Spreading of Water Waves

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Abstract

In this project, I will investigate the spreading problem of water waves and obtain a mathematical description in two-dimensional and three dimensional spaces which is involved with the classical Cauchy-Poisson problem in 19th century. In the first half part, I will mainly discuss the two-dimensional problem including how the mathematical model extracted from our real life which may refer to the derivation of governing equation and initial and boundary conditions, and use Maple to process the numerical calculation. Three-dimensional spreading problem will be studied in the rest of my dissertation. Basically, compared with two-dimensional problem, three-dimensional case will do the same work but in three-dimensional space. In the very last part, I will draw a conclusion about almost everything that has been achieved in my work.

Acknowledgements

I would like to express my gratefulness to my tutor, Professor Phil McIver, who has given me a great and kind help throughout my project.

Contents

1 Introduction of the project

According to what has been stated in [1], when a ship passing through or the wind blowing, it probably will result in the generation of waves on the surface of the sea. As we can see, the waves will spread out from the place where the initial disturbance happened and then gradually vanish as time grows. In this project, I will acquire a mathematical description of how water waves spread out on the surface from a given initial disturbance. This problem could be referred to as the classical Cauchy-Poisson problem which was first studied by these two mathematicians in 19th century, and it could be described as a wave-spreading problem which waves generated when a disturbance happened to the surface of water. Cauchy-Poisson problem has been attempted in both two-dimensional and three-dimensional spaces, inviscid and viscous, irrotational and rotational, finite depth and infinite depth of fluid with linear and nonlinear conditions. However, in my work, I will consider this problem in both two-dimensional and three-dimensional cases but in inviscid, incompressible, irrotational water with finite depth and linearized boundary conditions. In this paper, explicit solutions will be given in each situation and I will use the solutions that have been acquired to simulate the real phenomena of spreading of water wave with the help of Maple.

1.1 The mathematical model

During the whole project, we assume that the water is inviscid and incompressible so that the friction between fluid particles is neglected. Besides, the water is with a constant density, and the flow to be irrotational so that the flow may be described by a velocity potential ϕ which is a scalar function whose gradient equals to the velocity of the water. Moreover, from the description in [1], when the free surface (this is the name for the surface of the water) disturbed by some disturbance that may be caused by the wind or the passage of a ship, the restoring forces which consist of gravity and free surface tension will make the water go back to its equilibrium state. In such situation, the water waves have been generated by the water inertia and restoring forces. What's more, it might be a reasonable and proper assumption that wavelength is much bigger than its amplitude in the reality, in which case the surface tension could be neglected. It is therefore rational to build up a linear system to illustrate the water wave. The shape of the water wave may be described as the figure 1. Also, the linearised free-surface condition will be used in this project, however the condition is nonlinear in general situation.

1.2 The derivation of the governing equations

The derivation of governing equations is introduced in C. Linton and P. McIver's *Handbook of Mathematical Techniques for Wave/Structure Interactions* [2]. The equation for conservation of mass for an inviscid and incompressible fluid is

$$\nabla \cdot \mathbf{v} = 0$$

where $\mathbf{v}$ is the velocity field. Based on the former assumption that the water is irrotational so that the velocity field can be written as the gradient of a potential function, $\mathbf{v} = \nabla\phi$. Therefore the equation for conservation of mass becomes the Laplace's equation

$$\nabla^2\phi = 0.$$

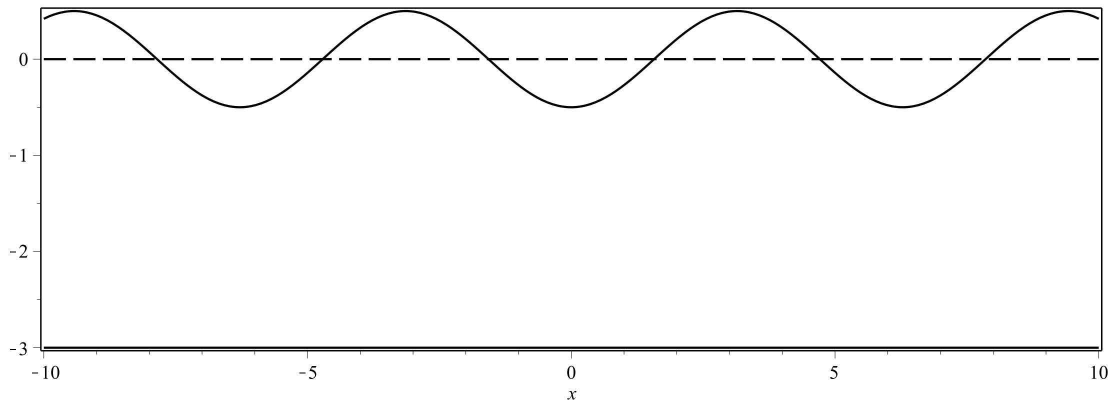

Figure 1: The rough shape of wave

As what is discussed in [2], the linearized boundary conditions that will be used first come from the nonlinear kinematic and dynamic free-surface boundary-conditions. We first denote

$$z = \eta(x, y, t)$$

as the vertical elevation of a particle on the surface of water, where (x, y, z) is Cartesian coordinate system with (x,y) plane described the undisturbed surface of water and axis z being perpendicular to the surface. The nonlinear kinematic free-surface boundary-condition is described as

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \eta}{\partial y}\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = \eta(x, y, t) \tag{1}$$

which states the vertical motion of the flow equals to the vertical velocity of the free surface. The dynamic free-surface boundary condition could be written as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + g\eta = 0 \quad \text{on} \quad z = \eta(x, y, t) \tag{2}$$

which provided by Bernoulli's equation, where g is the acceleration of gravity.
Based on the previous assumption that amplitude of the wave is sufficiently small compared with the wavelength so that the nonlinear kinematic and dynamic free-surface boundary-conditions in equations (1) and (2) can be linearized on $z = 0$. In this case, the nonlinear kinematic and dynamic free-surface boundary-conditions become

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on} \quad z = 0 \tag{3}$$

and

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on} \quad z = 0. \tag{4}$$

Combine equation (3) and the derivation of equation (4) together, we get the linearized boundary condition

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0. \tag{5}$$

Besides, having noticed that from equation (4), it can be obtained the expression of vertical elevation of the water surface, η which is

$$\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t} \quad \text{on} \quad z = 0. \tag{6}$$

2 The two-dimensional spreading problem

2.1 The mathematical model of two-dimensional problem

We consider a two-dimensional layer of water with a constant depth h but no limitation in both horizontal directions. Choose Cartesian coordinates (x, z) to describe the plane of two-dimensional layer of water with coordinate axis x or $z = 0$ being its free surface and z directed vertically upwards from the surface of the water. In this way, $z = -h$ becomes the bed of water. Thus, we get the water domain is bounded by $\mathbf{D} = \{(x, z) : x \in \mathbb{R}, -h < z < 0\}$. Then, an initial disturbance needs to be given to the free surface to cause the wave. It is assumed that the disturbance happens at time $t = 0$ and for $t < 0$, the water is at rest. We use a potential $\phi_0(x)$ and a free surface elevation $H_0(x)$ to describe the initial disturbance. At subsequent times the velocity potential $\phi(x, z, t)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad in\, \mathbf{D} \tag{2.1}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = -h, x \in \mathbb{R} \tag{2.2}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0,\ x \in \mathbb{R} \tag{2.3}$$

$$|\nabla\phi| \to 0 \quad \text{on} \quad |x| \to \infty,\ -h < z < 0 \tag{2.4}$$

$$\phi(x, 0, 0) = \phi_0(x) \quad \text{on} \quad z = 0 \tag{2.5}$$

$$\frac{\partial\phi}{\partial t}(x, 0, 0) = -gH_0(x) \quad \text{on} \quad z = 0 \tag{2.6}$$

- Equation (2.1) is the Laplace's equation which has been proved being satisfied by the velocity potential $\phi(x, z, t)$ everywhere in the water.
- Equation (2.2) illustrates that there is no flow through the bed of water.
- Equation (2.3) is the linearized free surface condition that has been given in (5) and it should be applied on the undisturbed free surface $z = 0$.
- Equation (2.4) means there is no motion at infinity.
- Equation (2.5) states that at $t = 0$ and $z = 0$, the velocity potential equals to a given potential.
- Equation (2.6) illustrates that $H_0(x)$ is the initial elevation of the free-surface.

It can be considered that equation (2.2) and (2.3) are the boundary conditions and equation (2.5) and (2.6) are the initial conditions.

2.2 The solution of two-dimensional problem

We will use Fourier transform to solve the governing equation which satisfies both boundary and initial conditions. The formula of Fourier transform may vary. I choose the one giving in [3]: Fourier transform in x of a function f is defined by

$$f^*(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

and the inverse version of Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)e^{ikx}dk.$$

It is worth mentioning that we apply the Fourier transform to this problem with respect to x, nevertheless, it also could be applied to t. They are just different ways to solve the problem, but it will be much more complex using the way of applying the transform with respect to t.

