Monte Carlo Simulation in Financial Valuation

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www.value-yield.com

Abstract

This paper uses Monte Carlo simulation of a simple equity growth model with resampling of historical financial data to estimate the probability distribution of the future equity, earnings and payouts of companies, which are then used to estimate the probability distribution of the future return on the stock and stock options. The model is used on the S&P 500 stock market index and the Coca-Cola company. The relation between USA government bonds, the S&P 500 index and the Dow Jones Venture Capital index (DJVC) is also studied and it is found that there is no consistent and predictable risk premium between USA government bonds and the S&P 500 and DJVC indices, but there is significant correlation between the monthly returns of the S&P 500 and DJVC indices.

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[1] If you find any parts difficult to understand, or if you encounter any errors, whether logical, mathematical, spelling, grammatical or otherwise, please mark the document and e-mail it to: Magnus (at) Hvass-Labs (dot) Org
[2] See last page for revision history.

Nomenclature

IID	Independent and identically distributed stochastic variables.
PDF	Probability Density Function.
CDF	Cumulative Distribution Function (Empirical).
v	Present value of future payouts, dividends and share-price.
g	Annual growth rate used in valuation.
d	Discount rate used in valuation.
k	Kilo, a factor 10^3
m	Million, a factor 10^6
b	Billion, a factor 10^9
∞	Infinity.
$P/Book$	Price-to-Book ratio: $P/Book = MarketCap/Equity = SharePrice/(Equity/Shares)$
$Shares$	Number of shares outstanding.
$SharePrice$	Price per share.
$MarketCap$	Market capitalization (also written market-cap): $MarketCap = Shares \cdot SharePrice$
$Equity$	Capital supplied by shareholders as well as retained earnings, not per-share.
$Earnings$	Earnings available for payout to shareholders, not per-share.
$Dividend$	Dividend payout, pre-tax, not per-share.
$Buyback$	Amount used for share buyback.
$Issuance$	Amount of share issuance.
$Net\ Buyback$	Share buyback net of issuance: $Net\ Buyback = Buyback - Issuance$
$Payout$	Net payout from company: $Payout = Dividends + Buyback - Issuance$
$Retain$	Part of earnings retained in company: $Retain = Earnings - Payout$
ROA	Return on Assets: $ROA = Earnings/Assets$
ROE	Return on Equity: $ROE = Earnings/Equity$
$=$	$a = b$ means that a equals b
$\geq$	$a \geq b$ means that a is greater than or equal to b
$\simeq$	$a \simeq b$ means that a is approximately equal to b
$\Rightarrow$	Implication: $A \Rightarrow B$ means that A implies B
$\Leftrightarrow$	Bi-implication: $A \Leftrightarrow B$ means that $A \Rightarrow B$ and $B \Rightarrow A$
$\cdot$	Multiplication: $a \cdot b$ means that a is multiplied by b
$\sum$	Summation: $\sum_{t=a}^{b} x_t = x_a + x_{a+1} + \cdots + x_b$
Π	Multiplication: $\prod_{t=a}^{b} x_t = x_a \cdot x_{a+1} \cdot \ldots \cdot x_b$
$Max(a,b)$	Maximum of a or b. Similarly for $Min(a,b)$
$\log$	Logarithmic function with base e (natural logarithm).
$\|a\|$	Absolute value of a (the sign of a is removed).
$\Pr[X = x]$	Probability of the stochastic variable X being equal to x.
$E[X]$	Expected (or mean) value of the stochastic variable X.
$Var[X]$	Variance of the stochastic variable X.
$Stdev[X]$	Standard deviation of the stochastic variable X: $Stdev[X] = \sqrt{Var[X]}$

Introduction

1. Introduction

Present value calculations are ubiquitous in finance. But the average of stochastic inputs cannot be used to calculate the average present value because the formula is non-linear. This is due to the so-called Jensen inequality. It is therefore necessary to know the probability distribution of the inputs in order to calculate the mean present value. The probability distribution is also useful when the average present value is misleading because it is unlikely to occur.

Monte Carlo simulation is the use of computers to simulate numerous outcomes of a mathematical model so as to estimate the probability distribution. This is useful when the model cannot be studied analytically. There are several problems in finance where Monte Carlo simulation is useful, see e.g. Glasserman [**1**].

This paper uses a simple equity growth model to simulate the future equity, earnings and payouts of companies, based on resampling of historical data for the return on equity and the fraction of earnings retained in the past. This is a reasonable model for companies whose earnings are related to their equity capital. The model can also estimate future share prices by multiplying the simulated equity with the historical distribution of the company's P/Book ratio (also called price-to-book or price-to-equity). The model can also estimate the annualized rate of return from buying shares at a given price and selling them after a given period or holding them for eternity. The model can also be used for valuing options.

The equity growth model may also be used on stock market indices by considering the constituent companies of the index as if they were one big conglomerate, which is done for the Standard & Poor's 500 (S&P 500) stock market index in section 6. Of particular interest may be the probability distribution of the future rate of return on the S&P 500 index which is given for an infinite holding period in section 6.11 and for finite holding periods in section 6.12. Options on the S&P 500 index are valued in section 6.13.

The historical yield on USA government bonds is studied in section 7. The Dow Jones Venture Capital (DJVC) index is studied in section 8. It is found that there is no consistent and predictable risk premium between USA government bonds and the S&P 500 and DJVC indices, but there is significant correlation between the monthly returns of the S&P 500 and DJCV indices. The Coca-Cola company is studied in section 9 including the probability distribution of its present value relative to the S&P 500 and DJVC indices.

1.1. Paper Overview

The paper is structured as follows:

- Section 2 gives formulas for present value calculations and the equity growth model.
- Section 3 gives formulas for stochastic present value calculations.
- Section 4 gives formulas for stock option valuation.
- Section 5 gives algorithms for Monte Carlo simulation of the equity growth model.
- Section 6 studies the S&P 500 stock market index.
- Section 7 studies USA government bonds and their historical relation to the S&P 500 index.
- Section 8 studies the Dow Jones Venture Capital (DJVC) index.
- Section 9 studies the Coca-Cola company.
- Section 10 is the conclusion.

1.2. Source Code & Data

The experiments in this paper have been implemented in the statistical programming language R which is freely available from the internet. The source-code and data-files are available at:

www.hvass-labs.org/people/magnus/publications/pedersen2013monte-carlo.zip

Time Usage

Executing this implementation on a consumer-level computer from the year 2011 typically requires only seconds or minutes for an experiment consisting of a thousand Monte Carlo simulations. Implementing a parallelized version or using another programming language might significantly decrease the time usage.

1.3. Internet Website

Automatically updated plots using the models of this paper on the S&P 500 are available on the internet:

www.value-yield.com

Theory

2. Present Value

The present value of the dividend for future year t is the amount that would have to be invested today with an annual rate of return d, also called the discount rate, so as to compound into becoming $Dividend_t$ after t years:

$$Present\ Value\ of\ Dividend_t \cdot (1+d)^t = Dividend_t \Leftrightarrow Present\ Value\ of\ Dividend_t = \frac{Dividend_t}{(1+d)^t}$$

Eq. 2-1

An eternal shareholder is defined here as one who never sells the shares and thus derives value from the shares only through the receipt of dividends. In Williams' theory of investment value [**2**], the value of a company to its eternal shareholders is defined as the present value of all future dividends.

Let v denote the present value of all future dividends prior to dividend tax and not per share. Assume the discount rate d is constant forever. The present value v is then:

$$v = \sum_{t=0}^{\infty} Present\ Value\ of\ Dividend_t = \sum_{t=0}^{\infty} \frac{Dividend_t}{(1+d)^t}$$

Eq. 2-2

2.1. Payout

Instead of paying dividends, companies may also buy back or issue shares. Share buyback net of issuance is:

$$Net\ Buyback_t = Buyback_t - Issuance_t$$

Eq. 2-3

The sum of dividends and net share buybacks is called payout:

$$Payout_t = Dividends_t + Net\ Buyback_t = Dividends_t + Buyback_t - Issuance_t$$

Eq. 2-4

The term *payout* is a misnomer for share buybacks net of issuance as argued by Pedersen [**3**] [**4**], because a share buyback merely reduces the number of shares outstanding which may have unexpected effects on the share-price and hence does not constitute an actual payout from the company to shareholders. A more accurate term for the combination of dividends and share buybacks would therefore be desirable but the term *payout* will be used here and the reader should keep this distinction in mind.

The part of the earnings being retained in the company is:

$$Retain_t = Earnings_t - Payout_t$$

Eq. 2-5

This is equivalent to:

$$Payout_t = Earnings_t - Retain_t$$

Eq. 2-6

The ratio of earnings being retained in the company is:

$$\frac{Retain_t}{Earnings_t} = \frac{Earnings_t - Payout_t}{Earnings_t} = 1 - \frac{Dividends_t + Net\ Buyback_t}{Earnings_t}$$
$$= 1 - \frac{Dividends_t}{Earnings_t} - \frac{Net\ Buyback_t}{Earnings_t}$$

Eq. 2-7

If the company has no excess cash, then the company will first have to generate earnings before making dividend payouts and share buybacks, so the present value is calculated with starting year $t = 1$:

$$v = \sum_{t=1}^{\infty} \frac{Payout_t}{(1+d)^t} = \sum_{t=1}^{\infty} \frac{Earnings_t - Retain_t}{(1+d)^t}$$

Eq. 2-8

2.2. Equity Growth Model

The company's equity is the capital supplied directly by shareholders and the accumulation of retained earnings. Earnings are retained for the purpose of investing in new assets that can increase future earnings.

Let $Equity_t$ be the equity at the end of year t and let $Retain_t$ be the part of the earnings that are retained in year t. The equity at the end of year t is then:

$$Equity_t = Equity_{t-1} + Retain_t$$

Eq. 2-9

The accumulation of equity is:

$$Equity_t = Equity_0 + \sum_{k=1}^{t} Retain_k$$

Eq. 2-10

The Return on Equity (ROE) is defined as a year's earnings divided by the equity at the end of the previous year. For year t this is:

$$ROE_t = \frac{Earnings_t}{Equity_{t-1}}$$

Eq. 2-11

This is equivalent to:

$$Earnings_t = Equity_{t-1} \cdot ROE_t \Leftrightarrow Equity_{t-1} = \frac{Earnings_t}{ROE_t}$$

Eq. 2-12

Using this with the definition of payout from Eq. 2-6 gives:

$$\begin{aligned} Payout_t = Earnings_t - Retain_t &= Equity_{t-1} \cdot ROE_t - Retain_t \\ &= Equity_{t-1} \cdot ROE_t \cdot \left(1 - \frac{Retain_t}{Earnings_t}\right) \end{aligned}$$

Eq. 2-13

The present value to eternal shareholders is then:

$$v = \sum_{t=1}^{\infty} \frac{Payout_t}{(1+d)^t} = \sum_{t=1}^{\infty} \frac{Equity_{t-1} \cdot ROE_t - Retain_t}{(1+d)^t}$$

Eq. 2-14

2.2.1. Equity Growth Rate

The equity grows by the rate g_t in year t which is calculated using the definition of $Equity_t$ from Eq. 2-9:

$$1 + g_t = \frac{Equity_t}{Equity_{t-1}} = \frac{Equity_{t-1} + Retain_t}{Equity_{t-1}} \Leftrightarrow g_t = \frac{Retain_t}{Equity_{t-1}}$$

Eq. 2-15

Using the definition of ROE from Eq. 2-12, this is equivalent to:

$$g_t = ROE_t \cdot \frac{Retain_t}{Earnings_t}$$

Eq. 2-16

The accumulation of equity is expressed as a sum in Eq. 2-10 but it can also be expressed as a product using this growth rate:

$$Equity_t = Equity_0 \cdot \prod_{\mathrm{k}=1}^{\mathrm{t}} \left(1 + ROE_t \cdot \frac{Retain_t}{Earnings_t}\right)$$

Eq. 2-17

2.2.2. Mean Growth Rate

Consider ROE and $Retain/Earnings$ to be stochastic variables and their product G is the equity growth rate from Eq. 2-16. The mean equity growth rate is then:

$$E[G] = E\left[ROE \cdot \frac{Retain}{Earnings}\right]$$

Eq. 2-18

The mean earnings also grow by this rate because earnings are derived from the equity. This follows from the definition of earnings from Eq. 2-12, the assumed independence of the stochastic variables ROE_t and $Equity_t$, and the identical distributions for ROE_t and ROE_{t+1} so their means are also identical:

$$\frac{E[Earnings_{t+1}]}{E[Earnings_t]} = \frac{E[ROE_{t+1} \cdot Equity_{t+1}]}{E[ROE_t \cdot Equity_t]} = \frac{E[ROE_{t+1}] \cdot E[Equity_{t+1}]}{E[ROE_t] \cdot E[Equity_t]} = \frac{E[Equity_{t+1}]}{E[Equity_t]} = E[G]$$

Eq. 2-19

A similar derivation can be made from Eq. 2-13 to show that the mean payout growth rate is also $E[G]$.

Independence

If ROE and $Retain$ are independent stochastic variables then the mean growth rate is:

$$E[G] = E\left[ROE \cdot \frac{Retain}{Earnings}\right] = E[ROE] \cdot E\left[\frac{Retain}{Earnings}\right]$$

Eq. 2-20

2.2.3. Normalized Equity

The accumulation of equity in Eq. 2-10 and Eq. 2-17 can be normalized by setting $Equity_0 = 1$ so it is independent of the starting equity. This also normalizes the earnings calculated in Eq. 2-12 and the payouts calculated in Eq. 2-13, which allows for Monte Carlo simulation based solely on the probability distributions for ROE and $Retain/Earnings$ so the results can easily be used with different starting equity.

2.2.4. Debt Change

Changes in the level of debt relative to equity are taken into account implicitly by the use of historical data for ROE and $Retain/Earnings$ which are affected by the change in debt levels. This is deemed acceptable because the changes are small and not permanent in these case studies. Large and permanent changes should be taken into account in the valuation models.

2.2.5. Scaled Retained Earnings

The sampled ROE and $Retain/Earnings$ can be scaled so as to decrease future growth. An example of this is given in section 9.4.

The payout consists of dividends and net share buybacks which are found by multiplying the earnings with samples of the historical $Dividend/Earnings$ and $Net\ Buyback/Earnings$. An adjustment is necessary if $Retain/Earnings$ is scaled because the total earnings must equal the retained and paid out parts.

Let $ScaleRetain$ be the scale applied to the retained earnings and let $ScalePayout$ be the scale used on dividends and net share buyback. Then the following must hold:

$$Earnings = ScaleRetain \cdot Retain + ScalePayout \cdot Payout$$

Eq. 2-21

This can be rearranged to find the payout scale:

$$ScalePayout = \frac{1 - ScaleRetain \cdot \frac{Retain}{Earnings}}{1 - \frac{Retain}{Earnings}}$$

Eq. 2-22

2.3. Equity Growth for Two Companies

Consider two companies with starting equity denoted by $Equity$ and $Equity'$ with $Equity > Equity'$ and whose annual equity growth rates are g and g' with $g < g'$. Because of the higher growth rate g', the lower $Equity'$ grows to exceed $Equity$ after n years, which is calculated as:

$$(1+g)^n \cdot Equity < (1+g')^n \cdot Equity' \Leftrightarrow n > \frac{\log\left(\frac{Equity'}{Equity}\right)}{\log\left(\frac{1+g}{1+g'}\right)}$$

Eq. 2-23

2.4. Market Capitalization

Let $Shares$ be the number of shares outstanding and let $SharePrice$ be the market-price per share. The market-cap (or market capitalization, or market value) is the total price for all shares outstanding:

$$MarketCap = Shares \cdot SharePrice$$

Eq. 2-24

The market-cap is frequently considered relative to the equity, which is also known as the price-to-book-value or P/Book ratio:

$$P/Book = \frac{MarketCap}{Equity} = \frac{SharePrice}{Equity/Shares}$$

Eq. 2-25

This is equivalent to:

$$MarketCap = P/Book \cdot Equity \Leftrightarrow SharePrice = P/Book \cdot Equity/Shares$$

Eq. 2-26

Because the starting equity is normalized to one in these Monte Carlo simulations, see section 2.2.3, it is often convenient to express the formulas involving the market-cap in terms of the P/Book ratio instead.

2.5. Share Issuance & Buyback

Share issuance and buyback changes the number of shares outstanding and hence affects the per-share numbers, such as earnings per share, equity per share and price per share.

The number of shares is normalized by setting $Shares_0 = 1$. If the share buyback and issuance occurs when the shares are priced at $MarketCap_t$ then the number of shares changes according to the following formula, see Pedersen [**3**] [**4**] for details:

$$Shares_t = Shares_{t-1} \cdot \left(1 - \frac{NetBuyback_t}{MarketCap_t}\right)$$

Eq. 2-27

Where $Net\ Buyback_t$ is defined in Eq. 2-3 and $MarketCap_t$ is calculated using Eq. 2-26 with Monte Carlo simulated equity and P/Book ratio:

$$MarketCap_t = Equity_t \cdot (P/Book)_t$$

Eq. 2-28

The number of shares is then used with the Monte Carlo simulated equity, earnings, etc. to find the per-share numbers. For example, the equity per share in year t is calculated as:

$$Equity\ Per\ Share_t = \frac{Equity_t}{Shares_t}$$

Eq. 2-29

The price per share in year t is calculated from Eq. 2-28 and Eq. 2-27:

$$SharePrice_t = \frac{MarketCap_t}{Shares_t} = \frac{Equity_t \cdot (P/Book)_t}{Shares_t}$$

Eq. 2-30

2.6. Value Yield

The value yield is defined as the discount rate which makes the market-cap equal to the present value:[3]

$$d = Value\ Yield \Leftrightarrow MarketCap = v$$

Eq. 2-31

This may be easier to understand if the notation makes clear that the present value v is a function of the discount rate d by writing the present value as $v(d)$. The value yield is then the choice of discount rate that causes the present value to equal the market-cap:

$$MarketCap = v(Value\ Yield)$$

Eq. 2-32

[3] The value yield is also called the Internal Rate of Return (IRR) but that may be confused with the concept of the Return on Equity (ROE) and is therefore not used here.

2.6.1. No Share Buyback & Issuance

First assume the company's payout consists entirely of dividends so there are no share buybacks and issuances which means the number of shares remains constant.

The value yield for an eternal shareholder must then satisfy the equation:

$$MarketCap = \sum_{t=1}^{\infty} \frac{Payout_t}{(1 + Value\ Yield)^t}$$

Eq. 2-33

For a shareholder who owns the shares and receives payouts for n years after which the shares are sold at a price of $MarketCap_n$, the value yield must satisfy the equation:

$$MarketCap = \sum_{t=1}^{n} \frac{Payout_t}{(1 + Value\ Yield)^t} + \frac{MarketCap_n}{(1 + Value\ Yield)^n}$$

Eq. 2-34

2.6.2. Share Buyback & Issuance

If the company makes share buybacks and/or issuances then the number of shares changes and the present value is calculated from the dividend per share instead of the total annual payout. For an eternal shareholder, the value yield must satisfy:

$$SharePrice = \sum_{t=1}^{\infty} \frac{Dividend_t / Shares_t}{(1 + Value\ Yield)^t}$$

Eq. 2-35

For a temporary shareholder who owns the shares and receives dividends for n years after which the shares are sold at a price of $SharePrice_n$, the value yield must satisfy the equation:

$$SharePrice = \sum_{t=1}^{n} \frac{Dividend_t / Shares_t}{(1 + Value\ Yield)^t} + \frac{SharePrice_n}{(1 + Value\ Yield)^n}$$

Eq. 2-36

2.6.3. Interpretation as Rate of Return

The value yield is the annualized rate of return on an investment over its life, given the current market price of that investment. This follows from the duality of the definition of the present value from section 2, in which the present value may be considered as the discounting of a future payout using a discount rate d, or equivalently the future payout may be considered the result of exponential growth of the present value using d as the growth rate. The choice of d that makes the present value equal to the market-cap is called the value yield. This interpretation also extends to multiple future payouts that spread over a number of years or perhaps continuing for eternity, where the present value is merely the sum of all those future payouts discounted at the same rate.

Note that the value yield is not the rate of return on reinvestment of future payouts, which will depend on the market price of the financial security at the time of such future payouts.

The value yield may be identical for different holding periods and different timing of payouts as shown in the examples below. Individual investors may have different preferences for when they would like to receive payouts even though the annualized rate of return is identical. This is known as a utility function and is ignored here because it differs amongst investors.

2.6.4. Use as Discount Rate

Because the value yield is the rate of return that can be obtained from an investment over a given holding period, the value yield can be used as the discount rate in calculating the present value of another investment so as to find their relative value. If the value yield changes with the holding period, as is the case in section 6.12, then the value yield must be chosen to match the period of the investment whose present value is being calculated. An example of such a present value calculation is given in section 9.6.

2.6.5. Payout Growth

If the payout grows by the rate g each year for eternity so that $Payout_t = Payout \cdot (1+g)^t$ then the value yield is calculated using Eq. 11-2:

$$MarketCap = \sum_{t=1}^{\infty} \frac{Payout \cdot (1+g)^t}{(1+Value\ Yield)^t} = \frac{Payout \cdot (1+g)}{Value\ Yield - g} \Leftrightarrow Value\ Yield = \frac{Payout}{MarketCap} \cdot (1+g) + g$$

Eq. 2-37

If the payout growth first starts next year so that $Payout_t = Payout \cdot (1+g)^{t-1}$ then Eq. 11-3 gives:

$$MarketCap = \sum_{t=1}^{\infty} \frac{Payout \cdot (1+g)^{t-1}}{(1+Value\ Yield)^t} = \frac{Payout}{Value\ Yield - g} \Leftrightarrow Value\ Yield = \frac{Payout}{MarketCap} + g$$

Eq. 2-38

In both cases the lower bound of the value yield is the growth rate g as $MarketCap$ approaches infinity.

2.6.6. Examples

Assume all payouts are zero $Payout_t = 0$ and that the market-cap grows to become $MarketCap_n = MarketCap \cdot (1+r)^n$ after n years. This corresponds to a bond paying all of its interest when the principal is returned (also called zero coupon). It follows from Eq. 2-34 that the value yield equals the interest rate r:

$$MarketCap = \sum_{t=1}^{n} \frac{0}{(1+Value\ Yield)^t} + \frac{MarketCap \cdot (1+r)^n}{(1+Value\ Yield)^n} \Leftrightarrow Value\ Yield = r$$

Eq. 2-39

Now assume $Payout_t = r \cdot MarketCap$ and $MarketCap_n = MarketCap$ which corresponds to a bond paying annual interest (also called coupons) of $r \cdot MarketCap$ and the principal is returned after n years. It follows from Eq. 2-34 and Eq. 11-5 that the value yield again equals the interest rate r:

$$MarketCap = \sum_{t=1}^{n} \frac{r \cdot MarketCap}{(1 + Value\ Yield)^t} + \frac{MarketCap}{(1 + Value\ Yield)^n} \Leftrightarrow Value\ Yield = r$$

Eq. 2-40

Now assume the payout grows by the rate g each year so that $Payout_t = r \cdot MarketCap \cdot (1+g)^t$ and the selling price after n years grows to become $MarketCap_n = MarketCap \cdot (1+g)^n$. The value yield is then calculated using Eq. 2-34 and Eq. 11-5:

$$MarketCap = \sum_{t=1}^{n} \frac{r \cdot MarketCap \cdot (1+g)^t}{(1 + Value\ Yield)^t} + \frac{MarketCap \cdot (1+g)^n}{(1 + Value\ Yield)^n} \Leftrightarrow Value\ Yield = r \cdot (1+g) + g$$

Eq. 2-41

This equals the value yield for eternal growth in Eq. 2-37 with $r = Payout/MarketCap$.

Now assume the payout first starts growing next year so that $Payout_t = r \cdot MarketCap \cdot (1+g)^{t-1}$ but the selling price after n years is still $MarketCap_n = MarketCap \cdot (1+g)^n$. The value yield is then:

$$MarketCap = \sum_{t=1}^{n} \frac{r \cdot MarketCap \cdot (1+g)^{t-1}}{(1 + Value\ Yield)^t} + \frac{MarketCap \cdot (1+g)^n}{(1 + Value\ Yield)^n} \Leftrightarrow Value\ Yield = r + g$$

Eq. 2-42

This equals the value yield for eternal growth in Eq. 2-38 with $r = Payout/MarketCap$.

2.6.7. Existence of Value Yield

The so-called Cauchy-Hadamard theorem states that the power series used in the calculation of present value, converges if the payout growth does not exceed exponential growth indefinitely, which is a reasonable assumption for payout growth in the real world. According to that theorem, the present value exists for some continuous range of discount rates. In case the power series is finite, meaning all payouts are zero after some year, then the present value exists for all choices of discount rate except $d = -100\%$. In case all payouts are non-negative then the value yield exists, that is, the discount rate can be chosen so as to make the present value equal the market-cap, because the present value decreases continuously and monotonically from a limit of infinity when $d = -100\%$ towards zero as the discount rate increases.

The value yield may not be well-defined when some of the payouts are negative. For example, if positive payouts are followed by negative payouts (corresponding to raising new capital) and then zero payouts for eternity, which may be the pattern of a company going bankrupt, then the present value does not change monotonically with a changing discount rate because the positive payouts may dominate the present value when the discount rate is sufficiently high and the negative payouts may dominate the present value when the discount rate is sufficiently low. This means there is a maximum present value when varying the discount rate. If the market-cap is higher than this maximum present value then the value yield is not well-defined.

The value yield is well-defined in the Monte Carlo simulations in this paper and is found using a numerical root-finding method which is commonly available in mathematical software packages.

2.7. Terminal Value

The present value of the equity growth model in Eq. 2-14 is defined from an infinite number of iterations, but the Monte Carlo simulation must terminate after a finite number of iterations. Estimating the present value is therefore done by separating v into $v_{Simulation}$ which is the present value of the payout in the years that have been Monte Carlo simulated, and $v_{Terminal}$ which is an approximation to the remaining value if the Monte Carlo simulation had been allowed to continue for an infinite number of iterations:

$$v = v_{Simulation} + v_{Terminal}$$

Eq. 2-43

2.7.1. Iterations Required

The number of iterations k required to achieve a given accuracy of $v_{Simulation}$ is now derived. To estimate the present value assume the starting payout is $Payout_0 = Payout$ and the annual payouts grow by g each for eternity so that $Payout_t = Payout \cdot (1+g)^t$. Let $v_{(m,n)}$ denote the partial present value where the year t is between m and n:

$$v_{(m,n)} = Payout \cdot \sum_{t=m}^{n} \left(\frac{1+g}{1+d}\right)^t$$

Eq. 2-44

Note that:

$$v = v_{(1,\infty)} = v_{(1,k)} + v_{(k+1,\infty)} \Leftrightarrow v_{(1,k)} = v_{(1,\infty)} - v_{(k+1,\infty)}$$

Eq. 2-45

The geometric series $v = v_{(1,\infty)}$ is reduced using Eq. 11-2:

$$v = v_{(1,\infty)} = Payout \cdot \sum_{t=1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = Payout \cdot \frac{1+g}{d-g}$$

Eq. 2-46

The terminal value consisting of the payouts in year $k+1$ and onwards is calculated using Eq. 11-4:

$$v_{(k+1,\infty)} = Payout \cdot \sum_{t=k+1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = Payout \cdot \left(\frac{1+g}{1+d}\right)^k \cdot \frac{1+g}{d-g}$$

Eq. 2-47

The limit of the terminal value is zero as k approaches infinity provided $g < d$:

$$\lim_{k \to \infty} v_{(k+1,\infty)} = \lim_{k \to \infty} Payout \cdot \left(\frac{1+g}{1+d}\right)^k \cdot \frac{1+g}{d-g} = 0$$

Eq. 2-48

The limit of $v_{(1,k)}$ approaches v as k approaches infinity:

$$\lim_{k \to \infty} v_{(1,k)} = \lim_{k \to \infty} \left(v_{(1,\infty)} - v_{(k+1,\infty)}\right) = v_{(1,\infty)} = v$$

Eq. 2-49

The number of iterations k required in order for $v_{(1,k)}$ to be within relative error $\epsilon > 0$ of the value v is:

$$\left|\frac{v - v_{(1,k)}}{v}\right| < \epsilon \Leftrightarrow \left|\frac{Payout \cdot \left(\frac{1+g}{1+d}\right)^k \cdot \frac{1+g}{d-g}}{Payout \cdot \frac{1+g}{d-g}}\right| < \epsilon \Leftrightarrow k > \frac{\log(\epsilon)}{\log\left(\frac{1+g}{1+d}\right)}$$

Eq. 2-50

In the derivation of this, the inequality is reversed because $\log((1+g)/(1+d)) < 0$ when $g < d$ which is assumed here.

If the payout growth rate is stochastic then the required number of iterations can be estimated using Eq. 2-50 with the mean growth rate. For example, in the equity growth model from section 2.2, the mean growth rate from Eq. 2-18 may be used.

If the discount rate is the unknown variable that is sought determined, as is the case in the Monte Carlo simulation in section 5.2, then the discount rate is unknown when k is calculated in Eq. 2-50. The discount rate may then be set to the expected discount rate or slightly above the mean growth-rate. Furthermore, k may be increased by a margin because the Monte Carlo simulation may produce a value $v_{(1,k)}$ that is significantly different from the average used to derive k here.

2.7.2. Mean Terminal Value

Let $Payout_k$ be the stochastic payout in year k which starts growing next year by the stochastic rate G independently of $Payout_k$. Note that a single growth rate is used for all years t but the choice of that growth rate is stochastic. Let d be a deterministic discount rate which is assumed to be greater than the growth rate $d > G$. The mean terminal value is calculated using Eq. 3-13 with additional discounting to calculate the present value today:

$$E[V_{Terminal}] = E\left[\frac{Payout_k}{(1+d)^k} \cdot \sum_{t=1}^{\infty} \frac{(1+G)^{t-1}}{(1+d)^t}\right] = \frac{E[Payout_k]}{(1+d)^k} \cdot E\left[\frac{1}{d-G}\right] \geq \frac{E[Payout_k]}{(1+d)^k} \cdot \frac{1}{d - E[G]}$$

Eq. 2-51

This would have been an equality if a new stochastic growth rate had been used each year and the growth rates were independent and identically distributed (IID), see section 3.3.1.

For the equity growth model in section 2.2, the mean growth rate $E[G]$ would be calculated from Eq. 2-18:

$$E[G] = E\left[ROE \cdot \frac{Retain}{Earnings}\right]$$

When the equity at the end of year $k-1$ is known, the expected payout can be calculated from Eq. 2-13:

$$E[Payout_k] = Equity_{k-1} \cdot E\left[ROE \cdot \left(1 - \frac{Retain}{Earnings}\right)\right]$$

Eq. 2-52

No Growth

If the equity at the end of year k is known and all earnings are being paid out so the equity and payouts do not grow, then the mean payout is:

$$E[Payout_{k+1}] = E[Equity_k \cdot ROE] = Equity_k \cdot E[ROE]$$

Eq. 2-53

As the mean payout is constant for eternity, the mean terminal value is:

$$E[V_{Terminal}] = E\left[\sum_{t=k+1}^{\infty} \frac{Payout_t}{(1+d)^t}\right] = E\left[\frac{Payout_{k+1}}{(1+d)^k} \cdot \sum_{t=1}^{\infty} \frac{1}{(1+d)^t}\right] = \frac{Equity_{k+1} \cdot E[ROE]}{(1+d)^k \cdot d}$$

Eq. 2-54

This approaches infinity as the discount rate d approaches zero, while the mean terminal value in Eq. 2-51 approaches infinity as the discount rate approaches the mean growth rate $E[G]$.

3. Stochastic Present Value

There may be uncertainty about future payouts in the present value calculation. There may also be uncertainty about the discount rate, which is the rate of return that can be obtained from the alternative investment used as the benchmark in calculating the present value. In such cases, the present value may be considered a stochastic variable V taking on values v according to some probability distribution $\Pr[V = v]$.

Using the mean payout growth and discount rates in a present value calculation does not result in the mean present value. This is because the present value formula is non-linear so Jensen's inequality applies.

3.1. Jensen's Inequality

In general, let X be a stochastic variable and let φ be a convex function, then Jensen's inequality states that:

$$\varphi(E[X]) \leq E[\varphi(X)]$$

Eq. 3-1

This becomes a strict inequality if φ is strictly convex and $Var[X] > 0$.

Let $\varphi(a) = 1/a$ be the reciprocal function which is strictly convex for $a > 0$ and assume $X > 0$, then according to Jensen's inequality:

$$\frac{1}{E[X]} \leq E\left[\frac{1}{X}\right]$$

Eq. 3-2

Let $\varphi(a) = a^t$ which is strictly convex for $t > 1$, then according to Jensen's inequality:

$$(E[X])^t \leq E[X^t]$$

Eq. 3-3

Let $\varphi(a) = a^{-t} = 1/a^t$ which is strictly convex for $t \geq 1$ and $a > 0$, then according to Jensen's inequality:

$$\frac{1}{(E[X])^t} \leq E\left[\frac{1}{X^t}\right]$$

Eq. 3-4

Let Y be another stochastic variable which is independent of X, then Eq. 3-2 gives:

$$\frac{E[Y]}{E[X]} \leq E[Y] \cdot E\left[\frac{1}{X}\right] = E\left[\frac{Y}{X}\right]$$

Eq. 3-5

This inequality also holds for powers of $t > 0$:

$$\left(\frac{E[Y]}{E[X]}\right)^t \leq \left(E\left[\frac{Y}{X}\right]\right)^t$$

Eq. 3-6

Jensen's inequality has been studied previously in present value calculations, see e.g. Newell and Pizer [**5**].

3.2. Stochastic Discount Rate

Let $Payout_t$ be a stochastic variable for the payout in year t. Let D be a stochastic variable for the discount rate assumed to be independent of $Payout_t$. Note that a single discount rate is used for all years t but the choice of that discount rate is stochastic. The present value V is the derived stochastic variable:

$$V = \sum_{t=1}^{\infty} \frac{Payout_t}{(1+D)^t}$$

Eq. 3-7

Using linearity of expectation, which applies to geometric series that converge, and Jensen's inequality from Eq. 3-4, gives:

$$E[V] = E\left[\sum_{t=1}^{\infty} \frac{Payout_t}{(1+D)^t}\right] = \sum_{t=1}^{\infty} E[Payout_t] \cdot E\left[\frac{1}{(1+D)^t}\right] \geq \sum_{t=1}^{\infty} \frac{E[Payout_t]}{(1+E[D])^t}$$

Eq. 3-8

That is, the mean present value may be underestimated when calculated from the mean discount rate.

3.2.1. Time-Varying Discount Rate

Let D_t be a stochastic variable for the discount factor in year t so the present value is:

$$V = \sum_{t=1}^{\infty} \frac{Payout_t}{D_t}$$

Eq. 3-9

Using Jensen's inequality in Eq. 3-2 gives:

$$E[V] = E\left[\sum_{t=1}^{\infty} \frac{Payout_t}{D_t}\right] = \sum_{t=1}^{\infty} E[Payout_t] \cdot E\left[\frac{1}{D_t}\right] \geq \sum_{t=1}^{\infty} \frac{E[Payout_t]}{E[D_t]}$$

Eq. 3-10

That is, the mean present value may be underestimated when it is calculated from the mean discount rates.

3.3. Stochastic Growth Rate

Let $Payout$ be a stochastic variable for the starting payout and let G be a stochastic variable for the payout growth which starts next year. Note that a single growth rate is used for all years t but the choice of that growth rate is stochastic. Assume the stochastic variables G and $Payout$ are independent. The payout in year t is the derived stochastic variable:

$$Payout_t = Payout \cdot (1+G)^{t-1}$$

Eq. 3-11

Let d be a deterministic discount rate greater than G. The present value V is a derived stochastic variable calculated using Eq. 11-3:

$$V = \sum_{t=1}^{\infty} \frac{Payout \cdot (1+G)^{t-1}}{(1+d)^t} = \frac{Payout}{d-G}$$

Eq. 3-12

The mean present value is calculated using Jensen's inequality in Eq. 3-2:

$$E[V] = E\left[\frac{Payout}{d-G}\right] = E[Payout] \cdot E\left[\frac{1}{d-G}\right] \geq \frac{E[Payout]}{d-E[G]}$$

Eq. 3-13

That is, the mean present value may be underestimated when calculated from the mean growth rate.

3.3.1. Time-Varying Growth Rate

Let G_t be the payout growth rate in year t which is assumed to start next year and be independent of the stochastic variable for the starting payout. The payout for year t is the derived stochastic variable:

$$Payout_t = Payout \cdot \prod_{k=1}^{t-1} (1+G_k)$$

Eq. 3-14

Where the accumulated growth is denoted:

$$\widehat{G_t} = \prod_{k=1}^{t-1} (1+G_k)$$

Eq. 3-15

Further assume the growth rates G_t are independent and identically distributed (IID) so they have identical means $E[G_t]$. The mean accumulated growth can then be reduced to:

$$E\left[\widehat{G_t}\right] = E\left[\prod_{k=1}^{t-1} (1+G_k)\right] = \prod_{k=1}^{t-1} (1+E[G_k]) = (1+E[G_t])^{t-1}$$

Eq. 3-16

The mean payout in year t is then:

$$E[Payout_t] = E[Payout] \cdot (1 + E[G_t])^{t-1}$$

Eq. 3-17

Using this with Eq. 11-3 and linearity of expectation, which applies to geometric series that converge, gives the mean present value:

$$E[V] = \sum_{t=1}^{\infty} \frac{E[Payout_t]}{(1+d)^t} = E[Payout] \cdot \sum_{t=1}^{\infty} \frac{(1 + E[G_t])^{t-1}}{(1+d)^t} = \frac{E[Payout]}{d - E[G_t]}$$

Eq. 3-18

Had the growth rates G_t not been IID then the mean present value could not have been reduced in this manner. Also note that Eq. 3-13 is an inequality because a single stochastic growth rate is used for all years and the exponentiation of that growth rate invokes Jensen's inequality.

3.4. Stochastic Discount & Growth Rates

If the discount rate and the payout growth rate are independent stochastic variables then it follows from Eq. 3-8 (for a time-constant discount rate) and Eq. 3-10 (for a time-varying discount rate) that the mean present value may be underestimated when calculated from the mean discount and growth rates. This is because Jensen's inequality is invoked for the stochastic discount rate regardless of whether the payout growth rate is stochastic or not.

3.5. Stochastic Value Yield

Let $Payout$ be a stochastic variable for the starting payout and let G be a stochastic variable for the payout growth which starts next year. Note that a single growth rate is used for all years t but the choice of that growth rate is stochastic. The payout in year t is a derived stochastic variable:

$$Payout_t = Payout \cdot (1 + G)^{t-1}$$

Eq. 3-19

Let D be a stochastic discount rate greater than G. The present value V is a derived stochastic variable calculated using Eq. 11-3:

$$V = \sum_{t=1}^{\infty} \frac{Payout \cdot (1+G)^{t-1}}{(1+D)^t} = \frac{Payout}{D - G}$$

Eq. 3-20

The value yield is a stochastic variable derived from $Payout$ and G which makes the present value equal to the market-cap:

$$MarketCap = \frac{Payout}{Value\ Yield - G} \Leftrightarrow Value\ Yield = \frac{Payout}{MarketCap} + G$$

Eq. 3-21

Because $MarketCap$ is a constant, the mean value yield is:

$$E[Value\ Yield] = \frac{E[Payout]}{MarketCap} + E[G]$$

Eq. 3-22

Note that the mean value yield approaches $E[G]$ as the market-cap approaches infinity.

3.5.1. Time-Varying Growth Rate

Let G_t be the stochastic payout growth rate in year t and assume the G_t's are IID. Let $\widehat{G_t}$ be the accumulated growth in year t as defined in Eq. 3-15.

The value yield is the discount rate that makes the present value equal to the market-cap, assuming it exists and is well-defined (see section 2.6.7):

$$MarketCap = Payout \cdot \sum_{t=1}^{\infty} \frac{\widehat{G_t}}{(1 + Value\ Yield)^t}$$

Eq. 3-23

This geometric series is assumed to converge, so we can select a single stochastic growth rate G' and use it to construct a geometric series that converges to the same value according to Eq. 11-3:

$$MarketCap = Payout \cdot \sum_{t=1}^{\infty} \frac{\widehat{G_t}}{(1 + Value\ Yield)^t} = Payout \cdot \sum_{t=1}^{\infty} \frac{(1 + G')^{t-1}}{(1 + Value\ Yield)^t} = \frac{Payout}{Value\ Yield - G'}$$

Eq. 3-24

This can be rearranged to derive the value yield from the stochastic variables $Payout$ and G':

$$Value\ Yield = \frac{Payout}{MarketCap} + G'$$

The mean value yield is then:

$$E[Value\ Yield] = \frac{E[Payout]}{MarketCap} + E[G']$$

Eq. 3-25

The probability distribution of G' is unknown and may have another distribution than G_t, for example, G_t may be discrete while G' is continuous. The mean payout growth rate is $E[G']$ and it is known from Eq. 3-17 that the mean payout growth rate also equals $E[G_t]$ so the mean value yield is:

$$E[Value\ Yield] = \frac{E[Payout]}{MarketCap} + E[G_t]$$

Eq. 3-26

This simple formula for the mean value yield is possible because the payout growth rates G_t are assumed to be IID. Also note that the mean value yield approaches $E[G_t]$ as the market-cap approaches infinity.

3.5.2. Equity Growth Model

In the equity growth model from section 2.2, the mean starting payout is calculated using Eq. 2-13 assuming the stochastic variables ROE and $Retain/Earnings$ are dependent on each other but independent of the starting equity:

$$E[Payout] = Equity \cdot E\left[ROE \cdot \left(1 - \frac{Retain}{Earnings}\right)\right]$$

Eq. 3-27

Using this in Eq. 3-26 with the mean payout growth rate from Eq. 2-18 gives the mean value yield:

$$\begin{aligned} E[Value\ Yield] &= \frac{Equity \cdot E\left[ROE \cdot \left(1 - \frac{Retain}{Earnings}\right)\right]}{MarketCap} + E\left[ROE \cdot \frac{Retain}{Earnings}\right] \\ &= \frac{E\left[ROE \cdot \left(1 - \frac{Retain}{Earnings}\right)\right]}{P/Book} + E\left[ROE \cdot \frac{Retain}{Earnings}\right] \end{aligned}$$

Eq. 3-28

Independent Stochastic Variables

If instead ROE and $Retain/Earnings$ are independent stochastic variables, then the mean payout in Eq. 3-27 becomes:

$$E[Payout] = Equity \cdot E[ROE] \cdot \left(1 - E\left[\frac{Retain}{Earnings}\right]\right)$$

Eq. 3-29

Using this and the mean payout growth rate from Eq. 2-20 with Eq. 3-26 gives the mean value yield:

$$\begin{aligned} E[Value\ Yield] &= \frac{Equity \cdot E[ROE] \cdot \left(1 - E\left[\frac{Retain}{Earnings}\right]\right)}{MarketCap} + E[ROE] \cdot E\left[\frac{Retain}{Earnings}\right] \\ &= \frac{E[ROE] \cdot \left(1 - E\left[\frac{Retain}{Earnings}\right]\right)}{P/Book} + E[ROE] \cdot E\left[\frac{Retain}{Earnings}\right] \end{aligned}$$

Eq. 3-30

3.5.3. Mean Value Yield as Discount Rate

The value yield of one investment can be used as the discount rate for calculating the present value of another investment, thus giving the value of one investment relative to the other. For example, the value yield of the S&P 500 stock market index is used as the discount rate in calculating the present value of the Coca-Cola company in section 9.6.1.

If the mean present value is wanted but not its probability distribution, then it would be desirable to use the mean value yield as the discount rate because it can be calculated from the simple Eq. 3-28 under certain assumptions. But using the mean value yield as the discount rate underestimates the mean present value due to Jensen's inequality as shown in section 3.2. So the probability distribution of the value yield is still needed.

3.6. Comparing Stochastic Value Yields

Comparing stochastic value yields is useful in estimating the probability that the value yield of one investment is greater than that of another investment with some risk premium. Let $Value\ Yield_{Company}$ and $Value\ Yield_{Index}$ denote the stochastic value yields of a company and a stock market index, respectively. The probability of the company's value yield being the greatest is denoted:

$$\Pr\left[Value\ Yield_{Company} > Value\ Yield_{Index} + Risk\ Premium\right]$$

Eq. 3-31

This is equivalent to:

$$\Pr\left[Value\ Yield_{Company} - Value\ Yield_{Index} - Risk\ Premium > 0\right]$$

Eq. 3-32

In this form, the probability can be calculated directly when the value yields result from Monte Carlo simulations, by simply counting the number of value yields that satisfy the condition and dividing by the total number of simulations. This way of calculating the probability also has the advantage of working for value yields that are statistically dependent, for example if the company is itself a part of the stock market index or if the value yields are dependent in some other way.

3.6.1. Alternative Calculation

There is another way of calculating the probability of one value yield being greater than another. Let vy denote a value yield taken on by the stochastic variable $Value\ Yield_{Company}$ so Eq. 3-31 becomes:

$$\begin{aligned}\Pr&\left[Value\ Yield_{Company} > Value\ Yield_{Index} + Risk\ Premium\right]\\ &= \sum_{vy} \Pr\left[Value\ Yield_{Company} = vy\right]\\ &\cdot \Pr\left[vy > Value\ Yield_{Index} + Risk\ Premium \middle| Value\ Yield_{Company} = vy\right]\end{aligned}$$

Eq. 3-33

If $Value\ Yield_{Index}$ and $Value\ Yield_{Company}$ are independent then this probability equals:

$$\sum_{vy} \Pr\left[Value\ Yield_{Company} = vy\right] \cdot \Pr[vy > Value\ Yield_{Index} + Risk\ Premium]$$

Eq. 3-34

The CDF for $Value\ Yield_{Index}$ is denoted $F_{Value\ Yield_{Index}}$ and defined as:

$$F_{Value\ Yield_{Index}}(x) = \Pr[Value\ Yield_{Index} < x]$$

Eq. 3-35

This means:

$$\Pr[Value\ Yield_{Index} + Risk\ Premium < vy] = F_{Value\ Yield_{Index}}(vy - Risk\ Premium)$$

Eq. 3-36

Using this with Eq. 3-34 gives:

$$\Pr\left[Value\ Yield_{Company} > Value\ Yield_{Index} + Risk\ Premium\right] \\ = \sum_{vy} \Pr\left[Value\ Yield_{Company} = vy\right] \cdot F_{Value\ Yield_{Index}}(vy - Risk\ Premium)$$

Eq. 3-37

This formula is only valid when $Value\ Yield_{Company}$ and $Value\ Yield_{Index}$ are independent otherwise the joint probability distribution must be used. The significantly simpler Eq. 3-32 works in both dependent and independent cases and is therefore recommended.

4. Option Valuation

An option is a financial instrument which gives the holder of the option the right but not the obligation to either buy (in case of a call option) or sell (in case of a put option) an underlying financial security at a predetermined exercise price (or strike price). Options have expiration dates after which they cannot be exercised. The period until the expiration date is known as the option's life or maturity period.

A call option will not get exercised if the share price is lower than the exercise price because the option holder could instead buy the share directly in the market at a lower price, so the exercise value is:

$$ExerciseValue_{Call} = \text{Max}\ (SharePrice - ExercisePrice, 0)$$

Eq. 4-1

Conversely, a put option will not get exercised if the share price is higher than the exercise price because the option holder could instead sell the share directly in the market at a higher price, so the exercise value is:

$$ExerciseValue_{Put} = \text{Max}\ (ExercisePrice - SharePrice, 0)$$

Eq. 4-2

The so-called intrinsic value of an option is the exercise value calculated using the current share price rather than the share price at the time of exercise. The difference between the intrinsic value and the option price is called the time value and is the premium that must be paid for the possibility that the share price will change sufficiently in the future so as to increase the exercise value of the option. In some cases the time value is almost zero as shown in Figure 63.

The profit (or loss) from exercising an option is the exercise value minus the option's price:

$$Profit = ExerciseValue - OptionPrice$$

Eq. 4-3

This is the profit from the perspective of an option buyer which may be made explicit by writing $Profit_{Buyer}$ instead. The profit for the option seller is described below.

When considering Monte Carlo simulated exercise values, the probability of a (positive) profit is calculated by counting the number of simulated profits that are positive and dividing them by the total number of simulated profits. The probability of profit is denoted:

$$\text{Pr}[Profit > 0] = \text{Pr}\ [ExerciseValue > OptionPrice]$$

Eq. 4-4

The present value of the exercise value is calculated using the discount rate d and the number of years t (possibly not an integer) until the option is exercised:

$$Present\ Value\ of\ ExerciseValue = \frac{ExerciseValue}{(1+d)^t}$$

Eq. 4-5

The value yield is the discount rate which makes the present value equal to the option's price:

$$OptionPrice = \frac{ExerciseValue}{(1+Value\ Yield)^t} \Leftrightarrow Value\ Yield = \left(\frac{ExerciseValue}{OptionPrice}\right)^{\frac{1}{t}} - 1$$

Eq. 4-6

If the exercise value is zero then the value yield is -1, which represents an annual rate of return that is a total loss. Call options are worthless when the market price for the share is below the exercise price, and put options are worthless when the share price is above the exercise price, see Eq. 4-1 and Eq. 4-2.

4.1. Seller's Profit

The formulas above are from the perspective of an option buyer. From the perspective of an option seller (also known as an option writer), the profit (or loss) is the option price minus the exercise value:

$$Profit_{Seller} = OptionPrice - ExerciseValue$$

Eq. 4-7

Using this with the definition of a buyer's profit from Eq. 4-3 shows that the buyer's profit is the seller's loss, and vice versa:

$$Profit_{Buyer} + Profit_{Seller} = 0 \Leftrightarrow Profit_{Buyer} = -Profit_{Seller}$$

Eq. 4-8

If the exercise value is zero then the option price is all profit for the seller. The exercise value of a call option does not have an upper bound so the profit to a call option seller is upper bounded by the option price but not lower bounded. The exercise value of a put option is both lower and upper bounded because the price of the underlying stock cannot be lower than zero, so the put option seller's profit is lower bounded by the option price minus the exercise price and upper bounded by the option price.

The probability of (positive) profit to the seller is defined from Eq. 4-7:

$$\Pr[Profit_{Seller} > 0] = \Pr[OptionPrice > ExerciseValue]$$

Eq. 4-9

Using this with Eq. 4-4 and assuming $\Pr[OptionPrice = ExerciseValue] = 0$ gives:

$$\Pr[Profit_{Seller} > 0] = 1 - \Pr\,[Profit_{Buyer} > 0]$$

Eq. 4-10

4.2. Value Estimation

The exercise value of an option depends on the stock price at the time of exercise which is inherently unpredictable. Several models exist for estimating future stock prices and hence the value of options, with a popular one being the Black-Scholes formula [**6**] [**7**] which assumes stock prices follow a random walk, a so-called geometric Brownian motion (GBM) with constant volatility, that stocks and options are correctly priced so there is no possibility for increasing expected returns without also increasing the risk, that there is unlimited borrowing ability, no transaction costs, interest rates are constant and known in advance, etc. Although several of these assumptions were criticized as being unrealistic by one of the original authors [**8**], the fundamental assumption of the Black-Scholes formula was not critiqued, namely that stock prices are assumed to follow a random walk in which changes to stock prices are independent log-normal distributed stochastic variables. Figure 1 shows the daily stock price returns for the S&P 500 stock market index for the period 1984-2011 and Figure 2 shows the yearly stock price returns for that period, both of which are clearly not log-normal distributed. Section 6.9 further shows that successive price changes of the S&P 500 are strongly correlated. This means the Black-Scholes model is oversimplified and becomes increasingly inaccurate with the option duration, as also noted by Buffett [**9**].

The so-called binomial model by Cox et al. [**10**] uses a different model to forecast stock prices but it is still based on assumptions similar to the Black-Scholes model, namely that stock prices follow a random walk with a simple probability distribution, capital markets are perfect, etc., and the model therefore exhibits problems that are similar to those of Black-Scholes.

Monte Carlo simulation can also be used to estimate the value of options by simulating future stock prices. This is typically done by simulating a stochastic process as a GBM variant, see e.g. Glasserman [**1**]. This paper takes another approach by first simulating the underlying financial variables of a company using probability distributions from historical data and then simulating the stock price as a function of these financial variables.

5. Monte Carlo Simulation

Monte Carlo simulation is computer simulation of a stochastic model repeated numerous times so as to estimate the probability distribution of the outcome of the stochastic model. This is useful when the probability distribution is not possible to derive analytically, either because it is too complex or because the stochastic variables of the model are not from simple, well-behaved probability distributions. Monte Carlo simulation allows for arbitrary probability distributions so that very rare events can also be modelled.

It was shown in section 3 that using the mean growth and discount rates will underestimate the present value. Monte Carlo simulation allows better estimation of the present value by repeating the calculation with the growth and discount rates selected at random according to their probability distribution.

5.1. Equity Growth Model

A single Monte Carlo simulation of the equity growth model consists of these steps:

1. Load historical financial data and determine probability distributions and dependencies.
2. Determine the required number of Monte Carlo iterations k using Eq. 2-50.
3. Set $Equity_0 = 1$ for normalization purposes.
4. For $t = 1$ to k: Sample ROE_t, $(Dividend/Earnings)_t$ and $(NetBuyback/Earnings)_t$ from historical data. Calculate $Earnings_t$ from Eq. 2-12, $Payout_t$ from Eq. 2-13, $Equity_t$ from Eq. 2-9, $Dividend_t = Earnings_t \cdot (Dividend/Earnings)_t$, and $Net\ Buyback_t = Earnings_t \cdot (Net\ Buyback/Earnings)_t$.

The probability distribution is found by repeating steps 3-4 and recording the resulting values.

5.2. Value Yield

Monte Carlo simulation of the value yield extends the simulation in section 5.1 by adding these steps:

5. Calculate the mean terminal value from Eq. 2-51.
6. Use a numerical optimization method to find the value yield in Eq. 2-32.

The probability distribution is found by repeating steps 3-6 and recording the resulting value yields.

5.3. P/Book

A single Monte Carlo simulation of future P/Book ratios consists of these steps:

1. Determine the probability distribution for the daily P/Book changes.
2. Initialize $(P/Book)_0$ with some value.
3. For $t = 1$ to k first select $Change_t$ from the P/Book change distribution and then calculate:
$$(P/Book)_t = (P/Book)_{t-1} \cdot Change_t$$

The probability distribution is found by repeating steps 2-3 and recording the resulting P/Book ratios.

5.4. Price

The future equity and P/Book ratios resulting from the Monte Carlo simulations in sections 5.1 and 5.3 can be multiplied so as to simulate the share price, see Eq. 2-26. This in turn may be used to value options.

5.5. Per Share

Calculating per-share numbers for equity, earnings, etc. consists of four steps:

1. The equity, earnings, share buyback net of issuance, etc. are Monte Carlo simulated as described in section 5.1.
2. The price (or market-cap) is Monte Carlo simulated as described in section 5.4.
3. The results of steps 1 and 2 are used with Eq. 2-27 and Eq. 2-28 to calculate the number of shares.
4. The equity, earnings, etc. from step 1 are divided by the number of shares from step 3 to find the per-share numbers.

5.5.1. Statistical Dependency

The ratio $NetBuyback/Earnings$ is sampled from the same year as ROE and $Dividend/Earnings$ but is assumed to be independent of the share-price. In reality, however, companies have had a tendency to increase share buybacks when the share price was high and increase share issuance when the share price was low, see Pedersen [**4**]. This historical tendency is ignored here but should ideally be modelled and simulated as well.

5.6. Present Value

Calculating the present value consists of first Monte Carlo simulating the payouts (sum of dividends and net share buybacks) in section 5.1 or simulating the dividend per share in section 5.5.

The discount rate is the rate of return that can be obtained from an alternative investment, thus giving the value of the payouts or dividends per share relative to that alternative investment. The discount rate may be sampled from the probability distribution of the value yield that can be obtained from the alternative investment or a sequence of compounded returns can be used as the discount rate.

The payouts and discount rate are used with Eq. 2-8 to calculate the present value of one Monte Carlo simulation and this is repeated to get the probability distribution.

Case Studies

6. S&P 500

The Standard & Poor's 500 stock market index (S&P 500) consists of 500 large companies traded on the stock markets in USA and operating in a wide variety of industries including energy and utility, financial services, health care, information technology, heavy industry, manufacturers of consumer products, etc. The S&P 500 index may be used as a proxy for the entire USA stock market as it covers about 75% of that market.[4]

6.1. Stock Market Forecasting

Estimating the stock market's future rate of return is important in many aspects of financial valuation.

6.1.1. Extrapolation of Total Return

A simple approach to forecasting future returns of the stock market is to extrapolate its historical total return, that is, the annual rate of return from capital gains and reinvestment of dividends. But this approach is flawed because it ignores the current stock market price relative to the future earnings of the underlying companies, hence implicitly assuming that the stock market is correctly priced relative to its future earnings.

6.1.2. Equity Risk Premium

Another approach to forecasting future stock market returns is to use an Equity Risk Premium (ERP) derived from historical observations of stock market returns relative to low-risk government bonds. Then adding the historical average ERP to the current yield on government bonds gives the expected future rate of return of the stock market. This is an attractive technique for estimating future stock market returns because it is simple and uses two known numbers to estimate an unknown number. Several variations to ERP estimation have been proposed for taking into account macro-economic indicators, supply and demand, etc., see e.g. Hammond et al. [**11**] and Damodaran [**12**]. Unfortunately, the fundamental idea of ERP is flawed, firstly because it is incorrect to use averages in valuations with non-linear calculations, see section 3, and secondly because the ERP changes unpredictably over time, see section 7.1.

6.1.3. Value Yield

This paper takes another approach to estimating future stock market returns. Instead of trying to forecast the stock market's future rate of return from its historical total return or relation to government bonds, the stock market's accumulation of equity capital and its earnings resulting from that capital is modelled and Monte Carlo simulated. Combined with the stock market's historical P/Book ratio it results in probability distributions for the stock market's annualized rate of return, or value yield, depending on the current stock price and the holding period of the stock. This circumvents the problem of determining the relationship between government bonds and the stock market, and through its use of historical data implicitly takes into account a wide range of potential changes in the macro economy, government bond yields, ERP, etc.

6.2. Historical Data

Table 1 shows the following historical financial data for the S&P 500 index during the period 1984-2011: Return on Assets (ROA), Return on Equity (ROE), dividends over earnings, net share buyback over earnings, earnings retaining, and equity over assets. The summary statistics are shown in Table 2 and suggest that

[4] S&P 500 Fact Sheet, retrieved April 11, 2013:
www.standardandpoors.com/indices/articles/en/us/?articleType=PDF&assetID=1221190434733

the earnings of the S&P 500 are related to its assets which in turn are related to its equity, and this means it is reasonable to use the equity growth model from section 2.2 to simulate the future equity and earnings.

Appendix 11.3 gives details about the compiling and limitations of this data, most notably that the calculations using e.g. equity and earnings of the S&P 500 companies are un-weighted because the weights used in the official S&P 500 index are proprietary, so the equity and earnings are merely the sum of the equity and earnings of all the companies in the S&P 500 index. This still provides a useful estimate of the rate of return to be expected from investing in a broad stock market index, it is just slightly different from the official S&P 500 index.

6.2.1. Sampling

Statistical dependencies of the data in Table 1 should be modelled because the data is being repeatedly sampled in the Monte Carlo simulations.

The relations between ROE and $Retain/Earnings$ when selected from the same year or time-shifted one year are shown in the scatter-plots of Figure 3. When ROE and $Retain/Earnings$ are selected from the same year they tend to increase together, while no such tendency is apparent when ROE and $Retain/Earnings$ are time-shifted from each other. Although the data set is small and this tendency may not be statistically significant, it seems more reasonable to select ROE and $Retain/Earnings$ from the same year when sampling the data in the Monte Carlo simulations.

The autocorrelations for ROE and $Retain/Earnings$ are shown in Figure 4 and Figure 5, which suggest that there is no statistically significant linear correlation over time. This means the historical data pairs ROE and $Retain/Earnings$ can be randomly sampled in the Monte Carlo simulations.

6.2.2. Mean Growth Rate

Because ROE and $Retain/Earnings$ are sampled from the same year, the mean equity growth rate $E[G]$ can be calculated using Eq. 2-18 with the ROE and $Retain/Earnings$ pairs from Table 1:

$$E[G] = E[ROE \cdot Retain/Earnings] \simeq 4.46\%$$

Eq. 6-1

The growth rate of the mean earnings and payouts is the same because they are derived from the equity.

If ROE and $Retain/Earnings$ were instead sampled independently then the mean growth rate would be calculated using Eq. 2-20 with the mean ROE and $Retain/Earnings$ from Table 2:

$$E[G] = E[ROE] \cdot E[Retain/Earnings] = 13.5\% \cdot 26.4\% \simeq 3.6\%$$

6.3. Curve Fittings

The results of the Monte Carlo simulations below are curve fitted so as to provide simple formulas for estimating various probability distributions from input variables such as the P/Book ratio. No theoretical justification is given for selecting those particular functions for the curve fittings. The guiding principle has merely been to select functions that are simple and provide a good fit to the observed data.

6.4. Earnings

The probability distribution for the future earnings of the S&P 500 stock market index can be estimated using the Monte Carlo simulation from section 5.1 which is repeated 1,000 times here.

6.4.1. Mean & Standard Deviation

Figure 6 and Figure 7 show log-normal Q-Q plots of the earnings in different future years resulting from these Monte Carlo simulations. The earnings are evidently not from a simple probability distribution but after several hundred years the Q-Q plots become somewhat linear which suggests the earnings eventually approach a log-normal distribution.

The earnings mean and standard deviation are shown in Figure 8 with the fitted functions:

$$E[Earnings_t] = e^{0.04366 \cdot t - 2.03288}$$

Eq. 6-2

$$Stdev[Earnings_t] = e^{0.04603 \cdot t - 3.03202}$$

Eq. 6-3

The growth rates for the mean and standard deviation are:

$$\frac{E[Earnings_{t+1}]}{E[Earnings_t]} - 1 = e^{0.04366} - 1 \simeq 4.46\%$$

Eq. 6-4

$$\frac{Stdev[Earnings_{t+1}]}{Stdev[Earnings_t]} - 1 = e^{0.04603} - 1 \simeq 4.71\%$$

Eq. 6-5

That is, the mean earnings increase about 4.46% per year and the standard deviation about 4.71%. The growth rate for the mean earnings is the same as for the mean equity, see section 6.2.2.

Adjusting for Normalized Equity

Because the equity starts at 1 in these Monte Carlo simulations, the stochastic variable for the earnings must be multiplied by the actual starting equity which is denoted $Equity$ here. The adjusted mean earnings are derived from Eq. 6-2 using linearity of the expectation operator:

$$E[Equity \cdot Earnings_t] = Equity \cdot E[Earnings_t] = Equity \cdot e^{0.04366 \cdot t - 2.03288}$$

Eq. 6-6

The standard deviation of the adjusted earnings is derived using Eq. 6-3 with a well-known property of the standard deviation and assuming $Equity > 0$:

$$Stdev[Equity \cdot Earnings_t] = |Equity| \cdot Stdev[Earnings_t] = Equity \cdot e^{0.04603 \cdot t - 3.03202}$$

Eq. 6-7

This assumes the number of shares remains constant. Share buyback and issuance is taken into account in section 6.10.3.

