Homework 1

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Source: Github repository

1 Task 1

Simulation link

Solution:

We can start by calculating $y(x)$ from given parametric equations:

$$\vec{r}(t) = \begin{bmatrix} 3t \\ 4t^2 + 1 \end{bmatrix} \tag{1}$$

Convert through expressing t in terms of x and substituting to y:

$$t = \frac{1}{3}x \tag{2}$$

$$y(x) = 4\left(\frac{1}{3}x\right)^2 + 1 \tag{3}$$

$$y(x) = \frac{4}{9}x^2 + 1 \tag{4}$$

Calculating velocity and acceleration can be done through differentiation:

$$\frac{dr}{dt}(t) = \vec{v}(t) = \begin{bmatrix} 3 \\ 8t \end{bmatrix} \tag{5}$$

$$v(t) = \sqrt{3^2 + 8^2t^2} = \sqrt{9 + 64t^2} \tag{6}$$

$$\frac{dv}{dt}(t) = \vec{a}(t) = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \tag{7}$$

$$a(t) = \sqrt{0^2 + 8^2} = 8 \tag{8}$$

Tangential acceleration can be calculating by taking the dot product of velocity and acceleration:

$$a_t(t) = ||\vec{v}(t)|| \cdot ||\vec{a}(t)|| = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8t \end{bmatrix} * \frac{1}{\sqrt{9 + 64t^2}} = \frac{64t}{\sqrt{9 + 64t^2}} \tag{9}$$

Simulation hint: we can find vectorized tangential acceleration by multiplying unit vector of velocity by scalar value of acceleration:

$$\vec{a}_t(t) = \frac{1}{||\vec{v}(t)||} \cdot \vec{v}(t) \cdot a_t(t) = \frac{64t}{9 + 64t^2} \cdot \frac{1}{\sqrt{9 + 64t^2}} \cdot \begin{bmatrix} 3 \\ 8t \end{bmatrix} \tag{10}$$

Normal acceleration is simply the difference between acceleration and tangential acceleration:

$$\vec{a}_n(t) = \vec{a}(t) - \vec{a}_t(t) \tag{11}$$

But for usual calculation without simulation we could do it this way, by taking the cross product of velocity and acceleration:

$$\vec{a}_n(t) = \frac{||\vec{v}(t) \times \vec{a}(t)||}{v(t)} = || \begin{bmatrix} 3 \\ 8t \end{bmatrix} \times \begin{bmatrix} 0 \\ 8 \end{bmatrix} || \cdot \frac{1}{\sqrt{9 + 64t^2}} = \tag{12}$$

$$|| \begin{bmatrix} 0 \\ 0 \\ -24t \end{bmatrix} || \cdot \frac{1}{\sqrt{9 + 64t^2}} = \frac{24t}{\sqrt{9 + 64t^2}} \tag{13}$$

We can find curvature using this formula:

$$k(t) = \frac{a_n}{v(t)^2} = \frac{24t}{\sqrt{9 + 64t^2}} \cdot \frac{1}{9 + 64t^2} = \frac{24t}{(9 + 64t^2)^{\frac{3}{2}}} \tag{14}$$

Answer:

