codes from small-dimensional spaces, such as the unitary group $U(n_T)$, where n_T is the number of transmit antennas. The construction is flexible, meaning that codes of various dimensions and rates (spectral efficiency) may be constructed. Additionally, because of the properties of the deployed mapping, the structure of the spherical codes applies to the Grassmannian constellations as well.**

I. INTRODUCTION

The use of multiple antennas at the transmitter and the receiver increases the spectral efficiency of wireless systems. The concept of space time coding has been proven to be an appropriate mechanism to mitigate the fading effects and exploit the spatial diversity gains available with the introduction of multiple transmit antennas.

A capacity and performance analysis of space time codes in non-coherent Rayleigh block flat fading channels reveals that the relevant coding space is the complex Grassmann manifold $G^{\mathbb{C}}_{n_T,T}$[1], where n_T corresponds to the number of transmit antennas and T is the coherence time (block length). In first, the code design corresponds to a sphere packing problem in the Grassmann manifold.

The homogeneous structure of the Grassmann manifold provides a natural connection to spherical codes, motivating the construction of space time codes from spherical codes. In [2] and [3] Henkel provides a differential geometric connection between space time codes and spherical codes, by mapping the tangent space of the sphere to the tangent space of the Grassmann manifold, given that both spaces are of same dimension. We will exploit this connection in the construction of our space time codes.

There are two reasons to use spherical codes as a building block for space time codes: first, the structure present at some spherical codes results in decreased receiver complexity which is not the case with arbitrary space time codes; second, "good" spherical codes (in terms of minimal distance) are expected to give "good" space time codes.

Due to the complexity, most of the space time codes in the literature deal with small dimensional coding spaces, typically the unitary group $U(n_T)$. However, additional coding gain is expected when increasing the block length, due to the increase of the minimal distance [3]. Additionally, the construction of non-coherent space time codes of higher rate (and thus higher spectral efficiency) requires storing and decoding of large constellations, which is not feasible even for moderate rates and block lengths.

The spherical code deployed here contains structure which allows for flexible constructions of space time codes of various dimensions (block lengths T) and rates. Furthermore, because of the specific construction, no codewords have to be stored at the receiver. Additionally, the code exploits the advantage of the increase of the dimensionality of the coding space, yet keeping the decoding complexity reasonable.

II. SYSTEM MODEL

The channel model is the block fading model initiated by Marzetta and Hochwald [4] with channel coefficients which are i.i.d. Rayleigh distributed and remain constant for T symbol periods before changing to a new independent realization. The number of transmit antennas is n_T and the number of receive antennas is n_R. The model is given by

$$\Psi = \Phi H + \sqrt{\frac{n_T}{\rho T}} W. \tag{1}$$

Φ is $T \times n_T$ matrix of transmitted signals, in our case point from the Grassmann manifold $G^{\mathbb{C}}_{n_T,T}$. This matrix representation is unique up to right multiplication by $n_T \times n_T$ unitary matrix. H is $n_T \times n_R$ matrix of fading coefficients, i.i.d complex Gaussian, $CN(0,1)$. W is $T \times n_R$ matrix of the additive noise, also i.i.d complex Gaussian, $CN(0,1)$. Ψ is $T \times n_R$ matrix of received signals. With this identification, ρ is the SNR at each receive antenna.

The columns of Φ represent a basis for the n_T dimensional subspace. The $n_T \times n_R$ channel matrix rotates and scales the basis vectors within the same n_T dimensional subspace. Therefore, in absence of noise, the information about the n_T-dimensional subspace can be drawn from the received matrix Ψ. This can be done for example by performing a QR decomposition of the received matrix Ψ. Since it is the noise that changes the signal subspace, it is obvious that the design problem of transmit constellations is a packing problem in the Grassmann manifold, i.e. the problem of packing n_T-dimensional planes in $\mathbb{C}^T$.

III. Differential Geometry Preliminaries

In this section we will give the necessary introduction to the Grassmann manifold. We will introduce the notion of tangent space and give the differential geometry relation between the Grassmann manifold and its tangent space at a point of the manifold.