6.5. Dividend

The probability distribution for the future dividends of the S&P 500 stock market index can be estimated using the Monte Carlo simulation from section 5.1 which is repeated 1,000 times here.

6.5.1. Mean & Standard Deviation

Figure 9 and Figure 10 show log-normal Q-Q plots of future dividends resulting from these Monte Carlo simulations. The dividends are evidently not from a simple probability distribution in the first 10 years but after about 20 years the Q-Q plots become somewhat linear and hence approach a log-normal distribution.

The dividend mean and standard deviation are shown in Figure 11 with the fitted functions:

$$E[Dividend_t] = e^{0.04367 \cdot t - 2.82092}$$

Eq. 6-8

$$Stdev[Dividend_t] = e^{0.04744 \cdot t - 4.33119}$$

Eq. 6-9

The growth rates for the mean and standard deviation are:

$$\frac{E[Dividend_{t+1}]}{E[Dividend_t]} - 1 = e^{0.04367} - 1 \simeq 4.46\%$$

Eq. 6-10

$$\frac{Stdev[Dividend_{t+1}]}{Stdev[Dividend_t]} - 1 = e^{0.04744} - 1 \simeq 4.86\%$$

Eq. 6-11

That is, the mean dividend increases about 4.46% per year and the standard deviation about 4.86%. The growth rate for the mean dividend is the same as for the mean equity, see section 6.2.2.

Adjusting for Normalized Equity

Because the equity is normalized to start at 1 in these Monte Carlo simulations, the stochastic variable for the dividends must be multiplied by the actual starting equity which is denoted $Equity$ here. The adjusted mean dividends are derived from Eq. 6-8 using linearity of expectation:

$$E[Equity \cdot Dividend_t] = Equity \cdot E[Dividendd_t] = Equity \cdot e^{0.04367 \cdot t - 2.82092}$$

Eq. 6-12

The standard deviation of the adjusted dividends is derived from Eq. 6-9 with a well-known property of the standard deviation and assuming $Equity > 0$:

$$Stdev[Equity \cdot Dividend_t] = |Equity| \cdot Stdev[Dividend_t] = Equity \cdot e^{0.04744 \cdot t - 4.33119}$$

Eq. 6-13

This assumes the number of shares remains constant. Share buyback and issuance is taken into account in section 6.10.3.

6.6. Share Buyback Net of Issuance

The probability distribution for the amounts used for share buybacks net of issuance for the S&P 500 stock market index can be estimated using the Monte Carlo simulation from section 5.1 which is repeated 1,000 times here.

6.6.1. Mean & Standard Deviation

Figure 12 and Figure 13 show log-normal Q-Q plots of future net share buybacks resulting from these Monte Carlo simulations. The net share buybacks are evidently not log-normal distributed in the first 10 years but after several hundred years the Q-Q plots become increasingly linear and hence approach a log-normal distribution.

The net share buyback mean and standard deviation are shown in Figure 14 with the fitted functions:

$$E[Net\ Buyback_t] = e^{0.04368 \cdot t - 3.62753}$$

Eq. 6-14

$$Stdev[Net\ Buyback_t] = e^{0.04453 \cdot t - 3.63738}$$

Eq. 6-15

The growth rates for the mean and standard deviation are:

$$\frac{E[Net\ Buyback_{t+1}]}{E[Net\ Buyback_t]} - 1 = e^{0.04368} - 1 \simeq 4.46\%$$

Eq. 6-16

$$\frac{Stdev[Net\ Buyback_{t+1}]}{Stdev[Net\ Buyback_t]} - 1 = e^{0.04453} - 1 \simeq 4.55\%$$

Eq. 6-17

That is, the mean net share buyback increases about 4.46% per year and the standard deviation about 4.55%. The growth rate for the mean net buyback is the same as for the mean equity, see section 6.2.2.

Adjusting for Normalized Equity

Because the equity is normalized to start at 1 in these Monte Carlo simulations, the stochastic variable for the net share buybacks must be multiplied by the actual starting equity which is denoted $Equity$ here. The adjusted mean net buybacks are derived from Eq. 6-14 using linearity of expectation:

$$E[Equity \cdot Net\ Buyback_t] = Equity \cdot E[Net\ Buyback_t] = Equity \cdot e^{0.04368 \cdot t - 3.62753}$$

Eq. 6-18

The standard deviation of the adjusted net buybacks is derived from Eq. 6-15 with a well-known property of the standard deviation and assuming $Equity > 0$:

$$Stdev[Equity \cdot Net\ Buyback_t] = |Equity| \cdot Stdev[Net\ Buyback_t] = Equity \cdot e^{0.04453 \cdot t - 3.63738}$$

Eq. 6-19

6.7. Payout

The probability distribution for the future payout (dividend and net share buyback) of the S&P 500 stock market index can be estimated using the Monte Carlo simulation from section 5.1 which is repeated 1,000 times here.

6.7.1. Mean & Standard Deviation

Figure 15 and Figure 16 show Q-Q log-normal plots of the payouts in different future years resulting from these Monte Carlo simulations. In the first 50 years the payouts are evidently not from a simple probability distribution but as the years increase the Q-Q plots become increasingly linear which suggests the payouts approach a log-normal distribution.

The payout mean and standard deviation increase over time as shown in Figure 17 where the fitted functions are:

$$E[Payout_t] = e^{0.04366 \cdot t - 2.43394}$$

Eq. 6-20

$$Stdev[Payout_t] = e^{0.04626 \cdot t - 3.51652}$$

Eq. 6-21

The growth rates for the mean and standard deviation are:

$$\frac{E[Payout_{t+1}]}{E[Payout_t]} - 1 = e^{0.04366} - 1 \simeq 4.46\%$$

Eq. 6-22

$$\frac{Stdev[Payout_{t+1}]}{Stdev[Payout_t]} - 1 = e^{0.04626} - 1 \simeq 4.73\%$$

Eq. 6-23

That is, the mean payouts increase about 4.46% per year and the standard deviation about 4.73%. The growth rate for the mean payouts is the same as for the mean equity, see section 6.2.2.

Adjusting for Normalized Equity

Because the equity starts at 1 in these Monte Carlo simulations, the stochastic variable for the payout must be multiplied by the actual starting equity which is denoted $Equity$ here. The adjusted mean payout is derived from Eq. 6-20:

$$E[Equity \cdot Payout_t] = Equity \cdot E[Payout_t] = Equity \cdot e^{0.04366 \cdot t - 2.43394}$$

Eq. 6-24

The standard deviation of the adjusted payout is derived using Eq. 6-21:

$$Stdev[Equity \cdot Payout_t] = |Equity| \cdot Stdev[Payout_t] = Equity \cdot e^{0.04626 \cdot t - 3.51652}$$

Eq. 6-25

This assumes the payout consists entirely of dividends so the number of shares remains constant. Share buyback and issuance is taken into account in section 6.10.

6.7.2. Payout Sum

The cumulative sum of payouts (dividends and net share buybacks) for the years 1 to t is:

$$PayoutSum_t = \sum_{k=1}^{t} Payout_k$$

Eq. 6-26

Figure 18 shows histograms for the payout sums resulting from the Monte Carlo simulations described above. Also shown are the fitted log-normal distributions which indicate that the probability distributions approximate the log-normal distribution as the years increase.

Assume $PayoutSum_t \sim \log \mathcal{N}(\mu, \sigma^2)$ for year $t > 1$ is a log-normal distributed stochastic variable. Figure 19 shows the parameters μ and σ when the year t ranges between 1 and 290. Also shown are the fitted functions:

$$\mu = \log(e^{0.0431754 \cdot t} - 1) + 0.7332384$$

Eq. 6-27

$$\sigma = 0.3310138 \cdot t^{-0.7443678} + 0.0118551 \cdot t^{0.6641084}$$

Eq. 6-28

Adjusting for Normalized Equity

The stochastic variable $PayoutSum_t$ must be multiplied by the actual starting equity which is denoted $Equity$ without a subscript, because the equity in these Monte Carlo simulations starts at one for normalization purposes. Using a well-known property of the log-normal distribution, the stochastic variable $Equity \cdot PayoutSum_t$ is also log-normal distributed with parameter μ derived from Eq. 6-27:

$$\mu = \log(e^{0.0431754 \cdot t} - 1) + 0.7332384 + \log(Equity)$$

Eq. 6-29

The parameter σ is the same as in Eq. 6-28.

6.8. Equity

The probability distribution for the future equity of the S&P 500 stock market index can be estimated using the Monte Carlo simulation from section 5.1 which is repeated 1,000 times here.

6.8.1. Log-Normal Parameters

Figure 20 and Figure 21 show Q-Q log-normal plots for the equity in different years resulting from these Monte Carlo simulations. As the years increase the Q-Q plots approach straight lines, thus indicating the equity is approximately log-normal distributed.

Let the starting equity be $Equity_0 = 1$ and assume $Equity_t \sim \log \mathcal{N}(\mu, \sigma^2)$ for year $t > 1$ is a log-normal distributed stochastic variable. Figure 22 shows the parameters μ and σ resulting from the Monte Carlo simulations when the year t ranges between 1 and 290. Also shown are the fitted functions:

$$\mu = 0.04312 \cdot t$$

Eq. 6-30

$$\sigma = 0.03251 \cdot \sqrt{t}$$

Eq. 6-31

For example, in year $t = 8$ the equity is estimated to be a log-normal distributed stochastic variable with parameters:

$$\mu = 0.04312 \cdot 8 \simeq 0.345$$

$$\sigma = 0.03251 \cdot \sqrt{8} \simeq 0.092$$

Figure 23 compares this distribution to the results of the Monte Carlo simulation.

Adjusting for Normalized Equity

The stochastic variable $Equity_t$ must be multiplied by the actual starting equity which is denoted $Equity$ without a subscript, because $Equity_0$ was set to equal one in these Monte Carlo simulations for normalization purposes. Using a well-known property of the log-normal distribution, the stochastic variable $Equity \cdot Equity_t$ is also log-normal distributed with parameter μ derived from Eq. 6-30:

$$\mu = 0.04312 \cdot t + \log(Equity)$$

Eq. 6-32

The parameter σ is the same as in Eq. 6-31.

6.8.2. Mean & Standard Deviation

Using the log-normal parameters μ and σ for $Equity_t$ from Eq. 6-30 and Eq. 6-31 with Eq. 11-6 and Eq. 11-7 gives:

$$E[Equity_t] = e^{\mu+\frac{\sigma^2}{2}} = 1.04462^t$$

Eq. 6-33

$$Stdev[Equity_t] = E[Equity_t] \cdot \sqrt{e^{\sigma^2} - 1} = E[Equity_t] \cdot \sqrt{1.033^t - 1}$$

Eq. 6-34

The mean equity grows by the rate:

$$\frac{E[Equity_{t+1}]}{E[Equity_t]} - 1 \simeq 4.45\%$$

Eq. 6-35

This almost equals the mean growth rate calculated directly from the financial data, see section 6.2.2, with the small difference likely being due to sampling and rounding error.

Adjusting for Normalized Equity

Because the equity starts at 1 in these Monte Carlo simulations, the stochastic variable $Equity_t$ must be multiplied by the actual starting equity which is denoted $Equity$ here. The adjusted mean equity is derived from Eq. 6-33:

$$E[Equity \cdot Equity_t] = Equity \cdot E[Equity_t] = Equity \cdot 1.04462^t$$

Eq. 6-36

The standard deviation of the adjusted equity is derived from Eq. 6-34:

$$Stdev[Equity \cdot Equity_t] = |Equity| \cdot Stdev[Equity_t] = Equity \cdot E[Equity_t] \cdot \sqrt{1.033^t - 1}$$

Eq. 6-37

This assumes the number of shares remains constant. Share buyback and issuance is taken into account in section 6.10.3.

6.9. Price

The probability distribution for the price of the S&P 500 stock market index at a given future time t can be estimated by multiplying the P/Book ratio with the equity at that time. Recall that $P/Book$ is just another notation for $MarketCap/Equity$, see Eq. 2-25, so multiplying this ratio with the equity results in the market-cap which is the price of all the shares:

$$MarketCap_t = (P/Book)_t \cdot Equity_t$$

Eq. 6-38

These terms and notations will be used interchangeably in the following.

6.9.1. P/Book Data

In the data-set for the S&P 500 index used here, the book-values are only known at the last trading day of each year during 1983-2011 so the daily book-values are interpolated from these. It is reasonable to use interpolated book-values because the actual book-values are usually quite stable during a year. The volatility in the P/Book ratio is caused by the volatility of the share-price. The daily share-price is available from another data-set[5]. Dividing the daily prices with the interpolated book-values estimates the P/Book ratio on each trading day from the beginning of 1984 to the end of 2011 as shown in Figure 24.

6.9.2. Constant P/Book

Assuming the P/Book ratio is constant, the future market-cap of the S&P 500 index is estimated from Eq. 6-38 by multiplying the constant P/Book ratio with the future equity at that time which is estimated in section 6.8.

6.9.3. Historical P/Book Distribution

Figure 25 shows the historical distribution of P/Book ratios for the S&P 500 index during the period 1984-2011, which is clearly not log-normal or any other simple distribution. If a future P/Book ratio is assumed to be chosen randomly from the historical distribution then the price of the S&P 500 index is estimated from Eq. 6-38 using samples of the historical P/Book ratio multiplied by samples of the equity distribution, which can either be Monte Carlo simulated or sampled from the distribution estimates in section 6.8.

Figure 26 shows histograms for the price distributions in different years, where each year's distribution results from multiplying one million samples of the historical P/Book ratios with samples of the Monte Carlo simulated equity in that year. The price distribution can be seen to approach a log-normal distribution as the years increase. These price distributions assume that the equity starts at 1 and the number of shares remains constant.

6.9.4. Simulated P/Book Distribution

Figure 27 shows the autocorrelation of the P/Book ratio for the S&P 500 index in the period 1984-2011. This shows that successive P/Book ratios are strongly correlated which should be taken into account when estimating future P/Book ratios.

Figure 28 shows the distribution of daily P/Book changes, that is, the P/Book of one day divided by the P/Book of the previous day, which is a narrow bell-shaped distribution with mean 1.000113 and standard deviation 0.01175446, but it is neither normal nor log-normal distributed (Figure 28 has a log-normal fitting). Also shown in that figure is a scatter-plot of the P/Book versus P/Book change for each day in the period 1984-2011, which exhibits dense clusters around certain P/Book ratios such as 1.8, 2.8 and 4.8, etc. Also note the outlier with a daily change of 0.8, that is, a decrease in the price of the S&P 500 index of about 20% which occurred on October 19, 1987.

Figure 29 shows the distribution of yearly P/Book changes with mean 1.095 and standard deviation 0.17 and which is clearly not log-normal distributed either. Also shown in that figure is the scatter-plot of the P/Book versus yearly P/Book change for each day in the period 1984-2011 which shows small spreads for P/Book ratios below 3 and large spreads for P/Book ratios above 3.

[5] http://research.stlouisfed.org/fred2/series/SP500/

A simple way of modelling the dependency of successive P/Book ratios is to perform a Monte Carlo simulation as outlined in section 5.3, where the daily P/Book change is sampled from the distribution in Figure 28. The distribution of P/Book changes is found by first separating the P/Book ratios into bins similar to the histogram in Figure 25 and for each bin assemble the corresponding P/Book changes. The Monte Carlo simulation starts with a given P/Book ratio and samples from the appropriate bin to get the P/Book change which is then multiplied with the P/Book ratio to get the P/Book ratio for the next day; this is repeated for each day being simulated. This can also be done using the yearly P/Book changes in Figure 29.

Figure 30 shows examples of the P/Book time-series resulting from these Monte Carlo simulations when the starting P/Book ratio is set to 2.3. Compare these to the historical time-series in Figure 24 which indicates that the simulated P/Book time-series are considerably more volatile. Figure 31 shows the autocorrelations for the P/Book time-series resulting from the Monte Carlo simulations. Compare these to the autocorrelation for the historical time-series in Figure 27 which again indicates that the simulated P/Book time-series are more volatile.

Figure 32 shows the histograms for the simulated P/Book ratios in future years. Figure 33 shows the corresponding CDF. Figure 34 shows Q-Q plots comparing the distributions for the simulated P/Book ratios from Figure 32 to the historical P/Book ratios from Figure 25, which indicates that the simulated distribution approaches the historical distribution as the years increase. However, there is a small offset in the simulated distribution because the simulated P/Book ratios cannot reach the lowest historical ratios around 1 which occurred during 1985, see Figure 24. This is because the prior years are not included in the data-set so the simulation can only sample the incremental P/Book changes for that period and not the decremental changes that would lead to the P/Book ratios around 1.

In these Monte Carlo simulations the starting P/Book ratio was set to 2.3. Had the starting P/Book ratio been different then the simulated P/Book ratios would have been affected in the short-term but in the long-term the P/Book distributions are unaffected by the starting P/Book ratio. This is because the simulation model is memory-less, meaning that only the current P/Book is considered when sampling the next P/Book, so the time-series would merely start at another P/Book ratio but otherwise exhibit similar long-term behaviour.

Figure 35 shows the price distribution in future years where each distribution is estimated from one million samples of the P/Book ratios resulting from the above Monte Carlo simulation and the equity resulting from the Monte Carlo simulations described in section 6.8. Compare these to the price distributions in Figure 26 where the P/Book ratio is instead sampled from its historical distribution. In the near future the distributions depend on the starting P/Book ratio for the Monte Carlo simulations, but as the years increase both price distributions approach the log-normal distribution with similar parameters, because the P/Book distribution resulting from Monte Carlo simulations approaches the historical P/Book distribution. Figure 36 shows the CDFs for the price of the S&P 500 index in future years where the near future again depends on the starting P/Book ratio. Figure 37 shows the price mean and standard deviation for different years and starting P/Book ratios, and subsets are shown in Table 4 and Table 5. These assume there is no change in the number of shares outstanding. Share buyback and issuance is taken into account in section 6.10.4.

6.10. Per Share

The future equity, earnings, dividends, etc. were Monte Carlo simulated in the above as total amounts, not per-share. Because a significant part of the annual payouts of the companies in the S&P 500 are in the form of share buybacks net of issuance, there are significant changes in the number of shares outstanding.

The future number of shares can be Monte Carlo simulated by first simulating the future equity as in section 6.8, the future net share buyback as in section 6.6, and the future P/Book ratio as in section 6.9.4, which are then combined using Eq. 2-27 so as to calculate the change in the number of shares.

6.10.1. Sampling Historical Data

Table 1 shows the historical data for share buybacks net of issuance for the S&P 500 companies, which is sampled along with ROE and $Retain/Earnings$ for use in the Monte Carlo simulation.

It is argued in section 6.2.1 that ROE and $Retain/Earnings$ are statistically dependent so they should be sampled from the same historical year. Furthermore, ROE and $Retain/Earnings$ depend on economic reality of the capital structure of companies, the return on equity capital, and the amount of earnings that can be retained for increasing the equity capital and productive assets.

Share buybacks do not depend on economic reality because share buybacks are at the discretion of a company's management. It was shown by Pedersen [**4**] that the companies in the S&P 500 have had a historical tendency to increase share buybacks when the P/Book ratio increased, which is the opposite of what the companies should have done. The reason is perhaps the common but erroneous belief that share buybacks are value neutral to eternal shareholders. It is quite possible that companies may significantly change their future share buybacks in relation to earnings which would make the historical data invalid for use in Monte Carlo simulation. Nevertheless, using the historical data for share buybacks seems superior to a mere guess, so it will be used here.

Another problematic aspect is that the simulated P/Book ratios are assumed to be independent of the simulated earnings. This is an oversimplification because changes in price and hence P/Book ratio may also depend on changes in earnings, but that is not modelled here.

6.10.2. Shares Outstanding

Figure 38 shows the probability distributions for the number of shares outstanding in different future years and with different starting P/Book ratios. The number of shares is normalized and starts at one at the beginning of the first year.

Figure 39 shows the mean and standard deviation for the number of shares outstanding with different starting P/Book ratios and for 300 years of simulations and Figure 40 shows a subset for the first 30 years.

Note that the rate of change seems almost independent of the starting P/Book ratios. This is because the P/Book ratios approach the same distribution after some years of simulation, see section 6.9.4.

6.10.3. Equity, Earnings and Dividends Per Share

Figure 41 shows the mean and standard deviation for the equity per share from Monte Carlo simulating the equity as described in section 6.8 and dividing by the simulated number of shares as described above. Similarly, Figure 42 shows it for the earnings per share and Figure 43 shows it for the dividends per share.

The growth rates for these means are almost identical at:

$$\frac{E[Equity\ Per\ Share_{t+1}]}{E[Equity\ Per\ Share_t]} - 1 \simeq 5.6\%$$

Eq. 6-39

Compare this to the growth rate of about 4.45% for the mean equity (not per share) in Eq. 6-35 so share buyback and issuance cause a growth difference of more than 1% (percentage points) per share, per year.

Mean & Standard Deviation

For the equity per share, the mean and standard deviation in year t are approximately:

$$E[Equity\ Per\ Share_t] \simeq 1.001 \cdot 1.056^t$$

Eq. 6-40

$$Stdev[Equity\ Per\ Share_t] = 0.059 \cdot 1.105^t$$

Eq. 6-41

For the earnings per share, the mean and standard deviation are approximately:

$$E[Earnings\ Per\ Share_t] \simeq 0.13 \cdot 1.056^t$$

Eq. 6-42

$$Stdev[Earnings\ Per\ Share_t] \simeq 0.044 \cdot 1.061^t$$

Eq. 6-43

For the dividend per share, the mean and standard deviation are approximately:

$$E[Dividend\ Per\ Share_t] \simeq 0.06 \cdot 1.056^t$$

Eq. 6-44

$$Stdev[Dividend\ Per\ Share_t] \simeq 0.009 \cdot 1.073^t$$

Eq. 6-45

Note that these curve fittings are independent of the starting P/Book ratio. The approximations are quite accurate except for the standard deviation of the equity per share which is not an exponential curve as demonstrated in Figure 45.

6.10.4. Price Per Share

The price per share is calculated using Eq. 2-30 with the Monte Carlo simulated equity, P/Book ratios and number of shares outstanding. Figure 44 shows the mean and standard deviation for the price per share and subsets are shown in Table 6 and Table 7. Compare these to Figure 37, Table 4 and Table 5 which are not adjusted for the changes in number of shares outstanding through share buybacks and issuance. From this it can be seen that the changes in the number of shares results in increased share prices over time.

The reason is that the amounts used for share buybacks are usually (but not always) greater than the amounts used for share issuances, so the net effect is a reduction in the number of shares over time and hence an increase in the book-value per share. If the future P/Book ratios are sampled from their historical distribution, the price per share increases over time.

Note that Figure 44 uses P/Book ratios that are Monte Carlo simulated from historical *annual* changes, while Figure 37 uses P/Book ratios that are simulated from historical *daily* changes. But the difference is negligible and the difference between these figures is mainly caused by the decreasing number of shares.

6.10.5. Adjusting for Normalized Equity

The Monte Carlo simulations described above were all normalized by setting the starting equity per share to one. The simulation results must therefore be multiplied by the actual starting equity.

For example, let the starting equity per share be USD 666.97 as it was on December 31, 2012 where the share price was USD 1426.19, which gives a P/Book ratio of about 2.14. Figure 45 shows the mean and standard deviation for the equity per share that result from the Monte Carlo simulations. Figure 46 shows it for earnings per share. Figure 47 shows it for dividend per share. Figure 48 shows it for price per share.

6.11. Value Yield, Eternal Shareholder

The value yield of the S&P 500 stock market index is the annual rate of return an investor would obtain from buying shares at the given price and owning the shares for eternity. The value yield can be used as the discount rate when calculating the present value of another investment so as to value that investment relative to the return that could be obtained from investing in the S&P 500 index. The actual value yield of the S&P 500 index is not known in advance because it depends on future dividends per share of the constituent companies of the S&P 500 index which are not known in advance. The probability distribution of the value yield is therefore estimated using the Monte Carlo simulation from section 5.2.

A Monte Carlo run consists of 1,000 simulations with the number of iterations (years) per simulation set to 300 as calculated from Eq. 2-50 for the S&P 500 data-set.

It is first assumed that all future earnings that are not retained are paid out as dividends so there is no change in the number of shares outstanding through share buyback and issuance. The value yield for this case is defined in Eq. 2-33. Then in section 6.11.4 below, future share buybacks and issuances are also Monte Carlo simulated and the value yields compared.

6.11.1. Log-Normal Parameters

Figure 49 shows the value yield histograms resulting from Monte Carlo runs with different P/Book ratios. Also shown are the fitted log-normal distributions. Figure 50 shows the Q-Q log-normal plots which appear to be approximately linear except for a few outliers that seem to be caused by error in the numerical optimization of the value yield and are therefore removed in subsequent experiments. The value yields appear to be approximately log-normal distributed.

Assume $Value\ Yield \sim \log \mathcal{N}(\mu, \sigma^2)$ is a log-normal distributed stochastic variable with parameters μ and σ. Figure 52 shows the parameters μ and σ resulting from a series of Monte Carlo runs when P/Book ranges between 0.2 and 10. Also shown are the fitted functions:

$$\mu = \frac{1.507}{(P/Book)^{0.411}} - 3.537$$

Eq. 6-46

$$\sigma = \frac{0.046}{(P/Book)^{0.432}} + 0.027$$

Eq. 6-47

For example, the P/Book ratio is about 2.3 at the time of this writing in late January 2013, which gives:

$$\mu = \frac{1.507}{2.3^{0.411}} - 3.537 \simeq -2.47$$

$$\sigma = \frac{0.046}{2.3^{0.432}} + 0.027 \simeq 0.06$$

So an investor who buys shares in the S&P 500 stock market index at a P/Book ratio of 2.3 can expect the value yield to be log-normal distributed with parameters $\mu \simeq -2.47$ and $\sigma \simeq 0.06$.

6.11.2. Mean & Standard Deviation

Figure 51 shows the Q-Q normal plots (not log-normal) which appear to be approximately linear except for a few outliers that again seem to be caused by error in the numerical optimization of the value yield and are therefore removed in subsequent experiments. The value yields appear to be approximately normal distributed.

Figure 53 shows the value yield mean and standard deviation resulting from a series of Monte Carlo runs when P/Book ranges between 0.2 and 10. Also shown are the fitted functions:

$$E[Value\ Yield] = \frac{9.1\%}{P/Book} + 4.4\%$$

Eq. 6-48

$$Stdev[Value\ Yield] = \frac{0.7\%}{(P/Book)^{1.34}} + 0.2\%$$

Eq. 6-49

For example, when $P/Book = 2.3$ the value yield mean and standard deviation is:

$$E[Value\ Yield] = \frac{9.1\%}{2.3} + 4.4\% \simeq 8.4\%$$

$$Stdev[Value\ Yield] = \frac{0.7\%}{2.3^{1.34}} + 0.2\% \simeq 0.4\%$$

So an investor who buys shares in the S&P 500 stock market index at a P/Book ratio of 2.3 can expect a value yield with mean about 8.4% and standard deviation about 0.4%.

Table 8 shows the minimum and maximum $E[Value\ Yield]$ and $Stdev[Value\ Yield]$ for each year in the period 1984-2011, calculated using Eq. 6-48 and Eq. 6-49 with each year's minimum and maximum P/Book ratios from Figure 24. During the period 1984-2011 the average $E[Value\ Yield]$ was about 8.3% and the average $Stdev[Value\ Yield]$ was about 0.4%.

6.11.3. Comparison to Analytic Mean

In the Monte Carlo simulations above, ROE and $Retain/Earnings$ were sampled in pairs from the same year because they are believed to be dependent in that manner. Using the ROE and $Retain/Earnings$ data pairs from Table 1 with Eq. 3-28 gives the mean value yield:

$$E[Value\ Yield] = \frac{E\left[ROE \cdot \left(1 - \frac{Retain}{Earnings}\right)\right]}{P/Book} + E\left[ROE \cdot \frac{Retain}{Earnings}\right] \simeq \frac{9.1\%}{P/Book} + 4.5\%$$

Eq. 6-50

This is slightly higher than the mean value yield in Eq. 6-48 derived from the Monte Carlo simulations. The difference is likely due to sampling and rounding error.

6.11.4. With Share Buyback & Issuance

Share buybacks and issuances were not included in the above Monte Carlo simulations of the value yield where the payout was assumed to consist entirely of dividends. Share buybacks and issuances can be simulated as in section 6.10 which results in probability distributions for dividends per share with the number of shares changing over time due to the share buyback and issuance. The value yield is then found for the dividends per share in Eq. 2-35, which results in the probability distributions shown in Figure 54. The Q-Q log-normal and normal plots are shown in Figure 55 and Figure 56, both of which are approximately linear and hence suggest the value yield can be approximated by both normal and log-normal distributions.

The difference between the two value yields, without and with share buybacks, is shown in Figure 57. The value yield without share buybacks is usually greater than the value yield with share buybacks, when the starting P/Book of the S&P 500 is below 2.5, which is around its historical average according to Table 3, and vice versa when the P/Book ratio is above 2.5. The explanation seems to be that the P/Book ratios in the distant future approach the historical distribution in these Monte Carlo simulations, so it is mainly the starting P/Book ratio that causes the difference in value yield when share buybacks are simulated.

Because the difference in value yield is usually negligible from simulating share buybacks, and because the future share buyback policies may change greatly, it is suggested that the value yield without share buybacks is used.

6.12. Value Yield, Temporary Shareholder

Consider the scenario where shares of the S&P 500 index are bought and held for n years after which they are sold. The shareholder will receive payouts during the holding period and will receive the proceeds from the sale of the shares. Let $MarketCap$ denote the current market-cap and let $MarketCap_n$ be the market-cap after n years where the shares are sold. Let $Payout_t$ be the payout in year t which is first assumed to consist entirely of dividends so there is no change in the number of shares outstanding through share buyback and issuance. The value yield is the discount rate that makes the present value of the payouts and final selling price equal to the current market-cap, see Eq. 2-34:

$$MarketCap = \sum_{t=1}^{n} \frac{Payout_t}{(1 + Value\ Yield)^t} + \frac{MarketCap_n}{(1 + Value\ Yield)^n}$$

Eq. 6-51

This is dominated by the present value of the payouts when the holding period n is large and it is dominated by the present value of the selling price when the holding period n is small.

The price after n years can be calculated from the P/Book ratio multiplied by the equity at that time:

$$MarketCap_n = (P/Book)_n \cdot Equity_n$$

Eq. 6-52

Note that in these calculations, the prices and payouts are all divided by the starting equity $Equity_0$ for normalization purposes. This means the starting price is actually the starting P/Book ratio instead. Also note that taxes for capital gains and dividends are ignored because they differ amongst shareholders.

It is first assumed that all future earnings that are not retained are paid out as dividends so there is no change in the number of shares outstanding through share buyback and issuance. Then in section 6.12.3 below, future share buybacks and issuances are also Monte Carlo simulated and the value yields compared.

6.12.1. Historical P/Book

Figure 58 shows the value yield distributions for different choices of starting P/Book ratios and years n. The shares are assumed to be sold after n years and the selling price is calculated by sampling the historical P/Book ratios of the S&P 500 index from Figure 24 and multiplying the sampled P/Book ratio with the equity for the n'th year resulting from the Monte Carlo simulations described in section 5.1.

As can be seen from the plots in Figure 58, when the starting P/Book is 1 and the shares are sold after two years then the mean value yield is more than 70%. That is, the mean annualized rate of return a shareholder would get from buying S&P 500 shares at a P/Book ratio of 1 and selling the shares after two years, is more than 0.7 (or 70%). After 5 years the mean value yield is more than 30% and after 10 years it is more than 20%. If the starting P/Book ratio had instead been 3 then the mean value yield for a 2 year holding period would be around zero. For a 5 year holding period the mean value yield would be about 4% and for a 10 year holding period it would be about 6%. If instead the starting P/Book ratio had been 5 then the mean value yield for a 2 year holding period would be negative by almost -25%. After 5 years the mean value yield would be about -6% and after 10 years it would be about zero. This shows that the mean value yield depends on the starting P/Book ratio and the duration of the holding period.

Regardless of the starting P/Book ratio, the standard deviation for the value yield is high if the shares are sold after only a few years, which is because the price of the S&P 500 index is more volatile than the annual payouts and the selling price dominates the value yield as calculated in Eq. 6-51 when the holding period is short. The standard deviation for the value yield decreases as the holding period increases.

6.12.2. Simulated P/Book

As shown in section 6.9.4, successive P/Book ratios of the S&P 500 index are strongly correlated. Figure 59 shows the value yield distributions when the selling price is calculated using P/Book ratios from the Monte Carlo simulations in section 6.9.4. As the holding period increases, the value yield distributions approach those from Figure 58 which used random samples of the historical P/Book distributions. This is because the simulated P/Book distribution approaches the historical distribution as the number of years increase. For short holding periods the value yield distributions are quite different. Figure 59 shows for a starting P/Book ratio of 1 and a holding period of 2 years that the mean value yield is about 40% compared to more than 70% in Figure 58. For a starting P/Book ratio of 3 and a 2 year holding period the mean value yield is about 4% compared to zero in Figure 58. For a starting P/Book ratio of 5 and a 2 year holding period the mean value yield is about -3% compared to -25% in Figure 58.

Although the value yield distributions in Figure 59 are obviously not normal, it is useful to plot the mean and standard deviations for varying P/Book ratios and holding periods. This is done in Figure 60 and subsets of the numbers are shown in Table 9 and Table 10. The standard deviation for the value yield has a clear tendency to decrease as the number of years increase and becomes almost independent of the starting P/Book ratio. The mean value yield is more complicated. If the starting P/Book ratio is lower than about 3 then the mean value yield decreases as the number of years increase. But if the starting P/Book ratio is around 3 or higher than 4 then the mean value yield increases as the number of years increase. Although there are also exceptions to this, for example as shown in Table 9, if the starting P/Book ratio is 4 then the mean value yield will decrease until the holding period is about 4 years after which it increases.

6.12.3. With Share Buyback & Issuance

Share buybacks and issuances were not included in the above Monte Carlo simulations of the value yield where the payout was assumed to consist entirely of dividends. Share buybacks and issuances can be simulated as in section 6.10 which results in probability distributions for dividends and selling price per share where the number of shares changes over time due to the share buyback and issuance. The value yield is then found from the dividends per share and selling price per share as defined in Eq. 2-36 which is reprinted here for convenience:

$$SharePrice = \sum_{t=1}^{n} \frac{Dividend_t / Shares_t}{(1 + Value\ Yield)^t} + \frac{SharePrice_n}{(1 + Value\ Yield)^n}$$

Figure 61 shows the mean and standard deviation for the value yields and Table 11 and Table 12 show subsets. These show similar tendencies as Figure 60 for the value yields without share buyback and issuance, but the value yields are not identical.

Figure 62 shows the mean and standard deviation for the difference between the two value yields; without and with share buybacks. On average, the share buybacks decrease the value yield when the starting P/Book is low, which shows as a positive value yield difference in Figure 62, and conversely the share buybacks increase the value yield when the starting P/Book is high, which shows as a negative value yield difference in Figure 62. This may seem counter-intuitive but the reason is that a shareholder will get a lower rate of return if future share buybacks are made at higher P/Book ratios than the shareholder paid, and conversely the shareholder will get a higher rate of return if future share buybacks are made at lower P/Book ratios than the shareholder paid. In these Monte Carlo simulations, the P/Book gradually approaches the historical distribution which has an average P/Book around 2.5, see Table 3, so the value yield is least affected from share buybacks and issuance when the starting P/Book ratio is around this average of 2.5, although Figure 62 shows that neither the mean nor standard deviation are exactly zero.

The value yield difference is negligible for starting P/Book ratios above 2 and for less than 30 simulation years, so the value yield without share buybacks can be used. But if the P/Book ratio is lower than 2 or the number of simulation years is very large then share buybacks should be taken into account.

6.12.4. Theoretical Implications

Share buybacks are commonly believed by scholars and practitioners to be a perfect substitute for dividends. This belief is disproven by these experiments because share buybacks and issuances were shown to affect the value yield. See Pedersen [**4**] for more on share buyback valuation.

6.13. Option Valuation

Option valuation for the S&P 500 stock market index is done by considering the options for the SPY Exchange Traded Fund (ETF) which mimics the S&P 500 index. The share price of the SPY ETF is 1/10 of the S&P 500 index and its financial data such as equity per share is therefore also 1/10 of that of the S&P 500 index. Figure 63 shows the so-called option chains for SPY call and put options expiring on December 18, 2015 as quoted on March 15, 2013. These are the options with the farthest expiration date that are publicly available at the time of this writing. For simplicity, transaction costs are ignored in this study and the option prices are the averages of the bid and ask quotes. In practice, an option buyer should use the asking prices in valuation and an option seller should use the bid prices. The difference between bid and ask prices (or the spread) is sometimes large. The options will be valued on their expiration date only (also known as European options) but the valuation method can be extended to cover the entire life of the option (known as American options), or an average of the last period of the option's life (known as Asian options), etc.

6.13.1. Simulated Share Prices

Figure 64 shows the probability distribution of SPY share prices that result from Monte Carlo simulations similar to those described in section 6.10.4 for the S&P 500 index, which take share buyback and issuance into account. The simulation period starts March 15, 2013 and ends December 18, 2015 which is the expiration date of the options being considered. This period has an estimated 696 trading days (about 252 trading days per calendar year and about 2.76 years in the period considered). The last known equity for the S&P 500 index was USD 661.93 per share on September 28, 2012. According to Eq. 6-39, the mean annual growth rate for the equity per share is 5.6% so the equity per share on March 15, 2013, which is about 6 months after the latest known equity, is expected to grow about 2.8% to become USD 680.46 per share, or USD 68.05 per share of the SPY ETF. The closing share price for SPY on March 14, 2013 was USD 156.32 so the P/Book ratio was about 2.3. This is the starting P/Book ratio for the Monte Carlo simulations of future P/Book ratios. The equity is simulated from its last known date on September 28, 2012 until September 28, 2015 and there is then an additional 81 days until the expiration date of the options on December 18, 2015. The simulated equity on September 28, 2015 is therefore multiplied by the expected growth rate for the remaining 81 days, which is about 1.2%.

6.13.2. Profit

Figure 65 shows the CDF for the profits on the SPY call options with different exercise prices, and Figure 66 shows the same for the put options. The profits are calculated for the option expiration date only, using Eq.

4-3 with the share prices from the Monte Carlo simulations described above and the option prices from Figure 63. Figure 67 shows the profit means and standard deviations for the SPY call options with various exercise prices, and Figure 68 shows the same for the put options. For the call options, the maximum profit mean occurs when the exercise price is at or below USD 100 per share where the profit mean is over USD 26 per share and the standard deviation is about USD 38 per share. The minimum profit mean is less than USD 1 per share which occurs when the exercise price is USD 250 per share. For the put options, the profit means are all negative which means the put options are all expected to result in a loss. The lowest mean loss on the put options occurs when the exercise price is USD 10 per share where the profit mean is just below zero and the standard deviation is zero, this is because the option price is almost zero, see Figure 63. The biggest mean loss on the put options is almost USD -32 per share which occurs when the exercise price is USD 250 per share.

6.13.3. Profit Probability

Figure 69 shows the probability of the profit being positive, that is, $\Pr[Profit > 0]$ for the call and put options. For example, for the SPY call option with an exercise price of USD 150 per share the probability of (positive) profit is about 0.59 (or 59%). For a call option with an exercise price at or below USD 110 the probability of profit is almost 0.72 (or 72%). For a call option with an exercise price of USD 250 the probability of profit is about 0.04 (or 4%). For a put option with an exercise price of USD 150 the probability of profit is about 0.068 (or 6.8%). For a put option with an exercise price at or below USD 100 the probability of profit is zero because the share prices in the Monte Carlo simulations were never less than USD 100, see Figure 64. For a put option with an exercise price of USD 250 the probability of profit is about 0.21 (or 21%).

6.13.4. Value Yield

Figure 70 shows the CDFs for the value yields of SPY call options for different exercise prices, and Figure 71 shows the same for the put options. The value yield is the annual rate of return that would be obtained from buying an option at the given price and holding it until the expiration date at which point it is exercised if it has positive value, otherwise it expires worthless. The value yield is calculated for the expiration date only, using Eq. 4-6 with the share prices resulting from the Monte Carlo simulations described above. In many of these simulations the call options become worthless because the simulated share prices are below the exercise price and the option price is then a loss to the buyer of the call option. This makes the value yield calculated from Eq. 4-6 equal to -1. A put option also has value yield -1 if it expires worthless, which happens when the share price is higher than the exercise price.

Figure 72 shows the value yield means and standard deviations for the call options and Figure 73 shows them for the put options. For the call options, the maximum value yield mean occurs when the exercise price is around USD 115 per share where the value yield mean is about 12.6% and the standard deviation is about 28%. For the put options, all the value yield means are negative because the put options are expected to cause a loss on average. The maximum value yield mean for the put options occurs when the exercise price is USD 250 per share where the value yield mean is about -18% and the standard deviation is around 24%.

6.13.5. Comparison of Puts & Calls

According to Figure 69 the call options with exercise prices at or below USD 165 have probabilities of profit above 0.5 (or 50%), while the put options all have profit probabilities below 0.21 (or 21%) and most put options have much lower or even zero profit probability.

Figure 67 shows that the mean profit is always positive for the call options and Figure 68 shows that the mean profit is always negative for the put options. The reason is that the Monte Carlo simulated share prices used in these option valuations, as shown in Figure 64, are often higher than the combined exercise and option prices.

If the equity growth model used in these Monte Carlo simulations is deemed a reasonable way of estimating the probability distribution of the future equity, earnings and payout for the S&P 500 index (and hence the SPY ETF), and if the simulation of future P/Book ratios from their historical probability distribution is also deemed reasonable, then the options must be mispriced: The call options must be underpriced and the put options must be overpriced.

6.13.6. Trading Strategies

An option buyer would want to maximize both the profit probability and mean profit while simultaneously minimizing the option price. An option seller would want to do the opposite. Both buyers and sellers would want to minimize the standard deviation of profits. This is known as a multi-objective optimization problem because several objectives must be optimized simultaneously.

Figure 74 shows the call option price versus profit mean and probability which both increase as the option price increases until the option price is slightly over USD 50, which corresponds to an exercise price of about USD 105, see Figure 63. When the option price is above USD 50 the profit mean and probability do not change (actually, the mean profit decreases slightly), so the call option would be more expensive but it would not lead to a higher mean profit or profit probability. An investor would want to invest as little capital as possible for the same profit mean and probability so the call options priced above USD 50 should not be bought. Figure 72 is an alternative way of reaching this conclusion by considering the mean value yield (annualized rate of return) that could be obtained from buying the different call options at their respective prices. An investor would want the highest rate of return on the capital used to purchase the call option and Figure 72 shows that the maximum value yield mean for a call option is when the exercise price is USD 110 where the value yield mean is about 12.6% and the standard deviation about 28%. This may be compared to the value yield estimates for buying the underlying stock as shown in Figure 60 for the S&P 500 index (which the SPY shares mimic) with various starting P/Book ratios and holding periods. In the option valuations here, the starting P/Book ratio is about 2.3 and the period before the options expire is 2.76 years. Interpolating the nearby values from Table 9 and Table 10 gives for the underlying stock a value yield mean about 10% and standard deviation about 8%. The value yield mean is slightly higher for the call option and the standard deviation is much greater with the stock having about 8% and the call option about 28%, which means the value yield of the call option may be significantly larger or smaller than the value yield of the stock. This is because an option is effectively a gearing of the stock.

Figure 75 shows put option price versus profit mean and probability. In this example the profit means are all negative and the profit probability is below 0.21 (or 21%) for all the put options. This means a loss is expected from buying these put options and they will give a corresponding profit to the option seller, as shown in section 4.1. A seller of put options would want to maximize the option price while minimizing the buyer's profit mean and probability. Figure 75 shows that the put option buyer's mean profit is inversely related to the profit probability, that is, the mean profit decreases when the profit probability increases. This makes it impossible to simultaneously minimize both the mean profit and probability. The seller of put options must therefore decide whether to sell a put option at a high price and with a comparatively high probability of loss, or sell a put option at a low price and with a comparatively small probability of loss. The profit probability for the put seller is almost 100% when the option price is less than about USD 5 which is when the exercise price is less than USD 105, see Figure 63. So it is better to sell the put option with exercise price USD 105 than the options with lower exercise prices because the seller will receive more in sales proceeds with the same profit probability. Because the profit probability and mean are inversely related, the choice of which put option to sell will depend on the individual seller and what the sales proceeds will be used for. If the seller cannot reinvest the sales proceeds elsewhere, then the seller may want to minimize the probability of loss upon exercising of the options and should therefore sell the options with exercise price USD 105 and option price USD 5 which have almost zero probability of a loss to the option seller. But if the seller can invest the sales proceeds elsewhere at a sufficiently high rate of return and the put option can only be exercised on its expiration date, then the put option with an exercise price of USD 250 will give the option seller USD 100 in sales proceeds to reinvest elsewhere. The probability of a loss to the option seller is about 21% upon exercising of such a put option, see Figure 75. But this probability as well as the magnitude of any loss both decrease if the sales proceeds are reinvested elsewhere at a sufficiently high rate of return until the put option is exercised.

7. USA Government Bonds

Figure 76 shows the historical yield on USA government bonds for the periods 1798-2012 (average annual yields) and 1962-2013 (average daily yields). During this period the bonds have had varying maturity period, terms and taxation. Table 13 shows basic statistics and a mean yield about 4.5-5% with standard deviation about 2%. Figure 77 shows the histograms of these historical bond yields and Figure 78 shows the autocorrelation plots with significant correlation between successive bond yields.

7.1. Equity Risk Premium

Section 6.1.2 discussed the so-called Equity Risk Premium (ERP), which is a simple way of forecasting future stock market returns from the historical average premium of stock market returns relative to low-risk government bond yields. There are different ways of calculating the ERP, a few of which are now explored.

7.1.1. Historical Data

The following compares the current yield of USA government bonds to returns of the S&P 500 stock-market index for monthly, annually and 10-year periods. Changes in bond-prices and dividends for the S&P 500 are ignored. The current yield on a bond is the annual rate of return a bond-buyer would receive if the bond is held until maturity, which may exceed the period used for comparison to the S&P 500. So the comparison is somewhat inaccurate. But if there are strong correlations between the current yield on USA government bonds and returns of the S&P 500 it should still show from this comparison.

Figure 79 shows for the period 1957-2013 the ERP calculated as the difference between the monthly returns of the S&P 500 stock-market index and the monthly yield on USA government bonds with 10 year maturity periods, which have been converted from annual to monthly yields using the formula:

$$Monthly\ Yield = (Annual\ Yield)^{1/12}$$

Eq. 7-1

Figure 79 also shows the histogram which is not normal distributed. Figure 80 shows the scatter-plot and cross-correlation for this data which show no significant correlations. This means there is no simple, linear relation between the monthly returns of the S&P 500 and the current yield on USA government bonds.

Figure 81 shows the ERP between the annual returns of the S&P 500 stock-market index and the current yield on USA government bonds. Figure 82 shows the same but for 10-year returns of the S&P 500. Figure 83 shows the scatter-plots for this data, where the annual returns of the S&P 500 are uncorrelated to the bond yields, while there is some correlation for the 10-year returns of the S&P 500 which has coefficient of determination R^2 about 0.41 and the scatter-plot shows the relation to be complex and non-linear.

All this corroborates the argument in section 6.1.2 that future stock market returns cannot be reliably estimated from merely adding the historical ERP to the current yield on long-term government bonds.

7.1.2. Value Yield Estimate

Figure 84 shows the historical government bond yields for the period 1984-2011 from Figure 76 along with the mean value yield for the S&P 500 stock market index as estimated from Eq. 6-48 using the P/Book data from Figure 24. Their difference is an estimate of the ERP for eternal shareholders which is also shown in the plot and ranges from about -2% around year 1985 to almost +8% in years 2009 and 2011. The arithmetic mean ERP is about 2.1%, geometric mean slightly less, harmonic mean about 4.8%, and standard deviation about 1.8%. The ERP is not normal distributed as shown in the histogram in Figure 85. Figure 86 shows the scatter-plot and cross-correlation of the S&P 500 mean value yields and the USA government bond yields during the period 1984-2011, which shows significant cross-correlation but not in a simple pattern that is predictable from this data alone.

Recall that section 6.12 showed that the value yield estimate of the S&P 500 index is affected by its current P/Book ratio and the holding period, so the estimated ERP is also affected by this and hence cannot be a single number.

For example, at the time of this writing in March 2013 the yield on USA government bonds with 10 year maturity is 1.94%. If the historical mean ERP of about 2.1% is merely added to the bond yield then it would suggest a mean value yield for the S&P 500 index about 4%, while in fact the P/Book ratio for the S&P 500 index is about 2.3 thus giving a mean value yield around 8.4% according to Eq. 6-48, which gives an ERP of 6.5%. There is currently much speculation in the news-media from even the most experienced and respected people in finance that if the interest rate increases then stock prices would decrease. But the current government bond yields around 2% are historically low according to Table 13 and the S&P 500 P/Book ratio and hence its mean value yield is also lower than its historical average. If both of these are to revert to their historical averages in the future then it would mean that the government bond yields should increase (bond prices decrease) and stock prices should increase (value yields decrease). But as shown with the many statistical studies in this paper it is not certain that this will happen.

8. Dow Jones Venture Capital Index

The Dow Jones Venture Capital (DJVC) / Sand Hill index shown in Figure 87 covers approximately 18,000 venture companies during the period 1991-2010. This is a so-called total return index which measures the return, before taxes and fees for the venture capital funds through which investments are typically made, from investing in all the venture companies of the index in proportion to their market-cap, and re-investing all payouts from public offerings and acquisitions into the remaining venture companies.

8.1. Monthly Returns

The monthly return of the DJVC index is calculated as:

$$Return_t = \frac{Index_t}{Index_{t-1}} - 1$$

Eq. 8-1

The monthly returns are shown in Figure 87, from which it can be seen that the volatility of the DJVC index increased during the years 1999-2003 which is known as the Dot-Com bubble where the market price increased greatly and then collapsed for both venture and established public companies. The index was also more volatile during a severe financial crisis in the years 2008-2009.

Figure 88 shows a histogram of the monthly returns which have arithmetic mean 1.65%, geometric mean 1.24% and a comparatively high standard deviation of 9%. The compounded annual return has geometric mean about 16%. Also shown in Figure 88 is the autocorrelation of monthly returns which shows no significant correlation between successive monthly returns.

8.2. Risk Premium

The monthly returns of the DJVC index are highly volatile and future returns of the index appear to be unpredictable from its prior returns. It would therefore be desirable to forecast future returns from the historical risk premium on another investment whose future return is known.

8.2.1. USA Government Bonds

Figure 89 shows the scatter plot of monthly returns of the DJVC index and the yield on USA government bonds with 10 year maturity, which have been converted from annual to monthly yields using Eq. 7-1. This shows the bond yields are uncorrelated with the monthly return experienced on the DJVC index.

Figure 89 also shows the cross-correlation between the DJVC index and government bond yields, that is, the correlation of the two time-series with varying time-lags. Although there are some statistically significant correlations, the correlation coefficient is at most 0.2 which is not enough to suggest a clear predictive relationship between the two time-series.

Figure 90 shows the scatter plots for one-year and ten-year returns of the DJVC index and USA government bond yields, which also do not have any statistically significant correlations.

This means the historical difference in returns of the DJVC index and USA government bonds cannot be used as a predictable risk premium in forecasting future returns of the DJVC index from the current yield on USA government bonds.

8.2.2. S&P 500

Figure 91 shows the scatter plot of monthly returns of the DJVC venture index and the S&P 500 stock-market index. Note that DJVC is a total return index while the S&P 500 index data used here is not because the monthly total return data for the S&P 500 index was not publicly available at the time of this writing. However, the scatter plot still shows strong correlation with the linear fit having coefficient of determination $R^2 = 0.75$ and the relation:

$$DJCV\ Monthly\ Return = S\&P\ 500\ Monthly\ Return \cdot 1.84 + 0.006 + \epsilon$$

Eq. 8-2

Where ϵ is the error term with standard deviation 0.045. The residuals are also shown in Figure 91 and are clearly not normal-distributed so this linear relation between DJVC and S&P 500 is only an approximation.

Figure 92 shows the QQ-plot of the monthly returns of the DJVC and S&P 500 which also show a non-linear relation. Also shown in that figure is the cross-correlation which only shows strong correlation between the monthly returns of the DJVC and S&P 500 indices when the time-lag is zero.

Figure 93 shows the highly volatile difference between the monthly changes of the DJVC and S&P 500 indices with arithmetic mean about 1.1% (percentage points) and standard deviation about 5.7% (percentage points). The difference is greatest in the period 1999-2002 which was the time of a large stock-market bubble that affected the DJVC relatively more than the S&P 500.

8.3. Simulating Future Returns

Figure 94 shows Monte Carlo simulation of the DJVC index normalized to begin at 1. First the future share prices of the S&P 500 are simulated as described in section 6.10.4, then for each month the return of the DJVC is calculated using Eq. 8-2 and the monthly returns are then compounded to obtain these plots.

Compare these plots to the historical DJVC index in Figure 87; the simulated plots are much more volatile than the historical plot and in some cases the simulations degenerate and approach zero as time increases. This model for simulating the future DJVC index therefore seems incorrect.

It is unclear what is causing this problem. Perhaps it is the simulations of S&P 500 returns that are too volatile. Figure 95 supports this notion as it shows the linear model from Eq. 8-2 applied to the historical S&P 500 returns which results in simulated DJVC returns that are somewhat similar to the historical DJVC returns in Figure 87; although the simulated returns are still not quite accurate, e.g. the peak around year 2000 should be higher than the peak around year 2007 if the historical tendencies were to be matched.

8.3.1. Sampling Historical Returns

A simpler method of simulating the future DJVC returns is to sample and compound historical monthly returns. As shown in Figure 88, consecutive monthly returns of the DJVC index appear to be uncorrelated so the historical data can be sampled without modelling time-dependence. Figure 96 shows examples of such sampling and compounding which are considerably more stable than the simulations in Figure 94.

However, there is no theoretical argument as to why the future returns of the DJVC index would merely be randomly sampled from the historical returns, so these simulations should be used with skepticism.

8.4. Discount Rate

Valuing an investment relative to the return that could be obtained from the DJVC index is done by simulating future returns of the DJVC index as described above and using the compounded return as the discount rate in the present value calculation. An example of this is given in section 9.6.

9. Coca-Cola

The company *Coca-Cola* (abbreviated as *Coke*) was incorporated in 1919 in USA but its origin is significantly older. The company produces beverages that are sold worldwide. In 2012 the company employed 150,000 people and had revenue of about USD 48b with net income of about USD 9b.[6] At the time of this writing in late March 2013, Coke's market-cap is about USD 180b and the latest known equity was almost USD 33b in December 2012, which means the P/Book ratio is currently about 5.5.

9.1. Simulation Settings

The company's future equity, earnings and payouts are Monte Carlo simulated using the equity growth model from section 2.2 which was also used to simulate the S&P 500 index in section 6. Although the same aspects could be studied here as was studied in section 6, this section focuses on the value yield of the Coca-Cola company and its relation to the value yield of the S&P 500 index, as well as the present value.

The ROE and $Retain/Earnings$ data from Table 14 is used in the Monte Carlo simulation described in section 5.2. Figure 97 shows that there is no significant correlation between the ROE and $Retain/Earnings$ data so they are sampled in pairs from the same year. The Monte Carlo simulation is repeated 1,000 times and each simulation is for 320 years, which is calculated using Eq. 2-50 so as to achieve sufficient numerical accuracy.

9.2. Value Yield

The value yield is the annualized rate of return a shareholder would get from buying Coke shares at the given price and holding the shares for eternity. Although the payouts for the first year are only expected to be about 5.3% of the market-cap, the retained earnings will cause future payouts to increase exponentially. For example, after 10 years the mean payout is 16% of the initial share price, after 50 years the mean payout is 22 times (2,200%) the initial share price, and after 100 years the mean payout is 58k (5,800,000%) times the initial share price. The exponentially growing payouts cause the value yield to be much greater than the first year's rate of return of about 5.3%. Figure 98 shows the probability distribution of the value yield which has mean 18.3% and standard deviation 1.5%. However, this rate of return is extremely high and unrealistic as shown next.

[6] Form 10-K annual report for 2012 filed with US SEC: www.sec.gov/Archives/edgar/data/21344/000002134413000007/a2012123110-k.htm

9.3. Equity Growth

On September 28, 2012, the equity of the S&P 500 stock market index was about USD 5,918b[7] and the equity of Coke was about USD 33b.[8] The mean equity growth rate of the S&P 500 index for the period 1984-2011 was 4.46% as calculated in Eq. 6-1. The mean equity growth rate for Coke is calculated using Eq. 2-18 with the data from Table 14 for the period 1993-2012:

$$E[G] = E[ROE \cdot Retain] \simeq 13.2\%$$

Eq. 9-1

If these mean equity growth rates are assumed to continue in the future, then the equity of Coke is expected to exceed the equity of the S&P 500 index after n years, which is calculated using Eq. 2-23:

$$104.46\%^n \cdot USD\ 5{,}918b < 113.2\%^n \cdot USD\ 33b \Leftrightarrow n > \frac{\log\left(\frac{USD\ 33b}{USD\ 5{,}918b}\right)}{\log\left(\frac{1 + 4.46\%}{1 + 13.2\%}\right)} \simeq 64.6$$

That is, after 65 years the equity of Coke is expected to exceed the equity of the S&P 500 index under these assumptions. This calculation ignored the fact that Coke is itself a part of the S&P 500 index which means the two cannot have different growth rates forever. If Coke grows to dominate the S&P 500 index then either the growth rate of Coke will decrease or the growth rate of the S&P 500 index will increase. However, it seems unrealistic that a beverage producer will grow to dominate the stock markets in USA so the growth assumptions must be wrong.

9.4. Decreasing ROE & Retain Ratios

In modelling the future growth of a company's equity, earnings and payouts, a number of factors can be taken into account, such as the expected change in the overall market size for the company's products, the company's expected market share, decreasing return on equity if competition is believed to increase, etc. Such adjustments rely at least partially on qualitative assessments. A simple way of reducing the future growth of Coke is used here for demonstration purposes and the reader is encouraged to experiment with other models and assumptions using the computer source-code and data-files described in section 1.2.

The mean equity growth rate for Coke was 13.2%, see Eq. 9-1, while it was 4.46% for the S&P 500 index, see Eq. 6-1. The future equity growth rate of Coke can be made to converge to that of the S&P 500 index by exponentially decreasing the ROE and $Retain/Earnings$ ratios in the Monte Carlo simulations. The total factor of reduction must converge to $4.46\%/13.2\% = 0.338$ so the reduction in ROE can be set to converge to e.g. 0.67 and the reduction in the $Retain/Earnings$ ratio is then made to converge to 0.5 because $0.67 \cdot 0.5 = 0.335$. The choice of how quickly ROE and $Retain/Earnings$ should be reduced as a function of the number of years t is somewhat arbitrarily set to a factor 0.95 per year for ROE and 0.9 for $Retain/Earnings$.

[7] Equity of the S&P 500 index is estimated as the equity per share (USD 661.93) multiplied by the divisor (8940.96). Data was retrieved early March 2013 from S&P Dow Jones Indices:
http://us.spindices.com/documents/additional-material/sp-500-eps-est.xls

[8] Form 10-Q for Coca-Cola filed with US SEC for the quarter ending September 28, 2012:
www.sec.gov/Archives/edgar/data/21344/000002134412000051/a2012092810q.htm

The Retain scale is:

$$ScaleRetain_t = 0.5 \cdot 0.9^t + 0.5$$

The ROE scale is:

$$ScaleROE_t = 0.33 \cdot 0.95^t + 0.67$$

The Retain scale is 0.95 in year $t = 1$, in year 10 it is about 0.67, in year 50 it has almost converged to 0.5. The ROE scale is 0.9835 in year 1, in year 10 it is about 0.87, in year 50 it is about 0.70, and it converges to 0.67 as t increases. The last ROE scale is also used in the estimation of the terminal value in Eq. 2-54.

Applying these scales to the sampled ROE and $Retain/Earnings$ ratios in the Monte Carlo simulations results in Coke's mean payout for the first year to be about 5.4% of the market-cap, which is about the same as in the non-scaled version in section 9.2 because the scale is almost 1. After 10 years the mean payout is about 12% of the initial share price, after 50 years the mean payout is about 88% of the initial share price, and after 100 years the mean payout is about 743% of the initial share price. Figure 99 shows the probability distribution of the value yield which has mean 12.6% and standard deviation 1.4%.

9.5. Comparison to S&P 500

The value yield of Coke can be compared to the value yield of the S&P 500 stock market index plus some risk premium so as to assess the probability that Coke offers a higher risk-adjusted rate of return. This probability is denoted:

$$Pr[Value\ Yield_{Coke} > Value\ Yield_{S\&P\ 500} + Risk\ Premium]$$

This is equivalent to:

$$Pr[Value\ Yield_{Coke} - Value\ Yield_{S\&P\ 500} - Risk\ Premium > 0]$$

Figure 100 shows the difference $Value\ Yield_{Coke} - Value\ Yield_{S\&P\ 500}$ where Coke's value yields result from the Monte Carlo simulations described in section 9.4 and the value yields for the S&P 500 index result from Monte Carlo simulations similar to those described in section 6.11. Coke is a part of the S&P 500 index so they are dependent in at least this way, but they have been assumed here to be independent for the sake of simplicity. Two plots are made in Figure 100 for different P/Book ratios. The left plot uses the P/Book ratios at the time of this writing in late March 2013 which were about 5.5 for Coke and about 2.3 for the S&P 500 index. The mean value yield difference is then about 4.1% with standard deviation 1.4%. The right plot uses a hypothetical case where Coke is somewhat cheaper with a P/Book ratio of 5 and the S&P 500 index is more expensive with a P/Book ratio of 3.5. The mean value yield difference is then 6.2% with standard deviation 1.5%.

Figure 101 shows the probability of Coke's value yield being greater than that of the S&P 500 index with varying risk premiums and for different P/Book ratios. Note how the curve shifts on the x-axis when the P/Book ratios change. For example, using the P/Book ratios at the time of this writing (left plot) gives a probability of almost 30% of Coke's value yield being greater than that of the S&P 500 index plus a risk premium of 5%. In the hypothetical case (right plot) where Coke is cheaper and S&P 500 is more expensive, the probability is almost 80% of Coke's value yield being greater than that of the S&P 500 index plus a risk premium of 5%.

