Homework 2

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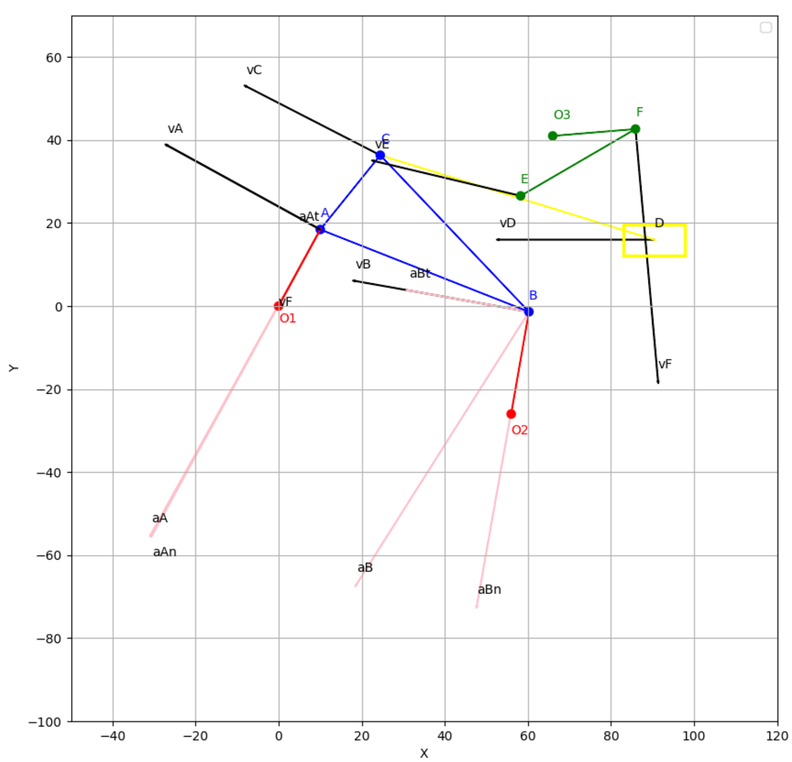
1 Task 1



1.1 Derivation of points positions

1. I assume that point O_1 is positioned in the (0, 0).

2. We can describe the position of point A as it rotates around point O_1 as follows:

$$\vec{r_A}(t) = O_1 A \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} \tag{1}$$

3. Position of O_2 is $(-c, a)$ as follows from task description.

4. We can describe the position of point B as intersection between two circles: first with center at A and radius AB and second with center at O_2 and radius OA_2:

$$\begin{cases} (x_B - x_A)^2 + (y_B - y_A)^2 = AB^2 \\ (x_B - x_{O_2})^2 + (y_B - y_{O_2})^2 = O_2A^2 \end{cases} \tag{2}$$

   These equations can be solved easily with solvers like *Sympy* for *Python*. I faced several problems with this step:

   (a) Mechanism does not work for all angles of ϕ. Distance between A and O_2 is at maximum:

$$(O_1A \cdot \cos(\phi) - a)^2 + (O_1A \cdot \sin(\phi) + c)^2 = (AB + O_2B)^2 \tag{3}$$

   This will happen when O_2BA will form a straight line. Wolframagic solution link We get 2 angles, but actually only one is important for us, because we rotate CCW starting from $\pi/3$. Limit angle will be ≈ 2.004 radians. Thus my simulation is limited for $\phi \in [\pi/3, 2.004]$.

   (b) We have 2 solutions from quadratic equations and have to choose one: I select the most right and top (greatest x and y coordinates) point because it looks nicer and closer to starting position from picture. I will follow this approach for other cases too.