A. The Grassmann manifold

Before we formally define the Grassmann manifold, it is necessary to briefly introduce the Stiefel manifold, since both stand in close connection. The (complex) Stiefel manifold $V^{\mathbb{C}}_{n_T,T}$ is the set of n_T orthonormal vectors in $\mathbb{C}^T$

$$V^{\mathbb{C}}_{n_T,T} := \{\Phi \in \mathbb{C}^{T\times n_T} | \Phi^H \Phi = I_{n_T}\} \tag{2}$$

where I_{n_T} is the $n_T \times n_T$ identity matrix.

Having defined the Stiefel manifold, the (complex) Grassmann manifold is the set of all n_T-dimensional linear subspaces of $\mathbb{C}^T$

$$G^{\mathbb{C}}_{n_T,T} := \{\langle\Phi\rangle | \Phi \in V^{\mathbb{C}}_{n_T,T}\}. \tag{3}$$

The Grassmann manifold carries the structure of a $U(T)$ homogeneous space

$$G^{\mathbb{C}}_{n_T,T} \cong U(T)/\begin{pmatrix} U(n_T) & 0 \\ 0 & U(T-n_T) \end{pmatrix}. \tag{4}$$

B. The Grassmann manifold and its tangent space

Being quotient spaces within the unitary group, the Grassmann manifold inherit its geometry. With the Lie group identification of the unitary group $U(T)$, its tangent space at the identity I_T is the corresponding Lie algebra $\mathfrak{u}(T)$, i.e. the set of skew-hermitian matrices. The exponential map maps the Lie algebra to the Lie group [5].

The tangent space at I_T may be decomposed into two complementary linear subspaces: the vertical space and the horizontal space. The vertical space consists of vectors tangent to the equivalence class $\langle I_T\rangle$ (set of of all unitary matrices whose first n_T columns span the same subspace as those of I_T)[6]. The horizontal space consists of tangent vectors at I_T orthogonal to the vertical space. The horizontal space provides a representation of tangents to the quotient space [6]. Vectors in the horizontal space (and thus tangents of the Grassmann manifold) at $\langle I_{T,n_T}\rangle = \langle \left(\begin{smallmatrix} I_{n_T} \\ 0 \end{smallmatrix} \right) \rangle$ are of the form

$$X = \begin{pmatrix} 0 & -B^H \\ B & 0 \end{pmatrix}, \; B \in \mathbb{C}^{(T-n_T)\times n_T}. \tag{5}$$

C. Parameterization of the Grassmann manifold

The exponential map provides connection between the tangent space and the Grassmann manifold, i.e. maps points from the tangent space (subset of the Lie algebra of the unitary group) to points from the Grassmann manifold (equivalence classes of the unitary group).

The length of the geodesic connecting $\langle\Psi\rangle = \langle I_{T,n_T}\rangle \in G^{\mathbb{C}}_{n_T,T}$ with $\langle\Phi\rangle = \langle(\exp X)\cdot I_{T,n_T}\rangle \in G^{\mathbb{C}}_{n_T,T}$ is expressed in terms of the space of tangents

$$r_G = \frac{1}{\sqrt{2}}||X||_F = ||B||_F. \tag{6}$$

The exponential map is, however, computationally inefficient. Fortunately, the representation of the tangents in the form (5), provides efficient computation of the exponential map. Given the horizontal tangent, the singular value decomposition of $B, B \in \mathbb{C}^{(T-n_T)\times n_T}$, reads [3], [6]

$$B = V\Sigma W^H \tag{7}$$

where $V = (V1, V2)$,$V \in U(T-n_T)$, $\Sigma = \left(\begin{smallmatrix} S \\ 0 \end{smallmatrix} \right)$ is the matrix of singular values of B in decreasing order, and $W \in U(n_T)$. If we denote $\Phi = \exp(X)\left(\begin{smallmatrix} I_{n_T} \\ 0 \end{smallmatrix} \right)$, it can be shown that [3], [6]

$$\Phi = \begin{pmatrix} W(\cos S)W^H \\ V1(\sin S)W^H \end{pmatrix}. \tag{8}$$

Φ is only one particular representative for the n_T dimensional subspace spanned by the columns of Φ, but the symmetry of the upper part of Φ should be noted. It is this particular representation that can relate the subspace with the coresponding vector from the tangent space, as we will see in the decoding procedure.