9.6. Present Value

The present value of Coke can be calculated by Monte Carlo simulating Coke's future payouts for n years, estimating a terminal value, and discounting all these with the compounded rate of return that can be obtained from alternative investments.

Let the discount-factor for year t be denoted D_t. The present value calculation from Eq. 2-8 is then:

$$v = \sum_{t=1}^{n} \frac{Payout_t}{D_t} + Terminal\ Value$$

Eq. 9-2

Coke's payouts are simulated as described in section 9.4. The number of simulation years n is 320 as calculated from the formulas in section 2.7.1 so as to achieve sufficient numerical precision.

9.6.1. Present Value Relative to S&P 500

Coke's future payouts can be valued relative to the return on an investment in the S&P 500 index by using the value yield for the S&P 500 index as the discount rate in the present value calculation. This is done by sampling a discount rate d from the log-normal distribution for S&P 500 value yields described in section 6.11.1 and setting the discount factor in Eq. 9-2 to $D_t = (1 + d)^t$. This present value calculation is repeated several times so as to estimate the probability distribution.

Figure 102 shows the resulting probability distributions for Coke's present value with different choices of P/Book ratios for the S&P 500 index and risk premiums. The CDFs for these probability distributions are shown in Figure 103. Note that these plots have been normalized by dividing the present values with Coke's current equity (denoted PV/Equity or equivalently PV/Book) and should therefore be compared to the company's P/Book ratio which is currently about 5.5.

For example, if the risk premium demanded in excess of the return from the S&P 500 index is 2% (percentage points) and the P/Book ratio for the S&P 500 index is 2.3 as it currently is, then Figure 103 shows that there is a probability of about 0.07 (or 7%) that the present value of Coke is less than its current market-cap to someone who can own Coke shares for eternity. The same plot shows that there is a probability of about 0.05 (or 5%) that the present value of Coke is greater than 11 which is twice the current market price of Coke (11 equals two times its current P/Book ratio of 5.5). This means there is a probability of about 0.88 (or 88%) that the present value of Coke lies between its current market price and twice that price.

Using present value calculations such as these, it is also possible to make a plot showing the probability of the present value being greater than the market-cap for varying risk premiums. Such a plot would be identical to those in Figure 101 that are made with value yields, which by definition is equivalent to comparing the present value to the market-cap.

9.6.2. Present Value Relative to DJVC

Coke's future payouts can also be valued relative to an investment in the DJVC index by using the compounded returns resulting from the Monte Carlo simulations described in section 8.3.1 as the discount factor D_t in Eq. 9-2. As noted in section 8.3.1, there is no theoretical argument that the future DJVC returns are merely random samples from the historical returns, so the following should be interpreted with skepticism and is meant for demonstration purposes. Although it is also possible to add a risk premium, it is not done here.

Compounded DJVC Returns

Figure 104 shows the resulting probability distribution for the present value of Coke when using the simulated DJVC returns as the discount rate. The distribution has a single present value above 800 but the plots have been limited to only show the present values below 50 to improve legibility. As above, the present values are normalized by dividing with Coke's equity and should therefore be compared to the P/Book ratio which is currently 5.5. The CDF plot shows that the probability is more than 0.6 (or 60%) that the present value of Coke is less than its current market-cap, and there is a probability of about 0.15 (or 15%) that the present value is greater than twice its current market-cap.

Annualized DJVC Returns

Now consider the annualized rate of return which is the exponential growth rate that causes the starting index level $Index_0$ to become $Index_n$ in n years and is calculated as:

$$Annualized\ Rate\ of\ Return = \left(\frac{Index_n}{Index_0}\right)^{1/n} - 1$$

Eq. 9-3

The annualized rate of return gives a smooth exponential increase from $Index_0$ to $Index_n$ while the compounded return gives a volatile increase, see Figure 106.

Figure 105 shows the probability distribution for the present value of Coke when the discount rate is the annualized rate of return of the DJVC index. The CDF plot shows that the probability is now about 0.9 (or 90%) that the present value of Coke is less than its current market-cap, and there is about zero probability that the present value is greater than twice its current market-cap.

Comparison of Compounded and Annualized DJVC Returns

Figure 107 shows there is much difference in the probability distributions of Coke's present value when the discount rate is either the compounded or annualized returns of the DJVC index. The reason is that present value calculations are affected by changes in the discount rate between years so that using the volatile compounded return as the discount rate gives a different present value than using the constant annualized return.

Ideally, the compounded return should be used as the discount rate because it more accurately accounts for the timing in the present value calculation, but this also requires for dependencies between Coke's payouts and the DJVC index to be modelled which is not done here. The same is true when using the value yield of the S&P 500 as the discount rate.

9.6.3. Comparison of S&P 500 and DJVC Discount Rates

The discount rates that were sampled from the S&P 500 value yield distribution ranged between 6.9% and 10.6% with arithmetic mean 8.5% and standard deviation 0.5%, which resulted in the present values of Coke (that is, PV/Equity) ranging between 6.1 and 28 with arithmetic mean 12.6 and standard deviation 3.5. No risk premium was used and the P/Book ratio of the S&P 500 index was 2.3. If a risk premium of 2% (percentage points) is used then the mean discount rate is 10.5% and the present value ranges between 3.8 and 18.4 with arithmetic mean 7.9 and standard deviation 1.9.

Using the simulated compounded returns of the DJVC index as the discount rate, resulted in the present values ranging between 0.6 and 809 with arithmetic mean 9.5 and standard deviation 36.9. Over the entire simulation period of 320 years the annualized rate of return of the DJVC index ranged between 6.2% and 21.3% with arithmetic mean 16% and standard deviation 2%. Using this annualized rate of return as the discount rate resulted in a present value between 1.7 and 22.9 with arithmetic mean 3.9 and standard deviation 1.3.

The DJVC returns were highly volatile as shown in Figure 108, which means that some simulations result in the returns of DJVC being much lower than the returns of S&P 500 for many years. In such cases the present values calculated using the compounded returns of DJVC as the discount rate are much greater than if the S&P 500 returns had been used as the discount rate, because the present value is dominated by the payouts that are nearest in time. This also explains why the mean present value of Coke is higher when calculated with the DJVC returns even though the mean compounded return is higher for the DJVC index than for the S&P 500 index.

This shows a severe weakness in using only the arithmetic mean of present values when evaluating investments where the probability distribution should be used instead. For example, although the mean present value of Coke is 7.9 when calculated with the S&P 500 returns as discount rate and a risk premium of 2%, Figure 103 shows that there is a probability of about 0.93 (or 93%) that the present value is greater than the current market-cap. Conversely, the mean present value is 9.5 when calculated relative to the DJVC returns, but Figure 104 shows that there is a probability of less than 0.4 (or 40%) that the present value is greater than the current market-cap.

9.6.4. Jensen's Inequality

Another reason for using the probability distribution in present value calculations is due to Jensen's inequality, which states that if the discount rate is a stochastic variable then the present value may be underestimated by using the mean discount rate in the present value calculation, see section 3.2.

Figure 109 shows the difference in present value when calculated using as discount rate samples of the S&P 500 value yield and its mean, and similarly when using as discount rate the DJVC compounded return and its mean. This shows the present value is significantly affected by using mean discount rates.

The S&P 500 value yield from section 9.6.1 is used here and its mean is about 10.5% including a 2% risk premium. Using this as the discount rate results in Coke's mean present value being about 7.75. Instead using samples of the S&P 500 value yield as the discount rate, gives a mean present value of about 7.81. This confirms Jensen's inequality in Eq. 3-8 that using a mean discount rate underestimates the mean present value. The small difference in present value is due to the comparatively low standard deviation of about 0.5% for the S&P 500 value yield.

The mean annualized rate of return of the DJVC index from section 9.6.2 is about 15.9%. Using this as the discount rate results in Coke's mean present value being about 3.70. Instead using samples of the DJVC annualized rate of return as the discount rate gives a mean present value of about 8.35. This again confirms Jensen's inequality that using a mean discount rate may underestimate the mean present value. The large difference here is due to the comparatively large standard deviation of about 2% for the annualized rate of return of the DJVC index.

Conclusion

10. Conclusion

This paper used a simple equity growth model combined with historical financial data in Monte Carlo simulation of the future equity, earnings and payouts of the S&P 500 stock market index and the Coca-Cola company. This estimated the probability distribution of the equity, earnings and payouts, which was in turn used to estimate the annualized rate of return with different holding periods, as well as stock option values. The distorting effect of using averages in present value calculations was also demonstrated.

The paper also studied the returns from the S&P 500 index in relation to USA government bonds and the Dow Jones Venture Capital (DJVC) index. It was found that there is no consistent and predictable risk premium between the monthly, one-year and 10-year returns of USA government bonds and the S&P 500 and DJVC indices, but there is significant statistical correlation between the monthly returns of the S&P 500 and DJVC indices during the period 1991-2010.

Several assumptions were made in this study and the results should be interpreted with some caution. The computer source-code and data files are provided on the internet so the reader can experiment with other assumptions, see section 1.2.

Appendix

11. Appendix

11.1. Geometric Series

The following is a well-known property of geometric series:

$$\sum_{t=0}^{\infty} x^t = \frac{1}{1-x} \quad \text{for } |x| < 1$$

Starting the summation at $t = 1$ gives:

$$\sum_{t=1}^{\infty} x^t = \sum_{t=0}^{\infty} x^t - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x} \quad \text{for } |x| < 1$$

So for a growth rate g and discount rate d with $g < d$ we have:

$$\sum_{t=0}^{\infty} \left(\frac{1+g}{1+d}\right)^t = \frac{1+d}{d-g}$$

Eq. 11-1

$$\sum_{t=1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = \frac{1+d}{d-g} - 1 = \frac{1+g}{d-g}$$

Eq. 11-2

$$\sum_{t=1}^{\infty} \frac{(1+g)^{t-1}}{(1+d)^t} = \frac{1}{1+g} \cdot \sum_{t=1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = \frac{1}{d-g}$$

Eq. 11-3

Instead starting the summation at $t = k + 1$ gives:

$$\sum_{t=k+1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = \sum_{t=1}^{\infty} \left(\frac{1+g}{1+d}\right)^{t+k} = \left(\frac{1+g}{1+d}\right)^k \cdot \sum_{t=1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = \left(\frac{1+g}{1+d}\right)^k \cdot \frac{1+g}{d-g}$$

Eq. 11-4

Summing from $t = 1$ to k gives:

$$\sum_{t=1}^{k} \left(\frac{1+g}{1+d}\right)^t = \sum_{t=1}^{\infty} \left(\frac{1+g}{1+d}\right)^t - \sum_{t=k+1}^{\infty} \left(\frac{1+g}{1+d}\right)^t = \frac{1+g}{d-g} \cdot \left(1 - \left(\frac{1+g}{1+d}\right)^k\right)$$

Eq. 11-5

11.2. Properties of Log-Normal Distribution

The mean of a log-normal distributed stochastic variable $X \sim \log \mathcal{N}(\mu, \sigma^2)$ is known to be:

$$E[X] = e^{\mu + \frac{\sigma^2}{2}}$$

Eq. 11-6

The standard deviation is known to be:

$$Stdev[X] = E[X] \cdot \sqrt{e^{\sigma^2} - 1}$$

Eq. 11-7

11.3. S&P 500 Data

Data for the S&P 500 stock market index has been collected by the staff at the Customized Research Department of Compustat. [9] The data was then used to calculate the various financial ratios. This appendix details the compromises made in collecting and making calculations on the data.

The S&P 500 stock market index consists of 500 companies that are weighted according to certain changes and events that affect their capitalization. The weights are proprietary and could not be obtained. Instead, the data for the individual companies in the S&P 500 index has merely been aggregated (summed).

In the period 1983-2011 the S&P 500 index consisted mostly of companies reporting their financial statements in USD currency. Of the 500 companies in the index an average of 497 companies used USD currency each year. To avoid distorting effects of non-USD currencies those would either have to be converted into USD or removed from the data. For simplicity and because so few companies reported in non-USD currency they were removed from the data.

The S&P 500 index is being studied here as if it was one big conglomerate. This means the financial data such as assets, equity, earnings, and market-cap are aggregated. This differs from the accounting used in actual conglomerates where the consolidated financial statements would adjust for inter-company, intra-conglomerate dependencies such as revenue and liabilities. Making such consolidated financial statements is a complex process requiring access to financial details of the companies in question which is only available to those companies and their auditors. The sum of financial data is deemed sufficiently accurate for this study.

The Compustat database contains an item named MKVALT, which is the market price for the common stock of a company, or market-cap as it is referred to here. To find the market-cap for the S&P 500 conglomerate, the MKVALT items should be summed for all companies in the S&P 500 index. However, prior to 1998 the MKVALT item does not exist in the Compustat database so the research staff at Compustat had to make a formula for estimating this by taking several factors into account, such as multiple share classes. The formula gave estimates of MKVALT that were reasonably close to the existing values in the Compustat database for the period 1998-2011 so the MKVALT estimates for the period 1983-2011 are used with sufficient confidence of their accuracy, which is deemed to be within the precision required for this study.

[9] The Compustat database was accessed through the facilities of the Collaborative Research Center 649 on Economic Risk at the Humboldt University of Berlin, Germany.

The equity per share of the S&P 500 index is calculated for the last day of each year from the known price per share, the aggregated equity and MKVALT estimates for the entire index (book-value is another name for the equity):

$$\frac{Equity}{Shares} = \frac{Price/Shares}{Price/Equity} = \frac{Price/Shares}{P/Book} = \frac{Price}{Shares} \cdot \frac{Equity}{MKVALT}$$

This calculation may have some error because the S&P 500 share price is for weighted constituent companies but MKVALT is un-weighted. The error is expected to be negligible.

The equity per share is estimated for each day of a year by linearly interpolating between the known equity values at year end. This is also used to estimate the P/Book ratios for all days of a year by dividing a known price per share for each day with the estimated equity per share for that day.

ROE is calculated using the reported net income available to common shareholders for a given year, divided by the equity at the end of the prior year. The constituent companies of the S&P 500 index change each year so the equity for the prior year may be for different companies than the net income of the current year. In the period 1983-2011, covering the data of this study, the number of changes to the constituent companies of the S&P 500 index was 24 companies per year on average with standard deviation 11. That is, on average less than 5% of the constituent companies of the S&P 500 index were changed each year, whose impact on the financial data is likely negligible as large companies typically remain in the S&P 500 index, and large companies dominate the aggregated financial data. So this way of calculating ROE for the S&P 500 conglomerate is considered satisfactory in terms of numerical precision. Similarly for ROA.

Compustat provides the data items PRSTKC and SSTK for the amount of share buyback and issuance, respectively, where the preferred and common stocks are combined. However, the other data items being considered in this study are for the common stock alone, which means comparisons and calculations made using these numbers for share buyback and issuance contain an error as the preferred stock is included. But the error is negligible because the preferred equity is only 4% of the common equity on average for the period 1983-2011.

Cash flows associated with tax benefits of stock options have been ignored as they are negligible.

Figures & Tables

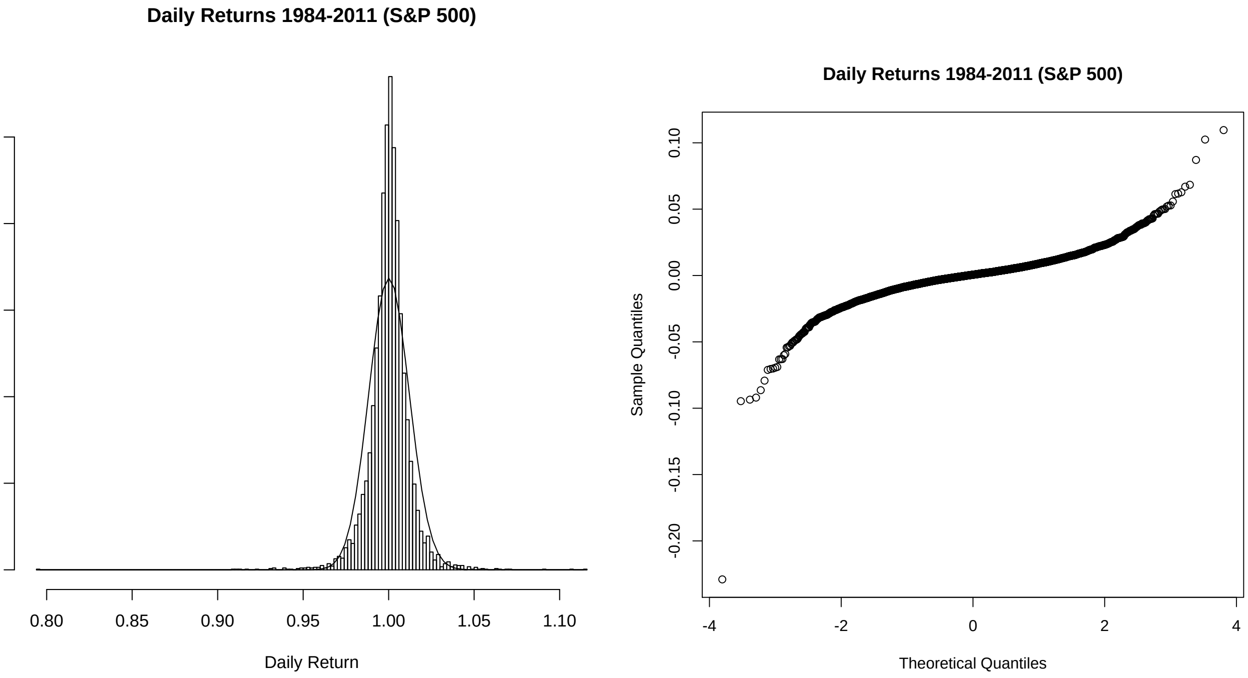


Figure 1: <u>Daily price returns</u> for the <u>S&P 500</u> stock market index during the period <u>1984-2011</u>, calculated as $Return_t = Price_{Day}/Price_{Day-1}$. The histogram (left) has a fitted log-normal PDF and the Q-Q plot (right) also compares to the log-normal PDF. This clearly shows that the distribution of daily price returns is not log-normal. The minimum price return is about 0.795 (corresponding to a change of -20.5%), the maximum is about 1.116 (a change of +11.6%), the mean is 1 (a change of 0%) and the standard deviation is 0.0118 (or 1.18%).

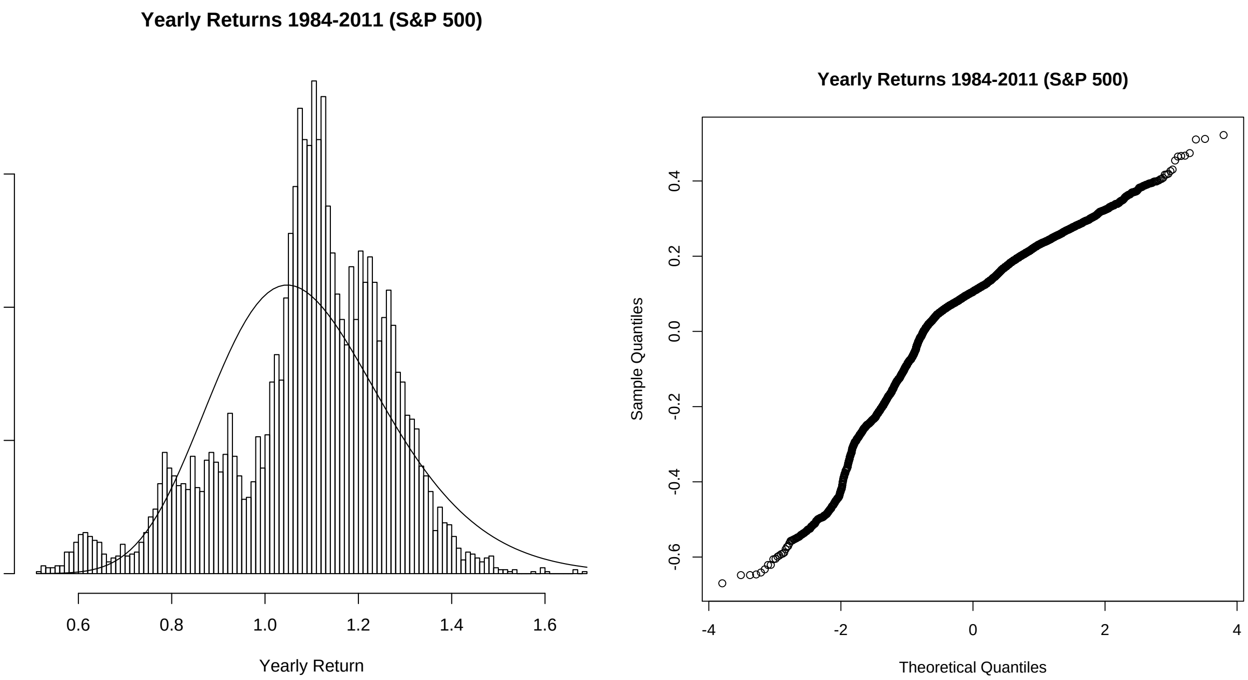


Figure 2: <u>Yearly price returns</u> for the <u>S&P 500</u> stock market index during the period <u>1984-2011</u>, calculated as $Return_t = Price_{Year}/Price_{Year-1}$. The histogram (left) has a fitted log-normal PDF and the Q-Q plot (right) also compares to the log-normal PDF. This clearly shows that the distribution of yearly price returns is not log-normal. The minimum price return is about 0.51 (corresponding to a change of -49%), the maximum is about 1.69 (a change of +69%), the mean is 1.095 (a change of 9.5%) and the standard deviation is 0.173 (or 17.3%).

Year	ROA	ROE	Dividends/Earnings	Net Buyback/Earnings	Retain/Earnings	Equity/Assets
1984	4.0%	13.7%	52.6%	(22.0)%	49.3%	28%
1985	3.0%	10.9%	45.0%	5.7%	29.2%	26%
1986	2.8%	10.9%	56.7%	14.1%	30.5%	25%
1987	3.2%	12.6%	61.0%	8.5%	27.4%	25%
1988	4.1%	16.7%	54.8%	17.8%	33.2%	21%
1989	3.2%	14.8%	48.2%	18.6%	41.6%	20%
1990	2.6%	12.6%	48.3%	10.0%	28.5%	20%
1991	1.7%	8.3%	56.7%	14.9%	28.9%	20%
1992	0.8%	4.0%	82.0%	(10.9)%	(49.3)%	18%
1993	1.8%	9.7%	170.6%	(21.3)%	29.3%	17%
1994	2.9%	16.8%	75.7%	(5.0)%	50.6%	18%
1995	3.1%	17.2%	43.9%	5.5%	35.9%	18%
1996	3.3%	18.7%	46.3%	17.8%	47.9%	18%
1997	3.2%	17.6%	37.6%	14.5%	35.9%	18%
1998	3.1%	17.9%	38.0%	26.0%	32.1%	17%
1999	3.3%	20.0%	38.9%	29.0%	45.7%	17%
2000	3.0%	17.5%	32.2%	22.1%	50.4%	18%
2001	1.2%	6.7%	33.0%	16.5%	5.9%	18%
2002	0.5%	2.9%	72.5%	21.6%	(140.8)%	16%
2003	2.4%	15.5%	165.8%	75.0%	49.8%	16%
2004	2.6%	15.7%	34.3%	15.9%	41.1%	17%
2005	2.9%	17.1%	34.4%	24.6%	26.6%	17%
2006	3.2%	18.7%	35.9%	37.5%	21.4%	17%
2007	2.4%	13.9%	30.6%	48.0%	(16.8)%	16%
2008	0.9%	5.3%	44.6%	72.2%	42.0%	16%
2009	1.9%	12.0%	98.7%	(40.7)%	47.5%	19%
2010	3.0%	15.8%	43.5%	9.1%	44.0%	21%
2011	3.2%	15.7%	29.7%	26.3%	28.3%	21%

Table 1: Financial data for the aggregated companies of the S&P 500 stock market index. Calculated using Eq. 2-11 for ROE and similarly for ROA, and Eq. 2-7 for $Retain/Earnings$. Statistics are shown in Table 2. This data is available for download from the internet as described in section 6.12.

	Mean	Stdev	Min	Max
ROA	2.6%	0.9%	0.5%	4.1%
ROE	13.5%	4.7%	2.9%	20.0%
Retain	26.4%	39.0%	(140.8)%	69.4%
Dividends/Earnings	56.6%	35.1%	29.7%	170.6%
Net Buyback/Earnings	17.0%	24.5%	-40.7%	75.0%
Equity/Assets	19.6%	3.6%	15.6%	28.8%

Table 2: Statistics for Table 1.

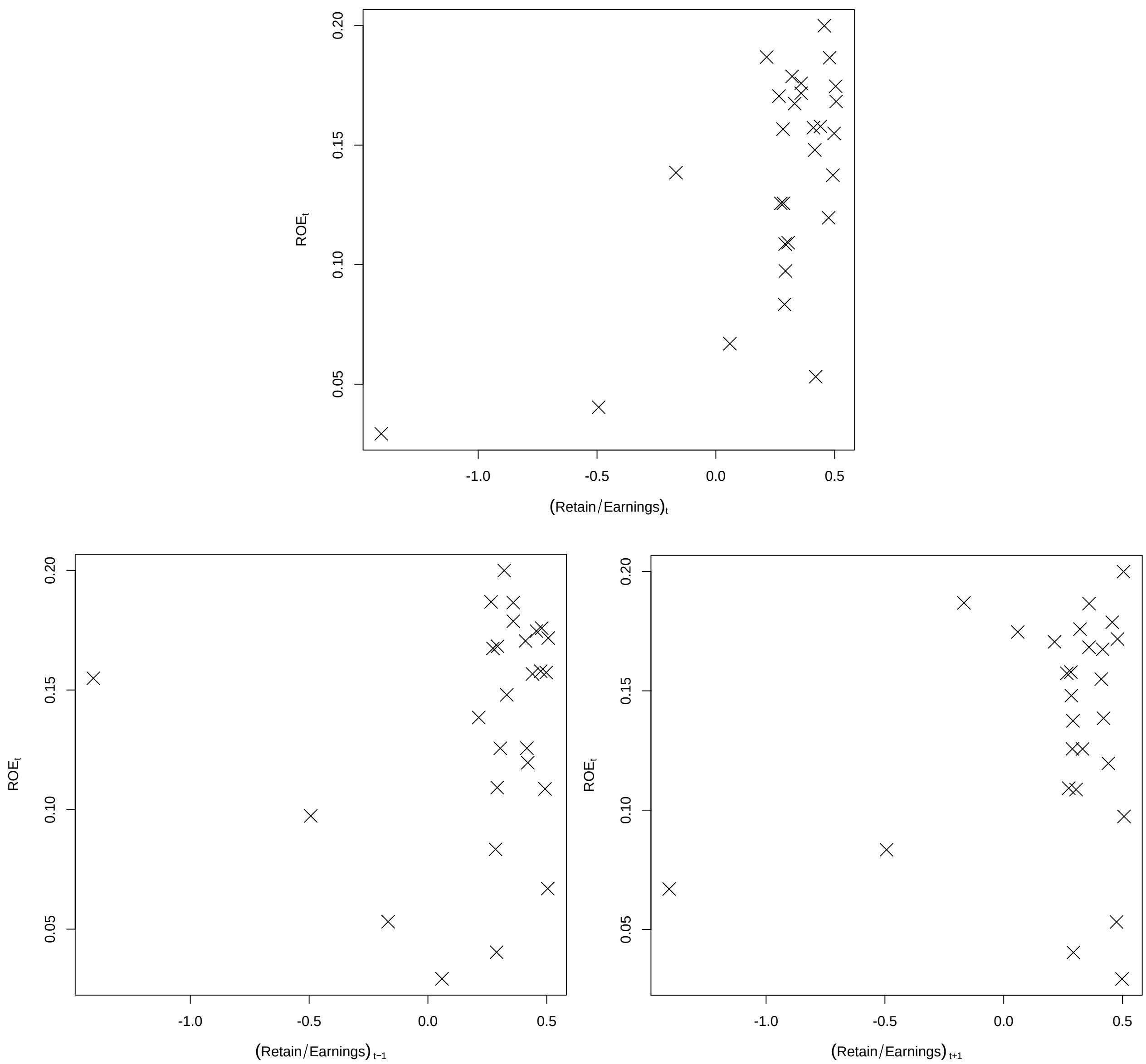


Figure 3: Relations between ROE and $Retain/Earnings$ for the S&P 500 stock market index using the data from Table 1. The upper plot is for $((Retain/Earnings)_t,\ ROE_t)$, the lower left plot is for $((Retain/Earnings)_{t-1},\ ROE_t)$ and the lower right plot is for $((Retain/Earnings)_{t+1},\ ROE_t)$, that is, the lower plots have shifted the $Retain/Earnings$ series one year backwards and forwards, respectively. All three plots show a cluster of $Retain/Earnings$ around 25-50% and ROE around 10-20%, but the outliers behave differently in these plots. For the two lower plots where the $Retain/Earnings$ series is time-shifted, the outliers do not show a clear tendency e.g. of $Retain/Earnings$ and ROE increasing together, as is evident in the upper plot for $((Retain/Earnings)_t,\ ROE_t)$ which is therefore used when sampling in Monte Carlo simulations.

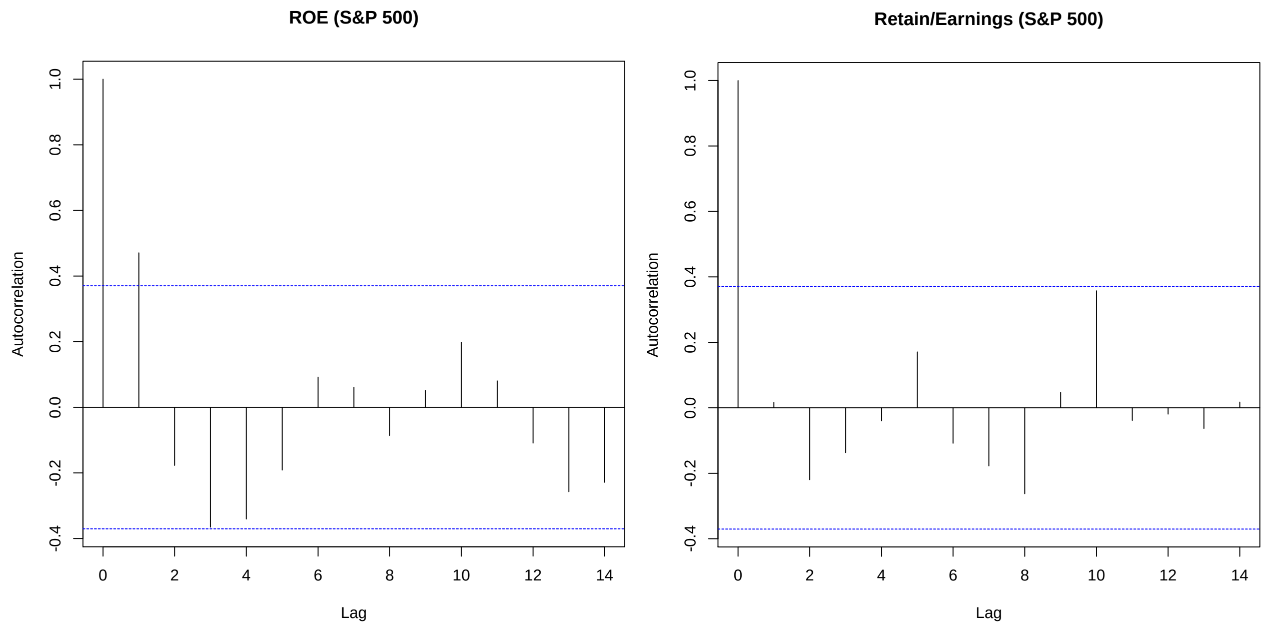


Figure 4: Autocorrelation of ROE and $Retain/Earnings$ with varying time lag for the S&P 500 stock market index using the data from Table 1. The left plot shows the autocorrelation of ROE with the dotted line being the level of statistical significance, which shows that ROE_t is obviously perfectly correlated with itself (that is, when $Lag = 0$), and there appears to be a positive correlation between ROE_t and ROE_{t+1} which is explored further in Figure 5, and an almost significant negative correlation between ROE_t and ROE_{t+3} and ROE_{t+4}. The right plot shows the autocorrelation of the $Retain/Earnings$ series whose largest autocorrelation occurs with a 10 year lag, but is not statistically significant.

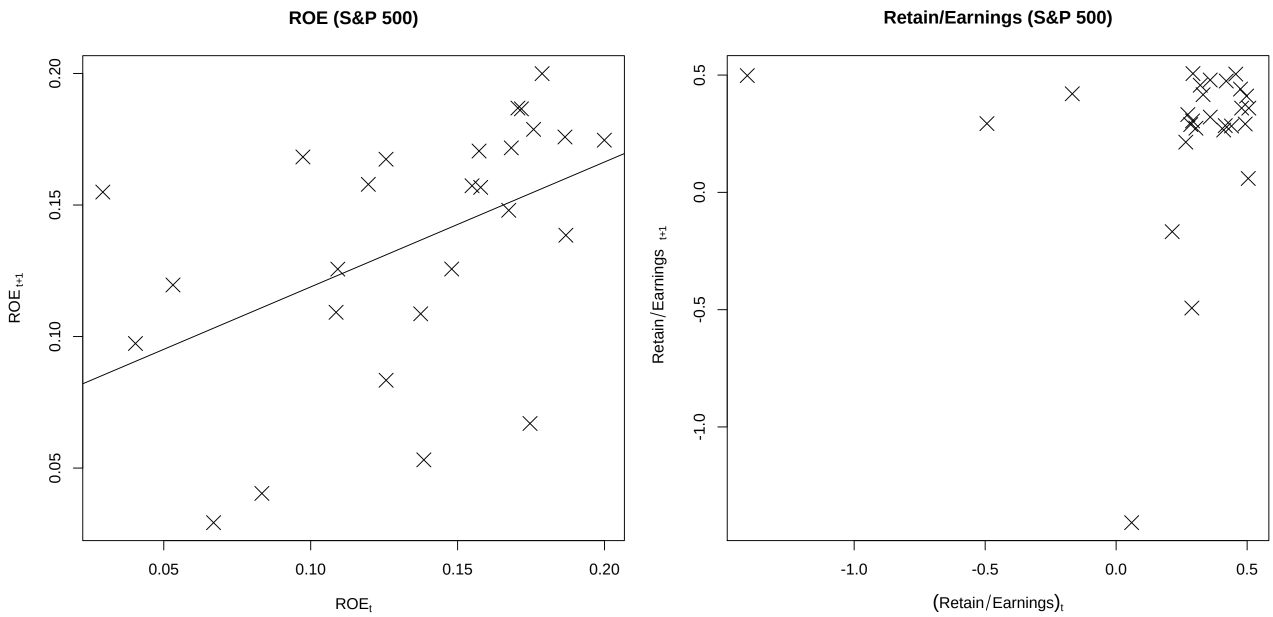


Figure 5: Autocorrelation of ROE and $Retain$ with a one-year time lag, for the S&P 500 stock market index using the data from Table 1. The left plot shows the relation between ROE_t and ROE_{t+1} whose linear fit has a low coefficient of determination $R^2 = 0.22$ and hence does not appear to be significantly linearly correlated. The right plot shows the relation between $Retain_t$ and $Retain_{t+1}$ which has a cluster around 25-50% but appears to be uncorrelated otherwise.

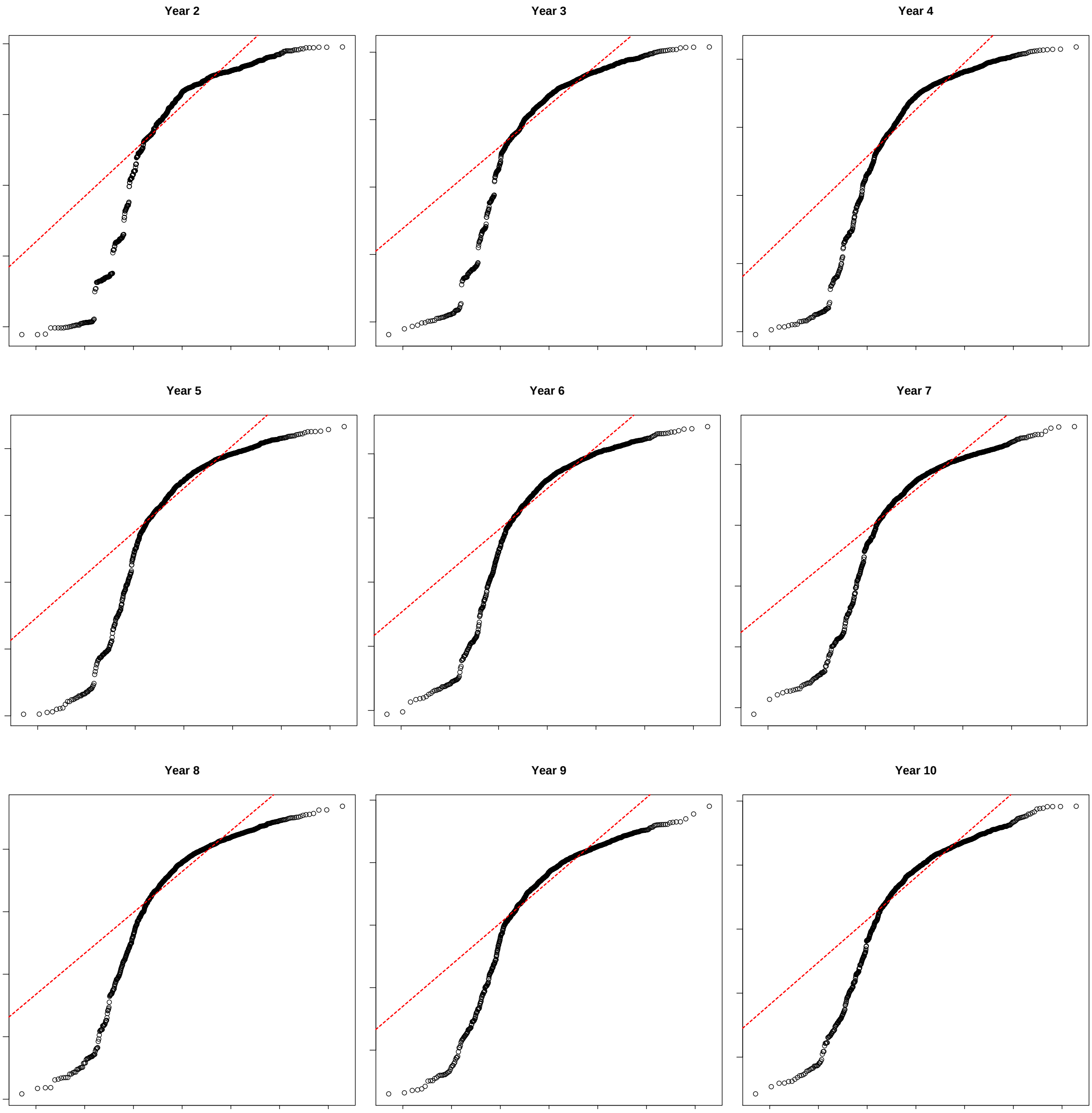


Figure 6: Q-Q log-normal plots of the earnings of the S&P 500 stock market index in different years, resulting from the Monte Carlo simulations described in section 6.4. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

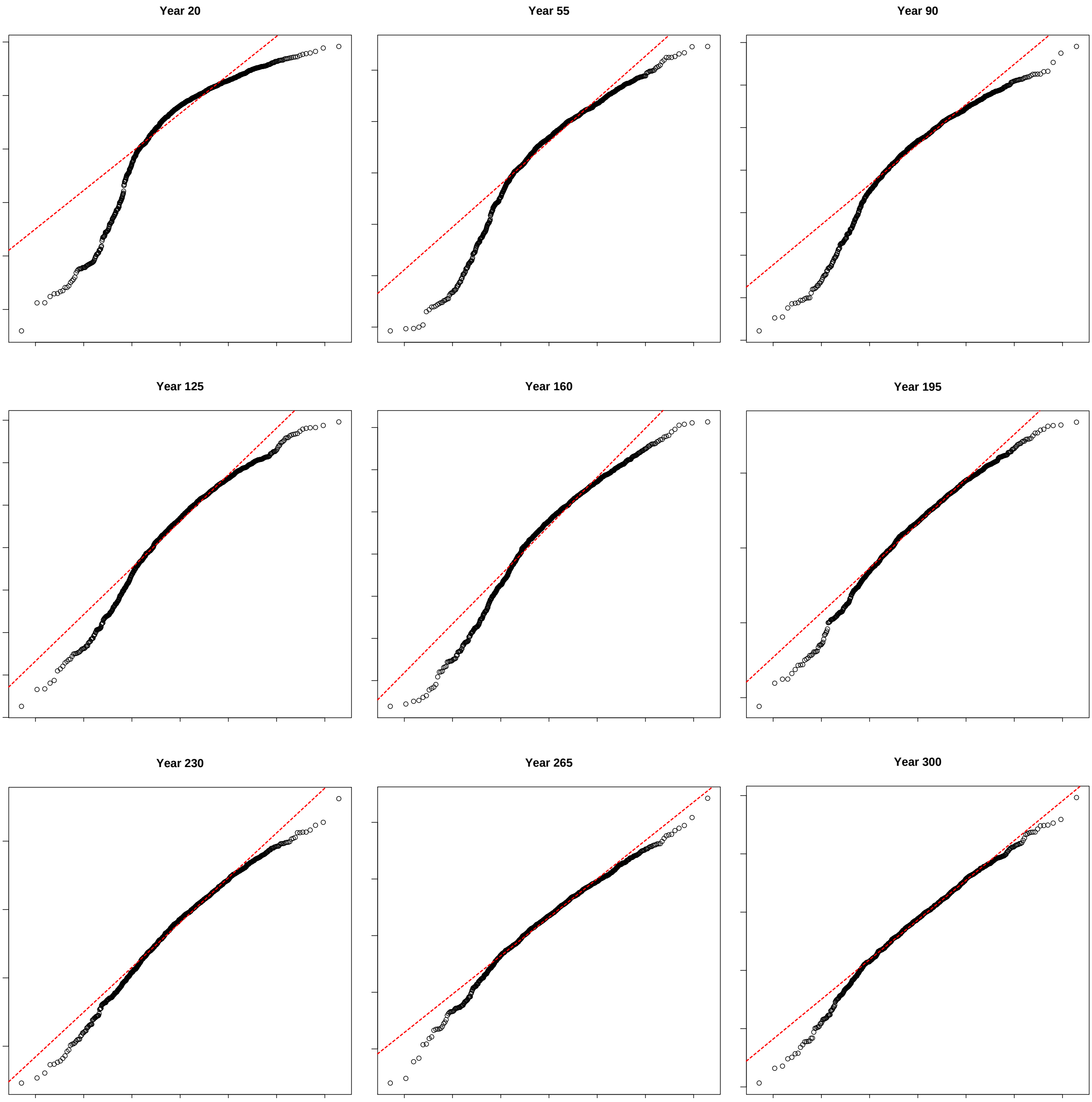
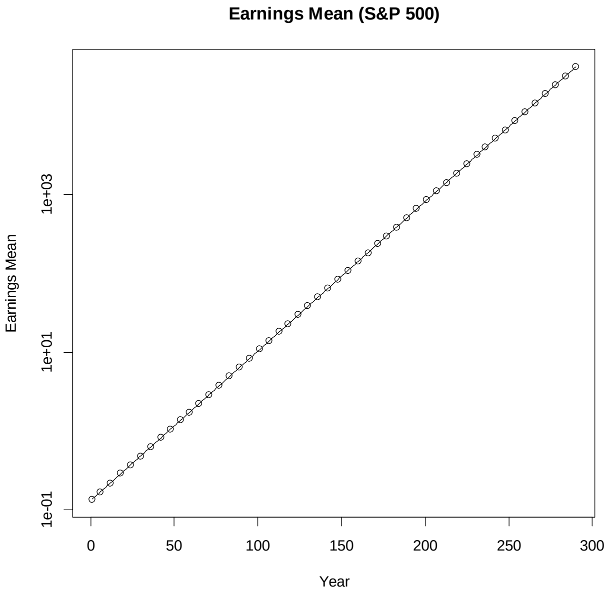


Figure 7: Q-Q log-normal plots of the earnings of the S&P 500 stock market index in different years, resulting from the Monte Carlo simulations described in section 6.4. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.



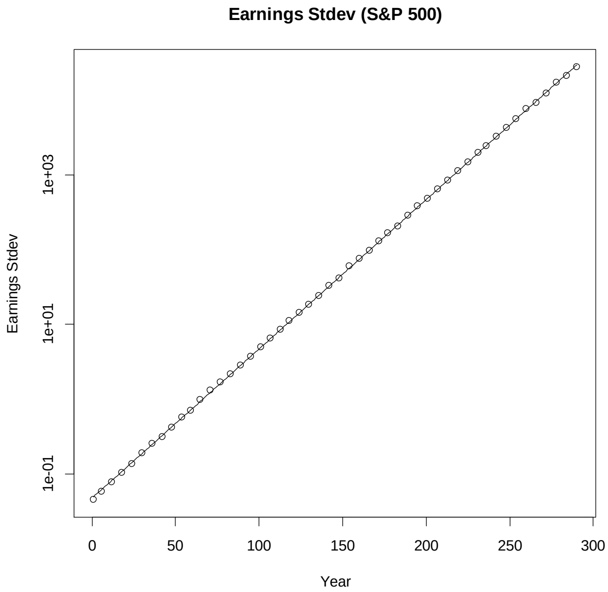


Figure 8: Earnings mean and standard deviation for the S&P 500 stock market index, resulting from a series of Monte Carlo simulations as described in section 6.4. The small circles are the results of the Monte Carlo simulations and the fitted lines are given in Eq. 6-2 and Eq. 6-3. The y-axes are logarithmic.

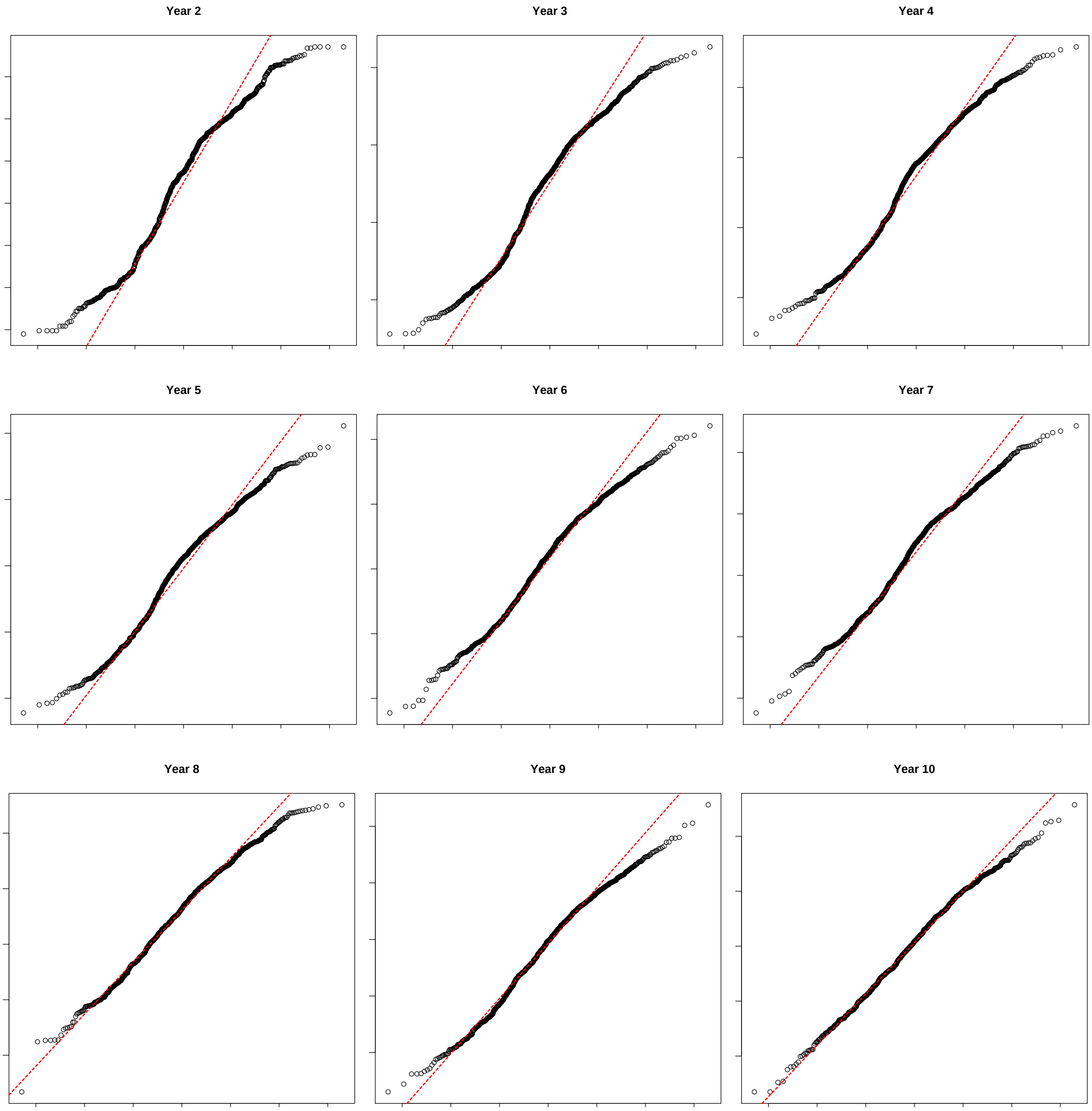


Figure 9: Q-Q log-normal plots of the <u>dividends</u> of the <u>S&P 500</u> stock market index in different years, resulting from the Monte Carlo simulations described in section 6.5. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

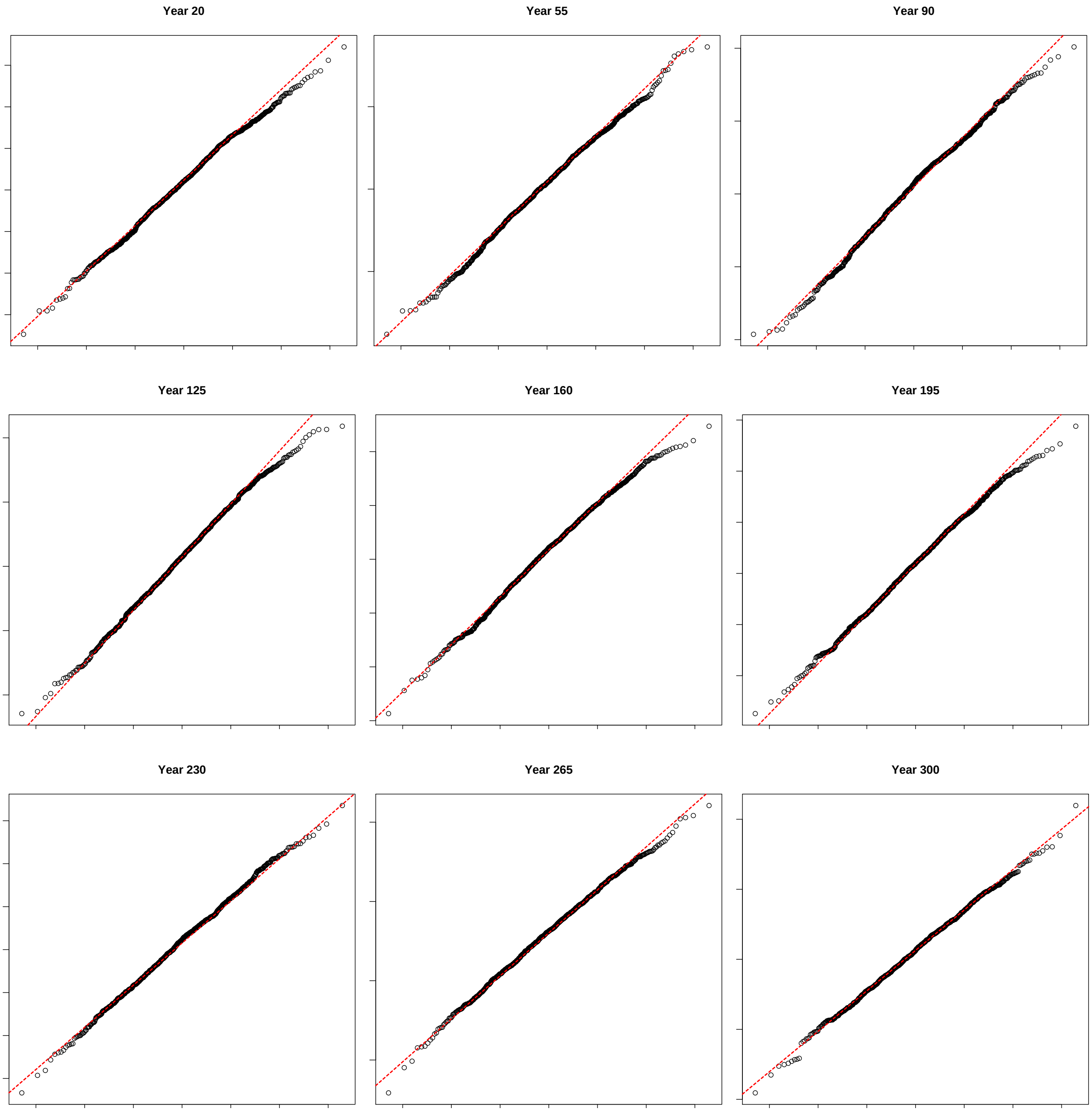


Figure 10: Q-Q log-normal plots of the dividends of the S&P 500 stock market index in different years, resulting from the Monte Carlo simulations described in section 6.5. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

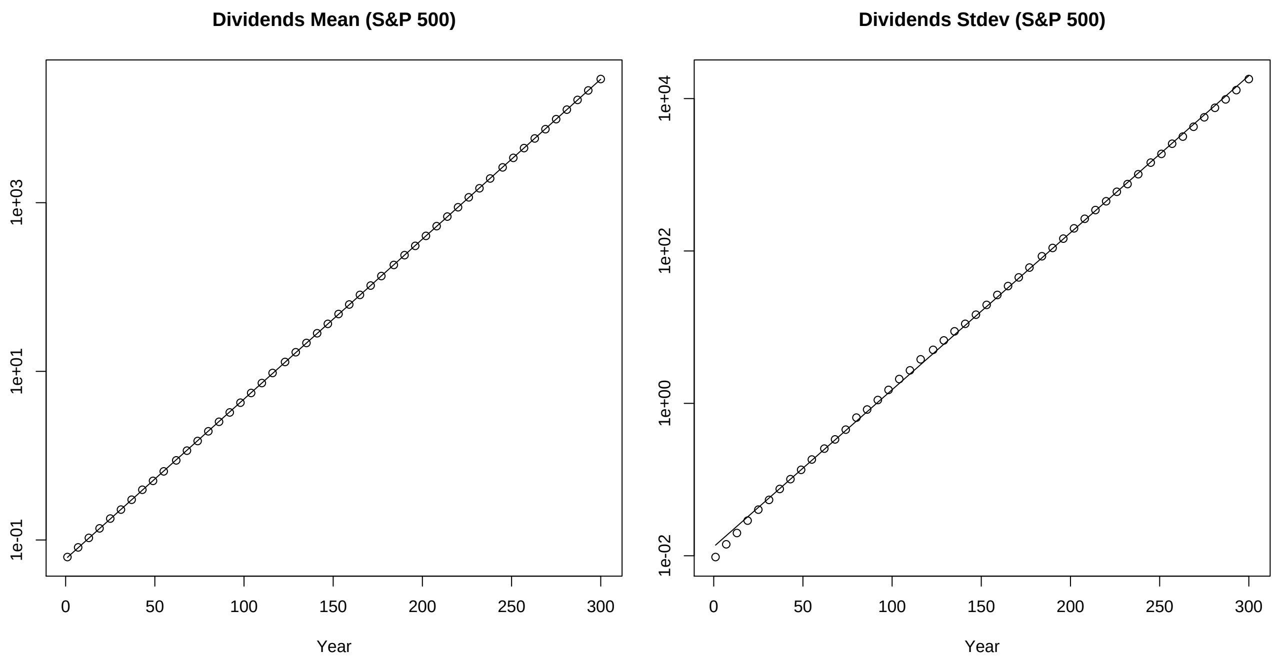


Figure 11: Dividend mean and standard deviation for the S&P 500 stock market index, resulting from a series of Monte Carlo simulations as described in section 6.5. The small circles are the results of the Monte Carlo simulations and the fitted lines are given in Eq. 6-8 and Eq. 6-9. The y-axes are logarithmic.

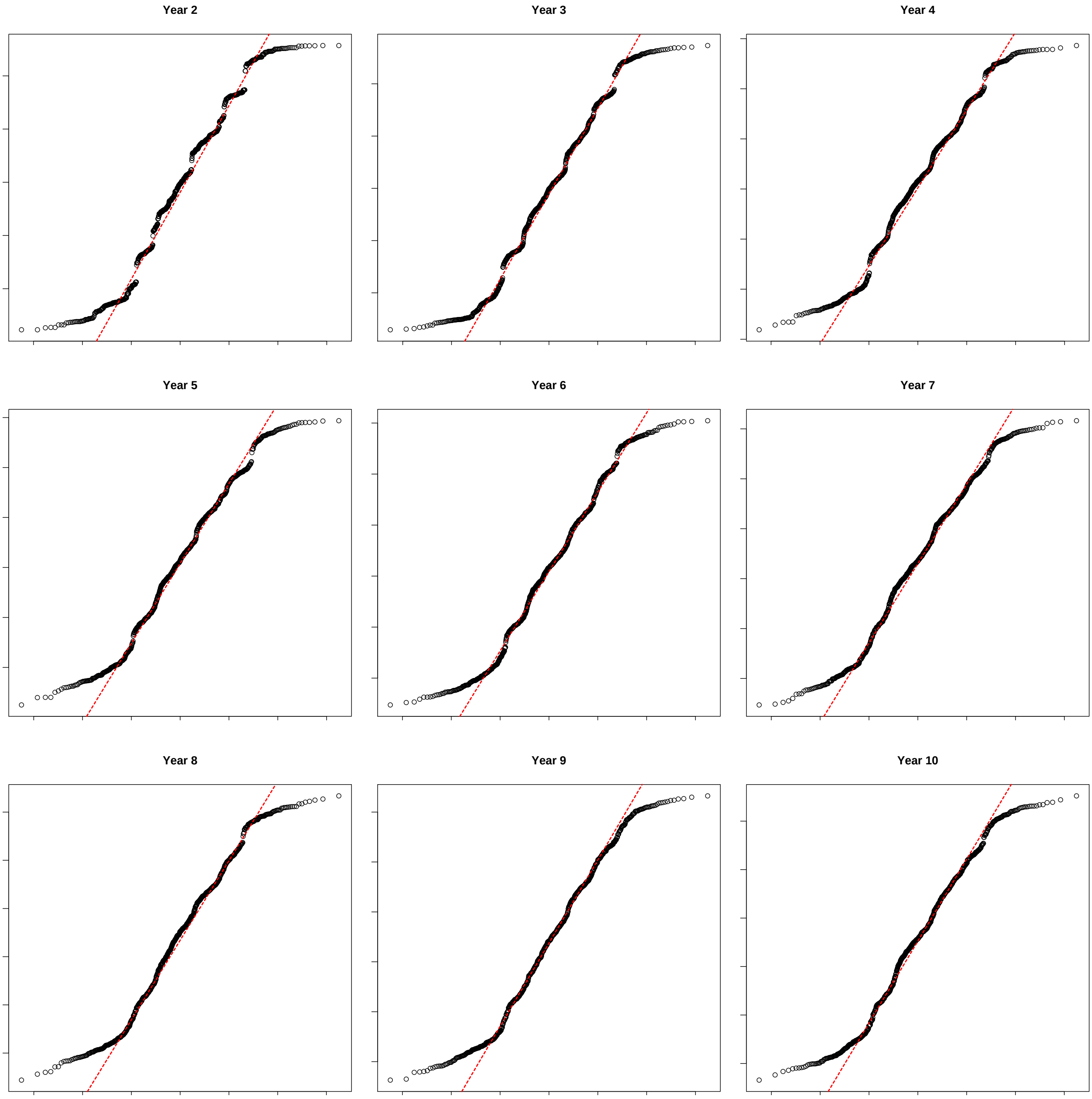


Figure 12: Q-Q log-normal plots of the <u>share buybacks net of issuance</u> for the <u>S&P 500</u> stock market index in different years, resulting from the Monte Carlo simulations described in section 6.6. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

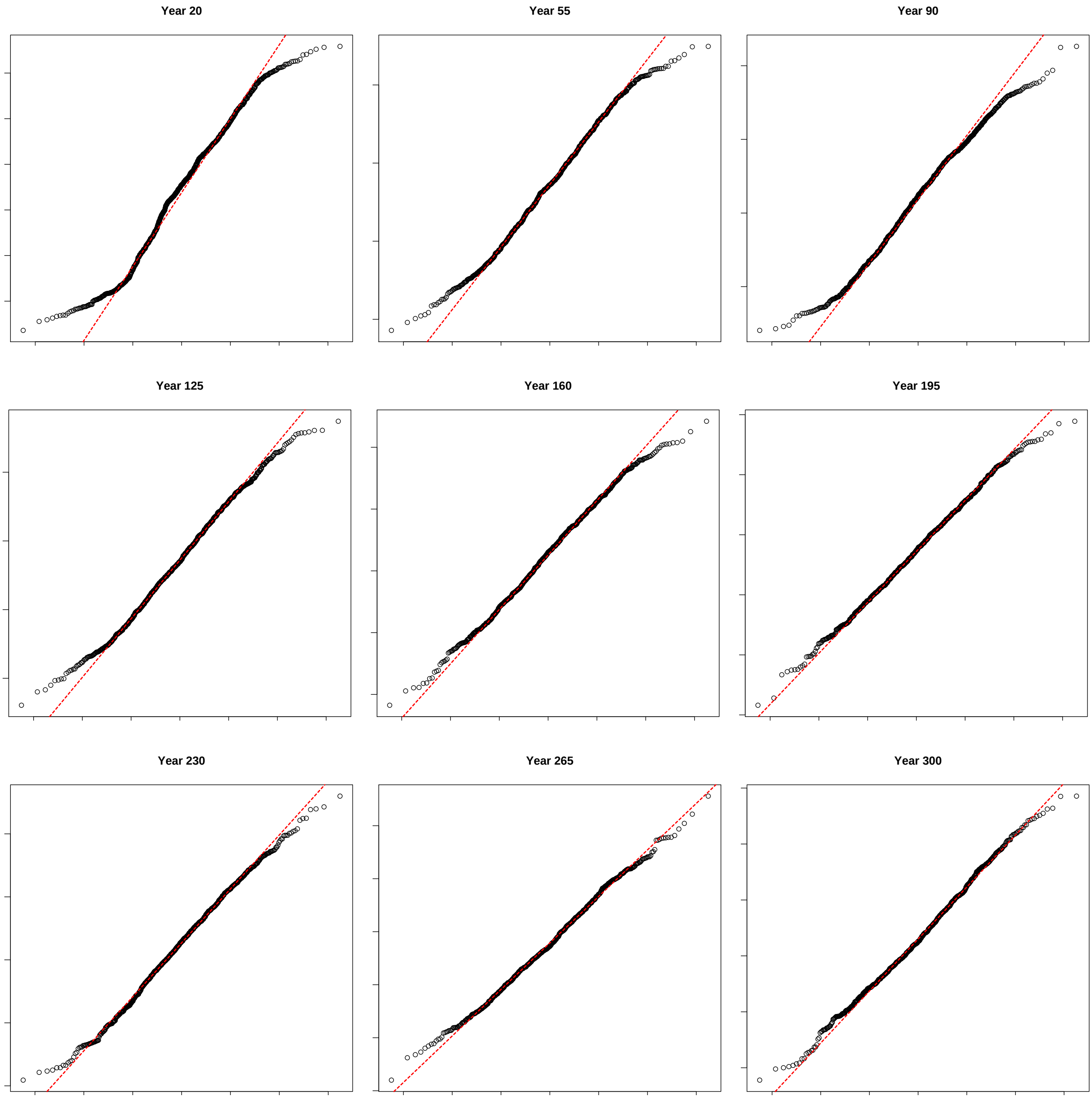


Figure 13: Q-Q log-normal plots of the share buybacks net of issuance for the S&P 500 stock market index in different years, resulting from the Monte Carlo simulations described in section 6.6. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

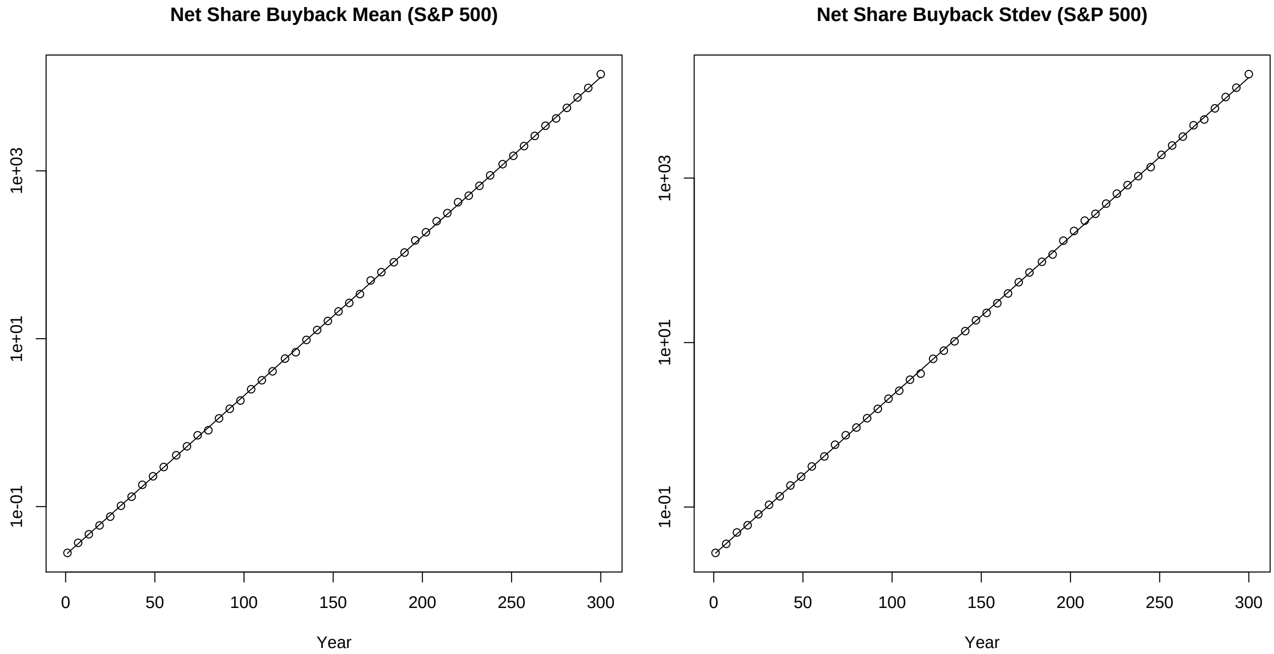


Figure 14: Net share buyback mean and standard deviation for the S&P 500 stock market index, resulting from a series of Monte Carlo simulations as described in section 6.6. The small circles are the results of the Monte Carlo simulations and the fitted lines are given in Eq. 6-14 and Eq. 6-15. The y-axes are logarithmic.