The first step of solving the problem is to apply the Fourier transform to the governing equation, boundary and initial conditions(we use ϕ^* to show the transformed version). To get the transformed governing equation, we need to apply the Fourier transform to every term in equation (2.1) which are as followings (here $\mathscr{F}$ denotes the Fourier transform):

$$\mathscr{F}[\phi(x,z,t)] = \int_{-\infty}^{\infty} \phi(x,z,t)e^{-ikx}dx = \phi^*(k,z,t)$$

$$\mathscr{F}[\phi_{xx}] = \int_{-\infty}^{\infty} \phi_{xx}e^{-ikx}dx = \int_{-\infty}^{\infty} e^{-ikx}d\phi_x = -k^2\phi^*$$

$$\mathscr{F}[\phi_{zz}] = \int_{-\infty}^{\infty} \phi_{zz}e^{-ikx}dx = \phi^*_{zz}$$

Thus, the governing equation becomes:

$$-k^2\phi^* + \phi^*_{zz} = 0$$

By using the same way, the boundary conditions are transformed into

$$\mathscr{F}[\phi_z] = \phi^*_z = 0 \quad \text{on} \quad z = -h$$

$$\mathscr{F}[\phi_{tt} + g\phi_z] = \phi^*_{tt} + g\phi^*_z = 0 \quad \text{on} \quad z = 0,$$

and initial conditions turn out to be

$$\mathscr{F}[\phi(x,0,0)] = \phi^*_0(x)$$

$$\mathscr{F}[\phi_t(x,0,0)] = -gH^*_0(x).$$

Therefore, the original two-dimensional problem is now with the form

$$-k^2\phi^* + \phi^*_{zz} = 0, \tag{2.7}$$

with the boundary conditions

$$\phi^*_z = 0 \quad \text{on} \quad z = -h \tag{2.8}$$

and

$$\phi^*_{tt} + g\phi^*_z = 0 \quad \text{on} \quad z = 0, \tag{2.9}$$

and the initial conditions

$$\phi^*(x,0,0) = \phi^*_0(x) \tag{2.10}$$

and

$$\phi_t^*(x,0,0) = -gH_0^*(x). \tag{2.11}$$

The general solution of equation (2.7) is

$$\phi^*(k,z,t) = A(k,t)e^{kz} + B(k,t)e^{-kz}.$$

Substitute this general solution into equation (2.8) so that we get

$$\phi_z^* = -kA(k,t)e^{-kh} - kB(k,t)e^{kh} = 0 \tag{2.12}$$

From the above equation, it can be acquired that $B = Ae^{-2kh}$. Substituting it into the general solution gives:

$$\phi^*(k,z,t) = A(k,t)e^{kz} + A(k,t)e^{-2kh}e^{-kz},$$

which also equals to

$$A(k,t)e^{-kh} \cdot 2\cosh(kz+kh).$$

If we let

$$C(k,t) = 2Ae^{-kh},$$

then we get

$$\phi^*(k,z,t) = C(k,t)\cosh(kz+kh). \tag{2.13}$$

We now substitute the form of $\phi(k,z,t)$ in equation (2.13) into the second boundary condition, equation (2,9) and get:

$$\phi_{tt}^* + g\phi_z^* = C_{tt}\cdot\cosh(kh) + gkC\cdot\sinh(kh) = 0 \quad \text{on} \quad z=0 \tag{2.14}$$

i.e.

$$C_{tt} + gkC\cdot\tanh(kh) = 0 \quad \text{on} \quad z=0 \tag{2.15}$$

The next job is to work out the expression of C(k,t) and substitute it back into equation (2.13) in order to get the solution of $\phi^*(k,z,t)$. We know that the general solution of equation (2.15) is

$$C(k,t) = E(k)\cos(\omega t) + F(k)\sin(\omega t), \tag{2.16}$$

where

$$\omega = \sqrt{gk\cdot\tanh(kh)}.$$

For getting $C(k,t)$, we only need to figure out the expressions of $E(k)$ and $F(k)$. Here, the initial conditions, equation (2.10) and (2.11) will be put into use. Now we substitute the equation (2.16) into the initial conditions:

$$\phi^*(k,0,0) = C(k,0)\cosh(kh) = E(k)\cosh(kh) = \phi_0^*(k)$$

i.e.

$$E(k) = \frac{\phi_0^*(k)}{\cosh(kh)} \tag{2.17}$$

and

$$\phi_t(k,0,0) = C_t(k,0)\cosh(kh) = \omega F(k)\cosh(kh) = -gH_0^*(k)$$

i.e.

$$F(k) = \frac{-gH_0^*(k)}{\omega\cosh(kh)}. \tag{2.18}$$

Substitute every term into the expression of $\phi^*(k,z,t)$. Here is the final solution of $\phi^*(k,z,t)$:

$$\phi^*(k,z,t) = C(k,t)\cosh(kz+kh)$$

where

$$C(k,t) = \frac{\phi_0^*(k)}{\cosh(kh)}\cos(\omega t) - \frac{-gH_0^*(k)}{\omega\cosh(kh)}\sin(\omega t)$$

and

$$\omega = \sqrt{gk\tanh(kh)}. \tag{2.19}$$

Note that this is the solution after performing Fourier transform. The last step that needs to be done is apply the inverse Fourier transform to the solution of $\phi^*(k,z,t)$ to get the final solution of $\phi(x,z,t)$. After doing so, we finally get the solution of two-dimensional problem,

$$\phi(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi^*(k,z,t)e^{ikx}dx. \tag{2.20}$$

2.3 Solution check

To make sure we get the right solution, we need to bring this solution back into the original problem to see if all the governing equation, boundary conditions and initial conditions still be satisfied. It is obvious that the governing equation is satisfied by this solution. We only need to check the boundary and initial conditions. For initial conditions:

$$\phi(x,0,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi^*(k,0,0)e^{ikx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_0^*(k)e^{ikx}dx = \phi_0(x),$$

and

$$\phi_t(x,0,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_t^*(k,0,0)e^{ikx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}-gH_0^*(k)e^{ikx}dx = -gH_0(x).$$

For boundary conditions:

$$\phi_z(x,z,t) = \frac{1}{2\pi}\cdot\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\phi^*(k,z,t)e^{ikx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_z^*(x,z,t)dx = 0 \quad \text{on} \quad z=-h,$$

and

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{-\omega^2\phi_0^*(k)}{\cosh(kh)}\cos(\omega t)+\frac{g\omega H_0^*(k)}{\cosh(kh)}\sin(\omega t)+g[\frac{\phi_0^*(k)}{\cosh(kh)}\cos(\omega t)-$$

$$\frac{-gH_0^*(k)}{\omega\cosh(kh)}\sin(\omega t)]\omega^2 k\sinh(kh)e^{ikx}dx=0 \quad \text{on} \quad z=0.$$

From the above check, we could see that every condition is satisfied by the solution we obtained.

2.4 Numerical calculation of the solution

The next work will be done with the help of the software Maple. The main works with Maple basically are:

1. Choose a specific value of h and some easy functions as initial conditions to plot the shape of initial water surface.
2. Plot the spreading animation of water wave in some time interval with these initial conditions.
3. Find a quicker way to plot animations.
4. Find more applicable functions that could be used as initial conditions.

For point 1, what I choose for h is let $h=1$. It can be vary, but h needs to be positive. Then, choose some specific functions as ϕ_0 and H_0 which are initial conditions and these functions need to have Fourier transformed version and have the shape like a wave. In my work, I divide the plotting into two situations. Situation one is let $\phi_0=0$ and H_0 equals to an nonzero function which has the above characteristics. The physical interpretation of this situation is equivalent to draw the free surface to a specific hight which may cause by the wind or the passage of a ship, and let it go to see how wave spreads out. The other one is just on the opposite. ϕ_0 equals to that nonzero function and $H_0=0$. This situation is like hitting the surface (e.g. with hand).
For the nonzero function, I would like to select

$$f(x)=\frac{1}{1+x^2}$$

and it's transformed version is

$$f^*(x)=\pi e^{-|k|}.$$

The shapes of these two functions are in figure 2 and figure 3. The reasons of choosing this function are that first it could be easier to start with these simple formed functions since it may save a lot of time when being calculated by computer and they also have the shape of wave that we could see from the figures at the same time.