IV. Construction of the Space Time Code

The design criteria for non-coherent channels requires maximization of the minimum chordal distance [1] or, in the equivalent geometric representation, solving the sphere packing problem in the Grassmann manifold with respect to the chordal distance. The chordal distance between two points (subspaces) $\langle\Phi\rangle$ and $\langle\Psi\rangle$ in the Grassmann manifold is defined as [7]

$$d_G(\langle\Phi\rangle, \langle\Psi\rangle) := \sqrt{\sin^2\theta_1 + \cdots + \sin^2\theta_{n_T}} \tag{9}$$

where $\theta_1, \ldots, \theta_{n_T}$ are the principle angles between the subspaces. However, the procedure of constructing non-coherent space time codes from spherical codes involves the notion of geodesic distance, as we will see later in this section. Nevertheless, although the geodesic distance is only an approximation of the chordal distance at the receiver, these two are locally equivalent, as shown in [3].

In the remaining part of this section we will first describe the construction of the spherical code, together with the decoding algorithm. Then we will construct the space time code based on the procedure described in [2]. Finally, we will discuss the specifics of the Grassmannian constellations obtained in this way.

A. Construction of the spherical code

The underlying spherical code is a modified version of the spherical code introduced in [8], [9]. The codewords are points from the unit sphere $S^D \in \mathbb{R}^{D+1}$. As shown on Fig. 1, the points of the codebook are "uniformly" distributed on equidistant *layers* obtained as a result of the intersection of

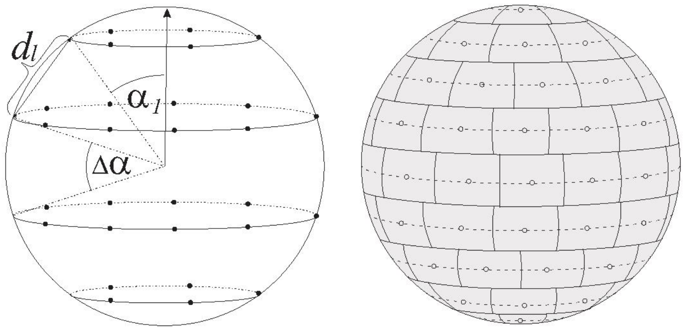

Fig. 1. a) Construction of the spherical code. b) Decoding regions of the spherical code

the sphere with parallel planes. The number of points on each layer is proportional to the layer's $D-1$ dimensional content (surface area). The choice of the layers is done in a way such that the obtained spherical code is antipodal. The procedure is recursive since the layers obtained in this way are also spheres, now of smaller dimension (one less) from the sphere they originate from. The same is done at every instance (dimension). For a code with given dimension and rate, an appropriate value of the parameter d_L is found in an iterative procedure, starting with an initial value d_{L_o} which is obtained by approximating the surface area of the k-dimensional unit radius sphere with the area covered by N D-dimensional curved hypercubes with sidelength equal to d_{L_o}.

The set of layers at some instance k of the recursive procedure may equivalently be described by the set of angles $\alpha_i^{(k)} \in [0, \pi]$, $\quad i = 1, 2, \ldots, n$ and $\alpha_{i+1}^{(k)} - \alpha_i^k = \Delta\alpha^k$.

Another way to describe the process of point distribution is by a tree with nodes denoting the layers of the code structure, where each codeword is represented as a path through the tree. According to this convention, the path (ordered sequence) $j_D, j_{D-1}, \cdots, j_1$ corresponds to a point from the codebook which belongs to the layer $L_j^{(D)}$ at stage D, $L_j^{(D-1)}$ at stage $D-1$, etc.

Decoding of the spherical code: This structure may be efficiently exploited by the receiver where the decision is based on successive "hard" decisions about affiliation to particular layers at each stage (dimension) of this procedure.

