

Homework 2

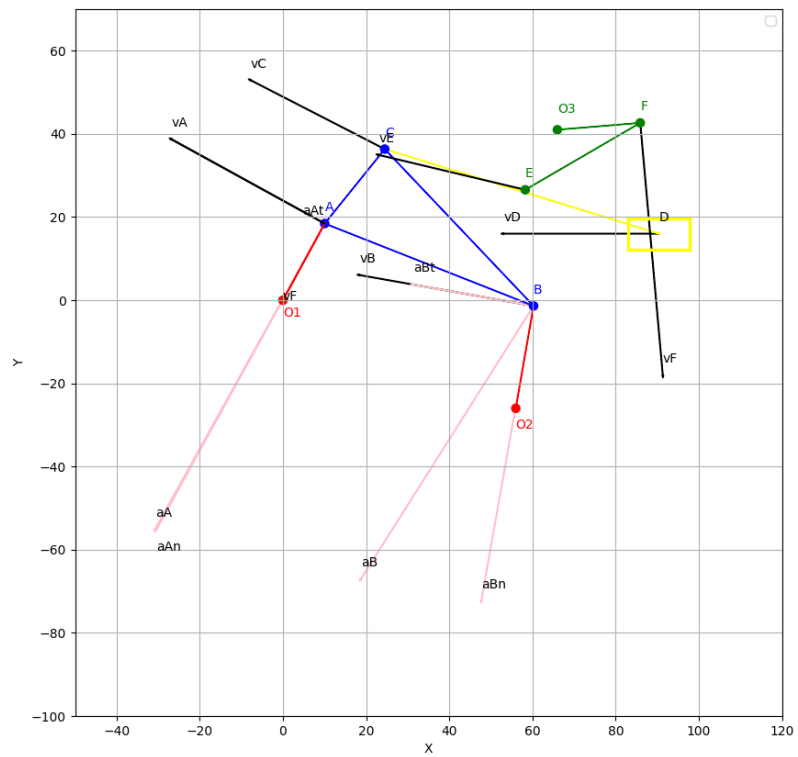
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Homework 2 webpage

1 Task 1

simulation



1.1 Derivation of points positions

1. I assume that point O_1 is positioned in the $(0, 0)$.
2. We can describe the position of point A as it rotates around point O_1 as follows:

$$\vec{r}_A(t) = O_1A \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \quad (1)$$

3. Position of O_2 is $(-c, a)$ as follows from task description.
4. We can describe the position of point B as intersection between two circles: first with center at A and radius AB and second with center at O_2 and radius O_2A :

$$\begin{cases} (x_B - x_A)^2 + (y_B - y_A)^2 = AB^2 \\ (x_B - x_{O_2})^2 + (y_B - y_{O_2})^2 = O_2A^2 \end{cases} \quad (2)$$

These equations can be solved easily with solvers like *Sympy* for *Python*. I faced several problems with this step:

- (a) Mechanism does not work for all angles of ϕ . Distance between A and O_2 is at maximum:

$$(O_1A \cdot \cos(\phi) - a)^2 + (O_1A \cdot \sin(\phi) + c)^2 = (AB + O_2B)^2 \quad (3)$$

This will happen when O_2BA will form a straight line. Wolframagic solution link We get 2 angles, between them will be unreachable angles.

$$\phi_{unreachable} \in (2 \cdot (3.1416n - 1.4395), 2 \cdot (3.1416n + 1.0049)), n \in \mathbb{Z} \quad (4)$$

- (b) We have 2 solutions from quadratic equations and have to choose one: I select the most right and top (greatest x and y coordinates) point because it looks nicer and closer to starting position from picture. I will follow this approach for other cases too.
5. We can describe the position of point C as intersection between two circles: first with center at B and radius BC and second with center at A and radius AB :

$$\begin{cases} (x_C - x_B)^2 + (y_C - y_B)^2 = BC^2 \\ (x_C - x_A)^2 + (y_C - y_A)^2 = AB^2 \end{cases} \quad (5)$$