1. $y(x) = \frac{4}{9}x^2 + 1$

2. $v(\vec{t}) = \begin{bmatrix} 3 \\ 8t \end{bmatrix}$

3. $a(\vec{t}) = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

4. $\boxed{a_t(t) = \frac{64t}{\sqrt{9+64t^2}}}$

5. $\boxed{a_n(t) = \frac{24t}{\sqrt{9+64t^2}}}$

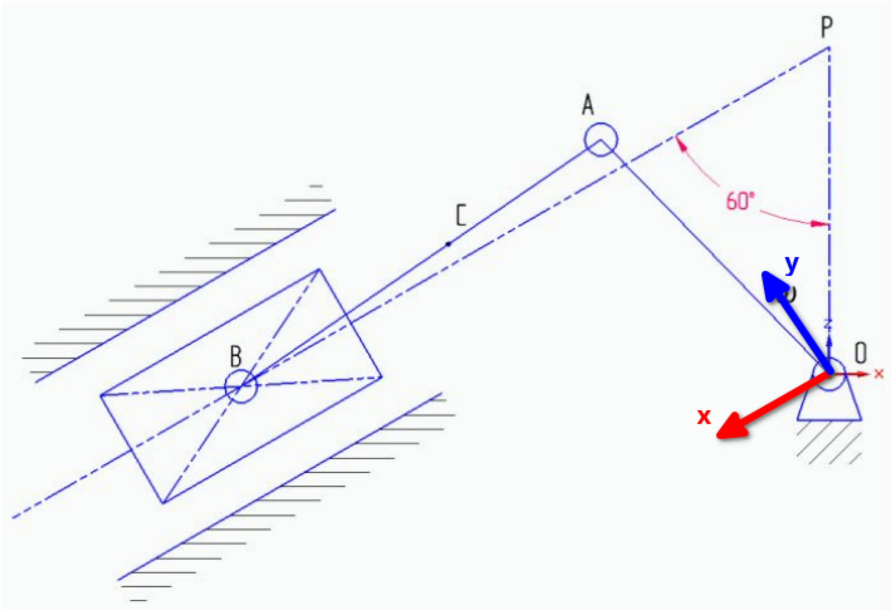
6. $\boxed{k(t) = \frac{24t}{(9+64t^2)^{\frac{3}{2}}}}$

2 Task 2

Simulation link

Solution:

First of all I decided to change the coordinate system to make the problem easier. I will use the following coordinate system:



Let's start with point A:

Its position can easily be described as parametric equation of a circle:

$$\vec{A(t)} = OA \cdot \begin{bmatrix} sin(\phi) \\ cos(\phi) \end{bmatrix} \tag{15}$$

Velocity can be calculated through given angular velocity:

$$\vec{v_A(t)} = \omega \cdot \begin{bmatrix} -cos(\phi) \\ sin(\phi) \end{bmatrix} \tag{16}$$

As angular acceleration is zero ($w = const$), we will have only normal acceleration:

$$\vec{a_A(t)} = \vec{\omega} \times (\vec{\omega} \times \vec{A(t)}) \tag{17}$$

Now we can describe point B depending on input angle ϕ:

$$\vec{B} = \begin{bmatrix} PB - OP \cdot \cos\frac{\pi}{3} \\ OP \cdot \cos\frac{\pi}{3} \end{bmatrix} = \ OA \cdot \begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix} + \ AB \cdot \begin{bmatrix} \sqrt{1 - \cos^2(\gamma)} \\ \cos(\gamma) \end{bmatrix} \tag{18}$$

In this formula we have two unknowns: γ and PB. We can find γ by using the fact that:

$$\cos(\gamma) = \frac{OP \cdot \cos\frac{\pi}{3} - OA\cos\phi}{AB} \tag{19}$$

Let me introduce very interesting fact:

$$\sin(arccos(x)) = \sqrt{1 - x^2} \tag{20}$$

Relax, won't bother you: explanation
We are ready to calculate x coordinate of point B:

$$x_B = OA \cdot \sin(\phi) + AB \cdot \sin(\gamma) \tag{21}$$

$$x_B = OA \cdot \sin(\phi) + AB \cdot \sqrt{1 - (\frac{OP \cdot \cos\frac{\pi}{3} - OA\cos\phi}{AB})^2} \tag{22}$$

$$x_B = OA \cdot \sin(\phi) + \sqrt{AB^2 - (OP \cdot \cos\frac{\pi}{3} - OA\cos\phi)^2} \tag{23}$$

That's it boom! We have found x coordinate of point B.

Velocity and acceleration will be simply calculated by differentiation of position and I did it using online calculators. The only valuable information is that they have only x components.

Calculation of point C:

We know that point C lies on the line AB and we know that $AB = 80, AC = 20$.