When plotting, what is worth noticing is that ϕ is only the potential function but not the real shape of water. However, from the previous discussion in introduction, we do know that η in equation (6) which is

$$\eta=-\frac{1}{g}\frac{\partial\phi}{\partial t}$$

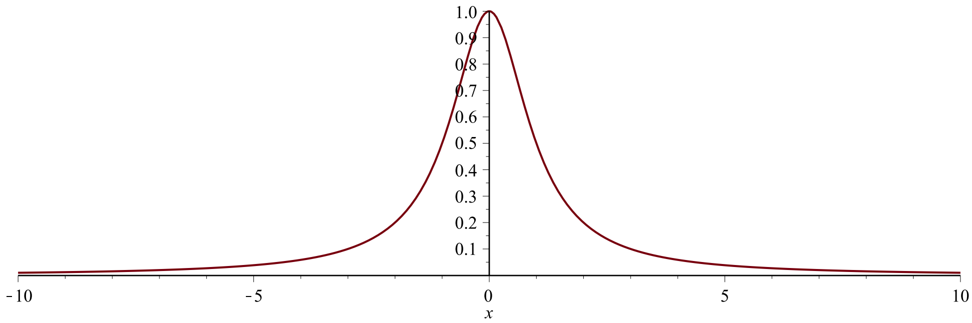

Figure 2: The function of $\frac{1}{1+x^2}$

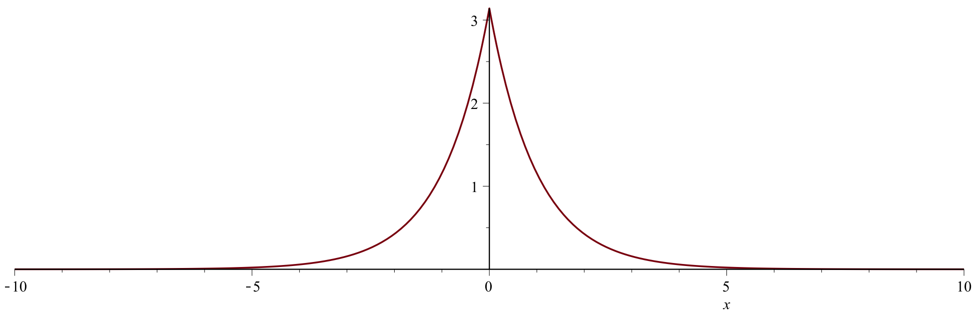

Figure 3: The function of $\pi e^{-|k|}$

is the elevation of free surface. Therefore, η is the function that needs to be plotted. Maple code please see appendix 1.

From the Maple code, we can see that there are actually only the transformed versions of initial conditions that are put into the commands. Because that are the only terms appeared in the expression of η. Figure 4 is the initial status($t = 0$) of free surface of situation one. Figure 5 is the plot of η in situation two at $t = 0.1$ that is because when $t = 0$ situation two gives $\eta = 0$. This makes sense since we give the free surface a disturbance at $t = 0$. At that point, free surface still stay at the state of rest, but at next time point the surface will be different.

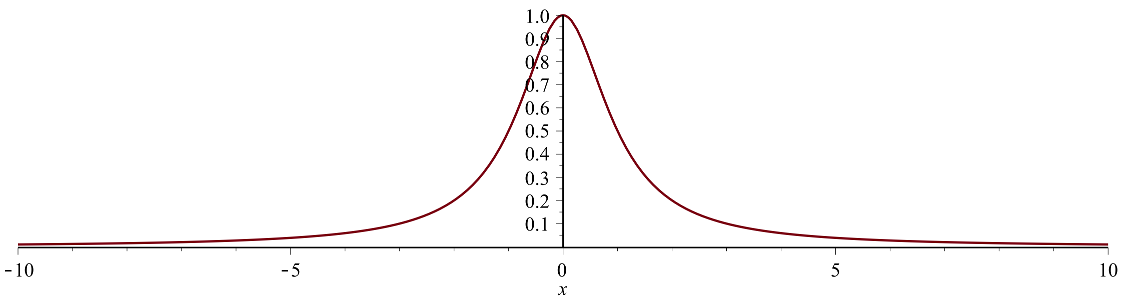

Figure 4: Initial status of situation one

Point 2, plotting animations is easier but more interesting. For both situations, I made the first 10 seconds of spreading animations. Maple codes please see appendix 2. Figure 6 to figure 12 are some plots of spreading animations at some specific time points in situation one and figure 13 to figure 20 are the plotps in situation two .

