III. Disjoint paths

1. Shortest paths

Let $D = (V, A)$ be a directed graph, and let $s, t \in V$.[1] A *path* is a sequence $P = (v_0, a_1, v_1, \ldots, a_m, v_m)$ where a_i is an arc from v_{i-1} to v_i for $i = 1, \ldots, m$ and where $v_0, \ldots, v_m$ all are different. The path P is called an $s-t$ *path* if $v_0 = s$ and $v_m = t$. The *length* of P is m. Here m is allowed to be 0. The *distance* from s to t is the minimum length of any $s-t$ path. (If no $s-t$ path exists, we set the distance from s to t equal to ∞.) A shortest $s-t$ path can easily be found by breadth-first search.

There is a trivial min-max relation characterizing the minimum length of an $s-t$ path. To this end, call a subset A' of A an $s-t$ *cut* if $A' = \delta^{\text{out}}(U)$ for some subset U of V satisfying $s \in U$ and $t \notin U$.[2] Throughout, *disjoint* means *pairwise disjoint*. Then the following was observed by Robacker [8]:

Theorem 1. *The minimum length of an $s-t$ path is equal to the maximum number of disjoint $s-t$ cuts.*

Proof. Trivially, the minimum is at least the maximum, since each $s-t$ path intersects each $s-t$ cut in an arc. To see equality, let d be the distance from s to t, and let U_i be the set of vertices at distance less than i from s, for $i = 1, \ldots, d$. Taking $C_i := \delta^{\text{out}}(U_i)$, we obtain disjoint $s-t$ cuts $C_1, \ldots, C_d$. ∎

2. Length functions

This can be generalized to the case where arcs have a certain 'length'. Let $l : A \to \mathbb{R}_+$, called a *length function*. For any path $P = (v_0, a_1, v_1, \ldots, a_m, v_m)$, let $l(P)$ be the length of P. That is:

$$l(P) := \sum_{i=1}^{m} l(a_i). \tag{1}$$

Now the *distance* from s to t (with respect to l) is equal to the minimum length of any $s-t$ path. If no $s-t$ path exists, the distance is ∞.

Then a weighted version of Theorem 1 is as follows:

Theorem 2. *Let $D = (V, A)$ be a directed graph, let $s, t \in V$, and let $l : A \to \mathbb{Z}_+$. Then the minimum length of an $s-t$ path is equal to the maximum number k of $s-t$ cuts $C_1, \ldots, C_k$ (repetition allowed) such that each arc a is in at most $l(a)$ of the cuts C_i.*

---

[1] A *directed graph* or *digraph* is a pair (V, A), where V is a finite set and $A \subseteq V \times V$. The elements of A are called the *arcs* of D. If $a = (u, v)$, then u is called the *tail* of a and v is called the *head* of a.

[2] $\delta^{\text{out}}(U)$ and $\delta^{\text{in}}(U)$ denote the sets of arcs leaving and entering U, respectively.

Proof. Again, the minimum is not smaller than the maximum, since if P is any $s-t$ path and $C_1, \ldots, C_k$ is any collection as described in the theorem:[3]

$$(2) \qquad l(P) = \sum_{a \in AP} l(a) \geq \sum_{a \in AP} (\text{ number of } i \text{ with } a \in C_i) = \sum_{i=1}^{k} |C_i \cap AP| \geq \sum_{i=1}^{k} 1 = k.$$

To see equality, let d be the distance from s to t, and let U_i be the set of vertices at distance less than i from s, for $i = 1, \ldots, d$. Taking $C_i := \delta^{\text{out}}(U_i)$, we obtain a collection $C_1, \ldots, C_d$ as required. ∎

3. Menger's theorem

In this section we study the maximum number k of disjoint paths in a graph connecting two vertices, or two sets of vertices.

Let $D = (V, A)$ be a directed graph and let S and T be subsets of V. A path is called an $S-T$ *path* if it runs from a vertex in S to a vertex in T.

Menger [7] gave a min-max theorem for the maximum number of disjoint $S-T$ paths. We follow the proof given by Göring [6].

Call a set of paths *vertex-disjoint* if no two of them have vertices in common. (Hence they also have no arcs in common.) A set C of vertices is called $S-T$ *disconnecting* if C intersects each $S-T$ path (C may intersect $S \cup T$).