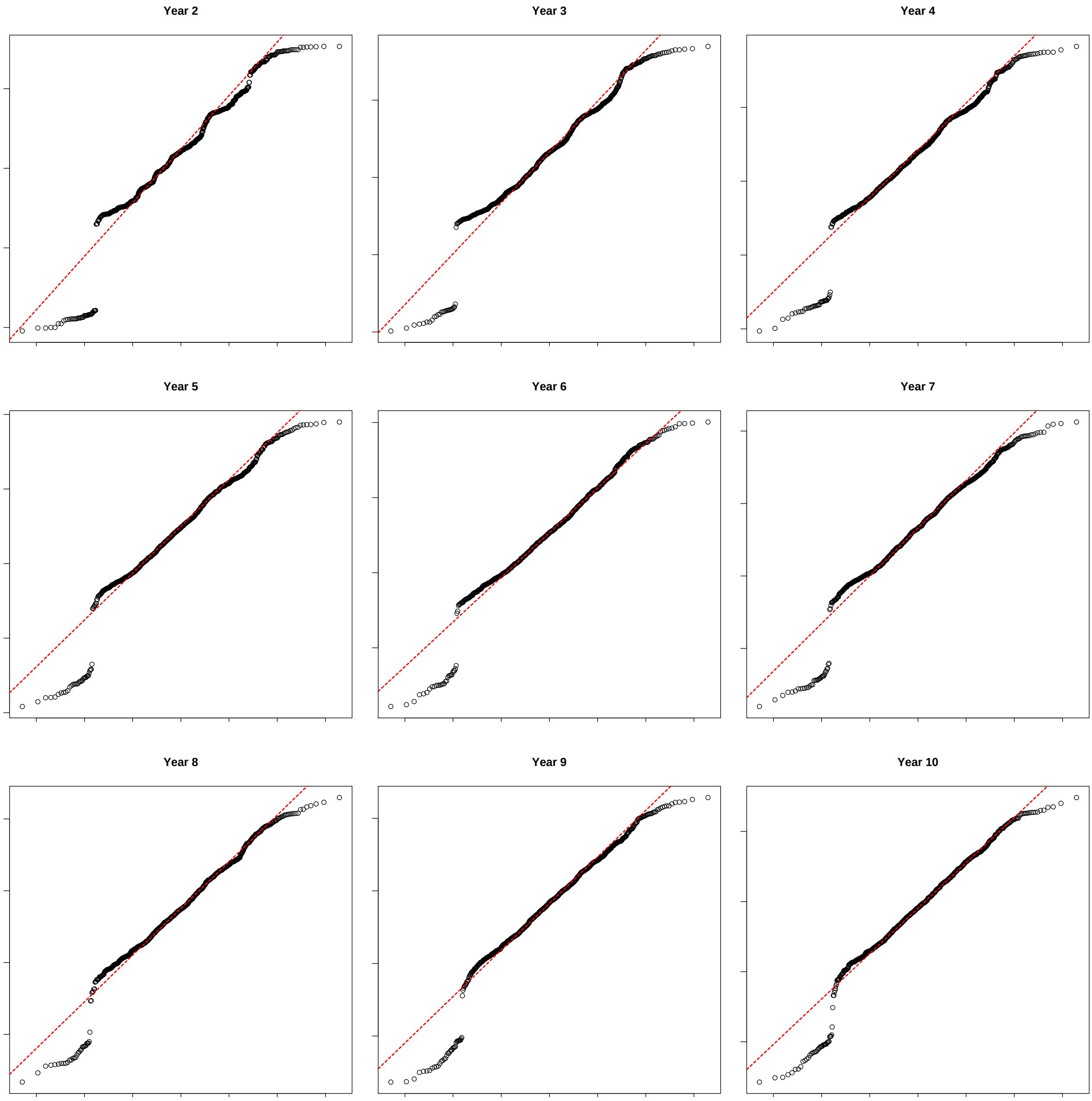


Figure 15: Q-Q log-normal plots of the <u>payouts</u> of the <u>S&P 500</u> stock market index in different years, resulting from the Monte Carlo simulations described in section 6.7. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

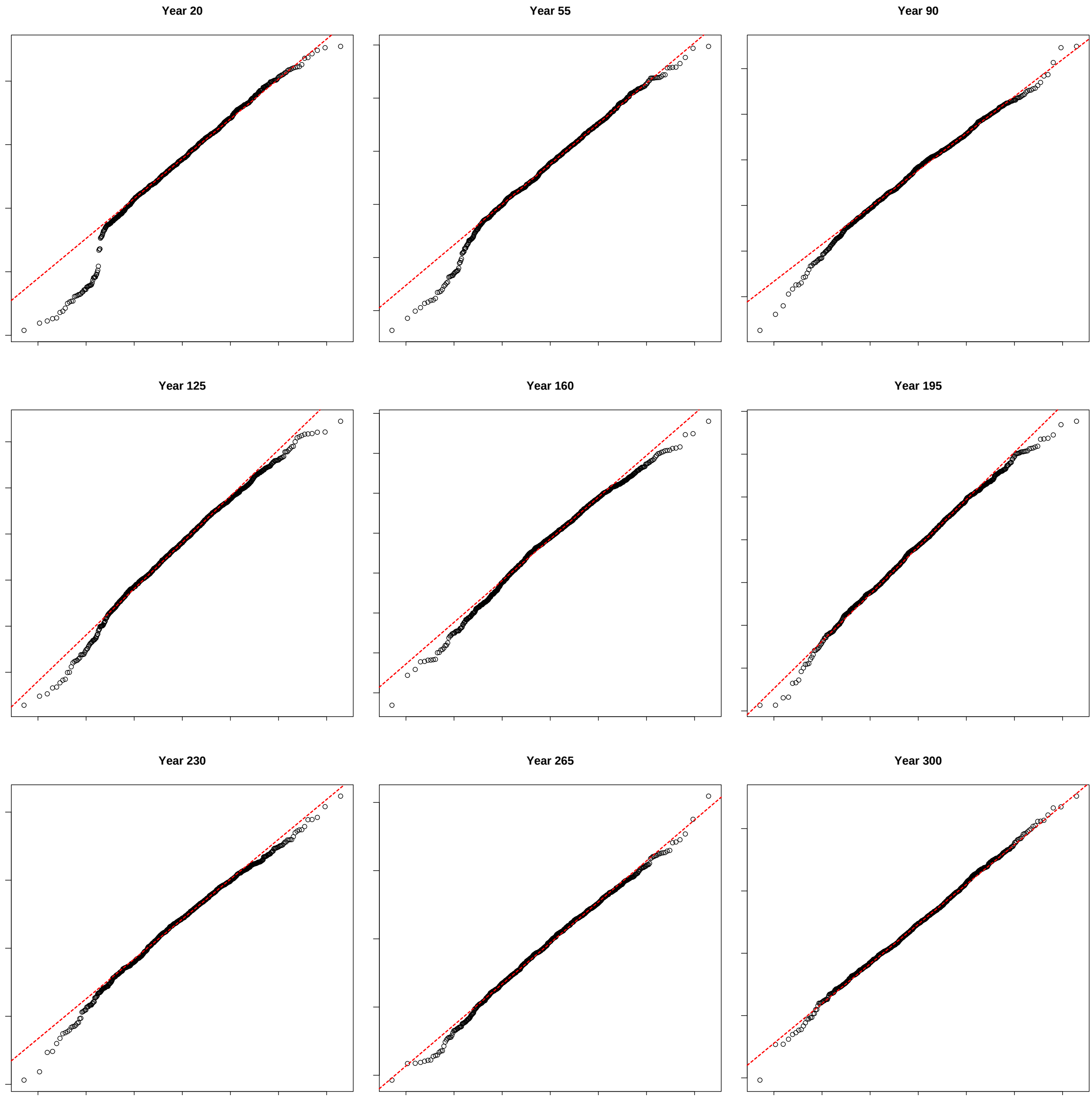


Figure 16: Q-Q log-normal plots of the <u>payouts</u> of the <u>S&P 500</u> stock market index in different years, resulting from the Monte Carlo simulations described in section 6.7. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

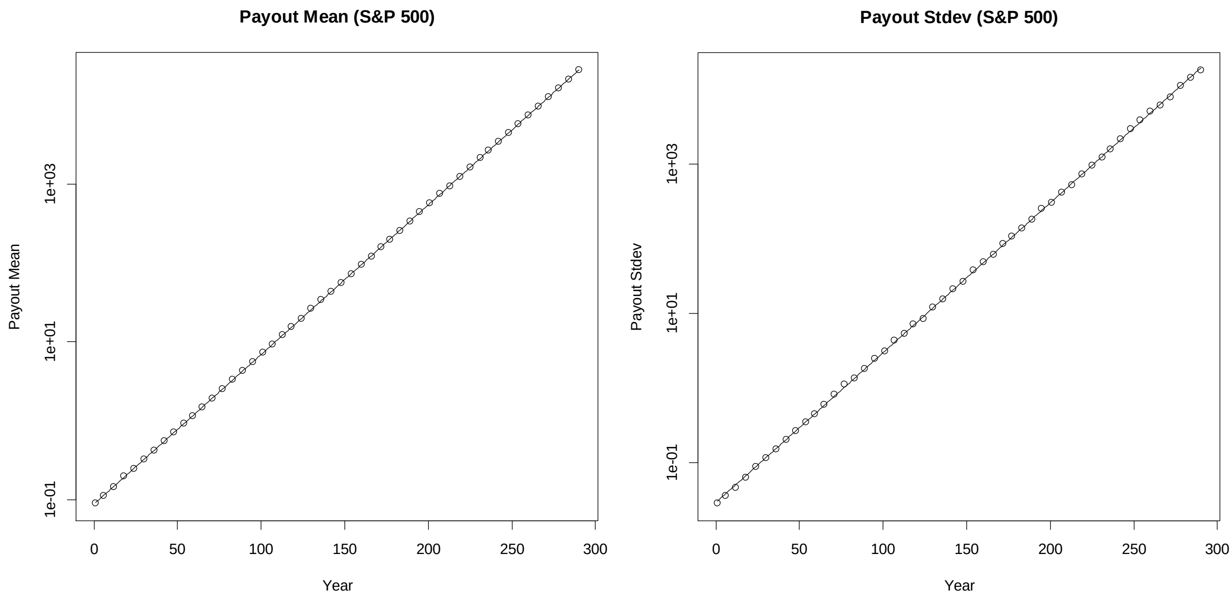


Figure 17: Payout mean and standard deviation for the S&P 500 stock market index, resulting from a series of Monte Carlo simulations as described in section 6.7. The small circles are the results of the Monte Carlo simulations and the fitted lines are given in Eq. 6-20 and Eq. 6-21. The y-axes are logarithmic.

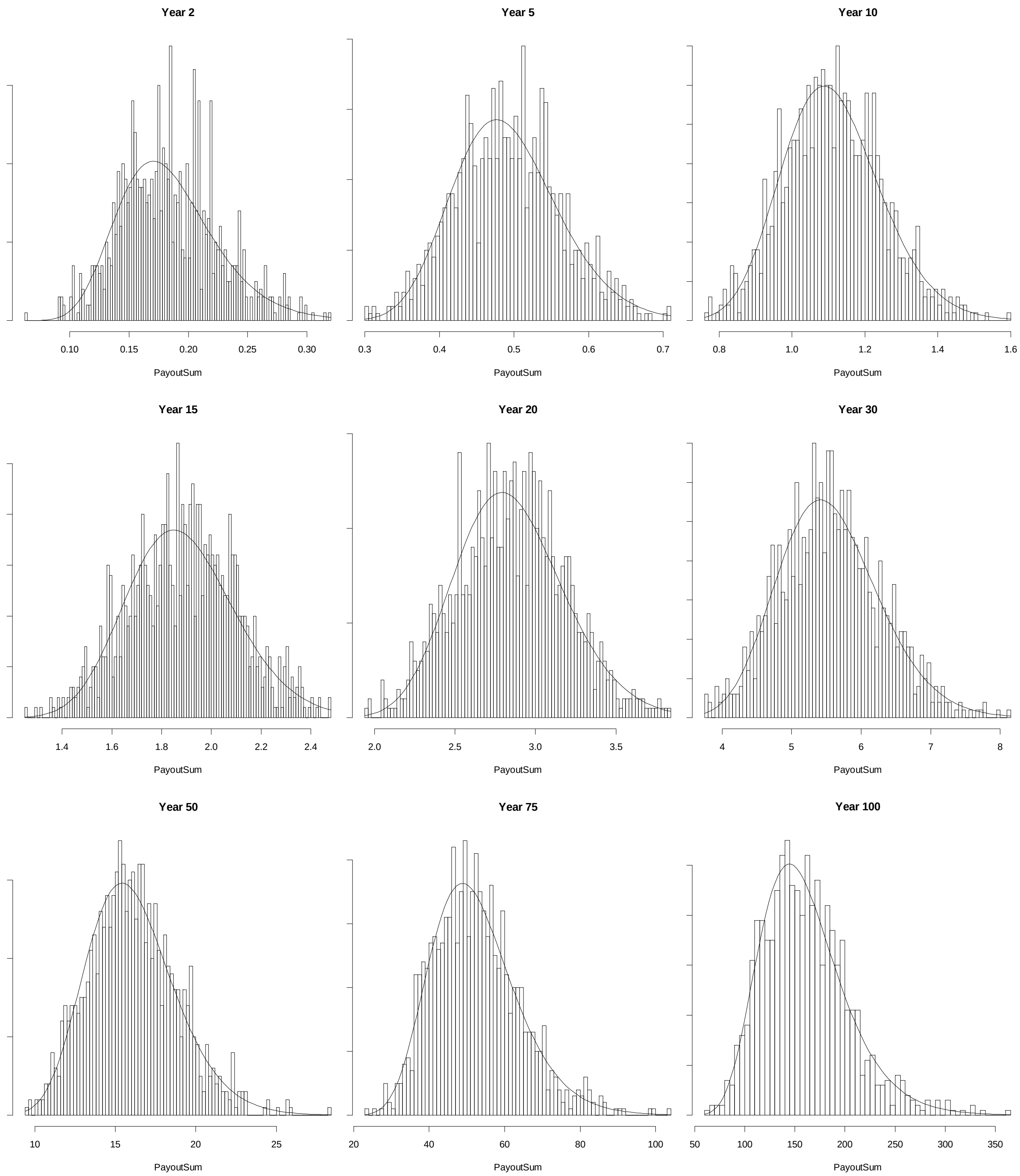


Figure 18: Histograms with fitted log-normal PDFs of the <u>cumulative payout sums</u> for the <u>S&P 500</u> stock market index in different years, resulting from the Monte Carlo simulations described in section 6.7.2.

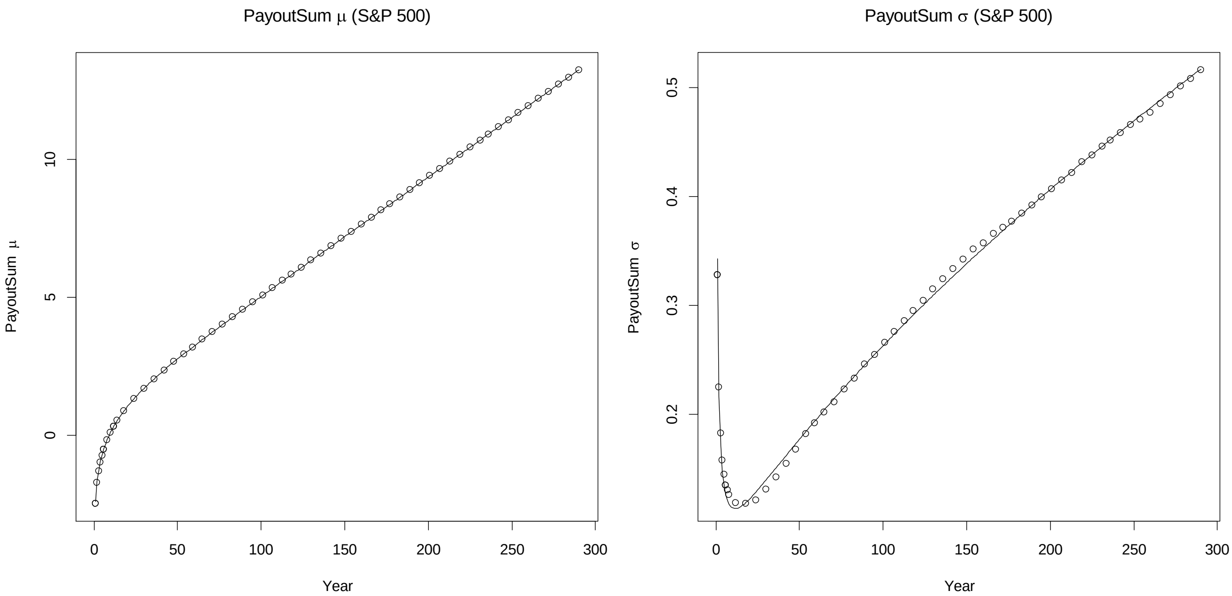


Figure 19: Log-normal probability distribution parameters μ and σ for the payout sum of the S&P 500 stock market index, resulting from a series of Monte Carlo runs as described in section 6.7.2. The small circles are the results of the Monte Carlo simulations and the fitted curves are given in Eq. 6-27 and Eq. 6-28.

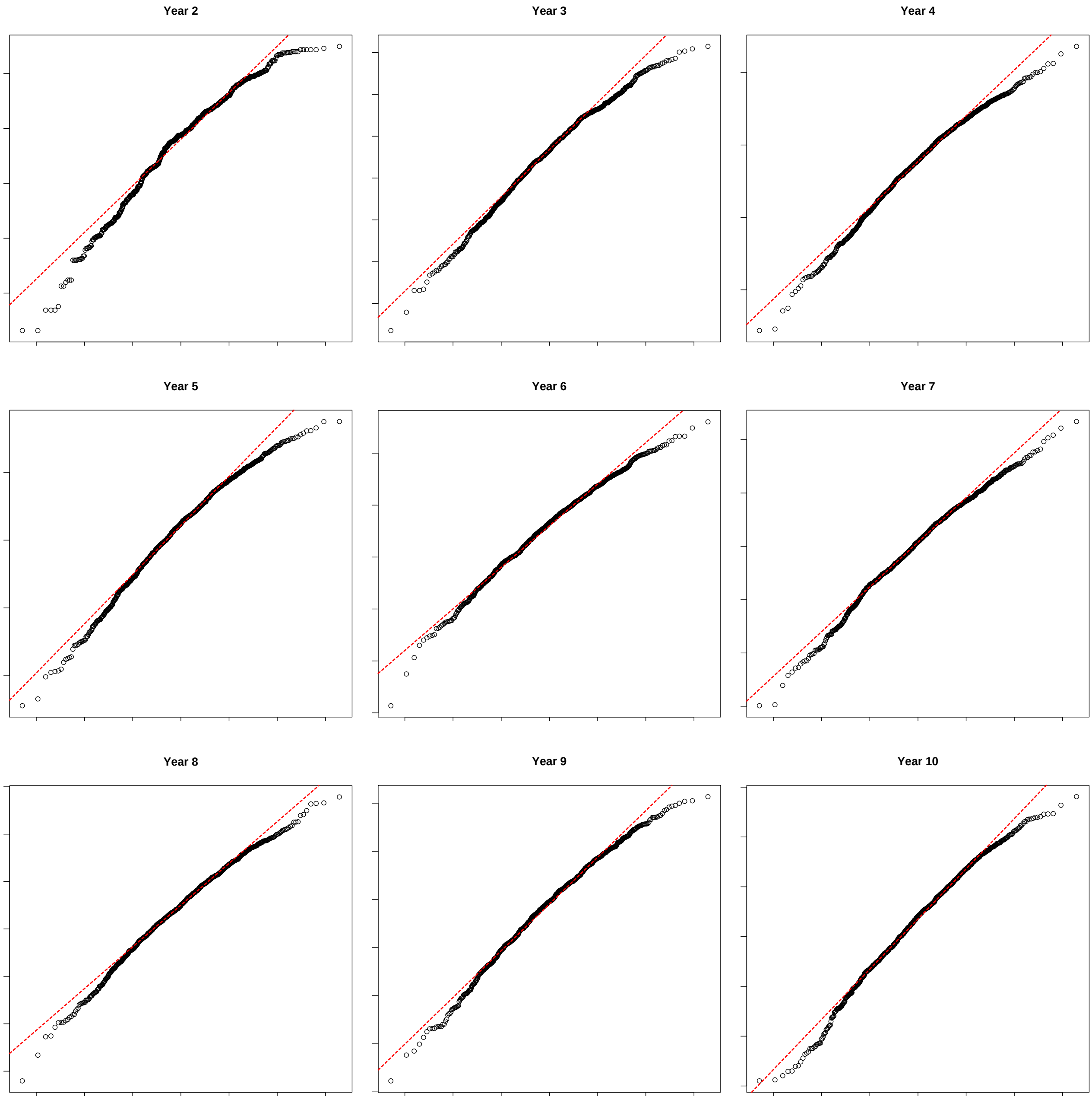


Figure 20: Q-Q log-normal plots of the equity of the S&P 500 stock market index in different years, resulting from the Monte Carlo simulations described in section 6.8. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.

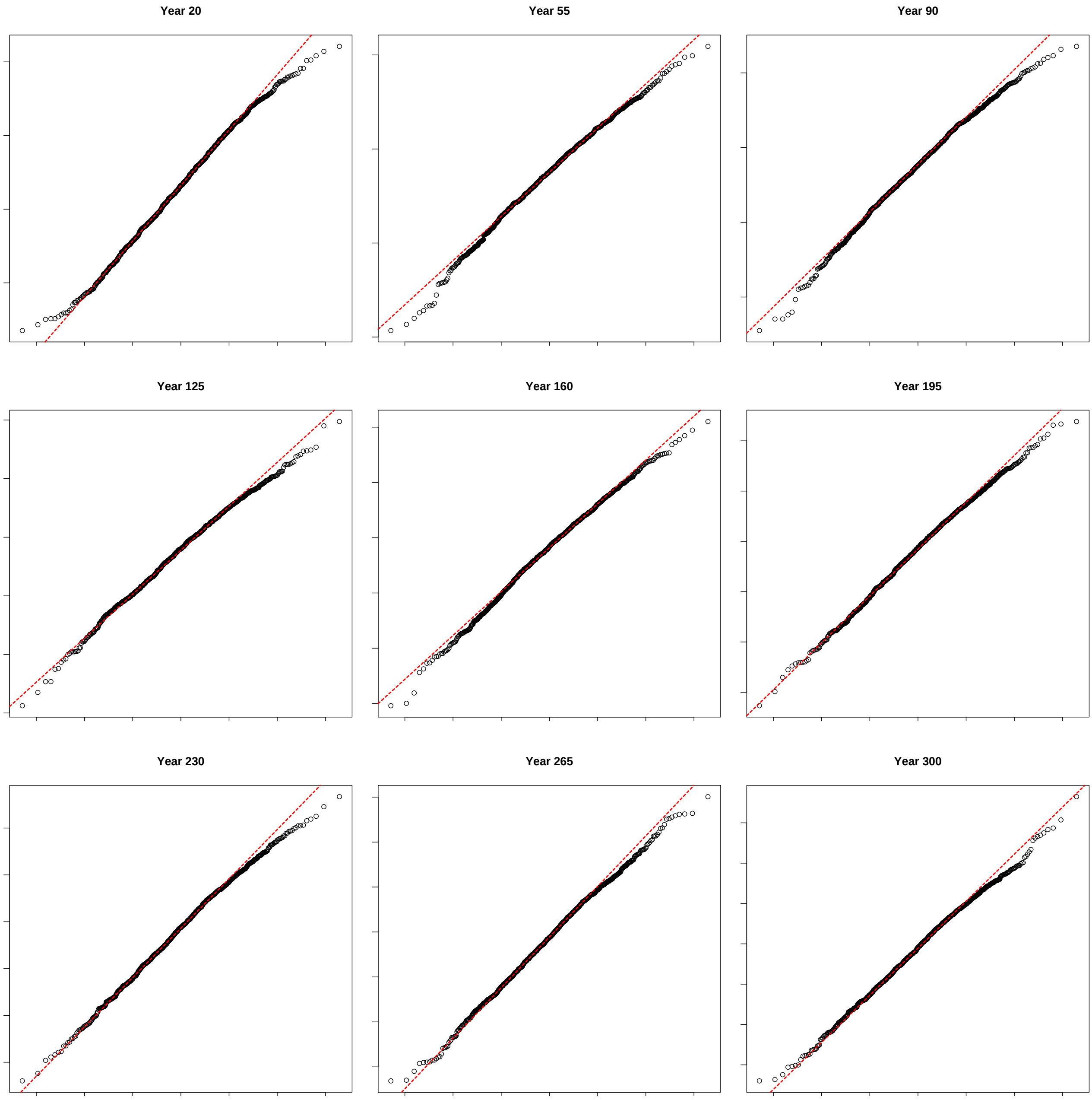
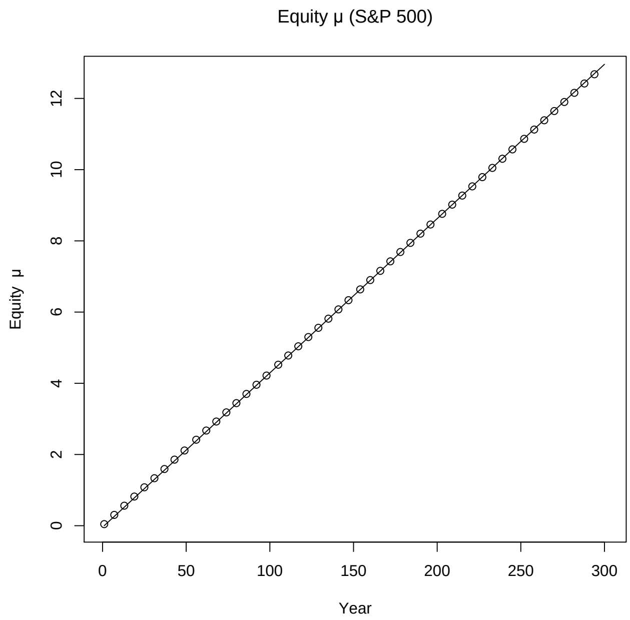


Figure 21: Q-Q log-normal plots of the equity of the S&P 500 stock market index in different years, resulting from the Monte Carlo simulations described in section 6.8. The x-axes are theoretical quantiles and the y-axes are sample quantiles. The red dotted lines pass through the first and third quartiles.



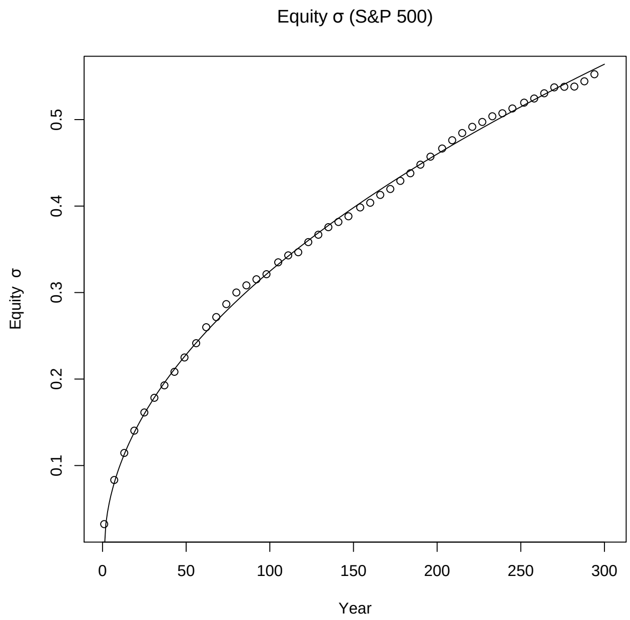


Figure 22: Log-normal probability distribution parameters μ and σ for the equity of the S&P 500 stock market index, resulting from a series of Monte Carlo runs as described in section 6.8.1. The small circles are the results of the Monte Carlo simulations and the fitted curves are given in Eq. 6-30 and Eq. 6-31.

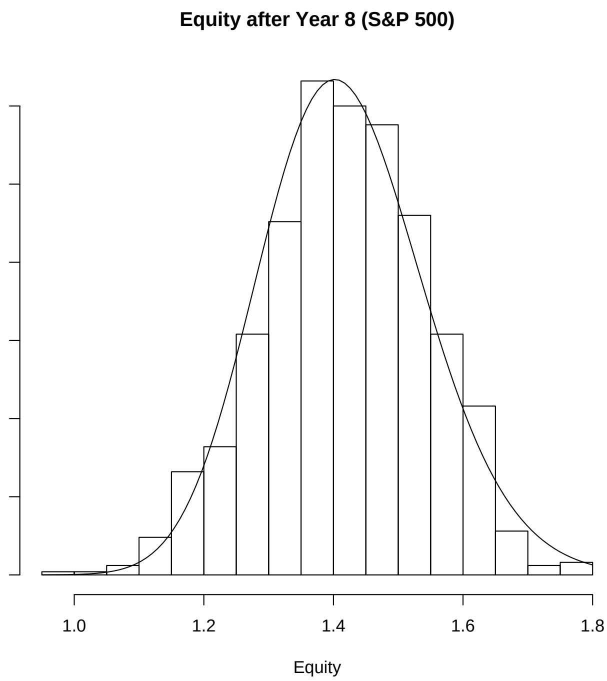


Figure 23: Distribution of equity for the S&P 500 stock market index after year 8. Also shown is the fitted log-normal PDF. The Monte Carlo simulation resulting in this plot is described in section 6.8. The Q-Q plot is shown in Figure 20.

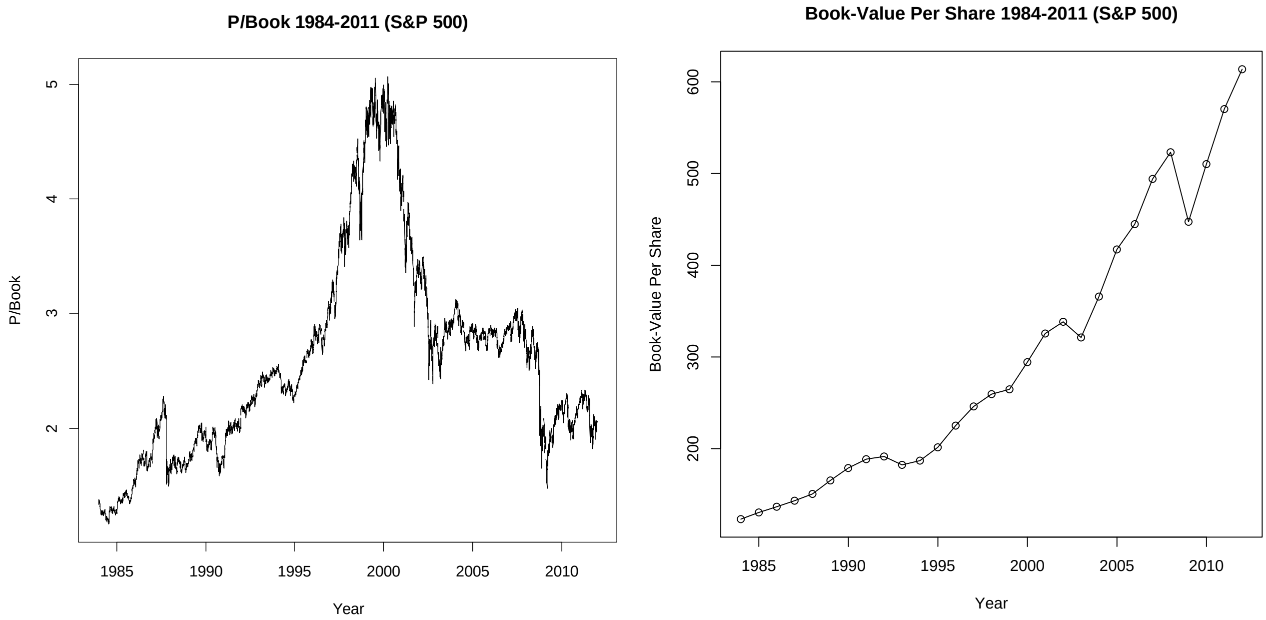


Figure 24: P/Book ratio (left) and book-value per share (right) for the S&P 500 stock market index during the period 1984-2011. The price observations are available daily while the book-value is interpolated from year-end observations which are depicted as circles in the plot to the right. As can be seen the book-value is increasing comparatively steadily and the volatility of the P/Book ratio is caused by the volatile share-price.

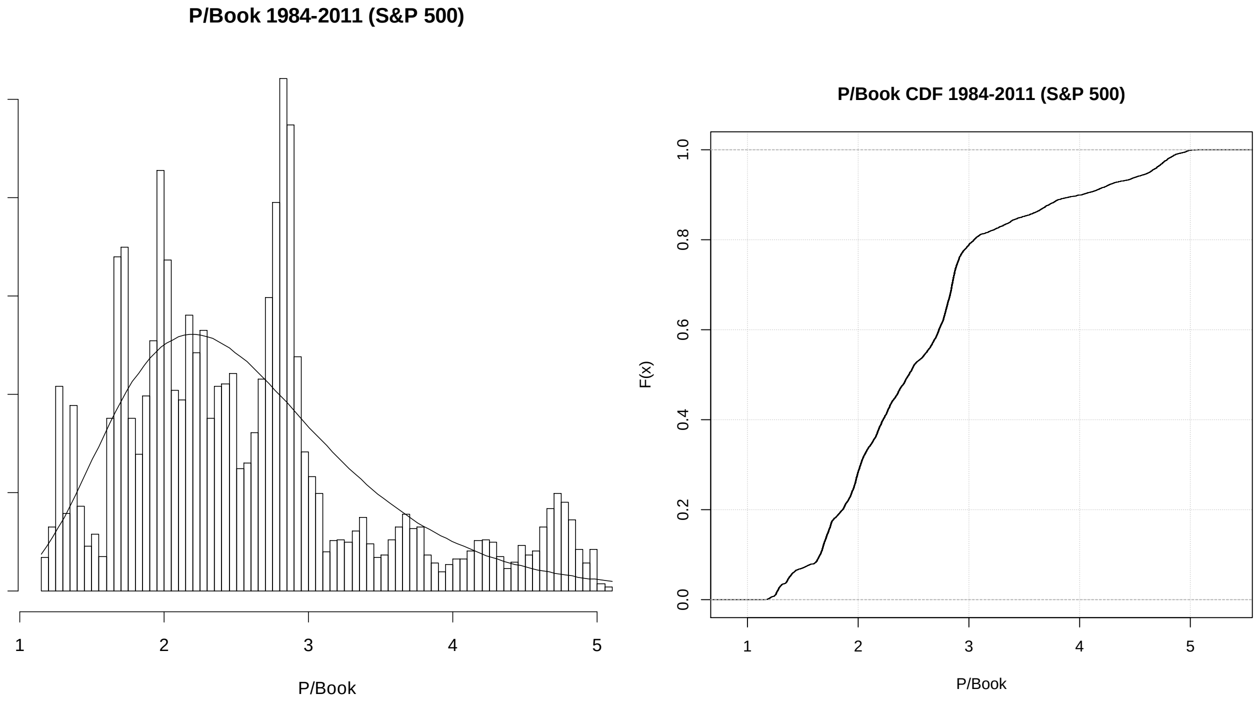


Figure 25: Histogram with fitted log-normal PDF (left) and CDF (right) for the daily P/Book ratios of the S&P 500 stock market index as shown in Figure 24.

Arithmetic Mean	Geometric Mean	Harmonic Mean	Stdev	Min	Max
2.56	2.45	2.32	0.88	1.16	5.07

Table 3: Statistics for the P/Book ratios of the S&P 500 stock market index during the period 1984-2011 as shown in Figure 24.

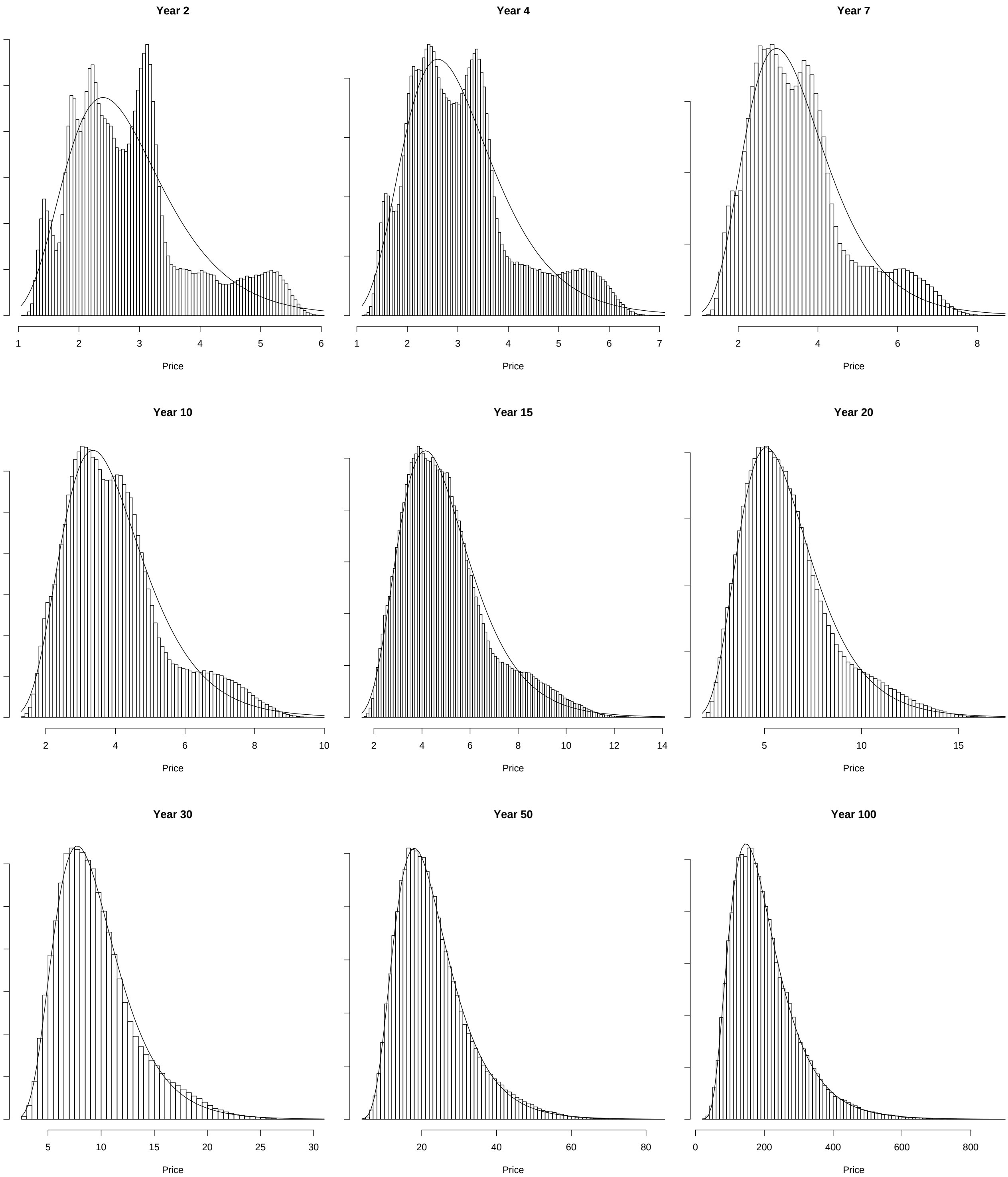


Figure 26: Price distribution for the S&P 500 stock market index when the price is calculated by sampling the P/Book ratio from its historical distribution shown in Figure 25 and multiplying the P/Book ratio with the equity sampled from the probability distribution resulting from the Monte Carlo simulations described in section 6.8. Also shown are the fitted log-normal PDFs which show that the log-normal distribution is approached as the number of years increase.

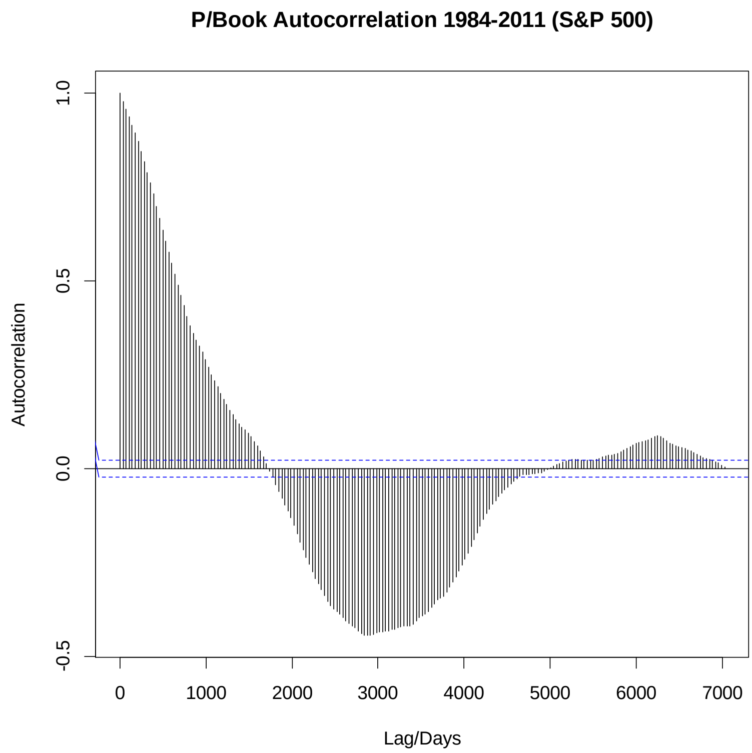
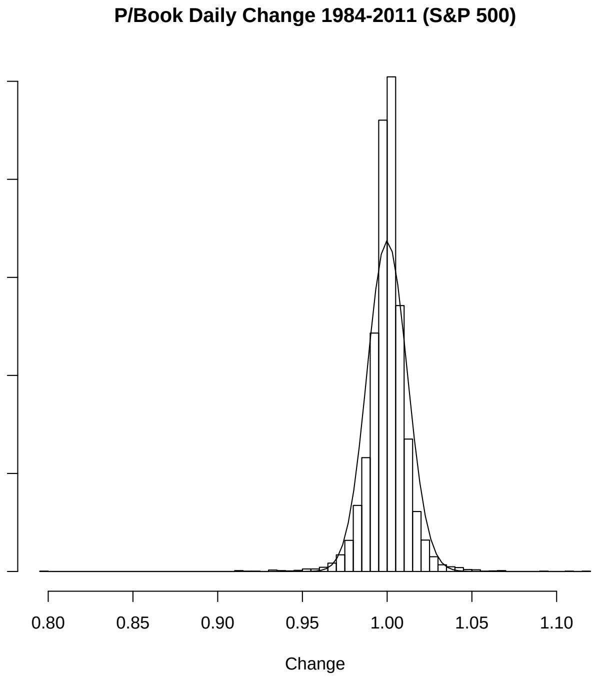


Figure 27: Autocorrelation of the P/Book ratios of the S&P 500 stock market index for the period 1984-2011.



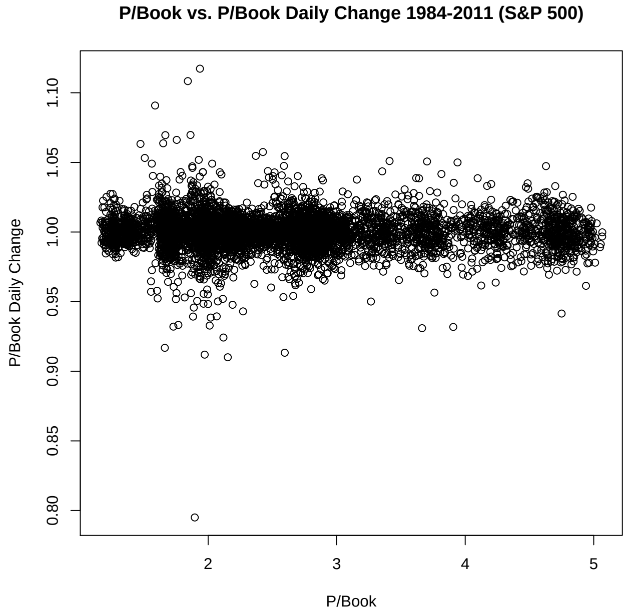
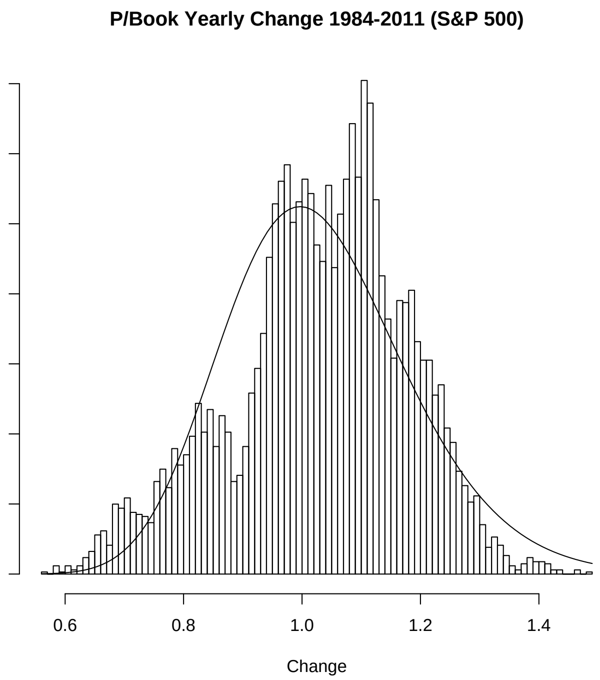


Figure 28: Histogram with fitted log-normal PDF (left) for the <u>daily P/Book changes</u> of the <u>S&P 500</u> stock market index in the period <u>1984-2011</u>, and scatter-plot (right) for the daily P/Book and P/Book changes. Note that the histogram is similar to Figure 1 for the daily price changes because the daily change in P/Book ratio is dominated by the price change as the equity (or book-value) is almost constant on successive days.



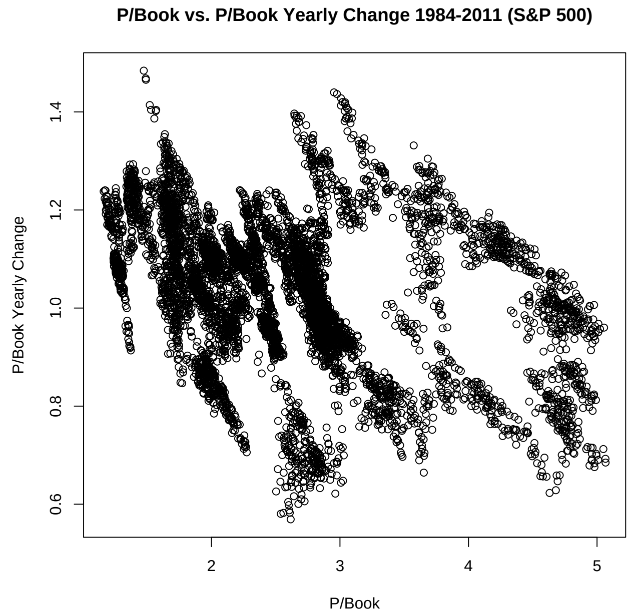


Figure 29: Histogram with fitted log-normal PDF (left) for the <u>yearly P/Book changes</u> of the <u>S&P 500</u> stock market index in the period <u>1984-2011</u>, and scatter-plot (right) for the yearly P/Book and P/Book changes. Note that the histogram is shaped similarly to Figure 2 for the yearly price changes because the yearly change in P/Book ratio is dominated by the price change as the equity (or book-value) changes comparatively little within a year, see Figure 24.

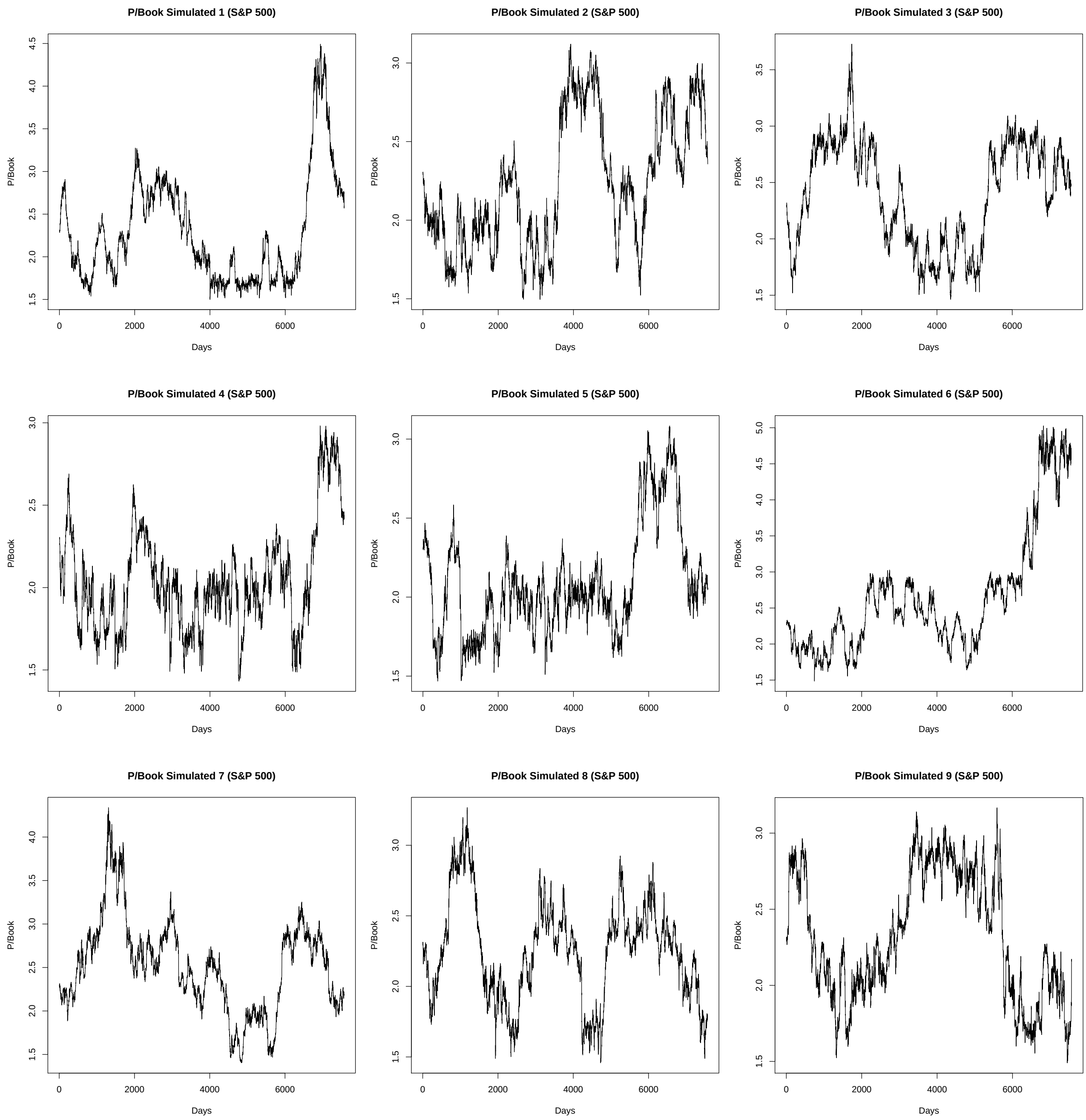


Figure 30: Simulated P/Book for the S&P 500 stock market index 30 years into the future (about 252 trading days per year), as described in section 6.9.4. The starting P/Book is set to 2.3.

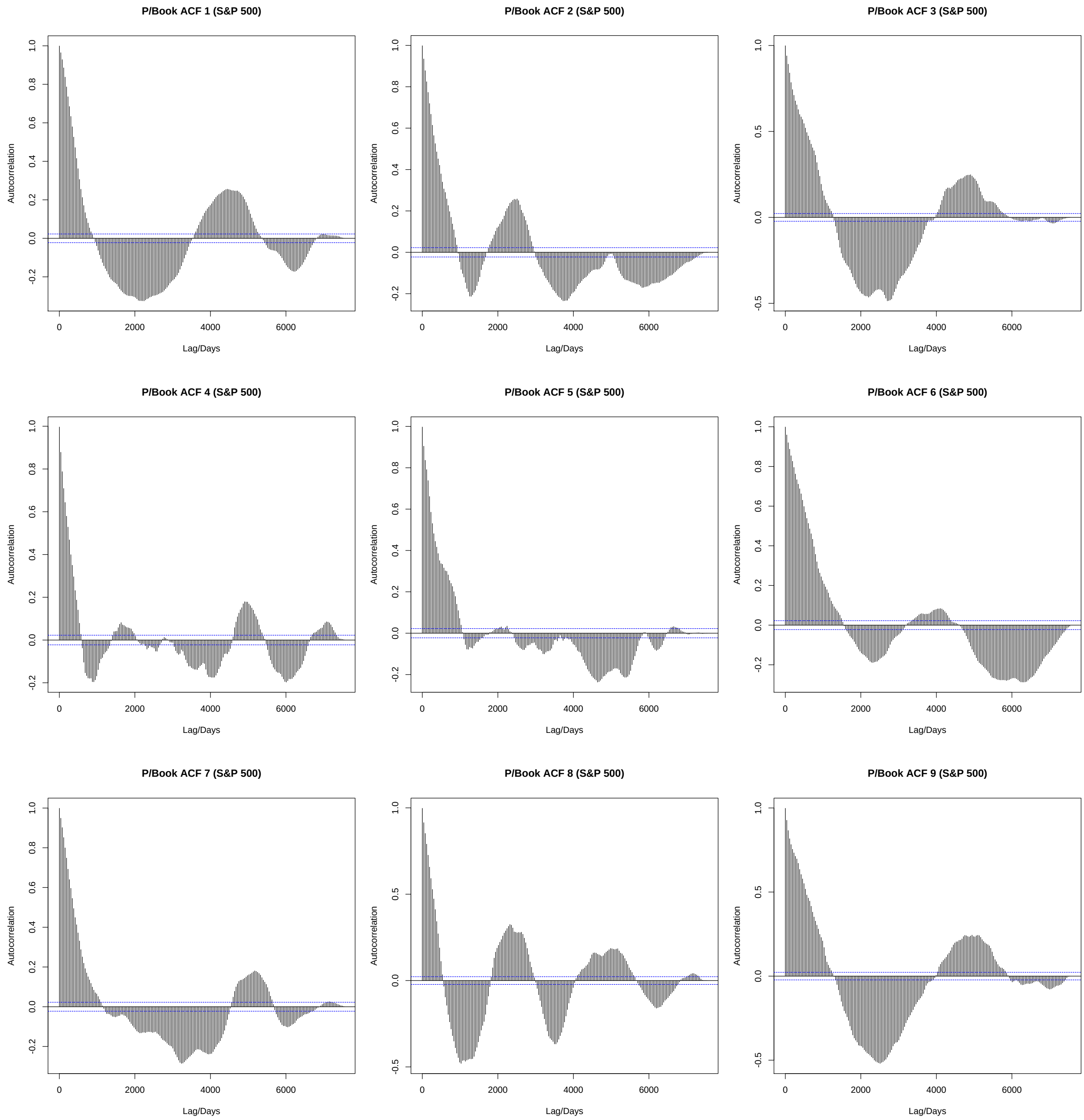


Figure 31: Autocorrelation for the simulated P/Book ratios from Figure 30 for the S&P 500 stock market index.

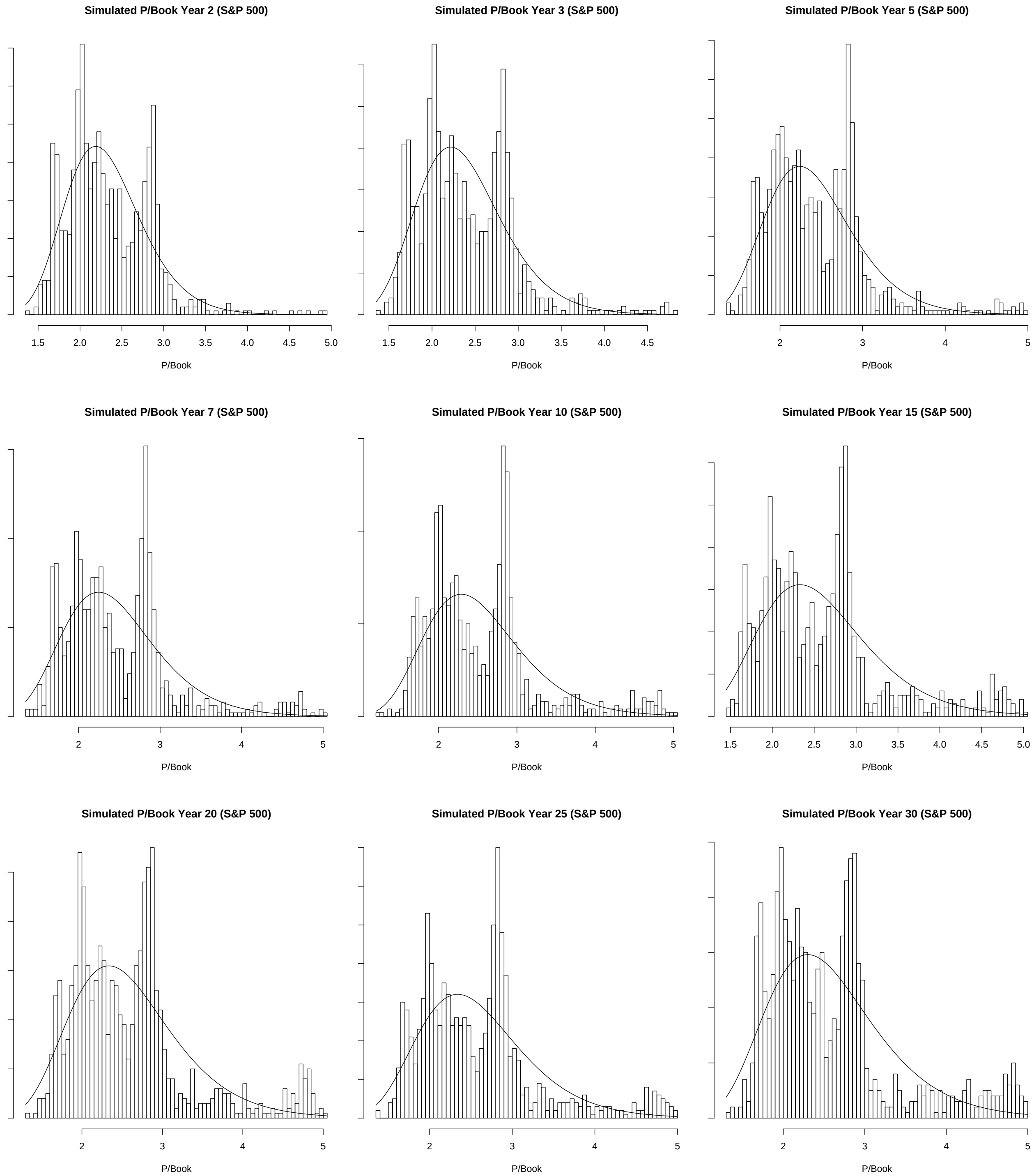


Figure 32: Distribution of P/Book ratios for the S&P 500 stock market index in different years that resulted from the Monte Carlo simulations described in section 6.9.4. Also shown are the fitted log-normal PDFs. The CDFs are shown in Figure 33.