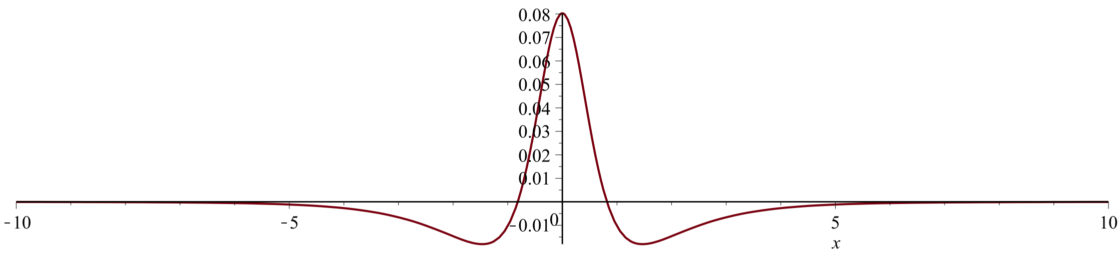

Figure 5: Initial status of situation two

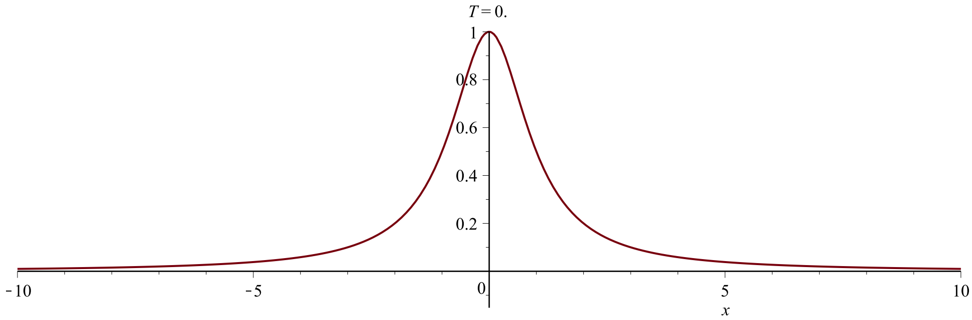

Figure 6: situation one at $t = 0$

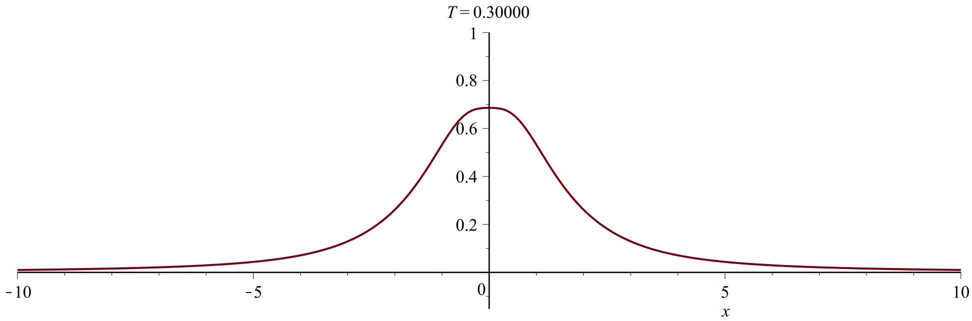

Figure 7: situation one at $t = 0.3$

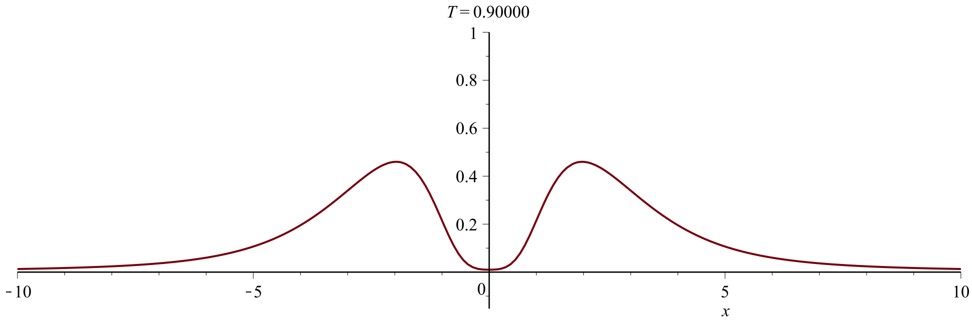

Figure 8: situation one at $t = 0.9$

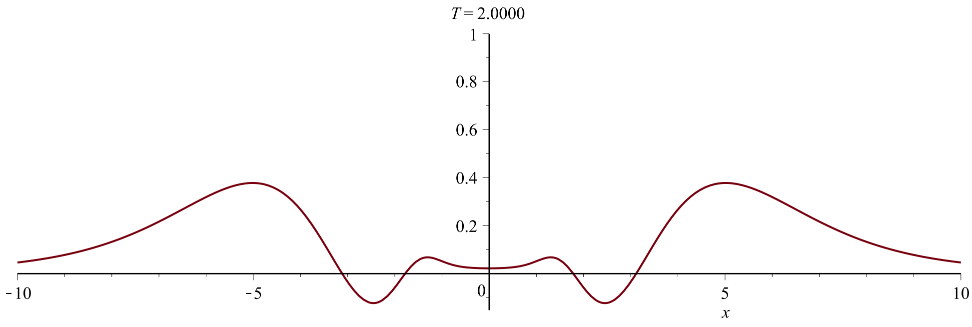

Figure 9: situation one at $t = 2$

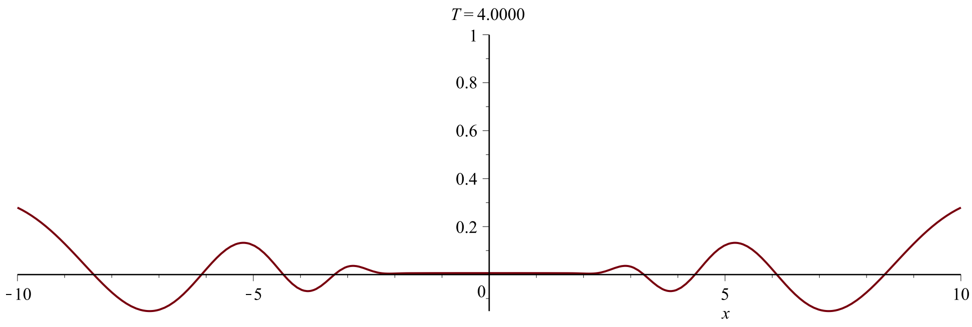

Figure 10: situation one at $t = 4$

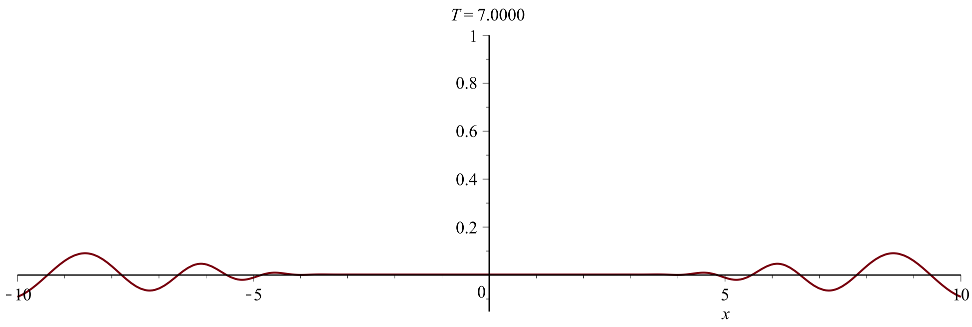

Figure 11: situation one at $t = 7$

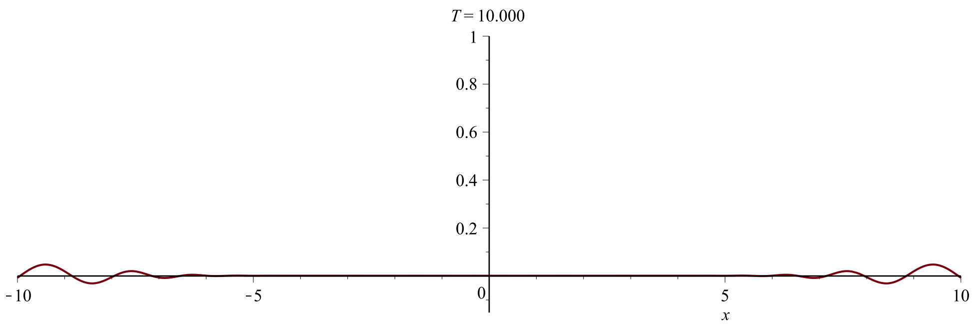

Figure 12: situation one at $t = 10$

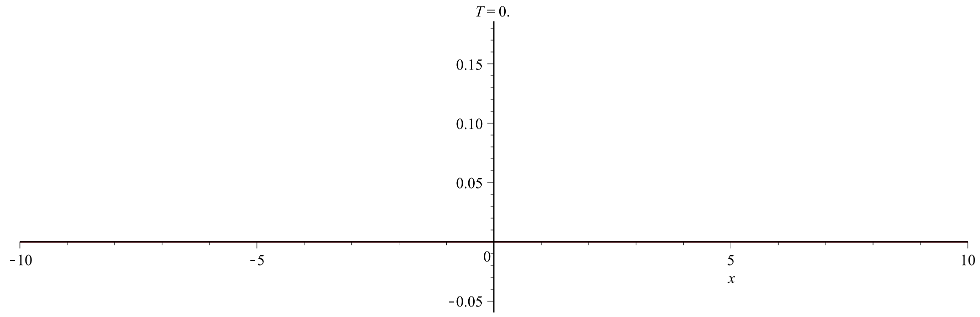

Figure 13: situation two at $t = 0$

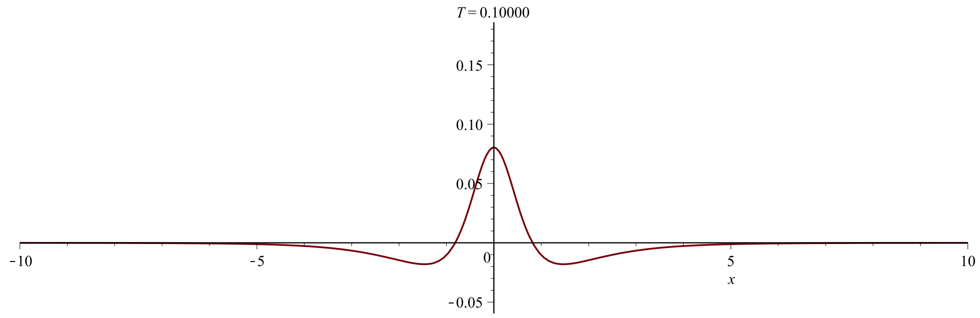

Figure 14: situation two at $t = 0.1$

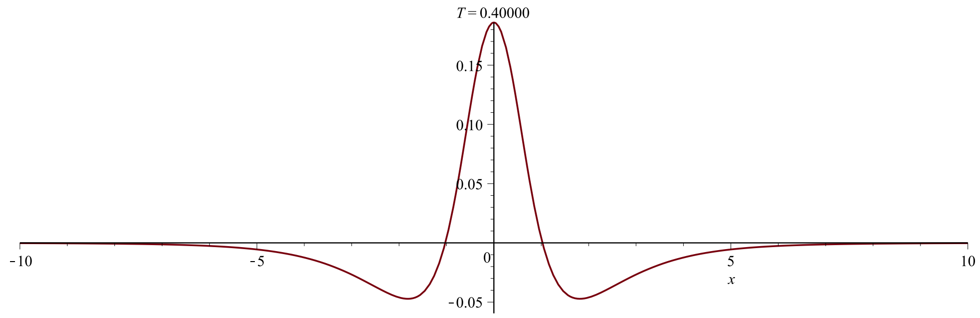

Figure 15: situation two at $t = 0.4$

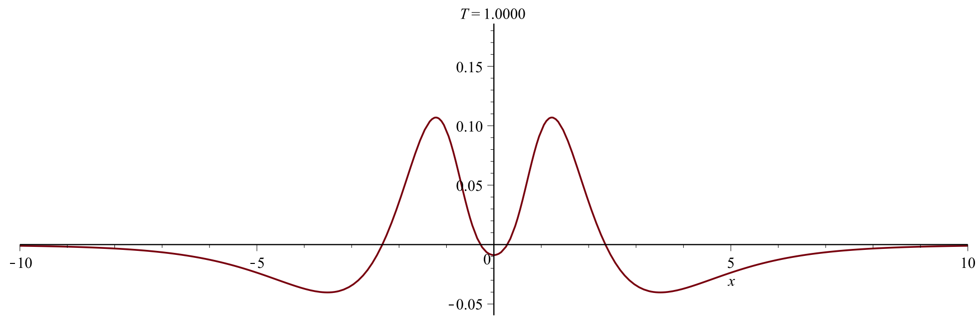

Figure 16: situation two at $t = 1$

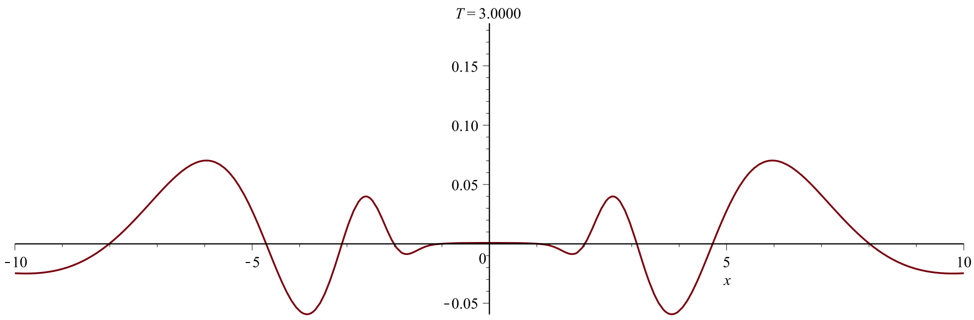

Figure 17: situation two at $t = 3$

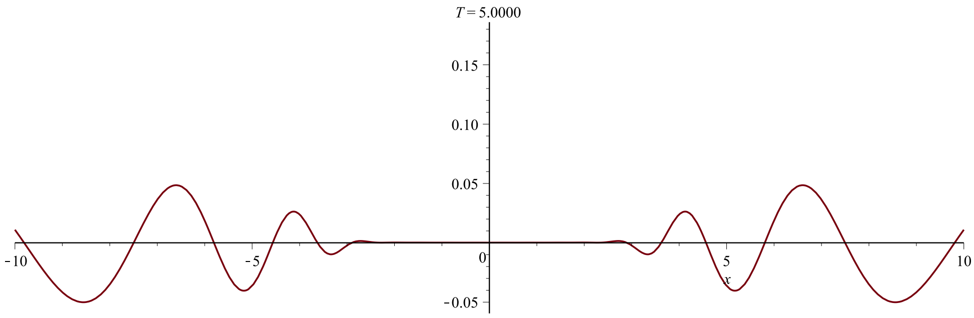

Figure 18: situation two at $t = 5$

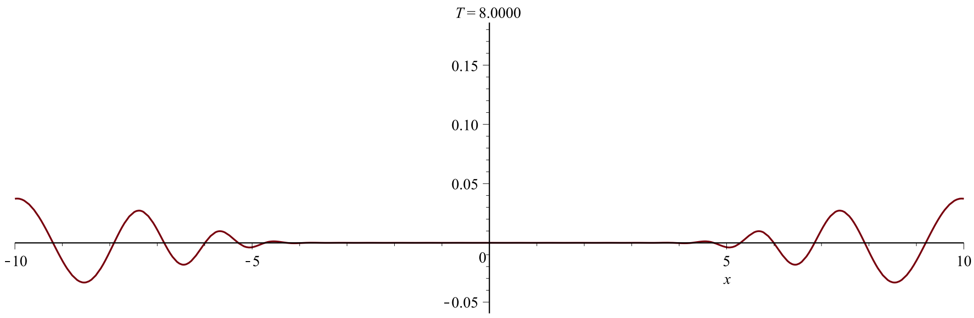

Figure 19: situation two at $t = 8$

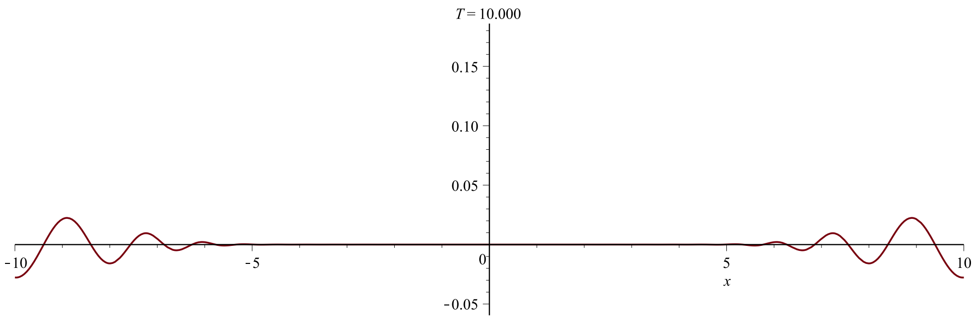

Figure 20: situation two at $t = 10$

From the above plots and the speed c of water waves which is given in [9] by

$$c^2 = (\frac{\omega}{k})^2 = \frac{g}{k}\tanh kh,$$

we could see that the longer the wavelength is the faster the wave will travel. Here, g is the the acceleration due to gravity, ω is the same with in equation (2.19), $k = \frac{2\pi}{\lambda}$ and λ is the wavelength,

When the animations of η being plot, they take the computer hours to finish plotting. Hence, it is necessary to find a faster way to plot which is point 3. Doing some transformation to the expression of η, like transforming η from a complex function into a real function so that it may save time in calculating the imaginary part. It has been proved to be effective by practice.
We could start with separating the integral in η into two parts, minus infinity to zero and zero to infinity. Next is to choose the real part of expression. We name the new form of η to η_1.

$$2\pi\eta_1 = \int_0^\infty \phi_t e^{ikx} dk + \int_{-\infty}^0 \phi_t e^{ikx} dk = 2\Re \int_0^\infty \phi_t e^{ikx} dk \tag{2.21}$$

i.e.

$$\eta_1 = \frac{1}{\pi g}\Re \int_0^\infty \phi_t e^{ikx} dk. \tag{2.22}$$

Here we use the knowledge:

$$\int_{-\infty}^0 f^*(k)dk = \int_\infty^0 f^*(-k)(-dk) = \int_0^\infty f^*(-k)dk$$

and

$$f^*(-k) = \overline{f^*(k)}.$$

The real difference between using η and η_1 in plotting animations is remarkable and it saves nearly a third of time. However, when comes to plotting the diagraphs of initial status, the difference is not that obvious. Plotting η_1 is less than 0.2 seconds faster than plotting η, a slight faster, but still a progress.

We already know that there are not only one functions that could be used as initial conditions. Except the one I choose, like:

$$f(x) = Ae^{-\beta x}, \beta > 0$$

which Fourier transformed version is

$$f^*(k) = \sqrt{\frac{\pi}{\beta}} A e^{\frac{\omega^2}{4\beta}}$$

