

Homework 1

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1 Task 1

Solution:

We can start by calculating $y(x)$ from given parametric equations:

$$\vec{r}(t) = \begin{bmatrix} 3t \\ 4t^2 + 1 \end{bmatrix} \quad (1)$$

Convert through expressing t in terms of x and substituting to y :

$$t = \frac{1}{3}x \quad (2)$$

$$y(x) = 4 \left(\frac{1}{3}x \right)^2 + 1 \quad (3)$$

$$y(x) = \frac{4}{9}x^2 + 1 \quad (4)$$

Calculating velocity and acceleration can be done through differentiation:

$$\frac{dr}{dt}(t) = \vec{v}(t) = \begin{bmatrix} 3 \\ 8t \end{bmatrix} \quad (5)$$

$$v(t) = \sqrt{3^2 + 8^2 t^2} = \sqrt{9 + 64t^2} \quad (6)$$

$$\frac{dv}{dt}(t) = \vec{a}(t) = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad (7)$$

$$a(t) = \sqrt{0^2 + 8^2} = 8 \quad (8)$$

Tangential acceleration can be calculating by taking the dot product of velocity and acceleration:

$$a_t(t) = \|\vec{v}(t)\| \cdot \|\vec{a}(t)\| = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8t \end{bmatrix} * \frac{1}{\sqrt{9+64t^2}} = \frac{64t}{\sqrt{9+64t^2}} \quad (9)$$

Simulation hint: we can find vectorized tangential acceleration by multiplying unit vector of velocity by scalar value of acceleration:

$$\vec{a}_t(t) = \frac{1}{\|\vec{v}(t)\|} \cdot \vec{v}(t) \cdot a_t(t) = \frac{64t}{9+64t^2} \cdot \frac{1}{\sqrt{9+64t^2}} \cdot \begin{bmatrix} 3 \\ 8t \end{bmatrix} \quad (10)$$

Normal acceleration is simply the difference between acceleration and tangential acceleration:

$$\vec{a}_n(t) = \vec{a}(t) - \vec{a}_t(t) \quad (11)$$

But for usual calculation without simulation we could do it this way, by taking the cross product of velocity and acceleration:

$$\vec{a}_n(t) = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{v(t)} = \left\| \begin{bmatrix} 3 \\ 8t \end{bmatrix} \times \begin{bmatrix} 0 \\ 8 \end{bmatrix} \right\| \cdot \frac{1}{\sqrt{9+64t^2}} = \quad (12)$$

$$\left\| \begin{bmatrix} 0 \\ 0 \\ -24t \end{bmatrix} \right\| \cdot \frac{1}{\sqrt{9+64t^2}} = \frac{24t}{\sqrt{9+64t^2}} \quad (13)$$

We can find curvature using this formula:

$$k(t) = \frac{a_n}{v(t)^2} = \frac{24t}{\sqrt{9+64t^2}} \cdot \frac{1}{9+64t^2} = \frac{24t}{(9+64t^2)^{\frac{3}{2}}} \quad (14)$$

Answer:

1. $y(x) = \frac{4}{9}x^2 + 1$

2. $\vec{v}(t) = \begin{bmatrix} 3 \\ 8t \end{bmatrix}$

3. $\vec{a}(t) = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

4. $a_t(t) = \frac{64t}{\sqrt{9+64t^2}}$

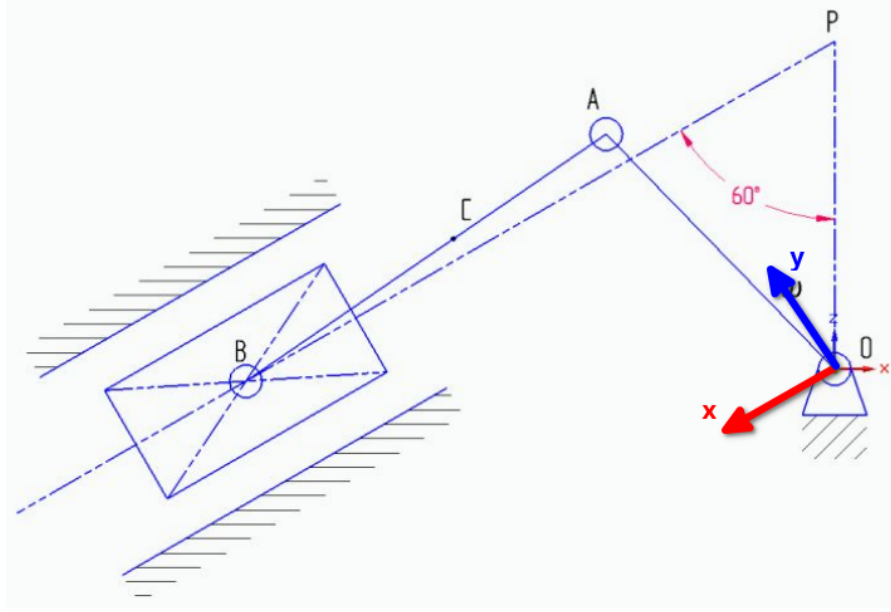
5. $a_n(t) = \frac{24t}{\sqrt{9+64t^2}}$