6. We can describe the position of point D as intersection between circle (center C radius CD) and a line because point D has only translational motion on the fixed axis:

$$\begin{cases} (x_D - x_C)^2 + (y_D - y_C)^2 = CD^2 \\ y_D = d \end{cases} \quad (6)$$

7. Point E is on CD segment and can be found as follows:

$$\vec{r}_E(t) = \vec{r}_C(t) + \frac{\vec{r}_D(t) - \vec{r}_C(t)}{CD} CE \quad (7)$$

8. Point F can be found as intersection of two circles: one with center E and radius EF and second with center O_3 and radius O_3F :

$$\begin{cases} (x_F - x_E)^2 + (y_F - y_E)^2 = EF^2 \\ (x_F - x_{O_3})^2 + (y_F - y_{O_3})^2 = O_3F^2 \end{cases} \quad (8)$$

1.2 Derivation of velocities for points

meme link

1. Velocity of point A :

$$\vec{v}_A(t) = \dot{\vec{r}}_A(t) = \omega \cdot \vec{r}_A(t) \quad (9)$$

2. Velocity of point B can be found as follows:

$$\vec{v}_B(t) = \dot{\vec{r}}_B(t) \quad (10)$$

3. Velocity of point C can be found as follows:

$$\vec{v}_C(t) = \dot{\vec{r}}_C(t) \quad (11)$$

4. Velocity of point D can be found as follows:

$$\vec{v}_D(t) = \dot{\vec{r}}_D(t) \quad (12)$$

5. Velocity of point E can be found as follows:

$$\vec{v}_E(t) = \dot{\vec{r}}_E(t) \quad (13)$$

6. Velocity of point F can be found as follows:

$$\vec{v}_F(t) = \dot{\vec{r}}_F(t) \quad (14)$$

1.3 Derivation of angular velocities

Let's start with meme:



1. We could use IC(instantaneous centre of zero velocity to simplify our calculations, but I will not do it because we already have all velocities and positions of points and it is enough.

2. For angular velocities we can use this formula:

$$\vec{v}_1(t) = \vec{v}_2(t) + \vec{\omega}(t) \times \vec{r}_{12}(t) \quad (15)$$

3. Angular velocity of O_1A :

(a) O_1A is a fixed point and we know from description that $\omega = 2$.

(b) But nevertheless:

$$\vec{v}_A(t) = \vec{v}_{O_1}(t) + \omega_{O_1A}(t) \times \vec{r}_{O_1A}(t) \quad (16)$$

$$\omega_{O_1A} = \frac{|\vec{v}_A(t) - \vec{v}_{O_1}(t)|}{|\vec{r}_{O_1A}(t)|} \quad (17)$$

(c) In simulation I compute both variants, and they almost equal with respect to precision of simulation and derivative computation algorithm (I use the easiest).

4. As all formulas are pretty much the same I will not comment others.

5. Angular velocity of O_2B :

$$\vec{v}_B(t) = \vec{v}_{O_2}(t) + \omega_{O_2B}(t) \times \vec{r}_{O_2B}(t) \quad (18)$$

$$\omega_{O_2B} = \frac{|\vec{v}_B(t) - \vec{v}_{O_2}(t)|}{|\vec{r}_{O_2B}(t)|} \quad (19)$$

6. Angular velocity of AB :

$$\vec{v}_B(t) = \vec{v}_A(t) + \omega_{AB}(t) \times \vec{r}_{AB}(t) \quad (20)$$

$$\omega_{AB} = \frac{|\vec{v}_B(t) - \vec{v}_A(t)|}{|\vec{r}_{AB}(t)|} \quad (21)$$

7. Angular velocity of BC and AC : As ABC form a fixed triangle, their angular velocities are equal:

$$\omega_{AB} = \omega_{BC} = \omega_{AC} \quad (22)$$

But in order to justify and prove it to myself I also compute them separately:

$$\vec{v}_C(t) = \vec{v}_B(t) + \omega_{BC}(t) \times \vec{r}_{BC}(t) \quad (23)$$

$$\omega_{BC} = \frac{|\vec{v}_C(t) - \vec{v}_B(t)|}{|\vec{r}_{BC}(t)|} \quad (24)$$

$$\vec{v}_C(t) = \vec{v}_A(t) + \omega_{AC}(t) \times \vec{r}_{AC}(t) \quad (25)$$

$$\omega_{AC} = \frac{|\vec{v}_C(t) - \vec{v}_A(t)|}{|\vec{r}_{AC}(t)|} \quad (26)$$