is also a good choice. For this function, it equips with all the necessary conditions and also could be controlled in rate of convergence by changing β.

2.5 Conclusion of two-dimensional problem

In two dimensional problem, we acquired two-dimensional mathematical model including governing equation, boundary conditions and initial conditions:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad \text{in } \mathbf{D}$$

$$\frac{\partial\phi}{\partial z}=0 \quad \text{on} \quad z=-h, x\in\mathbb{R}$$

$$\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}=0 \quad \text{on} \quad z=0,\ x\in\mathbb{R}$$

$$\left|\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial z}\right|\to 0 \quad \text{as} \quad |x|\to\infty,\ -h<z<0$$

$$\phi(x,0,0)=\phi_0(x) \quad \text{on} \quad z=0$$

$$\frac{\partial\phi}{\partial t}(x,0,0)=-gH_0(x) \quad \text{on} \quad z=0.$$

Besides, we have also got the solution of two-dimensional spreading problem with the help of Fourier transform:

$$\phi^*(k,z,t)=C(k,t)\cosh(kz+kh)$$

where

$$C(k,t)=\frac{\phi_0^*(k)}{\cosh(kh)}\cos(\omega t)-\frac{-gH_0^*(k)}{\omega\cosh(kh)}\sin(\omega t)$$

and

$$\omega=\sqrt{gk\tanh(kh)}.$$

After this, the initial status of free surface in two different situations and the spreading of water wave were plotted by Maple which seem quite vivid to the real situation.

3 The three-dimensional spreading problem

Three-dimensional problem is expanded from two-dimensional problem by using cylindrical polar coordinates but not Cartesian coordinates. Therefore, basically they have similar governing equation, boundary conditions and initial conditions, but using different coordinates to express.

3.1 The mathematical model of three-dimensional problem

By introducing three-dimensional problem, we should start by bringing in cylindrical polar coordinates, which are like the follows:

$$x=r\cos\theta, y=r\sin\theta, z=z$$

where r is the radius and θ is the angle with axis x. After gaining cylindrical polar coordinates, we need to rewrite the original spreading problem in two-dimensional space by changing variables through the new coordinates. Therefore, the Laplace's equation of three-dimensional problem takes the form:

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{3.1}$$

which could be simply acquired by direct transformation.
Then, we re-express the potential function ϕ through the new variables and separate variables by the substitution of $\phi = R(r)Z(z)\Theta(\theta)$. It leads to the Sturm-Liouville problem

$$\Theta'' + n^2\Theta = 0,\ \Theta'(0) = 0,$$

where n is a constant, and another problem of R and Z which is

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{Z''}{Z}=\lambda^2,$$

where λ is another constant. Solving these two problems gives the solution

$$\phi = \sum_{n=0}^{\infty} F_n(r,z,t)\cos n\theta, \tag{3.2}$$

where F_n $(n = 0, 1, 2...)$ are going to be determined.
Substitute (3.2) into (3.1) and divide by $\cos n\theta$ at the same time to get the new governing equation:

$$\frac{\partial^2 F_n}{\partial r^2}+\frac{1}{r}\frac{\partial F_n}{\partial r}-\frac{n^2}{r^2}F_n+\frac{\partial^2 F_n}{\partial z^2}=0. \tag{3.3}$$

Then we substitute the new form of ϕ into the original boundary and initial conditions.