6. $k(t) = \frac{24t}{(9+64t^2)^{\frac{3}{2}}}$

2 Task 2

Solution:

First of all I decided to change the coordinate system to make the problem easier. I will use the following coordinate system:



Now we can describe point B depending on input angle ϕ :

$$B = \begin{bmatrix} PB - OP \cdot \cos \frac{\pi}{3} \\ OP \cdot \cos \frac{\pi}{3} \end{bmatrix} = OA \cdot \begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix} + AB \cdot \begin{bmatrix} \sqrt{1 - \cos^2(\gamma)} \\ \cos(\gamma) \end{bmatrix} \quad (15)$$

In this formula we have two unknowns: γ and PB . We can find γ by using the fact that:

$$\cos(\gamma) = \frac{OP \cdot \cos \frac{\pi}{3} - OA \cos \phi}{AB} \quad (16)$$

Let me introduce very interesting fact:

$$\sin(\arccos(x)) = \sqrt{1 - x^2} \quad (17)$$

Relax, won't bother you: explanation

We are ready to calculate x coordinate of point B :

$$x_B = OA \cdot \sin(\phi) + AB \cdot \sin(\gamma) \quad (18)$$

$$x_B = OA \cdot \sin(\phi) + AB \cdot \sqrt{1 - \left(\frac{OP \cdot \cos \frac{\pi}{3} - OA \cos \phi}{AB}\right)^2} \quad (19)$$

$$x_B = OA \cdot \sin(\phi) + \sqrt{AB^2 - (OP \cdot \cos \frac{\pi}{3} - OA \cos \phi)^2} \quad (20)$$

That's it boom! We have found x coordinate of point B .

Velocity and acceleration will be simply calculated by differentiation of position and I did it using online calculators. The only valuable information is that they have only x components.

Calculation of point C :

We know that point C lies on the line AB and we know that $AB = 80$, $AC = 20$.

So point C is basically segment of line.

$$\vec{C} = \vec{A} + (\vec{B} - \vec{A}) \cdot \frac{AB}{AC} \quad (21)$$

Proportions will be the same for velocity and accelerations:

$$\vec{v}_C = \vec{v}_A + (\vec{v}_B - \vec{v}_A) \cdot \frac{AB}{AC} \quad (22)$$

$$\vec{a}_C = \vec{a}_A + (\vec{a}_B - \vec{a}_A) \cdot \frac{AB}{AC} \quad (23)$$

Tangential acceleration is aligned with velocity vector:

$$a_{Ct} = \vec{a}_C \cdot \frac{\vec{v}_C}{\|\vec{v}_C\|} \quad (24)$$

Normal acceleration is the difference between acceleration and tangential acceleration:

$$a_{Cn} = \vec{a}_C - a_{Ct} \quad (25)$$

That's it. We have found all the required values.

Answer:

1. $x_B(t) = OA \cdot \sin(\phi) + \sqrt{AB^2 - (OP \cdot \cos \frac{\pi}{3} - OA \cos \phi)^2}$
2. $v_B(t) = \frac{dx_B(t)}{dt}$

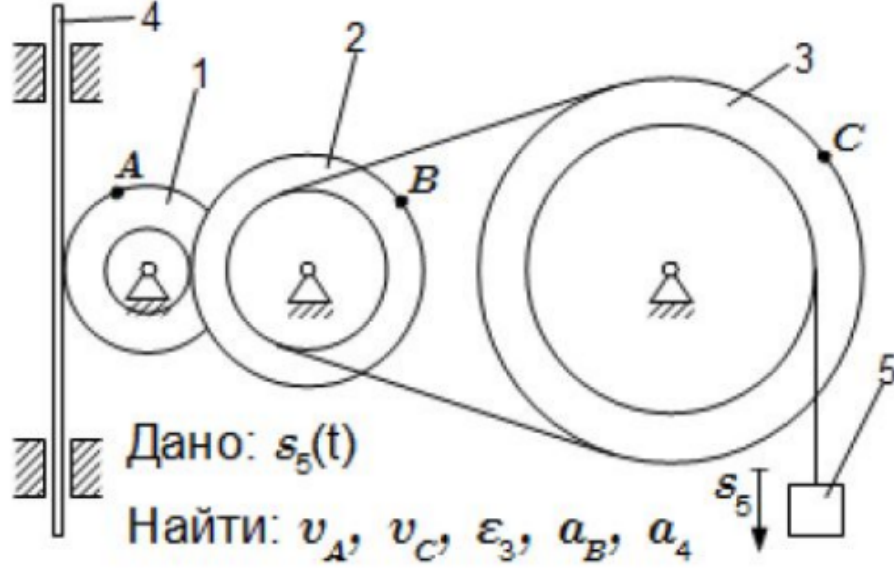
$$3. \quad a_B(t) = \frac{dv_B(t)}{dt}$$

$$4. \quad \vec{C}(t) = \vec{A} + (\vec{B} - \vec{A}) \cdot \frac{AB}{AC}$$

$$5. \quad \vec{v}_C(t) = \vec{v}_A + (\vec{v}_B - \vec{v}_A) \cdot \frac{AB}{AC}$$

$$6. \quad \vec{a}_C(t) = \vec{a}_A + (\vec{a}_B - \vec{a}_A) \cdot \frac{AB}{AC}$$