8. Angular velocity of CD :

$$\vec{v}_D(t) = \vec{v}_C(t) + \omega_{CD}(t) \times \vec{r}_{CD}(t) \quad (27)$$

$$\omega_{CD} = \frac{|\vec{v}_D(t) - \vec{v}_C(t)|}{|\vec{r}_{CD}(t)|} \quad (28)$$

9. Angular velocities of CE and ED : CDE are on the same "body" and their angular velocities are equal:

$$\omega_{CD} = \omega_{CE} = \omega_{ED} \quad (29)$$

10. Angular velocity of EF :

$$\vec{v}_F(t) = \vec{v}_E(t) + \omega_{EF}(t) \times r_{EF}(t) \quad (30)$$

$$\omega_{EF} = \frac{|\vec{v}_F(t) - \vec{v}_E(t)|}{|r_{EF}(t)|} \quad (31)$$

11. Angular velocity of O_3F :

$$\vec{v}_F(t) = \vec{v}_{O_3}(t) + \omega_{O_3F}(t) \times r_{O_3F}(t) \quad (32)$$

$$\omega_{O_3F} = \frac{|\vec{v}_F(t) - \vec{v}_{O_3}(t)|}{|r_{O_3F}(t)|} \quad (33)$$

1.4 Accelerations for A and B

1. Acceleration of A :

$$\vec{a}_A(t) = \dot{\vec{v}}_A(t) \quad (34)$$

We can mention that as ω is const, there will be no tangential acceleration for A . In simulation we can see very small tangential acceleration, but it is because of precision of simulation.

2. Tangential acceleration of A :

$$a_{At}(t) = \frac{\vec{a}_A(t) \cdot \vec{v}_A(t)}{|\vec{v}_A(t)|} r_A(t) \quad (35)$$

3. Normal acceleration of A :

$$a_{An}(t) = a_A(t) - a_{At}(t) \quad (36)$$

4. Acceleration of B :

$$\vec{a}_B(t) = \dot{\vec{v}}_B(t) \quad (37)$$

5. Tangential acceleration of B :

$$a_{Bt}(t) = \frac{\vec{a}_B(t) \cdot \vec{v}_B(t)}{|\vec{v}_B(t)|} r_B(t) \quad (38)$$

6. Normal acceleration of B :

$$a_{Bn}(t) = a_B(t) - a_{Bt}(t) \quad (39)$$

Answer:

All answers for this task were highlighted in the text with green background.

2 Task 2

simulation

2.1 Derivation of angle of rotation around Z

As ω_1 and ϵ_1 are given, we can convert them to $\phi(t)$:

$$\phi(t) = \phi_0 + \omega_1 \cdot t + \epsilon_1 \cdot \frac{t^2}{2} \quad (40)$$

This angle will determine the angle of rotation of cone A axis around Z axis.

2.2 Calculating axes of cones

1. For simplifications in simulation, I introduced several unit vectors for initial position.
2. Cone A axis (direction from base of cone to its vertex):

$$\vec{u}_A = \begin{bmatrix} 0 \\ -\cos \frac{\alpha_B - \alpha_A}{2} \\ -\sin \frac{\alpha_B - \alpha_A}{2} \end{bmatrix} \quad (41)$$

3. IC axis:

- (a) IC axis is axis of intersection between cone A and cone B because B is static and A moves without slipping.
- (b) All velocities of the points in A can easily be found using IC axis.
- (c) It is kinda obvious that this axis will be rotated around Z axis with angle $\phi(t)$ together with cone A .
- (d) It is enough to calculate initial vector and then rotate it around Z axis.

$$\vec{u}_{IC} = \begin{bmatrix} 0 \\ \sin(\alpha_B/2) \\ \cos(\alpha_B/2) \end{bmatrix} \quad (42)$$