$$\frac{\partial\phi}{\partial z}=0 \quad \text{on} \quad z=-h$$

turns out to be

$$\sum_{n=0}^{\infty}\frac{\partial F_n}{\partial z}\cos n\theta = 0 \quad \text{on} \quad z=-h.$$

In order to remove the term of θ, orthogonality is being applied. We multiply both sides by $\cos n\theta$ of the above equation, and then integrate from 0 to 2π with respect of θ to get:

$$\sum_{n=0}^{\infty}\frac{\partial F_n}{\partial z}\int_0^{2\pi}\cos n\theta\cos m\theta\, d\theta = 0 \quad \text{on} \quad z=-h \tag{3.4}$$

i.e

$$\frac{\partial F_n}{\partial z}=0 \quad \text{on} \quad z=-h. \tag{3.5}$$

Since we know that:

$$\int_0^{2\pi}\cos n\theta\cos m\theta d\theta = \begin{cases} 0 & \text{if } n\neq m \\ \pi & \text{if n=m}\end{cases}$$

For the second boundary condition:

$$\sum_{n=0}^{\infty}\frac{\partial^2 F_n}{\partial t^2}\cos n\theta + g\sum_{n=0}^{\infty}\frac{\partial F_n}{\partial z}\cos n\theta = 0 \quad \text{on} \quad z=0,$$

apply the same operation-multiply both sides by $\cos n\theta$ and integrate from 0 to 2π with respect of θ :

$$\sum_{n=0}^{\infty} \frac{\partial^2 F_n}{\partial t^2} \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta + g \sum_{n=0}^{\infty} \frac{\partial F_n}{\partial z} \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = 0 \quad \text{on} \quad z = 0$$

i.e.

$$\frac{\partial^2 F_n}{\partial t^2} + g \frac{\partial F_n}{\partial z} = 0 \quad \text{on} \quad z = 0.$$

Do the same transformation to initial conditions:

$$\phi(x, 0, 0) = \phi_0(x),$$

therefore the expression of F_n becomes

$$\sum_{n=0}^{\infty} F_n(r, 0, 0) \cos n\theta = \phi_0(r, \theta),$$

i.e.

$$\sum_{n=0}^{\infty} F_n(r, 0, 0) \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = \int_0^{2\pi} \phi_0(r, \theta) \cos m\theta \, d\theta,$$

and it follows by

$$F_n(r, 0, 0) = \frac{1}{\pi} \int_0^{2\pi} \phi_0(r, \theta) \cos m\theta \, d\theta,$$

noting it as $A_{0n}(r)$, and

$$\frac{\partial \phi}{\partial t}(x, 0, 0) = -gH_0(x)$$

comes to the form

$$\frac{\partial F_n}{\partial t}(r, 0, 0) \cos n\theta = -gH_0(r, \theta).$$

After applying the same transformation to the second initial condition, here we get

$$\frac{\partial F_n}{\partial t}(r, 0, 0) \int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta = -g \int_0^{2\pi} H_0(r, \theta) \cos m\theta \, d\theta$$

i.e.

$$\frac{\partial F_n}{\partial t}(r, 0, 0) = -g \int_0^{2\pi} H_0(r, \theta) \cos n\theta \, d\theta,$$

noting it as $-gB_{0n}(r)$. Hence, we get the three-dimensional problem with cylindrical polar coordinates:

$$\frac{\partial^2 F_n}{\partial r^2} + \frac{1}{r} \frac{\partial F_n}{\partial r} - \frac{n^2}{r^2} F_n + \frac{\partial^2 F_n}{\partial z^2} = 0, \tag{3.6}$$

with boundary conditions:

$$\frac{\partial F_n}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{3.7}$$

$$\frac{\partial^2 F_n}{\partial t^2} + g \frac{\partial F_n}{\partial z} = 0 \quad \text{on} \quad z = 0 \tag{3.8}$$

and initial conditions:

$$F_n(r, 0, 0) = A_{0n}(r) \tag{3.9}$$

$$\frac{\partial F_n}{\partial t}(r, 0, 0) = -gB_{0n}(r). \tag{3.10}$$

3.2 The solution of three-dimensional problem

In order to get the solution of three-dimensional problem, we need the help of Hankel transform which with the same form in [4]:

$$F_n^*(k) = \mathscr{H}(f(r)) = \int_0^\infty f(r)J_n(kr)r\,dr$$

where $J_n(kr)$ is Bessel function, and inverse Hankel transform is

$$f(r) = \mathscr{H}^{-1}(F_n^*(k)) = \int_0^\infty F_n^*(k)J_n(kr)k\,dk.$$

Using the similar way in two-dimensional problem, apply this transform to governing equation, initial conditions and boundary conditions to give the transformed problem.

Apply Hankel transform separately to the last term and first three terms in equation (3.6) to give:

$$\mathscr{H}(\frac{\partial^2 F_n}{\partial z^2}) = \frac{\partial^2}{\partial z^2}\int_0^\infty F_n(r,z,t)J_n(kr)r\,dr = F_{zz}^*,$$

and denote

$$\Delta_n F_n = \frac{\partial^2 F_n}{\partial r^2} + \frac{1}{r}\frac{\partial F_n}{\partial r} - \frac{n^2}{r^2}F_n = \frac{1}{r}\frac{\partial}{\partial r}(\frac{r\partial F_n}{\partial r}) - (\frac{n}{r})^2 F_n,$$

thus

$$\mathscr{H}(\Delta_n F_n) = \int_0^\infty [r^{-1}(r(F_n)_r)_r - n^2r^{-2}F_n]J_n(kr)r\,dr. \tag{3.11}$$

Simply integrate by parts to equation (3.11) gives

$$\mathscr{H}(\Delta_n F_n) = [r(F_n)_r J_n - rF_n(J_n)_r]\Big|_0^\infty + \int_0^\infty rF_n r^{-1}(r(J_n)_r)_r - n^2r^{-2}J_nF_n r\,dr,$$

having found that the first term equals to zero so that

$$\mathscr{H}(\Delta_n F_n) = \int_0^\infty rF_n\{\Delta_n J_n(kr)\}\,dr.$$

Based on the definition of Bessel function:

$$(\Delta_n + k^2)J_n(kr) = 0,$$

i.e

$$\mathscr{H}(\Delta_n F_n) = -k^2\mathscr{H}(F_n) = -k^2F_n^*,$$

the governing equation therefore becomes

$$(F_n)_{zz}^* - k^2F_n^* = 0. \tag{3.12}$$

Then, we apply the Hankel transform to boundary conditions to give

$$\frac{\partial F_n^*}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{3.13}$$

$$\frac{\partial^2 F_n^*}{\partial t^2} + g\frac{\partial F_n^*}{\partial z} = 0 \quad \text{on} \quad z = 0 \tag{3.14}$$

and to initial conditions to give

$$F_n^*(r,0,0) = A_{0n}^*(r) \tag{3.15}$$

$$\frac{\partial F_n^*}{\partial t}(r,0,0) = -gB_{0n}^*(r). \tag{3.16}$$

The general solution of the equation (3.12) is

$$F_n^*(k,z,t) = A(k,t)e^{kz} + B(k,t)e^{-kz}. \tag{3.17}$$

Substitute it into the boundary condition (3.13)

$$\frac{\partial F_n^*}{\partial z} = kA(k,t)e^{-kh} - kB(k,t)e^{kh} = 0 \quad \text{on} \quad z = -h$$

to get

$$B(k,t) = A(k,t)e^{-2kh}. \tag{3.18}$$

Bring it back to equation (3.17) so that the general solution comes to the form

$$\begin{aligned} F_n^*(k,z,t) &= A(k,t)e^{-kh} \cdot 2\cosh(kz+kh) \\ &= C(k,t)\cosh(kh+kz) \end{aligned} \tag{3.19}$$

where

$$C(k,t) = 2A(k,t)e^{-kh}.$$

Now, we use another boundary condition (3.14). Substituting equation (3.19) into it gives

$$\frac{\partial^2 F_n^*}{\partial t^2} + g\frac{\partial F_n^*}{\partial z} = C_{tt}\cosh(kh) + gkC\sinh(kh) = 0.$$

Divide by $\cosh(kh)$ on both sides at same time to get

$$C_{tt} + gkC\tanh(kh) = 0. \tag{3.20}$$

The general solution of the above equation is

$$C(k,t) = E(k)\cos\omega t + F(k)\sin\omega t \tag{3.21}$$

where

$$\omega = \sqrt{gk\tanh(kh)}$$

In order to get the expressions of $E(k)$ and $F(k)$, equation (3.21) should be substituted into both initial conditions.

$$\begin{aligned} C(k,0)\cosh(kh) = E(k)\cosh(kh) = A_{0n}^* \\ C_t(k,0)\cosh(kh) = \omega F(k)\cosh(kh) = B_{0n}^*, \end{aligned}$$

i.e.