Let $\hat{Y}$ be an input point at the spherical decoder and let Θ_D ,$\Theta_{D-1}, \ldots$: Θ_1 be the equivalent representation with generalized spherical coordinates. Then, the angle representation of the layers can be used for efficient decoding. Namely, at each stage (dimension $D, D-1, \ldots$), we can decide about the affiliation of the point $\hat{Y}$ to a particular layer by a simple scalar quantization (comparison with the angles of the layers) of the corresponding angle (in spherical coordinate representation). The decoding procedure can thus be represented by D successive scalar quantizations.

Obviously, the decoding procedure is suboptimal. However, it allows for efficient decoding of spherical codes of high dimension and various rate. Furthermore, the codewords do not need to be stored at the receiver, which is anyway not feasible for large codebooks. The decoder only has to perform the same indexing procedure as the encoder, following the same rules (using the same parameter of the sphere point distribution d_L). Obviously, very large constellations ($2^{(D+1)*rate}$ codewords) may be decoded.

B. Construction of the space time code

The $U(T)$-homogeneous structure of the Stiefel and the Grassmann manifold provides relation to spherical codes. Any code on a half sphere $S^D \in \mathbb{R}^{D+1}$ can be transformed into a code on the Grassmann manifold [2] where

$$D = \dim_{\mathbb{R}} G_{n_T,T}^{\mathbb{C}} = 2n_T(T - n_T) \qquad (10)$$

In this particular case, the spherical code is restricted to the northern hemisphere. The encoding procedure is the same one described in [2], shortly summarized in the following:

Encoding: First, we fix the north pole N of the D dimensional half-sphere; then we project the spherical code C_S on the tangent space $T_N(S^D)$ at the north pole (orthogonal projection) and scale to length equal to the geodesics emanating from N to the considered point of the spherical code C_S; in the next step, tangential code points from $T_N(S^D)$ are mapped to the tangent space of the manifold $T_{\langle I_{T,n_T}\rangle}(G_{n_T,T}^{\mathbb{C}})$ by choosing orthonormal bases in both spaces (this is allowed, since both tangent spaces are of same dimension); in the last step, we apply exponential map from $T_{\langle I_{T,n_T}\rangle}(G_{n_T,T}^{\mathbb{C}})$ to the Grassmann manifold to obtain the space time code (having on mind the quotient space representation). The exponential map is conducted in a computationally efficient way, as given by (7) and (8).

Grassmannian constellations with structure: The exponential map (and its inverse) preserve geodesics emanating from the point at which the tangent space is constructed (the north pole N in the case of the (half) sphere and the identity point $\begin{pmatrix} I_{n_T} \\ 0 \end{pmatrix}$ in the case of the Grassmann manifold.

With this, the circular (layer) structure of the spherical code will be transfered to the tangent space of the (half) sphere and thus to the tangent space of the Grassmann manifold. Finally, this structure will apply to the Grassmann manifold itself, meaning that points from the spherical code that belong to the same layer (sphere of dimension $D-1$) with particular latitude, will be equally distant (geodesic distance) from $\begin{pmatrix} I_{n_T} \\ 0 \end{pmatrix}$. The mapping, in general, distorts the chordal and geodesic distances between the points from the spherical code. However, since it preserves the geodesic distances to the fixed point where the tangent space is constructed, it also preserves the relative position of the points in the codewords, meaning that the structure is preserved. This imposes a certain "layer" structure to the Grassmannian constellation obtained with the mapping procedure, which is similar to the structure of the spherical code it originates from.

