Delaunay Triangulation

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Abstract

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$$\begin{aligned} \|a - m\| &= r \\ \|b - m\| &= r \\ \|c - m\| &= r \end{aligned}$$

$$\begin{aligned} \|a - m\|^2 &= r^2 \\ \|b - m\|^2 &= r^2 \\ \|c - m\|^2 &= r^2 \end{aligned}$$

$$\begin{aligned} \|\hat{m}\|^2 &= r^2 \\ \left\|\hat{b} - \hat{m}\right\| &= r^2 \\ \|\hat{c} - \hat{m}\| &= r^2 \end{aligned}$$

$$\begin{aligned} \left\|\hat{b}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= r^2 \\ \|\hat{c}\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= r^2 \end{aligned}$$

$$\begin{aligned} \left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= 0 \\ \|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= 0 \end{aligned}$$

$$\begin{aligned} \left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= 0 \\ \|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= 0 \end{aligned}$$

$$\begin{aligned} \left\langle \hat{b} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{b}\right\|^2 \\ \left\langle \hat{c} \middle| \hat{m} \right\rangle &= \frac{1}{2}\|\hat{c}\|^2 \end{aligned}$$

$$\begin{pmatrix} \hat{b} & \hat{c} \end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix} \left\|\hat{b}\right\|^2 \\ \|\hat{c}\|^2 \end{pmatrix}$$

$$\hat{m} = \frac{1}{2\det\begin{pmatrix}\hat{b} & \hat{c}\end{pmatrix}} \operatorname{adj}\begin{pmatrix}\hat{b} & \hat{c}\end{pmatrix}^{\mathrm{T}} \begin{pmatrix}\left\|\hat{b}\right\|^2 \\ \|\hat{c}\|^2\end{pmatrix}$$

$$\hat{m} = \frac{1}{2(u_x v_y - u_y v_x)} \begin{pmatrix} v_y(u_x^2 + u_y^2) - v_x(v_x^2 + v_y^2) \\ u_x(v_x^2 + v_y^2) - u_y(u_x^2 + u_y^2) \end{pmatrix}$$

2.2.2 Circumsphere

$$\begin{aligned}
\|a - m\| &= r \\
\|b - m\| &= r \\
\|c - m\| &= r \\
\|d - m\| &= r
\end{aligned}$$

$$\begin{aligned}
\|a - m\|^2 &= r^2 \\
\|b - m\|^2 &= r^2 \\
\|c - m\|^2 &= r^2 \\
\|d - m\|^2 &= r^2
\end{aligned}$$

$$\begin{aligned}
\|\hat{m}\|^2 &= r^2 \\
\left\|\hat{b} - \hat{m}\right\| &= r^2 \\
\|\hat{c} - \hat{m}\| &= r^2 \\
\left\|\hat{d} - \hat{m}\right\| &= r^2
\end{aligned}$$

$$\begin{aligned}
\left\|\hat{b}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= r^2 \\
\|\hat{c}\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= r^2 \\
\left\|\hat{d}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= r^2
\end{aligned}$$

$$\begin{aligned}
\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= 0 \\
\|\hat{c}\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= 0 \\
\left\|\hat{d}\right\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= 0
\end{aligned}$$

$$\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle = 0$$

$$\begin{aligned}\left\|\hat{c}\right\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= 0\\ \left\|\hat{d}\right\|^2 - 2\left\langle \hat{d} \middle| \hat{m} \right\rangle &= 0\end{aligned}$$

$$\begin{aligned}\left\langle \hat{b} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{b}\right\|^2\\ \left\langle \hat{c} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{c}\right\|^2\\ \left\langle \hat{d} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{d}\right\|^2\end{aligned}$$

$$\begin{pmatrix}\hat{b} & \hat{c} & \hat{d}\end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix}\left\|\hat{b}\right\|^2\\ \left\|\hat{c}\right\|^2\\ \left\|\hat{d}\right\|^2\end{pmatrix}$$

$$\hat{m} = \frac{1}{2\det\begin{pmatrix}\hat{b} & \hat{c} & \hat{d}\end{pmatrix}} \operatorname{adj}\begin{pmatrix}\hat{b} & \hat{c} & \hat{d}\end{pmatrix}^{\mathrm{T}} \begin{pmatrix}\left\|\hat{b}\right\|^2\\ \left\|\hat{c}\right\|^2\\ \left\|\hat{d}\right\|^2\end{pmatrix}$$

2.2.3 Circumscribed n-Sphere

$$\begin{aligned}&\left\|x_0 - m\right\| = r\\ &\left\|x_i - m\right\| = r\\ &\left\|x_i - m\right\|^2 = r^2\\ &\left\|\hat{x}_i - \hat{m}\right\|^2 = r^2\\ &\left\|\hat{x}_i\right\|^2 + \left\|\hat{m}\right\|^2 - 2\left\langle \hat{x}_i \middle| \hat{m} \right\rangle = r^2\\ &\left\|\hat{x}_i\right\|^2 - 2\left\langle \hat{x}_i \middle| \hat{m} \right\rangle = 0\\ &\left\langle \hat{x}_i \middle| \hat{m} \right\rangle = \frac{1}{2}\left\|\hat{x}_i\right\|^2\\ &\begin{pmatrix}\hat{x}_1 & \dots & \hat{x}_n\end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix}\left\|\hat{x}_1\right\|^2\\ \vdots\\ \left\|\hat{x}_n\right\|^2\end{pmatrix}\end{aligned}$$

Here, we should not try to build the adjoint. For higher dimensions, it is much more efficient to use a simple matrix solver, like LU decomposition or Cholesky.