5. We can describe the position of point C as intersection between two circles: first with center at B and radius BC and second with center at A and radius AB:

$$\begin{cases} (x_C - x_B)^2 + (y_C - y_B)^2 = BC^2 \\ (x_C - x_A)^2 + (y_C - y_A)^2 = AB^2 \end{cases} \tag{4}$$

6. We can describe the position of point D as intersection between circle (center C radius CD) and a line because point D has only translational motion on the fixed axis:

$$\begin{cases} (x_D - x_C)^2 + (y_D - y_C)^2 = CD^2 \\ y_D = d \end{cases} \tag{5}$$

7. Point E is on CD segment and can be found as follows:

$$\vec{r_E}(t) = \vec{r_C}(t) + \frac{\vec{r_D}(t) - \vec{r_C}(t)}{CD} CE \tag{6}$$

8. Point F can be found as intersection of two circles: one with center E and radius EF and second with center O_3 and radius O_3F:

$$\begin{cases} (x_F - x_E)^2 + (y_F - y_E)^2 = EF^2 \\ (x_F - x_{O_3})^2 + (y_F - y_{O_3})^2 = O_3F^2 \end{cases} \tag{7}$$

1.2 Derivation of velocities for points

meme link

1. Velocity of point A:

$$\vec{v_A(t)} = \dot{\vec{r_A}}(t) = \omega \cdot \vec{r_A}(t) \tag{8}$$

2. Velocity of point B can be found as follows:

$$\vec{v_B}(t) = \dot{\vec{r_B}}(t) \tag{9}$$

3. Velocity of point C can be found as follows:

$$\vec{v_C}(t) = \dot{\vec{r_C}}(t) \tag{10}$$

4. Velocity of point D can be found as follows:

$$\vec{v_D}(t) = \dot{\vec{r_D}}(t) \tag{11}$$

5. Velocity of point E can be found as follows:

$$\vec{v_E}(t) = \dot{\vec{r_E}}(t) \tag{12}$$

6. Velocity of point F can be found as follows:

$$\vec{v_F}(t) = \dot{\vec{r_F}}(t) \tag{13}$$

1.3 Derivation of angular velocities

Let's start with meme:

1. We could use IC(instanteneous centre of zero velocity to simplify our calculations, but I will not do it because we already have all velocities and positions of points and it is enough.

2. For angular velocities we can use this formula:

$$\vec{v_1}(t) = \vec{v_2}(t) + \vec{\omega}(t) \times \vec{r_{12}}(t) \tag{14}$$

3. Angular velocity of O_1A:

   (a) O_1A is a fixed point and we know from description that $\omega = 2$.

   (b) But nevertheless:

$$\vec{v_A}(t) = \vec{v_{O_1}}(t) + \vec{\omega_{O_1A}}(t) \times \vec{r_{O_1A}}(t) \tag{15}$$

$$\omega_{O_1A} = \frac{|\vec{v_A}(t) - \vec{v_{O_1}}(t)|}{|\vec{r_{O_1A}}(t)|} \tag{16}$$

   (c) In simulation I compute both variants, and they almost equal with respect to precision of simulation and derivative computation algorithm (I use the easiest).

4. As all formulas are pretty much the same I will not comment others.

5. Angular velocity of O_2B:

$$\vec{v_B}(t) = \vec{v_{O_2}}(t) + \vec{\omega_{O_2B}}(t) \times \vec{r_{O_2B}}(t) \tag{17}$$

$$\omega_{O_2B} = \frac{|\vec{v_B}(t) - \vec{v_{O_2}}(t)|}{|\vec{r_{O_2B}}(t)|} \tag{18}$$

6. Angular velocity of AB:

$$\vec{v_B}(t) = \vec{v_A}(t) + \vec{\omega_{AB}}(t) \times \vec{r_{AB}}(t) \tag{19}$$

$$\omega_{AB} = \frac{|\vec{v_B}(t) - \vec{v_A}(t)|}{|\vec{r_{AB}}(t)|} \tag{20}$$

7. Angular velocity of BC and AC: As ABC form a fixed triangle, their angular velocities are equal:

$$\omega_{AB} = \omega_{BC} = \omega_{AC} \tag{21}$$

   But in order to justify and prove it to myself I also compute them separately:

$$\vec{v_C}(t) = \vec{v_B}(t) + \vec{\omega_{BC}}(t) \times \vec{r_{BC}}(t) \tag{22}$$

$$\omega_{BC} = \frac{|\vec{v_C}(t) - \vec{v_B}(t)|}{|\vec{r_{BC}}(t)|} \tag{23}$$

$$\vec{v_C}(t) = \vec{v_A}(t) + \vec{\omega_{AC}}(t) \times \vec{r_{AC}}(t) \tag{24}$$

$$\omega_{AC} = \frac{|\vec{v_C}(t) - \vec{v_A}(t)|}{|\vec{r_{AC}}(t)|} \tag{25}$$