Decoding: The decoding procedure is the inverse encoding procedure. However, there are some specifics worth to be mentioned. Let Ψ be the received $T \times n_R$ matrix, as given by (1). The n_T-dimensional subspace detection can be done by performing QR decomposition of Ψ. Let $\hat{\Phi} \in \mathbb{C}^{T \times n_T}$ be the result of the decomposition (after deletion of the last $T - n_T$ columns). In the absence of noise, $\hat{\Phi}$ spans the same subspace as the transmitted matrix Φ. The result of the channel (after the decomposition) can be seen as a right multiplication by a unitary matrix $U_{ch} \in U(n_T)$. With this and (10), $\hat{\Phi}$ becomes

$$\hat{\Phi} = \begin{pmatrix} W(\cos S)W^H U_{ch} \\ V1(\sin S)W^H U_{ch} \end{pmatrix} \tag{11}$$

This helps us to perform the inverse mapping to the tangent space. If we remember, the corresponding tangent X is determined by the matrix B, (5), where $B = V_1 S W^H$. Thus, B can be read off from $\hat{\Phi}$. Additionally, U_{ch} can be found and $\hat{\Phi}U_{ch}^{-1}$ gives Φ. The same applies with the noise with the remark that the noise will change the n_T-dimensional subspace.

It should be noted that loss of performance is expected because of the properties of the inverse mapping, since only specific geodesic distances are preserved. However, the aim was to do the final decoding on the surface of the D-dimensional sphere, and not on the Grassmann manifold, in order to exploit the efficient decoding mechanism.

It is interesting to point out that, for smaller constellations that can be stored, it is possible to perform the decoding with the decoding algorithm for Grassmannian lattices, presented in [10]. In this context, our method can be seen a practical way of obtaining Grassmannian codes with certain structure (codewords distributed on layers). Once again, this is feasible for smaller size constellations and consequently there is a limit in the spectral efficiency in terms of sent bits per channel use.

V. RESULTS

It may be expected that "good" spherical codes would lead to "good" space time codes. In this extent, the Grassmannian constellations has first been compared to some of the best packings in $G^{\mathbb{R}}_{1,T}$ found in the literature (Sloane, [7]). This would correspond to the case of non-coherent channel with one transmit antenna, under the block Rayleigh fading assumption. The simulations show that, although the code is suboptimal in terms of code design and receiver implementation (suboptimal decoding), it compares well to the best Grassmannian packings, at least for the packings provided in [7]. The original spherical code has also been tested in another scenario (source coding of high-dimensional Gaussian sources) and the results show that it compares well to some of the best spherical codes designed for that purpose ([8], table 1) such as wrapped spherical codes introduced by Hamkins and Zeger, [11].

With this on mind, the next step was to test the performance in the multiple transmit antenna channel. Because of the specific construction and no storage requirements, codes of larger blocks (coherence time T) can be designed. Additionally, higher code rates can be achieved, leading to spectral efficient space time codes. Since not many results are present in the literature for codes with different coherence time T, the performance have been compared with some of the best codes known such as Cayley differential codes and TAST space time codes. We can see that the construction outperforms the Cayley codes for a certain range of SNR. However, the

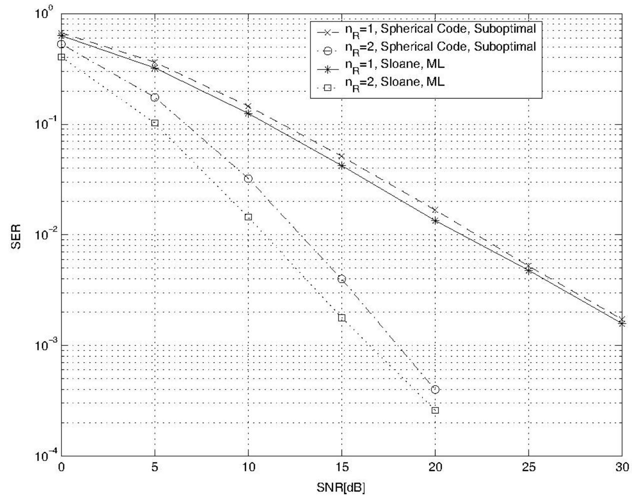

Fig. 2. Performance comparison of non-coherent space time codes: $n_T = 1, T = 6, \eta = 1$ bits per channel use.