2.3 Delaunay Triangulation and Tessellation

Triangulation, Tessellation, Simplicialization, Subdivision, Mesh Generation Two and Three Dimensions

3 Algorithms and Data Structures

3.1 Hash Map

3.2 Triangle Mesh

There are different schemes for storing a triangle. Information about neighbors seem to be important to improve efficiency but not necessary some algorithms.

3.3 Quad-Edge Data Structure

3.4 Radix Sort

3.5 Linear Morton Sort

3.6 Linear Floating-Point Quad-Tree

3.7 Bowyer-Watson Algorithm

Advantages: Easy structures. Low memory consumption. Possible in every dimension. Incremental construction at arbitrary point in domain. Disadvantages: Complexity is $\Theta\left(n^2\right)$. Algorithm needs bounding super triangle. Bad parallelization.

3.8 Guibas-Stolfi Incremental Algorithm

3.9 Guibas-Stolfi Divide-and-Conquer Algorithm

3.10 Dwyer Algorithm

4 Design and Implementation

4.1 API

4.2 Robustness

4.3 Incremental Algorithm

4.4 Divide-and-Conquer Algorithm

4.5 Multidimensional Tessellation

5 Tests and Testscenes

5.1 Uniform Rectangular Distribution

5.2 Gaussian Distribution

5.3 Robustness Tests

6 Benchmarks

7 Examples

7.1 Image Mosaic and Tessellation

SVG output is generated. We need the image as PNG or JPG. A seed will be generated automatically or given on command line. Hence, it should be returned as well. Also the triangle count can be defined.

7.2 Fluids for Ray Tracing

7.3 Finite Element Method

7.4 Pareto Frontiers

8 Results

9 Conclusions

References

Bowyer, A. (1981). "Computing Dirichlet Tessellations". In: *The Computer Journal* 24, pp. 162–166. DOI: 10.1093/comjnl/24.2.162.

Burnikel, Christoph (1998). *Delaunay Graphs by Divide and Conquer*. URL: https://pure.mpg.de/rest/items/item_1819432_4/component/file_2599484/content (visited on 11/07/2020).

Cignoni, P., C. Montani, and R. Scopigno (1998). "DeWall: A Fast Divide-and-Conquer Delaunay Triangulation Algorithm in E^d". In: *Computer-Aided Design* 30, pp. 333–341. DOI: 10.1016/S0010-4485(97)00082-1.

Dwyer, Rex A. (November 1987). "A Faster Divide-and-Conquer Algorithm for Constructing Delaunay Triangulations". In: *Algorithmica* 2, pp. 137–151. DOI: 10.1007/BF01840356.

Fuetterling, V., C. Lojewski, and F.-J. Pfreundt (2014). "High-Performance Delaunay Triangulation for Many-Core Computers". In: *High Performance Graphics* 2014, pp. 97–104. DOI: 10.2312/hpg.20141098.

Guibas, Leonidas and Jorge Stolfi (April 1985). "Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams". In: *ACM Transactions on Graphics* 4, pp. 74–123. DOI: 10.1145/282918.282923. URL: http://sccg.sk/~samuelcik/dgs/quad_edge.pdf (visited on 11/07/2020).

Katajainen, Jyrki and Markku Koppinen (April 1988). "Constructing Delaunay Triangulations by Merging Buckets in Quad-Tree Order". In: *Fundamenta Informaticae* 11, pp. 275–288.

Lee, D. T. and B. J. Schachter (1980). "Two Algorithms for Constructing a Delaunay Triangulation". In: *International Journal of Computer and Information Sciences* 9, pp. 219–242. DOI: 10.1007/BF00977785.

Lischinski, Dani (1993). *Incremental Delaunay Triangulation*. URL: http://www.karlchenofhell.org/cppswp/lischinski.pdf (visited on 11/07/2020).

Shewchuk, Jonathan Richard. *Triangle: Engineering a 2D Quality Mesh Generator and Delaunay Triangulator*. URL: https://people.eecs.berkeley.edu/~jrs/papers/triangle.pdf (visited on 11/07/2020).

Watson, D. F. (1981). "Computing the n-Dimensional Delaunay Tessellation with Application to Voronoi Polytopes". In: *The Computer Journal* 24, pp. 167–172. DOI: 10.1093/comjnl/24.2.167.