Theorem 3 (Menger's theorem (directed vertex-disjoint version))**.** *Let $D = (V, A)$ be a digraph and let $S, T \subseteq V$. Then the maximum number of vertex-disjoint $S-T$ paths is equal to the minimum size of an $S-T$ disconnecting vertex set.*

Proof. Obviously, the maximum does not exceed the minimum. Equality is shown by induction on $|A|$, the case $A = \emptyset$ being trivial.

Let k be the minimum size of an $S-T$ disconnecting vertex set. Choose $a = (u, v) \in A$. Let $D' := (V, A \setminus \{a\})$. If each $S-T$ disconnecting vertex set in D' has size at least k, then inductively there exist k vertex-disjoint $S-T$ paths in D', hence in D.

So we can assume that D' has an $S-T$ disconnecting vertex set C of size $\leq k-1$. Then $C \cup \{u\}$ and $C \cup \{v\}$ are $S-T$ disconnecting vertex sets of D of size k.

Now each $S-(C \cup \{u\})$ disconnecting vertex set B of D' has size at least k, as it is $S-T$ disconnecting in D. Indeed, each $S-T$ path P in D intersects $C \cup \{u\}$, and hence P contains an $S-(C \cup \{u\})$ path in D'. So P intersects B.

So by induction, D' contains k disjoint $S-(C \cup \{u\})$ paths. Similarly, D' contains k disjoint $(C \cup \{v\})-T$ paths. Any path in the first collection intersects any path in the second collection only in C, since otherwise D' contains an $S-T$ path avoiding C.

Hence, as $|C| = k-1$, we can pairwise concatenate these paths to obtain disjoint $S-T$ paths, inserting arc a between the path ending at u and the path starting at v. ∎

A consequence of this theorem is a variant on *internally vertex-disjoint* $s-t$ paths, that

[3] AP denotes the set of arcs traversed by P.

is, $s-t$ paths no two of which have a vertex in common except for s and t. A set U of vertices is called an $s-t$ *vertex-cut* if $s,t \notin U$ and each $s-t$ path intersects U.

Corollary 3a (Menger's theorem (directed internally vertex-disjoint version))**.** *Let $D=(V,A)$ be a digraph and let s and t be two nonadjacent vertices of D. Then the maximum number of internally vertex-disjoint $s-t$ paths is equal to the minimum size of an $s-t$ vertex-cut.*

Proof. Let $D' := D - s - t$ and let S and T be the sets of outneighbours of s and of inneighbours of t, respectively. Then Theorem 3 applied to D', S, T gives the corollary. ∎

In turn, Theorem 3 follows from Corollary 3a by adding two new vertices s and t and arcs (s,v) for all $v \in S$ and (v,t) for all $v \in T$.

Also an arc-disjoint version can be derived, where paths are *arc-disjoint* if they have no arc in common. Recall that a set C of arcs is an $s-t$ *cut* if $C = \delta^{\text{out}}(U)$ for some subset U of V with $s \in U$ and $t \notin U$.

Corollary 3b (Menger's theorem (directed arc-disjoint version))**.** *Let $D=(V,A)$ be a digraph and let $s,t \in V$. Then the maximum number of arc-disjoint $s-t$ paths is equal to the minimum size of an $s-t$ cut.*

Proof. Let $L(D)$ be the line digraph of D.[4] Let $S := \delta_A^{\text{out}}(s)$ and $T := \delta_A^{\text{in}}(t)$. Then Theorem 3 for $L(D), S, T$ implies the corollary. Note that a minimum-size set of arcs intersecting each $s-t$ path necessarily is an $s-t$ cut. ∎

The internally vertex-disjoint version of Menger's theorem can be derived in turn from the arc-disjoint version: make a digraph D' as follows from D: replace any vertex v by two vertices v', v'' and make an arc (v', v''); moreover, replace each arc (u,v) by (u'', v'). Then Corollary 3b for D', s'', t' gives Corollary 3a for D, s, t.

Similar theorems hold for *undirected* graphs. They can be derived from the directed case by replacing each undirected edge uw by two opposite arcs (u,w) and (w,u).

Exercises

3.1. Let $D=(V,A)$ be a directed graph and let $s, t_1, \ldots, t_k$ be vertices of D. Prove that there exist arc-disjoint paths $P_1, \ldots, P_k$ such that P_i is an $s-t_i$ path ($i=1,\ldots,k$) if and only if for each $U \subseteq V$ with $s \in U$ one has

$$(3) \qquad |\delta^{\text{out}}(U)| \geq |\{i \mid t_i \notin U\}|.$$

3.2. Let $\mathcal{A} = (A_1, \ldots, A_n)$ and $\mathcal{B} = (B_1, \ldots, B_n)$ be families of subsets of a finite set. Show that $\mathcal{A}$ and $\mathcal{B}$ have a common SDR if and only if for all $I, J \subseteq \{1, \ldots, n\}$ one has

$$(4) \qquad \Big|\bigcup_{i \in I} A_i \cap \bigcup_{j \in J} B_j\Big| \geq |I| + |J| - n.$$

[4] The *line digraph* of a digraph $D=(V,A)$ is the digraph with vertex set A and arcs set $\{(a,a') \mid a, a' \in A, \text{head}(a) = \text{tail}(a')\}$.