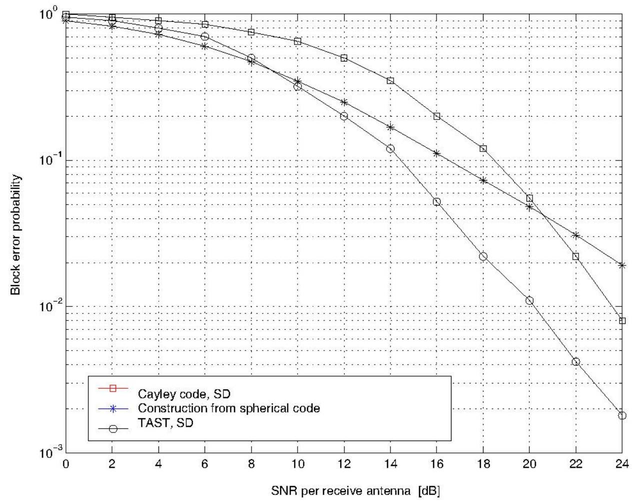

Fig. 3. Performance comparison of non-coherent space time codes: $n_T = 2, n_R = 2, T = 4, \eta = 2$ bits per channel use.

TAST constellation is still superior, particularly for higher SNR where seems that an error floor occurs. Nevertheless, it would be fare to mention that TAST codes rely on nonunitary constructions, which gives motivation for generalization of our procedure. It seems that, by releasing the unitary constraint, there might be some advantage regarding the diversity and the performance of the constellations.

Further investigation is needed to determine the reason for the high-SNR behavior of the scheme proposed here. We suspect that the reason might be in the suboptimal construction and decoding. As discussed, the exponential map (and its inverse) preserve only certain geodesic distances (emanating from the point where the tangent space is constructed). As a result, some of the distance properties of the spherical code might be lost (not transferred to the Grassmannian constellation obtained by the mapping), resulting in a constellation

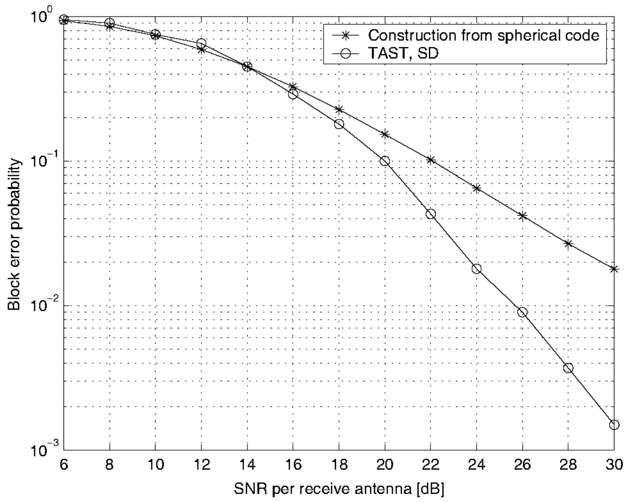

Fig. 4. Performance comparison of non-coherent space time codes: $n_T = 2, n_R = 2, T = 6, \eta = 4$ bits per channel use.

with smaller minimal chordal distance as expected. Another reason might be the possible noise amplification, due to the nonlinearity of the inverse mapping.

However, the possibility to construct codes for different (large) coherent times ($T >> n_T$), and different rate (spectral efficiency), gives reasons for future research.

VI. Conclusions and Future Work

In this paper, Grassmannian constellations constructed from novel spherical codes were presented. The spherical code has a structure which allows for construction and efficient decoding of high-dimensional (large block size) spectral-efficient constellations. The performance of the space time codes obtained in this way performs well compared to other non-coherent schemes in a certain SNR range.

However, further analysis is needed regarding the diversity and the influence of the suboptimal construction and decoding on the performance of the constellations. It is possible that small modifications in the construction procedure could lead to a denser packing in the Grasmannian manifold, and by this to a better performance of the constellation.

Another aspect would be the developement of an efficient decoding algorithm which relies on the same principle as the one for the spherical code, with the difference that operates directly on the Grassmann manifold. With this, the inverse mapping can be avoided. The motivation arises from the fact that, as discussed before, the "layer" structure of the spherical code applies to the Grassmannian constellation as well, due to the properties of the mapping.

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