8. Angular velocity of CD:

$$\vec{v_D}(t) = \vec{v_C}(t) + \vec{\omega_{CD}}(t) \times \vec{r_{CD}}(t) \tag{26}$$

$$\omega_{CD} = \frac{|\vec{v_D}(t) - \vec{v_C}(t)|}{|\vec{r_{CD}}(t)|} \tag{27}$$

9. Angular velocities of CE and ED: CDE are on the same "body" and their angular velocities are equal:

$$\omega_{CD} = \omega_{CE} = \omega_{ED} \tag{28}$$

10. Angular velocity of EF:

$$\vec{v_F}(t) = \vec{v_E}(t) + \vec{\omega_{EF}}(t) \times \vec{r_{EF}}(t) \tag{29}$$

$$\omega_{EF} = \frac{|\vec{v_F}(t) - \vec{v_E}(t)|}{|\vec{r_{EF}}(t)|} \tag{30}$$

11. Angular velocitiy of O_3F:

$$\vec{v_F}(t) = \vec{v_{O_3}}(t) + \vec{\omega_{O_3F}}(t) \times \vec{r_{O_3F}}(t) \tag{31}$$

$$\omega_{O_3F} = \frac{|\vec{v_F}(t) - \vec{v_{O_3}}(t)|}{|\vec{r_{O_3F}}(t)|} \tag{32}$$

1.4 Accelerations for A and B

1. Acceleration of A:

$$\vec{a_A}(t) = \dot{\vec{v_A}}(t) \tag{33}$$

We can mention that as ω is const, there will be no tangential acceleration for A. In simulation we can see very small tangential acceleration, but it is because of precision of simulation.

2. Tangential acceleration of A:

$$\vec{a_{At}}(t) = \frac{\vec{a_A}(t) \cdot \vec{v_A}(t)}{|\vec{v_A}(t)|}\vec{r_A}(t) \tag{34}$$

3. Normal acceleration of A:

$$\vec{a_{An}}(t) = \vec{a_A}(t) - \vec{a_{At}}(t) \tag{35}$$

4. Acceleration of B:

$$\vec{a_B}(t) = \dot{\vec{v_B}}(t) \tag{36}$$

5. Tangential acceleration of B:

$$\vec{a_{Bt}}(t) = \frac{\vec{a_B}(t) \cdot \vec{v_B}(t)}{|\vec{v_B}(t)|}\vec{r_B}(t) \tag{37}$$

6. Normal acceleration of B:

$$\vec{a_{Bn}}(t) = \vec{a_B}(t) - \vec{a_{Bt}}(t) \tag{38}$$

2 Task 2

2.1 Derivation of angle of rotation around Z

As ω_1 and ϵ_1 are given, we can convert them to $\phi(t)$:

$$\phi(t) = \phi_0 + \omega_1 \cdot t + \epsilon_1 \cdot \frac{t^2}{2} \tag{39}$$

This angle will determine the angle of rotation of cone A axis around Z axis.

2.2 Calculating axes of cones

1. For simplifications in simulation, I introduced several unit vectors for initial position.

2. Cone A axis (direction from base of cone to its vertex):

$$\vec{u_A} = \begin{bmatrix} 0 \\ -\cos\frac{\alpha_B - \alpha_A}{2} \\ -\sin\frac{\alpha_B - \alpha_A}{2} \end{bmatrix} \tag{40}$$

3. IC axis:

   (a) IC axis is axis of intersection between cone A and cone B because B is static and A moves without slippering.

   (b) All velocities of the points in A can easily be found using IC axis.

   (c) It is kinda obvious that this axis will be rotated around Z axis with angle $\phi(t)$ together will cone A.