4. Flows in networks

Other consequences of Menger's theorem concern 'flows in networks'. Let $D = (V, A)$ be a directed graph and let $s, t \in V$. A function $f : A \to \mathbb{R}$ is called an $s - t$ *flow* if:[5]

$$(5) \qquad \begin{array}{llcll} \text{(i)} & f(a) & \geq & 0 & \text{for each } a \in A; \\ \text{(ii)} & \displaystyle\sum_{a \in \delta^{\text{in}}(v)} f(a) & = & \displaystyle\sum_{a \in \delta^{\text{out}}(v)} f(a) & \text{for each } v \in V \setminus \{s, t\}. \end{array}$$

Condition (5)(ii) is called the *flow conservation law*: the amount of flow entering a vertex $v \neq s, t$ should be equal to the amount of flow leaving v.

The *value* of an $s - t$ flow f is, by definition:

$$(6) \qquad \text{value}(f) := \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a).$$

So the value is the net amount of flow leaving s. It can be shown that it is equal to the net amount of flow entering t.

Let $c : A \to \mathbb{R}_+$, called a *capacity function*. We say that a flow f is *under* c (or *subject to* c) if

$$(7) \qquad f(a) \leq c(a) \text{ for each } a \in A;$$

that is, if $f \leq c$. The *maximum flow problem* now is to find an $s - t$ flow under c, of maximum value.

To formulate a min-max relation, define the *capacity* of a cut $\delta^{\text{out}}(U)$ by:

$$(8) \qquad c(\delta^{\text{out}}(U)) := \sum_{a \in \delta^{\text{out}}(U)} c(a).$$

Then:

Proposition 1. *For every $s - t$ flow f under c and every $s - t$ cut $\delta^{\text{out}}(U)$ one has:*

$$(9) \qquad \text{value}(f) \leq c(\delta^{\text{out}}(U)).$$

Proof.

$$(10) \qquad \begin{aligned} \text{value}(f) &= \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) \\ &= \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) + \sum_{v \in U \setminus \{s\}} \Big(\sum_{a \in \delta^{\text{out}}(v)} f(a) - \sum_{a \in \delta^{\text{in}}(v)} f(a) \Big) \end{aligned}$$

[5] $\delta^{\text{out}}(v)$ and $\delta^{\text{in}}(v)$ denote the sets of arcs leaving v and entering v, respectively.

$$= \sum_{v \in U} \Big(\sum_{a \in \delta^{\text{out}}(v)} f(a) - \sum_{a \in \delta^{\text{in}}(v)} f(a) \Big) = \sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a)$$
$$\stackrel{\star}{\leq} \sum_{a \in \delta^{\text{out}}(U)} f(a) \stackrel{\star\star}{\leq} \sum_{a \in \delta^{\text{out}}(U)} c(a) = c(\delta^{\text{out}}(U)).$$

∎

It is convenient to note the following:

(11) $$\text{equality holds in (9)} \iff \begin{array}{l} \forall a \in \delta^{\text{in}}(U) : f(a) = 0 \text{ and} \\ \forall a \in \delta^{\text{out}}(U) : f(a) = c(a). \end{array}$$

This follows directly from the inequalities $\star$ and $\star\star$ in (10).

Now from Menger's theorem one can derive that equality can be attained in (9), which is a theorem of Ford and Fulkerson [4]:

Theorem 4 (max-flow min-cut theorem)**.** *For any directed graph $D = (V, A)$, $s, t \in V$, and $c : A \to \mathbb{R}_+$, the maximum value of an $s - t$ flow under c is equal to the minimum capacity of an $s - t$ cut. In formula:*

(12) $$\max_{\substack{f\ s-t\text{-flow} \\ f \leq c}} \text{value}(f) = \min_{\delta^{\text{out}}(U)\ s-t\text{-cut}} c(\delta^{\text{out}}(U)).$$

Proof. If c is integer-valued, the corollary follows from Menger's theorem by replacing each arc a by $c(a)$ parallel arcs. If c is rational-valued, there exists a natural number N such that $Nc(a)$ is integer for each $a \in A$. This resetting multiplies both the maximum and the minimum by N. So the equality follows from the case where c is integer-valued.