3 Task 3



Solution:

Given s_5 law of motion we can use it to propagate through all the connections of the system:

Obviously, we can observe that point on inner 3_{rd} wheel will have the same speed. Using this fact, we can get angular velocity of the 3_{rd} wheel:

$$\omega_3(t) = \frac{3t^2 - 6}{r_3} \quad (26)$$

Wheels 2 and 3 are connected using belt, so we can express angular velocity of the 2_{nd} wheel in terms of 3_{rd} wheel:

$$\omega_2(t) \cdot r_2 = \omega_3(t) \cdot R_3 \quad (27)$$

The same idea between 1_{st} and 2_{nd} wheels is the same:

$$\omega_1(t) \cdot r_1 = \omega_2(t) \cdot R_2 \quad (28)$$

These equations mainly give us everything to find required variables:

1. Velocity for point A (on outer radius of the wheel \Rightarrow uses R_1):

$$v_A(t) = \omega_1(t) \cdot R_1 = \frac{R_2}{r_1} \cdot \omega_2(t) \cdot R_1 = \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \omega_3(t) \cdot R_1 = \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot R_1 \quad (29)$$

At time $t = 2$ we have: $v_A(t) = \frac{8}{2} \cdot \frac{16}{6} \cdot \frac{6}{12} \cdot 4 = \frac{64}{3} \approx 21.33$

2. Velocity of point C (on outer radius of the wheel \Rightarrow uses R_3):

$$v_C(t) = \omega_3(t) \cdot R_3 = \frac{3t^2 - 6}{r_3} \cdot R_3 \quad (30)$$

At time $t = 2$ we have: $v_C(t) = \frac{6}{12} \cdot 4 = \frac{24}{3} = 8$

3. Angular acceleration of 3_{rd} wheel.

$$\epsilon_3(t) = \frac{dw_3(t)}{dt} = \frac{6t}{r_3} \quad (31)$$

At time $t = 2$ we have: $\epsilon_3(t) = \frac{12}{12} = 1$

4. Acceleration of B:

We can start by determining angular speed and angular acceleration of the 2_{nd} wheel

$$\omega_2(t) = \frac{R_3}{r_2} \cdot \omega_3(t) = \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \quad (32)$$

$$\epsilon_2(t) = \frac{dw_2(t)}{dt} = \frac{R_3}{r_2} \cdot \epsilon_3(t) = \frac{R_3}{r_2} \cdot \frac{6t}{r_3} \quad (33)$$

Now we can apply basic transformations to find linear components of accelerations:

$$a_{B\tau}(t) = \epsilon_2(t) \times R_2 = \frac{R_3}{r_2} \cdot \frac{6t}{r_3} \cdot R_2 \quad (34)$$

$$a_{Bn}(t) = \omega_2(t) \times (\omega_2(t) \times R_2) = \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot R_2 \quad (35)$$

$$a_B(t) = \sqrt{a_{B\tau}^2 + a_{Bn}^2} \quad (36)$$

At time $t = 2$ we have:

$$a_{B\tau}(2) = \frac{16}{6} \cdot \frac{12}{12} \cdot 8 = \frac{64}{3} \quad (37)$$

$$a_{Bn}(2) = \frac{16}{6} \cdot \frac{6}{12} \cdot \frac{16}{6} \cdot \frac{6}{12} \cdot 8 = \frac{128}{9} \quad (38)$$

$$a_B(2) = \sqrt{\frac{64^2}{3} + \frac{128^2}{9}} = 25.64 \quad (39)$$

5. Acceleration of rack 4:

We see that it is connected without slipping to the 1_{st} wheel. So we can express velocity of the bar in terms of angular velocity of the wheel:

$$v_4(t) = v_A(t) = \omega_1(t) \cdot R_1 = \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \frac{3t^2 - 6}{r_3} \cdot R_1 \quad (40)$$

$$a_4(t) = \frac{dv_4(t)}{dt} = \frac{R_2}{r_1} \cdot \frac{R_3}{r_2} \cdot \frac{6t}{r_3} \cdot R_1 \quad (41)$$

At time $t = 2$ we have:

$$a_4(2) = \frac{8}{2} \cdot \frac{16}{6} \cdot \frac{12}{12} \cdot 4 = \frac{128}{3} = 42.67 \quad (42)$$

Answer:

1. $v_A(2) = \frac{64}{3} \approx 21.33$
2. $v_C(2) = 8$
3. $\epsilon_3(2) = 1$
4. $a_{B\tau}(2) = \frac{64}{3}$
5. $a_{Bn}(2) = \frac{128}{9}$
6. $a_B(2) = 25.64$
7. $a_4(2) = 42.67$