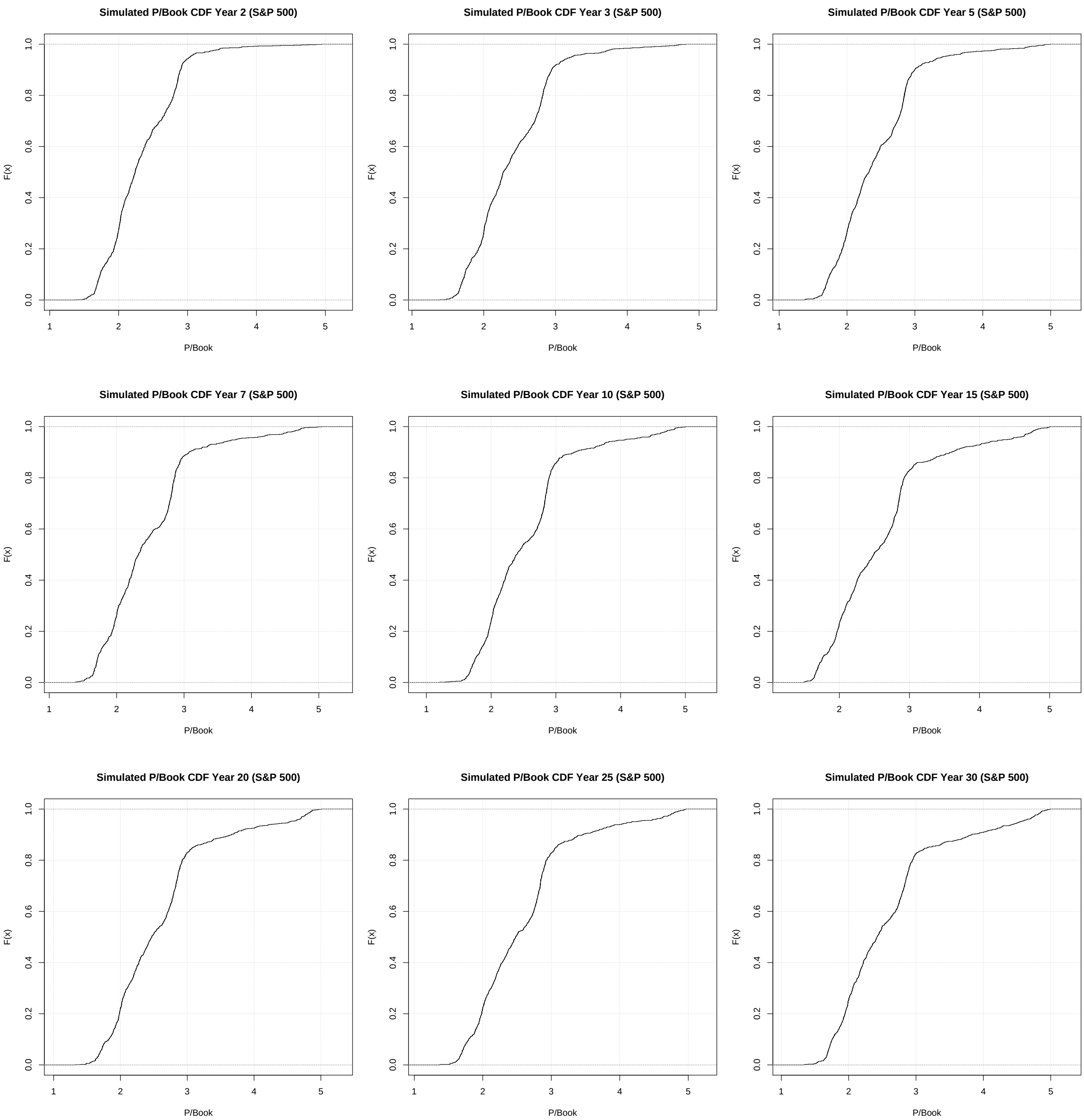


Figure 33: <u>CDF of P/Book</u> ratios for the <u>S&P 500</u> stock market index in different years that resulted from the Monte Carlo simulations described in section 6.9.4. The histograms are shown in Figure 32.

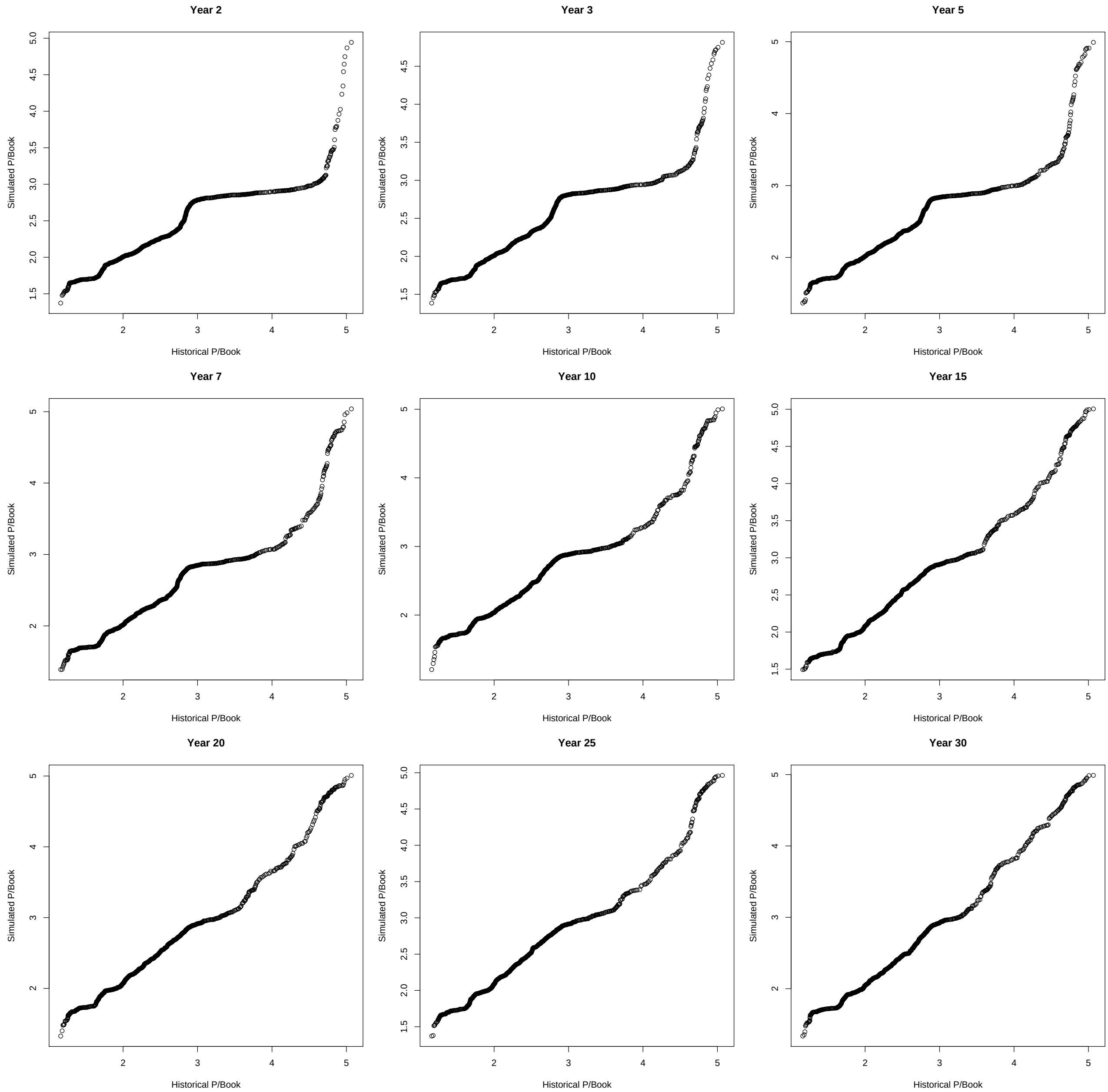


Figure 34: Q-Q plots comparing the distributions of the simulated P/Book ratios from Figure 32 to the historical P/Book ratios from Figure 25.

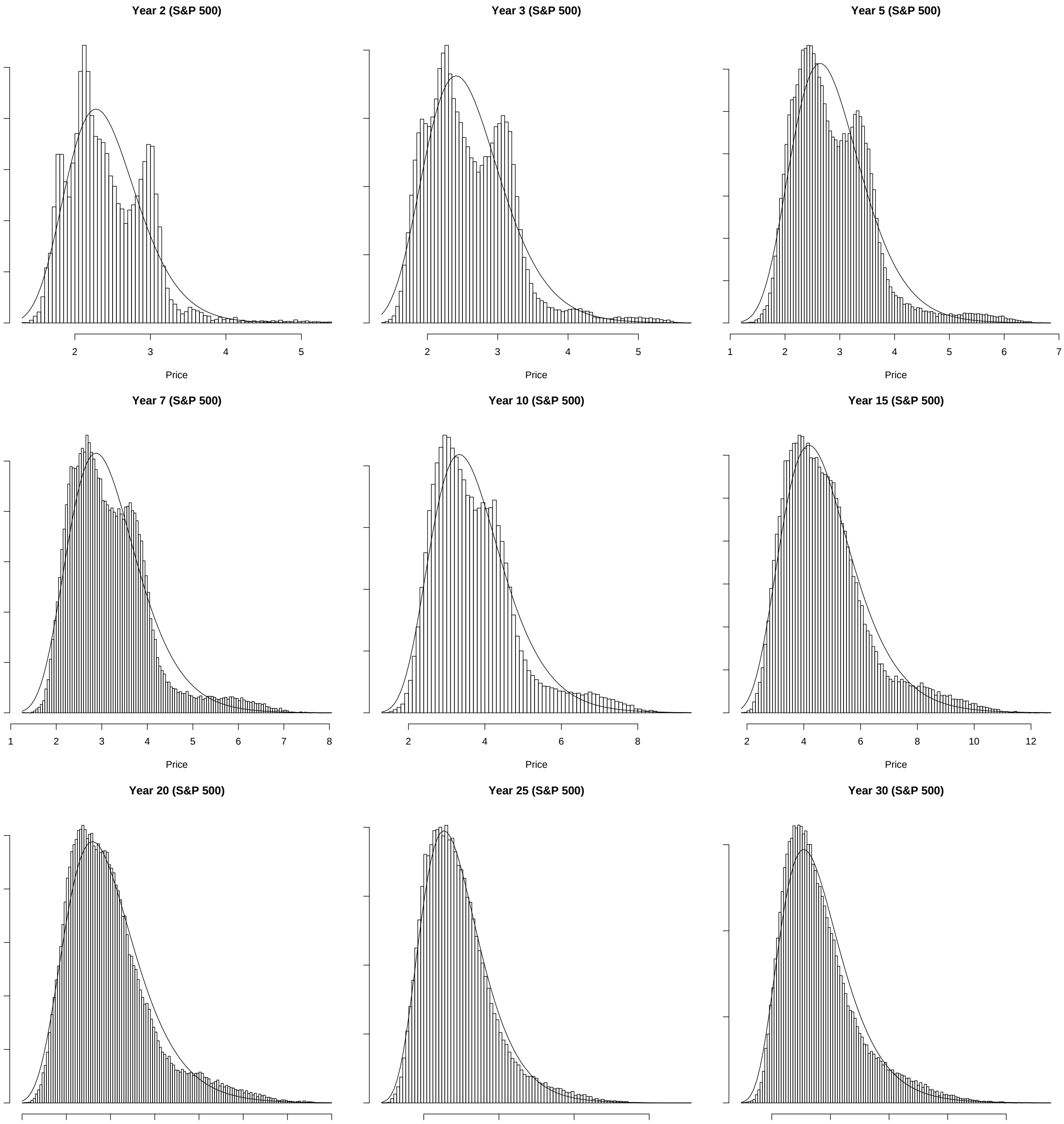


Figure 35: <u>Price distribution</u> for the <u>S&P 500</u> stock market index when the price is calculated by Monte Carlo simulating the P/Book ratio with a starting value of 2.3 as described in section 6.9.4, and multiplying with Monte Carlo simulated equity as described in section 6.8. Also shown are the fitted log-normal PDFs which show that log-normal distributions are approached as the number of years increase.

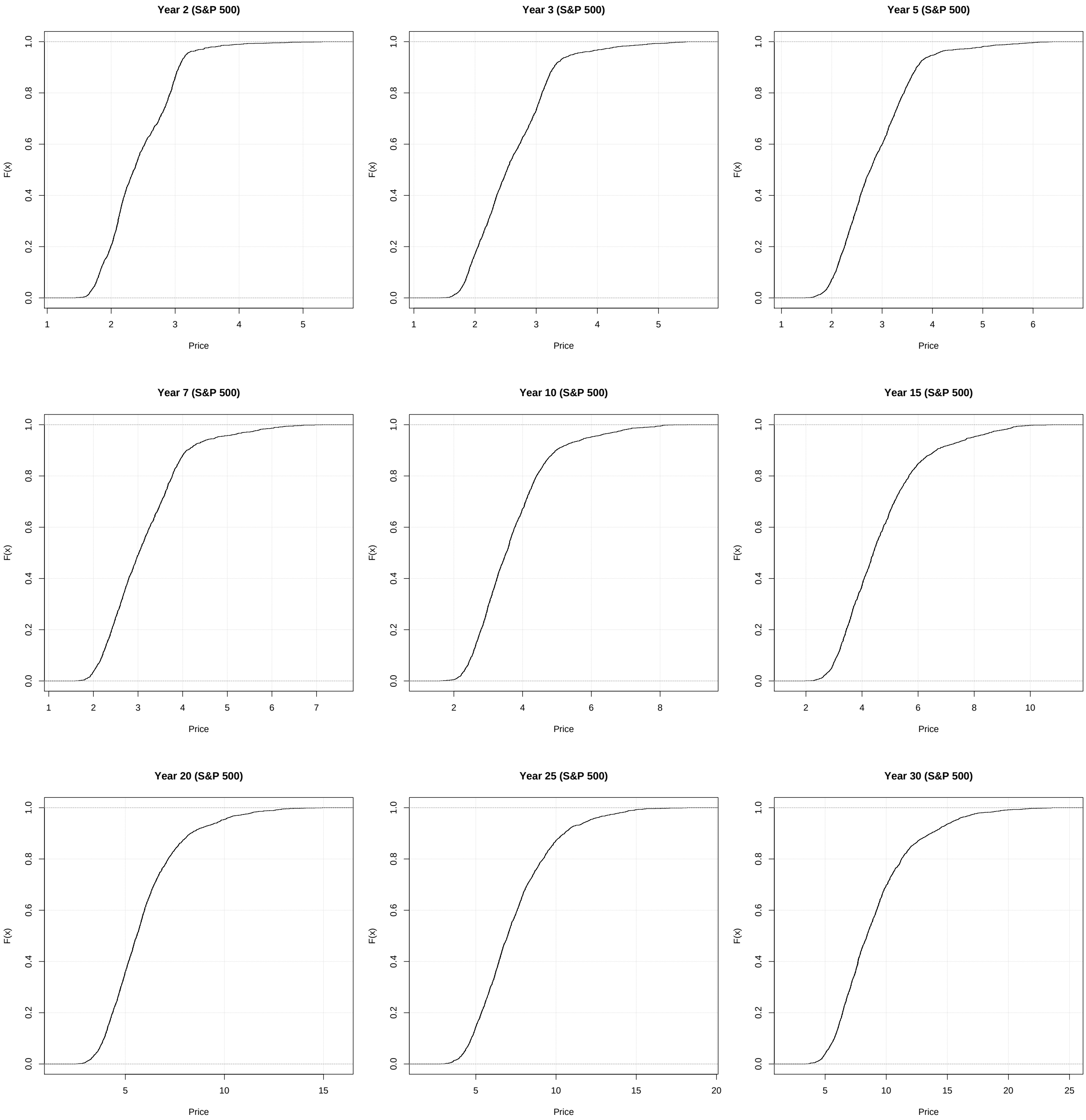


Figure 36: <u>CDF of the prices</u> for the <u>S&P 500</u> stock market index in different years where the price is calculated by Monte Carlo simulating the P/Book ratio as described in section 6.9.4 and multiplying with Monte Carlo simulated equity as described in section 6.8.

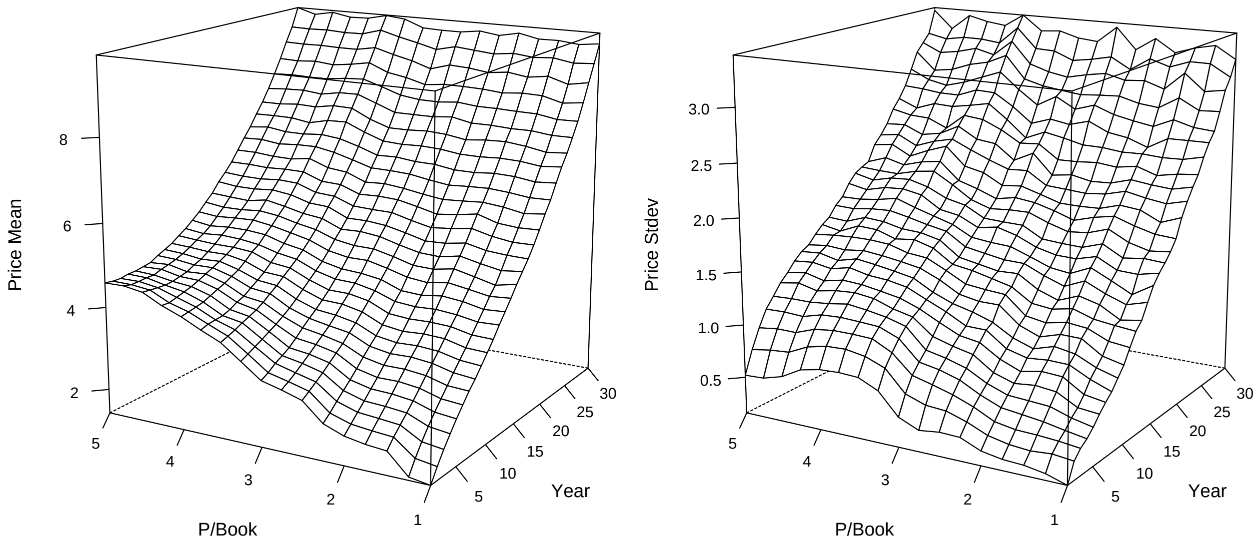


Figure 37: Mean and standard deviation for the future <u>price</u> of the <u>S&P 500</u> stock market index with different starting P/Book ratios. The price is calculated by Monte Carlo simulating the P/Book ratio as described in section 6.9.4 and multiplying with Monte Carlo simulated equity as described in section 6.8. Share buyback and issuance is not taken into account here but is taken into account in Figure 44. Subsets are shown in Table 4 and Table 5.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	1.42	1.99	2.12	2.71	2.99	3.70	4.13	4.53	4.62
	2	1.73	2.21	2.29	2.82	3.08	3.75	4.15	4.49	4.57
	4	2.30	2.56	2.63	3.08	3.32	3.90	4.20	4.50	4.56
	6	2.74	2.94	3.00	3.36	3.58	4.05	4.28	4.57	4.62
	8	3.15	3.34	3.37	3.65	3.84	4.25	4.43	4.66	4.72
	10	3.59	3.73	3.75	3.98	4.16	4.53	4.66	4.84	4.88
	12	4.00	4.14	4.15	4.35	4.50	4.83	4.93	5.10	5.11
	14	4.49	4.54	4.63	4.77	4.90	5.19	5.25	5.36	5.43
	16	5.00	5.02	5.09	5.15	5.31	5.61	5.63	5.71	5.77
	18	5.47	5.57	5.62	5.68	5.80	6.04	6.08	6.13	6.11
	20	6.05	6.09	6.14	6.26	6.32	6.48	6.57	6.56	6.52

Table 4: Subset of the price <u>means</u> from Figure 37.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	0.15	0.25	0.27	0.37	0.41	0.72	0.73	0.56	0.52
	2	0.34	0.35	0.40	0.51	0.58	0.90	0.90	0.75	0.75
	4	0.48	0.53	0.56	0.72	0.83	1.09	1.11	1.06	1.04
	6	0.60	0.70	0.72	0.91	1.02	1.22	1.26	1.25	1.25
	8	0.75	0.89	0.88	1.02	1.14	1.32	1.37	1.37	1.39
	10	0.92	1.01	1.06	1.16	1.26	1.45	1.49	1.48	1.54
	12	1.07	1.16	1.18	1.32	1.46	1.55	1.61	1.65	1.65
	14	1.30	1.31	1.38	1.48	1.57	1.68	1.75	1.74	1.83
	16	1.53	1.53	1.54	1.62	1.68	1.84	1.83	1.90	1.97
	18	1.69	1.74	1.72	1.81	1.90	2.02	2.01	2.03	2.10
	20	1.89	1.93	1.96	2.02	2.10	2.22	2.20	2.14	2.22

Table 5: Subset of the price <u>standard deviations</u> from Figure 37.

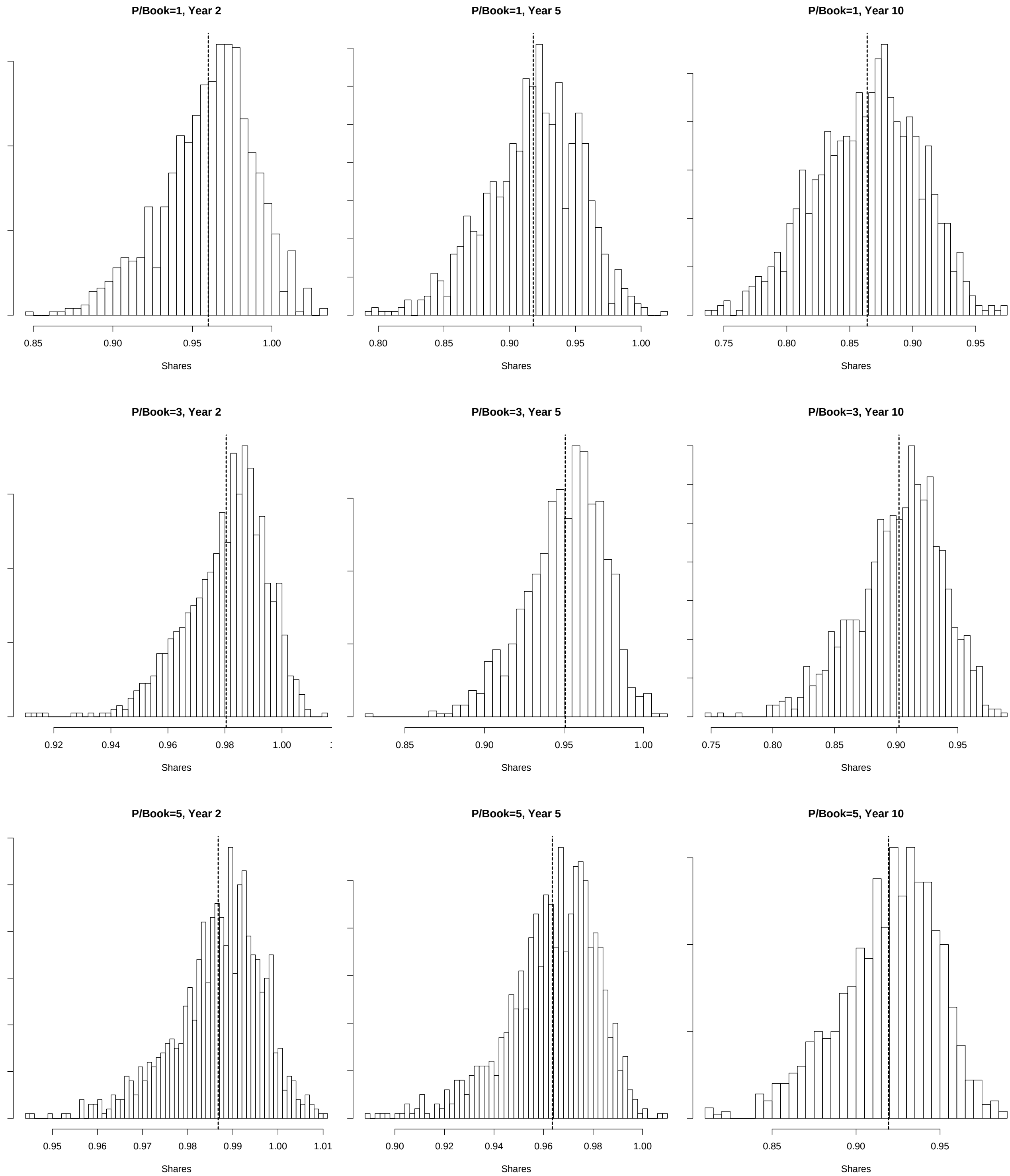


Figure 38: Shares outstanding for the S&P 500 stock market index for different starting P/Book ratios and after the given number of years. The means are shown as dotted lines. The number of shares is normalized to one at the beginning of the first year and the changes to the number of shares from share buyback and issuance are Monte Carlo simulated as described in section 6.10.

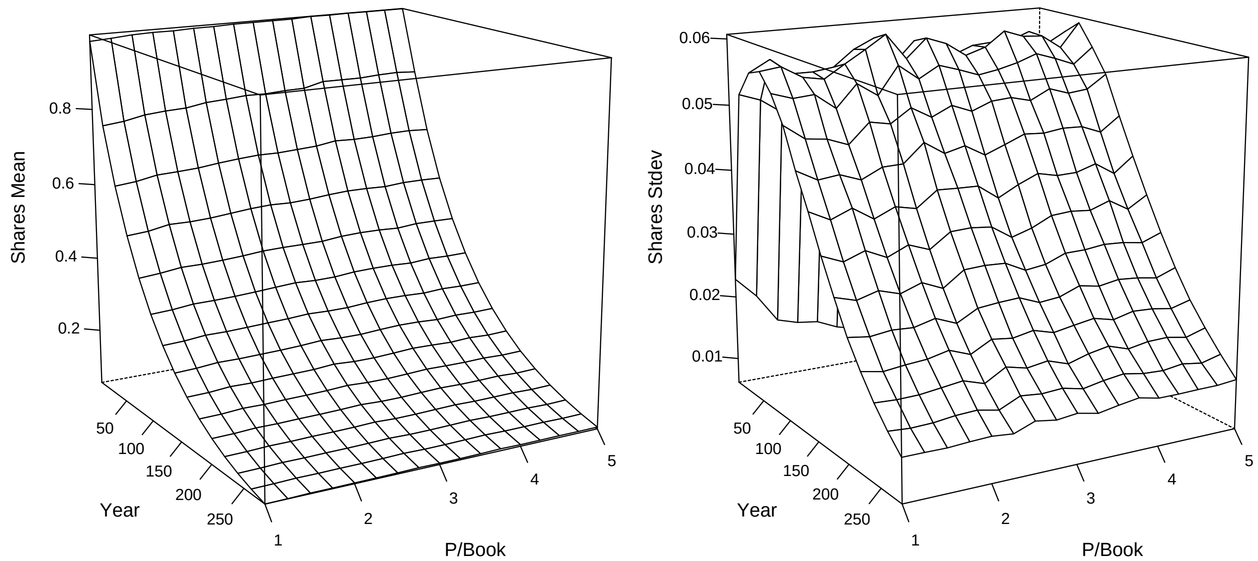


Figure 39: Mean and standard deviation for the number of <u>shares outstanding</u> for the <u>S&P 500</u> stock market index for different starting P/Book ratios and 300 years of simulations. The number of shares is normalized to one at the beginning of the first year and the changes to the number of shares from share buyback and issuance are Monte Carlo simulated as described in section 6.10. A subset is shown in Figure 40.

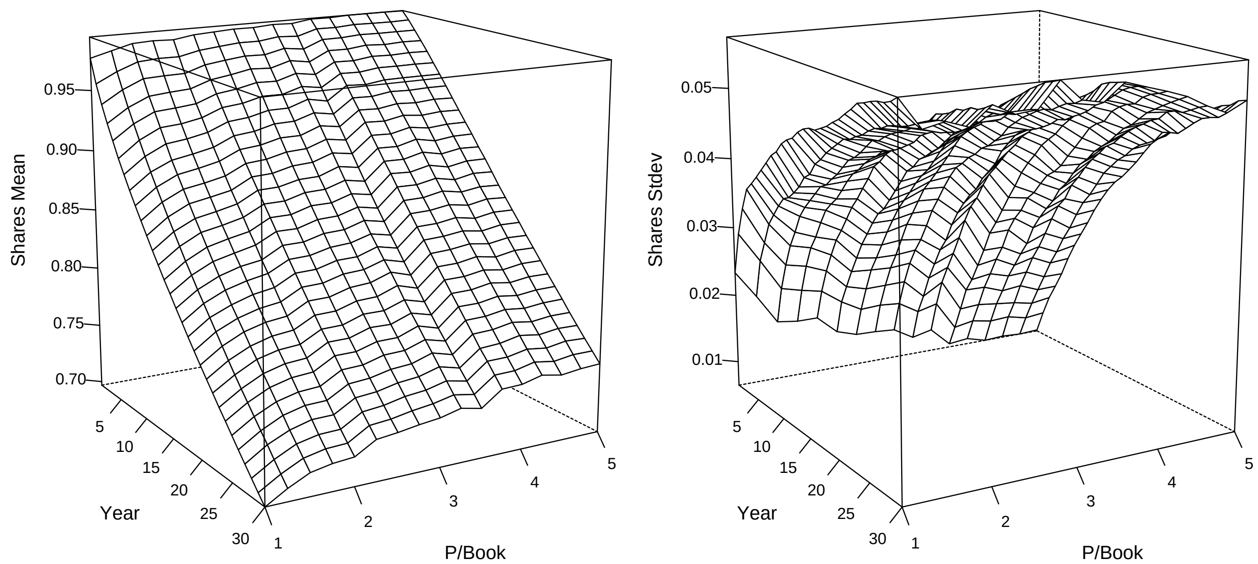


Figure 40: Mean and standard deviation for the number of <u>shares outstanding</u> for the <u>S&P 500</u> stock market index for different starting P/Book ratios and 30 years of simulations. The number of shares is normalized to one at the beginning of the first year and the changes to the number of shares from share buyback and issuance are Monte Carlo simulated as described in section 6.10. This is a subset of Figure 39.

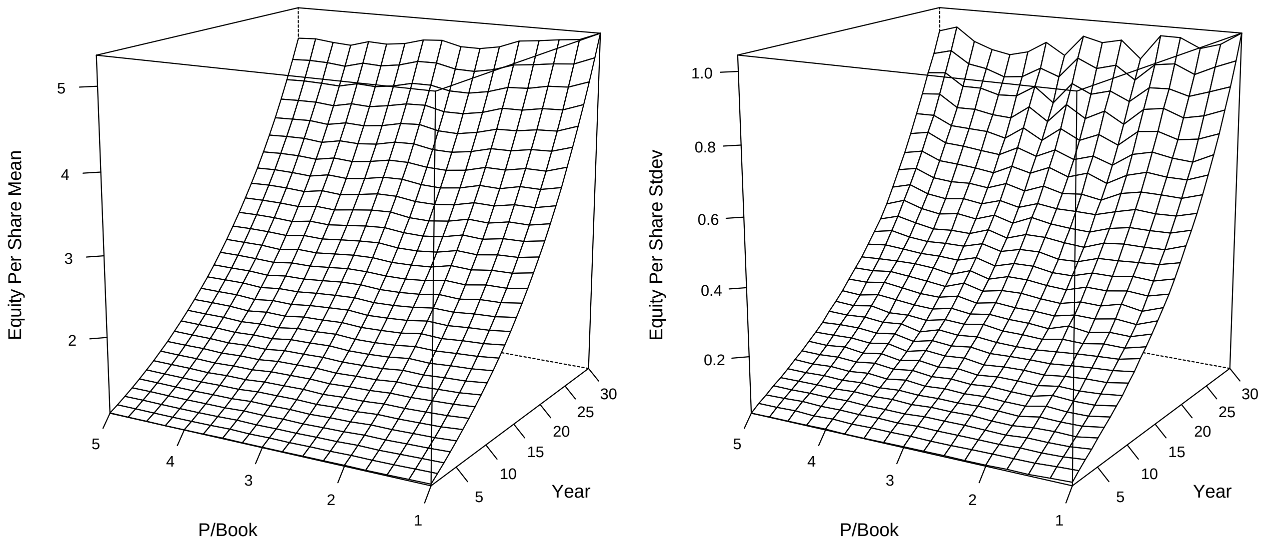


Figure 41: Mean and standard deviation for the <u>equity per share</u> for the <u>S&P 500</u> stock market index for different starting P/Book ratios and 30 years of simulations. These result from Monte Carlo simulating the equity as described in section 6.8 and dividing by the simulated number of shares as described in section 6.10.

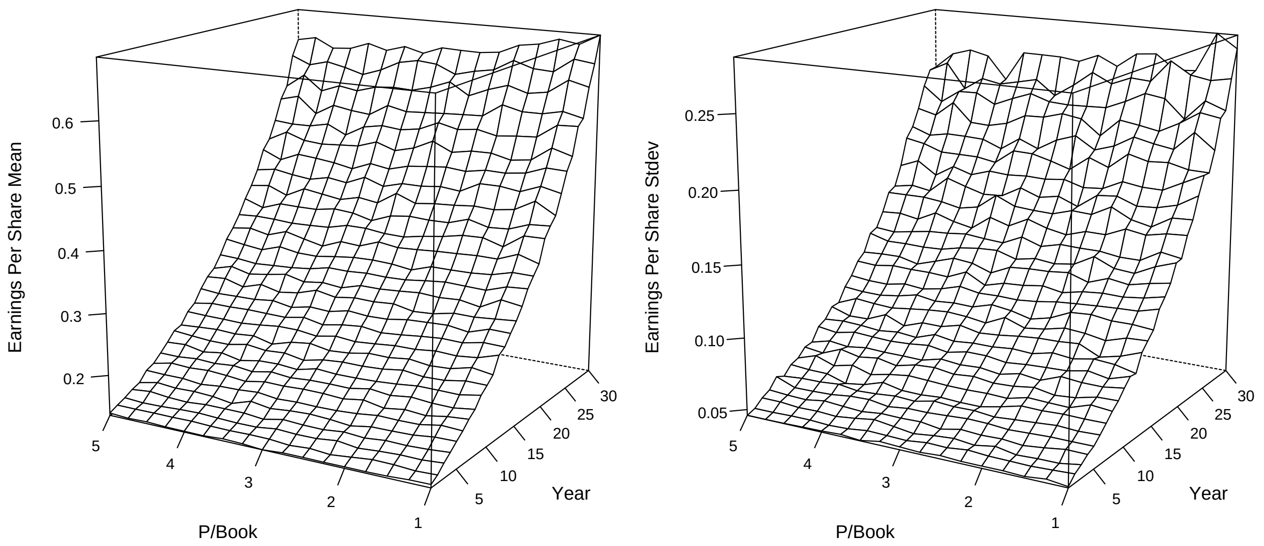
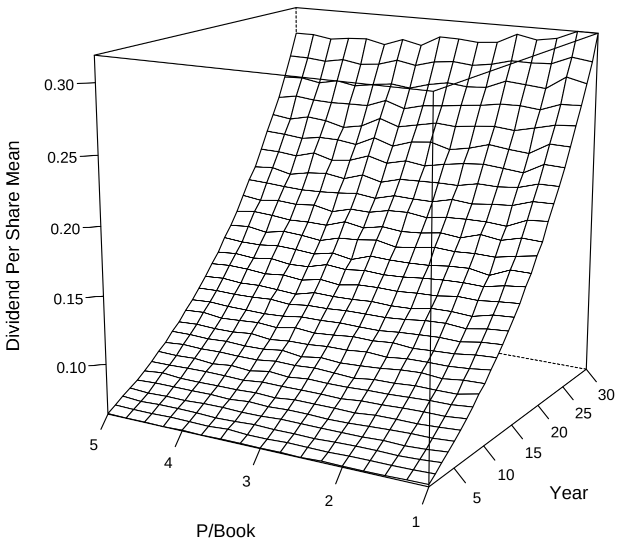


Figure 42: Mean and standard deviation for the <u>earnings per share</u> for the <u>S&P 500</u> stock market index for different starting P/Book ratios and 30 years of simulations. These result from Monte Carlo simulating the earnings as described in section 6.4 and dividing by the simulated number of shares as described in section 6.10.



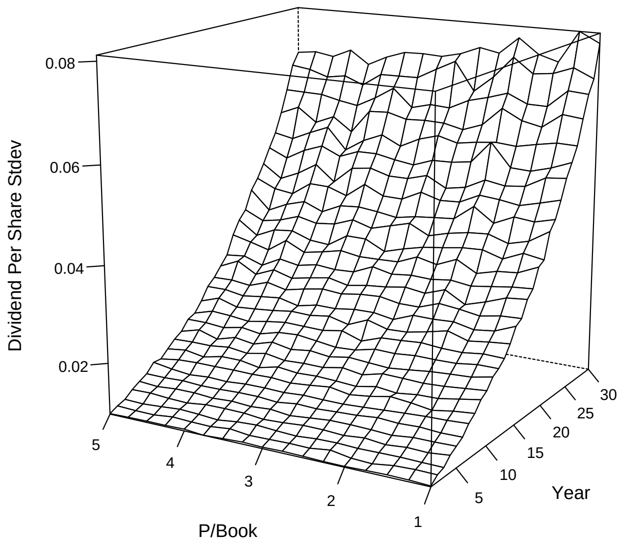


Figure 43: Mean and standard deviation for the <u>dividends per share</u> for the <u>S&P 500</u> stock market index for different starting P/Book ratios and 30 years of simulations. These result from Monte Carlo simulating the dividends as described in section 6.5 and dividing by the simulated number of shares as described in section 6.10.

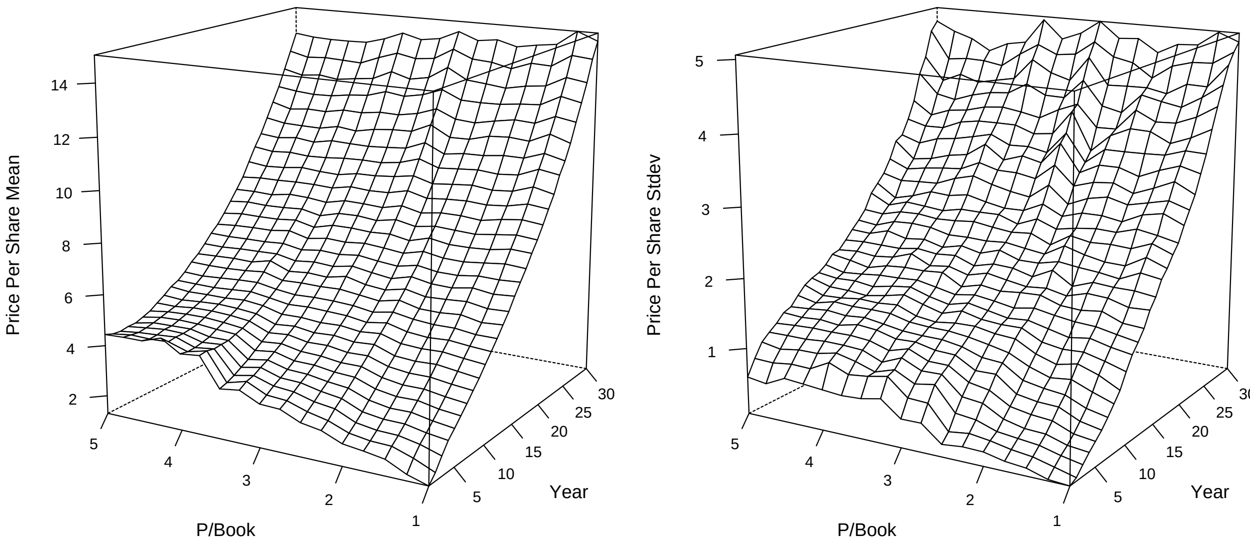


Figure 44: Mean and standard deviation for the future price per share of the S&P 500 stock market index with different starting P/Book ratios. The price per share is Monte Carlo simulated as described in section 6.10.4. Compare to Figure 37 which does not take into account changes in the number of shares through buybacks and issuance. Subsets are shown in Table 6 and Table 7.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	1.29	1.92	2.14	2.59	2.97	3.23	4.27	4.48	4.44
	2	1.61	2.23	2.37	2.85	3.16	3.32	4.21	4.42	4.33
	4	2.46	2.74	2.86	3.32	3.52	3.62	4.30	4.52	4.35
	6	3.05	3.31	3.42	3.76	3.89	3.99	4.53	4.62	4.46
	8	3.69	3.89	3.98	4.22	4.33	4.41	4.82	4.84	4.77
	10	4.38	4.56	4.58	4.75	4.82	4.88	5.17	5.12	5.18
	12	5.08	5.19	5.25	5.25	5.41	5.39	5.63	5.55	5.54
	14	5.84	5.95	5.86	5.87	6.14	6.03	6.26	6.11	6.06
	16	6.71	6.69	6.59	6.56	6.75	6.70	6.74	6.68	6.69
	18	7.57	7.47	7.43	7.34	7.45	7.49	7.47	7.39	7.31
	20	8.44	8.43	8.35	8.23	8.23	8.28	8.19	8.09	8.11

Table 6: Subset of the price per share means from Figure 44.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	0.05	0.16	0.25	0.23	0.46	0.62	0.63	0.68	0.59
	2	0.10	0.25	0.34	0.34	0.70	0.88	0.96	0.83	0.80
	4	0.32	0.44	0.53	0.72	0.96	0.98	1.11	1.09	1.08
	6	0.50	0.70	0.76	0.97	1.15	1.20	1.30	1.28	1.27
	8	0.75	0.97	1.03	1.23	1.32	1.35	1.42	1.44	1.43
	10	1.08	1.24	1.27	1.42	1.51	1.50	1.53	1.51	1.60
	12	1.38	1.50	1.55	1.56	1.65	1.66	1.70	1.67	1.71
	14	1.73	1.80	1.73	1.77	1.87	1.88	1.97	1.84	1.85
	16	2.07	1.98	2.05	2.02	2.12	2.09	2.10	2.14	2.08
	18	2.32	2.32	2.32	2.20	2.30	2.37	2.33	2.35	2.30
	20	2.65	2.72	2.63	2.53	2.54	2.58	2.60	2.52	2.63

Table 7: Subset of the price per share standard deviations from Figure 44.

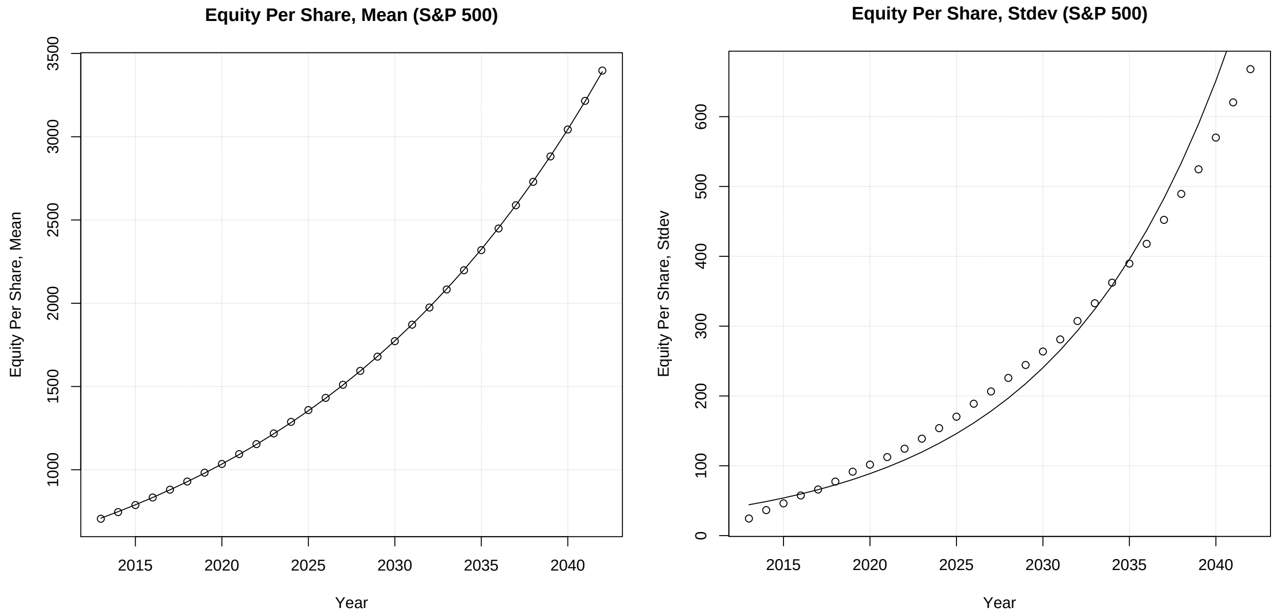


Figure 45: <u>Mean and standard deviation</u> for the <u>equity per share</u> of the <u>S&P 500</u> stock market index. The start equity per share was USD 666.97 on December 31, 2012. The future equity per share is Monte Carlo simulated as described in section 6.10.

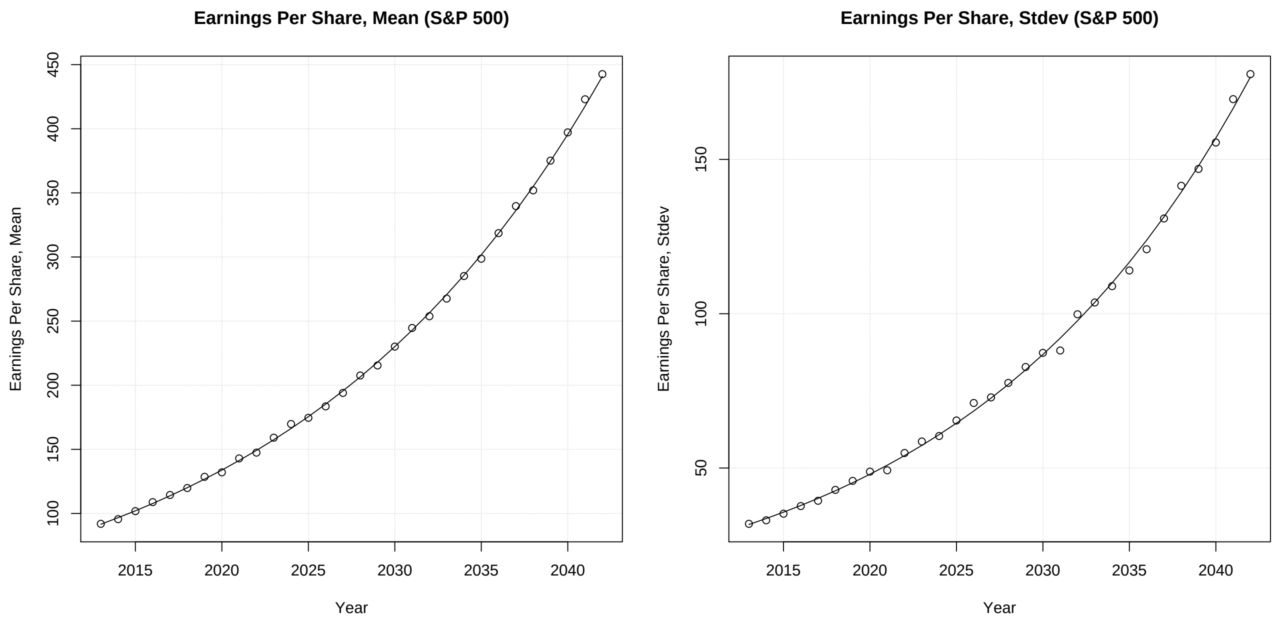


Figure 46: <u>Mean and standard deviation</u> for the <u>earnings per share</u> of the <u>S&P 500</u> stock market index. The start equity per share was USD 666.97 on December 31, 2012. The future earnings per share is Monte Carlo simulated as described in section 6.10.

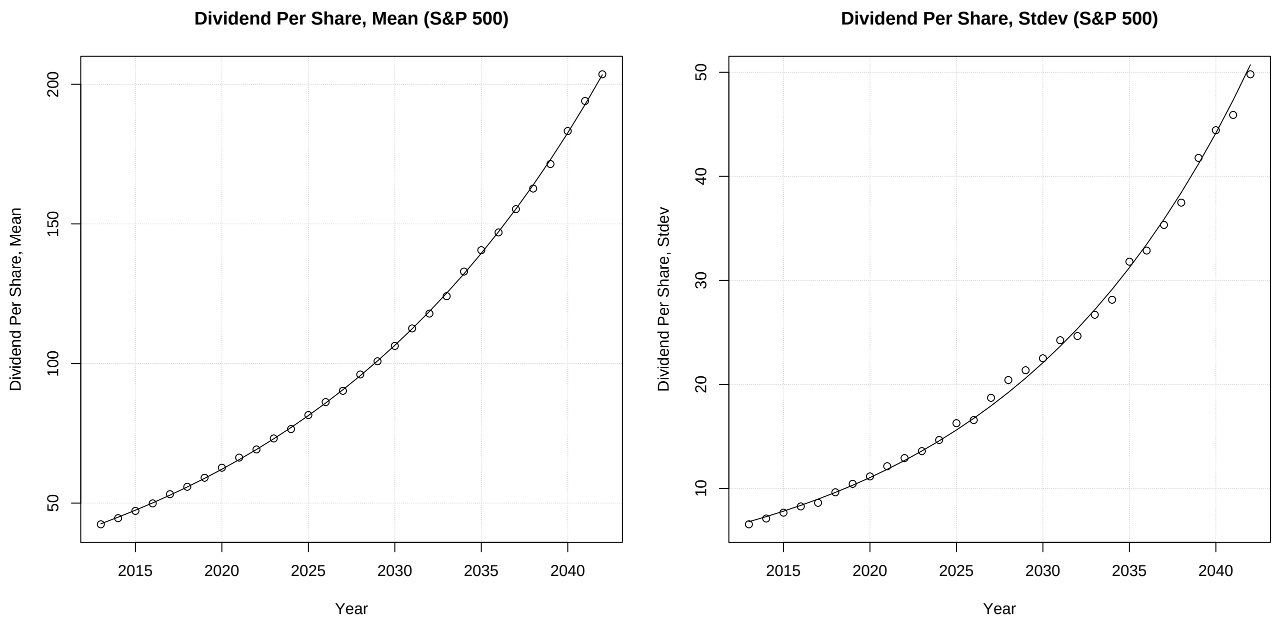


Figure 47: Mean and standard deviation for the dividend per share of the S&P 500 stock market index. The start equity per share was USD 666.97 on December 31, 2012. The future dividend per share is Monte Carlo simulated as described in section 6.10.

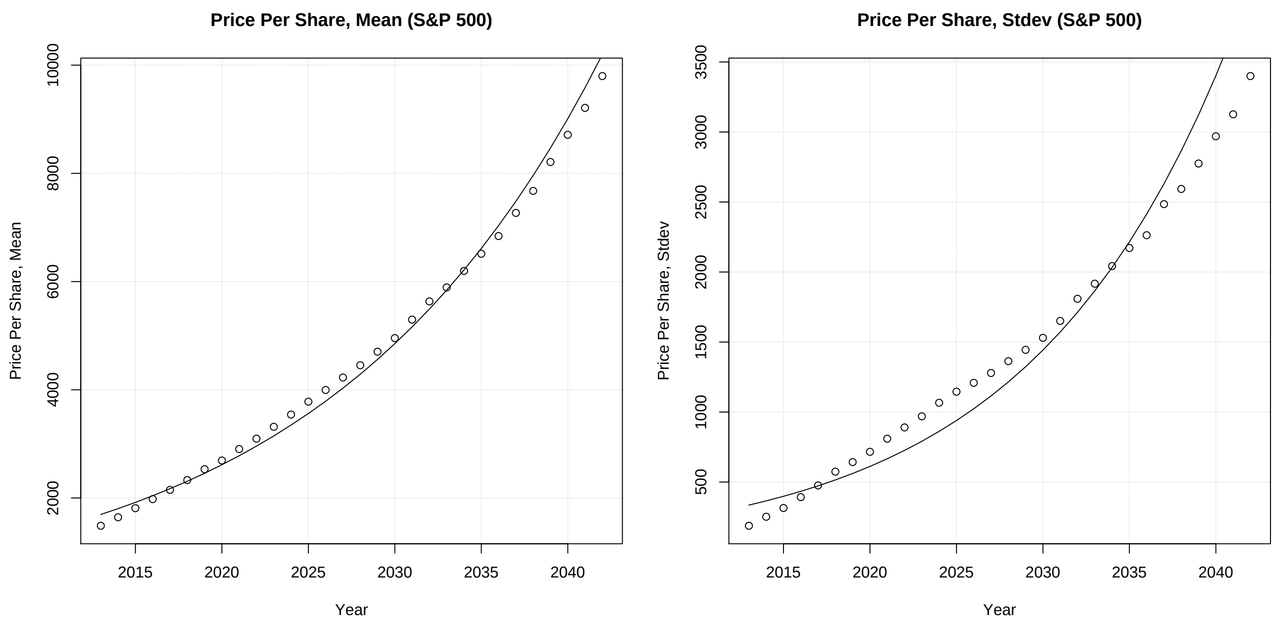


Figure 48: Mean and standard deviation for the price per share of the S&P 500 stock market index. The start equity per share was USD 666.97 on December 31, 2012. The future price per share is Monte Carlo simulated as described in section 6.10.

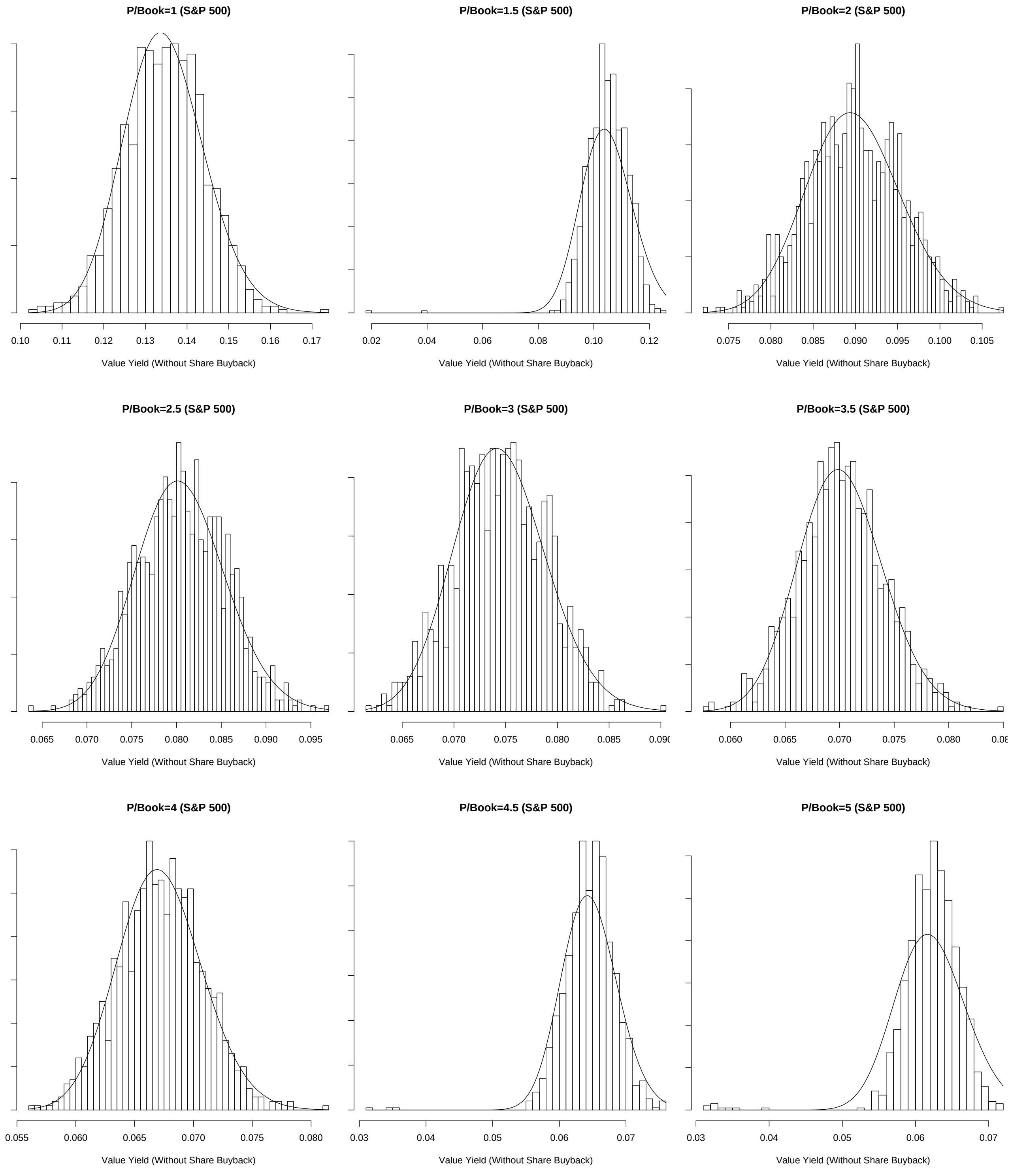


Figure 49: Value yield distributions for eternal shareholders of the S&P 500 stock market index with different starting P/Book ratios. These are without share buyback and issuance so there is no change in the number of shares and the payout consists entirely of dividends. The outliers are possibly caused by optimization errors when finding the value yields. Also shown are the fitted log-normal PDFs. The Monte Carlo simulations resulting in these plots are described in section 6.11. The Q-Q plots are shown in Figure 50 and Figure 51.

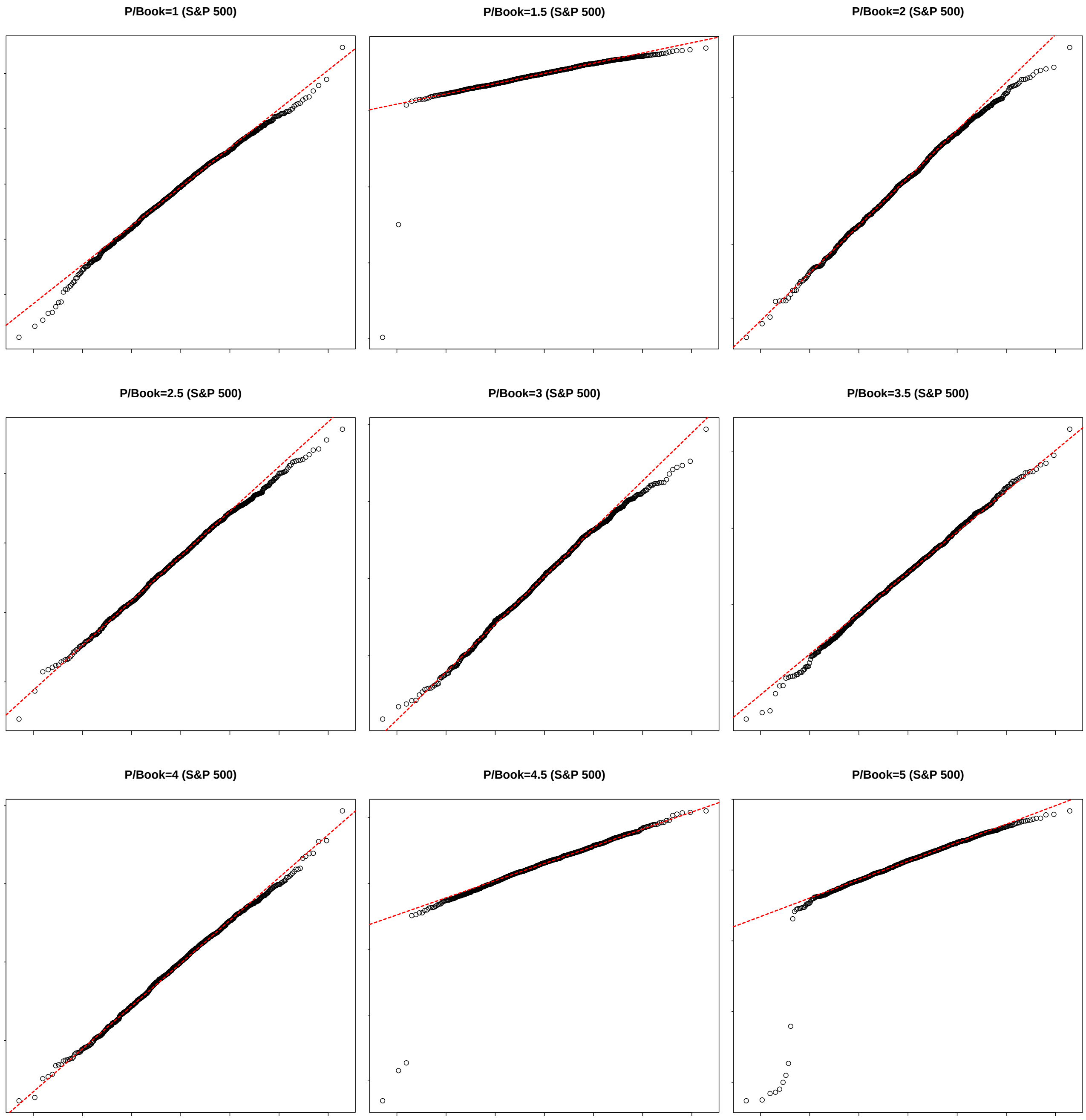


Figure 50: Q-Q log-normal plots for the value yield to eternal shareholders of the S&P 500 stock market index with different starting P/Book ratios. These are without share buyback and issuance so there is no change in the number of shares and the payout consists entirely of dividends. The outliers are possibly caused by optimization errors when finding the value yields. The Monte Carlo simulations resulting in these plots are described in section 6.11. The x-axes are theoretical quantiles and the y-axes are sample quantiles.