$$E(k) = \frac{A_{0n}^*}{\cosh(kh)} \tag{3.22}$$

$$F(k) = \frac{B_{0n}^*}{\omega\cosh(kh)}. \tag{3.23}$$

Here, we get the solution of the transformed governing equation (3.12)

$$F_n^*(k,z,t) = C(k,t)\cosh(kh+kz)$$

where

$$C(k,t) = \frac{A_{0n}^*}{\cosh(kh)}\cos(\omega t) - g\frac{B_{0n}^*}{\omega\cosh(kh)}\sin(\omega t)$$

and

$$\omega = \sqrt{gk\tanh(kh)}. \tag{3.24}$$

Apply the inverse Hankel transform to this solution gives

$$F_n(r,z,t) = \int_0^\infty F_n^*(k,z,t)J_n(kr)k\,dk \tag{3.25}$$

and therefore we obtain the expression of velocity potential ϕ.

$$\phi = \sum_{n=0}^{\infty}\int_0^\infty F_n^*(k,z,t)J_n(kr)k\,dk\cos n\theta. \tag{3.26}$$

3.3 Comparison of solutions between two-dimensional and three-dimensional problems

We can see from the solution of two-dimensional problem,

$$\phi^*(k,z,t) = C(k,t)\cosh(kz+kh)$$

where

$$C(k,t) = \frac{\phi_0^*(k)}{\cosh(kh)}\cos(\omega t) - \frac{gH_0^*(k)}{\omega\cosh(kh)}\sin(\omega t)$$

and

$$\omega = \sqrt{gk\tanh(kh)}, \tag{3.27}$$

that basically the solution of three-dimensional problem's governing equation has the almost same expression of $F_n^*(k,z,t)$ with $\phi^*(k,z,t)$ except for the initial conditions. But the differences between them are firstly, the form of inverse transform, what more notable is after gaining $F_n^*(k,z,t)$, three-dimensional problem uses parameters n and θ to expand the solution into three dimensional space, which is the essential difference.

3.4 Numerical calculation of the solution

Still similar with two-dimensional problem, we do the same work to three-dimensional problem using Maple: Plot the initial status of free surface and the spreading animation. In order to plot the graph, first of all, we need to choose a proper function as initial condition which has Hankel transformed version and the shape of wave. Here, I choose

$$f(r) = e^{-a^2r^2}r^n,$$

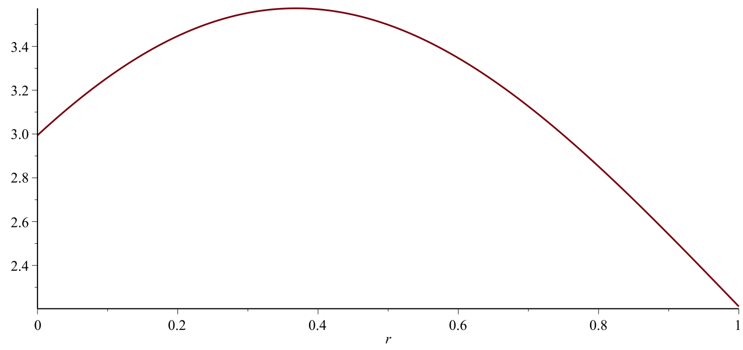

Figure 21: Situation one:η in two dimensional space

and its transformed version is

$$F^*(k) = \frac{k^n}{(2a^2)^{n+1}} e^{-\frac{k^2}{4a^2}}, \tag{3.28}$$

where a is a constant. We could change the rate of convergence by adjusting a, but for simplicity, we let $a = 1$. As before, we divide the plotting work into two situations: One is stand for giving free surface a initial disturbance like hitting the water, which is $A^*_{0n} = F^*(k)$ here. The other situation is let $B^*_{0n} = F^*(k)$ which means lift the surface to a specific elevation.

It is the same with two-dimensional problem that there is no practical meaning to plot a velocity potential ϕ. Hence, we need to plot the same function η in equation (6). Substitute equation (3.26) into equation (6) gives

$$\eta = -\frac{1}{g} \sum_{n=0}^{\infty} \int_0^{\infty} [\frac{A^*_{0n}}{\cosh(kh)} \cos(\omega t) - g \frac{B^*_{0n}}{\omega \cosh(kh)} \sin(\omega t)] \cosh(kh + kz) J_n(kr) k \, dk \cos n\theta. \tag{3.29}$$

Instead of plotting three dimensional graph, I plot the two dimensional one first to see if the function $F^*(k)$ will work as the initial condition. Maple codes are attached in appendix 3.

In situation two, if we let $t = 0$, there will show us an undisturbed free surface and the reason is as mentioned in two-dimensional problem: We give a disturbance at time point $t = 0$. At that point, nothing happens to the surface but the disturbance. However, at next time point, there will be waves spreading out.

It can be seen from the codes that if $\theta = 0$, where will be a two-dimensional plot. Since η is a quite complex function, we only calculate k from 0 to 5 and n from 0 to 1 just for simplifying the calculation. However, the value of k will affect the accuracy of the plots. The bigger of k is the longer of time will the calculation cost but more accurate the plot will be and vice versa. Thus, we need to figure out a proper value of k to make either the calculation not taking too long or the plot staying a well accuracy. After doing a few experiments, taking k as different values like $2, 3, 4, 5, 10$, it was finally found that 5 turns out to be a pretty good value for k. Figure 21 is the two-dimensional graph of η of r from 0 to 1 in situation one and figure 22 is the one situation two.

The graphs have been showing that function in (3.28) could be used as a proper initial condition. Therefore, we could use it in plotting three-dimensional initial status and the spreading animation. Maple codes are in appendix 4.

Because in three-dimensional problem, we are using cylindrical polar coordinates. In plotting, it should be switched back to Cartesian coordinates through building a suitable

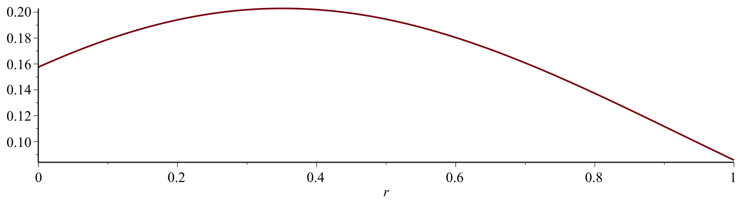

Figure 22: Situation two:η in two dimensional space

coordinate system which is the second code in plotting initial status(see appendix 4). Figure 23 to figure 30 come next are how the wave looks like at different time points in first situation including the initial status.

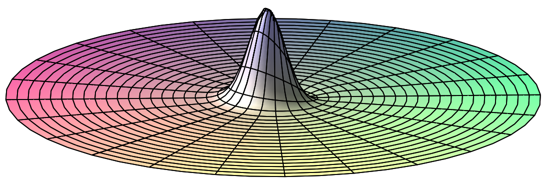

Figure 23: situation one at $t = 0$

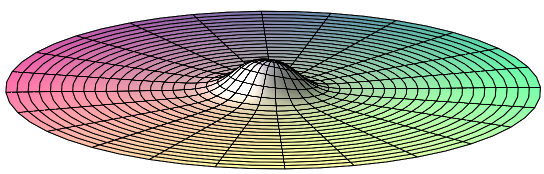

Figure 24: situation one at $t = 0.4$

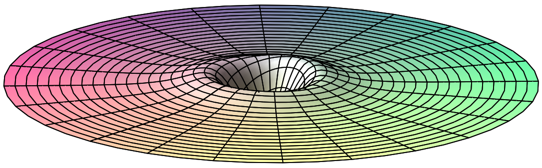

Figure 25: situation one at $t = 1$

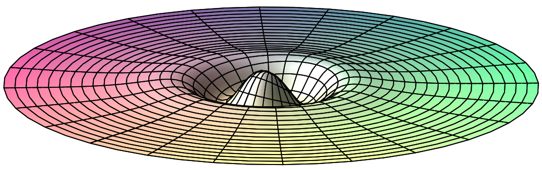

Figure 26: situation one at $t = 1.5$

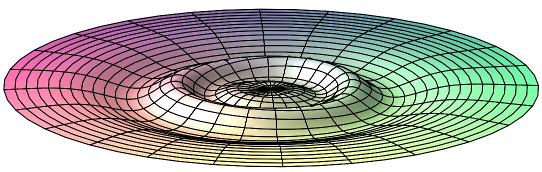

Figure 27: situation one at $t = 3$

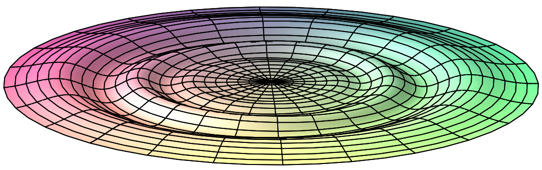

Figure 28: situation one at $t = 5$

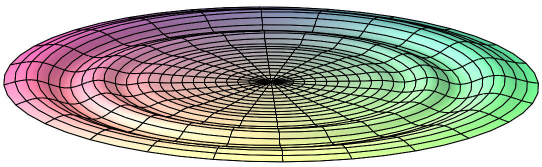

Figure 29: situation one at $t = 7$

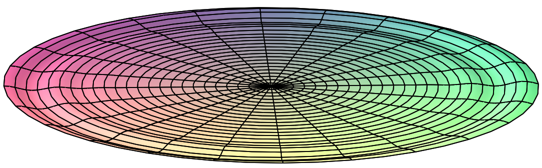

Figure 30: situation one at $t = 10$

3.5 Conclusion of three-dimensional problem

In this part of work, we derived the three-dimensional spreading problem from two-dimensional one including governing equation, initial and boundary conditions through using polar cylindrical coordinates. Then we solved the governing equation which is also a partial differential equation with the help of Hankel transform. The whole process of solving the problem and it's solution are very similar to two-dimensional problem's except for some little technics that were used. The solution is like the following