4. Radius of cone A axis:

- (a) I needed this unit vector to calculate position of point M as it is simpler to split transformation on parts and then combine them.
- (b) It is simply cone A axis rotated by 90 deg around X axis.

$$\vec{u}_{rA} = \begin{bmatrix} 0 \\ -\sin(\alpha_A/2) \\ \cos(\alpha_A/2) \end{bmatrix} \quad (43)$$

2.3 Angular velocity of cone A

1. Let state radius of cone A :

$$r_A = OM_0 \sin(\alpha_A/2) \quad (44)$$

2. It is widely known formula for velocity: $\vec{v} = \vec{\omega} \times \vec{r}$.
3. I discovered, that we can do reverse: $\vec{\omega} = \frac{\vec{r} \times \vec{v}}{||\vec{r}||^2}$ source.
4. In our case it would be radius vector from cone A center axis to IC axis.

$$\vec{r}_{centerA} = r_A \cdot R_z(\phi) \cdot \vec{u}_A \quad (45)$$

$$\vec{r} = \vec{r}_{centerA} - OM_0 \cdot R_z(\phi) \cdot \vec{u}_{rA} \quad (46)$$

$$\vec{v} = \dot{\vec{r}}_{centerA} \quad (47)$$

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{||\vec{r}||^2} \quad (48)$$

2.4 Angular acceleration of cone A

Without many words:

$$\vec{\epsilon} = \dot{\vec{\omega}} \quad (49)$$

2.5 Position of point M

1. We know that point M lies on the base of cone A
2. That means we can find its position by first transforming from origin to center of base of cone A and then rotating it around cone A axis on required angle.
3. Firstly, I needed to find the angle on which point M should be rotated. Imagine that circumference of cone B is flattened into the line. Then rotation of cone A on cone B is like the wheel with radius of cone A is rotating on the line.

$$\theta_A = \frac{OM_0 \cdot \phi}{OM_0 \cdot \sin(\alpha_A/2)} = \frac{\phi}{\sin(\alpha_A/2)} \quad (50)$$

This angle determines rotation of u_{rA} around u_A .

4. Then we can find position of point M :

$$\vec{r}_M = \vec{r}_{centerA} + R_z(\phi) \cdot R_{u_A}(\theta_A) \cdot \vec{u}_{rA} \quad (51)$$

2.6 Velocity and acceleration of point M

1. Velocity of point M :

$$\vec{v}_M = \dot{\vec{r}}_M \quad (52)$$

2. Acceleration of point M :

$$\vec{a}_M = \dot{\vec{v}}_M \quad (53)$$

3. Tangential acceleration of point M :

$$\vec{a}_{tM} = \vec{a}_M - \vec{v}_M \cdot \frac{\vec{v}_M}{||\vec{v}_M||} \quad (54)$$

Answer:

1. Angular velocity of cone A :

$$\begin{aligned} \vec{r}_{centerA} &= r_A \cdot R_z(\phi) \cdot \vec{u}_A \\ \vec{r} &= \vec{r}_{centerA} - OM_0 \cdot R_z(\phi) \cdot \vec{u}_{rA} \\ \vec{v} &= \dot{\vec{r}}_{centerA} \\ \vec{\omega} &= \frac{\vec{r} \times \vec{v}}{||\vec{r}||^2} \end{aligned}$$

2. Angular acceleration of cone A :

$$\vec{\epsilon} = \dot{\vec{\omega}}$$

3. Position of point M :

$$\begin{aligned} \vec{u}_{rA} &= \begin{bmatrix} 0 \\ -\sin(\alpha_A/2) \\ \cos(\alpha_A/2) \end{bmatrix} \\ \vec{r}_M &= \vec{r}_{centerA} + R_z(\phi) \cdot R_{u_A}(\theta_A) \cdot \vec{u}_{rA} \end{aligned}$$

4. Velocity of point M :

$$\vec{v}_M = \dot{\vec{r}}_M$$

5. Acceleration of point M :

$$\vec{a}_M = \dot{\vec{v}}_M$$