Delaunay Triangulation

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Abstract

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2.2.1 Circumcircles and Circumspheres

$$
\begin{aligned}
\|a - m\| &= r \\
\|b - m\| &= r \\
\|c - m\| &= r
\end{aligned}
$$

$$
\begin{aligned}
\|a - m\|^2 &= r^2 \\
\|b - m\|^2 &= r^2 \\
\|c - m\|^2 &= r^2
\end{aligned}
$$

$$
\begin{aligned}
\|\hat{m}\|^2 &= r^2 \\
\left\|\hat{b} - \hat{m}\right\| &= r^2 \\
\|\hat{c} - \hat{m}\| &= r^2
\end{aligned}
$$

$$
\begin{aligned}
\left\|\hat{b}\right\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= r^2 \\
\|\hat{c}\|^2 + \|\hat{m}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= r^2
\end{aligned}
$$

$$
\begin{aligned}
\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= 0 \\
\|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left\|\hat{b}\right\|^2 - 2\left\langle \hat{b} \middle| \hat{m} \right\rangle &= 0 \\
\|\hat{c}\|^2 - 2\left\langle \hat{c} \middle| \hat{m} \right\rangle &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{b} \middle| \hat{m} \right\rangle &= \frac{1}{2}\left\|\hat{b}\right\|^2 \\
\left\langle \hat{c} \middle| \hat{m} \right\rangle &= \frac{1}{2}\|\hat{c}\|^2
\end{aligned}
$$

$$
\begin{pmatrix} \hat{b} & \hat{c} \end{pmatrix}^{\mathrm{T}} \hat{m} = \frac{1}{2}\begin{pmatrix} \left\|\hat{b}\right\|^2 \\ \|\hat{c}\|^2 \end{pmatrix}
$$

$$
\hat{m} = \frac{1}{2\det\begin{pmatrix} \hat{b} & \hat{c} \end{pmatrix}} \operatorname{adj}\begin{pmatrix} \hat{b} & \hat{c} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \left\|\hat{b}\right\|^2 \\ \|\hat{c}\|^2 \end{pmatrix}
$$

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