If c is real-valued, we can derive the corollary from the case where c is rational-valued, by continuity and compactness arguments, as follows. Suppose that

(13) $$\max_{\substack{f\ s-t \text{ flow} \\ f \leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U)\ s-t \text{ cut}} c(\delta^{\text{out}}(U)).$$

(The maximum exists, as the set of $s - t$ flows f with $f \leq c$ is compact.)

Then we can choose a rational-valued $c' \leq c$ close enough to c such that

(14) $$\max_{\substack{f\ s-t \text{ flow} \\ f \leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U)\ s-t \text{ cut}} c'(\delta^{\text{out}}(U)).$$

So

(15) $$\max_{\substack{f\ s-t \text{ flow} \\ f \leq c'}} \text{value}(f) \leq \max_{\substack{f\ s-t \text{ flow} \\ f \leq c}} \text{value}(f) < \min_{\delta^{\text{out}}(U)\ s-t \text{ cut}} c'(\delta^{\text{out}}(U)).$$

This contradicts the above, as c' is rational. ∎

Moreover, one has (Dantzig [1]):

Corollary 4a (Integrity theorem)**.** *If c is integer-valued, there exists an integer-valued maximum-value flow $f \leq c$.*

Proof. Directly from Menger's theorem. ∎

Exercises

4.1. Let $D = (V, A)$ be a directed graph and let $s, t \in V$. Let $f : A \to \mathbb{R}_+$ be an $s-t$ flow of value β. Show that there exists an $s-t$ flow $f' : A \to \mathbb{Z}_+$ of value $\lceil \beta \rceil$ such that $\lfloor f(a) \rfloor \leq f'(a) \leq \lceil f(a) \rceil$ for each arc a.

5. Finding a maximum flow

Let $D = (V, A)$ be a directed graph, let $s, t \in V$, and let $c : A \to \mathbb{Q}_+$ be a 'capacity' function. We now describe the algorithm of Ford and Fulkerson [4] to find an $s - t$ flow of maximum value under c.

By *flow* we will mean an $s - t$ flow under c, and by *cut* an $s - t$ cut. A *maximum flow* is a flow of maximum value.

We now describe the algorithm of Ford and Fulkerson [5] to determine a maximum flow. We assume that $c(a) > 0$ for each arc a. First we give an important subroutine:

Flow augmenting algorithm

input: a flow f.
output: either (i) a flow f' with value$(f') >$ value(f),
or (ii) a cut $\delta^{\text{out}}(U)$ with $c(\delta^{\text{out}}(U)) =$ value(f).
description of the algorithm: For any pair $a = (v, w)$ define $a^{-1} := (w, v)$. Make an auxiliary graph $D_f = (V, A_f)$ by the following rule: for any arc $a \in A$,

(16) if $f(a) < c(a)$ then $a \in A_f$,
if $f(a) > 0$ then $a^{-1} \in A_f$.

So if $0 < f(a) < c(a)$ then both a and a^{-1} are arcs of A_f.

Now there are two possibilities:

(17) **Case 1:** *There exists an $s - t$ path in D_f.*
Case 2: *There is no $s - t$ path in D_f.*

Case 1: *There exists an $s - t$ path $P = (v_0, a_1, v_1, \ldots, a_k, v_k)$ in $D_f = (V, A_f)$.*
So $v_0 = s$ and $v_k = t$. We may assume that P is a simple path. As $a_1, \ldots, a_k$ belong to A_f, we know by (16) that for each $i = 1, \ldots, k$:

(18) either (i) $a_i \in A$ and $\sigma_i := c(a_i) - f(a_i) > 0$
or (ii) $a_i^{-1} \in A$ and $\sigma_i := f(a_i^{-1}) > 0$.

Define $\alpha := \min\{\sigma_1, \ldots, \sigma_k\}$. So $\alpha > 0$. Let $f' : A \to \mathbb{R}_+$ be defined by, for $a \in A$:

$$
(19) \qquad f'(a) := \begin{cases} f(a) + \alpha & \text{if } a = a_i \text{ for some } i = 1, \ldots, k; \\ f(a) - \alpha & \text{if } a = a_i^{-1} \text{ for some } i = 1, \ldots, k; \\ f(a) & \text{for all other } a. \end{cases}
$$

Then f' again is an $s - t$ flow under c. The inequalities $0 \le f'(a) \le c(a)$ hold because of our choice of α. It is easy to check that also the flow conservation law (5)(ii) is maintained.