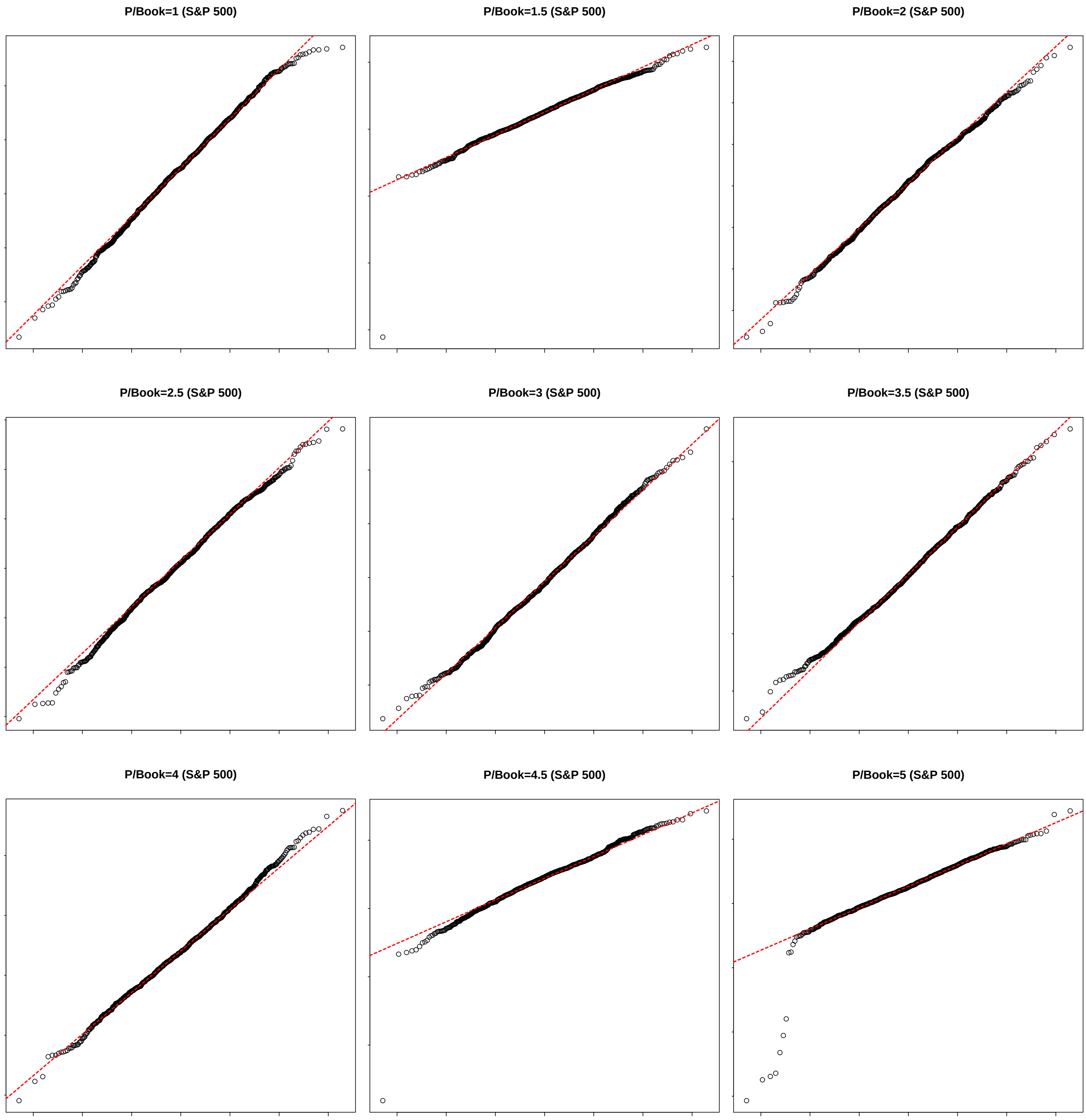


Figure 51: Q-Q normal plots for the value yield to eternal shareholders of the S&P 500 stock market index with different starting P/Book ratios. These are without share buyback and issuance so there is no change in the number of shares and the payout consists entirely of dividends. The outliers are possibly caused by optimization errors when finding the value yields. The Monte Carlo simulations resulting in these plots are described in section 6.11. The x-axes are theoretical quantiles and the y-axes are sample quantiles.

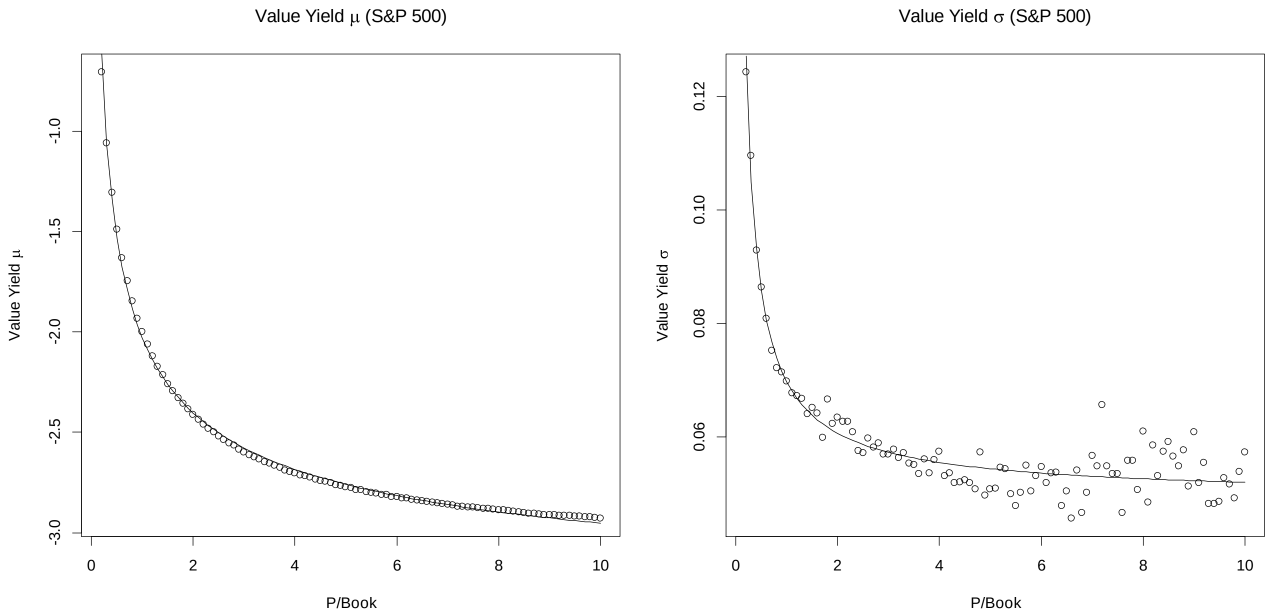


Figure 52: <u>Log-normal</u> probability distribution parameters <u>μ and σ</u> for the <u>value yield</u> to <u>eternal shareholders</u> of the <u>S&P 500</u> stock market index, resulting from a series of Monte Carlo runs with different starting P/Book ratios as described in section 6.11. The small circles are the results of the Monte Carlo simulations and the fitted curves are given in Eq. 6-46 and Eq. 6-47. These are <u>without share buyback and issuance</u> so there is no change in the number of shares and the payout consists entirely of dividends.

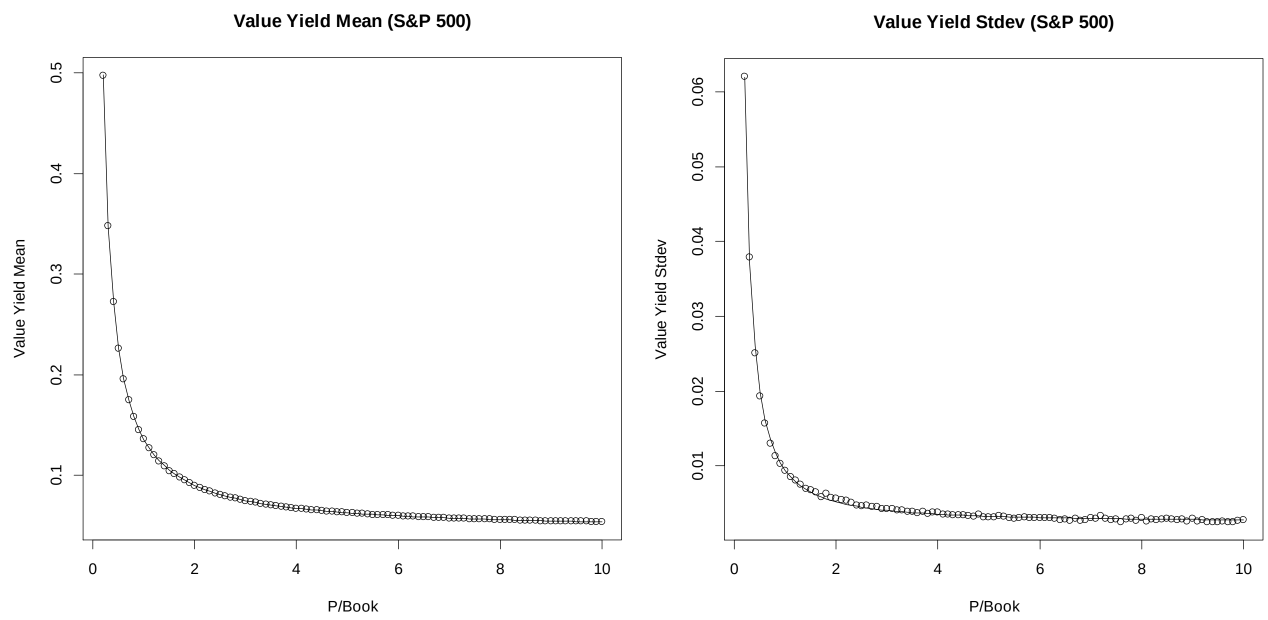


Figure 53: <u>Mean and standard deviation</u> for the <u>value yield</u> to <u>eternal shareholders</u> of the <u>S&P 500</u> stock market index, resulting from a series of Monte Carlo runs with different P/Book ratios as described in section 6.11. The small circles are the results of the Monte Carlo simulations and the fitted curves are given in Eq. 6-48 and Eq. 6-49. These are <u>without share buyback and issuance</u> so there is no change in the number of shares and the payout consists entirely of dividends.

		Year	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992
Value Yield	Mean	Min	10.9%	11.3%	10.2%	9.5%	8.4%	9.6%	8.8%	9.0%	8.5%	8.2%
		Max	12.5%	12.3%	11.9%	10.7%	10.4%	10.5%	10.2%	10.1%	9.9%	8.7%
	Stdev	Min	0.6%	0.7%	0.6%	0.5%	0.4%	0.5%	0.5%	0.5%	0.4%	0.4%
		Max	0.8%	0.8%	0.7%	0.6%	0.6%	0.6%	0.6%	0.6%	0.6%	0.5%

		Year	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Value Yield	Mean	Min	7.9%	8.0%	7.5%	7.2%	6.7%	6.3%	6.1%	6.2%	6.6%	6.9%
		Max	8.3%	8.4%	8.6%	7.9%	7.5%	6.9%	6.5%	6.6%	7.5%	8.2%
	Stdev	Min	0.4%	0.4%	0.4%	0.3%	0.3%	0.3%	0.3%	0.3%	0.3%	0.3%
		Max	0.4%	0.4%	0.4%	0.4%	0.4%	0.3%	0.3%	0.3%	0.4%	0.4%

		Year	2003	2004	2005	2006	2007	2008	2009	2010	2011
Value Yield	Mean	Min	7.2%	7.3%	7.5%	7.4%	7.4%	7.5%	8.2%	8.3%	8.4%
		Max	8.3%	7.7%	7.9%	7.9%	7.8%	10.3%	10.8%	9.2%	9.3%
	Stdev	Min	0.3%	0.4%	0.4%	0.4%	0.4%	0.4%	0.4%	0.4%	0.4%
		Max	0.4%	0.4%	0.4%	0.4%	0.4%	0.6%	0.6%	0.5%	0.5%

Table 8: Mean and standard deviation for the value yield to eternal shareholders of the S&P 500 stock market index, calculated using Eq. 6-48 and Eq. 6-49 with each year's minimum and maximum P/Book ratios from Figure 24.

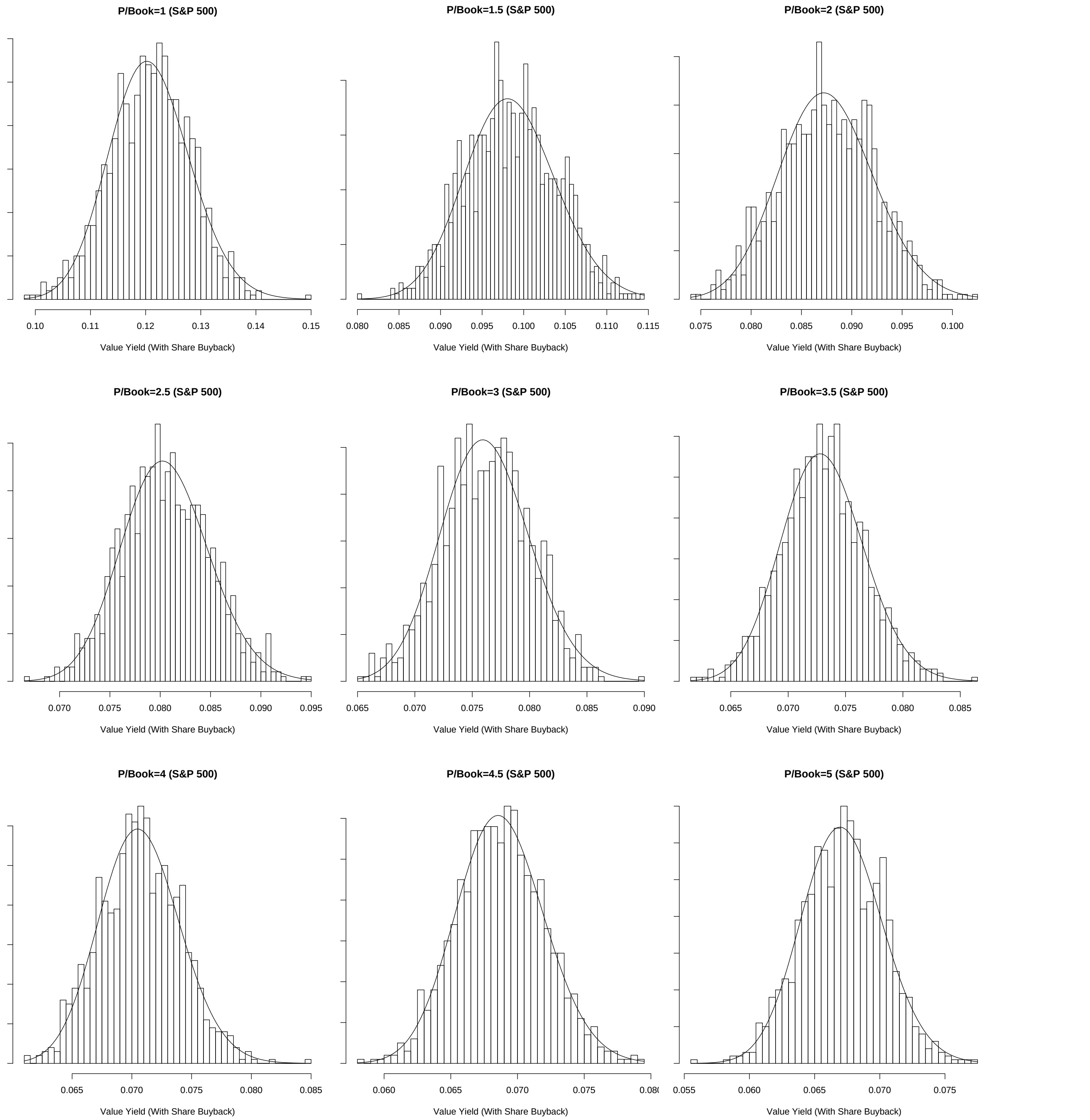


Figure 54: Value yield distributions for eternal shareholders of the S&P 500 stock market index with different starting P/Book ratios. These are with share buyback and issuance which affects the number of shares and hence the dividend per share. Also shown are the fitted log-normal PDFs. The Monte Carlo simulations resulting in these plots are described in section 6.11. The Q-Q plots are shown in Figure 55 and Figure 56.

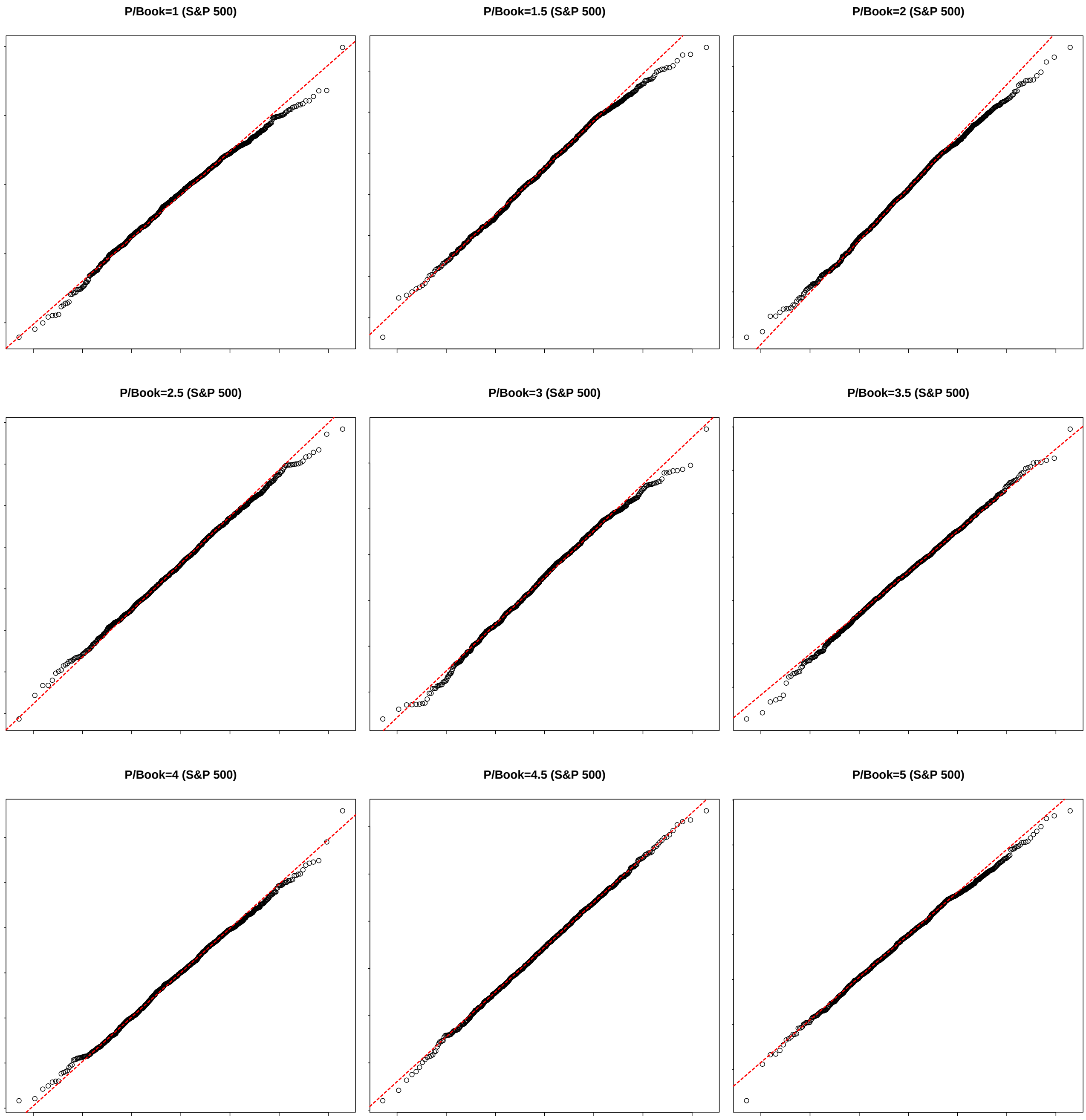


Figure 55: Q-Q log-normal plots for the value yield to eternal shareholders of the S&P 500 stock market index with different starting P/Book ratios. These are with share buyback and issuance which affects the number of shares and hence the dividend per share. The Monte Carlo simulations resulting in these plots are described in section 6.11. The x-axes are theoretical quantiles and the y-axes are sample quantiles.

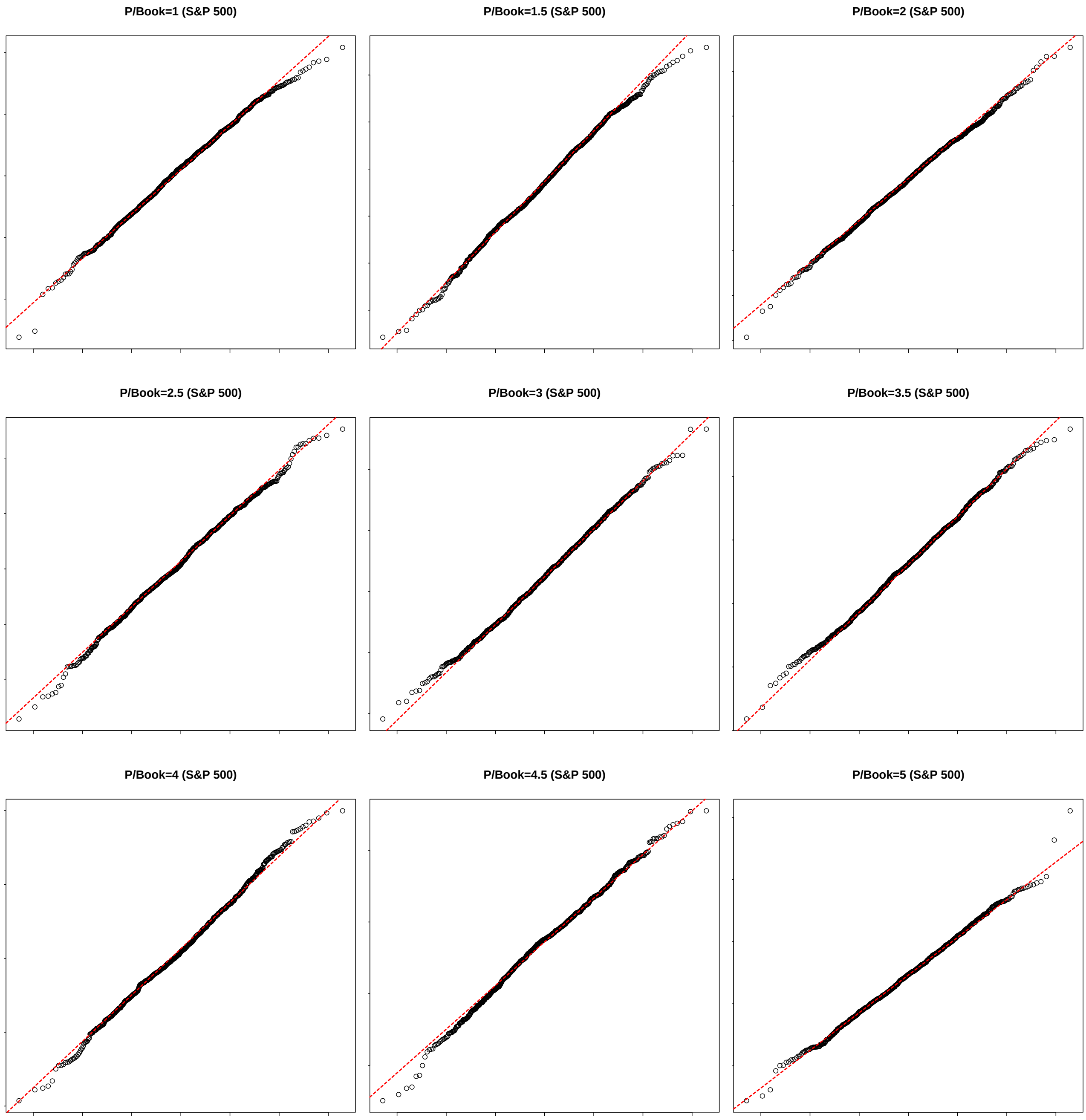


Figure 56: Q-Q normal plots for the value yield to eternal shareholders of the S&P 500 stock market index with different P/Book ratios. These are with share buyback and issuance which affects the number of shares and hence the dividend per share. The Monte Carlo simulations resulting in these plots are described in section 6.11. The x-axes are theoretical quantiles and the y-axes are sample quantiles.

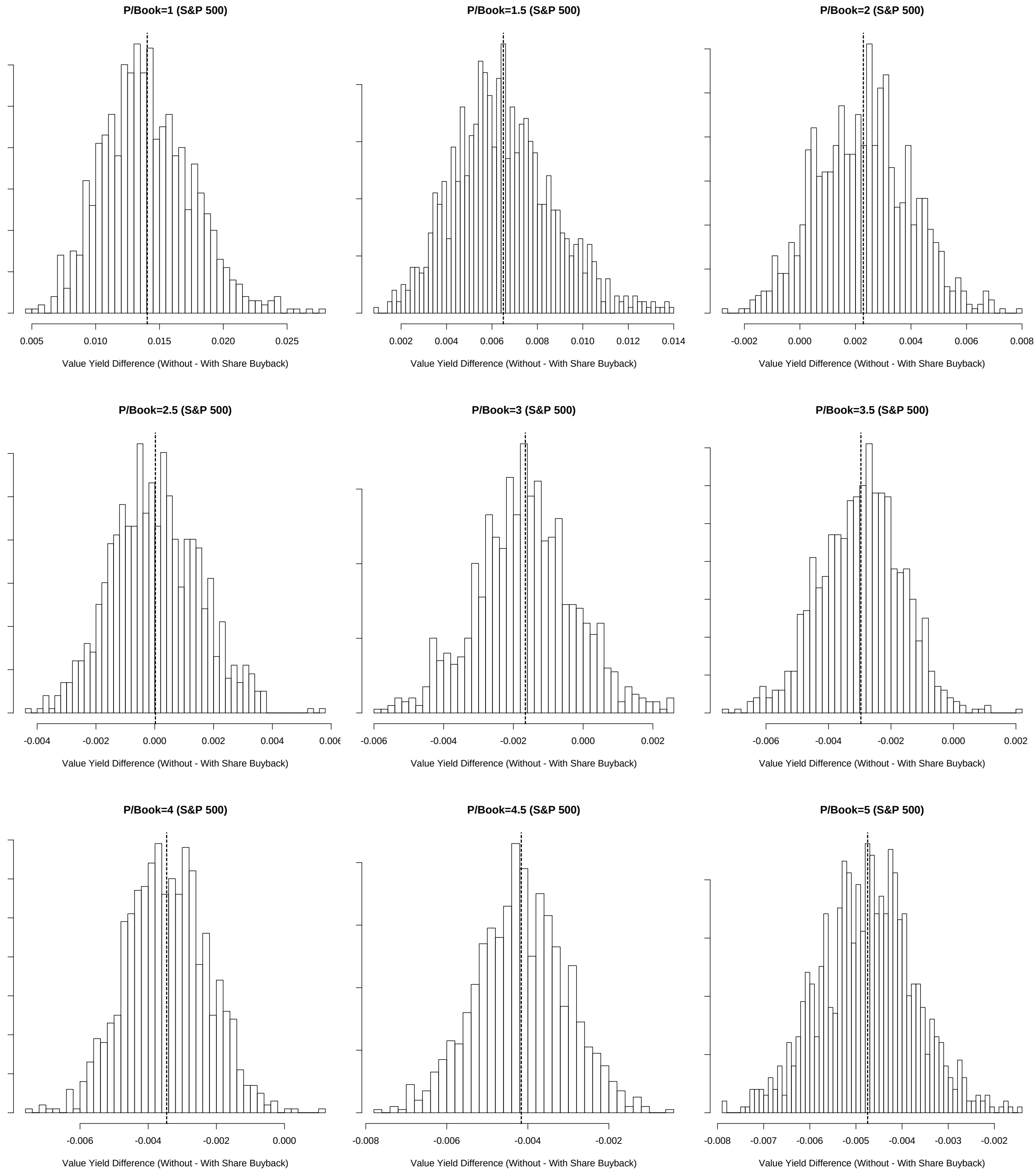


Figure 57: Distributions for the value yield differences to eternal shareholders of the S&P 500 stock market index, calculated as the value yield without share buybacks minus the value yield with share buybacks. The means are shown as dotted lines. The Monte Carlo simulations resulting in these plots are described in section 6.11.

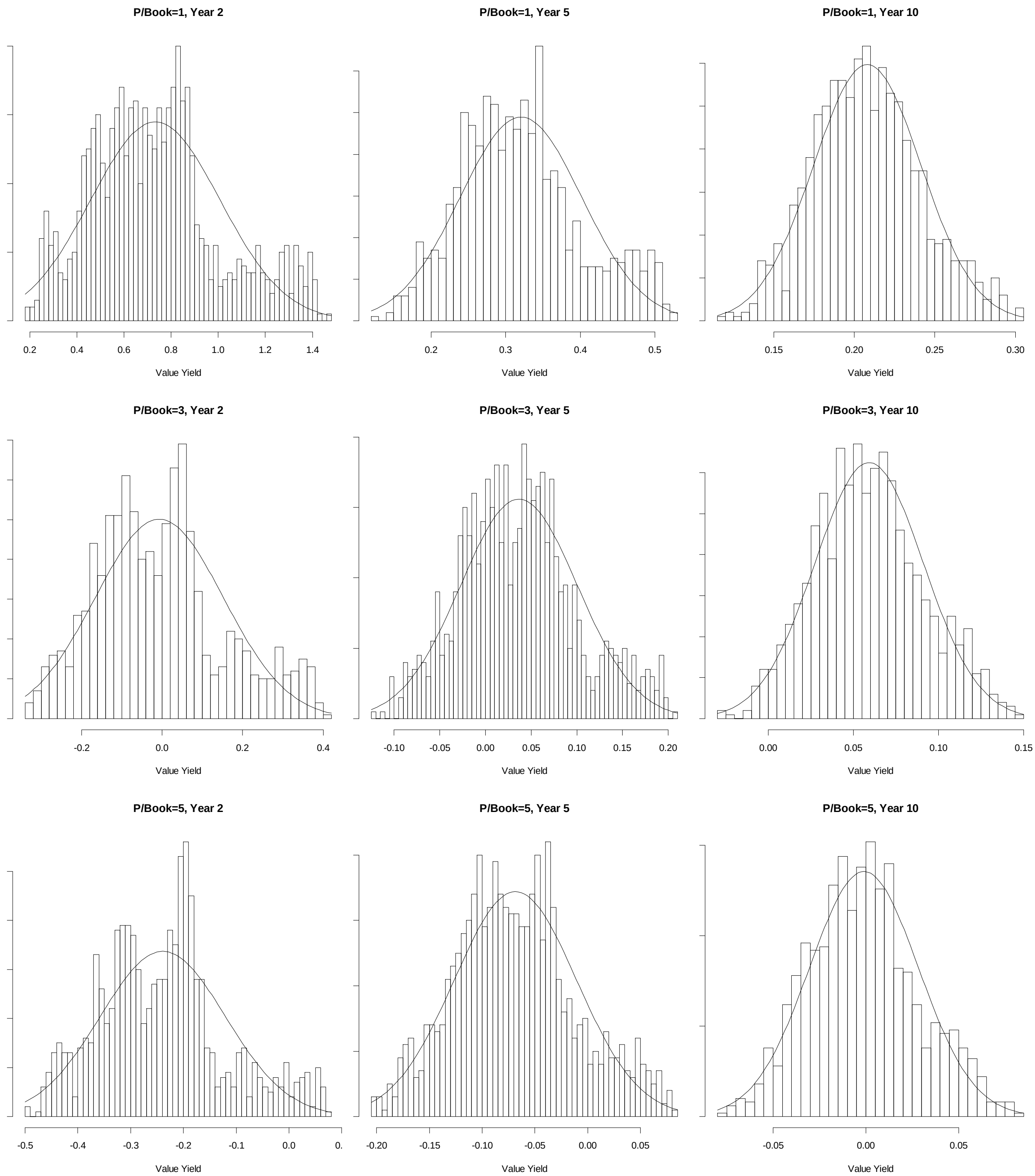


Figure 58: Value yield distributions for temporary shareholders of the S&P 500 stock market index for different starting P/Book ratios. The shares are sold after the given number of years and the selling price is calculated from Monte Carlo simulated equity multiplied by a P/Book ratio sampled from the historical distribution, as described in section 6.12.1. These are without share buyback and issuance so there is no change in the number of shares and the payout consists entirely of dividends. Also shown are the fitted normal PDFs.

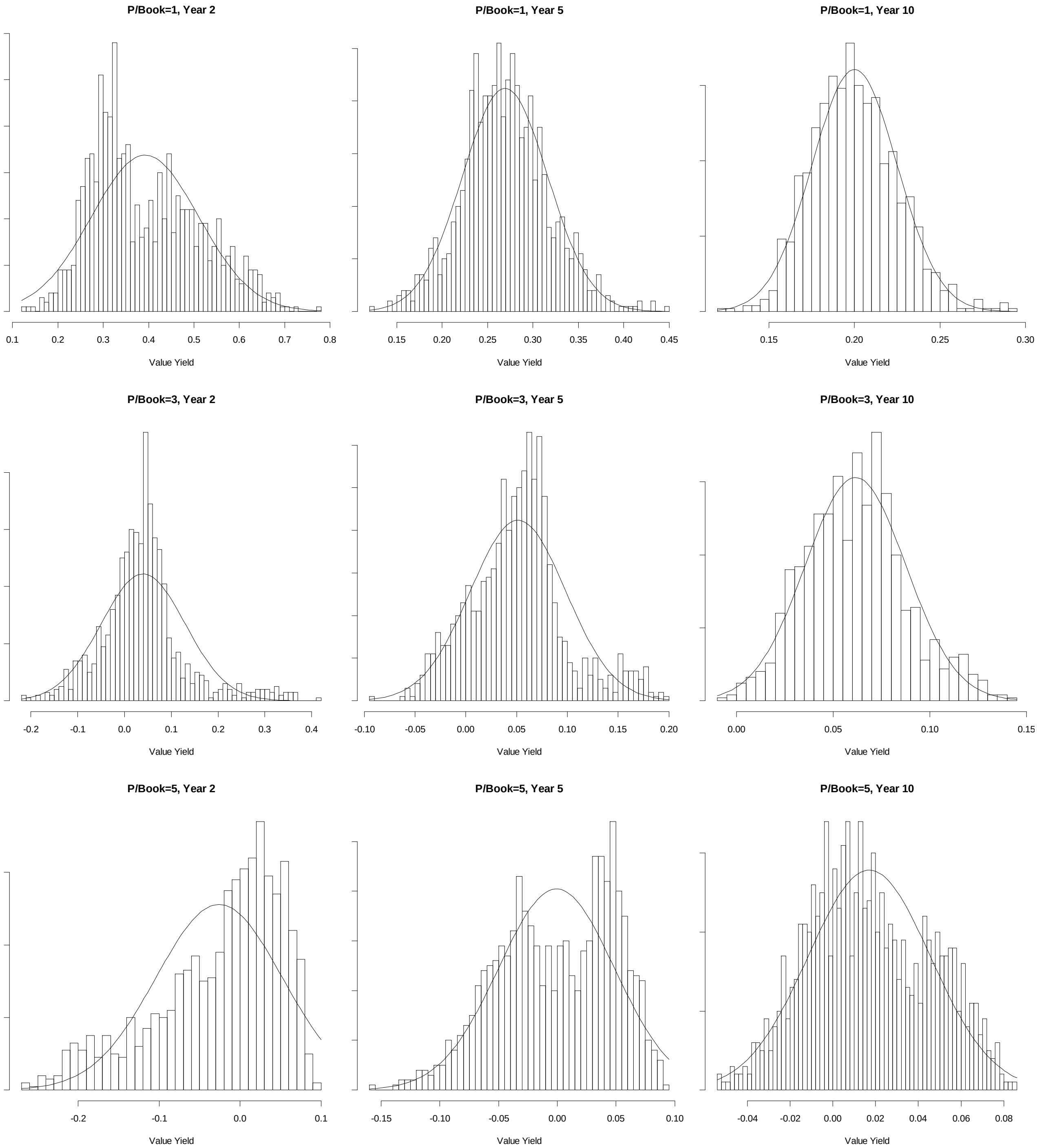


Figure 59: Value yield distributions for temporary shareholders of the S&P 500 stock market index for different starting P/Book ratios. The shares are sold after the given number of years and the selling price is calculated from the equity and P/Book ratios resulting from Monte Carlo simulations, as described in section 6.12.2. These are without share buyback and issuance so there is no change in the number of shares and the payout consists entirely of dividends. Also shown are the fitted normal PDFs.

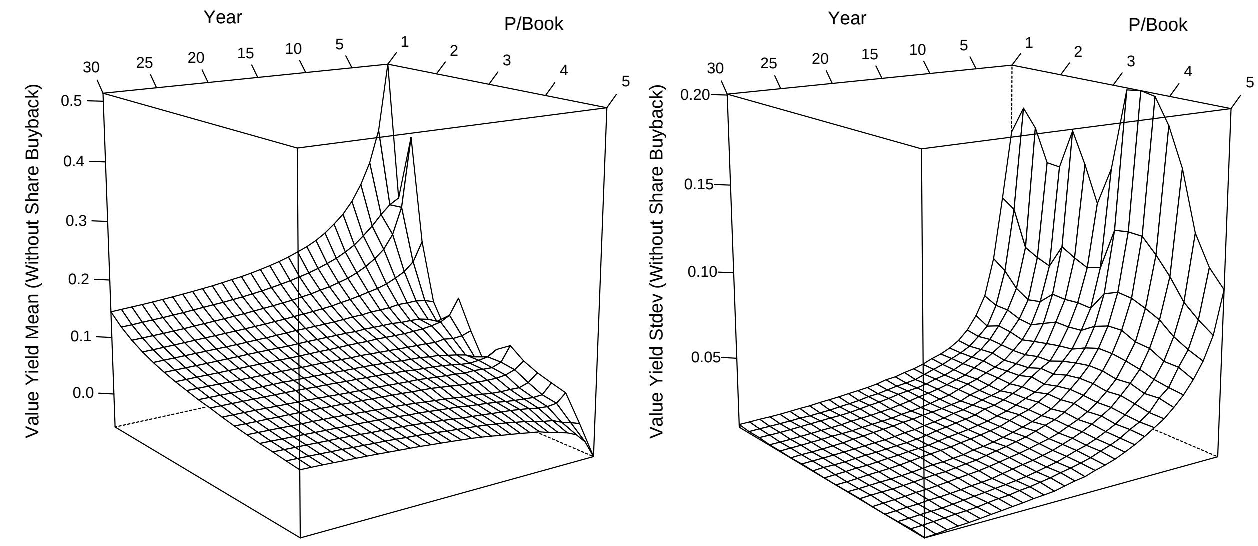


Figure 60: Value yield mean and standard deviation for temporary shareholders of the S&P 500 stock market index. The shares are bought at different P/Book ratios and sold after the given number of years. The selling price is calculated from the equity and P/Book ratios resulting from Monte Carlo simulations as described in section 6.12.2. These are without share buyback and issuance so there is no change in the number of shares and the payout consists entirely of dividends. Subsets of these figures are shown in Table 9 and Table 10.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	51%	39%	11%	13%	2.4%	7.5%	4.5%	2.5%	(6.4)%
	2	39%	26%	11%	9.8%	3.8%	5.3%	3.2%	0.9%	(2.7)%
	4	30%	19%	11%	8.5%	4.9%	4.4%	3.0%	1.4%	(0.6)%
	6	25%	17%	11%	8.3%	5.4%	4.7%	3.3%	1.9%	0.4%
	8	22%	15%	11%	8.1%	6.0%	4.7%	3.4%	2.3%	1.2%
	10	20%	14%	10%	8.1%	6.1%	4.9%	3.8%	2.8%	1.8%
	12	19%	14%	10%	8.2%	6.3%	5.2%	4.1%	3.2%	2.3%
	14	18%	13%	10%	8.2%	6.5%	5.4%	4.4%	3.5%	2.7%
	16	17%	13%	9.9%	8.2%	6.6%	5.6%	4.6%	3.8%	3.0%
	18	16%	12%	9.8%	8.1%	6.7%	5.7%	4.8%	4.0%	3.4%
	20	16%	12%	9.7%	8.1%	6.8%	5.9%	5.0%	4.3%	3.6%

Table 9: Subset of the value yield means from Figure 60.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	16%	17%	15%	16%	14%	20%	18%	13%	11%
	2	12%	9.1%	8.9%	9.7%	8.8%	13%	11%	9.1%	8.1%
	4	5.6%	5.2%	5.5%	5.9%	5.9%	6.8%	6.6%	6.0%	5.7%
	6	3.7%	3.8%	3.9%	4.2%	4.4%	4.8%	4.7%	4.5%	4.3%
	8	2.9%	2.9%	3.0%	3.2%	3.3%	3.6%	3.5%	3.6%	3.4%
	10	2.4%	2.4%	2.4%	2.5%	2.6%	2.8%	2.9%	2.9%	2.8%
	12	2.1%	2.1%	2.0%	2.1%	2.2%	2.3%	2.3%	2.4%	2.3%
	14	1.8%	1.8%	1.7%	1.9%	1.9%	2.0%	2.0%	2.0%	2.0%
	16	1.6%	1.6%	1.5%	1.7%	1.6%	1.7%	1.7%	1.8%	1.7%
	18	1.4%	1.4%	1.4%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%
	20	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.4%	1.4%

Table 10: Subset of the value yield standard deviations from Figure 60.

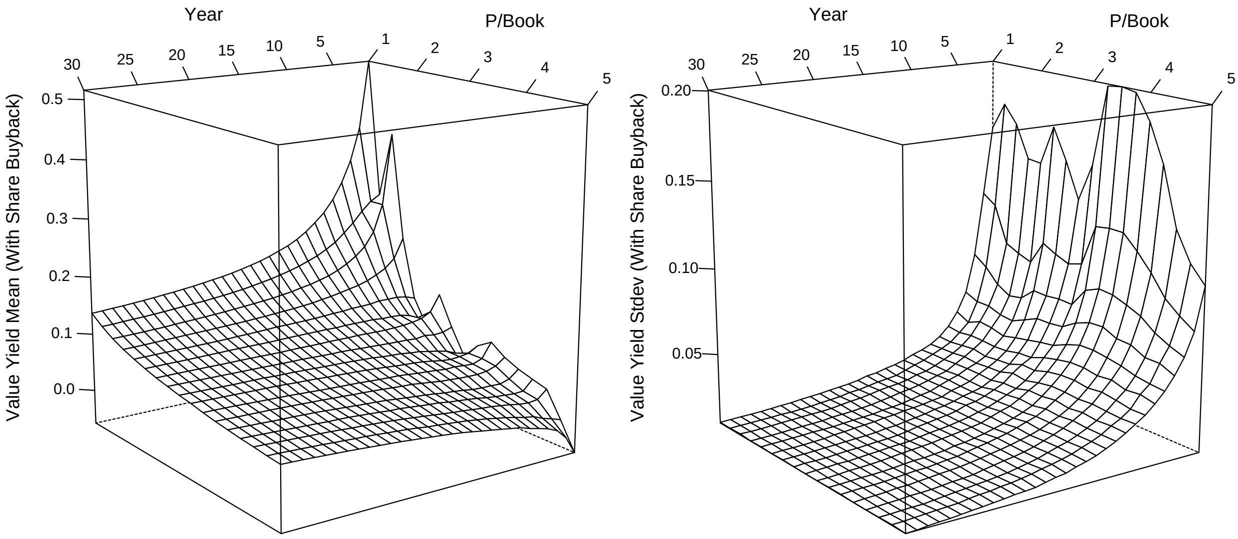


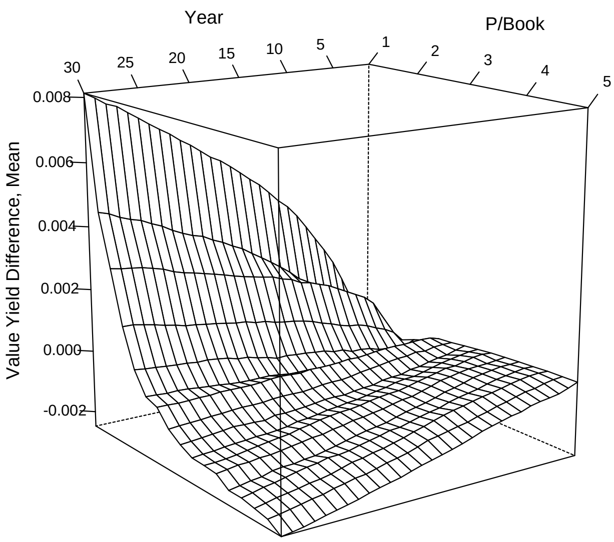
Figure 61: Value yield mean and standard deviation for temporary shareholders of the S&P 500 stock market index. The shares are bought at different P/Book ratios and sold after the given number of years. The selling price is calculated from the equity and P/Book ratios resulting from Monte Carlo simulations as described in section 6.12.3. These are with share buyback and issuance which affects the number of shares and hence the dividend and selling price per share. Subsets of these figures are shown in Table 11 and Table 12.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	52%	39%	10%	13%	3.1%	7.5%	4.6%	3.0%	(5.9)%
	2	40%	27%	11%	10%	4.0%	4.9%	3.2%	1.6%	(2.9)%
	4	30%	19%	11%	8.6%	5.1%	4.4%	2.8%	1.6%	(0.5)%
	6	25%	17%	11%	8.2%	5.7%	4.5%	2.9%	2.1%	0.3%
	8	22%	15%	11%	8.2%	6.1%	4.8%	3.3%	2.5%	1.2%
	10	20%	14%	10%	8.2%	6.2%	5.1%	3.7%	2.8%	1.7%
	12	18%	13%	10%	8.2%	6.5%	5.3%	4.0%	3.2%	2.3%
	14	17%	13%	10%	8.1%	6.7%	5.4%	4.4%	3.6%	2.8%
	16	16%	12%	10%	8.1%	6.7%	5.6%	4.6%	3.9%	3.1%
	18	16%	12%	10%	8.1%	6.9%	5.7%	4.8%	4.2%	3.5%
	20	15%	12%	10%	8.1%	6.9%	5.9%	5.1%	4.4%	3.8%

Table 11: Subset of the value yield means from Figure 61.

		P/Book								
		1	1.5	2	2.5	3	3.5	4	4.5	5
Year	1	16%	17%	15%	15%	15%	20%	18%	13%	10%
	2	12%	9.3%	8.7%	9.6%	9.6%	12%	11%	9.2%	7.9%
	4	5.9%	5.6%	5.2%	5.9%	6.0%	6.9%	6.5%	6.1%	5.7%
	6	3.9%	3.9%	3.8%	4.1%	4.4%	4.8%	4.6%	4.5%	4.4%
	8	3.0%	3.1%	3.1%	3.2%	3.3%	3.6%	3.5%	3.5%	3.5%
	10	2.6%	2.5%	2.6%	2.5%	2.7%	2.7%	2.8%	2.8%	2.8%
	12	2.2%	2.2%	2.2%	2.2%	2.2%	2.3%	2.3%	2.3%	2.3%
	14	1.9%	1.9%	1.8%	1.9%	1.9%	1.9%	2.0%	2.0%	2.0%
	16	1.7%	1.7%	1.6%	1.7%	1.7%	1.7%	1.7%	1.7%	1.7%
	18	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%
	20	1.4%	1.4%	1.3%	1.3%	1.3%	1.3%	1.3%	1.3%	1.4%

Table 12: Subset of the value yield standard deviations from Figure 61.



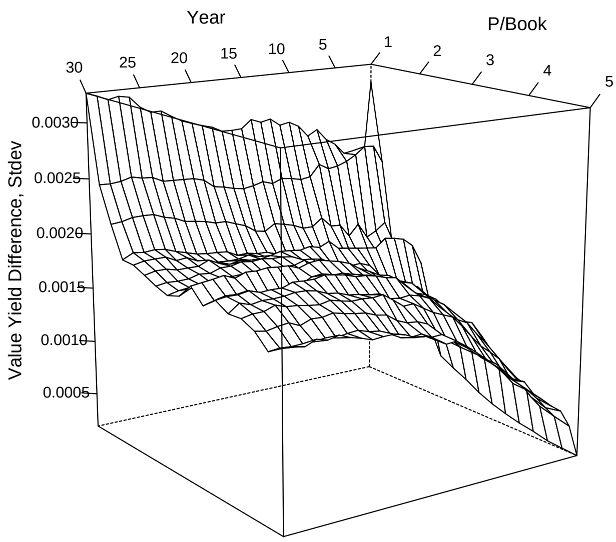


Figure 62: Mean and standard deviation for the value yield difference for temporary shareholders of the S&P 500 stock market index. The shares are bought at different P/Book ratios and sold after the given number of years. The value yield difference is calculated as the value yield without share buybacks minus the value yield with share buybacks. The Monte Carlo simulations resulting in these plots are described in section 6.12.

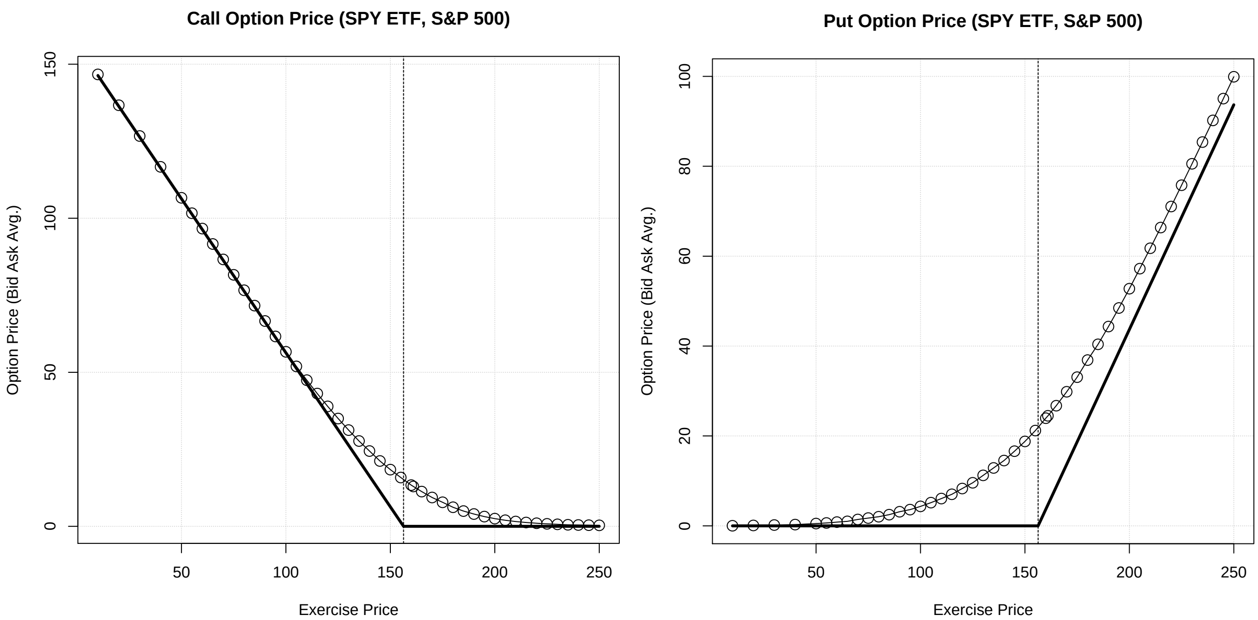


Figure 63: Option chains for the SPY ETF (S&P 500) for call options (left) and put options (right) expiring on December 18, 2015 as quoted on March 15, 2013. The option quotes are shown with circles that are connected with lines for clarity. The dashed lines show the closing share price on the previous trading day which was USD 156.32. The bold lines show the intrinsic values.[10]

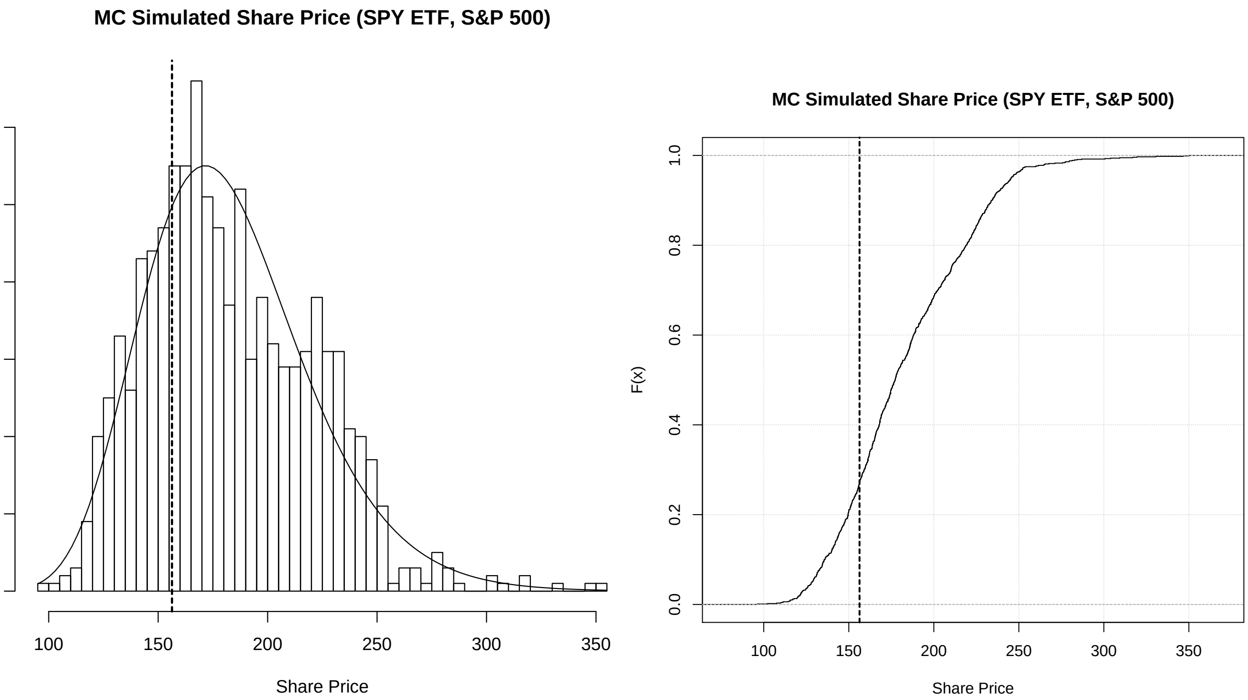


Figure 64: Histogram with fitted log-normal PDF (left) and CDF (right) of Monte Carlo simulated share prices for the SPY ETF (S&P 500) when the starting P/Book is 2.3 and the simulation period is 696 trading days, corresponding to the period between March 15, 2013 and December 18, 2015. The share price of SPY is 1/10 of the S&P 500 index. The closing share price on March 14, 2013 is shown as dashed lines.

[10] Data source for the SPY options: http://quote.morningstar.com/Option/Options.aspx?ticker=SPY

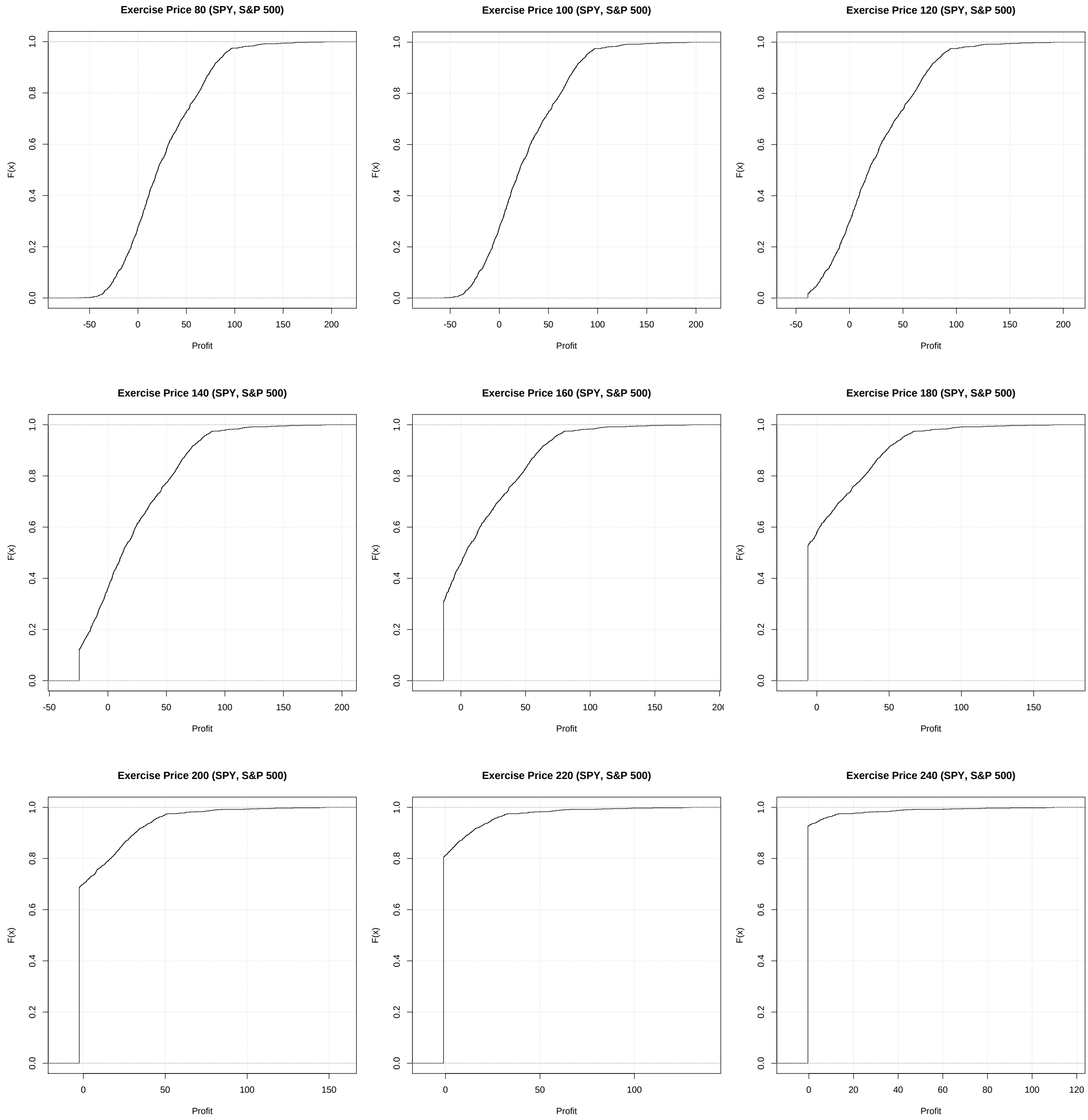


Figure 65: CDF for the <u>profits</u> on <u>call options</u> of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated for the expiration date only, using Eq. 4-3 with the share prices resulting from Monte Carlo simulations as described in section 6.13.

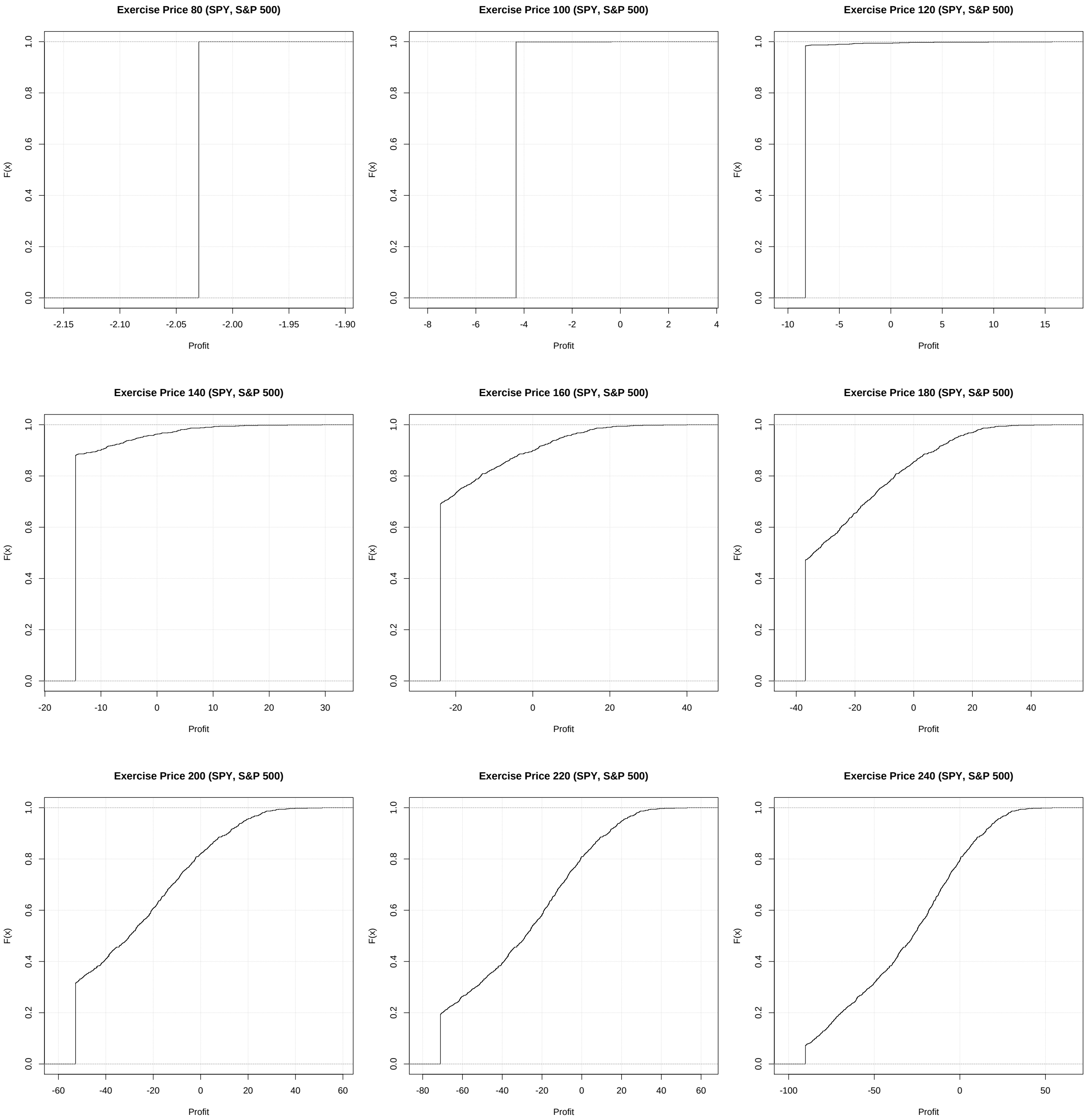


Figure 66: CDF for the profits on put options of SPY ETF (S&P 500) expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated for the expiration date only, using Eq. 4-3 with the share prices resulting from Monte Carlo simulations as described in section 6.13.