So point C is basically segment of line.

$$\vec{C} = \vec{A} + (\vec{B} - \vec{A}) \cdot \frac{AB}{AC} \tag{24}$$

Proportions will be the same for velocity and accelerations:

$$\vec{v_C} = \vec{v_A} + (\vec{v_B} - \vec{v_A}) \cdot \frac{AB}{AC} \tag{25}$$

$$\vec{a_C} = \vec{a_A} + (\vec{a_B} - \vec{a_A}) \cdot \frac{AB}{AC} \tag{26}$$

Tangential acceleration is aligned with velocity vector:

$$\vec{a_{Ct}} = \vec{a_C} \cdot \frac{\vec{v_C}}{||\vec{v_C}||} \tag{27}$$

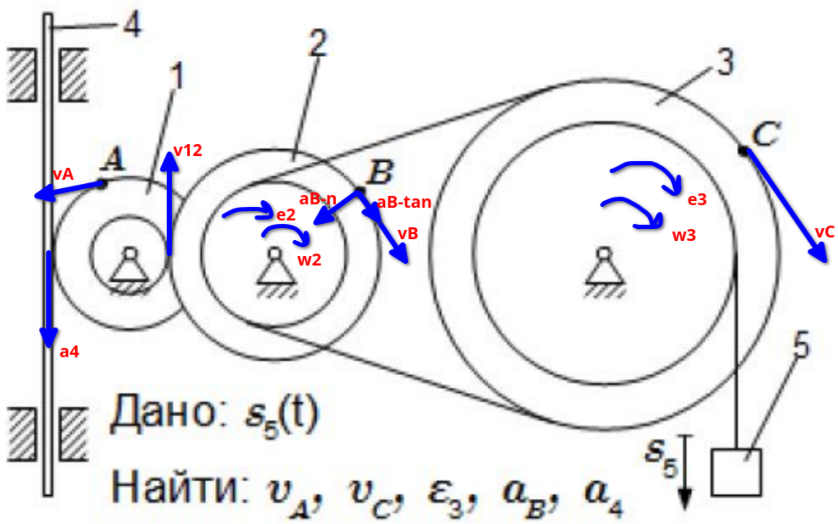
Normal acceleration is the difference between acceleration and tangential acceleration:

$$\vec{a_{Cn}} = \vec{a_C} - \vec{a_{Ct}} \tag{28}$$

That's it. We have found all the required values.
Answer:

1. $x_B(t) = OA \cdot \sin(\phi) + \sqrt{AB^2 - (OP \cdot \cos\frac{\pi}{3} - OA\cos\phi)^2}$
2. $v_B(t) = \frac{\mathrm{d}x_B(t)}{\mathrm{d}t}$
3. $a_B(t) = \frac{\mathrm{d}v_B(t)}{\mathrm{d}t}$
4. $\vec{C(t)} = \vec{A} + (\vec{B} - \vec{A}) \cdot \frac{AB}{AC}$
5. $\vec{v_C(t)} = \vec{v_A} + (\vec{v_B} - \vec{v_A}) \cdot \frac{AB}{AC}$
6. $\vec{a_C(t)} = \vec{a_A} + (\vec{a_B} - \vec{a_A}) \cdot \frac{AB}{AC}$

3 Task 3



Solution:

Given s_5 law of motion we can use it to propage through all the connections of the system:

Obviously, we can observe that point on inner 3_{rd} wheel will have the same speed. Using this fact, we can get angular velocity of the 3_{rd} wheel:

$$\omega_3(t) = \frac{3t^2 - 6}{r_3} \tag{29}$$

Wheels 2 and 3 are connected using belt, so we can express angular velocity of the 2_{nd} wheel in terms of 3_{rd} wheel:

$$\omega_2(t) \cdot r_2 = \omega_3(t) \cdot R_3 \tag{30}$$

The same idea between 1_{st} and 2_{nd} wheels is the same:

$$\omega_1(t) \cdot r_1 = \omega_2(t) \cdot R_2 \tag{31}$$

These equations mainly give us everything to find required variables:

1. Velocity for point A (on outer radius of the wheel $\implies$ uses R_1):

$$\begin{aligned} v_A(t) = \omega_1(t) \cdot R_1 = \frac{R_2}{r_1} \cdot \omega_2(t) \cdot R_1 = \ & \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \omega_3(t) \cdot R_1 = \\ & \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot R_1 \end{aligned} \tag{32}$$

At time $t = 2$ we have: $v_A(2) = \frac{8}{2} \cdot \frac{16}{6} \cdot \frac{6}{12} \cdot 4 = \frac{64}{3} \approx 21.33$
2. Velocity of point C (on outer radius of the wheel $\implies$ uses R_3):

$$v_C(t) = \omega_3(t) \cdot R_3 = \frac{3t^2 - 6}{r_3} \cdot R_3 \tag{33}$$

At time $t = 2$ we have: $v_C(2) = \frac{6}{12} \cdot 4 = \frac{24}{3} = 8$
3. Angular acceleration of 3_{rd} wheel.

$$\epsilon_3(t) = \frac{\mathrm{d}w_3(t)}{\mathrm{d}t} = \frac{6t}{r_3} \tag{34}$$

At time $t = 2$ we have: $\epsilon_3(2) = \frac{12}{12} = 1$
4. Acceleration of B:
We can start by determining angular speed and angular acceleration of the 2_{nd} wheel

$$\omega_2(t) = \frac{R_3}{r_2} \cdot \omega_3(t) = \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \tag{35}$$

$$\epsilon_2(t) = \frac{\mathrm{d}w_2(t)}{\mathrm{d}t} = \frac{R_3}{r_2} \cdot \epsilon_3(t) = \frac{R_3}{r_2} \cdot \frac{6t}{r_3} \tag{36}$$

Now we can apply basic transformations to find linear components of accelerations:

$$a_{B\tau}(t) = \epsilon_2(t) \times R_2 = \frac{R_3}{r_2} \cdot \frac{6t}{r_3} \cdot R_2 \tag{37}$$

$$a_{Bn}(t) = \omega_2(t) \times (\omega_2(t) \times R_2) = \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot R_2 \tag{38}$$

$$a_B(t) = \sqrt{a_{B\tau}^2 + a_{Bn}^2} \tag{39}$$

At time $t = 2$ we have:

$$a_{B\tau}(2) = \frac{16}{6} \cdot \frac{12}{12} \cdot 8 = \frac{64}{3} \tag{40}$$

$$a_{Bn}(2) = \frac{16}{6} \cdot \frac{6}{12} \cdot \frac{16}{6} \cdot \frac{6}{12} \cdot 8 = \frac{128}{9} \tag{41}$$

$$a_B(2) = \sqrt{\frac{64}{3}^2 + \frac{128}{9}^2} = 25.64 \tag{42}$$

5. Acceleration of rack 4:
We see that it is connected without slippering to the 1_{st} wheel. So we can express velocity of the bar in terms of angular velocity of the wheel:

$$v_4(t) = v_A(t) = \omega_1(t) \cdot R_1 = \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot R_1 \tag{43}$$

$$a_4(t) = \frac{\mathrm{d}v_4(t)}{\mathrm{d}t} = \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \frac{6t}{r_3} \cdot R_1 \tag{44}$$

At time $t = 2$ we have:

$$a_4(2) = \frac{8}{2} \cdot \frac{16}{6} \cdot \frac{12}{12} \cdot 4 = \frac{128}{3} = 42.67 \tag{45}$$

Answer:

1. $v_A(2) = \frac{64}{3} \approx 21.33$
2. $v_C(2) = 8$
3. $\epsilon_3(2) = 1$
4. $a_{B\tau}(2) = \frac{64}{3}$ (tangent to outer radius of $2nd$ wheel)
5. $a_{Bn}(2) = \frac{128}{9}$ (normal to outer radius of $2nd$ wheel)
6. $a_B(2) = 25.64$
7. $a_4(2) = 42.67$