Moreover,

$$
(20) \qquad \text{value}(f') = \text{value}(f) + \alpha,
$$

since either $(v_0, v_1) \in A$, in which case the outgoing flow in s is increased by α, or $(v_1, v_0) \in A$, in which case the ingoing flow in s is decreased by α.

Path P is called a *flow augmenting path.*

Case 2: *There is no $s - t$ path in $D_f = (V, A_f)$.*
Now define:

$$
(21) \qquad U := \{u \in V \mid \text{there exists a path in } D_f \text{ from } s \text{ to } u\}.
$$

Then $s \in U$ while $t \notin U$, and so $\delta^{\text{out}}(U)$ is an $s - t$ cut.

By definition of U, if $u \in U$ and $v \notin U$, then $(u, v) \notin A_f$ (as otherwise also v would belong to U). Therefore:

$$
(22) \qquad \begin{array}{l} \text{if } (u, v) \in \delta^{\text{out}}(U), \text{ then } (u, v) \notin A_f, \text{ and so (by (16)): } f(u, v) = c(u, v), \\ \text{if } (u, v) \in \delta^{\text{in}}(U), \text{ then } (v, u) \notin A_f, \text{ and so (by (16)): } f(u, v) = 0. \end{array}
$$

Then (11) gives:

$$
(23) \qquad c(\delta^{\text{out}}(U)) = \text{value}(f). \qquad \blacksquare
$$

This finishes the description of the flow augmenting algorithm. The description of the *(Ford-Fulkerson) maximum flow algorithm* is now simple:

Maximum flow algorithm

input: directed graph $D = (V, A), s, t \in V, c : A \to \mathbb{R}_+$.
output: a maximum flow f and a cut $\delta^{\text{out}}(U)$ of minimum capacity, with $\text{value}(f) = c(\delta^{\text{out}}(U))$.
description of the algorithm: Let f_0 be the 'null flow' (that is, $f_0(a) = 0$ for each arc a). Determine with the flow augmenting algorithm flows $f_1, f_2, \ldots, f_N$ such that $f_{i+1} = f_i'$, until, in the Nth iteration, say, we obtain output (ii) of the flow augmenting algorithm. Then we have flow f_N and a cut $\delta^{\text{out}}(U)$ with the given properties. $\blacksquare$

We show that the algorithm terminates, provided that all capacities are rational.

Theorem 5. *If all capacities $c(a)$ are rational, the algorithm terminates.*

Proof. If all capacities are rational, there exists a natural number K so that $Kc(a)$ is an integer for each $a \in A$. (We can take for K the l.c.m. of the denominators of the $c(a)$.)

Then in the flow augmenting iterations, every flow $f_i(a)$ and every α is a multiple of $1/K$. So at each iteration, the flow value increases by at least $1/K$. Since the flow value cannot exceed $c(\delta^{\text{out}}(s))$, we can have only finitely many iterations. ∎

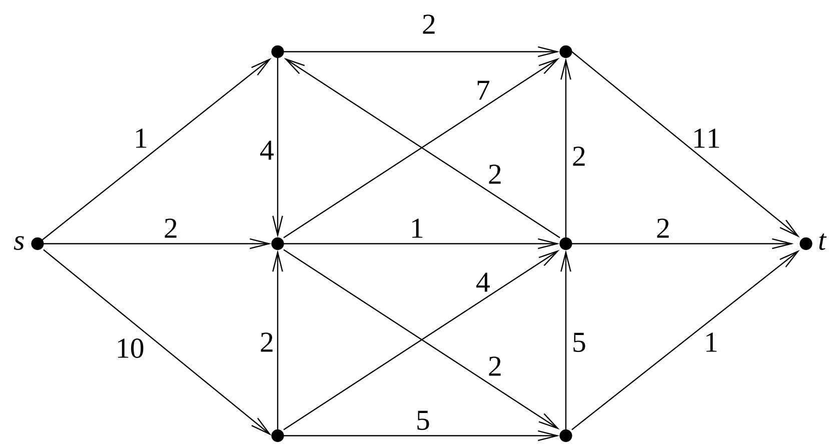
We note here that this theorem is not true if we allow general real-valued capacities. On the other hand, it was shown by Dinits [2] and Edmonds and Karp [3] that if we choose always a shortest path as flow augmenting path, then the algorithm has polynomially bounded running time (also in the case of irrational capacities).