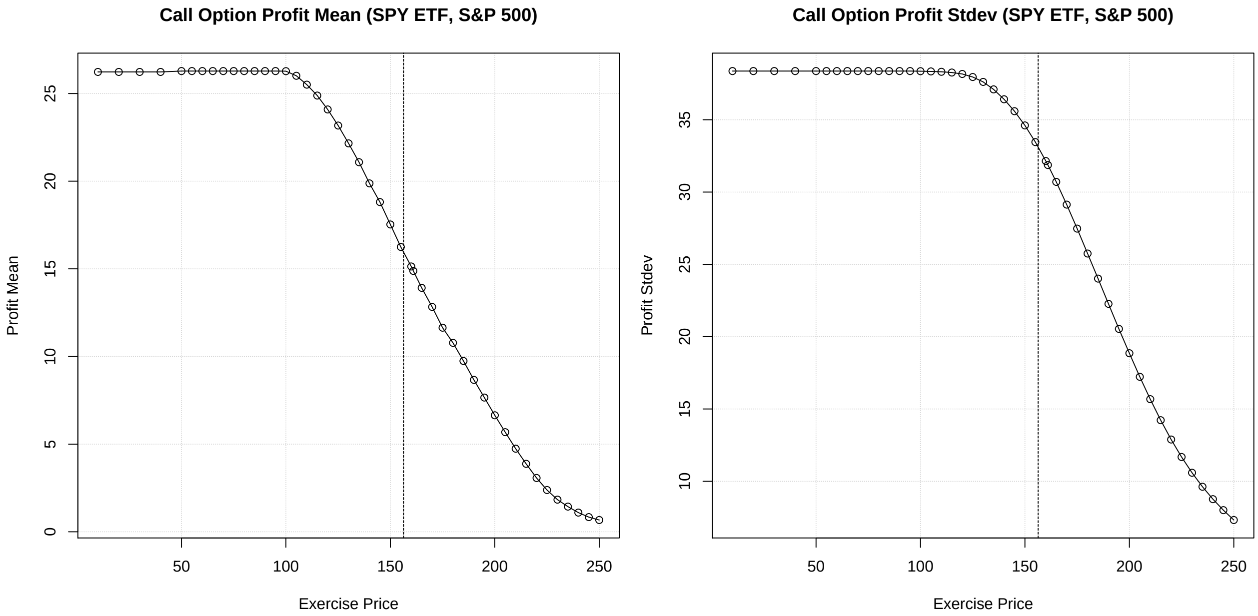


Figure 67: <u>Profit mean and standard deviation</u> for <u>call options</u> of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated for the expiration date only, using Eq. 4-3 with the share prices resulting from Monte Carlo simulations as described in section 6.13. The options are shown with circles that are connected with lines for clarity. The dashed lines show the closing share price on the previous trading day.

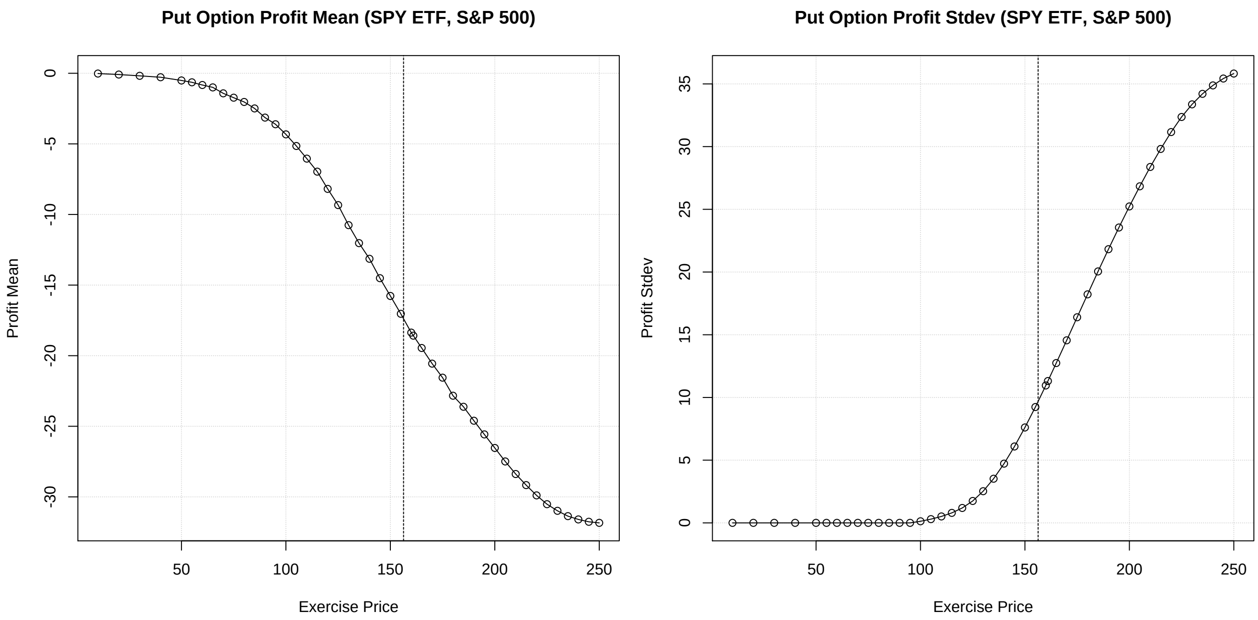


Figure 68: <u>Profit mean and standard deviation</u> for <u>put options</u> of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated for the expiration date only, using Eq. 4-3 with the share prices resulting from Monte Carlo simulations as described in section 6.13. The options are shown with circles that are connected with lines for clarity. The dashed lines show the closing share price on the previous trading day.

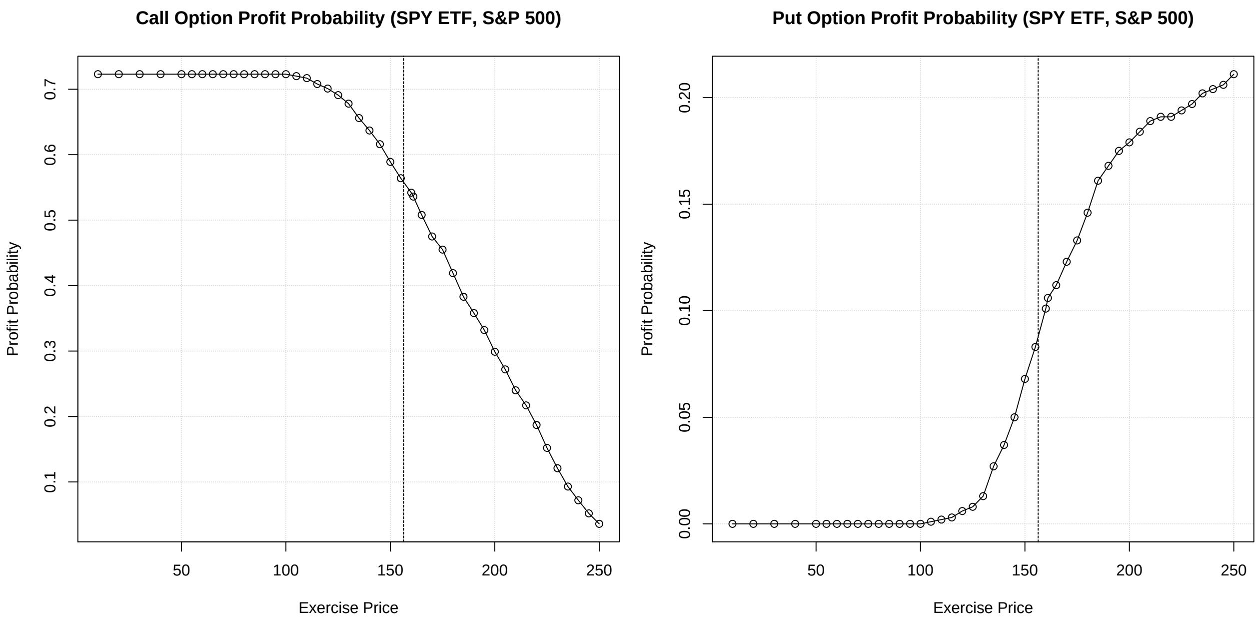


Figure 69: Profit probability $Pr[Profit > 0]$ for call options (left) and put options (right) of SPY ETF (S&P 500) expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated for the expiration date only, using share prices resulting from Monte Carlo simulations as described in section 6.13. The options are shown with circles that are connected with lines for clarity. The dashed lines show the closing share price on the previous trading day. Note that the scales of the y-axes (profit probability) are different in the two plots.

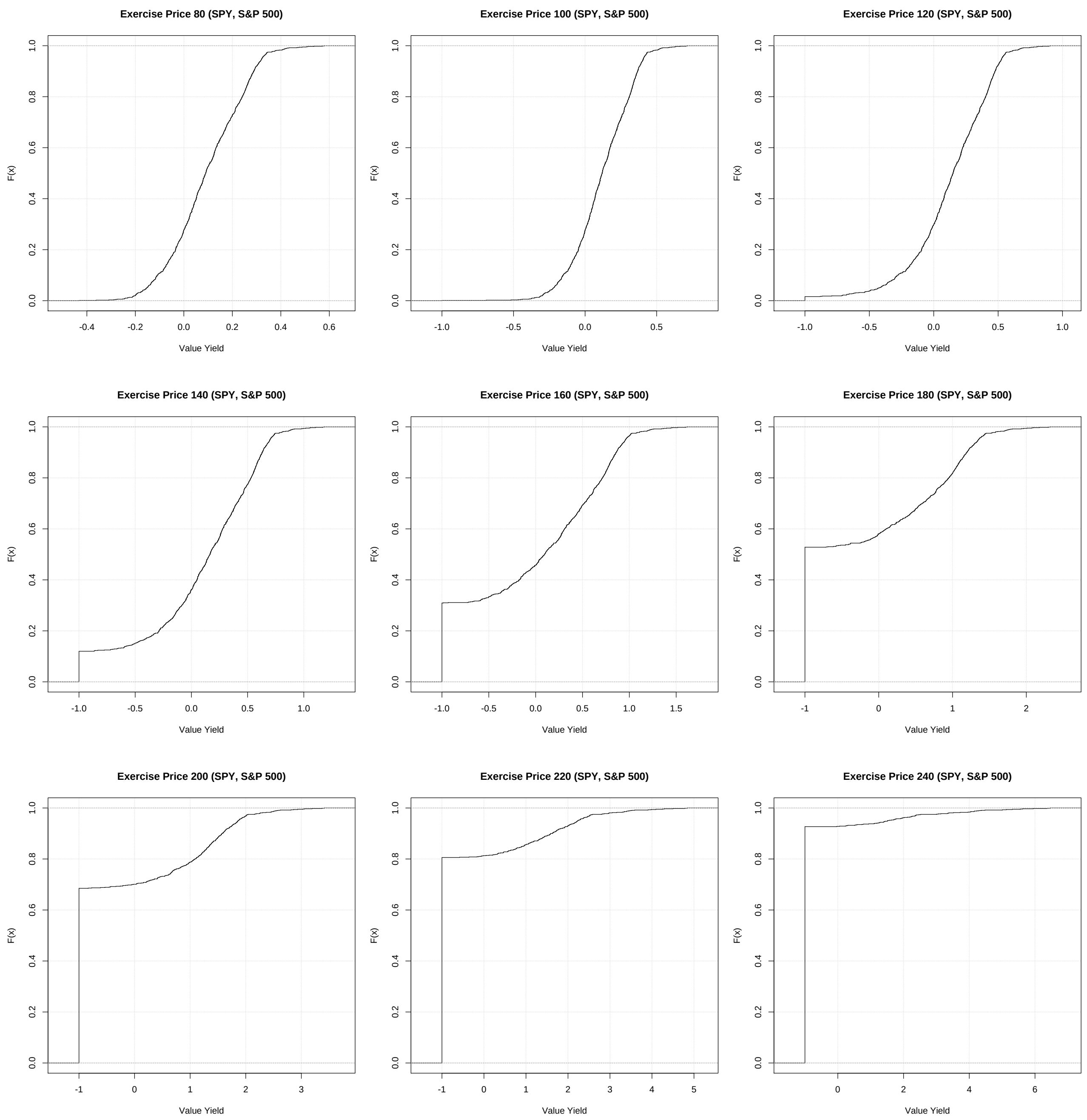


Figure 70: CDF for the value yields on call options of SPY ETF (S&P 500) expiring on December 18, 2015 as quoted on March 15, 2013. The value yields are calculated for the expiration date only, using Eq. 4-6 with the share prices resulting from Monte Carlo simulations as described in section 6.13.

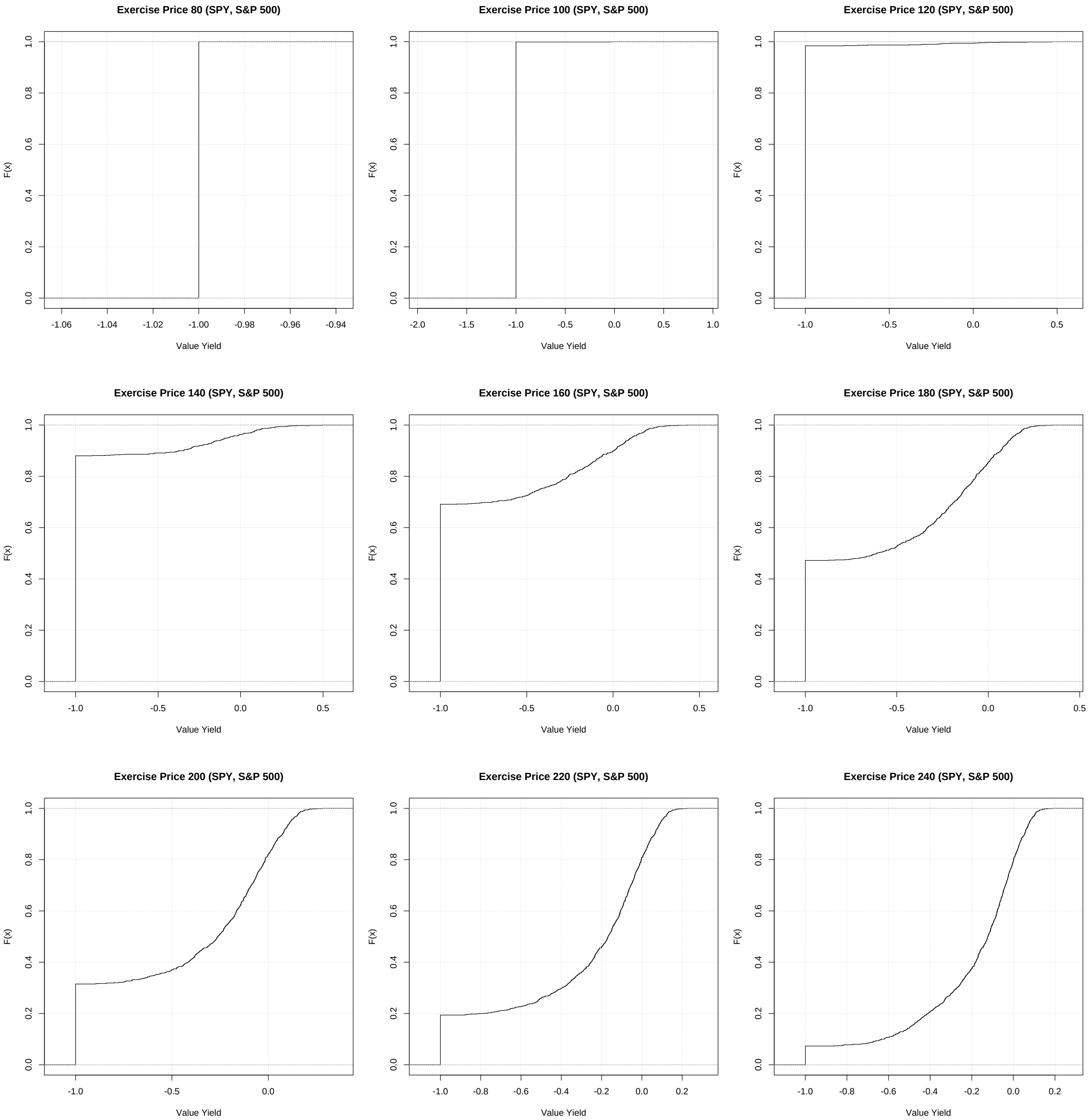


Figure 71: CDF for the value yields on put options of SPY ETF (S&P 500) expiring on December 18, 2015 as quoted on March 15, 2013. The value yields are calculated for the expiration date only, using Eq. 4-6 with the share prices resulting from Monte Carlo simulations as described in section 6.13.

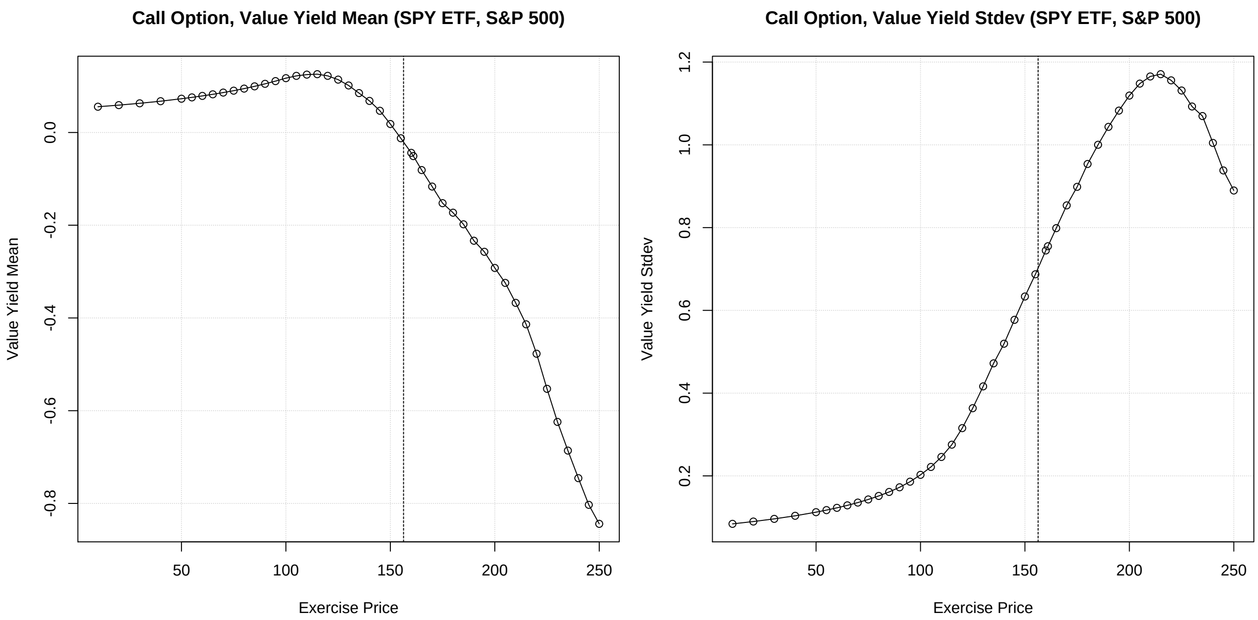


Figure 72: <u>Value yield mean and standard deviation</u> for <u>call options</u> of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The value yields are calculated for the expiration date only, using Eq. 4-6 and the share prices resulting from Monte Carlo simulations as described in section 6.13. The options are shown with circles that are connected with lines for clarity. The dashed lines show the closing share price on the previous trading day.

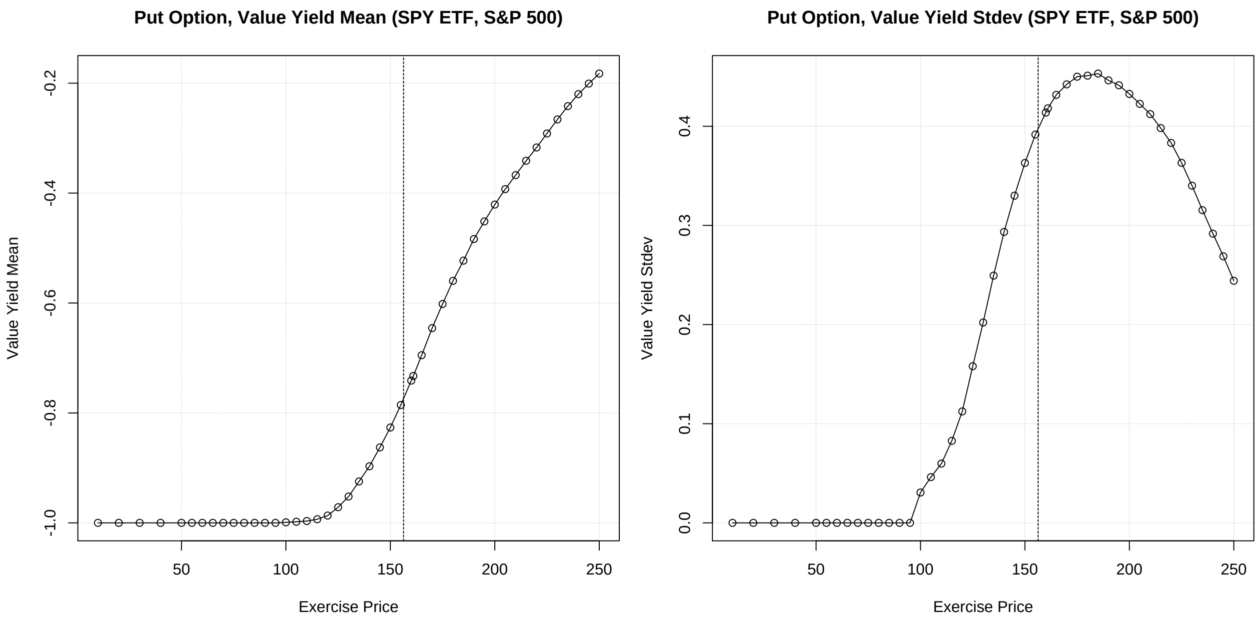


Figure 73: <u>Value yield mean and standard deviation</u> for <u>put options</u> of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The value yields are calculated for the expiration date only, using Eq. 4-6 and the share prices resulting from Monte Carlo simulations as described in section 6.13. The options are shown with circles that are connected with lines for clarity. The dashed lines show the closing share price on the previous trading day.

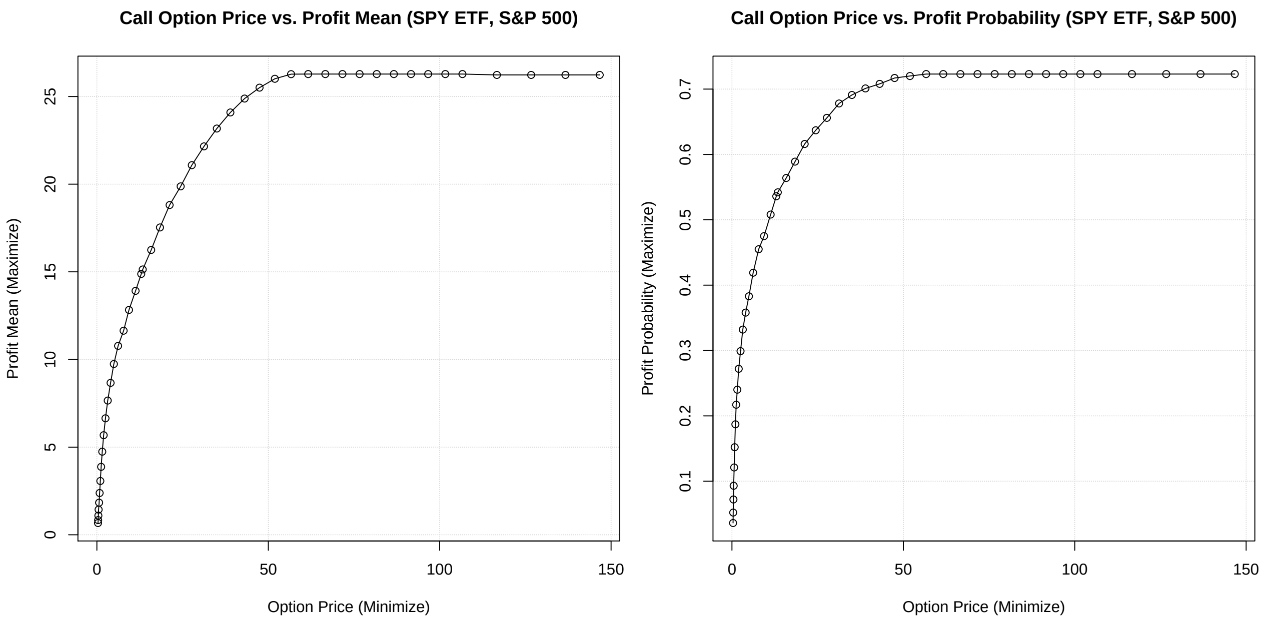


Figure 74: <u>Call option</u> price versus profit mean (left) and profit probability (right) for options of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated from Monte Carlo simulations as described in section 6.13. The buyer of a call option would want to maximize the profit mean and probability (y-axes) and minimize the option price (x-axes). The seller would want to do the opposite.

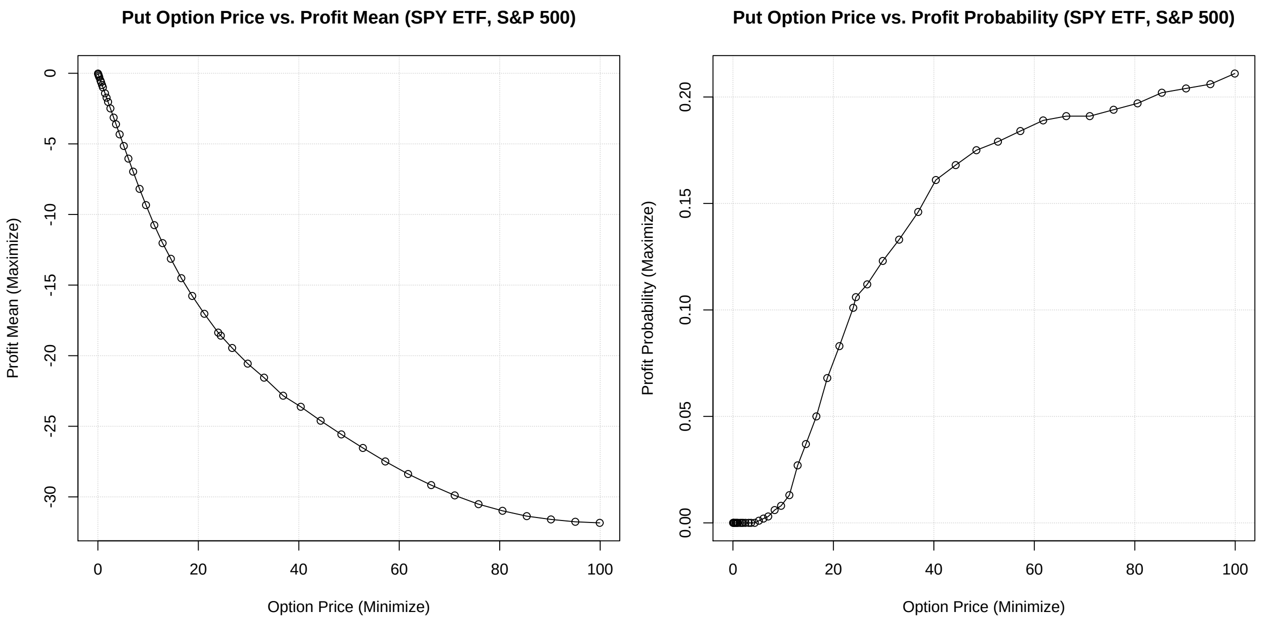


Figure 75: <u>Put option</u> price versus profit mean (left) and profit probability (right) for options of <u>SPY ETF (S&P 500)</u> expiring on December 18, 2015 as quoted on March 15, 2013. The profits are calculated from Monte Carlo simulations as described in section 6.13. The buyer of a call option would want to maximize the profit mean and probability (y-axes) and minimize the option price (x-axes). The seller would want to do the opposite.

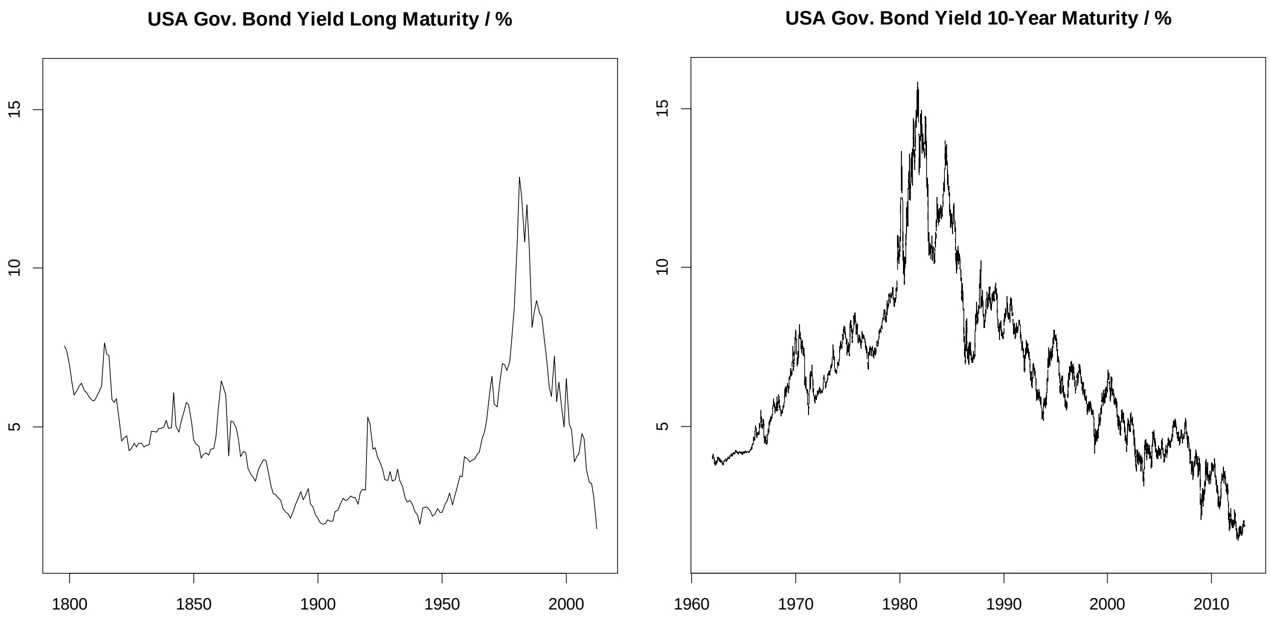


Figure 76: USA government bond yields for the period 1798-2012 (left plot, annual average, bonds have varying maturity, terms and taxation during this period) and for the period 1962-2013 (right plot, daily average, bonds have 10 year maturity period). Because of the annual averaging in the left plot the extreme bond yields are smoothened somewhat.[11]

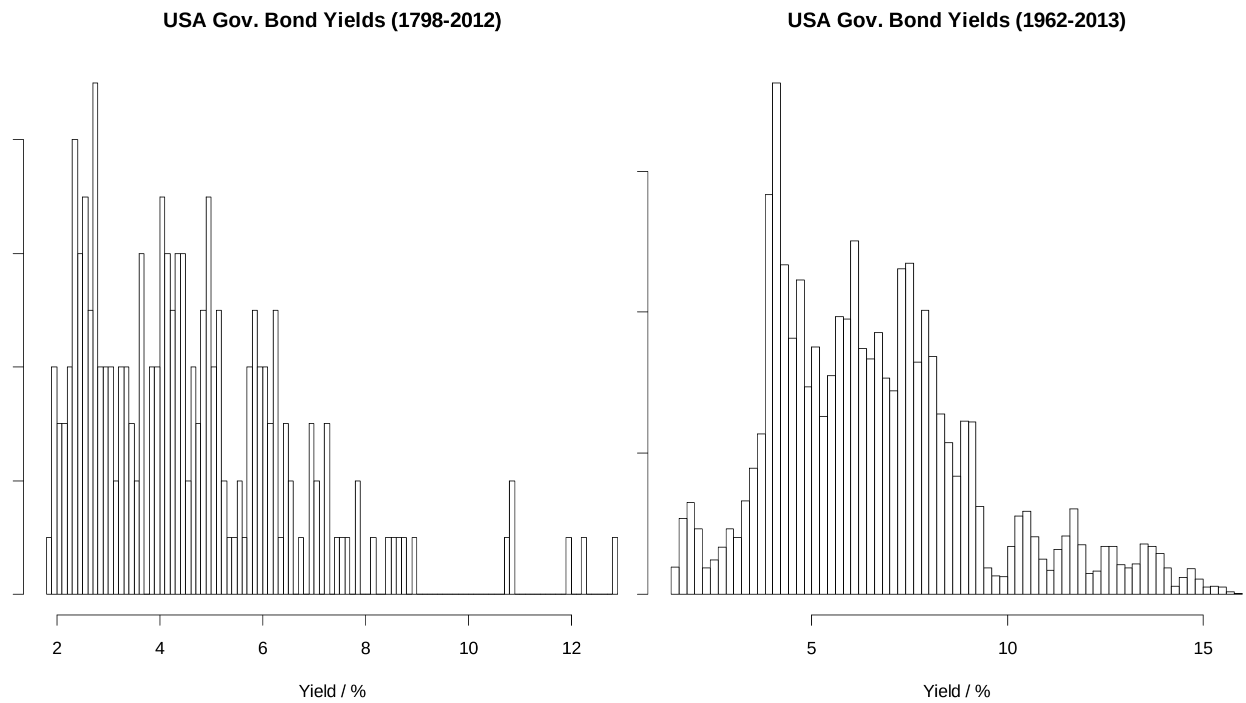


Figure 77: Histograms for the bond yields in Figure 76.

[11] Bond yields for the period 1798-2012 are from Homer & Sylla [**13**] tables 38, 46, 48, 51, 87, as well as from the Federal Reserve. Bond yields for the period 1962-2013 are also from the Federal Reserve: www.federalreserve.gov/releases/h15/data.htm

Period	Arithmetic Mean	Geometric Mean	Harmonic Mean	Stdev	Min	Max
1798-2012	4.6%	4.2%	3.8%	2.0%	1.8%	12.9%
1962-2013	6.6%	6.1%	5.5%	2.7%	1.4%	15.8%

Table 13: Statistics for the USA government bond yields in Figure 76. Because the bond yields for the period 1798-2012 are annual averages, the minimum and maximum yields for that period are inaccurate.

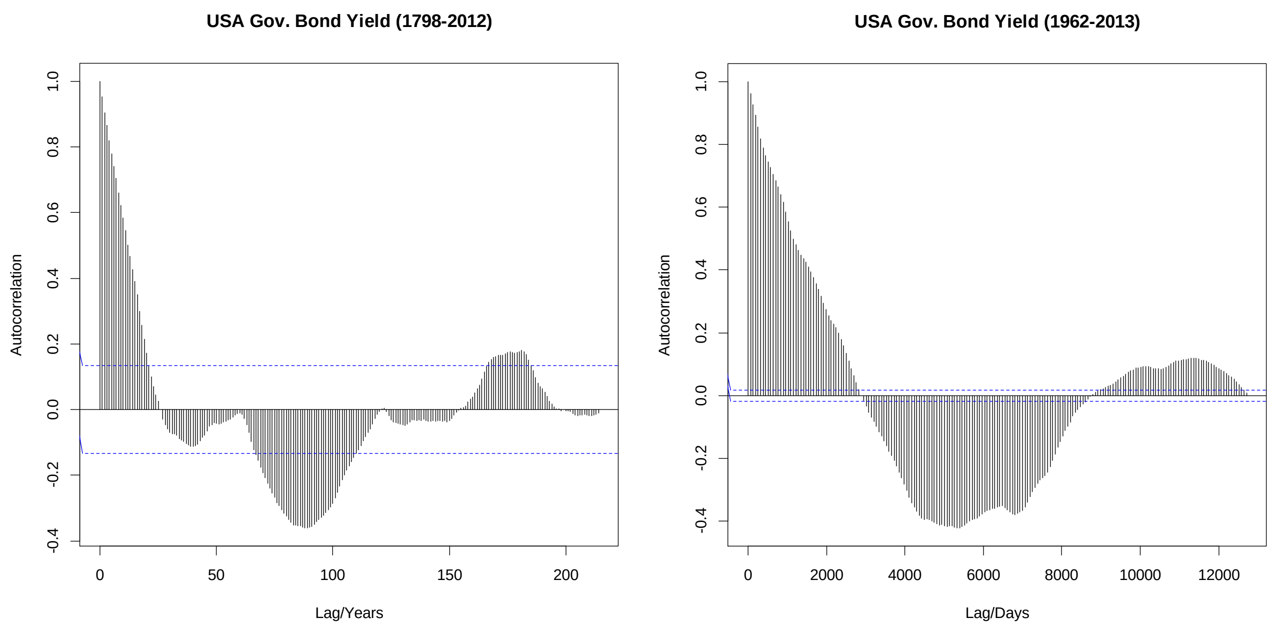


Figure 78: Autocorrelation for the USA government bond yields from Figure 76 which show significant correlation between successive bond yields. The reason the shapes of these plots differ is perhaps because the left plot is for annual averages and the right is for daily averages.

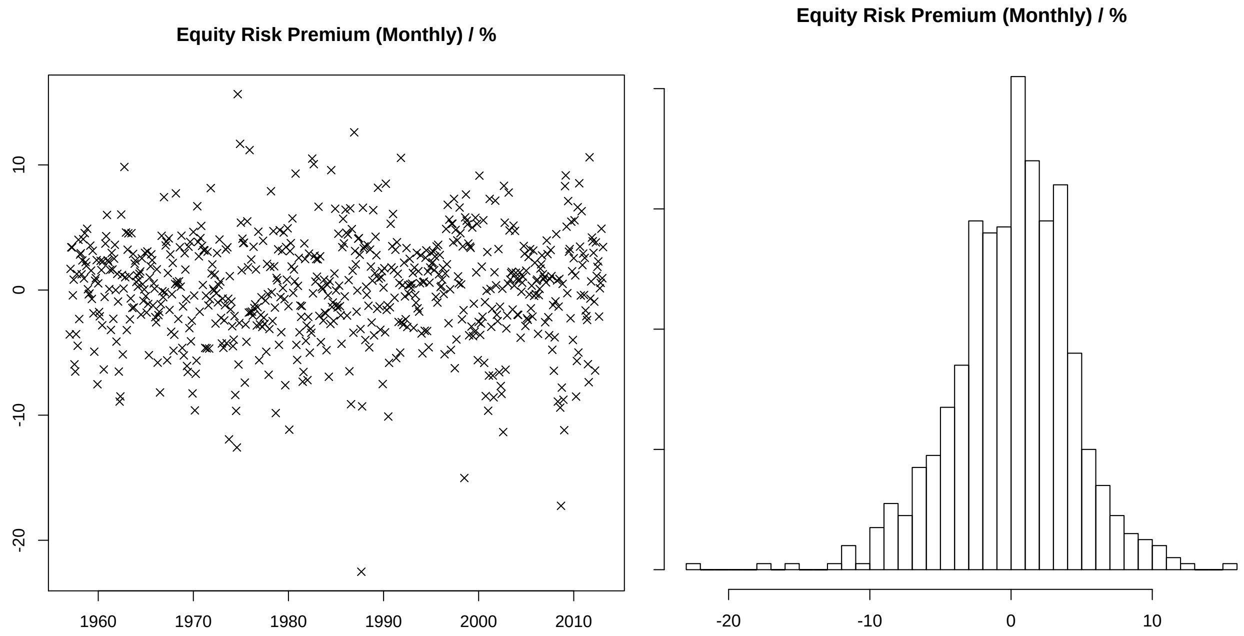


Figure 79: Equity risk premium for the period 1957-2013, calculated as the difference between the monthly yields of USA government bonds with 10 year maturity and the monthly returns of the S&P 500 stock market index. Left plot shows the time series and right plot shows the histogram.

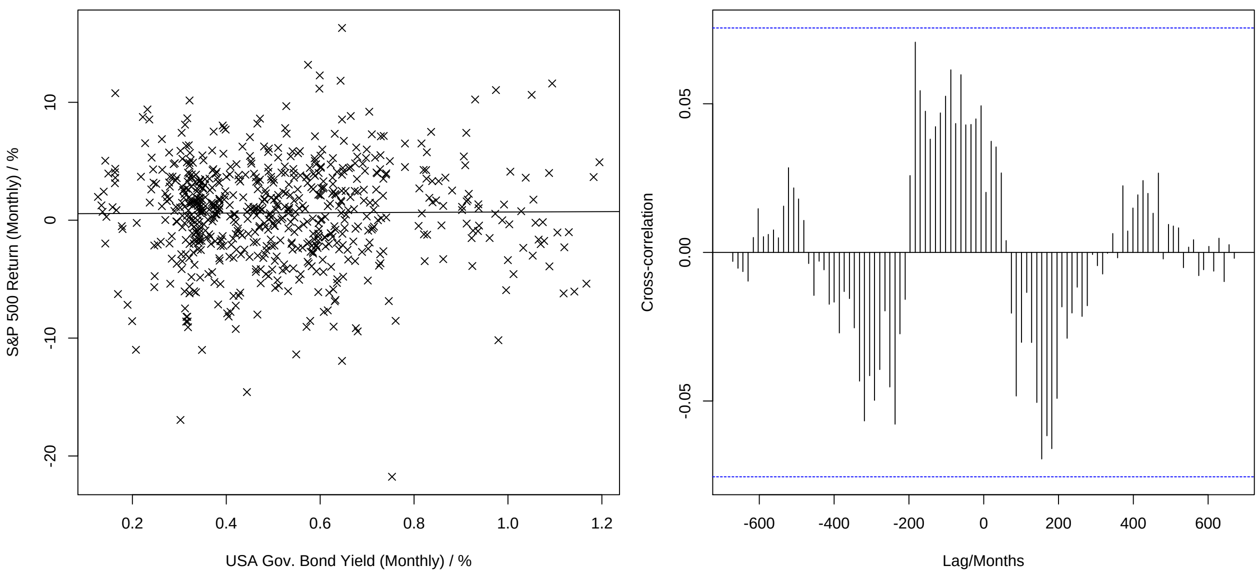


Figure 80: Scatter-plot with line-fit (left) of the monthly yields on USA government bonds with 10 year maturity versus the monthly returns of the S&P 500 stock market index. Data is for the period 1957-2013 from Figure 79. Right plot shows cross-correlation.

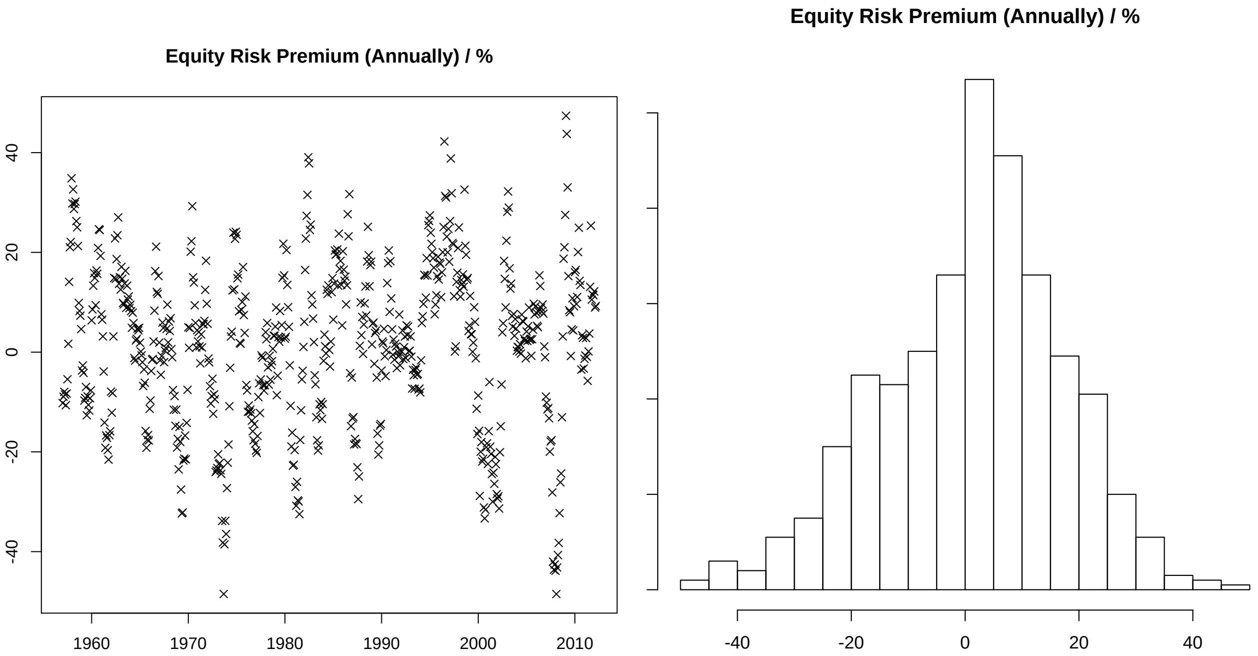


Figure 81: Equity risk premium for the period 1957-2013, calculated as the annual returns of the S&P 500 stock market index minus the annual yields of USA government bonds with 10 year maturity. Left plot shows the time series and right plot shows the histogram.

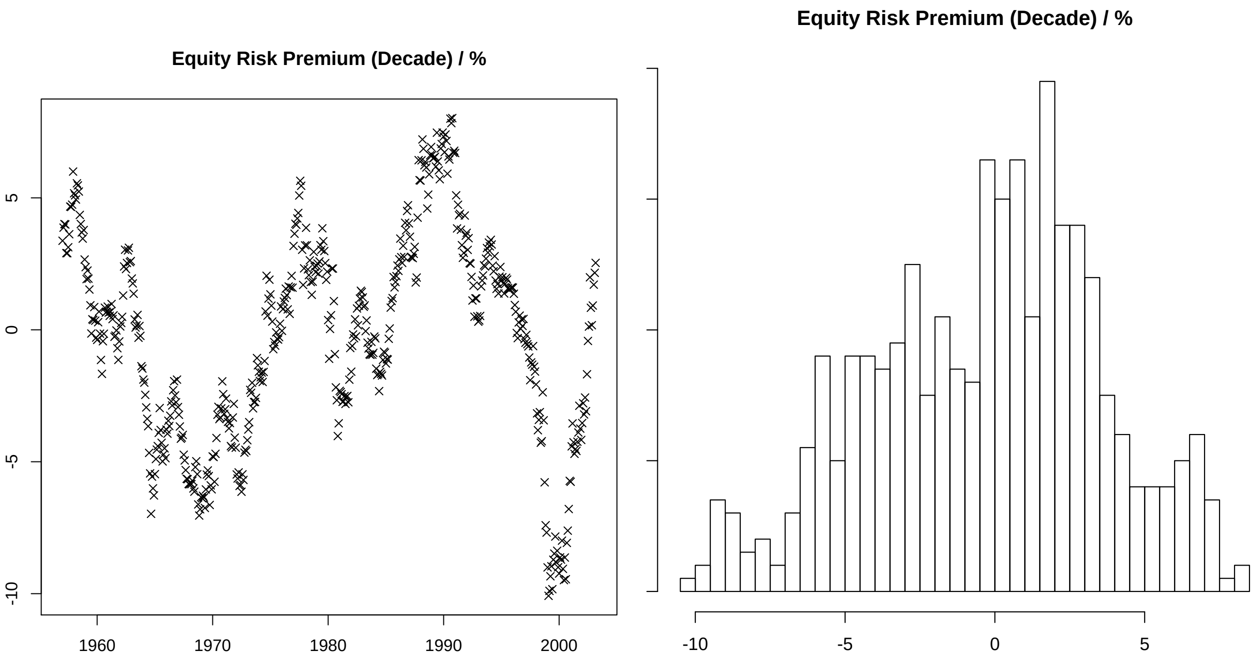


Figure 82: Equity risk premium for the period 1957-2013, calculated as the annualized returns of 10-year returns of the S&P 500 stock market index minus the annual yields of USA government bonds with 10 year maturity. Left plot shows the time series and right plot shows the histogram.

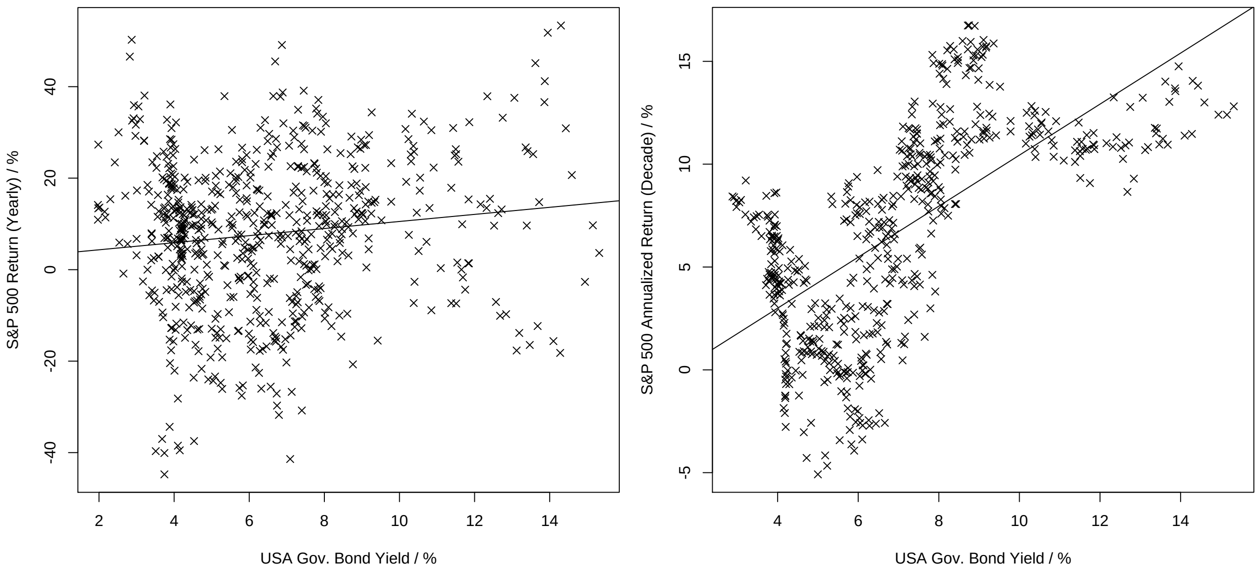


Figure 83: Scatter-plots with line-fits of the annual yield on USA government bonds with 10 year maturity versus the 1-year (left) and 10-year (right) returns of the S&P 500 stock market. Data is for the period 1957-2013 from Figure 81 and Figure 82.

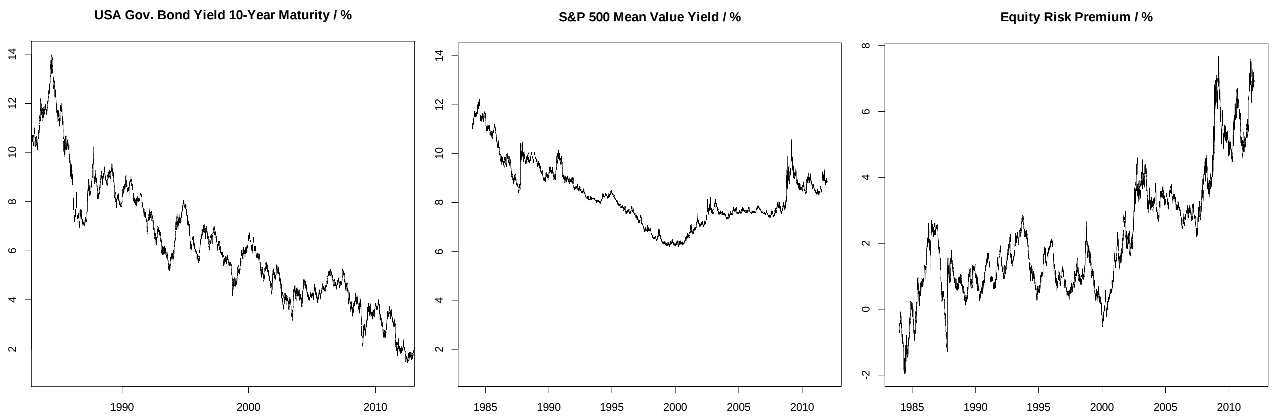


Figure 84: USA government bond yields with 10 year maturity (left plot), mean value yield for the S&P 500 stock market index (middle plot) estimated from Eq. 6-48 with the P/Book data from Figure 24. Their difference is the estimated mean Equity Risk Premium (ERP) (right plot).

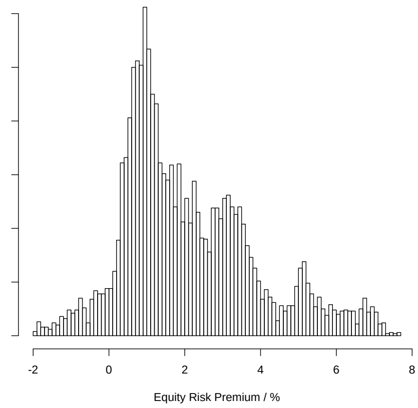


Figure 85: Histogram of the estimated mean Equity Risk Premium (ERP) in Figure 84.

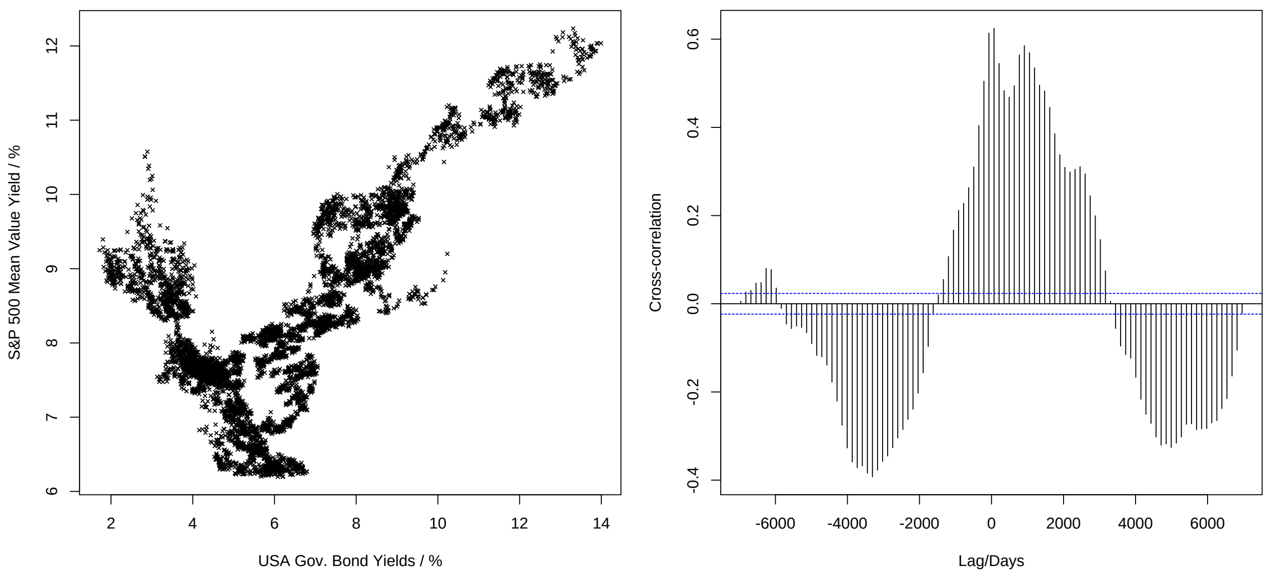
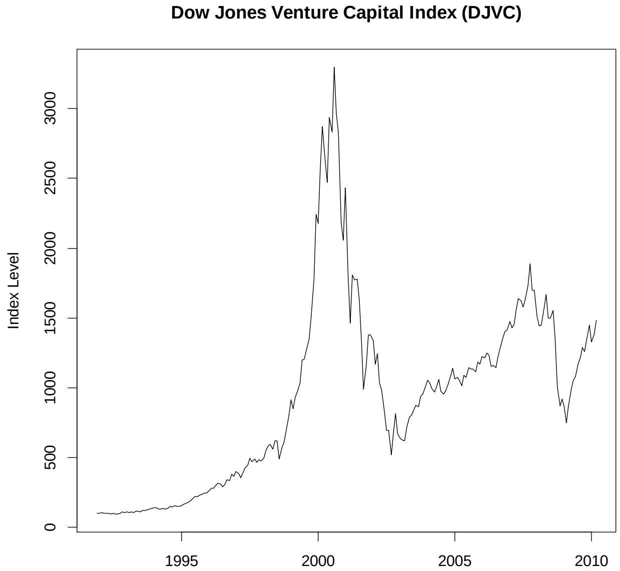


Figure 86: Scatter-plot (left) of <u>USA government bond yields</u> with 10 year maturity <u>versus</u> the estimated <u>mean value yield</u> of the <u>S&P 500</u> stock market index. Data is for the period 1984-2011 from Figure 84. Cross-correlation of the data (right plot).



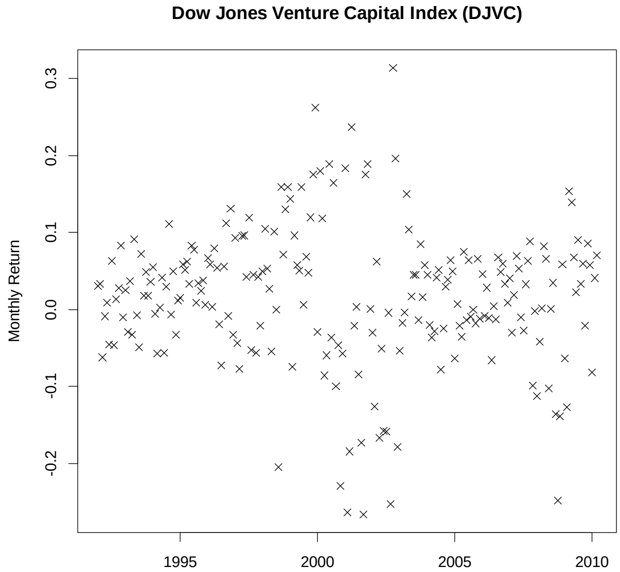
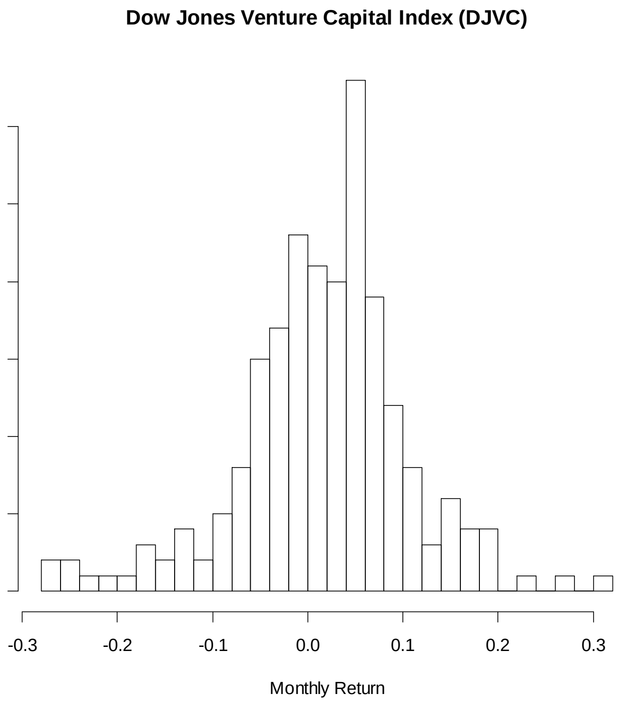


Figure 87: Dow Jones Venture Capital (DJVC) index, historical total return (left) and monthly returns (right).[12]



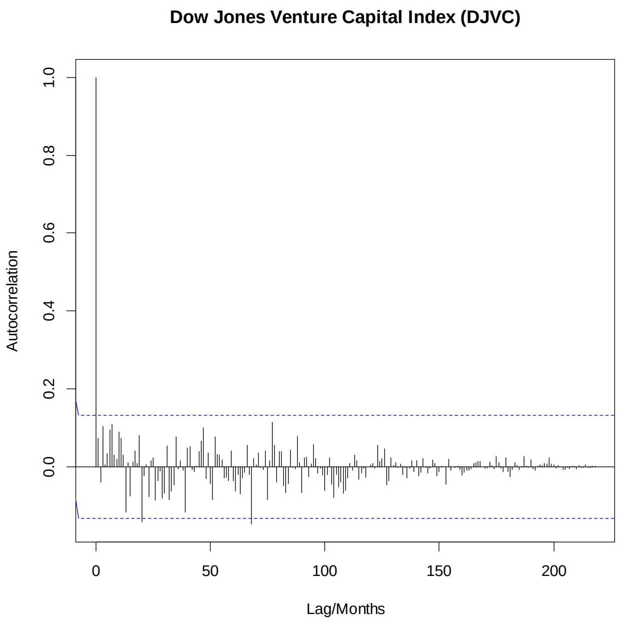


Figure 88: Dow Jones Venture Capital (DJVC) monthly returns histogram (left) and autocorrelation (right).

[12] Data retrieved from Sand Hill Econometrics in April 2013: www.sandhillecon.com/downloads/DJIndexOfVentureCapital-2010Q2.xls

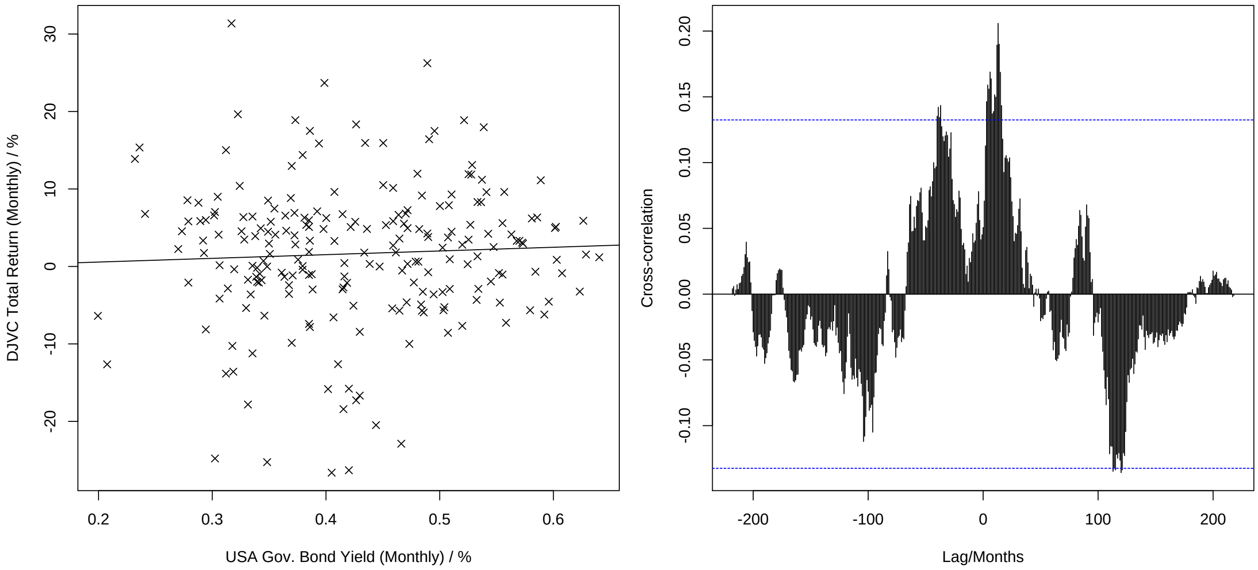


Figure 89: Scatter-plot (left) and cross-correlation (right) for the yields on USA government bonds with 10 year maturity versus the monthly returns of the Dow Jones Venture Capital (DJVC), during the period December 1991 to March 2010.