   (d) It is enough to calculate initial vector and then rotate it around Z axis.

$$\vec{u_{IC}} = \begin{bmatrix} 0 \\ \sin(\alpha_B/2) \\ \cos(\alpha_B/2) \end{bmatrix} \tag{41}$$

4. Radius of cone A axis:

   (a) I needed this unit vector to calculate position of point M as it is simplier to split transformation on parts and then combine them.

   (b) It is simply cone A axis rotated by 90 deg around X axis.

$$\vec{u_{rA}} = \begin{bmatrix} 0 \\ -\sin(\alpha_A/2) \\ \cos(\alpha_A/2) \end{bmatrix} \tag{42}$$

2.3 Angular velocity of cone A

1. Let state radius of cone A:

$$r_A = OM_0 \sin(\alpha_A/2) \tag{43}$$

2. It is widely known formula for velocity: $\vec{v} = \vec{\omega} \times \vec{r}$.

3. I discovered, that we can do reverse: $\vec{\omega} = \frac{\vec{r} \times \vec{v}}{||\vec{r}||^2}$ source.

4. In our case it would be radius vector from cone A center axis to IC axis.

$$\vec{r}_{centerA} = r_A \cdot R_z(\phi) \cdot \vec{u_A} \tag{44}$$
$$\vec{r} = \vec{r}_{centerA} - OM_0 \cdot R_z(\phi) \cdot \vec{u_{rA}} \tag{45}$$
$$\vec{v} = \dot{\vec{r}}_{centerA} \tag{46}$$
$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{||\vec{r}||^2} \tag{47}$$

2.4 Angular acceleration of cone A

Without many words:

$$\vec{\epsilon} = \dot{\vec{\omega}} \tag{48}$$

2.5 Position of point M

1. We know that point M lies on the base of cone A

2. That means we can find its position by first tranforming from origin to center of base of cone A and then rotating it around cone A axis on required angle.

3. Firstly, I needed to find the angle on which point M should be rotated. Imagine that circumference of cone B is flattened into the line. Then rotation of cone A on cone B is like the wheel with radius of cone A is rotating on the line.

$$\theta_A = \frac{OM_0 \cdot \phi}{OM_0 \cdot \sin(\alpha_A/2)} = \frac{\phi}{\sin(\alpha_A/2)} \tag{49}$$

   This angle determines rotation of u_{rA} around u_A.

4. Then we can find position of point M:

$$\vec{r}_M = \vec{r}_{centerA} + R_z(\phi) \cdot R_{u_A}(\theta_A) \cdot \vec{u}_{rA} \tag{50}$$

2.6 Velocity and accelerationi of point M

1. Velocity of point M:

$$\vec{v}_M = \dot{\vec{r}}_M \tag{51}$$

2. Acceleration of point M:

$$\vec{a}_M = \dot{\vec{v}}_M \tag{52}$$

3. Tangential acceleration of point M:

$$\vec{a}_{tM} = \vec{a}_M - \vec{v}_M \cdot \frac{\vec{v}_M}{||\vec{v}_M||} \tag{53}$$

Answer:

1. Angular velocity of cone A:

$$\begin{aligned} \vec{r}_{centerA} &= r_A \cdot R_z(\phi) \cdot \vec{u_A} \\ \vec{r} &= \vec{r}_{centerA} - OM_0 \cdot R_z(\phi) \cdot \vec{u_{rA}} \\ \vec{v} &= \dot{\vec{r}}_{centerA} \\ \vec{\omega} &= \frac{\vec{r} \times \vec{v}}{||\vec{r}||^2} \end{aligned}$$

2. Angular acceleration of cone A:

$$\vec{\epsilon} = \dot{\vec{\omega}}$$

3. Position of point M:

$$\begin{aligned} \vec{u_{rA}} &= \begin{bmatrix} 0 \\ -\sin(\alpha_A/2) \\ \cos(\alpha_A/2) \end{bmatrix} \\ \vec{r}_M &= \vec{r}_{centerA} + R_z(\phi) \cdot R_{u_A}(\theta_A) \cdot \vec{u}_{rA} \end{aligned}$$

4. Velocity of point M:

$$\vec{v}_M = \dot{\vec{r}}_M$$

5. Acceleration of point M:

$$\vec{a}_M = \dot{\vec{v}}_M$$