Note that the algorithm also implies the max-flow min-cut theorem (Theorem 4). Note moreover that in the maximum flow algorithm, if all capacities are integer, then the maximum flow found will also be integer-valued. So it also implies the integrity theorem (Corollary 4a).

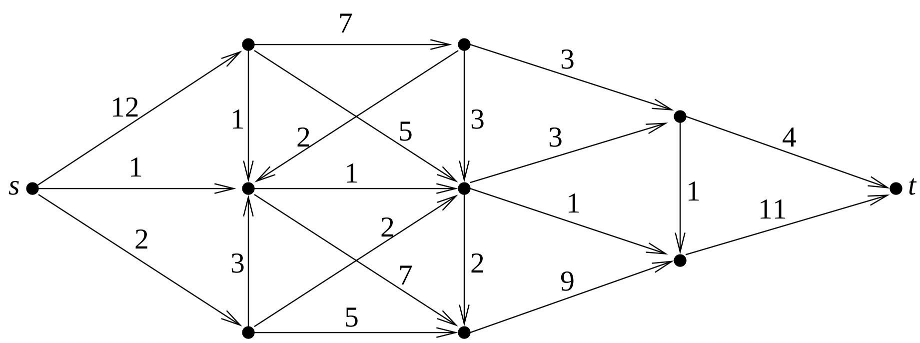
Exercises

5.1. Determine with the maximum flow algorithm an $s-t$ flow of maximum value and an $s-t$ cut of minimum capacity in the following graphs (where the numbers at the arcs give the capacities):

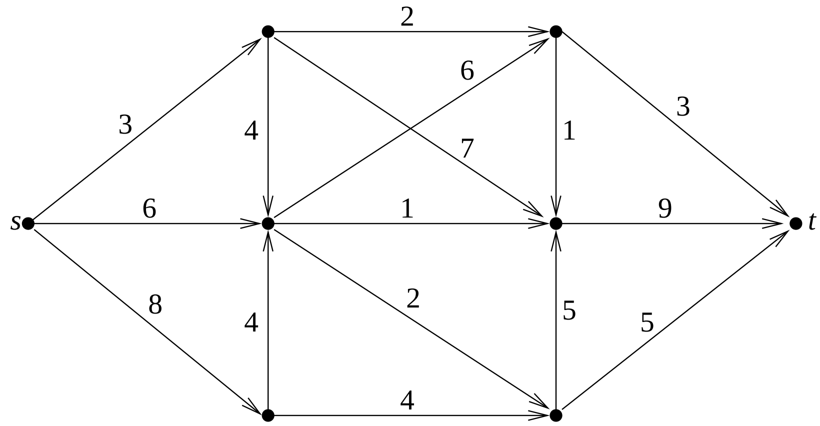
(i)



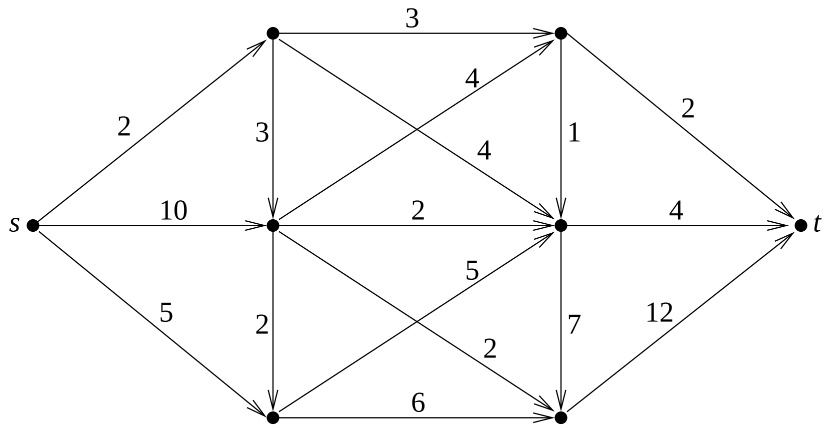
(ii)



(iii)



(iv)



5.2. Describe the problem of finding a maximum-size matching in a bipartite graph as a maximum integer flow problem.

5.3. Let $D = (V, A)$ be a directed graph, let $s, t \in V$ and let $f : A \to \mathbb{Q}_+$ be an $s-t$ flow of value b. Show that for each $U \subseteq V$ with $s \in U, t \notin U$ one has:

$$\sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a) = b. \tag{24}$$

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