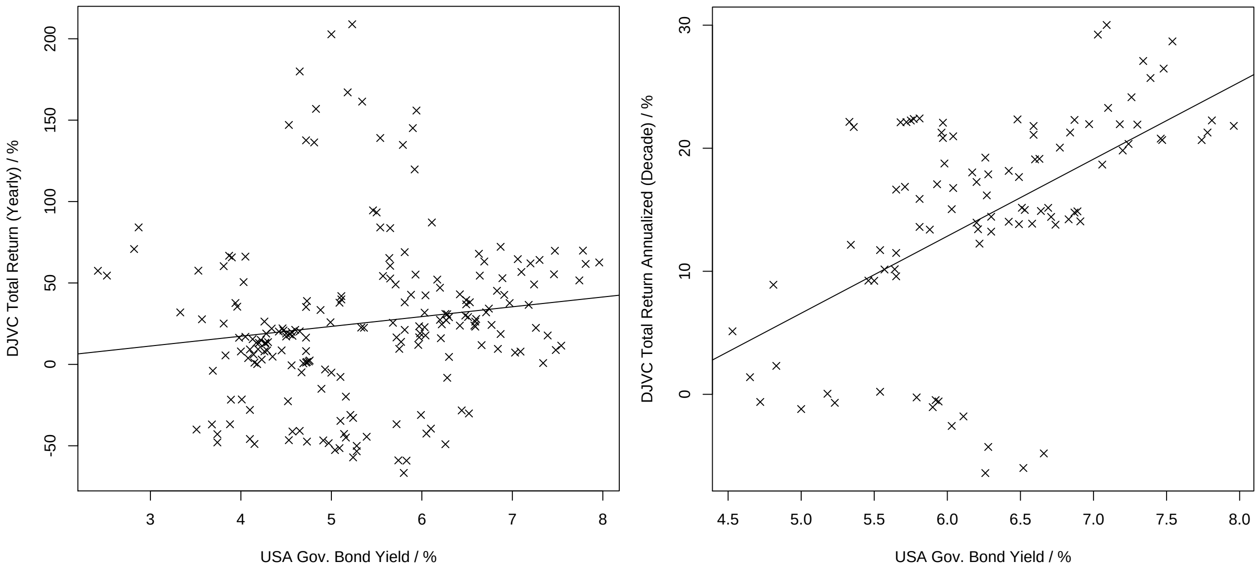


Figure 90: Scatter-plots for the yields on USA government bonds with 10 year maturity versus the one-year (left) and ten-year (right) returns of the Dow Jones Venture Capital (DJVC) index, during the period December 1991 to March 2010.

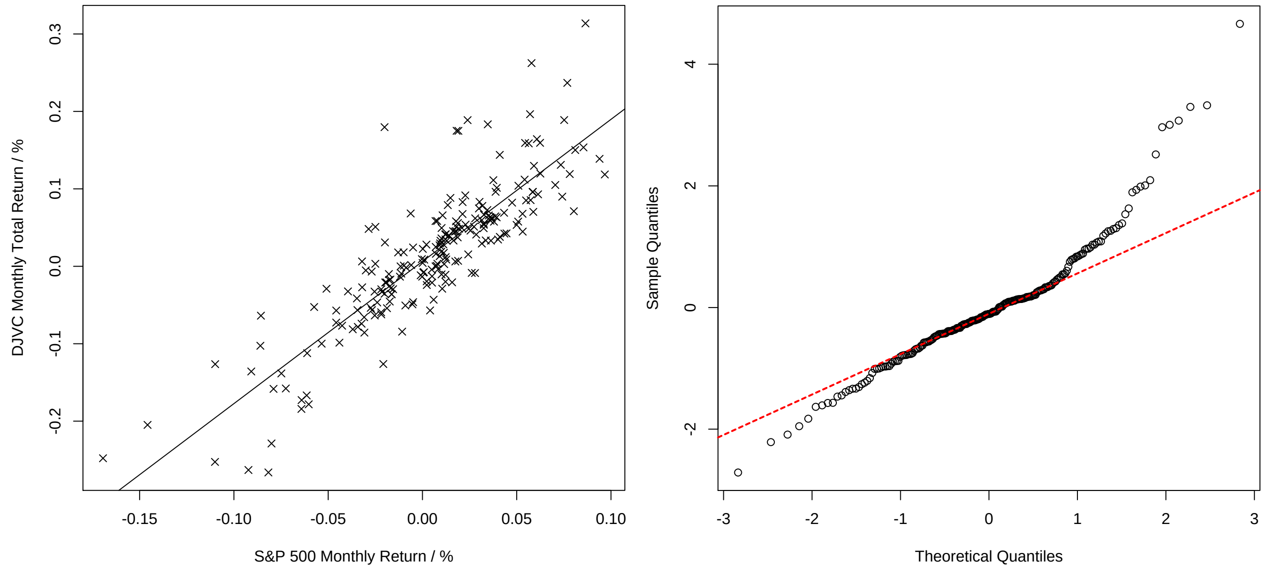


Figure 91: Scatter-plot with linear fit (left) and QQ-plot of the residuals from the linear fit (right) for the monthly returns of the S&P 500 stock market index versus the Dow Jones Venture Capital (DJVC) index, during the period December 1991 to March 2010.

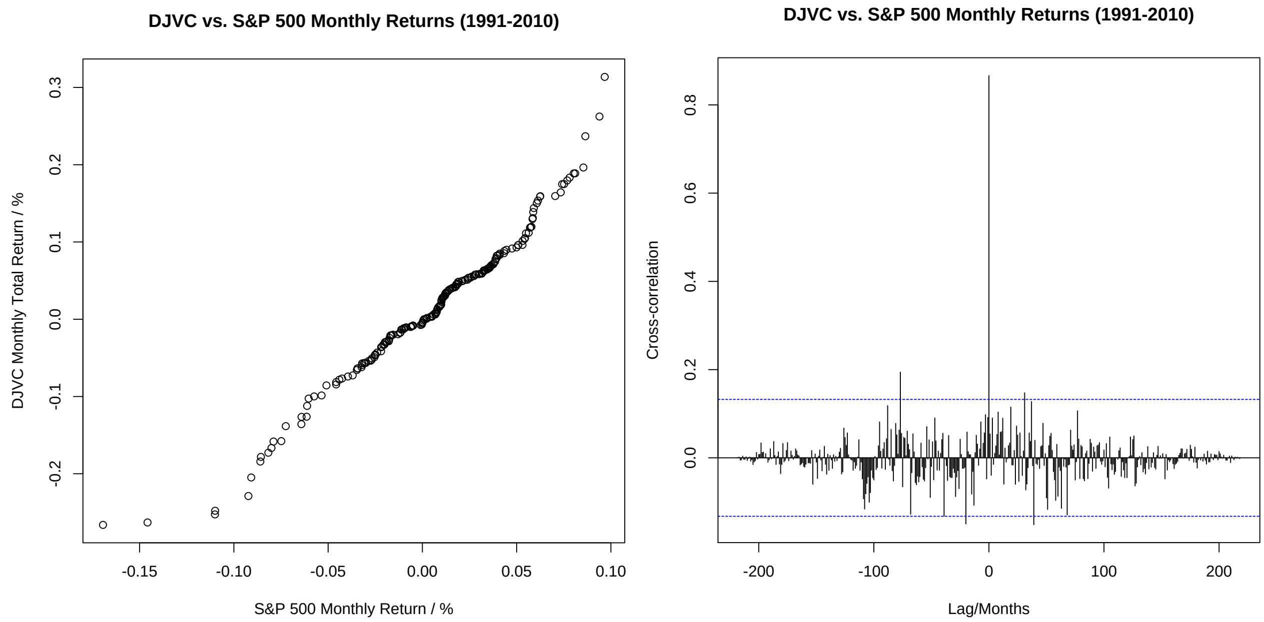


Figure 92: QQ-plot (left) and cross-correlation (right) for the monthly returns of the S&P 500 stock market index versus the Dow Jones Venture Capital (DJVC) index for the period December 1991 to March 2010.

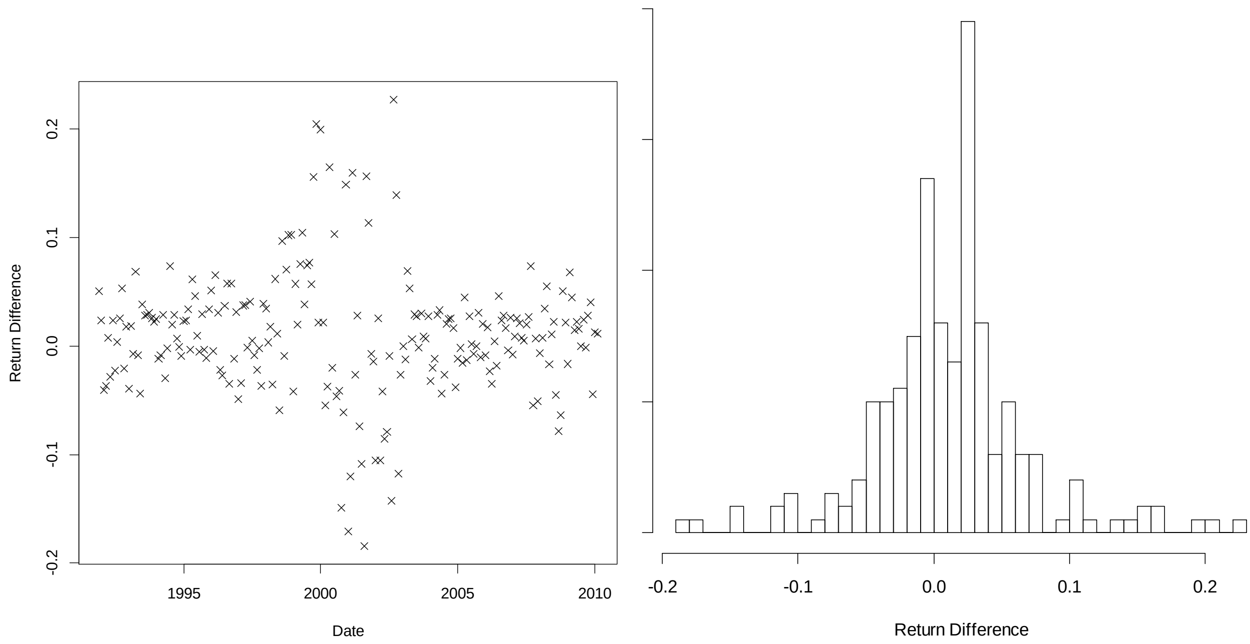


Figure 93: Difference between monthly returns of the Dow Jones Venture Capital (DJVC) index and the S&P 500 stock market index. Left plot shows the difference over time and right plot shows the histogram.

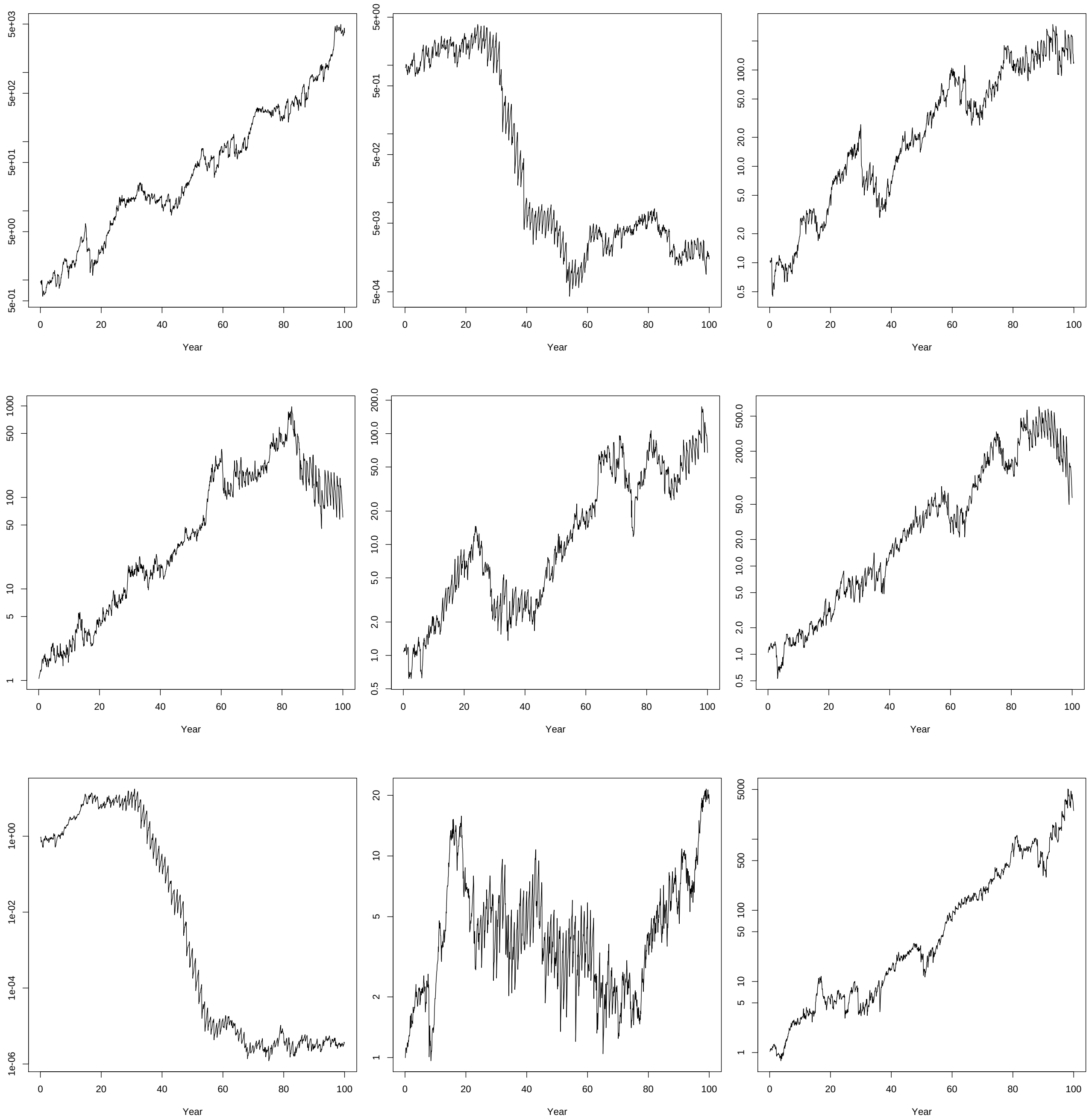


Figure 94: Monte Carlo simulated and compounded returns of the Dow Jones Venture Capital (DJVC) index as described in section 8.3. Normalized to begin at 1. The y-axes are logarithmic.

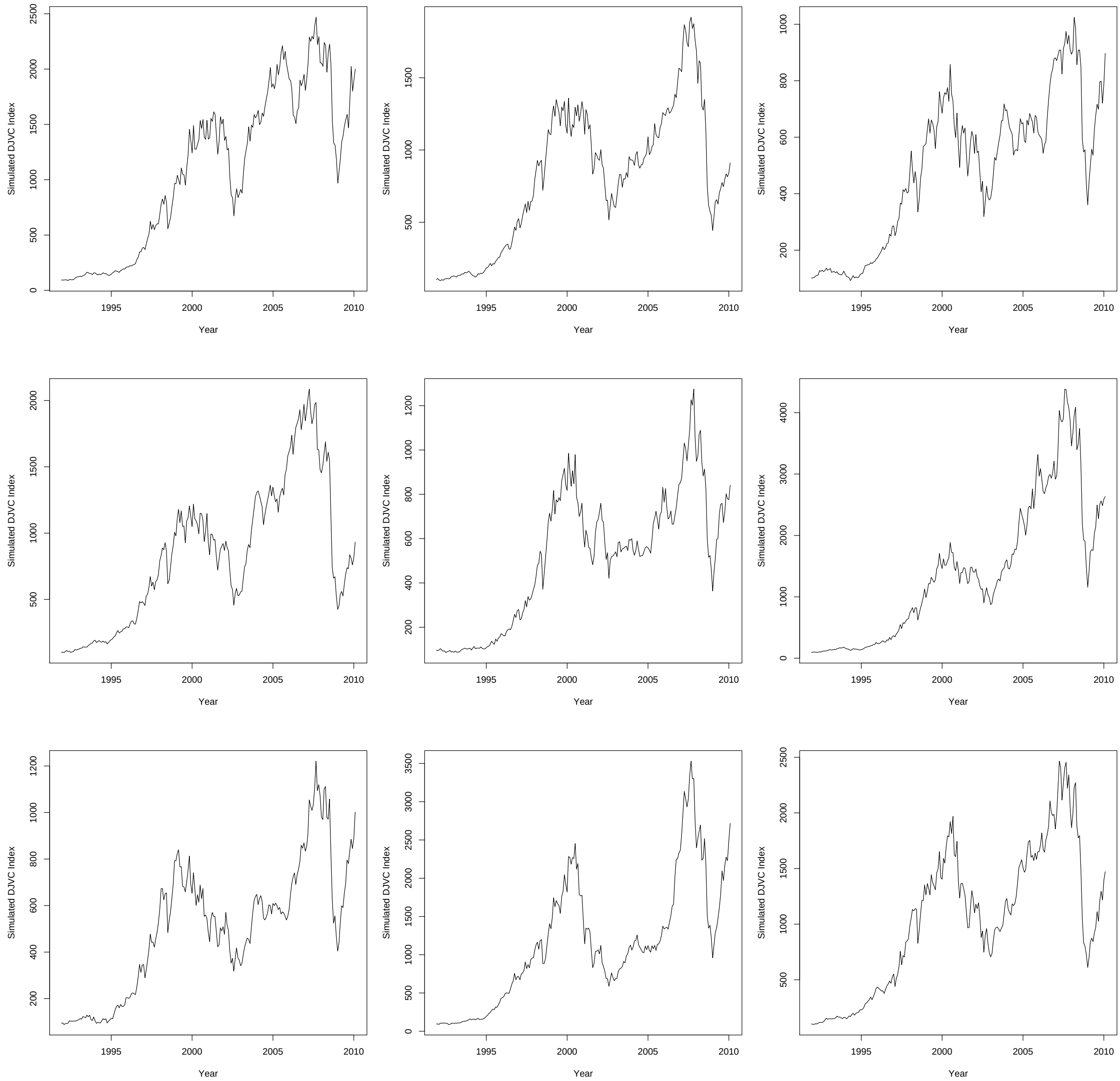


Figure 95: <u>Dow Jones Venture Capital (DJVC)</u> index <u>simulated from</u> the monthly <u>historical returns of the S&P 500</u> stock market index and the <u>linear model</u> in Eq. 8-2. Compare these plots to Figure 87 for the actual DJVC index during this period.

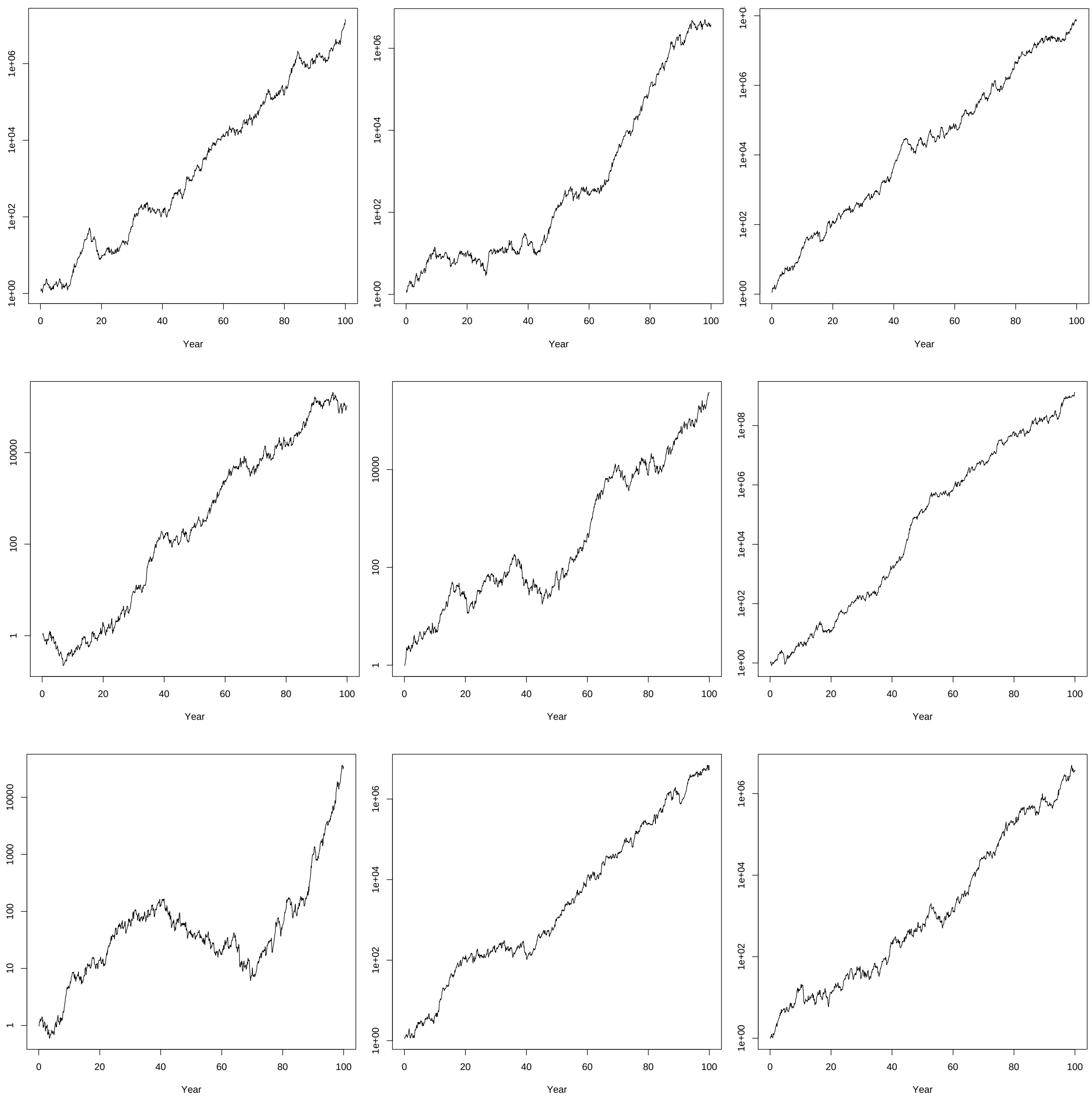


Figure 96: Compounded returns of the Dow Jones Venture Capital (DJVC) index which are simulated by randomly sampling and compounding the historical monthly returns of the DJVC index. Normalized to begin at 1. The y-axes are logarithmic.

Year	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
ROA	19.7%	21.2%	21.5%	23.2%	25.5%	20.9%	12.7%	10.1%	19.1%	13.6%
ROE	56.0%	55.7%	57.0%	64.8%	67.1%	48.6%	28.9%	22.9%	42.6%	26.8%
Dividend/Earnings	40.6%	39.4%	37.2%	35.7%	33.6%	41.9%	65.0%	77.4%	45.1%	65.1%
Net Buyback/Earnings	24.6%	44.0%	57.3%	40.0%	26.9%	35.7%	(6.3)%	(9.1)%	2.8%	19.1%
Retain/Earnings	34.8%	16.6%	5.6%	24.3%	39.5%	22.4%	41.3%	31.7%	52.0%	15.7%
Equity/Assets	38.1%	37.7%	35.8%	38.1%	43.1%	43.9%	44.0%	44.7%	50.7%	48.2%

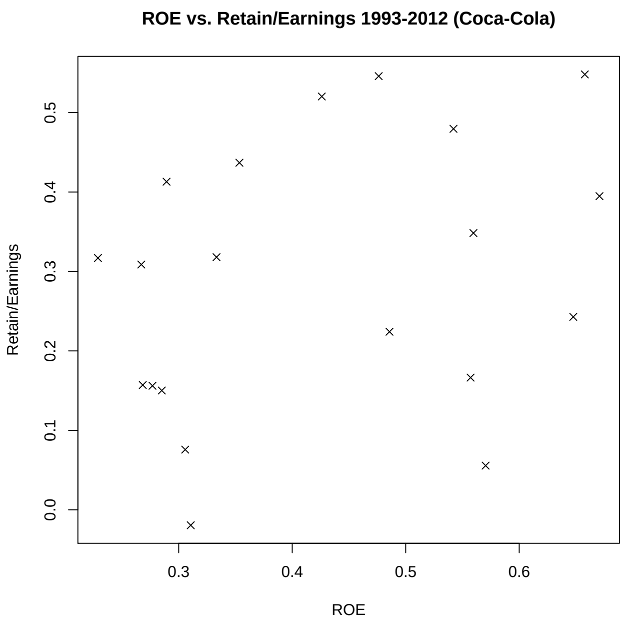
Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
ROA	31.7%	27.9%	15.5%	17.3%	20.0%	13.4%	16.8%	24.3%	11.8%	11.3%
ROE	65.8%	54.2%	30.6%	31.1%	35.3%	26.7%	33.3%	47.6%	27.7%	28.5%
Dividend/Earnings	27.9%	31.8%	55.0%	57.3%	52.7%	60.6%	55.7%	34.4%	50.1%	50.9%
Net Buyback/Earnings	17.3%	20.2%	37.5%	44.6%	3.7%	8.5%	12.5%	11.0%	34.3%	34.0%
Retain/Earnings	54.8%	48.0%	7.6%	(2%)	43.7%	30.9%	31.8%	54.6%	15.6%	15.0%
Equity/Assets	51.4%	50.7%	55.6%	56.5%	50.3%	50.5%	51.0%	42.5%	39.6%	38.1%

Table 14: Financial data for Coca-Cola. Calculated using Eq. 2-11 for ROE and similarly for ROA, and Eq. 2-7 for $Retain/Earnings$. Statistics are shown in Table 15. This data is available for download from the internet as described in section 1.2.[13]

	Mean	Geo. Mean	Stdev
ROA	18.9%	18.0%	5.9%
ROE	42.6%	41.8%	15.0%
Dividend/Earnings	47.9%	46.2%	13.3%
Net Buyback/Earnings	22.9%	-	18.1%
Retain/Earnings	29.2%	28.1%	17.0%
Equity/Assets	45.0%	45.1%	6.6%

Table 15: Statistics for Table 14.

[13] Form 10-K annual reports filed with US SEC:
www.sec.gov/cgi-bin/browse-edgar?action=getcompany&CIK=0000021344&type=10-K



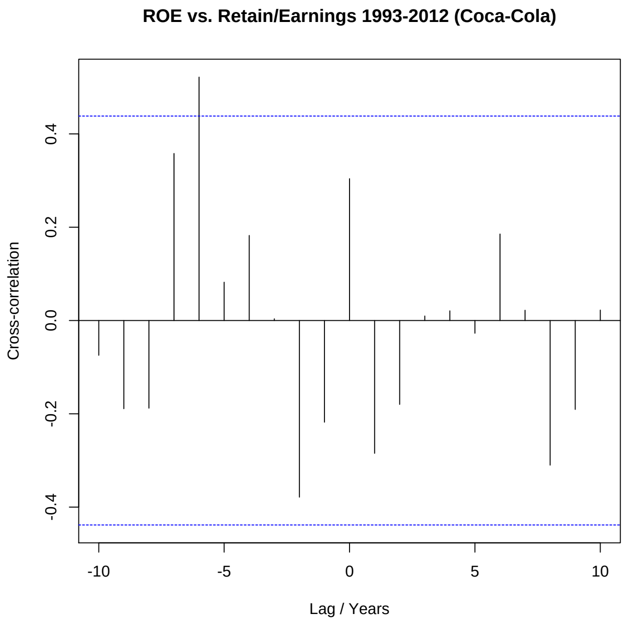


Figure 97: Scatter-plot (left) and cross-correlation (right) for the ROE and $Retain/Earnings$ ratios of Coca-Cola during the period 1993-2012. Data is shown in Table 14.

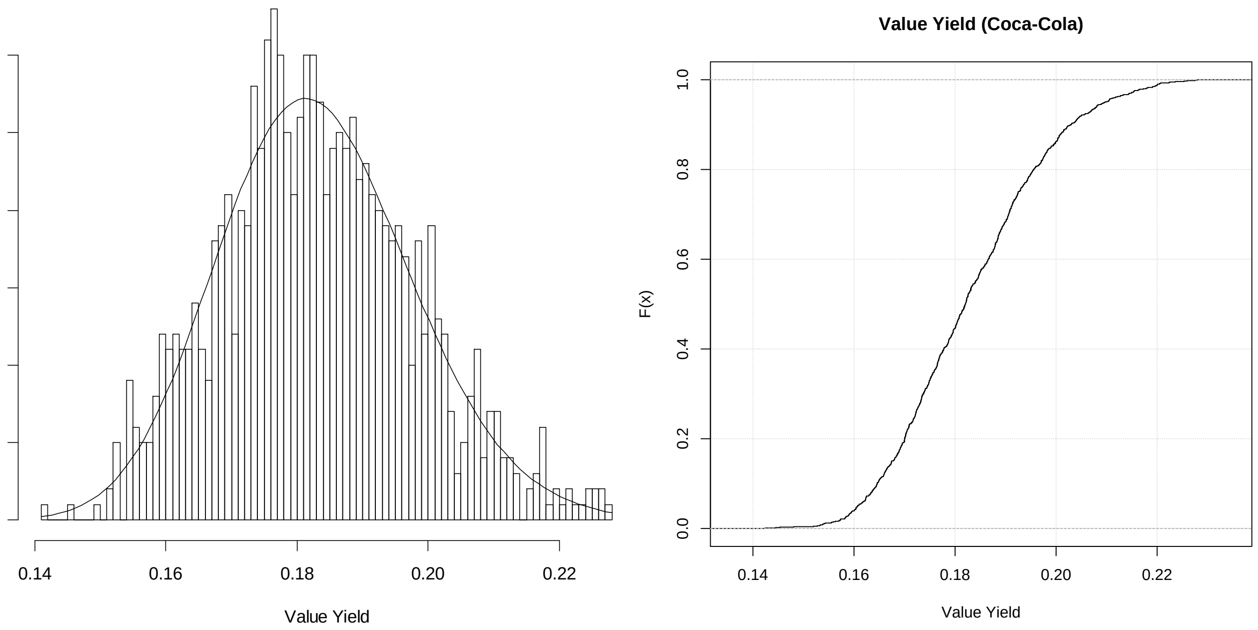


Figure 98: Histogram with fitted log-normal PDF (left) and CDF (right) for the value yield of the Coca Cola company when its P/Book ratio is 5.5 and its future payouts result from Monte Carlo simulations as described in section 9.2. The distribution converges to a log-normal PDF as the number of simulations increase.

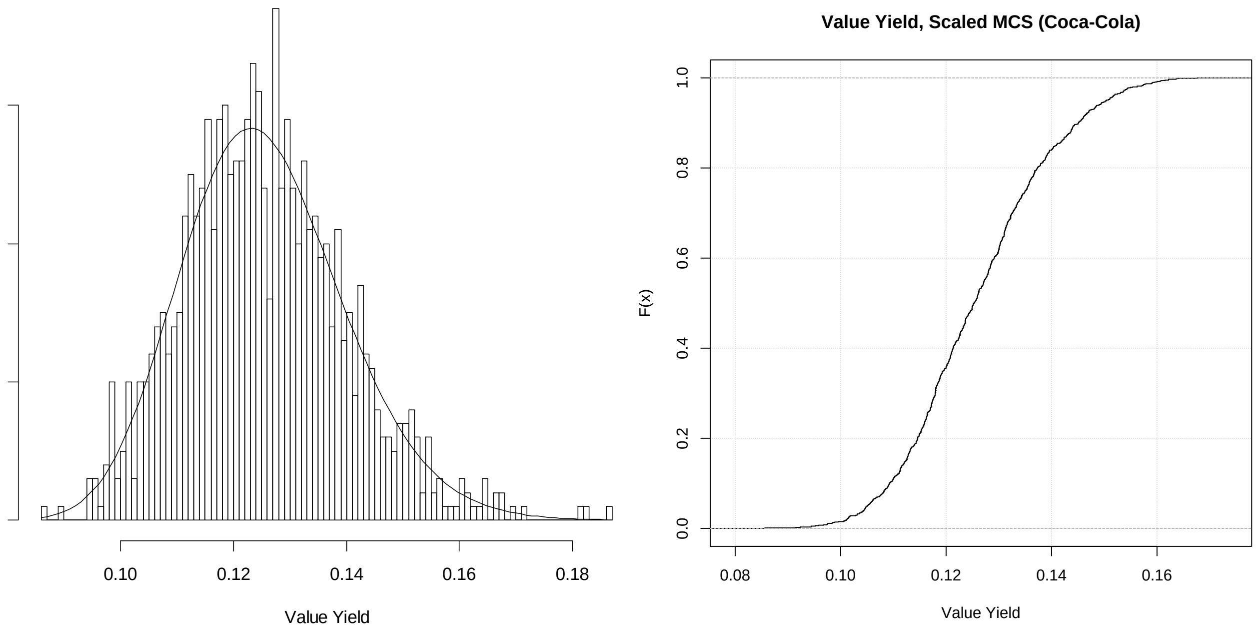


Figure 99: Histogram with fitted log-normal PDF (left) and CDF (right) for the value yield of the Coca Cola company when its P/Book ratio is 5.5 and its future payouts result from Monte Carlo simulations where the ROE and $Retain/Earnings$ ratios are decreased exponentially over time so as to make the equity growth rate converge to that of the S&P 500 index, as described in section 9.4. The distribution converges to a log-normal PDF as the number of simulations increase.

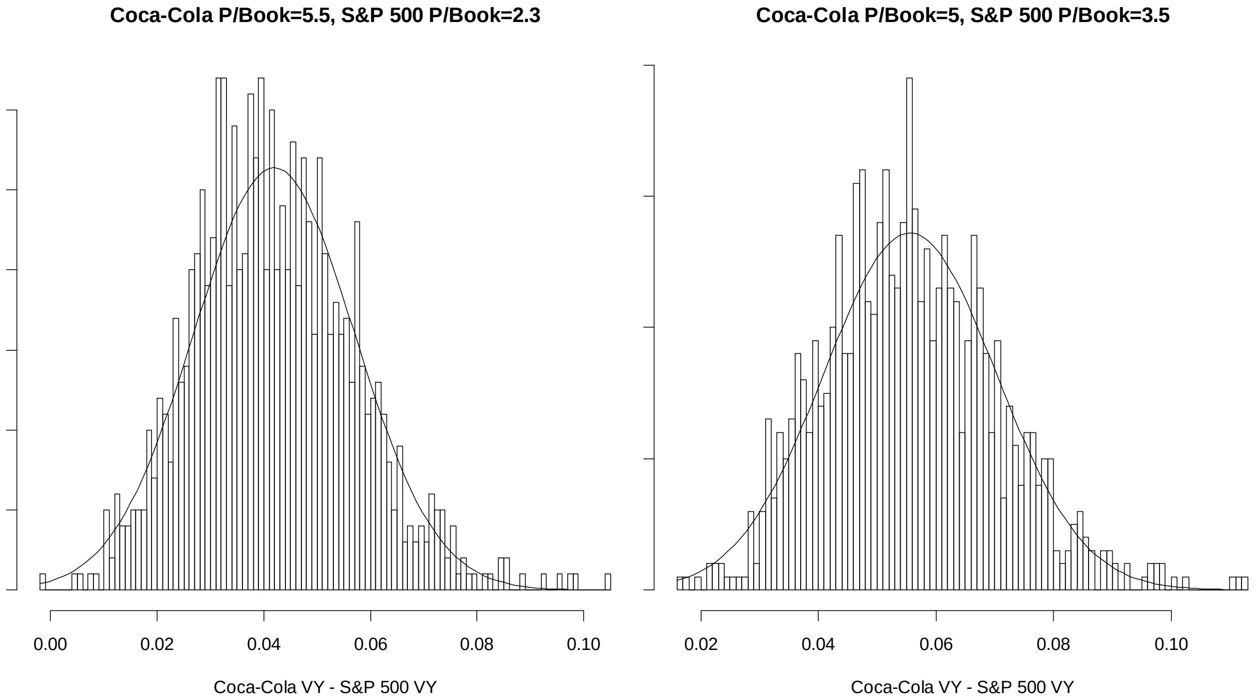


Figure 100: Difference between the value yield of Coca-Cola and the value yield of the S&P 500 stock market index for different P/Book ratios. The Monte Carlo simulations are described in section 9.5. Also shown are the fitted normal PDFs.

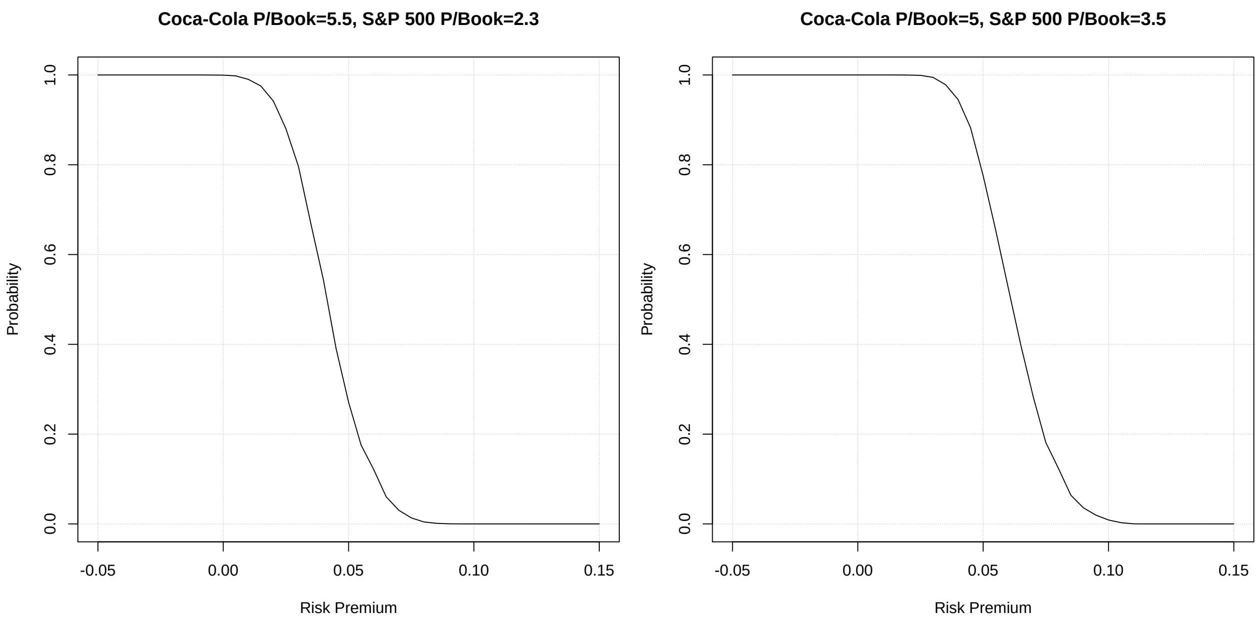


Figure 101: Probability of the value yield of Coca Cola being greater than the value yield of the S&P 500 stock market index for different P/Book ratios and with varying risk premiums: $\boldsymbol{Pr}[\boldsymbol{Value\ Yield_{Coke}} > Value\ Yiel\boldsymbol{d}_{S\&P\ 500} + \boldsymbol{Risk\ Premium}]$. The Monte Carlo simulations are described in section 9.5.

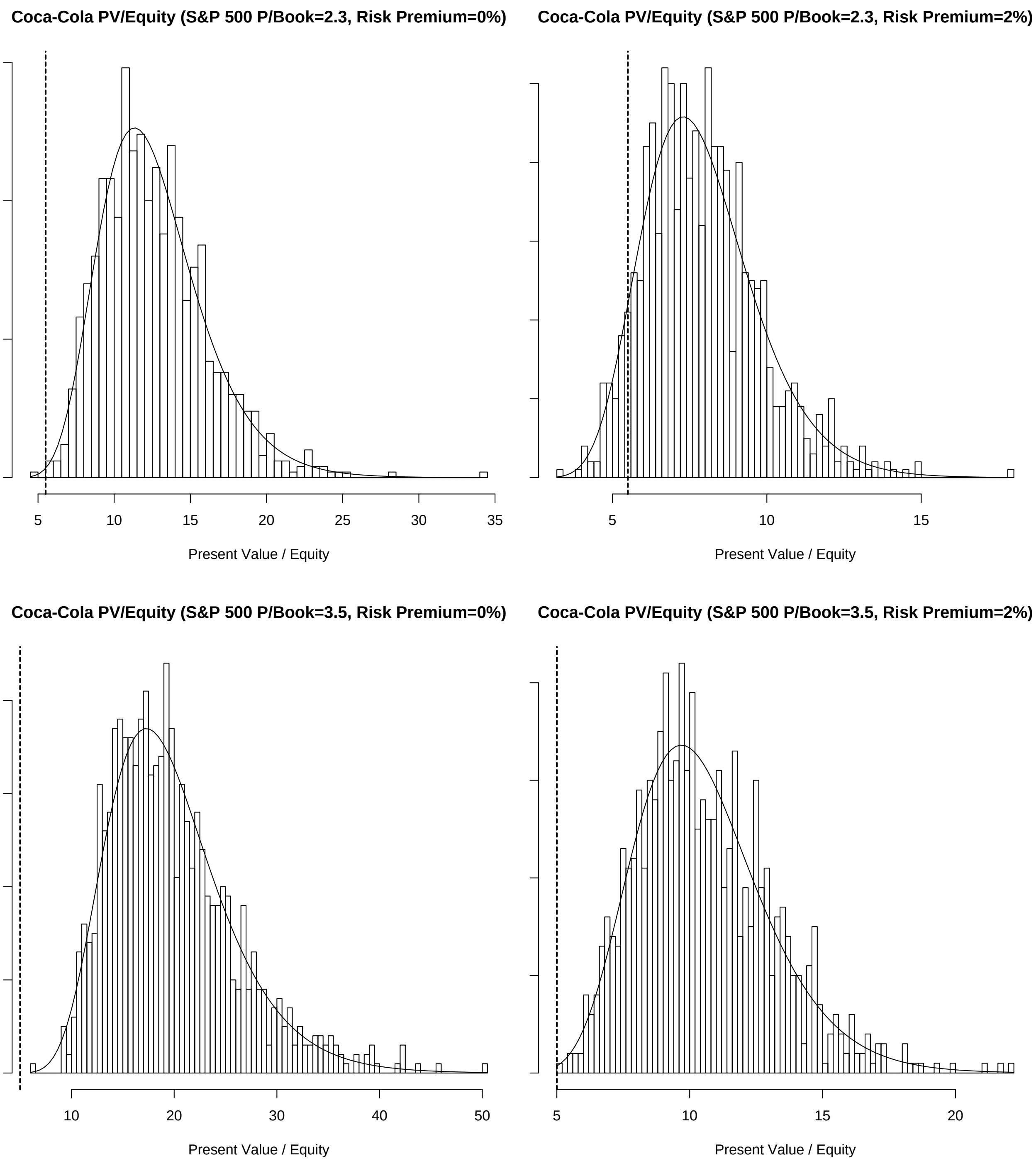


Figure 102: Probability distributions for the present value of Coca-Cola when the future payouts are Monte Carlo simulated as described in section 9.4, and the discount rates used in the present value calculations are sampled from the log-normal distribution of the value yield for the S&P 500 index described in section 6.11.1. The present values are normalized by dividing with the equity and should therefore be compared to the company's P/Book ratio which is currently about 5.5 and shown as dashed lines in these plots. Also shown are the fitted log-normal PDFs. The CDFs are shown in Figure 103.

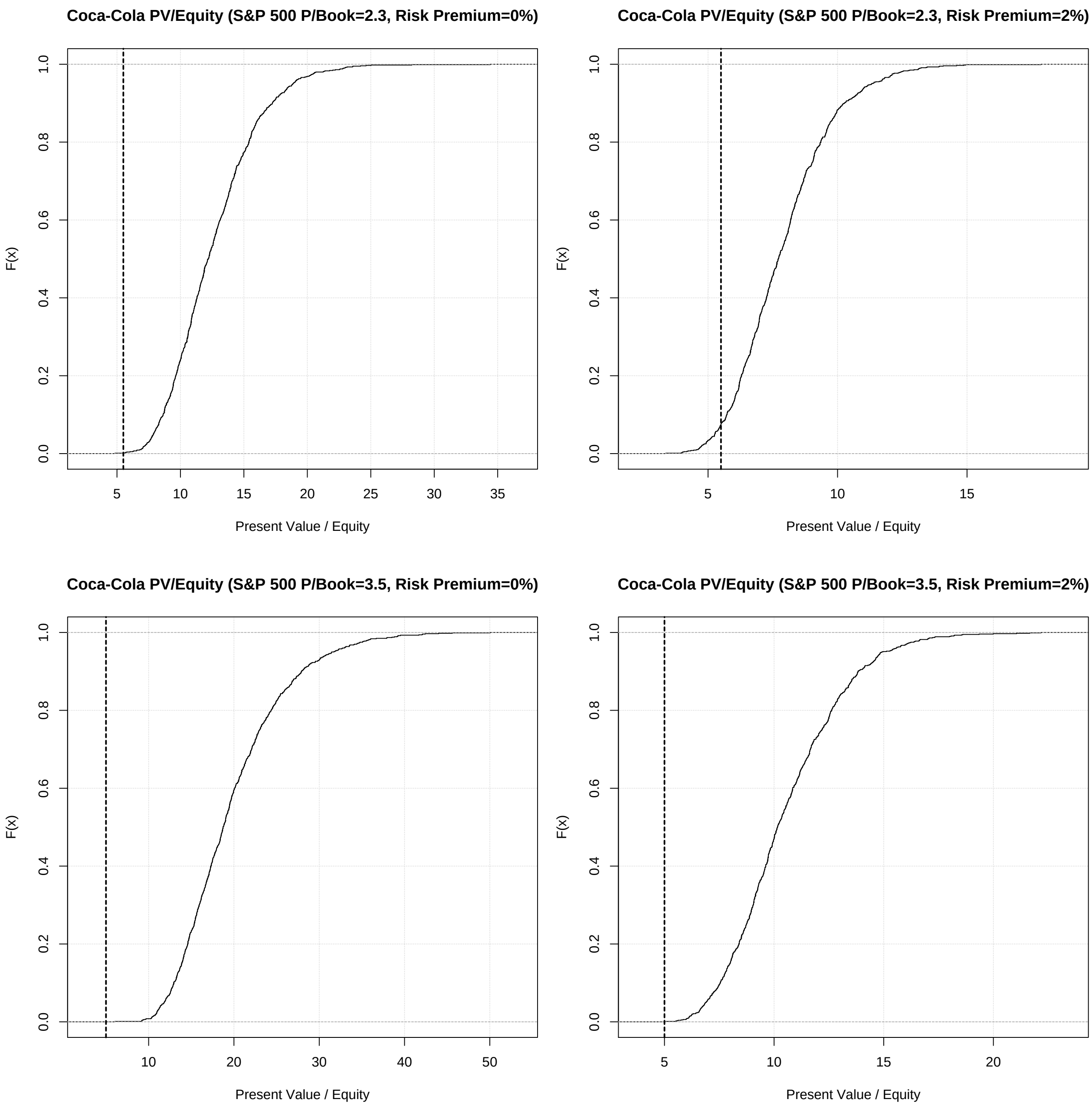


Figure 103: CDF for the present value of Coca-Cola when the future payouts are Monte Carlo simulated as described in section 9.4, and the discount rates used in the present value calculations are sampled from the log-normal distribution of the value yield for the S&P 500 index described in section 6.11.1. The present values are normalized by dividing with the equity and should therefore be compared to the company's P/Book ratio which is currently about 5.5 and shown as dashed lines in these plots. The histograms are shown in Figure 102.

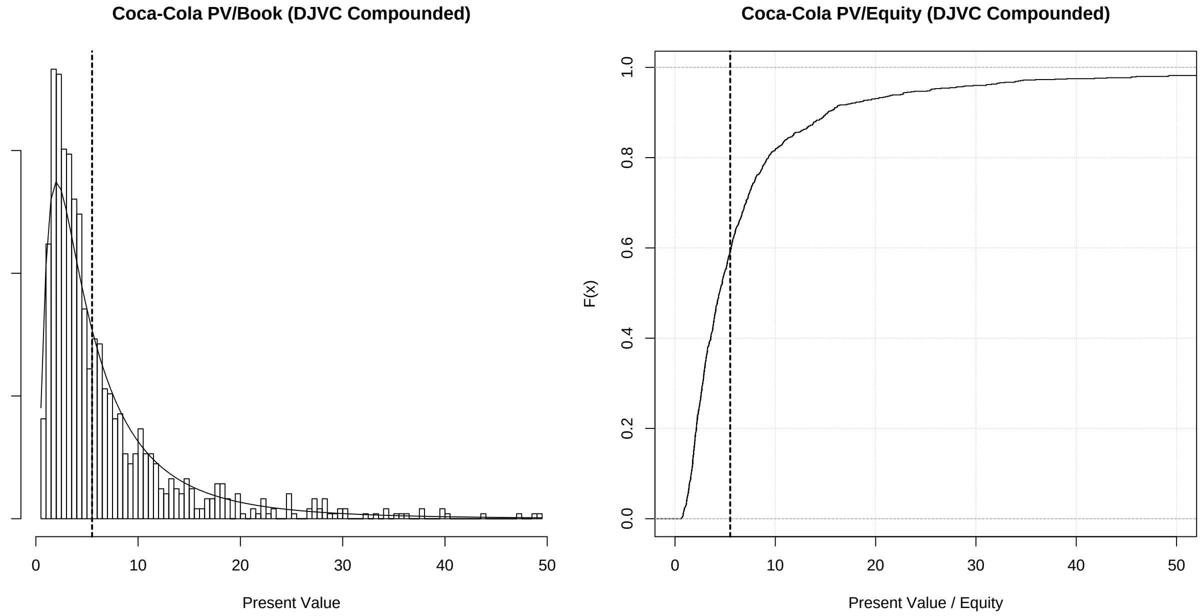


Figure 104: Histogram with fitted log-normal PDF (left) and CDF (right) for the present value of Coca-Cola when the future payouts are Monte Carlo simulated as described in section 9.4, and the discount rates used in the present value calculations are simulations of the compounded returns on the Dow Jones Venture Capital (DJVC) index as described in section 9.6.2. No risk premium is used here. The present values are normalized by dividing with the equity and should therefore be compared to the company's P/Book ratio which is currently about 5.5 and shown as dashed lines in these plots. There is a single present value above 800 in this distribution but the plots have been limited to only show present values below 50 to improve legibility.

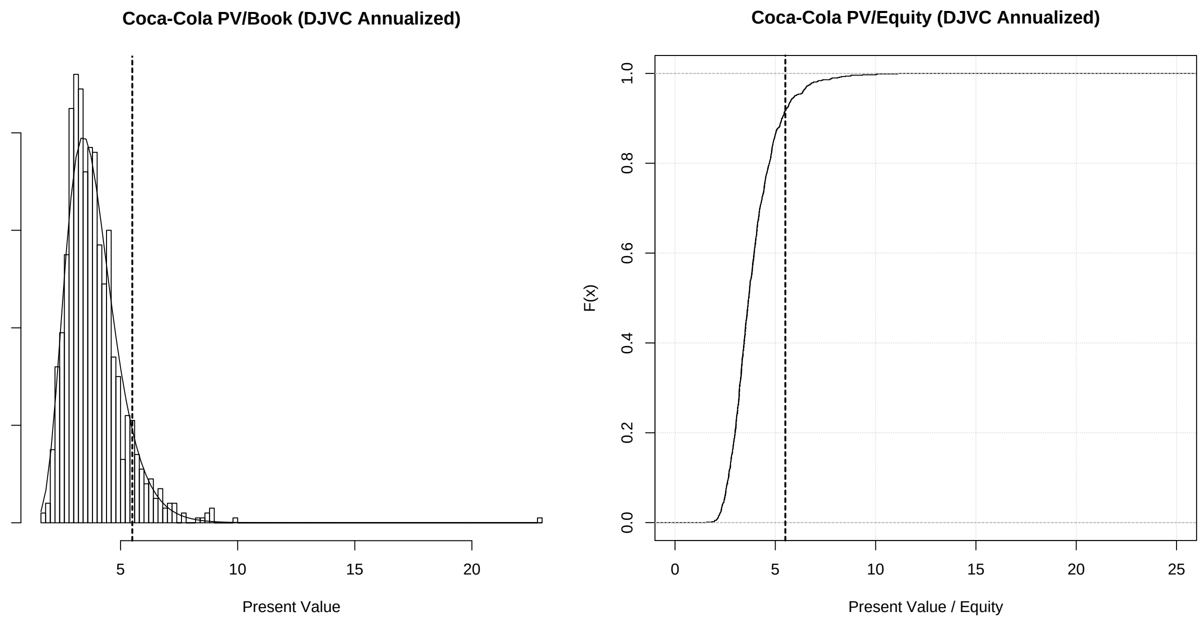


Figure 105: Histogram with fitted log-normal PDF (left) and CDF (right) for the present value of Coca-Cola when the future payouts are Monte Carlo simulated as described in section 9.4, and the discount rates used in the present value calculations are simulations of the annualized returns on the Dow Jones Venture Capital (DJVC) index as described in section 9.6.2. No risk premium is used here. The present values are normalized by dividing with the equity and should therefore be compared to the company's P/Book ratio which is currently about 5.5 and shown as dashed lines in these plots.

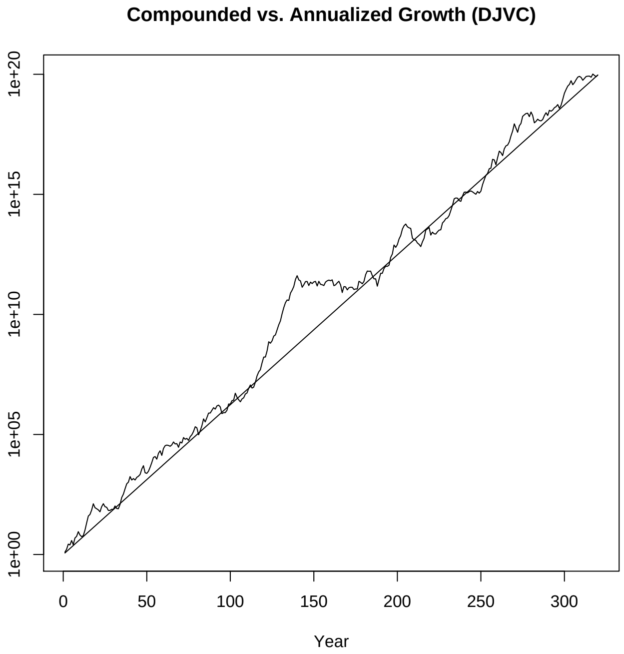


Figure 106: Comparison of <u>compounded (volatile line)</u> and <u>annualized (straight line)</u> growth which show identical results for the last year. The y-axis is logarithmic.

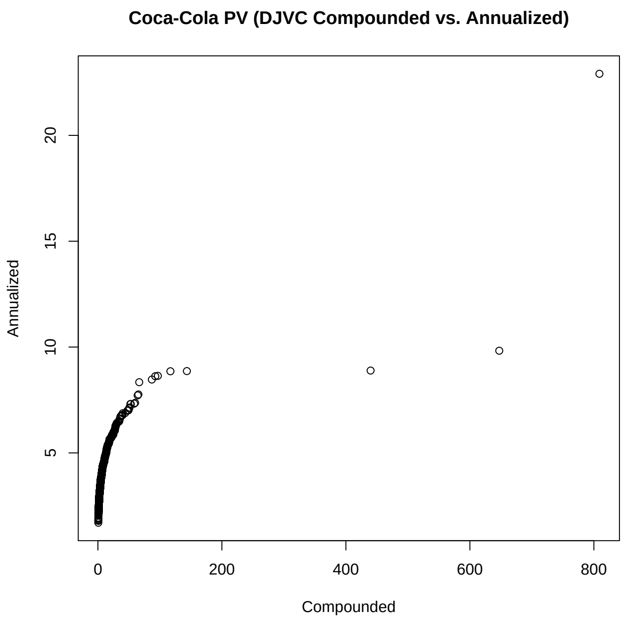


Figure 107: Q-Q plot comparing the probability distributions for the <u>present value</u> of <u>Coca-Cola</u> when the discount rate is the compounded (x-axis) and annualized (y-axis) returns of the DJVC index from Figure 104 and Figure 105.

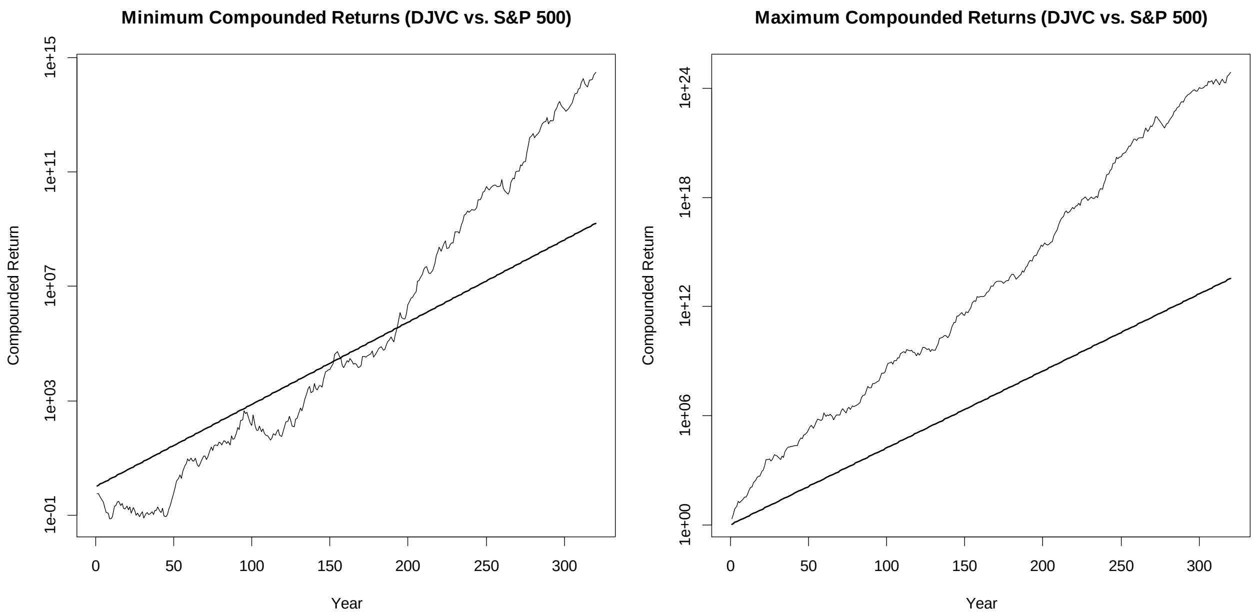


Figure 108: Simulated compounded returns for the S&P 500 index (straight lines) and the DJVC index (irregular lines). The left plot shows the lowest compounded returns and the right plot shows the highest compounded returns for the simulations in section 9.6, which are used as the discount rate in the calculation of the present value of Coca-Cola. The y-axes are logarithmic.

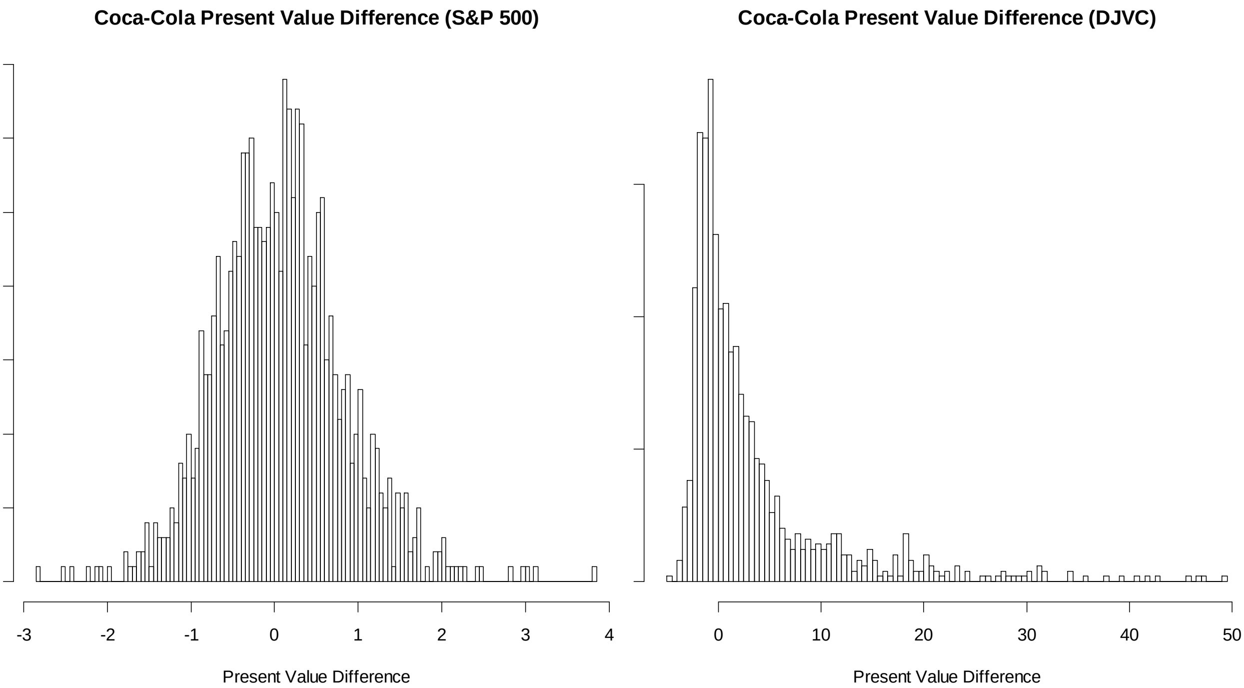


Figure 109: Present value of Coca-Cola with the discount rate being sampled from the stochastic value yield of the S&P 500 stock market index, minus the present value with the discount rate being the mean S&P 500 value yield (left plot). Similarly when the discount rate is being sampled from the annualized return of the DJVC venture index, minus the present value with the discount rate being the mean annualized return of the DJVC index (right plot, x-axis is limited at 50 for clarity). This shows the present value is significantly affected when calculated using the mean discount rate.

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13. Revision History

- 2014, February 21: Swapped axes on Figure 80, Figure 83, Figure 86, Figure 89, Figure 90. (160 pages)
- 2013, October 2: Extended section 6.10.3. (160 pages)
- 2013, September 28: Added several sections regarding share buyback and issuance. Rewrote section 6. Extended sections 7.1, 8.2. Added data to Table 1, Table 14. Added several figures. Changed $Equity_t$ indexing to start at zero instead of one. (159 pages)
- 2013, July 20: Replaced CDF plots in Figure 104 and Figure 105. Minor text revisions. (113 pages)
- 2013, June 4: Extended sections 7.1, 9.6. Added Table 4, Table 5, Figure 79, Figure 80, Figure 105, Figure 106, Figure 107. Minor clarifications and corrections. (113 pages)
- 2013, May 24: First edition. (109 pages)