$$\phi = \sum_{n=0}^{\infty} \int_0^{\infty} F_n^*(k,z,t) J_n(kr) k \, dk \cos n\theta,$$

where

$$F_n^*(k,z,t) = C(k,t)\cosh(kh + kz)$$

and

$$C(k,t) = \frac{A_{0n}^*}{\cosh(kh)}\cos(\omega t) - g\frac{B_{0n}^*}{\omega\cosh(kh)}\sin(\omega t)$$

and

$$\omega = \sqrt{gk\tanh(kh)}.$$

Then numerical calculations that run by Maple were done next. First of all, picked a proper function as initial condition. Then, plotted the two-dimensional graph of η which is in equation (2.9). Last, plotted the initial status of free surface and spreading animations which are seemed quite vivid and much more real than the plots in two-dimensional problem.

4 Conclusion

During the whole project, I investigated how water wave spreads out from a given initial disturbance in both two-dimensional and three-dimensional spaces which could be referred to Cauchy-Poisson problem in 19th century. I firstly considered a two-layer of water which is inviscid, incompressible and irrotational. Based on these assumptions and conservation of mass, it was found that the velocity potential ϕ which is the gradient of the velocity satisfies the Laplace's equation and therefore Laplace's equation comes to be the governing equation of both two-dimensional and three-dimensional problems. In this project, I also used linearized free surface conditions to a constant depth but infinite width of water. Give the free surface a specific initial disturbance and this disturbance happens and only happens at $t = 0$. Therefore, the free surface has its velocity and elevation at initial time point which are used as the initial conditions. After extracting the two-dimensional problem from reality, I started to solve the problem under the help of Fourier transform. Apply the transform to every equation in the problem with respect to x for eliminating the variable x so that the problem becomes a two-variable, (z,t) problem. Then, using some knowledge of differential equations to get the solution of transformed problem. At last, substitute the transformed solution ϕ^* into the inverse Fourier transform gives the expression of velocity potential ϕ. Next work was using software Maple to do the calculation of the solution. It is apparently that ϕ is not the displacement of free surface but η is. η was introduced in the first section which is a function of velocity potential ϕ.

Hence, η was plotted out in Maple. However, it is impossible to plot the wave without an initial disturbance. In this two-dimensional problem, I chose an easy-formed function which is like figure 2 that and figure 3 equipped with Fourier transformed version. It can be seen from the figures that both of them have the shape like a wave which is also a necessary condition of being an initial disturbance. After then, initial status of free surface and spreading animations were plotted in two situations. One is equivalent to hitting the surface of water and the other is like lifting the surface to a specific hight. The graphs and animations in both of the situations turned out to be quite vivid in the end.

Then, I came to the three-dimensional case which is expanded from the two-dimensional problem but expressed by cylindrical polar coordinates. No matter the governing equations or the initial and boundary conditions, they all look very similar. What's more, the ways of solving the problem are almost the same excluding Hankel transform but not Fourier transform was used in three-dimensional problem and a bit more technics were applied to it at the beginning. All the same with two-dimensional problem, after obtaining the solution, I used Maple to do the numerical calculations. But a different initial condition was chosen since this problem is with one more variable and it has to have the Hankel transformed version. At last, the three-dimensional animations showed a quite well and more vivid simulation of the spreading of waves.

In the future work, it will be concerned the comparison with the solutions that obtained from asymptotic methods. Asymptotic methods could be considered as representing the waves with right-going and left-going ones separately.

A Appendix 1

1. Situation one:
$$\phi_0^* = 0, \quad H_0^* = \pi e^{-|k|}.$$

```
restart;
g := 9.8;
h := 1;
t := 0;
z := 0;

phi0star := 0;
H0star := Pi*exp(-abs(k));
omega := sqrt(g*k*tanh(k*h));

phi[t] := -phi0star*omega*sin(omega*t)-g*H0star*cos(omega*t);
eta := -(int(phi[t]*exp(I*k*x), k = -infinity .. infinity))/(2*Pi*g);
plot(eta).
```

2. Situation two:
$$\phi_0^* = \pi e^{-|k|}, \quad H_0^* = 0.$$

```
restart;
g := 9.8;
h := 1;
t := 0.1;
z := 0;

omega := sqrt(g*k*tanh(k*h));
phi0star := Pi*exp(-abs(k));
H0star := 0;
phi[t] := -phi0star*omega*sin(omega*t)-g*H0star*cos(omega*t);

eta := -(int(phi[t]*exp(I*k*x), k = -infinity .. infinity))/(2*Pi*g);
plot(eta)


.
```

B Appendix 2

Maple codes: Situation one and two use the same codes

```
with(plots);
animate(plot, [(int(phi[t]*exp(I*k*x), k = 0 .. 10))/(2*Pi*g),x = -10 .. 10],
 T = 0 .. 10, frames = 101)
```

which mean the animations are from $t = 0$ to $t = 10$ and last for 10 seconds consist of the plots at 101 time points..

C Appendix 3

Situation one:

```
 > restart;
> g := 9.8;
> h := 1;
> t := 0.4;
> z := 0;
> omega := sqrt(g*k*tanh(k*h));
> A0nstar := 3*k^n*exp(-(1/4)*k^2)/2^(n+1);
> B0nstar := 0;
> nfinal := 1;
> Fnstar := A0nstar*cos(omega*t)-g*B0nstar*sin(omega*t)/omega;
> Fn := proc (r, z, t) options operator, arrow; evalf(int(Fnstar*Bessel
       J(n, k*r)*k,k = 0 .. infinity)) end proc;

> phi := sum(Fn*cos(n*theta), n = 0 .. nfinal);
> etafun := (r, theta) -> evalf(-(sum((int((-A0nstar*omega*sin(omega*t)
           -g*B0nstar*cos(omega*t))*BesselJ(n, k*r)*k, k = 0 .. 5))*
           cos(n*theta),n = 0 .. nfinal))/g) end proc
> plot(etafun(r, 0), r = 0 .. 1)
```

Situation two:

```
> restart;
> g := 9.8;
> h := 1;
> t := 0.1;
> z := 0;
> omega := sqrt(g*k*tanh(k*h));
> A0nstar := k^n*exp(-(1/4)*k^2)/2^(n+1);
> B0nstar := 0;
> nfinal := 1;
> Fnstar := A0nstar*cos(omega*t)-g*B0nstar*sin(omega*t)/omega;
> Fn := proc (r, z, t) options operator, arrow; evalf(int(Fnstar*Bessel
       J(n, k*r)*k,k = 0 .. infinity)) end proc;
>
> phi := sum(Fn*cos(n*theta), n = 0 .. nfinal);
>
> etafun := (r, theta) -> evalf(-(sum((int((-A0nstar*omega*sin(omega*t)
           -g*B0nstar*cos(omega*t))*BesselJ(n, k*r)*k, k = 0 .. 5))*
           cos(n*theta),n = 0 .. nfinal))/g) end proc;
```

D Appendix 4

Initial status (situation one and two use the same code):

```
> z := 'z';
> addcoords(z_cylindrical, [z, r, theta], [r*cos(theta), r*sin(theta),
            z]);
>
> plot3d(-(sum((int((-A0nstar*omega*sin(omega*t)-g*B0nstar*
        cos(omega*t))*BesselJ(n, k*r)*k, k = 0 .. 5))*cos(n*theta),
        n = 0 .. nfinal))/g,r = 0 .. 10, theta = 0 .. 2*Pi, coords =
        z_cylindrical, axes = boxed,scaling = constrained);
```

Spreading animation (situation one and two use the same code as well):

```
> with(plots);
> animate3d(-(sum((int((-A0nstar*omega*sin(omega*T)-g*B0nstar*
            cos(omega*T))*BesselJ(n, k*r)*k, k = 0 .. 5))*cos(n*theta),
            n = 0 .. nfinal))/g,r = 0 .. 10, theta = 0 .. 2*Pi,
            T = 0 .. 10, frames = 101,coords = z_cylindrical,
            axes = boxed);
```

E References

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