Delaunay Triangulation

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Abstract

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$$\begin{aligned}\|a-m\| &= r\\ \|b-m\| &= r\\ \|c-m\| &= r\end{aligned}$$

$$\begin{aligned}\|a-m\|^2 &= r^2\\ \|b-m\|^2 &= r^2\\ \|c-m\|^2 &= r^2\end{aligned}$$

$$\begin{aligned}\|\hat{m}\|^2 &= r^2\\ \left\|\hat{b}-\hat{m}\right\| &= r^2\\ \|\hat{c}-\hat{m}\| &= r^2\end{aligned}$$

$$\begin{aligned}\left\|\hat{b}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{b} \middle| \hat{m}\right\rangle &= r^2\\ \|\hat{c}\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} \middle| \hat{m}\right\rangle &= r^2\end{aligned}$$

$$\begin{aligned}\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m}\right\rangle &= 0\\ \|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m}\right\rangle &= 0\end{aligned}$$

$$\begin{aligned}\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m}\right\rangle &= 0\\ \|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m}\right\rangle &= 0\end{aligned}$$

$$\begin{aligned}\left\langle \hat{b} \middle| \hat{m}\right\rangle &= \frac{1}{2}\left\|\hat{b}\right\|^2\\ \left\langle \hat{c} \middle| \hat{m}\right\rangle &= \frac{1}{2}\|\hat{c}\|^2\end{aligned}$$

$$\begin{pmatrix}\hat{b} & \hat{c}\end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix}\left\|\hat{b}\right\|^2\\ \|\hat{c}\|^2\end{pmatrix}$$

$$\hat{m} = \frac{1}{2\det\begin{pmatrix}\hat{b} & \hat{c}\end{pmatrix}} \operatorname{adj}\begin{pmatrix}\hat{b} & \hat{c}\end{pmatrix}^{\mathrm{T}} \begin{pmatrix}\left\|\hat{b}\right\|^2 \\ \|\hat{c}\|^2\end{pmatrix}$$

$$\hat{m} = \frac{1}{2(u_x v_y - u_y v_x)} \begin{pmatrix} v_y(u_x^2 + u_y^2) - v_x(v_x^2 + v_y^2) \\ u_x(v_x^2 + v_y^2) - u_y(u_x^2 + u_y^2) \end{pmatrix}$$

2.2.2 Circumsphere

$$\begin{aligned}
\|a - m\| &= r \\
\|b - m\| &= r \\
\|c - m\| &= r \\
\|d - m\| &= r
\end{aligned}$$

$$\begin{aligned}
\|a - m\|^2 &= r^2 \\
\|b - m\|^2 &= r^2 \\
\|c - m\|^2 &= r^2 \\
\|d - m\|^2 &= r^2
\end{aligned}$$

$$\begin{aligned}
\|\hat{m}\|^2 &= r^2 \\
\left\|\hat{b} - \hat{m}\right\| &= r^2 \\
\|\hat{c} - \hat{m}\| &= r^2 \\
\left\|\hat{d} - \hat{m}\right\| &= r^2
\end{aligned}$$

$$\begin{aligned}
\left\|\hat{b}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= r^2 \\
\|\hat{c}\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= r^2 \\
\left\|\hat{d}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= r^2
\end{aligned}$$

$$\begin{aligned}
\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= 0 \\
\|\hat{c}\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= 0 \\
\left\|\hat{d}\right\|^2 - 2\left\langle \hat{c} | \hat{m} \right\rangle &= 0
\end{aligned}$$

$$\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle = 0$$

$$\begin{aligned}\|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= 0\\ \left\|\hat{d}\right\|^2 - 2\left\langle \hat{d} \middle| \hat{m} \right\rangle &= 0\end{aligned}$$

$$\begin{aligned}\left\langle \hat{b} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{b}\right\|^2\\ \left\langle \hat{c} \middle| \hat{m} \right\rangle &= \frac{1}{2}\|\hat{c}\|^2\\ \left\langle \hat{d} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{d}\right\|^2\end{aligned}$$

$$\begin{pmatrix}\hat{b} & \hat{c} & \hat{d}\end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix}\left\|\hat{b}\right\|^2\\ \|\hat{c}\|^2\\ \left\|\hat{d}\right\|^2\end{pmatrix}$$

$$\hat{m} = \frac{1}{2\det\begin{pmatrix}\hat{b} & \hat{c} & \hat{d}\end{pmatrix}} \operatorname{adj}\begin{pmatrix}\hat{b} & \hat{c} & \hat{d}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\left\|\hat{b}\right\|^2\\ \|\hat{c}\|^2\\ \left\|\hat{d}\right\|^2\end{pmatrix}$$

2.2.3 Circumscribed n-Sphere

$$\begin{aligned}&\|x_0 - m\| = r\\ &\|x_i - m\| = r\\ &\|x_i - m\|^2 = r^2\\ &\|\hat{x}_i - \hat{m}\|^2 = r^2\\ &\|\hat{x}_i\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{x}_i \middle| \hat{m} \right\rangle = r^2\\ &\|\hat{x}_i\|^2 - 2\left\langle \hat{x}_i \middle| \hat{m} \right\rangle = 0\\ &\left\langle \hat{x}_i \middle| \hat{m} \right\rangle = \frac{1}{2}\|\hat{x}_i\|^2\\ &\begin{pmatrix}\hat{x}_1 & \dots & \hat{x}_n\end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix}\|\hat{x}_1\|^2\\ \vdots\\ \|\hat{x}_n\|^2\end{pmatrix}\end{aligned}$$

Here, we should not try to build the adjoint. For higher dimensions, it is much more efficient to use a simple matrix solver, like LU decomposition or Cholesky.

2.3 Delaunay Triangulation and Tessellation

Triangulation, Tessellation, Simplicialization, Subdivision, Mesh Generation Two and Three Dimensions

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